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Methods and applications of distributed and decentralized Model Predictive Control

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I'm back.
Michael Jeffrey Jordan

Abstract

This thesis deals with the theoretical development of distributed and decentralized control algorithms based on Model Predictive Control (MPC) for linear systems subject to constraints on inputs and states. In all the presented techniques, the basic idea consists in considering the coupling terms among the subsystems as disturbances to be rejected. Part of this disturbance is assumed to be known over all the prediction horizon, while the remaining one is considered unknown but bounded. To reject this second term, a robust approach is implemented using polytopic invariant sets.

A regulation problem for distributed control is initially described, together with some practical solutions needed to deal with implementation issues. A continuous-time version of the proposed approach is also provided. Secondly, two different distributed solutions to the tracking problem are given. In the first one, developed for tracking piecewise constant setpoints, a fictitious reference signal is used to guarantee feasibility. The second one, instead, can be used for tracking constant setpoints and relies on the description of the dynamic system in the so-called "velocity-form", which allows one to insert an integral action in the closed-loop. The properties of systems described in velocity-form are then investigated for the centralized case. Finally, the results derived for the centralized systems in velocity-form are used to develop a completely decentralized approach with integral action for tracking piecewise constant references. Several simulation examples are reported to show the performances of the proposed algorithms.

Summary and publications

In recent years, process plants, electrical, communication and traffic networks and manufacturing systems have been characterized by an increasing complexity. Usually, they result to be composed by a huge number of relatively small or medium-scale subsystems interacting via inputs, states or outputs. The design of a centralized controller for the whole large-scale system is often a difficult challenge due to possible limitations, for instance, in computing capabilities or communication bandwidth. Reliability and robustness of the overall control system could represent additional reasons for avoiding centralized controllers.

According to these motivations, remarkable efforts have been put in the research field of hierarchical, decentralized and distributed control. Among all the possible solutions, particularly interesting appear to be those based on Model Predictive Control (MPC). Several reasons can be listed to support this claim: first of all, MPC is a multivariable optimal control technique and, among the advanced control methods, it is probably the only one really adopted by the industrial world and able to cope with operational constraints on inputs, states and outputs. Secondly, in the last years fundamental properties such as stability of the closed-loop systems and robustness with respect to several classes of external disturbances have been proved for many different MPC formulations. Lastly, when controlling large-scale systems in a distributed or decentralized framework, the values of inputs, states and outputs are explicitly computed over all the prediction horizon at each sampling time by a local controller designed with MPC and they can be used as

information to be transmitted to other local controllers to coordinate their actions. This data transmission can greatly simplify the design of a distributed control system and can allow one to obtain performances close to those of a centralized controller.

In this thesis, we present a number of distributed control algorithms based on the Distributed Predictive Control (DPC) approach. They are designed for linear systems subject to constraints on inputs and states, described in state space form and decomposable in several non-overlapping subsystems.

The main idea of DPC is that, at every sampling time, each subsystem transmits to its neighbors the reference trajectories of its inputs and states over all the prediction horizon. Moreover, by adding proper additional constraints to the MPC formulation, all the local controllers are able to guarantee that the real values of their inputs and states lie in a specified invariant neighbor of the corresponding reference trajectories. In this way, each subsystem has to solve an MPC problem where the reference trajectories received from the other controllers represent a disturbance known over all the prediction horizon, while the differences between the reference and the real values of its neighbors' inputs and states can be seen as an unknown bounded disturbance to be rejected. To this end, a robust tube-based MPC formulation is adopted and implemented using the theory of polytopic invariant sets.

This thesis also presents a completely decentralized version of DPC, named DePC (Decentralized Predictive Control) for strongly decoupled systems. In this case, no transmission of information is required, as all the interactions via inputs and states are seen as unknown disturbances to be rejected.

The thesis is structured as follows. Chapter 1 provides an overview of the most common centralized MPC algorithms for regulation, tracking and disturbance rejection, as well as a short description of the main solutions for distributed and decentralized control, with particular attention to those based on MPC.

In Chapter 2, the basic formulation of DPC for regulation of discrete-time systems is presented [BC1]. Efficient techniques for the computation of polytopic robust invariant sets, for the initialization of reference trajectories and for the online solutions to possible large and unexpected disturbances are also shown [J4], in order to provide both the theoretical results and some practical hints useful for real industrial applications. Since the discrete-time domain does not allow to consider the process inter-sampling behavior in the MPC optimization problem, in Chapter 3 a continuous-time version of DPC is illustrated [J3].

Chapters 4 and 5 present two extensions of DPC for the solution of the tracking problem: the first one, see [C3, J3], can be used to track piecewise constant targets. The main improvement with respect to the standard DPC formulation consists in including among the optimization variables of the MPC problem the value of the reference point that each subsystem really tracks at each sampling time. An additional term in the cost function is also considered to penalize the distance of this extra optimization variable from the desired external set point. The second solution [C1] is based on inserting an integral action in the closed-loop by rewriting the initial system in the so called “velocity-form”. This formulation guarantees rejection of constant disturbances and can be used to track constant targets: in fact, recursive feasibility can not be proved if the value of the reference signal is changed.

In Chapter 6, to overcome the limitation of being able to manage only constant references for systems in velocity-form, a thorough study of their properties for centralized control is described for both the case of nominal [C2] and disturbed [J1] systems. The obtained results allow one to use the velocity-form also in presence of varying set points. Once again, time-varying external references are handled adding as optimization variable the set point really tracked at each time instant. The cost function term weighting its distance from the external one guarantees an asymptotic convergence of the first one to the latter.

The outcomes of the previous developments for centralized control of systems in velocity-form have been exploited to develop a decentralized control method for tracking, called DePC [C4], and presented in Chapter 7. It is able to manage piecewise constant set points and to reject external disturbances, asymptotically deleting their effects in case they are constant.

Eventually, some conclusions are drawn in Chapter 8.

List of publications

International journals

J1. G. Betti, M. Farina, and R. Scattolini. A robust MPC algorithm for offset-free tracking of constant reference signals. *IEEE Transactions on Automatic Control*, 58(9):2394-2400, 2013.

J2. M. Farina, G. Betti, L. Giulioni, and R. Scattolini. An approach to distributed predictive control for tracking - theory and applications. *IEEE Transactions on Control Systems Technology*, in press.

J3. M. Farina, G. Betti, and R. Scattolini. Distributed predictive control of continuous-time systems. *Systems and Control Letters* (submitted).

J4. G. Betti, M. Farina, and R. Scattolini. Implementation issues and applications of a distributed predictive control algorithm. *Journal of Process Control* (submitted).

International conferences proceedings

C1. G. Betti, M. Farina, and R. Scattolini. Distributed predictive control for tracking constant references. In *American Control Conference (ACC), 2012*, pages 6364-6369.

C2. G. Betti, M. Farina, and R. Scattolini. An MPC algorithm for offset-free tracking of constant reference signals. In *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, pages 5182-5187.

C3. M. Farina, G. Betti, and R. Scattolini. A solution to the tracking problem using distributed predictive control. In *European Control Conference (ECC), 2013*, pages 4347-4352.

C4. G. Betti, M. Farina, and R. Scattolini. Decentralized predictive control for tracking constant references. In *Decision and Control (CDC), 2013 IEEE 52st Annual Conference on*, pages 5228-5233.

Book chapters

BC1. G. Betti, M. Farina, and R. Scattolini. Distributed MPC: a noncooperative approach based on robustness concepts. In J.M. Maestre and R.R. Negenborn, editors, *Distributed Model Predictive Control made easy*, volume 69 of *Intelligent Systems, Control and Automation: Science and Engineering*, pages 421 - 435. Springer Netherlands, 2014.

Contents

Part I

Introduction

1 Introduction

This chapter introduces the basic ideas underlying robust MPC based on polytopic invariant sets and a short review on distributed and decentralized predictive control. The goal is to provide in a simple and compact form the main techniques used in the following chapters, as well as to clearly describe the scientific context of this thesis. To improve the readability of the thesis, all the information about the adopted notation has been listed at the beginning of this Chapter.

1.1 Notation

The symbols and recurring terms used throughout the thesis are listed below.

- The discrete-time index is denoted by k, and the dependence on time of the variables of discrete-time systems is denoted by a subscript, e.g. x_k, u_{k+1}. The continuous-time index is denoted by t, and the dependence on time of the variables of continuous-time systems is represented in brackets, e.g. $x(t)$, $u(t)$.

- For a discrete-time signal s_k and $a, b \in N$, $a \leq b$, we denote $(s_a; s_{a+1}; \ldots; s_b)$ with $s_{[a:b]}$. For a continuous-time variable $s(t)$ and a given time interval $\mathcal{T} \subseteq \mathbb{R}_+$ ($\mathcal{T}$ can be open or closed), the trajectory $s(t)$ with $t \in \mathcal{T}$ is denoted by $s(\mathcal{T})$.

- The short-hand $v = (v_1, \ldots, v_s)$ denotes a column vector with s (not necessarily scalar) components $v_1, \ldots, v_s$.

- Only when distributed or decentralized techniques are discussed, the large-scale system (centralized) matrices and vectors are written in bold. In all the other cases, standard fonts are used to indicate also possibly non-scalar elements.

- The expression $\|x\|_Q^2$ stands for $x^[T]Qx$, where x is a column vector and x^T is the transpose of x.

- I_α stands for an identity matrix of order α. Where clear from the context, 0 is used to represent a matrix of proper dimensions with all its elements equal to zero.

- All the eigenvalues of a Schur matrix have absolute value strictly less than 1. A matrix is said to be Hurwitz if all its eigenvalues have negative real part.

- $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ are the maximum and the minimum eigenvalues of a matrix, respectively.

- A polyhedron is a subset of $\mathbb{R}^n$ defined by the intersection of a finite number of closed half-spaces, i.e. a polyhedron $\mathbb{X}$ is a set defined by a finite number of inequalities: $\mathbb{X} \triangleq \{x \in \mathbb{R}^n | f_i'x \leq g_i, i \in \mathbb{I}\}$ where $f_i \in \mathbb{R}^n$, $g_i \in \mathbb{R}$ and $\mathbb{I}$ is a finite index set. A polytope is a bounded polyhedron.

- Given two sets $\mathbb{A}$ and $\mathbb{B}$, their Minkowski sum (set addition) is $\mathbb{A} \oplus \mathbb{B} \triangleq \{a + b | a \in \mathbb{A}, b \in \mathbb{B}\}$. Their Minkowski, or Pontryagin, difference (set subtraction) is $\mathbb{A} \ominus \mathbb{B} \triangleq \{a | a \oplus \mathbb{B} \in \mathbb{A}\}$. We denote by $\bigoplus_{i=1}^M P_i$ the Minkowski sum of the sets $\{P_1, ..., P_M\}$.

- A generic p-norm ball centered at the origin in the $\mathbb{R}^{dim}$ space is defined as follows: $\mathbb{B}_{p,\varepsilon}^{(dim)}(0) := \{x \in \mathbb{R}^{dim} : \|x\|_p \leq \varepsilon\}$.

- Given a generic compact set $\mathbb{L}$, $\mathbb{H} = \text{box}(\mathbb{L})$ is the smallest hyperrectangle containing $\mathbb{L}$ with faces perpendicular to the cartesian axis.

- $\text{int}(\mathbb{X})$ denotes the interior of set $\mathbb{X}$.

- A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{K}_\infty$ function if and only if *i)* $\alpha(0) = 0$, *ii)* it is strictly increasing and *iii)* $\alpha(s) \to +\infty$ as $s \to +\infty$.

1.2 Model Predictive Control

In this Section the fundamentals of Model Predictive Control are illustrated. MPC is probably the most widely used advanced control technique for control of industrial plants (see, e.g., [127,128]). Its main features, that made it particularly suited for several applications, are:

- the control problem is reformulated as an optimization one, in which it is possible to include different and possibly conflicting goals.
- In the control problem formulation it is possible to explicitly consider constraints on inputs, outputs and states. This is achieved by predicting the evolution of the system as a function of an admissible sequence of future control inputs.
- It is possible to design the controller also using empirical process models obtained, for instance, through step or impulse responses of the system.

An in-depth presentation is beyond the scope of this thesis: for detailed discussions, the reader is referred to the textbooks [20, 95, 136] and to the survey papers [56, 105].

1.2.1 Standard MPC formulation

Consider a time-invariant, linear, discrete-time system described by

$$x_{k+1} = Ax_k + Bu_k \tag{1.1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the input vector and the pair (A, B) is assumed to be reachable. The states and the inputs have to fulfill the constraints $x_k \in \mathbb{X} \subset \mathbb{R}^n$ and $u_k \in \mathbb{U} \subset \mathbb{R}^m$. Given a prediction horizon of duration equal to $N \in \mathbb{I}^+$ time steps, the goal at time step k is to compute the sequence of N control variables $u_{[k:k+N-1]} = (u_k, u_{k+1}, \ldots, u_{k+N-1})$ that minimizes the finite horizon cost function

$$V(x_k, u_{[k:k+N-1]}) = \sum_{\nu=0}^{N-1}(\|x_{k+\nu}\|_Q^2 + \|u_{k+\nu}\|_R^2) + \|x_{k+N}\|_P^2 \tag{1.2}$$

where $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{n\times n}$ are positive definite matrices. Note that the definite positiveness of matrix Q is not strictly required [136], and here is asked only to simplify the following proofs. The term $\|x_{k+\nu}\|_Q^2 + \|u_{k+\nu}\|_R^2$ represents the stage cost where matrices Q and R are design parameters, while matrix P appearing in the terminal cost $\|x_{k+N}\|_P^2$ has to be carefully selected in order to guarantee the convergence of the control algorithm. For the same reason, an additional terminal constraint $x_{k+N} \in \mathbb{X}_f$ is required as well, where the properties of the set $\mathbb{X}_f \subseteq \mathbb{X}$ will be specified later on.

If the sets $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{X}_f$ are polytopes, it is easy to verify that, given the current value of the state x_k, the optimization problem

$$\begin{aligned} \min_{u_{[k:k+N-1]}} \quad & V(x_k, u_{[k:k+N-1]}) \\ & s.t. \\ x_{k+\nu} \in \mathbb{X} \quad & \forall\, \nu = 0, \ldots, N-1 \\ u_{k+\nu} \in \mathbb{U} \quad & \forall\, \nu = 0, \ldots, N-1 \\ x_{k+N} & \in \mathbb{X}_f \\ x_{k+1} & = Ax_k + Bu_k \end{aligned} \tag{1.3}$$

can be written as a quadratic programming problem, i.e., as an optimization problem of the form

$$\begin{aligned} \min_{z} \quad & \frac{1}{2} z' \Upsilon z + \varpi' z \\ & s.t. \\ & \Lambda z \leq \sigma \end{aligned} \tag{1.4}$$

whose solutions can be computed with well known and computationally efficient algorithms [19].

Let us denote by $u_{[k:k+N-1]|k} = (u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k})$ the optimal control sequence computed at time k. According to the receding horizon (or moving horizon) principle, the rationale underlying MPC is to use only the first element of the optimal sequence, i.e. $u_{k|k}$, and to solve again the optimization problem (1.3), referred to the horizon $[k+1, \ldots, k+N+1]$, at the next time step.

As the control input is the outcome of an optimization procedure solved at each time step k, the control law is an implicit state-feedback one. In order to prove that:

i) the optimization problem results to be feasible also at time step $k+1$;

ii) the convergence of the closed-loop system is guaranteed;

an auxiliary state-feedback control law $K_a x$, not used to compute the control law, but necessary for the mathematical proofs and for the design of the controller, together with a proper pair of weighting matrix P and terminal set $\mathbb{X}_f$, must be selected. The most common choices are the following ones:

1. *Zero Terminal Constraint (ZTC)*: $P = 0$, $K_a = 0$ and $\mathbb{X}_f = \{0\}$.

2. *Quasi-Infinte LQ (QILQ)*: P is the unique positive definite solution of the Riccati algebraic (stationary) equation $P = A'PA + Q - A'PB(R + B'PB)^{-1}B'PA$; K_a is the infinite horizon LQR (linear quadratic regulator) gain: $K_a = (R + B'PB)^{-1}B'PA$; $\mathbb{X}_f$ is chosen such that $\mathbb{X}_f \subseteq \mathbb{X}$, $K_a\mathbb{X}_f \subseteq \mathbb{U}$ and $(A + BK_a)\mathbb{X}_f \subseteq \mathbb{X}_f$.

3. *Arbitrary Stabilizing Gain (ASG)*: K_a is selected as an arbitrary stabilizing gain (using, e.g., eigenvalue assignment techniques); P is the solution of the Lyapunov equation $(A + BK_a)'P(A + BK_a) - P = -(Q + K_aRK_a')$; $\mathbb{X}_f$ is chosen such that $\mathbb{X}_f \subseteq \mathbb{X}$, $K_a\mathbb{X}_f \subseteq \mathbb{U}$ and $(A + BK_a)\mathbb{X}_f \subseteq \mathbb{X}_f$.

Note that a possible way to select $\mathbb{X}_f$ in QILQ and ASG is to define it as the ellipsoidal invariant set $\mathbb{X}_f = \{x|x'Px \leq o\}$ with the positive scalar o small enough to guarantee $\mathbb{X}_f \subseteq \mathbb{X}$ and $K_a\mathbb{X}_f \subseteq \mathbb{U}$. Then, it can be transformed in a polytopic positively invariant set as explained in [5].

A sketch of the recursive feasibility and of the convergence is now provided for all the three proposed choices. Let assume to be at generic time instant k and that the optimal control sequence is $u_{[k:k+N-1]|k} = (u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k})$, associated to the optimal value of the objective function $V_o(x_k)$. Applying the control input $u_{k|k}$, the system reaches the state $x_{k+1|k} = Ax_k + Bu_{k|k}$. If at time $k + 1$ the sequence $u_{[k+1:k+N-1]|k}$ of the control sequence computed at time k is applied to the system, the state at time $k + N$ is $x_{k+N|k} \in \mathbb{X}_f$ and a feasible trajectory is followed. Therefore, thanks to the properties of $\mathbb{X}_f$, the control sequence $\tilde{u}_{[k+1:k+N]} = (u_{k+1|k}, u_{k+2|k}, \ldots, K_a x_{k+N|k})$ is an admissible solution to the optimization problem at time $k + 1$. The corresponding cost function value $\tilde{V}(x_{k+1|k}, \tilde{u}_{[k+1:k+N]})$ is suboptimal, i.e. by definition it holds that $V_o(x_{k+1|k}) \leq \tilde{V}(x_{k+1|k}, \tilde{u}_{[k+1:k+N]})$.

Considering all the terms of the cost function, it is possible to see that

$$\begin{aligned}
&\tilde{V}(x_{k+1|k}, \tilde{u}_{[k+1:k+N]}) - V_o(x_k) = -\|x_k\|_Q^2 - \|u_{k|k}\|_R^2 \\
&+ \|x_{k+N|k}\|_Q^2 + \|K_a x_{k+N|k}\|_R^2 - \|x_{k+N|k}\|_P^2 + \|(A+BK_a)x_{k+N|k}\|_P^2 = \\
&- \|x_k\|_Q^2 - \|u_{k|k}\|_R^2 + \|x_{k+N|k}\|^2_{Q+K_a'RK_a-P+(A+BK_a)'P(A+BK_a)} \quad (1.5)
\end{aligned}$$

Since ZTC guarantees $x_{k+N|k} = 0$ while QILP and ASG, recalling that the stationary Riccati equation can be also written as $P = Q + K_a'RK_a + (A+BK_a)'P(A+BK_a)$, lead to $Q + K_a'RK_a - P + (A+BK_a)'P(A+BK_a) = 0$, in all cases we have

$$\tilde{V}(x_{k+1|k}, \tilde{u}_{[k+1:k+N]}) - V_o(x_k) = -\|x_k\|_Q^2 - \|u_{k|k}\|_R^2 \quad (1.6)$$

The suboptimality of $\tilde{V}(x_{k+1|k}, \tilde{u}_{[k+1:k+N]})$ allows one to write

$$V_o(x_{k+1|k}) - V_o(x_k) \leq -(\|x_k\|_Q^2 + \|u_{k|k}\|_R^2) \quad (1.7)$$

When the state is different from zero, the cost function is therefore a monotonic strictly decreasing positive function. Therefore, $V_o(x_k) - V_o(x_{k+1|k}) \to 0$. But $V_o(x_k) - V_o(x_{k+1|k}) \geq \|x_k\|_Q^2 + \|u_{k|k}\|_R^2$, thus $(\|x_k\|_Q^2 + \|u_{k|k}\|_R^2) \to 0$ as well. The definite positiveness of matrices Q and R implies that inputs and states converge to the origin.

1.2.2 Robust tube-based MPC

If the system to be controlled is affected by an external unknown (but bounded) disturbance, using the standard technique on the nominal system [85, 148], i.e. the system obtained by the real one neglecting the disturbance, could easily lead to the loss of the stability properties and/or to constraints violation [125]. As it has been clarified in [63], nominal MPC can be nonrobust even with respect to arbitrarily small disturbances. For this reason, attention has been recently focused on the development of MPC algorithms ensuring desired robustness properties, see, e.g., [11,79,80,103,129]. This activity has led to the development of two broad classes of algorithms: one is based on a min-max formulation of the optimization problem that defines MPC, see [86,100,129,147]. The other one relies on the a priori evaluation of the disturbance effect over the prediction horizon and the enforcement of tighter and tighter constraints to the predicted state trajectories, see [30, 62, 82, 84]. Among the latter ones, the so-called "tube-based" method discussed in [81, 107] has received much attention for its simplicity and in view of the fact that it requires an on-line computational load comparable to that of nominal MPC. In the reminder of this Section, the approach described in [107] will be presented more in detail, since it plays a fundamental role in the DPC control scheme.

Consider a linear, discrete-time system under control described by

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1.8}$$

where $x_k \in \mathbb{X} \subset \mathbb{R}^n$, $u_k \in \mathbb{U} \subset \mathbb{R}^m$ and $w_k \in \mathbb{W} \subset \mathbb{R}^n$ is an unknown but bounded disturbance. $\mathbb{U}$, $\mathbb{X}$ and $\mathbb{W}$ are polytopes containing the origin in their interior. The nominal system corresponding to (1.8) is

$$\hat{x}_{k+1} = A\hat{x}_k + B\hat{u}_k \tag{1.9}$$

Consider a control gain K selected in such a way that $F = A + BK$ is Schur and the robust positive invariant (RPI) set $\mathbb{Z}$ verifying $F\mathbb{Z} \oplus \mathbb{W} \subseteq \mathbb{Z}$ (details on how to compute $\mathbb{Z}$ as a polytopic set will be given in Section 1.3). It can be easily proved that if $x_k - \hat{x}_k \in \mathbb{Z}$ and if the real system is controlled with

$$u_k = \hat{u}_k + K(x_k - \hat{x}_k) \tag{1.10}$$

then $x_{k+1} - \hat{x}_{k+1} \in \mathbb{Z}$ for all $w_k \in \mathbb{W}$. Therefore, the input u_k to system (1.8) is computed as the sum of two terms: namely, by the nominal input $\hat{u}_k$ obtained as the solution of a standard MPC optimization problem solved considering the nominal model (1.9) and by the corrective term $K(x_k - \hat{x}_k)$, which has the role of keeping the real system state as close as possible to that of the nominal system. The fact that $x_{k+1} - \hat{x}_{k+1} \in \mathbb{Z}$, anyway, is guaranteed only if $x_k - \hat{x}_k \in \mathbb{Z}$: x_k is the measured, current state of the real system and therefore can not be instantaneously changed. On the contrary, $\hat{x}_k$ is the current state of a fictitious, non-existing system and can therefore be initialized arbitrarily. The tube-based MPC considers the nominal system (1.9) with tighter constraints and its main innovation relies in adding its current state to the set of optimization variables.

The cost function to be minimized is

$$V(\hat{x}_k, \hat{u}_{[k:k+N-1]}) = \frac{1}{2}\sum_{\nu=0}^{N-1}(\|\hat{x}_{k+\nu}\|_Q^2 + \|\hat{u}_{k+\nu}\|_R^2) + \frac{1}{2}\|\hat{x}_{k+N}\|_P^2 \tag{1.11}$$

where $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{n\times n}$ are positive definite matrices. As in the standard MPC case, the simplest choice for parameters K and P consists in synthesizing K using the LQR criterion and computing P as the corresponding solution of the stationary Lyapunov equation. The quadratic programming problem to be solved at each time step is

$$
\begin{aligned}
\min_{\hat{x}_k, \hat{u}_{[k:k+N-1]}} \quad & V(\hat{x}_k, \hat{u}_{[k:k+N-1]}) \\
& s.t. \\
x_k - \hat{x}_k &\in \mathbb{Z} \\
\hat{x}_{k+\nu} &\in \hat{\mathbb{X}} \quad \forall\, \nu = 0, \ldots, N-1 \\
\hat{u}_{k+\nu} &\in \hat{\mathbb{U}} \quad \forall\, \nu = 0, \ldots, N-1 \\
\hat{x}_{k+N} &\in \hat{\mathbb{X}}_f \\
\hat{x}_{k+1} &= A\hat{x}_k + B\hat{u}_k
\end{aligned}
\tag{1.12}
$$

where $\hat{\mathbb{X}} = \mathbb{X} \ominus \mathbb{Z}$, $\hat{\mathbb{U}} = \mathbb{U} \ominus K\mathbb{Z}$ and $\hat{\mathbb{X}}_f$ is such that $\hat{\mathbb{X}}_f \subset \hat{\mathbb{X}}$, $K\hat{\mathbb{X}}_f \subset \hat{\mathbb{U}}$ and $F\hat{\mathbb{X}}_f \subset \hat{\mathbb{X}}_f$. It is assumed that $\mathbb{W}$ is small enough to guarantee that $\mathbb{Z} \subset \mathrm{int}(\mathbb{X})$ and $K\mathbb{Z} \subset \mathrm{int}(\mathbb{U})$. Once the optimal pair $(\hat{x}_{k|k}, \hat{u}_{[k:k+N-1]|k})$ is computed, the control action for the real system is given by Equation (1.10), i.e., $u_k = \hat{u}_{k|k} + K(x_k - \hat{x}_{k|k})$.

Setting the design parameters as described, the proof of recursive feasibility turns out to be very similar to that of the standard MPC. Specifically, at time $k+1$ it holds that $x_{k+1} - \hat{x}_{k+1|k} \in \mathbb{Z}$. Denoting $\tilde{\hat{u}}_{[k+1:k+N]} = (\hat{u}_{[k+1:k+N-1]|k}, K\hat{x}_{k+N|k})$ the properties of $\hat{\mathbb{X}}_f$ guarantee that the tuple $(\hat{x}_{k+1|k}, \tilde{\hat{u}}_{[k+1:k+N]})$ is a (suboptimal) feasible solution to the optimization problem (1.12).

As for the convergence proof, the corresponding (suboptimal) cost function $\tilde{V}(\hat{x}_{k+1|k}, \tilde{\hat{u}}_{[k+1:k+N]})$ is characterized by

$$V_o(x_{k+1}) \le \tilde{V}(\hat{x}_{k+1|k}, \tilde{u}_{[k+1:k+N]})$$

where $V_o(x_{k+1})$ is the optimal value of the cost function at time $k+1$ obtained using $(\hat{x}_{k+1|k+1}, \hat{u}_{[k+1:k+N]|k+1})$. As in the standard case,

$$\tilde{V}(\hat{x}_{k+1|k}, \tilde{u}_{[k+1:k+N]}) - V_o(x_k) = -(\|\hat{x}_{k|k}\|_Q^2 + \|\hat{u}_{k|k}\|_R^2)$$

thus

$$V_o(x_{k+1}) - V_o(x_k) \le -(\|\hat{x}_{k|k}\|_Q^2 + \|\hat{u}_{k|k}\|_R^2)$$

The cost function turns out to be a monotonic strictly decreasing positive function, therefore $V_o(x_k) - V_o(x_{k+1|k}) \to 0$. Being

$$V_o(x_k) - V_o(x_{k+1|k}) \ge \|\hat{x}_{k|k}\|_Q^2 + \|\hat{u}_{k|k}\|_R^2$$

we have that $(\|\hat{x}_{k|k}\|_Q^2 + \|\hat{u}_{k|k}\|_R^2) \rightarrow 0$ as well. Since Q and R are positive definite, the nominal inputs and states converge to zero. Since the real states are always constrained in an invariant neighborhood defined by $\mathbb{Z}$ with respect to the nominal ones, they will be steered to the neighborhood of the origin $\mathbb{Z}$. Note that, for the same reason, the tighter constraints represented by $\hat{\mathbb{X}}$ and $\hat{\mathbb{U}}$, ensure the fulfillment of the real constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ at all the time instants.

1.2.3 MPC for tracking piecewise constant references

For practical application purposes, model predictive controllers must be able not only to regulate the system state to zero, but also to handle non-zero target steady states which can be provided by a steady state target optimizer [112]. The standard solution to this problem consists in changing the system state coordinates, i.e. shifting the system state to the desired steady state [114]. Unfortunately, the new target steady state could be unreachable and, moreover, after such shifting, feasibility may not be guaranteed. The latter problem can be solved by re-calculating the control horizon and the terminal set for the new target steady state, but the complexity of this procedure does not allow one to perform the recalculation on-line. This problem has motivated several solutions proposed in the literature [10, 16, 31, 58, 121, 141].

Recently, an effective MPC algorithm for tracking was proposed in [6, 52, 88, 89]. The main ingredients are: *i)* the online computation of the actual target to be really tracked at each time instant, *ii)* the penalization of the deviation between the artificial steady state and the desired one at the optimization problem level. The controller steers the system to any admissible target steady state while satisfying the system constraints; if the desired target is not admissible, the system is steered to the closest admissible steady state.

The system to be controlled is

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{1.13}$$

where $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the input and $y_k \in \mathbb{R}^m$ is the output. The number of outputs is assumed to be equal to the number of inputs only for the sake of simplicity: extensions to non-square systems are shown in the provided references. Inputs and states are subject to constraints defined by compact, convex polyhedra $\mathbb{X}$ and

$\mathbb{U}$ containing the origin in their interior and defined by

$$\mathbb{X} = \{x \in \mathbb{R}^n \ : \ A_x x \leq b_x\}$$

and

$$\mathbb{U} = \{u \in \mathbb{R}^m \ : \ A_u u \leq b_u\}$$

Let $\bar{y}$ be a generic set-point with $(\bar{x}, \bar{u})$ representing the related steady-state pair, i.e.

$$\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} A - I_n & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \bar{y} = S^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \bar{y} = M\bar{y} = \begin{bmatrix} M_x \\ M_u \end{bmatrix} \bar{y} \quad (1.14)$$

where it is assumed that the system matrix S has full rank and therefore can be inverted. The set of all the admissible set-points is denoted as

$$\mathbb{Y} = \{\bar{y} = C\bar{x} + D\bar{u} \ : \ \bar{x} \in \mathbb{X}, \bar{u} \in \mathbb{U}, (A - I_n)\bar{x} + B\bar{u} = 0\} \subseteq \mathbb{R}^m$$

As in the regulation cases, an auxiliary state-feedback control law and a proper terminal invariant set are required to ensure recursive feasibility. For a generic target $\bar{y}$, the auxiliary control law is given by

$$u = K(x - \bar{x}) + \bar{u} = Kx + \begin{bmatrix} -K & K \end{bmatrix} M\bar{y} = Kx + L\bar{y} \qquad (1.15)$$

which guarantees that the system is steered to the steady-state $(\bar{x}, \bar{u})$ provided that $A + BK$ is Schur. Since the system is subject to constraints, the main goal is to find a set such that the constraints are fulfilled when the auxiliary control law is used. From now on, we define the set of all the λ-admissible set points $\bar{y}^\lambda$ as

$$\mathbb{Y}_\lambda = \{\bar{y}^\lambda = C\bar{x}^\lambda + D\bar{u}^\lambda \ : \ \bar{x}^\lambda \in \lambda\mathbb{X}, \bar{u}^\lambda \in \lambda\mathbb{U}, (A - I_n)\bar{x}^\lambda + B\bar{u}^\lambda = 0\}$$

with $\lambda \in (0, 1]$.

To find this invariant set, consider the extended state $q = (x, \bar{y}^\lambda)$ and its closed loop dynamics when the auxiliary control law (1.15) is used to reach $\bar{x}^\lambda$

$$q_{k+1} = \begin{bmatrix} x_{k+1} \\ \bar{y}^\lambda \end{bmatrix} = \begin{bmatrix} A + BK & BL \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x_k \\ \bar{y}^\lambda \end{bmatrix} = A_q q_k \qquad (1.16)$$

Now select a positive scalar $\lambda \leq 1$, and define a set $\mathbb{Q}_\lambda$ given by all the values of q such that: *i)* the steady state $(\bar{x}^\lambda, \bar{u}^\lambda) = M\bar{y}^\lambda$ lies inside the set $\lambda\mathbb{X} \times \lambda\mathbb{U} \subseteq \mathbb{X} \times \mathbb{U}$; *ii)* the state x lies inside $\mathbb{X}$ and the auxiliary control law (1.15) corresponding to x belongs to $\mathbb{U}$. The set

$$\mathbb{Q}_\lambda = \{q = (x, \bar{y}^\lambda) \; : u = Kx + L\bar{y}^\lambda \in \mathbb{U}, x \in \mathbb{X}, \\ \bar{x}^\lambda = M_x\bar{y}^\lambda \in \lambda\mathbb{X}, \bar{u}^\lambda = M_u\bar{y}^\lambda \in \lambda\mathbb{U}\} \quad (1.17)$$

is a polyhedron defined by the inequalities

$$\begin{bmatrix} A_x & 0 \\ A_uK & A_uL \\ 0 & A_xM_x \\ 0 & A_uM_u \end{bmatrix} \begin{bmatrix} x \\ \bar{y}^\lambda \end{bmatrix} \leq \begin{bmatrix} b_x \\ b_u \\ \lambda b_x \\ \lambda b_u \end{bmatrix} \quad (1.18)$$

The actual constraints for the system correspond to $\mathbb{Q}_1$, i.e. to $\mathbb{Q}_\lambda$ with $\lambda = 1$. The need of tightening the constraints for the steady-state comes from the method used for computing the invariant set, as it will be explained later. An invariant set Ω^q for the extended state q, i.e. such that $A_q\Omega^q \subseteq \Omega^q$, is said to be an admissible invariant set for tracking if the extended state fulfills all the constraints, which means $\Omega^q \subseteq W_1$. The biggest among all possible admissible invariant sets for tracking is said to be the maximal invariant set for tracking (MIST), and it can be proved to be defined by [59]

$$O^q_\infty = \{q \; : \; A^i_q q \in \mathbb{Q}_1, \; \forall\, i \geq 0\} \quad (1.19)$$

Unfortunately, in general it is not possible to compute O^q_∞, because it might be not finitely determined by a finite set of constraints. On the contrary, choosing $\lambda \in (0, 1)$, it is possible to state that the MIST with respect to $\mathbb{Q}_\lambda$

$$O^{q,\lambda}_\infty = \{q \; : \; A^i_q q \in \mathbb{Q}_\lambda, \; \forall\, i \geq 0\} \quad (1.20)$$

is a finitely determined polytope, and therefore computable [59]. Note that, since λ can be chosen arbitrarily close to 1, the obtained invariant set is arbitrarily close to the real maximal invariant set O^q_∞.

After having chosen arbitrarily $\lambda \in (0, 1)$, let's assume that we want to track a desired λ-admissible target $\bar{y}^{\lambda,d}$ corresponding to the couple $(\bar{x}^{\lambda,d}, \bar{u}^{\lambda,d}) = M\bar{y}^{\lambda,d}$. The MPC for tracking considers as decision variable an artificial λ-admissible reference $\bar{y}^{\lambda,a}$, with $(\bar{x}^{\lambda,a}, \bar{u}^{\lambda,a}) = M\bar{y}^{\lambda,a}$, and the deviation between the artificial steady state $\bar{x}^{\lambda,a}$ and the desired steady state $\bar{x}^{\lambda,d}$ is penalized. This penalization guarantees that $\bar{x}^{\lambda,a}$ asymptotically reaches $\bar{x}^{\lambda,d}$. In order to present the control technique in the clearest possible way, we will consider only the case of

λ-admissible targets. Anyway it is easy to show that, if we have to track a generic non-λ-admissible target $\bar{y}^d$ corresponding to $(\bar{x}^d, \bar{u}^d) = M\bar{y}^d$, possibly non-admissible for system (1.13) because outside the set $\mathbb{Y}$, the penalization of the distance between $\bar{x}^{\lambda,a}$ and $\bar{x}^d$ makes the system evolve to a λ-admissible steady state such that its deviation with the desired (although possibly unreachable) steady state $\bar{x}^d$ is minimized.

The cost function over the prediction horizon N is

$$\begin{aligned} V(x_k, \bar{y}^{\lambda,d}, u_{k:k+N-1}, \bar{y}^{\lambda,a}) = \sum_{\nu=0}^{N-1} (\|x_{k+\nu} - \bar{x}^{\lambda,a}\|_Q^2 + \|u_{k+\nu} - \bar{u}^{\lambda,a}\|_R^2) \\ + \|x_{k+N} - \bar{x}^{\lambda,a}\|_P^2 + \|\bar{x}^{\lambda,a} - \bar{x}^{\lambda,d}\|_T^2 \end{aligned} \quad (1.21)$$

where the matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, $P \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ are all assumed to be positive definite. The input sequence over all the prediction horizon $u_{[k:k+N-1]}$ and $\bar{y}^{\lambda,a}$, i.e., the target really tracked at time k, are the decision variables, while the current state x_k and the desired target $\bar{y}^{\lambda,d}$ are parameters of the cost function. The MPC problem to be solved is

$$\begin{aligned} \min_{s, u_{[k:k+N-1]}} \; & V(x_k, \bar{y}^{\lambda,d}, u_{k:k+N-1}, \bar{y}^{\lambda,a}) \\ & s.t. \\ x_{k+\nu} \in \mathbb{X} \quad & \forall\, \nu = 0, \dots, N-1 \\ u_{k+\nu} \in \mathbb{U} \quad & \forall\, \nu = 0, \dots, N-1 \\ (x_{k+N}, \bar{y}^{\lambda,a}) \in & \, O_\infty^{w,\lambda} \\ x_{k+1} = & \, Ax_k + Bu_k \end{aligned} \quad (1.22)$$

The proofs of recursive feasibility of the MPC problem, of asymptotic stability of the controller and of convergence of the closed-loop system to $\bar{y}^{\lambda,d}$ are based on three main steps.

1. Prove that, if at the initial time instant feasibility holds for a certain value of $\bar{y}^{\lambda,a}$, with standard arguments it is possible to demonstrate that keeping $\bar{y}^{\lambda,a}$ constant at next time steps guarantees recursive feasibility while the system is steered towards $\bar{y}^{\lambda,a}$.

2. Prove that, if the system reaches an equilibrium point $\bar{y}^{\lambda,a} \neq \bar{y}^{\lambda,d}$, there exists a suboptimal input sequence that decreases the value of the cost function and that steers the system from $\bar{y}^{\lambda,a}$ to $\bar{y}^{\lambda,d}$ while fulfilling all the constraints. In other

words, the only closed-loop stable equilibrium point compatible with the proposed minimization problem is the one corresponding to $\bar{y}^{\lambda,a} = \bar{y}^{\lambda,d}$.

3. From the previous steps infer that the system asymptotically converges to the unique equilibrium point for the closed-loop system, where $\bar{y}^{\lambda,a} = \bar{y}^{\lambda,d}$.

The mathematical details, here skipped to avoid repetitions, can be found in Chapter 6 for systems enlarged with integrators.

1.3 Invariant sets

Given an autonomous system of the form

$$x_{k+1} = Fx_k \tag{1.23}$$

where $x_k \in \mathbb{R}^n$ and F is Schur, a set $\mathbb{P}$ is defined to be positively invariant if $Fx_k \in \mathbb{P}$ for all $x_k \in \mathbb{P}$, i.e., if $F\mathbb{P} \subseteq \mathbb{P}$. Assume that the system is affected by a disturbance and is described by

$$x_{k+1} = Fx_k + w_k \tag{1.24}$$

where $w_k \in \mathbb{W} \subset \mathbb{R}^n$ is an unknown bounded external disturbance. A set $\mathbb{P}$ is defined to be robustly positively invariant (RPI) if $Fx_k + w_k \in \mathbb{P}$ for all $x_k \in \mathbb{P}$ and $w_k \in \mathbb{W}$, i.e., if $F\mathbb{P} \oplus \mathbb{W} \subseteq \mathbb{P}$.

Invariant sets play an important role in control of constrained systems [15]. In fact, if the constraints are violated at any time instant, serious consequences may arise: for example, physical components may be damaged or saturation may cause a loss of closed-loop stability [8, 65, 157]. Considering in particular the case of MPC, as shown in the previous Sections, invariant sets can be required to prove recursive feasibility of the control algorithm or to design robust controllers.

1.3.1 Polytopic approximation of the minimal robust invariant set

For a system with dynamic equation (1.24), it is often necessary to compute the minimal RPI (mRPI) $\mathbb{P}_\infty$, which is the RPI set in $\mathbb{R}^n$ that is contained in every closed RPI set of (1.24). For instance, this need arises when a target set in robust time-optimal control has to be computed [106], when the properties of the maximal RPI have to be studied [78] or when robust predictive controllers have to be designed [81, 107] (as explained in Section 1.2.2).

Consider the discrete-time, linear, time-invariant system (1.24), where it is assumed that matrix $F \in \mathbb{R}^{n \times n}$ is Schur and set $\mathbb{W}$ is a polytope containing the origin in its interior. It is possible to demonstrate that the mRPI set $\mathbb{P}_\infty$ exists, is compact, contains the origin in its interior and is given by [78]

$$\mathbb{P}_\infty = \bigoplus_{i=0}^{\infty} F^i \mathbb{W} \tag{1.25}$$

Since $\mathbb{P}_\infty$ is a Minkowski sum of infinitely many terms, it is generally impossible to obtain an explicit characterization of it. However, if there exist an integer $\phi \geq 1$ and a scalar $\alpha \in [0, 1)$ such that $F^\phi = \alpha I_n$, then $\mathbb{P}_\infty = (1 - \alpha)^{-1} \bigoplus_{i=0}^{\phi-1} F^i \mathbb{W}$ [78]. Therefore, if F is nilpotent with index ϕ, i.e., if $F^\phi = 0$, it is possible to compute

$$\mathbb{P}_\infty = \bigoplus_{i=0}^{\phi-1} F^i \mathbb{W} \tag{1.26}$$

In [131, 132] the assumption that there exist an integer $\phi \geq 1$ and a scalar $\alpha \in [0, 1)$ such that $F^\phi = \alpha I_n$ is relaxed, and replaced by the assumption that there exists an integer $\phi \geq 1$ and a scalar $\alpha \in [0, 1)$ that satisfy $F^\phi \mathbb{W} \subseteq \alpha \mathbb{W}$. In this case we can no longer compute $\mathbb{P}_\infty$ in a finite number of steps, but we can solve the problem of computing an RPI set $\mathbb{P}(\alpha, \phi)$ that contains the mRPI set and that is as close as desired to it. In fact, it can be proved that

$$\mathbb{P}(\alpha, \phi) = (1 - \alpha)^{-1} \bigoplus_{i=0}^{\phi-1} F^i \mathbb{W} \tag{1.27}$$

is a convex, compact, polytopic RPI set of (1.24) containing $\mathbb{P}_\infty$. The approximation error decreases as α increases. Summarizing, the computation of the polytopic RPI outer approximation to the mRPI can be done as described in the next algorithm.

Algorithm 1.1 ***Computation of the RPI outer approximation of the mRPI***

1. *Choose arbitrarily* $\alpha \in [0, 1)$.
2. *Starting from* $\phi = 1$*, increase the integer* ϕ *until a value* $\phi \geq 1$ *is found such that* $F^{\phi}\mathbb{W} \subseteq \alpha\mathbb{W}$.
3. *Evaluate* $\mathbb{P}(\alpha, \phi)$ *using* (1.27).

Note that, due to the need of resorting to Minkowski sums, finding the approximation to the mRPI set can result in an extremely computationally demanding procedure. Finally, it is worth recalling that the same authors extended the approach also to continuous-time systems [133].

1.3.2 Maximal output admissible set

The standard MPC algorithms for regulation and tracking presented above usually require the computation of invariant terminal sets related to the auxiliary state-feedback control law and guaranteeing that constraints on states, inputs and outputs are fulfilled during the evolution of the system. An important contribution to the computation of such invariant set is presented in [59], where the notion of maximal output admissible set (MOAS) is introduced.

Consider an autonomous, discrete-time, time-invariant linear system

$$\begin{aligned} x_{k+1} &= F x_k \\ y_k &= C x_k \end{aligned} \tag{1.28}$$

where $x_k \in \mathbb{R}^n$ is the state and the output $y_k \in \mathbb{R}^p$ has to satisfy an output constraint $y_k \in \mathbb{Y}$ for all $k \geq 0$. For such a system, a set $\mathbb{O} \subset \mathbb{R}^n$ is said to be output admissible if $x_0 \in \mathbb{O}$ implies that $y_k \in \mathbb{Y}$ for all $k > 0$, i.e. if $CF^k x_0 \in \mathbb{Y}$ for all $k > 0$. The maximal output admissible set $\mathbb{O}_\infty$ is the biggest one, namely the set of all initial conditions x_0 for which $CF^k x_0 \in \mathbb{Y}$ holds for all $k > 0$. We underline that output admissible sets are not necessarily positively invariant, but it is easy to show that the maximal output admissible set is always positively invariant [95]. In addition, we stress the fact that computing the MOAS for system (1.28) can solve several different problems. For instance, if a terminal set for standard MPC with constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ is required, one can apply the results of [59] to the system

$$\begin{aligned} x_{k+1} &= (A+BK)x_k \\ y_k &= \begin{bmatrix} I_n \\ K \end{bmatrix} x_k \end{aligned} \tag{1.29}$$

imposing $y_k = (x_k, u_k) \in \mathbb{Y} = \mathbb{X} \times \mathbb{U}$.

If $\mathbb{Y} = \{y \in \mathbb{R}^m \ : \ f_i' y \leq g_i, i \in \mathbb{I}\}$ is a convex polytope containing the origin and, referring to system (1.28), the pair (F, C) is observable and the matrix F is strictly or simply stable (eigenvalues with absolute value equal to one are admitted), then $\mathbb{O}_\infty$ results to be convex, bounded and a neighbor of the origin. Moreover, if F is asymptotically stable, then $\mathbb{O}_\infty$ is described by a finite set of constraints and can be computed as follows.

Algorithm 1.2 ***Computation of the MOAS***

1. *Initialize $t = 0$.*
2. *For each $i \in \mathbb{I}$, maximize $J_i(x) = f_i' C F^{t+1} x$ with respect to x subject to $f_j' C F^h x \leq g_j$ for all $j \in \mathbb{I}$ and for all $h = 0, 1, \dots, t$. Denote with J_i^* the maximum values of $J_i(x)$.*
3. *If $J_i^* \leq g_i$ for all $i \in \mathbb{I}$, then stop and let $t^* = t$. Otherwise, set $t = t + 1$ and go to step 2.*
4. *Define the maximal output admissible set as*

$$\mathbb{O}_\infty = \{x \in \mathbb{R}^n \ : \ f_i' C F^t x \leq g_i, i \in \mathbb{I}, 0 \leq t \leq t^*\}$$

If the pair (F, C) is non-observable, it is possible to show that $\mathbb{O}_\infty(F, C, \mathbb{Y}) = \mathbb{O}_\infty(F_1, C_1, \mathbb{Y}) \times \mathbb{R}^{n-q}$, where the observable pair (F_1, C_1) can be obtained changing the state coordinates so that (F, C) takes the form

$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \ F = \begin{bmatrix} F_1 & 0 \\ F_3 & F_2 \end{bmatrix}$$

being $C_1 \in \mathbb{R}^{p \times q}$, $F_1 \in \mathbb{R}^{q \times q}$, $F_2 \in \mathbb{R}^{(n-q)\times(n-q)}$ and $F_3 \in \mathbb{R}^{(n-q)\times q}$. $\mathbb{O}_\infty$ turns out to be a set with infinite extent in those directions belonging to the unobservable subspace.

A second important case is the one where matrix F is not asymptotically stable but has d simple eigenvalues in 1. In this case, one can always find a coordinates change which transforms (F, C) in the form

$$C = \begin{bmatrix} C_S & C_L \end{bmatrix}, \; F = \begin{bmatrix} F_S & 0 \\ 0 & I_d \end{bmatrix}$$

where $F_S \in \mathbb{R}^{(n-d)\times(n-d)}$ is asymptotically stable and the partitioning of C and F is dimensionally consistent. Then define

$$\hat{C} = \begin{bmatrix} C_L & C_S \\ C_L & 0 \end{bmatrix}$$

and $\mathbb{Y}(\epsilon) = \{y \in \mathbb{R}^p \; : \; f_i' y \leq g_i - \epsilon, i \in \mathbb{I}\}$, where ϵ can be arbitrarily chosen such that $0 < \epsilon \leq \epsilon_0$ being $\epsilon_0 = -\max\{-g_i, i \in \mathbb{I}\}$. It can be demonstrated that $\mathbb{O}_\infty(F, \hat{C}, \mathbb{Y} \times \mathbb{Y}(\epsilon))$ is finitely determined and contained in $\mathbb{O}_\infty(F, C, \mathbb{Y})$. Thus, $\mathbb{O}_\infty(F, \hat{C}, \mathbb{Y} \times \mathbb{Y}(\epsilon))$ can be used to approximate $\mathbb{O}_\infty(F, C, \mathbb{Y})$, and the precision of the approximation increases as $\epsilon \to 0$. It is easy to see that using the set (1.18) with $\lambda < 1$ for computing the MIST, as explained in Section 1.2.3, is equivalent to adopting the set $\mathbb{Y} \times \mathbb{Y}(\epsilon)$ with $\epsilon > 0$.

1.4 Distributed and decentralized MPC

In the last years, the size of systems to be controlled is continuously increasing, mainly due to the advances in technology and telecommunications. Common examples are smart electrical grids, networks of sensors and actuators, big chemical plants and water or traffic managing systems [22, 25, 110, 115, 116, 161, 163, 164].

Since the sixties, researchers have been developing decentralized and distributed methods with guaranteed closed-loop stability [37, 38, 94, 151, 152, 154] to overcome the difficulties that arise when centralized techniques are applied to large-scale systems. Specifically, non-trivial problems can be caused, for instance, by possible limitations due to computing capabilities or communication bandwidth. Reliability and robustness of the overall control system could represent additional reasons for avoiding centralized solutions and for adopting distributed or decentralized ones. In the latter cases, in fact, the original system is controlled by several local agents, each one taking into account only a small portion of the whole system.

In this thesis, we assume that the decomposition of the initial large-scale system into several small-scale interacting subsystems is given. In-depth analysis on how to decompose the overall input and output vectors into input-output pairs or groups while minimizing mutual influences among subsystems can be found in [26, 61, 66–68, 75, 117, 136, 142, 156, 171]. Note that sometimes, in order to obtain a proper decomposition, it can be useful to apply state transformations or permutations to the initial system [94, 152, 154].

In the framework of distributed and decentralized control, an increasing attention has been devoted to algorithms based on optimization and receding horizon control. When using these techniques, in fact, the values over all the prediction horizon of inputs, states and output intrinsically computed at each sampling time by a local controller designed with MPC can be used as data to be broadcast to the neighbors. This information can simplify the design of a distributed control system and can allow one to obtain performances close to those of a centralized controller. In the following Sections, some control architectures proposed for decentralized and distributed MPC will be listed. We refer to [21, 136, 144] for additional details.

Techniques for distributed solution of centralized MPC [46, 55, 93, 170] will not be covered: in these cases, the optimization problem is decomposed into smaller subproblems and, at each sampling time, the local controllers solve the centralized optimization problem using iterative decomposition algorithms (e.g., dual decomposition, price coordination). Optimality and feasibility are generally guaranteed only when the convergence of the decomposition algorithm (within each sampling time) is achieved.

1.4.1 Decentralized control based on MPC

Often large-scale systems are controlled by resorting to decentralized techniques. In these cases, a non-overlapping decomposition of inputs u and controlled outputs y is needed, in order to group such variables into disjoint sets. Then, the obtained sets are coupled to produce non-overlapping pairs (or groups, if multivariable local control systems are used). Eventually, for each input-output pair (or group), a local controller is designed with the aim of regulate its own subsystem in a completely independent fashion. In Fig. 1.1, an example of decentralized control for a system constituted by two subsystems is given. It can be seen that the dynamics of each subsystem is influenced by the inputs

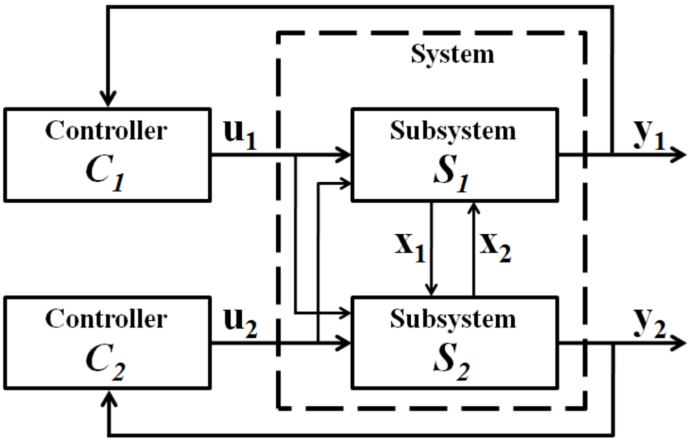

Figure 1.1: Decentralized control scheme of a system constituted by two subsystems.

and the states of the other one. In designing decentralized controllers, the interactions due to coupling inputs and/or states are neglected, so that the synthesis of the control system becomes trivial. On the other hand, asymptotic stability of the overall large-scale system can be attained only when the interactions are weak. Strong (neglected) coupling terms, in fact, can lead to poor performances and instability of the whole system, as explained, e.g., in [41,169], where the effects of fixed modes is studied. Conditions to obtain stability using decentralized controllers can be found in [9,40,70–72,94,143,145,152,153,155].

As for decentralized predictive controllers, only few contributions can be found in the literature. This is probably due to the fact that extending to the decentralized case the standard stability analysis of the centralized MPC, based on the idea of using the optimal cost as a Lyapunov function, is not an easy task. Examples of decentralized MPC where the local control law have been computed by neglecting the coupling terms among subsystems are reported in [1,45]. In [102], a decentralized MPC control method for nonlinear systems is presented. The dynamics is assumed to be affected by an external decaying disturbance, and stability is guaranteed, despite the effects of the exogenous disturbance and of the interactions among subsystems, by including a contraction constraint in the optimization problem. Coupling terms among subsystems are seen as disturbances to be rejected via a robust approach in [130], where Input to State Stability (ISS) [77] is used to prove stability. Also in [138, 139] interactions among subsystems are considered as disturbances to be rejected using the robust tube-based MPC algorithm. Linear discrete-time systems in a plug-and-play framework are considered: specifically, the problem of plugging subsystems into (or unplugging subsystems from) an existing plant without spoiling overall stability and constraint satisfaction is addressed.

1.4.2 Distributed control based on MPC

In Fig. 1.2 an example of a distributed controller is depicted. The difference with respect to the decentralized case is that some information is transmitted among the local regulators. In this way, each controller has some information about the behaviour of the neighbor systems. As already stated, when MPC is used to design the local controllers, the information exchange usually consists of the predicted control or state variables over the prediction horizon.

A first classification of distributed MPC algorithms can be made on the basis of the communication network topology: fully connected algorithms require information transmission from any local regulator to all the others; partially connected algorithms, on the contrary, are based on information transmission from any local regulator to a certain subset of the neighboring ones. In [137] a discussion of positive and negative sides of partially connected algorithms can be found: generally, it is possible to say that they are suited for weakly coupled subsystems, because in this case a reduction in the information exchange does not greatly affect the performance of the system. A second classification can be made based on the communication protocol: noniterative algorithms are characterized by a single transmission of information within each sampling time, while iterative algorithms allow local controllers to transmit and receive information many times within the sampling time. Obviously, local control systems working with an iterative transmission protocol can take decisions using an higher amount of information: often, iterations are even used for allowing a global consensus on the actions to be taken within the sampling interval [166, 167]. About this point, in fact, we can distinguish between independent, or non-cooperating, algorithms, where each local regulator aims to minimize a local performance index, and cooperating algorithms, where each local controller minimizes a global cost function.

Several distributed solutions based on MPC have been proposed in the literature. State feedback non-cooperating, noniterative al-

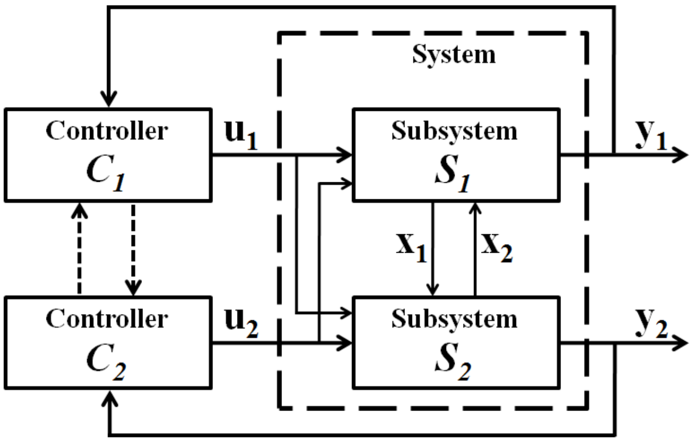

Figure 1.2: Distributed control scheme of a system constituted by two subsystems.

gorithms for discrete-time linear system are presented in [21], while in [43,83] independent, iterative, fully connected methods are described for discrete-time unconstrained linear systems represented by input-output models. An iterative, cooperating method for discrete-time linear systems is shown in [166]: interestingly, a global optimum is reached when the iterative procedure converges, but recursive feasibility and closed-loop stability are anyway guaranteed also if the information exchange is stopped at any intermediate iteration. In [2–4] a partially connected, noniterative, non-cooperating algorithm for discrete-time linear systems is described. Communication failures are taken into consideration, and conditions for a-posteriori stability analysis are given also in case of malfunctions. The control technique proposed in [73] can be classified as partially connected, noniterative and non-cooperating as well. It is suited for nonlinear, discrete-time systems and it is based on the idea of considering the effects of the interconnections among the subsystems as disturbances. Each local controller transmits the values of the predicted state trajectories so that the others can predict the disturbances they have to reject. The control inputs are computed using a min-max approach where local cost functions in worst-case scenario are minimized. Convergence to a set is proved. Another non-cooperating, noniterative, partially connected algorithm, this time for nonlinear continuous-time systems, is presented in [44]. Proofs of feasibility and convergence are based on the theory explained in [109], and rely on the assumptions that the interactions among subsystems are weak and that the actual inputs and states trajectories do not differ too much from the predicted values. The latter requirement is imposed with a proper consistency constraint added to the optimization problem. In [50, 140] two different non-cooperating, noniterative, partially connected algorithms for linear discrete-time systems are proposed. In both cases, interactions among subsystems are considered as disturbances to be rejected, and to this end the robust MPC based on tubes is exploited. Command governors strategies for distributed supervision and control of large-scale systems are proposed in [23, 24, 162], where dynamically coupled interconnected linear systems possibly affected by bounded disturbances and subject to local and global constraints are considered. Finally, an experimental implementation of distributed MPC is described in [108].

Part II

Solutions to the regulation problem

2 Distributed Predictive Control

In this chapter the state feedback Distributed Predictive Control (DPC) algorithm originally proposed in [50] is sketched and extended. The overall system to be controlled is assumed to be composed by a number of interacting subsystems $\mathcal{S}_i$ with non-overlapping states, linear dynamics and possible state and control constraints. The dynamics of each subsystem can depend on the state and input variables of the other subsystems, and joint state and control constraints can be considered. A subsystem $\mathcal{S}_i$ is said to be a neighbor of subsystem $\mathcal{S}_j$ if the state and/or control variables of $\mathcal{S}_i$ influence the dynamics of $\mathcal{S}_j$ or if a joint constraint on the states and/or on the inputs of $\mathcal{S}_i$ and $\mathcal{S}_j$ must be fulfilled.

DPC has been developed with the following rationale: at each sampling time, the subsystem $\mathcal{S}_i$ sends to its neighbors information about its future state $\tilde{x}_i$ and input $\tilde{u}_i$ reference trajectories, and guarantees that its actual trajectories x_i and u_i lie within certain bounds in the neighborhood of the reference ones. Therefore, these reference trajectories are known exogenous variables for the neighboring subsystems to be suitably compensated, while the differences $x_i - \tilde{x}_i$, $u_i - \tilde{u}_i$ are regarded as unknown bounded disturbances to be rejected. In this way, the control problem is set in the framework of robust MPC, and the tube-based approach inspired by [107] is used to formally state and solve a robust MPC problem for each subsystem.

The highlights of DPC are the following.

- It is not necessary for each subsystem to know the dynamical models governing the other subsystems (not even the ones of its neighbors), leading to a non-cooperative approach.

- The transmission of information is limited (DPC is non-iterative and requires a neighbor-to-neighbor, i.e., partially connected, communication network), in that each subsystem needs only to know the reference trajectories of its neighbors.
- Its rationale is similar to the MPC algorithms often employed in industry: reference trajectories, tailored on the dynamics of the system under control, are used.
- Convergence and stability properties are guaranteed under mild assumptions.
- The method can be extended to cope with the output feedback case.

2.1 Statement of the problem and main assumption

Consider a linear, discrete-time system described by the following state-space model:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \tag{2.1}$$

where $\mathbf{x}_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state and $\mathbf{u}_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the input, both subject to constraints.

Letting $\mathbf{x}_k = (x_k^{[1]}, \ldots, x_k^{[M]})$, $\mathbf{u}_k = (u_k^{[1]}, \ldots, u_k^{[M]})$, system (2.1) can be decomposed in a set of M dynamically coupled non-overlapping subsystems, each one described by the following state-space model:

$$x_{k+1}^{[i]} = A_{ii}x_k^{[i]} + B_{ii}u^{[i]}(k) + \sum_{j=1, j\neq i}^{M} \{A_{ij}x_k^{[j]} + B_{ij}u_k^{[j]}\} \tag{2.2}$$

where $x_k^{[i]} \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$, $n = \sum_{i=1}^{M} n_i$, and $u_k^{[i]} \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$, $m = \sum_{i=1}^{M} m_i$, are the state and input vectors of the i-th subsystem $\mathcal{S}_i$ $(i = 1, \ldots, M)$, and the sets $\mathbb{X}_i$ and $\mathbb{U}_i$ are convex neighborhoods of the origin.

The subsystem $\mathcal{S}_j$ is said to be a dynamic neighbor of the subsystem $\mathcal{S}_i$ if and only if the state or the input of $\mathcal{S}_j$ affects the dynamics of $\mathcal{S}_i$ i.e., iff $A_{ij} \neq 0$ or $B_{ij} \neq 0$. The symbol $\mathcal{N}_i$ denotes the set of dynamic neighbors of $\mathcal{S}_i$ (which excludes i).

Note that the matrices $\mathbf{A}$ and $\mathbf{B}$ of system (2.1) have block entries A_{ij} and B_{ij} respectively, and that $\mathbb{X} = \prod_{i=1}^{M} \mathbb{X}_i$ and $\mathbb{U} = \prod_{i=1}^{M} \mathbb{U}_i$ are convex by convexity of $\mathbb{X}_i$ and $\mathbb{U}_i$, respectively.

The states and inputs of the subsystems can be subject to coupling static constraints described in collective form by

$$H_s(\mathbf{x}_k, \mathbf{u}_k) \leq 0$$

where $s = 1, \dots, n_c$. H_s is a constraint on $\mathcal{S}_i$ if $x^{[i]}$ and/or $u^{[i]}$ are arguments of H_s, while $\mathbb{C}_i = \{s \in \{1, \dots, n_c\} : H_s \text{ is a constraint on } i\}$ denotes the set of constraints on $\mathcal{S}_i$. Subsystem $\mathcal{S}_j$ is a constraint neighbor of subsystem $\mathcal{S}_i$ if there exists $\bar{s} \in \mathbb{C}_i$ such that $x^{[j]}$ and/or $u^{[j]}$ are arguments of $H_{\bar{s}}$, while $\mathcal{H}_i$ is the set of the constraint neighbors of $\mathcal{S}_i$. Finally, for all $s \in \mathbb{C}_i$, a function is defined $h_{s,i}(x^{[i]}, u^{[i]}, \mathbf{x}, \mathbf{u}) = H_s(\mathbf{x}, \mathbf{u})$, where $x^{[i]}$ and $u^{[i]}$ are not arguments of $h_{s,i}(a, b, \cdot, \cdot)$. When $\mathbb{X} = \mathbb{R}^n$, $\mathbb{U} = \mathbb{R}^m$ and $n_c = 0$ the system is unconstrained. In general $\mathcal{S}_j$ is called a neighbor of $\mathcal{S}_i$ if $j \in \mathcal{N}_i \cup \mathcal{H}_i$. In line with these definitions, the communication topology which will be assumed from now on is a neighbor-to-neighbor one. Indeed, we require that information is transmitted from subsystem $\mathcal{S}_j$ to subsystem $\mathcal{S}_i$ if $\mathcal{S}_j$ is a neighbor of $\mathcal{S}_i$.

The algorithm proposed in this Chapter is based on MPC concepts and aims to solve, in a distributed fashion, the regulation problem for the described network of subsystems, while guaranteeing constraint satisfaction. Towards this aim, the following main assumption on decentralized stabilizability is introduced.

Assumption 2.1 *There exists a block diagonal matrix* $\mathbf{K} = diag(K_1, \dots, K_M)$*, with* $K_i \in \mathbb{R}^{m_i \times n_i}$*,* $i = 1, \dots, M$ *such that:*

i) $\mathbf{A} + \mathbf{BK}$ *is Schur.*

ii) $F_{ii} = (A_{ii} + B_{ii}K_i)$ *is Schur,* $i = 1, \dots, M$*.*

□

2.2 Description of the approach

In DPC, at any time instant k, each subsystem $\mathcal{S}_i$ transmits to the subsystems having $\mathcal{S}_i$ as neighbor its future state and input reference trajectories (to be later defined) $\tilde{x}^{[i]}_{k+\nu}$ and $\tilde{u}^{[i]}_{k+\nu}$, $\nu = 0, \dots, N-1$, respectively, referred to the whole prediction horizon N. Moreover, by adding suitable constraints to its MPC formulation, $\mathcal{S}_i$ is able to guarantee that, for all $k \geq 0$, its real trajectories lie in specified time invariant neighborhoods of the reference trajectories, i.e, $x^{[i]}_k \in \tilde{x}^{[i]}_k \oplus \mathbb{E}_i$

and $u_k^{[i]} \in \tilde{u}_k^{[i]} \oplus \mathbb{E}_i^u$, where $0 \in \mathbb{E}_i$ and $0 \in \mathbb{E}_i^u$. In this way, the dynamics (2.2) of $\mathcal{S}_i$ can be reformulated as

$$x_{k+1}^{[i]} = A_{ii} x_k^{[i]} + B_{ii} u_k^{[i]} + \sum_{j \in \mathcal{N}_i} (A_{ij} \tilde{x}_k^{[j]} + B_{ij} \tilde{u}_k^{[j]}) + w_k^{[i]} \tag{2.3}$$

where

$$w_k^{[i]} = \sum_{j \in \mathcal{N}_i} \{A_{ij}(x_k^{[j]} - \tilde{x}_k^{[j]}) + B_{ij}(u_k^{[j]} - \tilde{u}_k^{[j]})\} \in \mathbb{W}_i \tag{2.4}$$

and

$$\mathbb{W}_i = \bigoplus_{j \in \mathcal{N}_i} \{A_{ij}\mathbb{E}_j \oplus B_{ij}\mathbb{E}_j^u\} \tag{2.5}$$

As already discussed, the main idea behind DPC is that each subsystem solves a robust MPC optimization problem considering that its dynamics is given by (2.3), where the term $\sum_{j \in \mathcal{N}_i} (A_{ij} \tilde{x}_{k+\nu}^{[j]} + B_{ij} \tilde{u}_{k+\nu}^{[j]})$ can be interpreted as an input known in advance over the prediction horizon $\nu = 0, \ldots, N-1$ to be suitably compensated and $w_k^{[i]}$ is a bounded disturbance to be rejected.

By definition, $w_k^{[i]}$ represents the uncertainty of the future actions that will be carried out by the dynamic neighbors of subsystem $\mathcal{S}_i$. Therefore the local MPC optimization problem to be solved at each time instant by the controller embedded in subsystem $\mathcal{S}_i$ must minimize the cost associated to $\mathcal{S}_i$ for any possible uncertainty values, i.e., without having to make any assumption on strategies adopted by the other subsystems, provided that their future trajectories lie in the specified neighborhood of the reference ones. Such conservative but robust local strategies adopted by each subsystem can be interpreted, from a dynamic non-cooperative game theoretic perspective, as max-min strategies, i.e., the strategies that maximize the worst case utility of $\mathcal{S}_i$ (for more details see, e.g., [150]).

To solve local robust MPC problems (denoted i-DPC problems), the algorithm proposed in [107] has been selected in view of the facts that no burdensome min-max optimization problem is required to be solved on-line, and that it naturally provides the future reference trajectories $\tilde{x}_k^{[i]}$ and $\tilde{u}_k^{[i]}$, as it will be clarified later in this chapter. According to [107], a nominal model of subsystem $\mathcal{S}_i$ associated to equation (2.3) must be defined to compute predictions

$$\hat{x}_{k+1}^{[i]} = A_{ii} \hat{x}_k^{[i]} + B_{ii} \hat{u}_k^{[i]} + \sum_{j \in \mathcal{N}_i} (A_{ij} \tilde{x}_k^{[j]} + B_{ij} \tilde{u}_k^{[j]}) \tag{2.6}$$

while the control law to be used for $\mathcal{S}_i$ is

$$u_k^{[i]} = \hat{u}_k^{[i]} + K_i(x_k^{[i]} - \hat{x}_k^{[i]}) \tag{2.7}$$

where K_i, $i = 1, \dots, M$, must be chosen to satisfy Assumption 2.1.

Letting $z_k^{[i]} = x_k^{[i]} - \hat{x}_k^{[i]}$, in view of (2.3), (2.6) and (2.7) one has

$$z_{k+1}^{[i]} = F_{ii} z_k^{[i]} + w_k^{[i]} \tag{2.8}$$

where $w_k^{[i]} \in \mathbb{W}_i$. Since $\mathbb{W}_i$ is bounded and F_{ii} is Schur, there exists a robust positively invariant (RPI) set $\mathbb{Z}_i$ for (2.8) such that, for all $z_k^{[i]} \in \mathbb{Z}_i$, then $z_{k+1}^{[i]} \in \mathbb{Z}_i$. Given $\mathbb{Z}_i$, define two sets, neighborhoods of the origin, $\Delta\mathbb{E}_i$ and $\Delta\mathbb{E}_i^u$, $i = 1, \dots, M$ such that $\Delta\mathbb{E}_i \oplus \mathbb{Z}_i \subseteq \mathbb{E}_i$ and $\Delta\mathbb{E}_i^u \oplus K_i\mathbb{Z}_i \subseteq \mathbb{E}_i^u$, respectively.

Finally, define the function $\hat{h}_{s,i}$ such that the constraint $\hat{h}_{s,i}(\hat{x}_k^{[i]}, \hat{u}_k^{[i]}, \tilde{\mathbf{x}}_k, \tilde{\mathbf{u}}_k) \leq 0$ guarantees that $h_{s,i}(x_k^{[i]}, u_k^{[i]}, \mathbf{x}_k^*, \mathbf{u}_k^*) \leq 0$ for all $x_k^{[i]} \in \hat{x}_k^{[i]} \oplus \mathbb{Z}_i$, $u_k^{[i]} \in \hat{u}_k^{[i]} \oplus K_i\mathbb{Z}_i$, $\mathbf{x}_k^* \in \tilde{\mathbf{x}}_k \oplus \prod_{i=1}^M \mathbb{E}_i$, and $\mathbf{u}_k^* \in \tilde{\mathbf{u}}_k \oplus \prod_{i=1}^M \mathbb{E}_i^u$.

2.2.1 The online phase: the i-DPC optimization problems

At any time instant k, we assume that each subsystem $\mathcal{S}_i$ knows the future reference trajectories of its neighbors $\tilde{x}_{k+\nu}^{[j]}$, $\tilde{u}_{k+\nu}^{[j]}$, $\nu = 0, \dots, N-1$, $j \in \mathcal{N}_i \cup \mathcal{H}_i \cup \{i\}$ and, with reference to its nominal system (2.6) only, solves the following i-DPC problem.

$$\min_{\hat{x}_k^{[i]}, \hat{u}_{[k:k+N-1]}^{[i]}} V_i^N(\hat{x}_k^{[i]}, \hat{u}_{[k:k+N-1]}^{[i]}) = \sum_{\nu=0}^{N-1} (\|\hat{x}_{k+\nu}^{[i]}\|_{Q_i^o}^2 + \|\hat{u}_{k+\nu}^{[i]}\|_{R_i^o}^2) + \|\hat{x}_{k+N}^{[i]}\|_{P_i^o}^2 \tag{2.9}$$

subject to (2.6), to

$$x_k^{[i]} - \hat{x}_k^{[i]} \in \mathbb{Z}_i \tag{2.10}$$

to

$$\hat{x}_{k+\nu}^{[i]} - \tilde{x}_{k+\nu}^{[i]} \in \Delta\mathbb{E}_i \tag{2.11}$$

$$\hat{u}_{k+\nu}^{[i]} - \tilde{u}_{k+\nu}^{[i]} \in \Delta\mathbb{E}_i^u \tag{2.12}$$

$$\hat{x}_{k+\nu}^{[i]} \in \hat{\mathbb{X}}_i \subseteq \mathbb{X}_i \ominus \mathbb{Z}_i \tag{2.13}$$

$$\hat{u}_{k+\nu}^{[i]} \in \hat{\mathbb{U}}_i \subseteq \mathbb{U}_i \ominus K_i\mathbb{Z}_i \tag{2.14}$$

for $\nu = 0, \ldots, N-1$, to the coupling state constraints

$$\hat{h}_{s,i}(\hat{x}_k^{[i]}, \hat{u}_k^{[i]}, \tilde{\mathbf{x}}_k, \tilde{\mathbf{u}}_k) \leq 0 \tag{2.15}$$

for all $s \in \mathbb{C}_i$ and to the terminal constraint

$$\hat{x}_{k+N}^{[i]} \in \hat{\mathbb{X}}_i^F \tag{2.16}$$

Note that constraints (2.10), (2.11) and (2.12) are used to guarantee the boundedness of the equivalent disturbance $w_k^{[i]}$. In fact in the i-DPC problem, for $\nu = 0$, constraints (2.10), (2.11), and (2.12) imply that $x_k^{[i]} - \tilde{x}_k^{[i]} \in \Delta\mathbb{E}_i \oplus \mathbb{Z}_i \subseteq \mathbb{E}_i$ and $u_k^{[i]} - \tilde{u}_k^{[i]} \in \Delta\mathbb{E}_i^u \oplus K_i\mathbb{Z}_i \subseteq \mathbb{E}_i^u$, which in turns guarantees that $w_k^{[i]} \in \mathbb{W}_i$. This, in view of the invariance property of (2.8), implies that $x_{k+1}^{[i]} - \hat{x}_{k+1}^{[i]} \in \mathbb{Z}_i$ and, since (2.11) and (2.12) are imposed over the whole prediction horizon, it follows by induction that $w_{k+\nu}^{[i]} \in \mathbb{W}_i$ for all $\nu = 0, \ldots, N-1$ and $x_{k+\nu}^{[i]} - \hat{x}_{k+\nu}^{[i]} \in \mathbb{Z}_i$ for all $\nu = 1, \ldots, N$.

In (2.9), Q_i^o, R_i^o, and P_i^o are positive definite matrices and represent design parameters, whose choice is discussed in Section 2.4 to guarantee stability and convergence properties, while $\hat{\mathbb{X}}_i^F$ in (2.16) is a nominal terminal set whose properties will be discussed in the next Section.

At time k, let the pair $(\hat{x}_{k|k}^{[i]}, \hat{u}_{[k:k+N-1]|k}^{[i]})$ be the solution to the i-DPC problem and define by $\hat{u}_{k|k}^{[i]}$ the input to the nominal system (2.6). Then, according to (2.7), the input to the system (2.2) is

$$u_k^{[i]} = \hat{u}_{k|k}^{[i]} + K_i(x_k^{[i]} - \hat{x}_{k|k}^{[i]}) \tag{2.17}$$

Denoting by $\hat{x}_{k+\nu|k}^{[i]}$ the state trajectory of system (2.6) stemming from $\hat{x}_{k|k}^{[i]}$ and $\hat{u}_{k:k+N-1|k}^{[i]}$, at time k it is also possible to compute $\hat{x}_{k+N|k}^{[i]}$ and $K_i\hat{x}_{k+N|k}^{[i]}$. In DPC, these values incrementally define the trajectories of the reference state and input variables to be used at the next time instant $k+1$, that is

$$\tilde{x}_{k+N}^{[i]} = \hat{x}_{k+N|k}^{[i]}\,,\ \tilde{u}_{k+N}^{[i]} = K_i\hat{x}_{k+N|k}^{[i]} \tag{2.18}$$

Note that the only information to be transmitted consists in the reference trajectories update (2.18). More specifically, at time step k, subsystem $\mathcal{S}_i$ computes $\tilde{x}_{k+N}^{[i]}$ and $\tilde{u}_{k+N}^{[i]}$ according to (2.18) and transmits their values to all the subsystems having $\mathcal{S}_i$ as neighbor, before proceeding to the next time step.

2.3 Convergence results and properties of DPC

In order to state the main theoretical contribution of the paper, define the set of admissible initial conditions $\mathbf{x}_0 = (x_0^{[1]}, \dots, x_0^{[M]})$ and initial reference trajectories $\tilde{x}^{[j]}_{[0:N-1]}$, $\tilde{u}^{[j]}_{[0:N-1]}$, for all $j = 1 \dots, M$ as follows.

Definition 2.1 *Letting* $\mathbf{x} = (x^{[1]}, \dots, x^{[M]})$, *denote by*

$$\begin{aligned}\mathbb{X}^N := \; & \{\mathbf{x} : \textit{ if } x_0^{[i]} = x^{[i]} \textit{ for all } i = 1, \dots, M \\ & \textit{then } \exists (\tilde{x}^{[1]}_{[0:N-1]}, \dots, \tilde{x}^{[M]}_{[0:N-1]}), (\tilde{u}^{[1]}_{[0:N-1]}, \dots, \tilde{u}^{[M]}_{[0:N-1]}), \\ & (\hat{x}^{[1]}_{0/0}, \dots, \hat{x}^{[M]}_{0/0}), (\hat{u}^{[1]}_{[0:N-1]}, \dots, \hat{u}^{[M]}_{[0:N-1]}) \textit{ such that } (2.2) \\ & \textit{and } (2.10)\text{-}(2.16) \textit{ are satisfied for all } i = 1, \dots, M\}\end{aligned}$$

the feasibility region for all the i-DPC problems. Moreover, for each $\mathbf{x} \in \mathbb{X}^N$, *let*

$$\begin{aligned}\tilde{\mathbb{X}}_{\mathbf{x}} := \; & \{(\tilde{x}^{[1]}_{[0:N-1]}, \dots, \tilde{x}^{[M]}_{[0:N-1]}), (\tilde{u}^{[1]}_{[0:N-1]}, \dots, \tilde{u}^{[M]}_{[0:N-1]}) : \\ & \textit{if } x_0^{[i]} = x^{[i]} \textit{ for all } i = 1, \dots, M \textit{ then } \exists \\ & (\hat{x}^{[1]}_{0/0}, \dots, \hat{x}^{[M]}_{0/0}), (\hat{u}^{[1]}_{[0:N-1]}, \dots, \hat{u}^{[M]}_{[0:N-1]}) \textit{ such that } (2.2) \\ & \textit{and } (2.10)\text{-}(2.16) \textit{ are satisfied for all } i = 1, \dots, M\}\end{aligned}$$

be the region of feasible initial reference trajectories.

Assumption 2.2 *Letting* $\hat{\mathbb{X}} = \prod_{i=1}^M \hat{\mathbb{X}}_i$, $\hat{\mathbb{U}} = \prod_{i=1}^M \hat{\mathbb{U}}_i$ *and* $\hat{\mathbb{X}}^F = \prod_{i=1}^M \hat{\mathbb{X}}_i^F$, *it holds that:*

i) $\hat{H}_s^{[i]}(\hat{\mathbf{x}}) \leq 0$ *for all* $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$, *for all* $s \in \mathbb{C}_i$, *for all* $i = 1, \dots, M$, *where* $\hat{H}^{[i]}$ *is such that* $\hat{H}_s^{[i]}(\hat{\mathbf{x}}) = \hat{h}_{s,i}(\hat{x}^{[i]}, K_i^{aux}\hat{x}^{[i]}, \hat{\mathbf{x}}, \mathbf{K}^{aux}\hat{\mathbf{x}})$ *for all* $s \in \mathbb{C}_i$, *for all* $i = 1, \dots, M$.

ii) $\hat{\mathbb{X}}^F \subseteq \hat{\mathbb{X}}$ *is an invariant set for* $\hat{\mathbf{x}}^+ = (\mathbf{A} + \mathbf{B}\mathbf{K}^{aux})\hat{\mathbf{x}}$;

iii) $\hat{\mathbf{u}} = \mathbf{K}^{aux}\hat{\mathbf{x}} \in \hat{\mathbb{U}}$ *for any* $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$;

iv) for all $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$ *and, for a given constant* $\kappa > 0$,

$$\mathbf{V}^F\left(\hat{\mathbf{x}}^+\right) - \mathbf{V}^F\left(\hat{\mathbf{x}}\right) \leq -(1+\kappa)\mathbf{l}(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \tag{2.19}$$

where $\mathbf{V}^F(\hat{\mathbf{x}}) = \sum_{i=1}^M \|\hat{x}^{[i]}\|^2_{P_i^o}$ *and* $\mathbf{l}(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \sum_{i=1}^M (\|\hat{x}^{[i]}\|^2_{Q_i^o} + \|\hat{u}^{[i]}\|^2_{R_i^o})$.

□

In Section 2.4 we will show how to properly select matrices Q_i^o, R_i^o, and P_i^o to guarantee stability and convergence properties. It will also be explained how to choose the terminal set $\hat{\mathbb{X}}_i^F$ in order to satisfy Assumption 2.2.

Assumption 2.3 *Given the sets $\mathbb{E}_i$, $\mathbb{E}_i^u$, and the RPI sets $\mathbb{Z}_i$ for equation* (2.8)*, there exists a real positive constant $\bar{\rho}_E > 0$ such that $\mathbb{Z}_i \oplus \mathbb{B}_{\bar{\rho}_E}^{(n_i)}(0) \subseteq \mathbb{E}_i$ and $K_i\mathbb{Z}_i \oplus \mathbb{B}_{\bar{\rho}_E}^{(m_i)}(0) \subseteq \mathbb{E}_i^u$ for all $i = 1, \dots, M$, where $\mathcal{B}_{\bar{\rho}_E}^{(dim)}(0)$ is a ball of radius $\bar{\rho}_E > 0$ centered at the origin in the $\mathbb{R}^{dim}$ space.* □

Proper ways to select the design parameters satisfying Assumption 2.3 are presented in the following section.

Theorem 2.1 *Let Assumptions 2.1-2.3 be satisfied and let $\Delta\mathbb{E}_i$ and $\Delta\mathbb{E}_i^u$ be neighborhoods of the origin satisfying $\Delta\mathbb{E}_i \oplus \mathbb{Z}_i \subseteq \mathbb{E}_i$ and $\Delta\mathbb{E}_i^u \oplus K_i\mathbb{Z}_i \subseteq \mathbb{E}_i^u$. Then, for any initial reference trajectories in $\tilde{\mathbb{X}}_{\mathbf{x}_0}$, the trajectory $\mathbf{x}_k$, starting from any initial condition $\mathbf{x}_0 \in \mathbb{X}^N$, asymptotically converges to the origin while fulfilling all the constraints.* ■

Proofs can be found in the Appendix.

2.3.1 Properties of DPC

Some further properties of DPC are in order.

Optimality issues. Global optimality of the interconnected closed loop system cannot be guaranteed using DPC. This, on the one hand, is due to the inherent conservativeness of robust algorithms and, on the other hand, is due to the game theoretic characterization of DPC. Namely, the provided solution to the control problem can be cast as a max-min solution of a dynamic non-cooperative game (see, e.g., [150]) where all the involved subsystems aim to optimize local cost functions which are different from each other: therefore, different and possibly conflicting goals inevitably imply suboptimality. Differently from suboptimal distributed MPC algorithms discussed in [136], whose solutions can be regarded as Nash solutions of non-cooperative games and which possibly lead to instability of the closed-loop system, the convergence of the DPC algorithm can be guaranteed, see Theorem 2.1.

Robustness. As already discussed, the algorithm presented in this chapter basically relies on the tube-based robust MPC algorithm proposed in [107]. Namely, robustness is here used to cope with uncertainties on the input and state trajectories of the neighboring subsystems.

More specifically, the difference between the reference trajectories of the neighboring subsystems and the real ones is regarded as a disturbance, which is known to be bounded in view of suitable constraints imposed in the optimization problem. Anyway, the described approach can be naturally extended for coping also with standard additive disturbances in the interconnected models (2.2), i.e., in case the interconnected perturbed systems are described by the dynamic equations

$$x_{k+1}^{[i]} = A_{ii}x_k^{[i]} + B_{ii}u_{[i]}(k) + \sum_{j\in\mathcal{N}_i}\{A_{ij}x_k^{[j]} + B_{ij}u_k^{[j]}\} + d_k^{[i]} \tag{2.20}$$

where $d_k^{[i]} \in \mathbb{D}_i \subset \mathbb{R}^{n_i}$ is an unknown bounded disturbance and the set $\mathbb{D}_i$ is a convex neighborhood of the origin. In this case, we have again

$$x_{k+1}^{[i]} = A_{ii}x_k^{[i]} + B_{ii}u_k^{[i]} + \sum_{j\in\mathcal{N}_i}(A_{ij}\tilde{x}_k^{[j]} + B_{ij}\tilde{u}_k^{[j]}) + w_k^{[i]} \tag{2.21}$$

where this time the disturbance acting on subsystem i is

$$w_k^{[i]} = \sum_{j\in\mathcal{N}_i}\{A_{ij}(x_k^{[j]} - \tilde{x}_k^{[j]}) + B_{ij}(u_k^{[j]} - \tilde{u}_k^{[j]})\} + d_k^{[i]} \in \mathbb{W}_i \tag{2.22}$$

and

$$\mathbb{W}_i = \bigoplus_{j\in\mathcal{N}_i}\{A_{ij}\mathbb{E}_j \oplus B_{ij}\mathbb{E}_j^u\} \oplus \mathbb{D}_i \tag{2.23}$$

The overall disturbed collective system, letting $\mathbf{d}_k = (d_k^{[1]}, \ldots, d_k^{[M]})$ with $\mathbf{d}_k \in \mathbb{D} = \prod_{i=1}^M \mathbb{D}_i \subset \mathbb{R}^n$ can be written as

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{d}_k \tag{2.24}$$

and can be steered to a neighboring of the origin.

Output feedback. The approach that has been previously described for coping with unknown exogenous additive disturbances has been employed in [49] for designing a DPC algorithm for output feedback control. Specifically, assume that the input and output equations of the system are the following

$$\begin{array}{rcl} x_{k+1}^{o[i]} & = & A_{ii}x_k^{o[i]} + B_{ii}u_k^{[i]} + \sum_{j\in\mathcal{N}_i}\{A_{ij}x_k^{o[j]} + B_{ij}u_k^{[j]}\} \\ y_k^{[i]} & = & C_i x_k^{o[i]} \end{array} \tag{2.25}$$

where the state which is not directly available is here denoted as $x_k^{o[i]}$ for reasons that will become clearer later on. Denote with $x_k^{[i]}$ the estimate of $x_k^{o[i]}$, for all $i = 1, \ldots, M$. To estimate the state of (2.25) we employ a decentralized Luenberger-like observer of the type

$$x_{k+1}^{[i]} = A_{ii}x_k^{[i]} + B_{ii}u_{k+1}^{[i]} + \sum_{j\in\mathcal{N}_i}\{A_{ij}x_k^{[j]} + B_{ij}u_{k+1}^{[j]}\} - L_i(y_k^{[i]} - C_i x_k^{[i]}) \quad (2.26)$$

Assume that the decentralized observer is convergent i.e., $\mathbf{A} + \mathbf{LC}$ is Schur, where $\mathbf{C} = \mathrm{diag}(C_1, \ldots, C_M)$ and $\mathbf{L} = \mathrm{diag}(L_1, \ldots, L_M)$. Under this assumption it is possible to guarantee that the estimation error for each subsystem is bounded, i.e., $x_k^{o[i]} - x_k^{[i]} \in \Sigma_i$ for all $i = 1, \ldots, M$. In this way (2.26) exactly corresponds to the perturbed system (2.21), where $d_k^{[i]} = -L_i C_i (x_k^{o[i]} - x_k^{[i]})$ is regarded as a bounded disturbance, i.e., $d_k^{[i]} \in \mathbb{D}_i = -L_i C_i \Sigma_i$. From this point on, the output feedback control problem is solved as a robust state feedback problem applied to the system (2.21). Details on this approach can be found in [49], where a condition and a constructive method are derived to compute the sets Σ_i in such a way that $\Sigma = \prod_{i=1}^{M} \Sigma_i$ is an invariant set for the interconnected observer error.

Tracking. As it will be shown in the next chapters, the DPC method can be extended also to the problem of tracking desired output signals.

2.4 Implementation issues

2.4.1 The discretization method

In most of the control applications, the model of the plant is developed in the continuous-time starting from physical laws. In this framework, the sparse structure of the model clearly represents physical connections (such as mass or energy flows) among the subsystems, each one described by the linear (or linearized) model

$$\dot{x}^{[i]}(t) = A_{ii}^c x^{[i]}(t) + B_{ii}^c u^{[i]}(t) + \sum_{j\in\mathcal{N}_i}\{A_{ij}^c x^{[j]}(t) + B_{ij}^c u^{[j]}(t)\} \quad (2.27)$$

where the notation is coherent (mutatis mutandis) with the one adopted in (2.2). Unfortunately, the sparse, zero-nonzero pattern of the system (zero-nonzero matrices A_{ij}^c, B_{ij}^c) is lost when the exact ZOH (Zero-Order-Hold), Backward Euler, or bilinear transformations are used to discretize the system, while it is preserved only by the Forward Euler (FE) transformation. However, it is well known that with FE some important properties of the underlying continuous time system can be lost; for example stability is maintained only for very small sampling times, which can be inadvisable in many digital control applications.

In distributed and decentralized control techniques based on MPC, where discrete-time models are mainly utilized, the loss of sparsity can easily result in an increase of the controller complexity. For these reasons, in order to improve the performance of FE and to maintain sparsity, a new discretization method called Mixed Euler ZOH ($ME-ZOH$) has been proposed in [33, 34], and its properties have been studied in [48]. In synthesis, ME-ZOH allows one to compute the matrices of (2.2) starting from the continuous-time model (2.27) as follows

$$A_{ii}(h) = e^{A^c_{ii}h} \tag{2.28}$$

$$A_{ij}(h) = \int_0^h e^{A^c_{ii}t} dt A^c_{ij}, \quad j \neq i \tag{2.29}$$

$$B_{ij}(h) = \int_0^h e^{A^c_{ii}t} dt B^c_{ij}, \quad \forall\, i, j \tag{2.30}$$

where h is the adopted sampling time. It is apparent that the zero-nonzero structure of the matrices A^c_{ij} and B^c_{ij} is maintained together with some important properties, see [48].

2.4.2 Computation of the state-feedback gain and of the weighting matrices

The design of the block diagonal matrix $\mathbf{K}$ satisfying Assumption 2.1, and the computation of the positive-definite block diagonal matrix $\mathbf{P}^o = \text{diag}(P^o_1, \ldots, P^o_M)$, $P^o_i \in R^{n_i \times n_i}$ can be done resorting to Linear Matrix Inequalities (LMIs) [18], see the Appendix and [12] for additional details.

Algorithm 2.1 ***Computation of the state-feedback gain and of the weighting matrices - Method 1***

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1. *Define* $\mathbf{K} = \mathbf{Y}\mathbf{S}^{-1}$ *and* $\mathbf{P}^o = \mathbf{S}^{-1}$, *and solve for* $\mathbf{Y}$ *and* $\mathbf{S}$ *the following LMI*

$$\begin{bmatrix} \mathbf{S} & \mathbf{S}\mathbf{A}^T + \mathbf{Y}^T\mathbf{B}^T \\ \mathbf{A}\mathbf{S} + \mathbf{B}\mathbf{Y} & \mathbf{S} \end{bmatrix} \succ 0 \tag{2.31}$$

with the additional constraints

$$S_{ij} = \mathbf{0} \quad \forall i, j = 1, ..., M \quad (i \neq j) \tag{2.32}$$

$$Y_{ij} = \mathbf{0} \quad \forall i, j = 1, ..., M \quad (i \neq j) \tag{2.33}$$

where $S_{ij} \in \mathbb{R}^{(n_i+m_i)\times(n_j+m_j)}$, $Y_{ij} \in \mathbb{R}^{n_i\times(n_j+m_j)}$ *are the blocks outside the diagonal of* $\mathbf{S}$ *and* $\mathbf{Y}$*, respectively. Finally, denoting by* S_{ii} *and* Y_{ii} *the block diagonal elements of* $\mathbf{S}$ *and* $\mathbf{Y}$*, respectively, the requirement that each block* K_i *must be stabilizing for its i-th subsystem (recall again Assumption* (2.1)*), translates in the following set of conditions*

$$\begin{bmatrix} S_{ii} & S_{ii}A_{ii}^T + Y_{ii}^T B_{ii}^T \\ A_{ii}S_{ii} + B_{ii}Y_{ii} & S_{ii} \end{bmatrix} \succ 0 \tag{2.34}$$

In conclusion, the computation of $\mathbf{K}$ *and* $\mathbf{P}^o$ *calls for the solution of the set of LMI's* (2.77),(2.78), (2.79) *and* (2.80)*, which can be easily found with suitable available software (e.g., YALMIP [92]).*

2. *Once* $\mathbf{K}$ *and* $\mathbf{P}^o$ *are available the parameters* Q_i^o *and* R_i^o *must be chosen to satisfy* (2.19)*. To this end, define* $\bar{\mathbf{Q}} = \mathbf{P}^o - (\mathbf{A}+\mathbf{BK})^T\mathbf{P}^o(\mathbf{A}+\mathbf{BK})$*, choose an arbitrarily small positive constant* κ *and two block diagonal matrices* $\mathbf{Q} = diag(Q_1^o, \dots, Q_M^o)$ *and* $\mathbf{R} = diag(R_1^o, \dots, R_M^o)$*, with* $Q_i^o \succ 0 \in R^{n_i\times n_i}$ *and* $R_i^o \succ 0 \in R^{m_i\times m_i}$*. Then proceed as follows:*

   - *if*
   $$\bar{\mathbf{Q}} - (\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K})(1+\kappa) \succ 0 \tag{2.35}$$
   set $\mathbf{Q}^o = \mathbf{Q}$ *and* $\mathbf{R}^o = \mathbf{R}$*;*
   - *otherwise set* $\mathbf{Q} = \eta\mathbf{Q}$ *and* $\mathbf{R} = \eta\mathbf{R}$*, with* $0 < \eta < 1$*, and repeat the procedure until* (2.35) *is fulfilled. Once* $\mathbf{Q}$ *and* $\mathbf{R}$ *satisfying* (2.35) *have been found, set* $\mathbf{Q}^o = \mathbf{Q}$ *and* $\mathbf{R}^o = \mathbf{R}$.

   Finally, extract from $\mathbf{Q}^o$ *and* $\mathbf{R}^o$ *the submatrices* Q_i^o *and* R_i^o *of appropriate dimensions.*

A second possibility is discussed in [50], and requires that a set of control gains K_i, $i = 1, \dots, M$, verifying Assumption 2.1 are given.

Algorithm 2.2 ***Computation of the the weighting matrices - Method 2***

1. *Define* $F_{ij} = A_{ij} + B_{ij}K_j,\ i,j = 1,\dots,M,$ *and let* ν_i *denote the number of dynamic neighbors of subsystem* i *plus* 1. *It is well known that, if* $\sqrt{\nu_i}F_{ii}$ *is Schur, then for any* $Q_i = Q_i^T \succ 0$ *there exists a matrix* $P_i = P_i^T \succ 0$ *satisfying*
$$\nu_i F_{ii}^T P_i F_{ii} - P_i = -Q_i \tag{2.36}$$
Define the matrix $\mathcal{M}^Q \in \mathbb{R}^{M\times M}$ *with entries* μ_{ij}^Q
$$\mu_{ii}^Q = -\lambda_m(Q_i),\ i = 1,\dots,M \tag{2.37a}$$
$$\mu_{ij}^Q = \nu_j \|F_{ji}^T P_j F_{ji}\|_2,\ i,j = 1,\dots,M \text{ with } i \neq j \tag{2.37b}$$
2. *If* $\mathcal{M}^Q$ *is Hurwitz, define the values of* $p_i,\ i = 1,\dots,M$ *as the entries of the strictly positive vector* $\mathbf{p}$ *satisfying* $\mathcal{M}^Q\mathbf{p} \prec 0$.
3. *Set* $P_i^o = p_i P_i$.
4. $Q_i^o \succ 0$, $R_i^o \succ 0$, *and* κ_i *are chosen in such a way that*
$$rank([B_{ii}^T\ R_i^{oT}]^T) = m_i \tag{2.38}$$
$$(1+\kappa_i)(Q_i^o + (K_i)^T R_i^o R_i) \leq \tilde{Q}_i \tag{2.39}$$
where
$$\tilde{Q}_i = p_i Q_i - \sum_{j=1}^{M} p_j \nu_j F_{ji}^T P_j F_{ji} \tag{2.40}$$
5. *Set* $\kappa = \min(\kappa_1,\dots,\kappa_M)$.

2.4.3 Computation of the RPI sets and of the terminal sets

Two of the main issues in DPC are to verify, for all $i = 1,\dots,M$ that *i)* $\mathbb{E}_i \supseteq \mathbb{Z}_i \oplus \Delta\mathbb{E}_i$, $\mathbb{E}_i^u \supseteq K_i\mathbb{Z}_i \oplus \Delta\mathbb{E}_i^u$, $\mathbb{Z}_i \subset \mathbb{X}_i$ and $K_i\mathbb{Z}_i \subset \mathbb{U}_i$, and *ii)* $\hat{\mathbb{X}}_i^F \subseteq \hat{\mathbb{X}}_i$ and $K_i\hat{\mathbb{X}}_i^F \subseteq \hat{\mathbb{U}}_i$.

Concerning *i)*, remember that $\mathbb{Z}_i$ is the RPI set for equation (2.8) where the disturbance term w_k^i lies in the set $\mathbb{W}_i$, which depends on sets $\mathbb{Z}_j$, $j \in \mathcal{N}_i$. For this reason, the problem can not be tackled by considering each subsystem separately. In this section we propose some alternative solutions to *i)*.

Furthermore, to verify *ii)* we simply set $\hat{\mathbb{X}}_i^F = \alpha \mathbb{Z}_i$ for all $i = 1, \ldots, M$, for an arbitrary and sufficiently small $\alpha \in (0, 1)$. Finally, remark that an algorithm for obtaining a polytopic invariant outer approximation of the minimal RPI set has been presented in Chapter 1 [131, 132].

The first technique to compute the RPI sets $\mathbb{Z}_i$ is based on an empirical simplified distributed reachability analysis procedure, which has obtained remarkable results in several applications. We will use rectangular sets (i.e., through the *box* operation) to greatly simplify the set-theoretical computations (e.g., the Minkowski sums), at the price of slightly more conservative results. Anyway, the same procedure could be performed without resorting to the *box* operation, provided that the number of states of the submodels is small enough.

Algorithm 2.3 ***Computation of the RPI sets - Method 1***

1. *For all $i = 1, \ldots, M$, arbitrarily choose hyperrectangles $\Delta\mathbb{E}_i$ and $\Delta\mathbb{E}_i^u$.*
2. *Initialize $\mathbb{Z}_i = \bigoplus_{j \in \mathcal{N}_i} \{box(A_{ij}\Delta\mathbb{E}_j) \oplus box(B_{ij}\Delta\mathbb{E}_j^u)\}$ for all $i = 1, \ldots, M$.*
3. *For all $i = 1, \ldots, M$, compute $\mathbb{Z}_i^+ = box(F_{ii}\mathbb{Z}_i) \oplus \{\bigoplus_{j \in \mathcal{N}_i} \{box(A_{ij}\mathbb{Z}_j) \oplus box(B_{ij}K_j\mathbb{Z}_j)\}\} \oplus \{\bigoplus_{j \in \mathcal{N}_i} \{box(A_{ij}\Delta\mathbb{E}_j) \oplus box(B_{ij}\Delta\mathbb{E}_j^u)\}\}$.*
4. *If $\mathbb{Z}_i^+ \subseteq \mathbb{Z}_i$ for all $i = 1, \ldots, M$ then go to step 5: by definition, the hyperrectangles $\mathbb{Z}_i$ actually correspond to the required RPI sets. Otherwise set $\mathbb{Z}_i = \mathbb{Z}_i^+$ and repeat step 3.*
5. *If $\mathbb{Z}_i \subset \mathbb{X}_i$ and $K_i\mathbb{Z}_i \subset \mathbb{U}_i$ then stop. Otherwise set $\Delta\mathbb{E}_i = \gamma\Delta\mathbb{E}_i$, $\Delta\mathbb{U}_i = \gamma\Delta\mathbb{U}_i$, with $\gamma \in (0, 1)$, and go to step 2.*

A second possibility for computing the RPI sets $\mathbb{Z}_i$ consists in solving a linear programming (LP) problem. For all $i = 1, \ldots, M$, we define sets $\mathbb{E}_i$, $\mathbb{E}_i^u$, $\Delta\mathbb{E}_i$ and $\Delta\mathbb{E}_i^u$ as hypercubes, centred on the origin, with faces perpendicular to the cartesian axis and the scalars $e_i = \|\mathbb{E}_i\|_\infty$, $e_i^u = \|\mathbb{E}_i^u\|_\infty$, $\Delta e_i = \|\Delta\mathbb{E}_i\|_\infty$ and $\Delta e_i^u = \|\Delta\mathbb{E}_i^u\|_\infty$ corresponding to a half of the edge of each hypercube. Define also x_i^∞ and u_i^∞ as the infinity norms (i.e., a half of the edges) of the biggest cubes, centred on the origin, inscribed inside of $\mathbb{X}_i$ and $\mathbb{U}_i$, respectively.

If we define $w_i^\infty = \sum_{j \in \mathcal{N}_i} \{\|A_{ij}\|_\infty e_j + \|B_{ij}\|_\infty e_j^u\}$, using the properties of norm operators it is possible to state that $w_i^\infty \geq \|\mathbb{W}_i\|_\infty$, where $\mathbb{W}_i$ is the set containing the real disturbance affecting subsystem i and $\|\mathbb{W}_i\|_\infty$ corresponds to a half of the edge of the smallest hypercube centered on the origin having faces perpendicular to the cartesian axis and containing $\mathbb{W}_i$. To compute the RPI set $\mathbb{Z}_i$ for (2.8) (see [131, 132]) we use the hypercube $\mathbb{W}_i^\infty$ having infinity norm w_i^∞, i.e., $\mathbb{Z}_i = \frac{1}{1-\alpha_i} \bigoplus_{l=0}^{s_i-1} F_{ii}^l \mathbb{W}_i^\infty$, where s and $\alpha_i \in [0,1)$ must fulfill $F_{ii}^{s_i} \mathbb{W}_i^\infty \subseteq \alpha_i \mathbb{W}_i^\infty$. The latter is verified if $\|F_{ii}\|_\infty^{s_i} \leq \alpha_i$ in view of properties of the norm operator and of the hypercubes. In addition, remark that $\mathbb{Z}_i$ is contained inside the hypercube having infinity norm $\gamma_i w_i^\infty$, where $\gamma_i = 1/(1-\alpha_i)\sum_{l=0}^{s_i-1} \|F_{ii}\|_\infty^l$. These considerations suggest the following procedure for computing $\mathbb{Z}_i$.

Algorithm 2.4 ***Computation of the RPI sets - Method 2***

1. *For all $i = 1, \dots, M$, arbitrarily choose parameters α_i.*
2. *For all $i = 1, \dots, M$, compute s_i such that $\|F_{ii}\|_\infty^{s_i} \leq \alpha_i$ and than evaluate $\gamma_i = (1-\alpha_i)^{-1} \sum_{l=0}^{s_i-1} \|F_{ii}\|_\infty^l$.*
3. *Solve the following linear programming problem.*

$$\min_{o_v} \rho \tag{2.41}$$

subject to

$$\rho \geq \gamma_i w_i^\infty \quad \forall i = 1, \dots, M \tag{2.42}$$
$$\gamma_i w_i^\infty + \Delta e_i \leq e_i \quad \forall i = 1, \dots, M \tag{2.43}$$
$$\|K_i\|_\infty \gamma_i w_i^\infty + \Delta u_i \leq e_i^u \quad \forall i = 1, \dots, M \tag{2.44}$$
$$\gamma_i w_i^\infty \leq x_i^\infty \quad \forall i = 1, \dots, M \tag{2.45}$$
$$\|K_i\|_\infty \gamma_i w_i^\infty \leq u_i^\infty \quad \forall i = 1, \dots, M \tag{2.46}$$
$$\Delta e_i \geq \Delta \bar{e}_i \quad \forall i = 1, \dots, M \tag{2.47}$$
$$\Delta u_i \geq \Delta \bar{u}_i \quad \forall i = 1, \dots, M \tag{2.48}$$

where $o_v = (\Delta e_1, e_1, \Delta u_1, e_1^u, \dots, \Delta e_M, e_M, \Delta u_M, e_M^u) \in \mathcal{R}^{4M}$ contains only strictly positive elements. $\Delta \bar{e}_i$ and $\Delta \bar{u}_i$ are arbitrary positive parameters to be used in order to have sets $\Delta \mathbb{E}_i$ and $\Delta \mathbb{E}_i^u$ bigger than a prescribed size.

4. *Compute $\mathbb{Z}_i = (1-\alpha_i)^{-1} \bigoplus_{l=0}^{s_i-1} F_{ii}^l \mathbb{W}_i^\infty$.*

In the proposed optimization problem, the objective function combined together with constraints (2.42) aims at minimizing the largest RPI set.

Constraints (2.45) and (2.46) guarantee the existence of sets $\hat{\mathbb{X}}_i$ and $\hat{\mathbb{U}}_i$ for all the subsystems. Lastly, constraints (2.43) and (2.44), if the LP problem turns out to be feasible, allow one to find the hypercubes $\mathbb{E}_i$, $\Delta\mathbb{E}_i$, $\mathbb{E}_i^u$ and $\Delta\mathbb{E}_i^u$ such that $\mathbb{E}_i \supseteq \mathbb{Z}_i \oplus \Delta\mathbb{E}_i$ and $\mathbb{E}_i^u \supseteq K_i\mathbb{Z}_i \oplus \Delta\mathbb{E}_i^u$.

Note that these first two methods, with trivial modifications, can be also applied to systems where the subsystems are affected by an exogenous disturbance $d_k^{[i]} \in \mathbb{D}_i \subset \mathbb{R}^{n_i}$ and have a dynamics described by equation 2.20.

Specifically, Algorithm 2.3 has to be modified by initializing $\mathbb{Z}_i = \bigoplus_{j\in\mathcal{N}_i}\{\text{box}(A_{ij}\Delta\mathbb{E}_j) \oplus \text{box}(B_{ij}\Delta\mathbb{E}_j^u)\} \oplus \text{box}(\mathbb{D}_i)$ and computing

$$\begin{aligned}\mathbb{Z}_i^+ = \text{box}(F_{ii}\mathbb{Z}_i) \oplus \{&\bigoplus_{j\in\mathcal{N}_i}\{\text{box}(A_{ij}\mathbb{Z}_j) \oplus \text{box}(B_{ij}K_j\mathbb{Z}_j)\}\}\oplus \\ &\{\bigoplus_{j\in\mathcal{N}_i}\{\text{box}(A_{ij}\Delta\mathbb{E}_j) \oplus \text{box}(B_{ij}\Delta\mathbb{E}_j^u)\}\}) \oplus \text{box}(\mathbb{D}_i)\end{aligned}$$

for all $i = 1, \ldots, M$.

As for Algorithm 2.4, it is sufficient to define $w_i^\infty = \sum_{j\in\mathcal{N}_i}\{\|A_{ij}\|_\infty e_j + \|B_{ij}\|_\infty e_j^u\} + d_i^\infty$, where $d_i^\infty = \|\mathbb{D}_i\|_\infty$ is equal to a half of the edge of the smallest hypercube with faces perpendicular to the cartesian axis and containing the set $\mathbb{D}_i$.

A last option is the following [50].

Algorithm 2.5 ***Computation of the RPI sets - Method 3***

1. *Assume that $\mathbb{E}_i$ can be equivalently represented in one of the following two ways:*

$$\begin{aligned}\mathbb{E}_i &= \{\varepsilon_i \in \mathbb{R}^{n_i} | \varepsilon_i = \Xi_i\delta_i \text{ where } \|\delta_i\|_\infty \le l_i\} \\ &= \{\varepsilon_i \in \mathbb{R}^{n_i} | f_{i,r}^T\varepsilon_i \le l_i \text{ for all } r\}\end{aligned} \tag{2.49}$$

   where $\delta_i \in \mathbb{R}^{n_{\delta_i}}$, $\Xi_i \in \mathbb{R}^{n_i\times n_{\delta_i}}$, $f_{i,r} \in \mathbb{R}^{n_i}$, and $r = 1, \ldots, \bar{r}_i$ for all $i = 1, \ldots, M$. The constants $l_i \in \mathbb{R}_+$, appearing in both equivalent definitions, can be regarded as scaling factors.
   Define the shape of the polyhedra with a proper setting of matrices Ξ_i and vectors $f_{i,r}$, $i = 1, \ldots, M$.

2. *Assuming that F_{ii} is diagonalizable for all $i = 1, \ldots, M$ (which is always possible since K_is are design parameters) define N_i, $i = 1, \ldots, M$, such that $F_{ii} = N_i^{-1}\Lambda_i N_i$, where $\Lambda_i = diag(\lambda_{i,1}, \ldots, \lambda_{i,n_i})$, where $\lambda_{i,j}$ is the j-th eigenvalue of F_{ii}.*

Define also

$$f_i = \begin{bmatrix} f_{i,1}^T \\ \vdots \\ f_{i,\bar{r}_i}^T \end{bmatrix}$$

$i = 1, \dots, M$. Then, compute the matrix $\mathcal{M}^P \in \mathbb{R}^{M \times M}$ whose entries μ_{ij}^P are

$$\mu_{ii}^P = -1,\ i = 1, \dots, M \tag{2.50a}$$

$$\mu_{ij}^P = \|f_i N_i^{-1}\|_\infty (\|N_i A_{ij} \Xi_j\|_\infty + \|N_i B_{ij} K_j \Xi_j\|_\infty) \frac{1}{1 - \max_{j=1,\dots,n_i} |\lambda_{i,j}|},$$
$$i, j = 1, \dots, M \text{ with } i \neq j \tag{2.50b}$$

3. *If $\mathcal{M}^P$ is Hurwitz, define the values of l_i, $i = 1, \dots, M$, as the entries of the strictly positive vector $\mathbf{l}$ satisfying $\mathcal{M}^P \mathbf{l} < 0$. For its computation use the following procedure:*
   - *If the system is irreducible [47], $\mathbf{l}$ is the Frobenius eigenvector of matrix $\mathcal{M}^P$.*
   - *If the system is reducible:*
     a) *Since the system is reducible there exists a permutation matrix $\mathbf{H}$ (where $\mathbf{H}^T = \mathbf{H}^{-1}$) such that $\tilde{\mathcal{M}}^P = \mathbf{H}\mathcal{M}^P\mathbf{H}^T$ is lower block triangular, with block elements M_{ij}, whose diagonal blocks M_{ii} are irreducible. Let $\boldsymbol{\omega}_i$ (strictly positive element-wise) be the Frobenius eigenvector of $\mathbf{M}_{ii}$, associated to the eigenvalue $\lambda_i < 0$.*
     b) *Set $\alpha_1 = 1$ and define recursively α_i, for $i > 1$, such that $\alpha_i |\lambda_i v_i| > |\sum_{j=1}^{i-1} \alpha_j M_{ij} v_j|$ element-wise (v_i is the number of dynamic neighbors of subsystem i).*
     c) *Define $\boldsymbol{\omega} = [\alpha_1 \boldsymbol{\omega}_1^T, \dots, \alpha_M \boldsymbol{\omega}_M^T]^T$, and $\mathbf{v}_\mathcal{M} = \mathbf{H}^T \boldsymbol{\omega}$. From the definition of α_is, it follows that $\tilde{\mathcal{M}}^P \boldsymbol{\omega} < 0$, and so $\mathcal{M}^P \mathbf{v}_\mathcal{M} = \mathbf{H}^T\mathbf{H}\mathcal{M}^P\mathbf{H}^T\mathbf{H}\mathbf{v}_\mathcal{M} = \mathbf{H}^T\tilde{\mathcal{M}}^P\boldsymbol{\omega} < 0$.*
4. *Compute $\mathbb{E}_i^u = K_i \mathbb{E}_i$, for all $i = 1, \dots, M$.*
5. *Compute $\mathbb{W}_i = \bigoplus_{j \in \mathcal{N}_i} \{A_{ij}\mathbb{E}_j \oplus B_{ij}\mathbb{E}_j^u\}$, for all $i = 1, \dots, M$.*
6. *For all $i = 1, \dots, M$, compute $\mathbb{Z}_i$ as the polytopic RPI outer δ-approximation of the minimal RPI (mRPI) set for (2.8), as shown in [132].*

7. *For all $i = 1, \dots, M$, the sets $\Delta\mathbb{E}_i$, can be taken as any polytope satisfying $\Delta\mathbb{E}_i \oplus \mathbb{Z}_i \subseteq \mathbb{E}_i$, and finally $\Delta\mathbb{E}_i^u = K_i \Delta\mathbb{E}_i$.*

If $\mathcal{M}^P$ defined in (2.50) is not Hurwitz, the previous algorithm can not be applied.

We remark that the algorithms presented for computing the sets only provide sufficient conditions, meaning that other criteria and algorithms can be devised and adopted, which can be even more efficient, especially when applied to specific case studies. In addition, the effectiveness of the proposed methods strongly depends on some arbitrary initial choices, i.e., in Algorithm 2.5 matrices Ξ_i (or equivalently vectors f_i), $i = 1, \dots, M$, defining the shape of sets $\mathbb{E}_i$ and in Algorithm 2.3 the shapes and dimension of the sets $\Delta\mathbb{E}_i$ and $\Delta\mathbb{E}_i^u$, $i = 1, \dots, M$. If the selected method results to be inapplicable for a given choice, a trial-and-error procedure is suggested, in order to find a suitable initial choice (which is nevertheless not guaranteed to exist) guaranteeing the applicability of the selected method.

2.4.4 Generation of the reference trajectories (for systems without coupling constraints)

The DPC algorithm assumes that an initial feasible solution or, more specifically, an initial reference trajectory, exists. This problem can be cast as a purely offline design problem.

On the other hand, disturbances of unexpected entity could occur during the ordinary system operation, alterating the system's condition (i.e., by producing constraint violation, e.g., $x_{k+1}^{[i]} - \hat{x}_{k+1|k}^{[i]} \notin \mathbb{Z}_i$) with possible serious consequences on the future solution (e.g., concerning feasibility) of the control problems. Once this condition is detected by a given system $\mathcal{S}_i$, it must be broadcast to all other subsystems through an event-based emergency iterative transmission, and an extra-ordinary reset operation requires the recalculation of new suitable state and output reference trajectories for all subsystems.

The simplest solution consists (according with the approach suggested in [44]) in generating these trajectories using a centralized controller. This has the drawback that a centralized controller must be designed together with the distributed ones, and that it must be kept activated while the system is running in order to recover the proper functioning of the process if unpredicted external disturbances affect the plant. Obviously, the need of a centralized "hidden" supervisor greatly reduces the advantages of utilizing a distributed control scheme.

In this section, we present two different methods for the generation of the trajectories, useful both for offline reference trajectory generation (i.e., performed at time $k = 0$) and for extra-ordinary reset operations, requiring number of iterative information exchanges among neighbors. The first method (i.e., Algorithm 2.7) is applicable when $\mathbf{x} \in \hat{\mathbb{X}}^F$, while the second one (i.e., Algorithm 2.8) can be used when $\mathbf{x} \notin \hat{\mathbb{X}}^F$. Therefore, at each time step k, we first need a procedure to check whether $\mathbf{x}_k \in \hat{\mathbb{X}}^F$, i.e., that $x_k^{[i]} \in \hat{\mathbb{X}}_i^F$ for all $i = 1, \dots, M$.

To this purpose we define some useful notation: denote with $\mathcal{G} = (\mathbb{V}, \mathbb{A})$ the connected, undirected communication graph supporting the distributed control architecture for system (2.1). $\mathbb{V}$ is the set of M nodes, each corresponding to a subsystem, while $\mathbb{A}$ is the set of undirected arcs connecting the nodes (given two nodes $i, j \in \mathbb{V}$, there exists an undirected arc - of unitary length - $i \leftrightarrow j \in \mathbb{A}$ if and only if $j \in \mathcal{N}_i$ or $i \in \mathcal{N}_j$). We denote by P^s_{max} the longest among all the shortest paths linking all the possible pairs of nodes in $\mathbb{V}$. P^s_{max} can be computed, for instance, using the Floyd-Warshall algorithm [17, 54, 124]. P^s_{max} represent the maximum number of hops required for sending information from a node to all other vertices. The following procedure has to be executed.

Algorithm 2.6 ***Algorithm for evaluating whether*** $\mathbf{x}_k \in \hat{\mathbb{X}}^F$

1. *For all* $i = 1, \dots, M$*, initialize* $\mu_i = 1$ *if* $x_k^{[i]} \in \hat{\mathbb{X}}_i^F$ *or* $\mu_i = 0$ *if* $x_k^{[i]} \notin \hat{\mathbb{X}}_i^F$*. Set* $\nu = 0$*.*
2. *Receive* μ_j *from all* $j \in \mathcal{N}_i$ *and from all* $j : i \in \mathcal{N}_j$*. Set* $\nu = \nu + 1$*.*
3. *For all* $i = 1, \dots, M$*, set* $\mu_i = \min_{j:(i \leftrightarrow j) \in \mathbb{A} \bigcup \{i\}}(\mu_j)$*. If* $\nu < P^s_{max}$ *go to step 2. If* $\nu = P^s_{max}$ *go to step 4.*
4. *For all* $i = 1, \dots, M$*, if* $\mu_i = 1$*, then controller* i *can conclude that* $\mathbf{x}_k \in \hat{\mathbb{X}}^F$*. Otherwise, it holds that* $\mathbf{x}_k \notin \hat{\mathbb{X}}^F$*.*

Note that, after P^s_{max} iterations, it holds that $\mu_i = \mu_j$ for all $i, j = 1, \dots, M$.

We now present the two distributed techniques for generating the trajectories $\tilde{x}^{[i]}_{[k:k+N-1]}$ and $\tilde{u}^{[i]}_{[k:k+N-1]}$ that each subsystem has to transmit to its neighbors. The first one, to be used when the whole state $\mathbf{x}_k$ is inside $\hat{\mathbb{X}}^F$, is based on the auxiliary control law, and guarantees to find a solution. It requires N transmissions of information from each subsystem to its neighbors. The second one, instead, is an optimization-based procedure which has been proved to be very effective when $\mathbf{x}_k \notin \hat{\mathbb{X}}^F$. If a solution is found, the latter provides also the minimum prediction horizon length N such that a reference trajectory exists for all subsystems.

Algorithm 2.7 ***Computation of the reference trajectories - Method 1 (*** $\mathbf{x} \in \hat{\mathbb{X}}^F$ ***)***

---

1. *For all* $i = 1, \dots, M$, *initialize* $\tilde{x}_k^{[i]} = x_k^{[i]}$ *and* $\tilde{u}_k^{[i]} = K_i \tilde{x}_k^{[i]}$.

2. *Receive* $\tilde{x}_k^{[j]}$ *and* $\tilde{u}_k^{[j]}$ *from the neighbors* ($j \in \mathcal{N}_i$). *If* $N = 1$ *stop. If* $N \geq 2$, *set* $\nu = 0$ *and then go to step 3.*

3. *For all* $i = 1, \dots, M$, *update the state reference trajectory as* $\tilde{x}_{k+\nu+1}^{[i]} = A_{ii}\tilde{x}_{k+\nu}^{[i]} + B_{ii}\tilde{u}_{k+\nu}^{[i]} + \sum_{j \in \mathcal{N}_i} \{A_{ij}\tilde{x}_{k+\nu}^{[j]} + B_{ij}\tilde{u}_{k+\nu}^{[j]}\}$ *and set* $\tilde{u}_{k+\nu+1}^{[i]} = K_i \tilde{x}_{k+\nu+1}^{[i]}$.

4. *Receive* $\tilde{x}_{k+\nu+1}^{[j]}$ *and* $\tilde{u}_{k+\nu+1}^{[j]}$ *from the neighbors* ($j \in \mathcal{N}_i$). *If* $\nu = N - 1$ *stop. Else, set* $\nu = \nu + 1$ *and go to step* 3.

This first algorithm is very intuitive, because it exploits the properties of the invariant terminal set $\hat{\mathbb{X}}^F$. The N transmissions of information allow a distributed state evolution equal to the one that would result from applying the centralized auxiliary state feedback law to the entire system.

Algorithm 2.8 ***Computation of the reference trajectories - Method 2 (*** $\mathbf{x} \notin \hat{\mathbb{X}}^F$ ***)***

---

1. *For all* $i = 1, \dots, M$, *set* $\nu = 0$, *arbitrarily define sets* $\mathbb{B}_i$ *containing the origin (their role will be later specified), initialize* $\tilde{x}_k^{[i]} = x_k^{[i]}$ *and receive* $\tilde{x}_k^{[j]}$ *for all* $j \in \mathcal{N}_i$ *and for all* j *such that* $B_{ji} \neq 0$.

2. *For all* $i = 1, \dots, M$, *initialize* $\tilde{u}_{k-1}^{[i]}$ *solving the following quadratic programming (QP) problem*

$$\min_{\tilde{u}_{k-1}^{[i]}} \|\hat{\tilde{x}}_{k+1}^{0[ii]}\|^2 + \sum_{j: B_{ji} \neq 0} \frac{\|B_{ji}\|_2^2}{\|B_{ii}\|_2^2} \|\hat{\tilde{x}}_{k+1}^{0[ji]}\|^2 \tag{2.51}$$

subject to

$$\hat{\tilde{x}}^{0[ii]}_{k+1} = A_{ii}\tilde{x}^{[i]}_{k} + B_{ii}\tilde{u}^{[i]}_{k-1} + \sum_{j\in\mathcal{N}_i} A_{ij}\tilde{x}^{[j]}_{k} \tag{2.52}$$

$$\hat{\tilde{x}}^{0[ji]}_{k+1} = A_{jj}\tilde{x}^{[j]}_{k} + B_{ji}\tilde{u}^{[i]}_{k-1} + A_{ji}\tilde{x}^{[i]}_{k} \tag{2.53}$$

$$\tilde{u}^{[i]}_{k-1} \in \hat{\mathbb{U}}_i \tag{2.54}$$

$$\hat{\tilde{x}}^{[ii]}_{k+1} \in \hat{\mathbb{X}}_i \tag{2.55}$$

3. *For all* $i = 1, \dots, M$
   - *if* $\nu = 0$ *receive* $\tilde{u}^{[j]}_{k-1}$ *for all* $j \in \mathcal{N}_i$*;*
   - *if* $\nu \geq 1$ *receive* $\tilde{x}^{[j]}_{k+\nu}$ *for all* $j \in \mathcal{N}_i$*.*

4. *For all* $i = 1, \dots, M$*, for all* $j : B_{ij} \neq 0$ *compute* $\lambda^{[ij]}_{k+\nu} = A_{ii}\tilde{x}^{[i]}_{k+\nu} + B_{ii}\tilde{u}^{[i]}_{k+\nu-1} + \sum_{z\in\mathcal{N}_i\setminus\{j\}}\{A_{iz}\tilde{x}^{[z]}_{k+\nu} + B_{iz}\tilde{u}^{[z]}_{k+\nu-1}\}$.

5. *For all* $i = 1, \dots, M$*, for all* $j : B_{ji} \neq 0$*, receive* $\lambda^{[ji]}_{k+\nu}$*.*

6. *For all* $i = 1, \dots, M$*, compute* $\tilde{u}^{[i]}_{k+\nu}$ *solving the following QP problem*

$$\min_{\tilde{u}^{[i]}_{k+\nu}} \|\hat{\tilde{x}}^{[ii]}_{k+\nu+1}\|^2 + \sum_{j:B_{ji}\neq 0} \frac{\|B_{ji}\|_2^2}{\|B_{ii}\|_2^2}\|\hat{\tilde{x}}^{[ji]}_{k+\nu+1}\|^2 \tag{2.56}$$

   subject to

$$\hat{\tilde{x}}^{[ii]}_{k+\nu+1} = A_{ii}\tilde{x}^{[i]}_{k+\nu} + B_{ii}\tilde{u}^{[i]}_{k+\nu} + \sum_{j\in\mathcal{N}_i}\{A_{ij}\tilde{x}^{[j]}_{k+\nu} + B_{ij}\tilde{u}^{[j]}_{k+\nu-1}\} \tag{2.57}$$

$$\hat{\tilde{x}}^{[ji]}_{k+\nu+1} = A_{ji}\tilde{x}^{[i]}_{k+\nu} + B_{ji}\tilde{u}^{[i]}_{k+\nu} + \lambda^{[ji]}_{k+\nu} \tag{2.58}$$

$$\tilde{u}^{[i]}_{k+\nu} \in \hat{\mathbb{U}}_i \tag{2.59}$$

$$\tilde{u}^{[i]}_{k+\nu} - \tilde{u}^{[i]}_{k+\nu-1} \in \mathbb{B}_i \tag{2.60}$$

$$\hat{\tilde{x}}^{[ii]}_{k+\nu+1} \in \hat{\mathbb{X}}_i \ominus \bigoplus_{j\in\mathcal{N}_i} B_{ij}\mathbb{B}_j \tag{2.61}$$

7. *For all* $i = 1, \dots, M$*, receive* $\tilde{u}^{[j]}_{k+\nu}$ *for all* $j \in \mathcal{N}_i$*.*

8. *For all* $i = 1, \dots, M$*, update the state reference trajectory as* $\tilde{x}^{[i]}_{k+\nu+1} = A_{ii}\tilde{x}^{[i]}_{k+\nu} + B_{ii}\tilde{u}^{[i]}_{k+\nu} + \sum_{j\in\mathcal{N}_i}\{A_{ij}\tilde{x}^{[j]}_{k+\nu} + B_{ij}\tilde{u}^{[j]}_{k+\nu}\}$.

9. *If* $x_{k+\nu+1} \in \mathbb{X}_i^F$ *for all* $i = 1, \dots, M$*, then* $N = \nu + 1$ *and stop. Else, set* $\nu = \nu + 1$ *and go to step* 3.

The second algorithm aims at iteratively finding feasible inputs using one-step predictions. Each controller minimizes a cost function including both the norm of the state variable of subsystem $\mathcal{S}_i$ and the term $\frac{\|B_{ji}\|_2^2}{\|B_{ii}\|_2^2}\|\hat{\tilde{x}}^{[ji]}_{k+\nu+1}\|^2$, limiting the possible negative effect of the inputs of subsystem $\mathcal{S}_i$ on the state of subsystem $\mathcal{S}_j$, for all j such that $B_{ji} \neq 0$. The importance of this factor becomes greater as the coupling strength through inputs increases. The one-step prediction equation (2.57) is affected only by the errors on its neighbors' current inputs but, at the same time, such error is bounded using constraint (2.60) (and the uncertainty drops with the decrease of the size of the arbitrary set $\mathbb{B}_i$). The fulfillment of constraints on future states is required to each subsystem only with respect to its own states, see constraint (2.61), which is coherent to the idea of having weak coupling terms among the subsystems, such that a robust approach for managing their interactions can be used. Finally note that the check of the stopping criterion in step 9) requires Algorithm 2.6 to be applied; to reduce the iterations required, one can check $\mathbf{x}_{k+\nu+1} \in \mathbb{X}^F$ only after $\bar{N}$ iterations and, in case, only periodically.

2.5 Simulation examples

In this section, we present some simulation examples concerning widely-used and standard case studies in the context of distributed control.

2.5.1 Temperature control

We aim at regulating the temperatures T_A, T_B, T_C and T_D of the four rooms of the building represented in Figure 2.1. The first apartment is made by rooms A and B, while the second one by rooms C and D. Each room is equipped with a radiator supplying heats q_A, q_B, q_C and q_D. The heat transfer coefficient between rooms A - C and B - D is $k_1^t = 1\ \mathrm{W\,m^{-2}\,K^{-1}}$, the one between rooms A - B and C - D is $k_2^t = 2.5\ \mathrm{W\,m^{-2}\,K^{-1}}$, and the one between each room and the external environment is $k_e^t = 0.5\ \mathrm{W\,m^{-2}\,K^{-1}}$. The nominal external temperature is $\bar{T}_E = 0$ °C and, for the sake of simplicity, solar radiation is not considered. The volume of each room is $V = 48\ \mathrm{m}^3$, and the wall surfaces between the rooms are all equal to $s_r = 12\ \mathrm{m}^2$, while those of the external walls are equal to $s_e = 24$ m^2. Air density and heat capacity are $\rho = 1.225$ kg m^{-3} and $c = 1005$ J kg^{-1} K^{-1}, respectively.

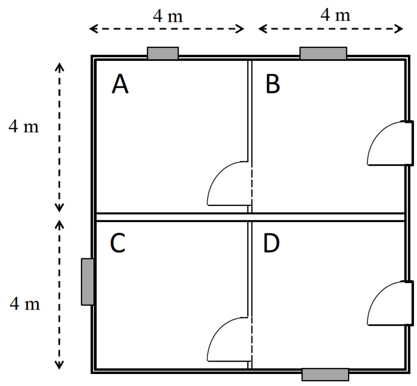

Figure 2.1: Schematic representation of a building with two apartments.

Letting $\phi = \rho c V$, the dynamic model is the following:

$$\begin{aligned}
\phi\frac{dT_A}{dt} &= s_r k_2^t(T_B - T_A) + s_r k_1^t(T_C - T_A) + s_e k_e^t(T_E - T_A) + q_A \\
\phi\frac{dT_B}{dt} &= s_r k_2^t(T_A - T_B) + s_r k_1^t(T_D - T_B) + s_e k_e^t(T_E - T_B) + q_B \\
\phi\frac{dT_C}{dt} &= s_r k_1^t(T_A - T_C) + s_r k_2^t(T_D - T_C) + s_e k_e^t(T_E - T_C) + q_C \\
\phi\frac{dT_D}{dt} &= s_r k_1^t(T_B - T_D) + s_r k_2^t(T_C - T_D) + s_e k_e^t(T_E - T_D) + q_D
\end{aligned} \tag{2.62}$$

Letting $i = \{A, B, C, D\}$, the considered equilibrium point is: $q_i = \bar{q} = 20 s_e k_e^t$ W, with $T_i = \bar{T} = 20$ °C in correspondence of $\bar{T}_E = 0$ °C. Let $\delta T_i = T_i - \bar{T}$, $\delta T_E = T_E - \bar{T}_E$, $\delta q_i = (q_i - \bar{q})/\phi$. In this way, denoting $\sigma_1 = s_r k_1^t/\phi$, $\sigma_2 = s_r k_2^t/\phi$, $\sigma_3 = s_e k_e^t/\phi$, $\sigma = \sigma_1 + \sigma_2 + \sigma_3$, $\mathbf{x} = (\delta T_A, \delta T_B, \delta T_C, \delta T_D)$, $\mathbf{u} = (\delta q_A, \delta q_B, \delta q_C, \delta q_D)$ and $\mathbf{d} = [\sigma_3\ \sigma_3\ \sigma_3\ \sigma_3]^T \delta T_E$ the previous model is rewritten in the continuous-time state space representation $\dot{\mathbf{x}}(t) = \mathbf{A_c}\mathbf{x}(t) + \mathbf{B_c}\mathbf{u}(t) + \mathbf{d}(t)$, where

$$\mathbf{A_c} = \begin{bmatrix} -\sigma & \sigma_2 & \sigma_1 & 0 \\ \sigma_2 & -\sigma & 0 & \sigma_1 \\ \sigma_1 & 0 & -\sigma & \sigma_2 \\ 0 & \sigma_1 & \sigma_2 & -\sigma \end{bmatrix}, \ \mathbf{B_c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The discrete-time system of the form (2.24) (with $n = 4$ and $m = 4$) is obtained by mE-ZOH discretization with sampling time $h = 10$ s. The partition of inputs and states is:

$$x^{[1]} = \begin{bmatrix} \delta T_A & \delta T_B \end{bmatrix}^T, \ u^{[1]} = \begin{bmatrix} \delta q_A & \delta q_B \end{bmatrix}^T$$

$$x^{[2]} = \begin{bmatrix}\delta T_C & \delta T_D\end{bmatrix}^T,\ u^{[2]} = \begin{bmatrix}\delta q_C & \delta q_D\end{bmatrix}^T$$

The constraints on the inputs and the states of the linearized system have been chosen as:

$$x^{[1]}_{min} = \begin{bmatrix}-5 & -5\end{bmatrix}^T,\ x^{[1]}_{max} = \begin{bmatrix}5 & 5\end{bmatrix}^T$$

$$x^{[2]}_{min} = \begin{bmatrix}-5 & -5\end{bmatrix}^T,\ x^{[2]}_{max} = \begin{bmatrix}5 & 5\end{bmatrix}^T$$

$$u^{[1]}_{min} = \begin{bmatrix}-0.038 & -0.038\end{bmatrix}^T,\ u^{[1]}_{max} = \begin{bmatrix}0.030 & 0.030\end{bmatrix}^T$$

$$u^{[2]}_{min} = \begin{bmatrix}-0.038 & -0.038\end{bmatrix}^T,\ u^{[2]}_{max} = \begin{bmatrix}0.030 & 0.030\end{bmatrix}^T$$

The real external temperature has been assumed to randomly vary between -10 °C and 10 °C. Matrices K_i and P_i representing a feasible solution to (2.77),(2.78), (2.79) and (2.80) are:

$$\begin{aligned} K_1 &= K_2 = \begin{bmatrix}-0.0986 & -0.0005\\ -0.0005 & -0,0986\end{bmatrix}\\ P_1 &= P_2 = \begin{bmatrix}2.17\cdot 10^6 & 1\\ 1 & 2.17\cdot 10^6\end{bmatrix}\end{aligned}$$

The weighting matrices used in the simulations are $Q^o_1 = Q^o_2 = R^o_1 = R^o_2 = I_2$. Algorithm 2.3 for computing the RPI sets has been used, while the initial reference trajectories have been generated using Algorithm 2.8. The results of the simulations, performed

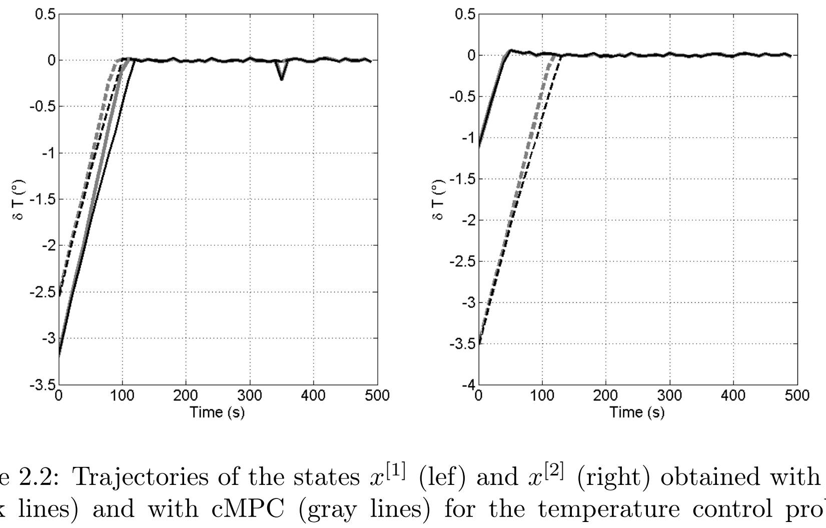

Figure 2.2: Trajectories of the states $x^{[1]}$ (lef) and $x^{[2]}$ (right) obtained with DPC (black lines) and with cMPC (gray lines) for the temperature control problem. Solid lines: δT_A and δT_C; dashed lines: δT_B and δT_D.

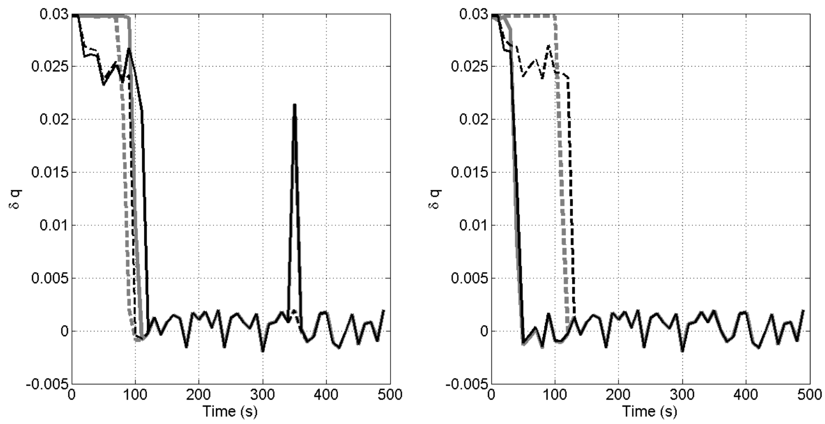

Figure 2.3: Inputs $u^{[1]}$ (left) and $u^{[2]}$ (right) obtained with DPC (black lines) and with cMPC (gray lines) for the temperature control problem. Solid lines: δq_A and δq_C; dashed lines: δq_B and δq_D.

using the continuous-time process model, are shown in Figure 2.2, while the values of the input variables are depicted in Figure 2.3. To show the capability of Algorithm 2.7 to recover the reference trajectories, a sudden decrease of temperature T_A is forced at $t = 350$ s (it could represent, for instance, the opening of a door). In both figures a comparison between DPC and centralized MPC (cMPC) is provided, showing only a small performance degradation.

2.5.2 Four-tanks system

A benchmark case often used to assess the effectiveness of distributed control algorithms is the four-tanks system schematically drawn in Figure 2.4, originally described in [74] and then utilized, for instance, in [7, 108].

The goal is to regulate the levels h_1, h_2, h_3 and h_4 of the four tanks. The manipulated inputs are the voltages of the two pumps v_1 and v_2. We assumed that a bounded unknown disturbance $w = (w_1, w_2)$ affects the applied voltages, i.e., that the real input to the plant is $\mathbf{u} = (v_1 + w_1, v_2 + w_2)$. The parameters γ_1 and $\gamma_2 \in (0, 1)$ represent the fraction of the water that flows inside the lower tanks, and are kept

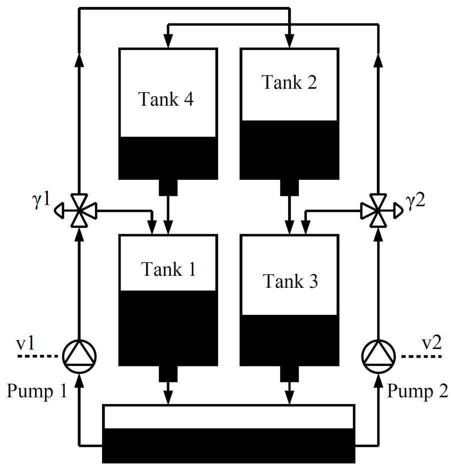

Figure 2.4: Schematic representation of a four-tanks system.

fixed during the simulations. The dynamics of the system is given by

$$\begin{aligned} \frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_4}{A_4}\sqrt{2gh_4} + \frac{\gamma_1 k_1}{A_1}v_1 \\ \frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{(1-\gamma_1)k_1}{A_2}v_1 \\ \frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{a_2}{A_2}\sqrt{2gh_2} + \frac{\gamma_2 k_2}{A_3}v_2 \\ \frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_2)k_2}{A_4}v_2 \end{aligned} \tag{2.63}$$

where A_i and a_i are the cross-section of Tank i and the cross section of the outlet hole of Tank i, respectively. The coefficients k_1 and k_2 represent the conversion parameters from the voltage applied to the pump to the flux of water. The values of the parameters, taken from [74], are: $A_1 = A_4 = 28 \quad \text{cm}^2$, $\quad A_2 = A_3 = 32 \quad \text{cm}^2$, $\quad a_1 = a_4 = 0.071 \quad \text{cm}^2$, $a_2 = a_3 = 0.057 \ \text{cm}^2$, $k_1 = 3.35 \ \text{cm}^3\,\text{V}^{-1}\,\text{s}^{-1}$, $k_2 = 3.33 \ \text{cm}^3\,\text{V}^{-1}\,\text{s}^{-1}$, $\gamma_1 = 0.7$, $\gamma_2 = 0.6$. The considered equilibrium point is $\bar{v}_1 = \bar{v}_2 = 3$ V, $\bar{h}_1 = 12.263$ cm, $\bar{h}_2 = 1.409$ cm, $\bar{h}_3 = 12.783$ cm and $\bar{h}_4 = 1.634$ cm. Letting $\delta h_l = h_l - \bar{h}_l, \quad l = 1,2,3,4$ and $\delta v_i = v_i - \bar{v}_i, \quad i = 1,2$, $\mathbf{x} = (\delta h_1, \delta h_2, \delta h_3, \delta h_4)$, $\mathbf{u} = (\delta v_1, \delta v_2)$, $\mathbf{d} = \mathbf{B}(w_1, w_2)$, linearizing system (2.63) around the considered equilibrium point and discretizing it using mE-ZOH with sampling time $h = 1$ s, we obtain a linear system of the form (2.24), where

$$\mathbf{A} = \begin{bmatrix} 0.984 & 0 & 0 & 0.044 \\ 0 & 0.967 & 0 & 0 \\ 0 & 0.033 & 0.989 & 0 \\ 0 & 0 & 0 & 0.957 \end{bmatrix}, \; \mathbf{B} = \begin{bmatrix} 0.083 & 0 \\ 0.031 & 0 \\ 0 & 0.062 \\ 0 & 0.047 \end{bmatrix}$$

Inputs and states are partitioned as:

$$x^{[1]} = \begin{bmatrix} \delta h_1 & \delta h_2 \end{bmatrix}^T, \; u^{[1]} = \delta v_1$$

$$x^{[2]} = \begin{bmatrix} \delta h_3 & \delta h_4 \end{bmatrix}^T, \; u^{[2]} = \delta v_2$$

The constraints on the inputs and the states of the linearized system have been chosen as:

$$x^{[1]}_{min} = \begin{bmatrix} -12.263 & -1.409 \end{bmatrix}^T, \; x^{[1]}_{max} = \begin{bmatrix} 40 & 40 \end{bmatrix}^T + x^{[1]}_{min}$$

$$x^{[2]}_{min} = \begin{bmatrix} -12.783 & -1.634 \end{bmatrix}^T, \; x^{[2]}_{max} = \begin{bmatrix} 40 & 40 \end{bmatrix}^T + x^{[2]}_{min}$$

$$u^{[1]}_{min} = u^{[2]}_{min} - 3, \; u^{[1]}_{max} = u^{[2]}_{max} = 3$$

The disturbances $w_{1,2}$ on the applied voltages are assumed to randomly vary between -0.01 V and 0.01 V. Matrices K_i and P_i satisfying the LMI conditions are:

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.772 & -0.181 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.778 & -0.250 \end{bmatrix} \\ P_1 &= \begin{bmatrix} 48.3 & -1 \\ -1 & 59.3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 166.8 & 1.94 \\ 1.94 & 70.7 \end{bmatrix} \end{aligned}$$

The weighting matrices are $Q_1^o = Q_2^o = I_2$ and $R_1^o = R_2^o = 1$. To compute the RPI sets, the Algorithm 2.3 has been used, and the initial reference trajectories have been designed using Algorithm 2.7. The simulation results, obtained using the continuous-time nonlinear model, are reported in Figure 2.5, while in Figure 2.6 the applied real voltages are shown. In addition to the external disturbance (w_1, w_2), included in the robust controller design, at time $t = 100$ s an unpredicted impulse equal to 2 V has been applied to the first pump. Then, the reference trajectories have been re-generated online to recover the nominal operating conditions with algorithm 2.7. The performances are close to the ones obtained with centralized MPC.

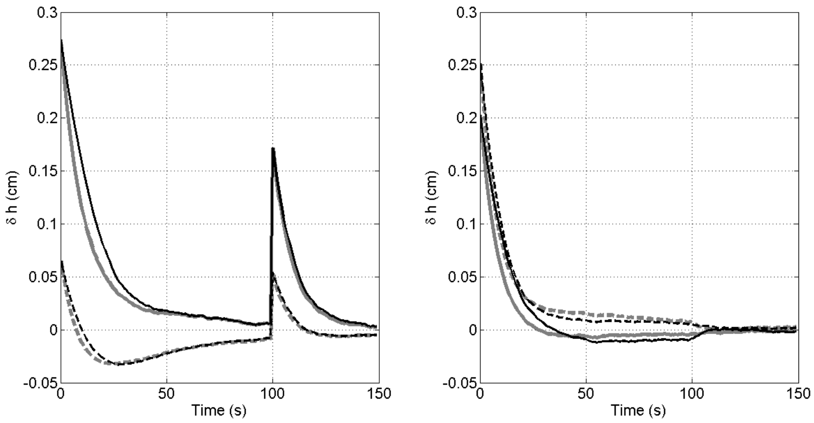

Figure 2.5: Trajectories of the states $x^{[1]}$ (lef) and $x^{[2]}$ (right) obtained with DPC (black lines) and with cMPC (gray lines) for controlling the four-tanks system. Solid lines: x_1 and x_3; dashed lines: x_2 and x_4.

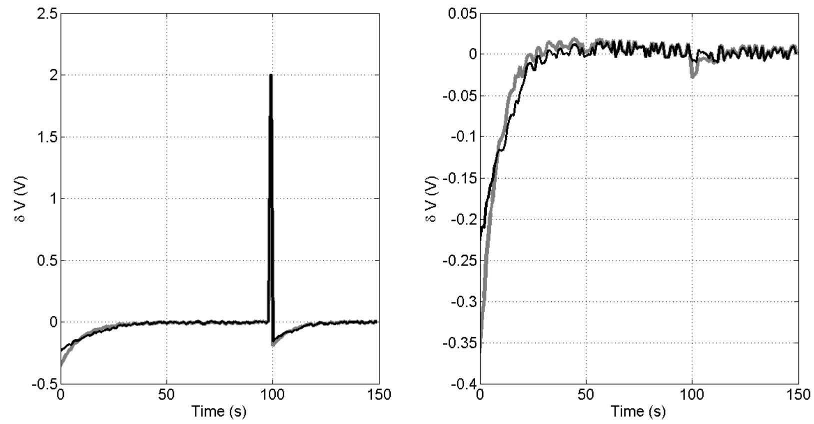

Figure 2.6: Inputs $u^{[1]}$ (lef) and $u^{[2]}$ (right) obtained with DPC (black lines) and with cMPC (gray lines) for controlling the four-tanks system.

2.5.3 Cascade coupled flotation tanks

Consider the level control problem of flotation tanks discussed in [158]. The system consists of five tanks connected in cascade with control valves between the tanks (Figure 2.7). A flow of pulp q enters into the first tank. The goal is to keep the levels y_i, $i = 1, \ldots, 5$, stable in all the tanks. The manipulated inputs are the signals to the valves v_i, $i = 1, \ldots, 5$. The mathematical model describing the dynamics of the

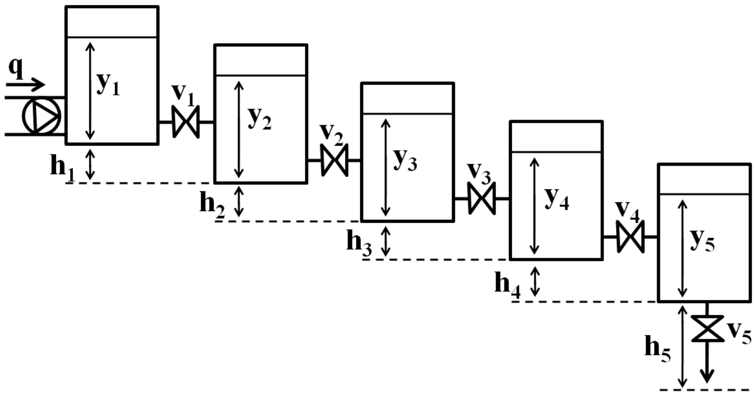

Figure 2.7: Schematic representation of the flotation tanks.

levels inside the five tanks is [158]:

$$
\begin{aligned}
\pi r^2 \tfrac{dy_1}{dt} &= q - k_1 v_1 \sqrt{y_1 - y_2 + h_1} \\
\pi r^2 \tfrac{dy_2}{dt} &= k_1 v_1 \sqrt{y_1 - y_2 + h_1} - k_2 v_2 \sqrt{y_2 - y_3 + h_2} \\
\pi r^2 \tfrac{dy_3}{dt} &= k_2 v_2 \sqrt{y_2 - y_3 + h_2} - k_3 v_3 \sqrt{y_3 - y_4 + h_3} \\
\pi r^2 \tfrac{dy_4}{dt} &= k_3 v_3 \sqrt{y_3 - y_4 + h_3} - k_4 v_4 \sqrt{y_4 - y_5 + h_4} \\
\pi r^2 \tfrac{dy_5}{dt} &= k_4 v_4 \sqrt{y_4 - y_5 + h_4} - k_5 v_5 \sqrt{y_5 + h_5}
\end{aligned}
\qquad (2.64)
$$

where r is radius of the tanks, k_i, $i = 1, \ldots, 5$ are the valves coefficients and h_i, $i = 1, \ldots, 5$ are the physical height differences between subsequent tanks. We set $r = 1$ m, $k_i = 0.1$ m$^{2.5}$/Vs, $i = 1, \ldots, 5$ and $h_i = 0.5$ m, $i = 1, \ldots, 5$. The nominal value for the inlet flow is $\bar{q} = 0.1$ m^3 s^{-1} and we assume that it is affected by an uncertainty $w = \pm 0.5\%$ randomly varying with the time. We consider the equilibrium point where $\bar{y}_i = 2$ m, $i = 1, \ldots, 5$, and, correspondingly, $\bar{v}_i = 1.4142$ m, $i = 1, \ldots, 4$ and $\bar{v}_5 = 0.6325$ V. Let $\delta y_i = y_i - \bar{y}_i$, $i = 1, \ldots, 5$, $\delta v_i = v_i - \bar{v}_i$, $i = 1, \ldots, 5$, $\mathbf{x} = (\delta y_1, \delta y_2, \delta y_3, \delta y_4, \delta y_5)$, $\mathbf{u} = (\delta v_1, \delta v_2, \delta v_3, \delta v_4, \delta v_5)$ and $\mathbf{d} = \mathbf{B}_d w$. The discrete-time linearized system corresponding to system (2.64), with mE-ZOH discretization and a sampling time $h = 5$ s, has the form (2.24), where $\mathbf{B}_d = [1.4714\ \ 0\ \ 0\ \ 0\ \ 0]^T$ and

$$\mathbf{A} = \begin{bmatrix} 0.853 & 0.147 & 0 & 0 & 0 \\ 0.136 & 0.727 & 0.136 & 0 & 0 \\ 0 & 0.136 & 0.727 & 0.136 & 0 \\ 0 & 0 & 0.136 & 0.727 & 0.136 \\ 0 & 0 & 0 & 0.157 & 0.969 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -0.104 & 0 & 0 & 0 & 0 \\ 0.096 & -0.096 & 0 & 0 & 0 \\ 0 & 0.096 & -0.096 & 0 & 0 \\ 0 & 0 & 0.096 & -0.096 & 0 \\ 0 & 0 & 0 & 0.111 & -0.248 \end{bmatrix}$$

The partitions of inputs and states, for $i = 1, \ldots, 5$ is:

$$x^{[i]} = \delta y_i,\ u^{[i]} = \delta v_i$$

The constraints on the inputs and the states of the linearized system, for $i = 1, \ldots, 5$, have been set as:

$$x^{[i]}_{min} = -1,\ x^{[i]}_{max} = 1,\ u^{[i]}_{min} = -\bar{v}_i,\ u^{[i]}_{max} = 3 - \bar{v}_i$$

terms K_i and P_i solving the LMI conditions are:

$$K_1 = 0.287, \quad K_2 = K_3 = K_4 = 0.143, \quad K_5 = 0.776$$

$$P_1 = 1.18, \quad P_2 = 1.07, \quad P_3 = 1.05, \quad P_4 = 1, \quad P_5 = 1$$

The weighting matrices, for $i = 1, \ldots, 5$, are $Q^o_i = R^o_i = 1$. To compute the RPI sets the Algorithm 2.4 has been used, while the initial reference trajectories have been designed using Algorithm 2.8. In Figure 2.8 we show the simulation results, obtained using the continuous-time nonlinear model, i.e., depicting the state and input of the first tank, directly affected by the external flow q. Figure 2.9 and Figure 2.10 report, respectively, the states and the inputs of the remaining four tanks. Also in this case, only minor differences arise between the centralized and the distributed solutions. At time $t = 300$ s, a disturbance of magnitude $w = 0.1\ \mathrm{m}^3\,\mathrm{s}^{-1}$ is applied to the plant, and the distributed control system reacts generating from scratch the reference trajectories (i.e., with Algorithm 2.7).

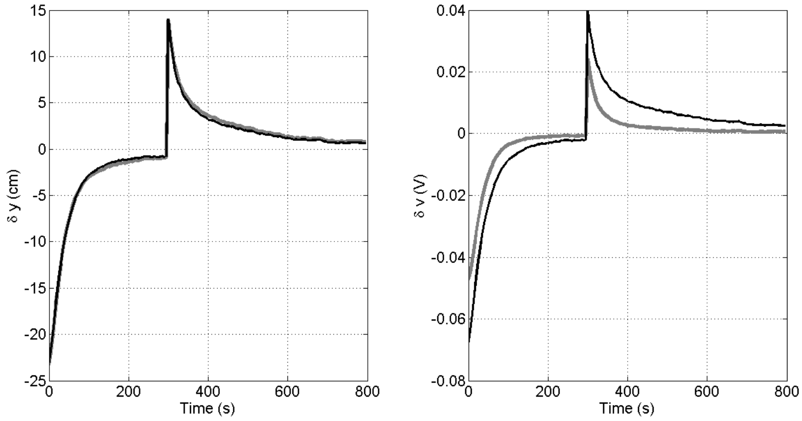

Figure 2.8: Trajectories of the state $x^{[1]}$ (left) and of the input $u^{[1]}$ (right) obtained with DPC (black lines) and with cMPC (gray lines) for the control of the floating tanks.

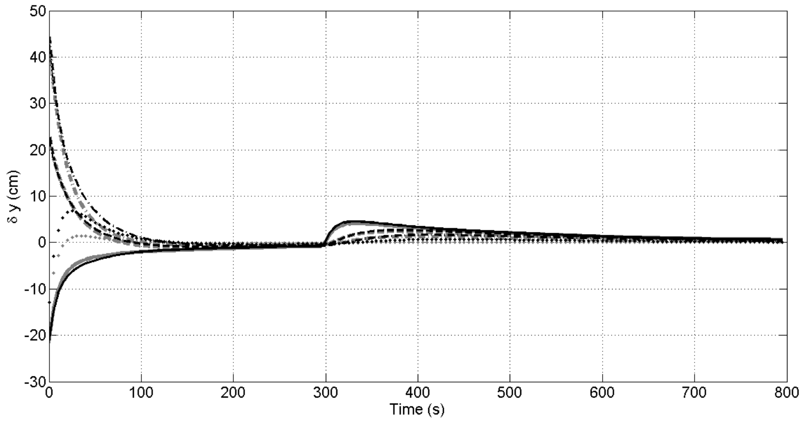

Figure 2.9: Trajectories of the states $x^{[2]}$ (solid lines), $x^{[3]}$ (dashed lines), $x^{[4]}$ (dash-dot lines) and $x^{[5]}$ (dotted lines) obtained with DPC (black lines) and with cMPC (gray lines) for the control of the floating tanks.

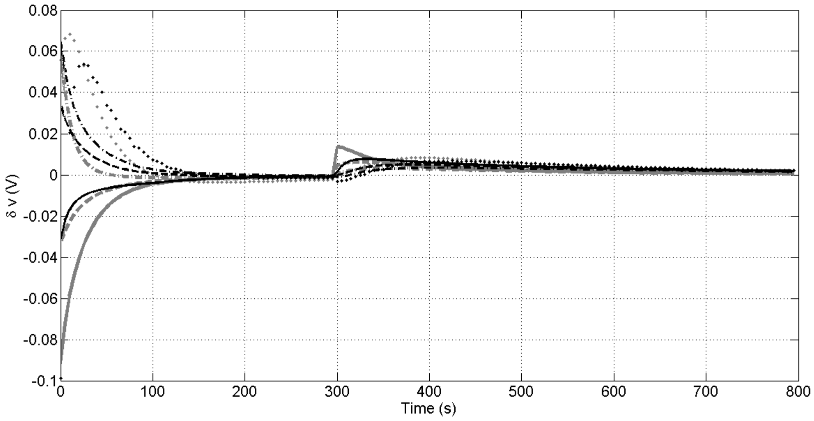

Figure 2.10: Inputs $u^{[2]}$ (solid lines), $u^{[3]}$ (dashed lines), $u^{[4]}$ (dash-dot lines) and $u^{[5]}$ (dotted lines) obtained with DPC (black lines) and with cMPC (gray lines) for the control of the floating tanks.

2.5.4 Reactor-separator process

The DPC algorithm has been used for control of the reactor-separator process already considered in [91, 159] and shown in Figure 2.11. The

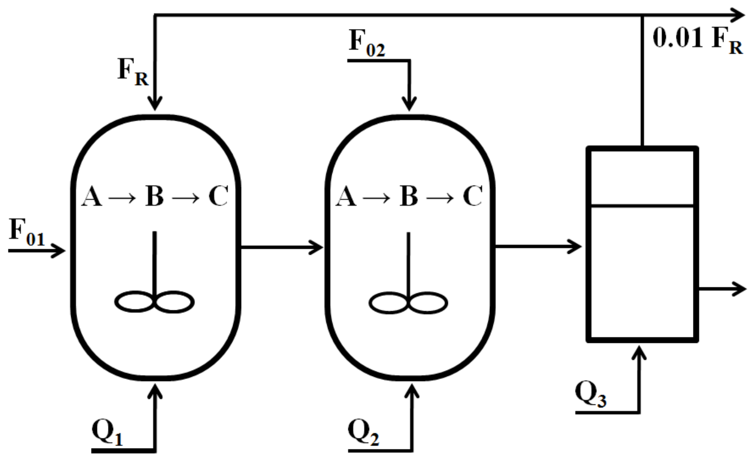

Figure 2.11: Schematic representation of the reactor-separator process.

plant consists of three subsystems, i.e. two reactors and a separator. The reactant A is inserted in the two reactors, where it is converted to product B, with a side product C; a significant recirculation from the separator to the first reactor makes the system heavily coupled .

The model, whose equations can be found in [91], is derived under the assumption of hydraulic equilibrium, so that each subsystem has three states: $x^{[i]} = (x_{Ai},\, x_{Bi},\, T_i)$ where x_{Ai} and x_{Bi} are the mass fractions of A and B in the vessel i, while T_i is its temperature; each subsystem has the heat input Q_i as control variable. Referring again to the model reported in [91], the plant parameters used in the experiments reported below are summarized in Table 1.

Table 1: Parameters used in the reactor-separator process.

Parameter	Description	Nominal value
F_{10}	effluent flow rate vessel 1	8.3 $\mathrm{kg\,s^{-1}}$
F_{02}	effluent flow rate vessel 2	0.5 $\mathrm{kg\,s^{-1}}$
F_R	recycle flow rate	40 $\mathrm{kg\,s^{-1}}$
V_1	volume vessel 1	89.4 $\mathrm{m^3}$
V_2	volume vessel 2	90 $\mathrm{m^3}$
V_3	volume vessel 3	13.27 $\mathrm{m^3}$
k_1	pre-exp. value for react. 1	0.336 $\mathrm{s^{-1}}$
k_2	pre-exp. value for react. 2	0.089 $\mathrm{s^{-1}}$
E_1/R	norm. act. energy for react. 1	-100 K
E_2/R	norm. act. energy for react. 2	-150 K
x_{A10}	mass fract. of A in ext. streams	1
x_{B10}	mass fract. of B in ext. streams	0
$T_{10,20}$	feed stream temperatures	313 K
ΔH_1	heat of reaction for reaction 1	-40 $\mathrm{kJ\,kg^{-1}}$
ΔH_2	heat of reaction for reaction 2	-50 $\mathrm{kJ\,kg^{-1}}$
C_p	heat capacity	2.5 $\mathrm{kJ\,kg^{-1}}$
ρ	solution density	0.15 $\mathrm{kg\,m^{-3}}$
α_A	relative volatility of A	3.5
α_B	relative volatility of B	1.1
α_C	relative volatility of C	0.5

Correspondingly, with $\bar{Q}_{1,2,3} = 10\ \mathrm{kJ\,s^{-1}}$, the equilibrium $\bar{x}_{A1} = 0.6$, $\bar{x}_{B1} = 0.352$, $\bar{T}_1 = 327$ K, $\bar{x}_{A2} = 0.536$, $\bar{x}_{B2} = 0.4$, $\bar{T}_2 = 328.4$ K, $\bar{x}_{A3} = 0.285$, $\bar{x}_{B3} = 0.565$, $\bar{T}_3 = 328.5$ K has been computed and the linearized model around this steady state has been derived and discretized with sampling time $\Delta = 0.1$ s (see Figure 2.12). For the design of DPC, first the gains K_i, $i = 1, 2, 3$ have been designed for the pairs (A_{ii}, B_{ii}) according to the LQ criterion with $Q_i = I_3$ and $R_i = 10^{-6}$ for all i, which allows to verify Assumption 2.1. The weighting matrices and the sets have been chosen in order to satisfy Assumptions 2.2 and 2.3. We set $N = 10$.

The capability of regulating the state trajectories of the linearized system to the origin has been tested in simulation, in face of a perturbation of magnitude $\Delta x^{[i]} = (\Delta x_{Ai}, \Delta x_{Bi}, \Delta T_i)$ where $\Delta x_{Ai} = -0.05$, $\Delta x_{Bi} = -0.05$ and $\Delta T_i = -5$ for all i with respect to the equilibrium condition at time $t = 0$ s. Note that the constraints on the absolute input variables $Q_i \in [0, 50]$ for all i, see [159], result in the following constraints on the deviation of Q_i with respect to $\bar{Q}_i$: $\Delta Q_i \in [-10, 40]$.

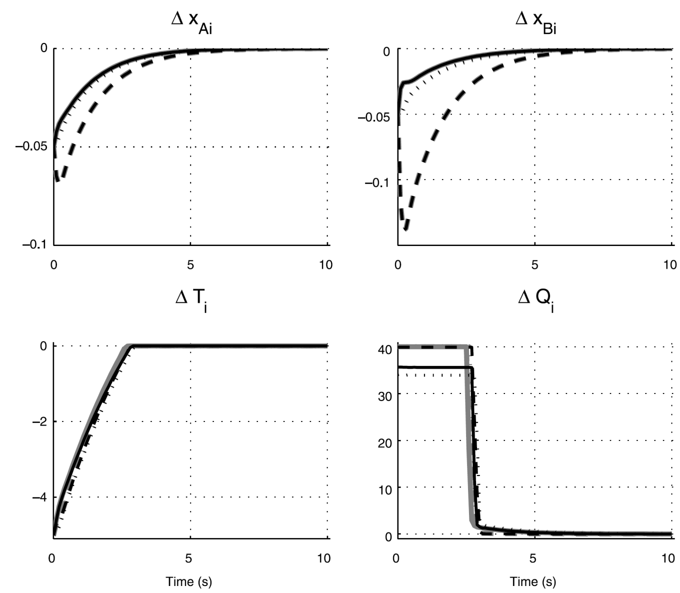

Figure 2.12: State and input variables with DPC (black lines) and with cMPC (grey lines) for all subsystems ($i = 1$: solid lines; $i = 2$, dotted lines, $i = 3$: dashed lines).

The dynamics of Δx_{Ai}, Δx_{Bi}, ΔT_i and the inputs ΔQ_i with respect to the equilibrium conditions are shown in Figure 2.12, and compared with the ones obtained applying a centralized MPC controller (cMPC). Note that ΔQ_i saturate at values which are not on the boundary of the feasibility set, since the robustness arguments used to define DPC make the constraints more conservative than the ones used in cMPC. This slightly degrades the performance of DPC with respect to cMPC in the regulation of T_i.

2.6 Conclusions

In this Chapter, starting from the theoretical results presented in [50], the DPC algorithm has been presented in a shape useful for practical implementation. All the offline design phase has been carefully studied, in order to propose algorithms as simple as possible. Some techniques for facing faults due to unpredicted external disturbances have been described as well. Several simulation results based on processes taken from the literature have been presented, showing the effectiveness of the approach, which leads to performances close to the ones obtained using a centralized solution. In the next chapters, we will extend the proposed approach to continuous-time systems and to the tracking problem.

2.7 Appendix

2.7.1 Proof of Theorem 2.1

For the sake of completeness, we report the proof of Theorem 2.1 [50].

The collective problem

Define the collective vectors $\hat{\mathbf{x}}_k = (\hat{x}_k^{[1]}, \ldots, \hat{x}_k^{[M]})$, $\tilde{\mathbf{x}}_k = (\tilde{x}_k^{[1]}, \ldots, \tilde{x}_k^{[M]})$, $\tilde{\mathbf{u}}_k = (\tilde{u}_k^{[1]}, \ldots, \tilde{u}_k^{[M]})$, $\hat{\mathbf{u}}_k = (\hat{u}_k^{[1]}, \ldots, \hat{u}_k^{[M]})$, $\mathbf{w}_k = (w_k^{[1]}, \ldots, w_k^{[M]})$, $\mathbf{z}_k = (z_k^{[1]}, \ldots, z_k^{[M]})$, and the matrices $\mathbf{A}^* = \text{diag}(A_{11}, \ldots, A_{MM})$, $\mathbf{B}^* = \text{diag}(B_{11}, \ldots, B_{MM})$, $\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{A}^*$, $\tilde{\mathbf{B}} = \mathbf{B} - \mathbf{B}^*$. Collectively, we write equations (2.3) and (2.6) as

$$\mathbf{x}_{k+1} = \mathbf{A}^*\mathbf{x}_k + \mathbf{B}^*\mathbf{u}_k + \tilde{\mathbf{A}}\tilde{\mathbf{x}}_k + \tilde{\mathbf{B}}\tilde{\mathbf{u}}_k + \mathbf{w}_k \tag{2.65}$$

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}^*\hat{\mathbf{x}}_k + \mathbf{B}^*\hat{\mathbf{u}}_k + \tilde{\mathbf{A}}\tilde{\mathbf{x}}_k + \tilde{\mathbf{B}}\tilde{\mathbf{u}}_k \tag{2.66}$$

In view of (2.7), $\mathbf{u}_k = \hat{\mathbf{u}}_k + \mathbf{K}(\mathbf{x}_k - \hat{\mathbf{x}}_k)$, from (2.8),

$$\mathbf{z}_{k+1} = (\mathbf{A}^* + \mathbf{B}^*\mathbf{K})\mathbf{z}_k + \mathbf{w}_k \tag{2.67}$$

Minimizing (2.9) for all $i = 1, \ldots, M$ is equivalent to minimize

$$\mathbf{V}^{N*}(\mathbf{x}_k) = \min_{\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_{[k:k+N-1]}} \mathbf{V}^N(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_{[k:k+N-1]}) \tag{2.68}$$

subject to the dynamic constraints (2.66) and

$$\mathbf{x}_k - \hat{\mathbf{x}}_k \in \mathbb{Z} = \prod_{i=1}^{M} Z_i \tag{2.69a}$$

$$\hat{\mathbf{x}}_{k+\nu} - \tilde{\mathbf{x}}_{k+\nu} \in \mathbb{E} = \prod_{i=1}^{M} E_i \tag{2.69b}$$

$$\hat{\mathbf{u}}_{k+\nu} - \tilde{\mathbf{u}}_{k+\nu} \in \tilde{\mathbb{U}} = \prod_{i=1}^{M} U_i \tag{2.69c}$$

$$\hat{\mathbf{x}}_{k+\nu} \in \hat{\mathbb{X}} \tag{2.69d}$$

$$\hat{\mathbf{u}}_{k+\nu} \in \hat{\mathbb{U}} \tag{2.69e}$$

$$\mathbf{H}(\hat{\mathbf{x}}_{k+\nu}, \hat{\mathbf{u}}_{k+\nu}, \tilde{\mathbf{x}}_{k+\nu}, \tilde{\mathbf{u}}_{k+\nu}) \leq 0 \tag{2.69f}$$

for $\nu = 0, \ldots, N-1$, and the terminal constraint

$$\hat{\mathbf{x}}_{k+N} \in \hat{\mathbb{X}}^F \tag{2.70}$$

In (2.69), $\mathbf{H}$ collects all the constraints (2.15) and, by i) in Assumption 2.2, $\mathbf{H}(\hat{\mathbf{x}}, \mathbf{K}\hat{\mathbf{x}}, \hat{\mathbf{x}}, \mathbf{K}\hat{\mathbf{x}}) \leq 0$ for all $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$. The collective cost function $\mathbf{V}^N$ is defined as

$$\mathbf{V}^N(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_{[k:k+N-1]}) = \sum_{\nu=0}^{N-1} \mathbf{l}(\hat{\mathbf{x}}_{k+\nu}, \hat{\mathbf{u}}_{k+\nu}) + \mathbf{V}^F(\hat{\mathbf{x}}_{k+N})$$

We also define

$$\mathbf{V}^{N,0}(\hat{\mathbf{x}}_k) = \min_{\hat{\mathbf{u}}_{[k:k+N-1]}} \mathbf{V}^N(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_{[k:k+N-1]}) \tag{2.71}$$

subject to (2.66), (2.69b)-(2.70).

Feasibility

From Definition 2.1, it collectively holds that

$$\begin{aligned}\mathbb{X}^N = \;& \{\mathbf{x} : \text{ if } \mathbf{x}_0 = \mathbf{x} \text{ then } \exists \tilde{\mathbf{x}}_{[0:N-1]}, \tilde{\mathbf{u}}_{[0:N-1]}, \hat{\mathbf{x}}_{0/0}, \hat{\mathbf{u}}_{[0,N-1]} \\ & \text{such that (2.66), (2.69) and (2.70) are satisfied}\}\end{aligned}$$

and that, for each point of the feasibility set $\mathbf{x} \in \mathbb{X}^N$,

$$\begin{aligned}\tilde{\mathbb{X}}_{\mathbf{x}} := \;& \{(\tilde{\mathbf{x}}_{[0:N-1]}, \tilde{\mathbf{u}}_{[0:N-1]}) : \text{ if } \mathbf{x}_0 = \mathbf{x} \text{ then } \exists \hat{\mathbf{x}}_{0/0}, \hat{\mathbf{u}}_{[0,N-1]} \\ & \text{such that (2.66), (2.69) and (2.70) are satisfied}\}\end{aligned}$$

At time k, $\mathbf{x}_k \in \mathbb{X}^N$ and $(\tilde{\mathbf{x}}_{[k:k+N-1]}, \tilde{\mathbf{u}}_{[k:k+N-1]}) \in \tilde{\mathbb{X}}_{\mathbf{x}_k}$. The optimal nominal input and state sequences obtained by minimizing the collective MPC problem are $\hat{\mathbf{u}}_{[k:k+N-1]/k} = \{\hat{\mathbf{u}}_{k/k}, \ldots, \hat{\mathbf{u}}_{k+N-1/k}\}$ and $\hat{\mathbf{x}}_{[k:k+N]/k} = \{\hat{\mathbf{x}}_{k/k}, \ldots, \hat{\mathbf{x}}_{k+N/k}\}$, respectively. Finally, recall that $\tilde{\mathbf{x}}_{k+N} = \hat{\mathbf{x}}_{k+N/k}$ and $\tilde{\mathbf{u}}_{k+N} = \mathbf{K}\hat{\mathbf{x}}_{k+N/k}$.

Define $\hat{\mathbf{u}}_{k+N/k} = \mathbf{K}\hat{\mathbf{x}}_{k+N/k}$ and compute $\hat{\mathbf{x}}_{k+N+1/k}$ according to (2.66) from $\hat{\mathbf{x}}_{k+N/k}$ where $\hat{\mathbf{u}}_{k+N} = \hat{\mathbf{u}}_{k+N/k}$. We obtain

$$\begin{aligned}\hat{\mathbf{x}}_{k+N+1/k} &= \mathbf{A}^*\hat{\mathbf{x}}_{k+N/k} + \mathbf{B}^*\hat{\mathbf{u}}_{k+N/k} + \tilde{\mathbf{A}}\tilde{\mathbf{x}}_{k+N} + \tilde{\mathbf{B}}\tilde{\mathbf{u}}_{k+N} \\ &= (\mathbf{A} + \mathbf{B}\mathbf{K})\hat{\mathbf{x}}_{k+N/k}\end{aligned}$$

since $\tilde{\mathbf{x}}_{k+N} = \hat{\mathbf{x}}_{k+N/k}$ and $\tilde{\mathbf{u}}_{k+N} = \hat{\mathbf{u}}_{k+N/k}$. In view of constraint (2.70) and Assumption 2.2, $\hat{\mathbf{u}}_{k+N/k} \in \hat{\mathbb{U}}$ and $\hat{\mathbf{x}}_{k+N+1/k} \in \hat{\mathbb{X}}^F$. Therefore, they satisfy (2.69d), (2.69e) and (2.70). Also, according to Assumption 2.2, (2.19) holds. We also define the input sequence $\mathbf{u}_{[k+1:k+N]/k}$ and the state sequence $\mathbf{x}_{[k+1:k+N+1]/k}$ stemming from the initial condition $\hat{\mathbf{x}}_{k+1/k}$ and the input sequence $\mathbf{u}_{[k+1:k+N]/k}$. In view of the feasibility of the i-DPC problem at time k, we have that $\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k} \in \mathbb{Z}$, $\hat{\mathbf{x}}_{k+\nu/k} - \tilde{\mathbf{x}}_{k+\nu} \in \mathbb{E}$, and $\hat{\mathbf{u}}_{k+\nu/k} - \tilde{\mathbf{u}}_{k+\nu} \in \tilde{\mathbb{U}}$ for all $\nu = 1, \ldots, N-1$. Note also that $\hat{\mathbf{x}}_{k+N/k} - \tilde{\mathbf{x}}_{k+N} = 0 \in \mathbb{E}$ and $\hat{\mathbf{u}}_{k+N/k} - \tilde{\mathbf{u}}_{k+N} = 0 \in \tilde{\mathbb{U}}$ by (2.18). Furthermore, since $\tilde{\mathbf{x}}_{k+N} = \hat{\mathbf{x}}_{k+N/k} \in \hat{\mathbb{X}}^F$, from (2.70) it holds that $\mathbf{H}(\hat{\mathbf{x}}_{k+N/k}, \hat{\mathbf{u}}_{k+N/k}, \tilde{\mathbf{x}}_{k+N}, \tilde{\mathbf{u}}_{k+N}) \leq 0$ from i) of Assumption 2.2. Therefore $\mathbf{x}_{[k+1:k+N+1]/k}$ and $\mathbf{u}_{[k+1:k+N]/k}$ are feasible at $k+1$, since (2.69) and (2.70) are satisfied. This proves that $\mathbf{x}_k \in \mathbb{X}^N$ and $(\tilde{\mathbf{x}}_{[k:k+N-1]}, \tilde{\mathbf{u}}_{[k:k+N-1]}) \in \tilde{\mathbb{X}}_{\mathbf{x}_k}$ implies that $\mathbf{x}_{k+1} \in \mathbb{X}^N$ and $(\tilde{\mathbf{x}}_{[k+1:k+N]}, \tilde{\mathbf{u}}_{[k+1:k+N]}) \in \tilde{\mathbb{X}}_{\mathbf{x}_{k+1}}$.

Convergence of the optimal cost function

By optimality, $\mathbf{V}^{N,0}(\hat{\mathbf{x}}_{k+1/k}) \leq \mathbf{V}^N(\hat{\mathbf{x}}_{k+1/k}, \mathbf{u}_{[k+1:k+N]/k})$, where

$$\mathbf{V}^N(\hat{\mathbf{x}}_{k+1/k}, \mathbf{u}_{[k+1:k+N]/k}) = \sum_{\nu=1}^{N} \mathbf{l}(\hat{\mathbf{x}}_{k+\nu/k}, \hat{\mathbf{u}}_{k+\nu/k}) + \mathbf{V}^F\left(\hat{\mathbf{x}}_{k+N+1/k}\right)$$

Therefore we compute that

$$\begin{aligned}\mathbf{V}^{N,0}(\hat{\mathbf{x}}_{k+1/k}) - \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{k/k}) \leq &-\mathbf{l}(\hat{\mathbf{x}}_{k/k}, \hat{\mathbf{u}}_{k/k}) + \mathbf{l}(\hat{\mathbf{x}}_{k+N/k}, \hat{\mathbf{u}}_{k+N/k}) + \\ &+ \mathbf{V}^F\left(\hat{\mathbf{x}}_{k+N+1/k}\right) - \mathbf{V}^F\left(\hat{\mathbf{x}}_{k+N/k}\right)\end{aligned} \quad (2.72)$$

and, in view of (2.19), it follows that

$$\mathbf{V}^{N,0}(\hat{\mathbf{x}}_{k+1/k}) - \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{k/k}) \leq -(\|\hat{\mathbf{x}}_{k/k}\|^2_{\mathbf{Q}^o} + \|\hat{\mathbf{u}}_{k/k}\|^2_{\mathbf{R}^o}) \quad (2.73)$$

Since $\mathbf{Q}^o$ and $\mathbf{R}^o$ are positive definite matrices $\hat{\mathbf{x}}_{k/k} \to 0$ and $\hat{\mathbf{u}}_{k/k} \to 0$ as $k \to \infty$.

Finally, recall that the state $\mathbf{x}_k$ evolves according to

$$\mathbf{x}_{k+1} = (\mathbf{A} + \mathbf{B}\mathbf{K})\,\mathbf{x}_k + \mathbf{B}\left(\hat{\mathbf{u}}_{k/k} - \mathbf{K}\hat{\mathbf{x}}_{k/k}\right)$$

By asymptotic convergence to zero of the nominal state and input signals $\hat{\mathbf{x}}_{k/k}$ and $\hat{\mathbf{u}}_{k/k}$ respectively, we obtain that $\mathbf{B}\left(\hat{\mathbf{u}}_{k/k} - \mathbf{K}\hat{\mathbf{x}}_{k/k}\right)$ is an asymptotically vanishing term. Since also $(\mathbf{A} + \mathbf{B}\mathbf{K})$ is Schur by Assumption 2.1, we obtain that $\mathbf{x}_k \to 0$ as $k \to +\infty$.

2.7.2 Proof of Algorithm 2.1

In this Section, the approach based on Linear Matrix Inequalities [18], for computing the state-feedback gain and the weighting matrices is described in detail.

First recall that the closed-loop system composed by (2.1) and by the state feedback $\mathbf{u}_k = \mathbf{K}\mathbf{x}_k$ control law is stable if and only if there exist a matrix $\mathbf{P}^o \in \mathbb{R}^{n \times n}$ such that

$$\begin{cases} \mathbf{P}^o \succ 0 \\ (\mathbf{A} + \mathbf{B}\mathbf{K})^T \mathbf{P}^o (\mathbf{A} + \mathbf{B}\mathbf{K}) - \mathbf{P}^o \prec 0 \end{cases} \tag{2.74}$$

Now define the positive definite matrix $\mathbf{S}$, such that $\mathbf{P}^o = \mathbf{S}^{-1}$ and rewrite conditions (2.74) as

$$\begin{cases} \mathbf{S} \succ 0 \\ (\mathbf{A}^T + \mathbf{K}^T\mathbf{B}^T)\mathbf{S}^{-1}(\mathbf{A} + \mathbf{B}\mathbf{K}) - \mathbf{S}^{-1} \prec 0 \end{cases} \tag{2.75}$$

Pre- and post-multiplying (2.75) by $\mathbf{S}$, and defining matrix $\mathbf{Y} \in \mathbb{R}^{m \times n}$ such that $\mathbf{K} = \mathbf{Y}\mathbf{S}^{-1}$, (2.75) can be written as

$$\begin{cases} \mathbf{S} \succ 0 \\ (\mathbf{S}\mathbf{A}^T + \mathbf{Y}^T\mathbf{B}^T)\mathbf{S}^{-1}(\mathbf{A}\mathbf{S} + \mathbf{B}\mathbf{Y}) - \mathbf{S} \prec 0 \end{cases} \tag{2.76}$$

By using the Schur complement transformation [173], this expression can be transformed in the following LMIs

$$\begin{bmatrix} \mathbf{S} & \mathbf{S}\mathbf{A}^T + \mathbf{Y}^T\mathbf{B}^T \\ \mathbf{A}\mathbf{S} + \mathbf{B}\mathbf{Y} & \mathbf{S} \end{bmatrix} \succ 0 \tag{2.77}$$

whose solution $\mathbf{S}$ and $\mathbf{Y}$ allows to compute $\mathbf{K} = \mathbf{Y}\mathbf{S}^{-1}$ and $\mathbf{P}^o = \mathbf{S}^{-1}$. Moreover, since $\mathbf{P}^o$ is required to have a block diagonal structure, the additional constraints have to be included into the LMI problem:

$$S_{ij} = 0 \quad \forall i, j = 1, \dots, M \quad (i \neq j) \tag{2.78}$$

where $S_{ij} \in \mathbb{R}^{n_i \times n_j}$ are the blocks of $\mathbf{S}$ outside the diagonal, and $S_{ii} \in \mathbb{R}^{n_i \times n_i}$ are the diagonal blocks.

Analogously, in order to have also $\mathbf{K}$ block diagonal, $\mathbf{Y}$ must be block diagonal as well:

$$Y_{ij} = 0 \quad \forall i, j = 1, \ldots, M \quad (i \neq j) \tag{2.79}$$

where $Y_{ij} \in \mathbb{R}^{m_i \times n_j}$ are the blocks of $\mathbf{Y}$ outside the diagonal.

Finally, each block K_i must be stabilizing for its i-th subsystem (recall again Assumption 2.2), which translates in the following condition for each subsystem:

$$\begin{bmatrix} S_{ii} & S_{ii}\bar{A}_{ii}^T + Y_{ii}^T \bar{B}_{ii}^T \\ \bar{A}_{ii}S_{ii} + \bar{B}_{ii}Y_{ii} & S_{ii} \end{bmatrix} \succ 0 \tag{2.80}$$

3 Continuous-time DPC

With some notable exceptions, see e.g. [32, 44, 90, 146], the majority of the distributed control algorithms proposed so far have been developed for discrete-time systems, possibly obtained from an underlying continuous-time model, see also the Distributed Predictive Control (DPC) technique presented in Chapter 2. The discrete-time framework is particularly suitable for the design of distributed MPC, since it also allows to easily develop methods based on distributed optimization approaches, see e.g. [7, 27, 28, 42, 76, 134], or on agent negotiation, see e.g. [97, 98]. On the other hand, it does not allow to consider the process inter-sampling behavior in the optimization problem underlying any MPC algorithm; for this reason, the development of new and effective MPC methods for continuous-time systems is of interest.

In this Chapter, the DPC algorithm is formulated in a continuous-time framework. Also the continuous-time DPC is based on a non-iterative scheme where the future state and control reference trajectories are transmitted among neighboring systems, i.e. systems with direct couplings through their state or control variables, and the differences between these trajectories and the true ones are interpreted as disturbances to be rejected by a proper robust control method. The continuous-time approach is characterized by a higher complexity, but it has the positive side of considering in the optimization problem the system behavior at all time instants.

3.1 Partitioned continuous-time systems

Consider a process made by M interacting systems described by the continuous-time linear model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{3.1}$$

where $\mathbf{x}(t) \in \mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbf{u}(t) \in \mathbb{U} \subseteq \mathbb{R}^m$ are the state and input vectors, respectively, both subject to constraints.

Letting $\mathbf{x}(t) = (x^{[1]}(t), \ldots, x^{[M]}(t))$ and $\mathbf{u}(t) = (u^{[1]}(t), \ldots, u^{[M]}(t))$, the dynamics of each subsystem is given by

$$\dot{x}^{[i]}(t) = A_{ii}x^{[i]}(t) + B_{ii}u^{[i]}(t) + \sum_{j\neq i, j\in M} \{A_{ij}x^{[j]}(t) + B_{ij}u^{[j]}(t)\} \tag{3.2}$$

where $x^{[i]}(t) \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ and $u^{[i]}(t) \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ are the state and input vectors, respectively, of the i-th system ($i = 1, \ldots, M$), and it holds that $n = \sum_{i=1}^{M} n_i$ and $m = \sum_{i=1}^{M} m_i$. The sets $\mathbb{X}_i$ and $\mathbb{U}_i$ defining the constraints are convex neighborhoods of the origin and we have that $\mathbb{X} = \prod_{i=1}^{M} \mathbb{X}_i$ and $\mathbb{U} = \prod_{i=1}^{M} \mathbb{U}_i$, which are convex due to the convexity of $\mathbb{X}_i$ and $\mathbb{U}_i$. The subsystems' matrices A_{ij} and B_{ij}, $i, j = 1, \ldots, M$ are the block entries of the matrices $\mathbf{A}$ and $\mathbf{B}$ describing the dynamics of the large-scale system (3.1). In the following, subsystem j will be defined as a neighbor of subsystem i if and only if $A_{ij} \neq 0$ and/or $B_{ij} \neq 0$, and $\mathcal{N}_i$ will denote the set of neighbors of subsystem i (which excludes i).

Concerning systems (3.2), the following stabilizability assumption is introduced.

Assumption 3.1 *There exist matrices $\bar{K}_i \in R^{m_i \times n_i}$, $i = 1, \ldots, M$, such that $\bar{F}_{ii} = (A_{ii} + B_{ii}\bar{K}_i)$ are Hurwitz.* □

We define $\bar{\mathbf{K}}$ =diag$(\bar{K}_1, \ldots, \bar{K}_M)$.

As for the collective system (3.1), the following assumption on decentralized stabilizability is made.

Assumption 3.2 *There exists a block-diagonal matrix $\mathbf{K}^c$, defined as $\mathbf{K}^c$ =diag$(K_1^c, \ldots, K_M^c)$ with $K_i^c \in R^{m_i \times n_i}$, $i = 1, \ldots, M$, such that:*

i) $\mathbf{A} + \mathbf{B}\mathbf{K}^c$ *is Hurwitz.*

ii) $F_{ii} = (A_{ii} + B_{ii}K_i^c)$ *is Hurwitz,* $i = 1, \ldots, M$.

□

Note that Assumption 3.2 implies Assumption 3.1 which can be trivially satisfied by setting $\bar{K}_i = K_i^c$. However, since $\bar{K}_i$ and K_i^c play different roles in the design algorithm to be presented, it can be useful to allow them to be different for performance enhancement.

3.2 DPC for continuous-time systems

The continuous-time distributed control law described in the following is composed by two terms, the first one is a standard state feedback, while the second one is computed by an MPC-based distributed control algorithm running with sampling period T and at sampling times $t_k = kT$, $k \in \mathbb{N}$. For simplicity of notation, given the sampling instant t_k, the time instant $t_k + hT$ will be denoted by t_{k+h}.

3.2.1 Models: perturbed, nominal and auxiliary

In order to set up the proposed distributed control method, it will be assumed that at any time instant t_k each subsystem i transmits to its neighbors its continuous-time future state and input reference trajectories $\tilde{x}^{[i]}(t)$ and $\tilde{u}^{[i]}(t)$, $t \in [t_k, t_{k+N-1})$. Moreover, by adding suitable constraints to the MPC formulation, each subsystem will be able to guarantee that its state and control trajectories lie in specified time-invariant neighborhoods of the reference trajectories, i.e, for all $t \in [t_k, t_{k+N-1})$, $x^{[i]}(t) \in \tilde{x}^{[i]}(t) \oplus \mathcal{E}_i$ and $u^{[i]}(t) \in \tilde{u}^{[i]}(t) \oplus \mathcal{U}_i$, where $0 \in \mathcal{E}_i$ and $0 \in \mathcal{U}_i$. It is possible to rewrite (3.2) as the perturbed model

$$\dot{x}^{[i]}(t) = A_{ii}x^{[i]}(t)+B_{ii}u^{[i]}(t)+\sum_{j\in\mathcal{N}_i}(A_{ij}\tilde{x}^{[j]}(t)+B_{ij}\tilde{u}^{[j]}(t))+w^{[i]}(t) \quad (3.3)$$

where the term $\sum_{j\in\mathcal{N}_i}(A_{ij}\tilde{x}^{[j]}(t) + B_{ij}\tilde{u}^{[j]}(t))$ can be interpreted as a disturbance, known in advance over the future prediction horizon of length $(N-1)T$ (i.e., for all $t \in [t_k, t_{k+N-1})$), to be suitably compensated. On the other hand,

$$w^{[i]}(t) = \sum_{j\in\mathcal{N}_i} \left(A_{ij}(x^{[j]}(t) - \tilde{x}^{[j]}(t)) + B_{ij}(u^{[j]}(t) - \tilde{u}^{[j]}(t))\right) \in \mathbb{W}_i \quad (3.4)$$

is a bounded unknown disturbance (i.e., $\mathbb{W}_i = \bigoplus_{j\in\mathcal{N}_i}\{A_{ij}\mathcal{E}_i \oplus B_{ij}\mathcal{U}_i\}$) to be rejected.

For the statement of the individual MPC sub-problems, hereafter denoted i-DPC problem, we rely on a continuous-time version, see [51], of the robust MPC algorithm presented in [107] for constrained discrete-time linear systems with bounded disturbances. As a preliminary step, define the i-th subsystem nominal model obtained from equation (3.3) by neglecting the disturbance $w^{[i]}(t)$:

$$\dot{\hat{x}}^{[i]}(t) = A_{ii}\hat{x}^{[i]}(t) + B_{ii}\hat{u}^{[i]}(t) + \sum_{j\in\mathcal{N}_i}(A_{ij}\tilde{x}^{[j]}(t) + B_{ij}\tilde{u}^{[j]}(t)) \tag{3.5}$$

The control law for the i-th perturbed subsystem (3.3) is given by

$$u^{[i]}(t) = \hat{u}^{[i]}(t) + K_i^c(x^{[i]}(t) - \hat{x}^{[i]}(t)) \tag{3.6}$$

where K_i^c is the feedback gain satisfying Assumption 3.2. Letting $z^{[i]}(t) = x^{[i]}(t) - \hat{x}^{[i]}(t)$, from equations (3.3) and (3.6), one obtains

$$\dot{z}^{[i]}(t) = F_{ii}z^{[i]}(t) + w^{[i]}(t) \tag{3.7}$$

where $w^{[i]}(t) \in \mathbb{W}_i$. Since $\mathbb{W}_i$ is bounded and F_{ii} is Hurwitz, it is possible to define the robust positively invariant (RPI) set Z_i for (3.7) (see, for example, [160] and [133]) such that, for all $z^{[i]}(t_k) \in Z_i$, then $z^{[i]}(t) \in Z_i$ for all $t \geq t_k$. Here, we assume that the sets Z_i are such that there exist non-empty sets $\hat{\mathbb{X}}_i \subseteq \mathbb{X}_i \ominus Z_i$ and $\hat{\mathbb{U}}_i \subseteq \mathbb{U}_i \ominus K_i^c Z_i$. Given Z_i, define the neighborhoods of the origin E_i and U_i, $i = 1, \dots, M$ such that $E_i \oplus Z_i \subseteq \mathcal{E}_i$ and $U_i \oplus K_i^c Z_i \subseteq \mathcal{U}_i$, respectively.

Assume also that it is possible to guarantee that, for suitably defined sets $\bar{\mathcal{E}}_i$ and $\bar{\mathcal{U}}_i$ and for $t \in [t_{k+N-1}, t_{k+N})$, $\tilde{x}^{[i]}(t) \in \bar{\mathcal{E}}_i$ and $\tilde{u}^{[i]}(t) \in \bar{\mathcal{U}}_i$. Then, with reference to the time interval $t \in [t_{k+N-1}, t_{k+N}]$, it is worth defining an auxiliary "decentralized" model, obtained from equation (3.5) by neglecting the known disturbance term:

$$\dot{\bar{x}}^{[i]}(t) = A_{ii}\bar{x}^{[i]}(t) + B_{ii}\bar{u}^{[i]}(t) \tag{3.8}$$

Similarly to (3.6), the term $\hat{u}^{[i]}(t)$ is set as follows

$$\hat{u}^{[i]}(t) = \bar{u}^{[i]}(t) + \bar{K}_i(\hat{x}^{[i]}(t) - \bar{x}^{[i]}(t)) \tag{3.9}$$

where $\bar{K}_i$ is the feedback gain satisfying Assumption 3.1. Letting $s^{[i]}(t) = \hat{x}^{[i]}(t) - \bar{x}^{[i]}(t)$, from (3.5), (3.8) and (3.9) one has

$$\dot{s}^{[i]}(t) = \bar{F}_{ii}s^{[i]}(t) + \bar{w}^{[i]}(t) \tag{3.10}$$

where $\bar{w}^{[i]}(t) = \sum_{j\in\mathcal{N}_i}\{A_{ij}\tilde{x}^{[j]}(t) + B_{ij}\tilde{u}^{[j]}(t)\} \in \bar{\mathbb{W}}_i$ and $\bar{\mathbb{W}}_i = \bigoplus_{j\in\mathcal{N}_i}\{A_{ij}\bar{\mathcal{E}}_j \oplus B_{ij}\bar{\mathcal{U}}_j\}$. Since $\bar{\mathbb{W}}_i$ is bounded and $\bar{F}_{ii}$ is Hurwitz, it is possible to define a further robust positively invariant (RPI) set S_i

for (3.10). The sets $\bar{\mathcal{E}}_i$ and $\bar{\mathcal{U}}_i$ must satisfy $\bar{\mathcal{E}}_i \oplus S_i \subseteq \hat{\mathbb{X}}_i$, $\bar{\mathcal{U}}_i \oplus \bar{K}_i S_i \subseteq \hat{\mathbb{U}}_i$, $S_i \subseteq E_i$ and $\bar{K}_i S_i \subseteq U_i$.

Although not strictly necessary for the derivation of the properties of the proposed DPC method, for simplicity in the following it will be assumed that the feed-forward term $\bar{u}^{[i]}(t)$ is a piecewise constant signal, i.e., $\bar{u}^{[i]}(t) = \bar{u}^{[i]}(t_{k+N-1})$ for all $t \in [t_{k+N-1}, t_{k+N})$.

3.2.2 Statement of the i-DPC problems

At any time instant t_k, given the future reference trajectories $\tilde{x}^{[j]}(t)$, $\tilde{u}^{[j]}(t)$, $t \in [t_k, t_{k+N-1})$, $j \in \mathcal{N}_i \cup \{i\}$, for system $i = 1, \dots M$ we define the following i-DPC problem

$$\min_{\hat{x}^{[i]}(t_k), \hat{u}^{[i]}([t_k, t_{k+N-1})), \bar{x}^{[i]}(t_{k+N-1}), \bar{u}^{[i]}(t_{k+N-1})} V_i^N \tag{3.11}$$

subject to (3.5), (3.8), to

$$x^{[i]}(t_k) - \hat{x}^{[i]}(t_k) \in Z_i \tag{3.12}$$

$$\hat{x}^{[i]}(t) - \tilde{x}^{[i]}(t) \in E_i \tag{3.13}$$

$$\hat{u}^{[i]}(t) - \tilde{u}^{[i]}(t) \in U_i \tag{3.14}$$

$$\hat{x}^{[i]}(t) \in \hat{\mathbb{X}}_i \tag{3.15}$$

$$\hat{u}^{[i]}(t) \in \hat{\mathbb{U}}_i \tag{3.16}$$

for all $t \in [t_k, t_{k+N-1})$, to

$$\hat{x}^{[i]}(t_{k+N-1}) - \bar{x}^{[i]}(t_{k+N-1}) \in S_i \tag{3.17}$$

$$\bar{x}^{[i]}(t) \in \bar{\mathcal{E}}_i \tag{3.18}$$

$$\bar{u}^{[i]}(t) \in \bar{\mathcal{U}}_i \tag{3.19}$$

for all $t \in [t_{k+N-1}, t_{k+N})$, and to the terminal constraint

$$\bar{x}^{[i]}(t_{k+N}) \in \bar{\mathbb{X}}_i^F \tag{3.20}$$

where $\bar{\mathbb{X}}_i^F$ is a terminal set related to the i-th nominal subsystem (3.8), specified in the following section.

The cost function V_i^N is

$$\begin{aligned} V_i^N = & \frac{1}{2} \int_{t_k}^{t_{k+N-1}} (\|\hat{x}^{[i]}(t)\|^2_{\hat{Q}_i} + \|\hat{u}^{[i]}(t)\|^2_{\hat{R}_i}) dt + \frac{1}{2} \|\hat{x}^{[i]}(t_{k+N-1})\|^2_{\hat{P}_i} + \\ & + \lambda \left(\frac{1}{2} \int_{t_{k+N-1}}^{t_{k+N}} (\|\bar{x}^{[i]}(t)\|^2_{\bar{Q}_i} + \|\bar{u}^{[i]}(t)\|^2_{\bar{R}_i}) dt + \frac{1}{2} \|\bar{x}^{[i]}(t_{k+N})\|^2_{\bar{P}_i} \right) \end{aligned} \tag{3.21}$$

where λ is a positive constant and the symmetric, positive definite matrices $\hat{Q}_i$, $\bar{Q}_i$, $\hat{R}_i$, $\bar{R}_i$, $\hat{P}_i$, and $\bar{P}_i$ are design parameters and are chosen as specified later.

Denoting by

$$X_i(t_k) = (\hat{x}^{[i]}(t_k), \hat{u}^{[i]}([t_k, t_{k+N-1})), \bar{x}^{[i]}(t_{k+N-1}), \bar{u}^{[i]}(t_{k+N-1}))$$

the arguments of the cost function V_i^N, the optimal solution to the i-DPC problem at time t_k is the 4-uple

$$X_i(t_k|t_k) = (\hat{x}^{[i]}(t_k|t_k), \hat{u}^{[i]}([t_k, t_{k+N-1})|t_k), \bar{x}^{[i]}(t_{k+N-1}|t_k), \bar{u}^{[i]}(t_{k+N-1}|t_k))$$

The signal $\hat{x}^{[i]}(t|t_k)$, $t \in [t_k, t_{k+N-1}]$ (respectively $\bar{x}^{[i]}(t|t_k)$, $t \in [t_{k+N-1}, t_{k+N}]$) is the solution to (3.5) (respectively to (3.8)) obtained with $\hat{x}^{[i]}(t_k|t_k)$ as initial conditionand $\hat{u}([t_k, t_{k+N-1})|t_k)$ as input sequence (respectively with $\bar{x}^{[i]}(t_{k+N-1}|t_k)$ as initial condition and $\bar{u}(t_{k+N-1}|t_k)$ as constant input). According to (3.6), the control law for the system (3.2), for $t \in [t_k, t_{k+1})$, is given by

$$u^{[i]}(t) = \hat{u}^{[i]}(t|t_k) + K_i^c(x^{[i]}(t) - \hat{x}^{[i]}(t|t_k)) \qquad (3.22)$$

Finally, the reference trajectories $\tilde{x}^{[i]}(t)$ and $\tilde{u}^{[i]}(t)$ are incrementally defined as follows. Specifically, for $t \in [t_{k+N-1}, t_{k+N})$, we set

$$\tilde{x}^{[i]}(t) = \bar{x}^{[i]}(t|t_k) \qquad (3.23a)$$

$$\tilde{u}^{[i]}(t) = \bar{u}^{[i]}(t|t_k) \qquad (3.23b)$$

Then, these pieces of trajectories are transmitted to the subsystems j such that $i \in \mathcal{N}_j$, i.e., which need their knowledge to compute the future predictions $\hat{x}^{[j]}(t)$. Remark that, by solely transmitting $\bar{x}^{[i]}(t_{k+N-1}|t_k)$ and $\bar{u}^{[i]}(t_{k+N-1}|t_k)$, the whole $\bar{x}^{[i]}(t|t_k)$, $t \in [t_{k+N-1}, t_{k+N})$, can be exactly reconstructed by subsystem j provided that j knows the dynamical model governing the subsystem i.

3.2.3 Properties of DPC

In order to establish the main stability and convergence properties of the proposed distributed control law, the following definition and assumption must be introduced. The set of admissible initial conditions $\mathbf{x}(t_0) = (x^{[1]}(t_0), \ldots, x^{[M]}(t_0))$ and initial reference trajectories $\tilde{x}^{[j]}(t)$, $\tilde{u}^{[j]}(t)$, for all $j = 1 \ldots, M$ and $t \in [t_0, t_{N-1})$, are defined as follows.

Definition 3.1 *Letting* $\mathbf{x} = (x^{[1]}, \ldots, x^{[M]})$, *denote by*

$$\begin{aligned}\mathbb{X}^N := \; & \{\mathbf{x} : \textit{ if } x^{[i]}(t_0) = x^{[i]} \textit{ for all } i = 1, \ldots, M \textit{ then } \exists \\ & (\tilde{x}^{[1]}(t), \ldots, \tilde{x}^{[M]}(t)), (\tilde{u}^{[1]}(t), \ldots, \tilde{u}^{[M]}(t)) \\ & \textit{for all } t \in [t_0, t_{N-1}), X_i(t_0|t_0) \textit{ such that } (3.5), (3.8), \\ & \textit{and } (3.12)\text{- } (3.20) \textit{ are satisfied for all } i = 1, \ldots, M\}\end{aligned}$$

the feasibility region for all the i-DPC problems. Moreover, for each $\mathbf{x} \in \mathbb{X}^N$, *let*

$$\begin{aligned}\tilde{\mathbb{X}}_{\mathbf{x}} := \; & \{(\tilde{x}^{[1]}(t), \ldots, \tilde{x}^{[M]}(t)), (\tilde{u}^{[1]}(t), \ldots, \tilde{u}^{[M]}(t)) \textit{ for all } t \in [t_0, t_{N-1}) : \\ & \textit{if } x^{[i]}(t_0) = x^{[i]} \textit{ for all } i = 1, \ldots, M \textit{ then } \exists X_i(t_0|t_0) \textit{ such that} \\ & (3.5), (3.8), (3.12)\text{- } (3.20) \textit{ are satisfied for all } i = 1, \ldots, M\}\end{aligned}$$

be the region of feasible initial reference trajectories.

Assumption 3.3 *Given the sets* $\mathcal{E}_i$, $\mathcal{U}_i$, *and the RPI sets* Z_i *for equation* (3.7), *there exists a real positive constant* $\bar{\rho}_E > 0$ *such that* $Z_i \oplus \mathcal{B}^{(n_i)}_{\bar{\rho}_E}(0) \subseteq \mathcal{E}_i$ *and* $K_i^c Z_i \oplus \mathcal{B}^{(m_i)}_{\bar{\rho}_E}(0) \subseteq \mathcal{U}_i$ *for all* $i = 1, \ldots, M$. □

Then, it is possible to state the following result (see the Appendix for the proof).

Theorem 3.1 *Let Assumptions 3.1, 3.2, and 3.3 be satisfied; then, there exist computable design parameters* $\lambda, \hat{Q}_i, \bar{Q}_i, \hat{R}_i, \bar{R}_i, \hat{P}_i, \bar{P}_i$ *such that, for any initial reference trajectories in* $\tilde{\mathbb{X}}_{\mathbf{x}(t_0)}$, *the trajectory* $\mathbf{x}(t)$, *starting from any initial condition* $\mathbf{x}(t_0) \in \mathbb{X}^N$, *asymptotically converges to the origin.* ■

A detailed discussion on how to select the design parameters and the sets of interest is reported in the following section.

Remark 3.1 *In the optimization problem* (3.11) *it has been assumed that* $\hat{u}^{[i]}([t_k, t_{k+N-1}))$ *is a generic function of time. However, for computational reasons, it is usually more convenient to resort to parameterized functions and to optimize with respect to the corresponding parameters.*

3.3 Tuning of the design parameters

In this section we show how to compute design parameters which guarantee that Theorem 3.1 holds.

3.3.1 Choice of the control gains K_i^c, $\overline{K}_i$

The control laws (3.6), (3.9) require the knowledge of the gains K_i^c and $\overline{K}_i$ satisfying Assumptions 3.1 and 3.2. While the terms $\overline{K}_i$ can be computed with any standard synthesis method provided that the pair (A_{ii}, B_i) is stabilizable, the computation of $\mathbf{K}^c$ =diag$(K_1^c, \dots, K_M^c)$ is more difficult, since both a collective and a number of local stability conditions must be fulfilled. For instance, this problem can be easily tackled in a centralized fashion by defining two block diagonal matrices $\mathbf{S}$ =diag$(S_1, \dots, S_M)$, $S_i \in \mathbb{R}^{n_i, n_i}$, and $\mathbf{Y}$ =diag$(Y_1, \dots, Y_M)$, $Y_i \in \mathbb{R}^{m_i, n_i}$, and by solving the following set of LMI's, see [172]

$$\begin{cases} \mathbf{S} \succ 0 \\ S_i \succ 0, i = 1, \dots, M \\ \mathbf{S}\mathbf{A}^T + \mathbf{A}\mathbf{S} + \mathbf{Y}^T\mathbf{B}^T + \mathbf{B}\mathbf{Y} \prec 0 \\ S_i A_{ii}^T + A_{ii} S_i + Y_i^T B_{ii}^T + B_{ii} Y_i \prec 0, i = 1, \dots, M \end{cases} \tag{3.24}$$

Then, $\mathbf{K}^c = \mathbf{Y}\mathbf{S}^{-1}$ is the required stabilizing block diagonal matrix.

3.3.2 Choice of $\bar{Q}_i$, $\bar{R}_i$, $\bar{P}_i$, $\bar{\mathbb{X}}_i^F$

In order to define matrices $\bar{Q}_i$, $\bar{R}_i$, $\bar{P}_i$, and the invariant set $\bar{\mathbb{X}}_i^F$, we must preliminarily define the auxiliary control law for the system (3.8), which must be consistent with the simplifying assumption that $\bar{u}^{[i]}(t)$ is piecewise constant. Assuming that the terminal constraint
$\bar{x}^{[i]}(t_{k+N}) \in \bar{\mathbb{X}}_i^F$ is verified, we define the auxiliary control law, to be applied to system (3.8) for all $t \in [t_{k+N}, t_{k+N+1})$, as

$$\bar{u}^{[i]}(t) = \bar{u}^{[i]}(t_{k+N}) = K_i^d \bar{x}^{[i]}(t_{t+N}) \tag{3.25}$$

where the gain K_i^d must stabilize the continuous-time system (3.8). Denoting, for all $\eta \in [0, T]$, $A_{ii}^{zoh}(\eta) = e^{A_{ii}\eta}$ and $B_{ii}^{zoh}(\eta) = \int_0^\eta e^{A_{ii}(\eta-\nu)} B_{ii} d\nu$ and given $\bar{x}^{[i]}(t_{k+N})$, for all $t \in [t_{k+N}, t_{k+N+1}]$ one has

$$\begin{aligned} \bar{x}^{[i]}(t) &= F_{ii}^{zoh}(t - t_{k+N}) \bar{x}^{[i]}(t_{k+N}) \\ \bar{x}^{[i]}(t_{k+N+1}) &= F_{ii}^d \bar{x}^{[i]}(t_{k+N}) \end{aligned} \tag{3.26}$$

where $F_{ii}^{zoh}(\eta) = A_{ii}^{zoh}(\eta) + B_{ii}^{zoh}(\eta) K_i^d$ and $F_{ii}^d = F_{ii}^{zoh}(T)$. Therefore, the gains K_i^d can be computed with any standard stabilization method to guarantee that F_{ii}^d is Schur. This procedure allows also one to resort to the results reported in [101], Lemma 1, which turn out to be useful in the following developments. Specifically, given the symmetric weighting matrices $\bar{Q}_i > 0$ and $\bar{R}_i > 0$ appearing in (3.21) and which can be chosen as free design parameters, define two constants $\gamma_{i1} > 0$, $\gamma_{i2} > 0$ in such a way that

$$\gamma_{i1} > \lambda_M(\bar{Q}_i) \tag{3.27a}$$

$$\gamma_{i2} > T\|K_i^d\|^2\lambda_M(\bar{R}_i) \tag{3.27b}$$

Furthermore, define a matrix Q_i^* in such a way that $\lambda_m(Q_i^*) > \gamma_{i1}$. Let the symmetric matrix $\bar{P}_i$ be the unique positive definite solution of the following Lyapunov equation

$$(F_{ii}^d)^T\bar{P}_iF_{ii}^d - \bar{P}_i + \tilde{Q}_i = 0 \tag{3.28}$$

where $\tilde{Q}_i = \int_0^T (F_{ii}^{zoh}(\eta))^T Q^* F_{ii}^{zoh}(\eta)d\eta + \gamma_{i2}I$. Then, for each pair of sets $\bar{\mathcal{E}}_i$, $\bar{\mathcal{U}}_i$, it is proven in [101] that there exist a sampling period $T \in [0, +\infty)$ and a constant $c_i > 0$ such that the set

$$\bar{\mathbb{X}}_i^F(K_i^d, T) = \{\bar{x}^{[i]} \mid \|\bar{x}^{[i]}\|_{\bar{P}_i}^2 \leq c_i\} \tag{3.29}$$

satisfies, for all $\bar{x}^{[i]}(t_{k+N}) \in \bar{\mathbb{X}}_i^F$ and for all $t \in [t_{k+N}, t_{k+N+1})$, the conditions

$$\bar{x}^{[i]}(t) \in \bar{\mathcal{E}}_i,\ K_d\bar{x}^{[i]}(t_{k+N}) \in \bar{\mathcal{U}}_i \tag{3.30a}$$

$$\|\bar{x}^{[i]}(t_{k+N+1})\|_{\bar{P}_i}^2 - \|\bar{x}^{[i]}(t_{k+N})\|_{\bar{P}_i}^2 \leq -\gamma_{i1}\int_{t_{k+N}}^{t_{k+N+1}} \|\bar{x}^{[i]}(\eta)\|^2 d\eta - \gamma_{i2}\|\bar{x}^{[i]}(t_{k+N})\|^2 \tag{3.30b}$$

Letting

$$\bar{l}_i(\bar{x}^{[i]}(t), \bar{u}^{[i]}(t)) = \frac{\lambda}{2}\int_t^{t+T} (\|\bar{x}^{[i]}(\eta)\|_{\bar{Q}_i}^2 + \|\bar{u}^{[i]}(\eta)\|_{\bar{R}_i}^2)d\eta \tag{3.31a}$$

$$\bar{V}_i^F(\bar{x}^{[i]}(t)) = \frac{\lambda}{2}\|\bar{x}^{[i]}(t)\|_{\bar{P}_i}^2 \tag{3.31b}$$

from the definition of $\gamma_{i1} > 0$, $\gamma_{i2} > 0$ and $\bar{V}_i^F$, and recalling (3.25), (3.31b) implies that $\bar{x}^{[i]}(t_{k+N+1}) \in \bar{\mathbb{X}}_i^F$ and

$$\bar{V}_i^F(\bar{x}^{[i]}(t_{k+N+1})) - \bar{V}_i^F(\bar{x}^{[i]}(t_{k+N})) \leq -\frac{\lambda}{2}\int_{t_{k+N}}^{t_{k+N+1}} (\|\bar{x}^{[i]}(\eta)\|_{\bar{Q}_i}^2 + \|\bar{u}^{[i]}(\eta)\|_{\bar{R}_i}^2)d\eta \leq -\bar{l}_i(\bar{x}^{[i]}(t_{k+N}), \bar{u}^{[i]}(t_{k+N})) \tag{3.32}$$

Therefore, since properties (3.30a)- (3.32) are required to establish the main properties of the method (see the proof of Theorem 3.1), for any pair $\bar{Q}_i, \bar{R}_i$ it is required to choose the weights $\bar{P}_i$ in (3.21) according to (3.28) and the terminal set $\bar{\mathbb{X}}_i^F$ in (3.20) according to (3.29).

3.3.3 Choice of $\hat{Q}_i$, $\hat{R}_i$, $\hat{P}_i$, λ

The symmetric, positive definite matrices $\hat{Q}_i$, $\hat{R}_i$ can be freely chosen according to specific design criteria, while, in order to guarantee the stability properties of Theorem 3.1, given an arbitrary constant $\alpha > 1$, the matrix $\hat{P}_i$ must be computed to satisfy the following Lyapunov equation:

$$\bar{\Phi}_x^{[i]}(T)^T \hat{P}_i \bar{\Phi}_x^{[i]}(T) - \hat{P}_i + \mathcal{Q}_x^{[i]} + \alpha I_n = 0 \tag{3.33}$$

where

$$\mathcal{Q}_x^{[i]} = \int_0^T \bar{\Phi}_x^{[i]}(\eta)^T \hat{Q}_i \bar{\Phi}_x^{[i]}(\eta) + \bar{\Phi}_u^{[i]}(\eta)^T \hat{R}_i \bar{\Phi}_u^{[i]}(\eta) d\eta$$

and, for all $\eta = [0, T]$, $\bar{\Phi}_x^{[i]}(\eta) = e^{\bar{F}_{ii}\eta}$ and $\bar{\Phi}_u^{[i]}(\eta) = \bar{K}_i \bar{\Phi}_x^{[i]}(\eta)$. For the tuning of scalar λ, there exists a positive number $\bar{\lambda} > 0$ such that, if $\lambda \geq \bar{\lambda}$, then the convergence of the scheme is guaranteed. For a numerical assessment of $\bar{\lambda}$, see the discussion in Section 3.6

3.4 Simulation example

Consider the problem of regulating the levels y_i, $i = 1, \ldots, 5$ of the five flotation tanks system discussed in [158], see also Chapter 2, where a flow of pulp q enters into the first one. The tanks are connected in cascade with control valves between subsequent reservoirs (Figure 3.1), and the manipulated inputs are the signals to the valves v_i, $i = 1, \ldots, 5$. We refer the reader to Chapter 2 for details about the dynamic model of the system, its parameters, the considered equilibrium point and the constraints on inputs and states. For the sake of simplicity, in this case external disturbances are not considered.

Let $\delta y_i = y_i - \bar{y}_i$, $i = 1, \ldots, 5$, $\delta v_i = v_i - \bar{v}_i$, $i = 1, \ldots, 5$, $\mathbf{x} = (\delta y_1, \delta y_2, \delta y_3, \delta y_4, \delta y_5)$ and $\mathbf{u} = (\delta v_1, \delta v_2, \delta v_3, \delta v_4, \delta v_5)$.

The partitions of inputs and states, for $i = 1, \ldots, 5$ is $x^{[i]} = \delta y_i$, $u^{[i]} = \delta v_i$.

The weighting matrices, for $i = 1, \ldots, 5$, are $\bar{Q}_i = \bar{R}_i = 1$ and $\hat{Q}_i = \hat{R}_i = 1$.

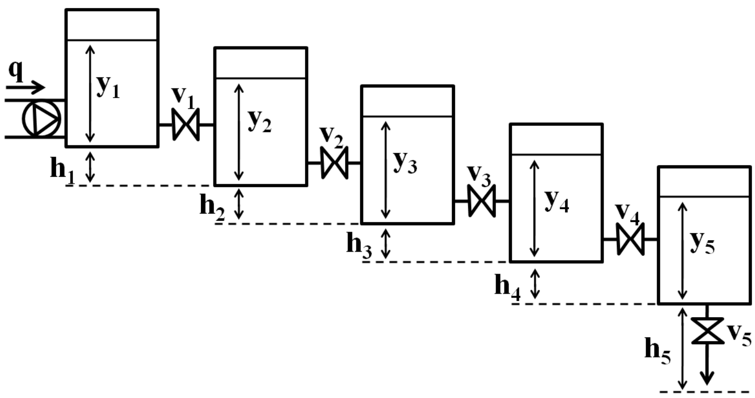

Figure 3.1: Schematic representation of the cascade coupled flotation tanks.

Note that, since the subsystems have all one state, the RPI sets can be computed easily. In fact, for a scalar system having dynamics $\dot{x}(t) = \lambda x(t) + w(t)$ with $w \in \mathbb{W} = \{w \in \mathbb{R} | -b \leq w \leq b\}$, we have that the RPI set $\mathbb{Z}$ is defined as $\mathbb{Z} = \{x \in \mathbb{R} \mid b/\lambda \leq x \leq -b/\lambda\}$ [133]. In this way, the computation of the RPI sets $\mathbb{Z}_i$ can be solved via a linear programming problem. The terms of the cost function, moreover, can be computed easily solving standard integrals. For subsystems with more than one state, symbolic calculus tools should be used.

In Figure 3.2 the states trajectories, obtained using the continuous-time nonlinear model in simulation are depicted. Figure 3.3 shows the applied inputs. In both cases, a comparison with a standard centralized discrete-time MPC algorithm is provided.

3.5 Conclusions

In this Chapter we have presented a non-cooperative distributed predictive control algorithm for continuous-time systems based on robust MPC, whose convergence properties have been proved. A realistic case study has been used for testing the performance of the algorithm. After the in-depth study of the regulation problem presented in first two chapters, in the next ones we will present some solutions to the tracking problem.

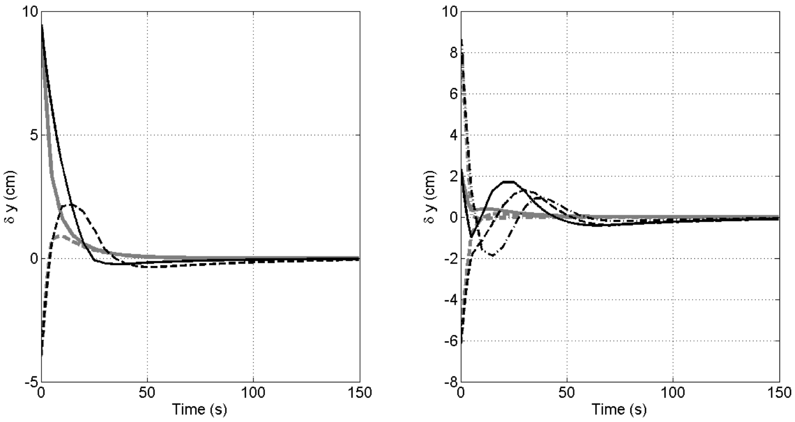

Figure 3.2: Trajectories of the state $x^{[1]}$ (solid lines), $x^{[2]}$ (dashed lines), on the left, and $x^{[3]}$ (solid lines), $x^{[4]}$ (dashed lines), $x^{[5]}$ (dash-dot lines), on the right, obtained with DPC (black lines) and with cMPC (gray lines) for the control of the floating tanks.

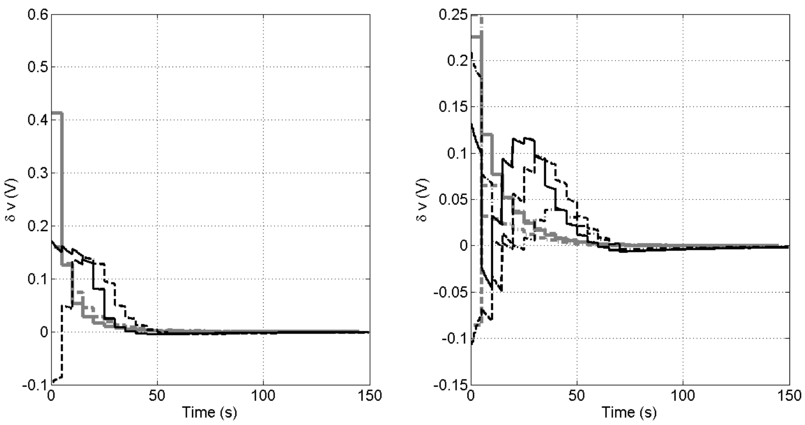

Figure 3.3: Inputs $u^{[1]}$ (solid lines), $u^{[2]}$ (dashed lines), on the left, and $u^{[3]}$ (solid lines), $u^{[4]}$ (dashed lines), $u^{[5]}$ (dash-dot lines), on the right, used with DPC (black lines) and with cMPC (gray lines) for the control of the floating tanks.

3.6 Appendix

3.6.1 Recursive feasibility

First we prove that, given for all $i = 1, \dots, M$, an optimal feasible solution $X_i(t_k|t_k)$ to (3.11) at time t_k, the 4-uple

$$X_i(t_{k+1}|t_k) = (\hat{x}^{[i]}(t_{k+1}|t_k), (\hat{u}^{[i]}([t_{k+1}, t_{k+N-1})|t_k), \bar{u}^{[i]}(t|t_k) + \\ + \bar{K}_i(\hat{x}^{[i]}(t|t_k) - \bar{x}^{[i]}(t|t_k)), t \in [t_{k+N-1}, t_{k+N})), \bar{x}^{[i]}(t_{k+N}|t_k), K_i^d \bar{x}^{[i]}(t_{k+N}|t_k)) \tag{3.34}$$

is a feasible solution to (3.11) at time t_{k+1}. Recall that, according to (3.23), after the solution to (3.11) is computed at time t_k, each subsystem j transmits $\tilde{x}^{[j]}([t_{k+N-1}, t_{k+N})) = \bar{x}^{[j]}([t_{k+N-1}, t_{k+N})|t_k)$ and $\tilde{u}^{[j]}([t_{k+N-1}, t_{k+N})) = \bar{u}^{[i]}([t_{k+N-1}, t_{k+N})|t_k)$ to all the subsystems i satisfying $j \in \mathcal{N}_i$.

Importantly, in (3.34), for $t \in [t_{k+N-1}, t_{k+N})$, the trajectory $\hat{x}^{[i]}(t|t_k)$ is computed, by subsystem i, according to system (3.5), with $\hat{u}^{[i]}(t) = \bar{u}^{[i]}(t|t_k) + \bar{K}_i(\hat{x}^{[i]}(t|t_k) - \bar{x}^{[i]}(t|t_k))$. Therefore it results that, for all $t \in [t_{k+N-1}, t_{k+N})$

$$\begin{aligned} \dot{\hat{x}}^{[i]}(t|t_k) &= A_{ii}\hat{x}^{[i]}(t|t_k) + B_{ii}\left(\bar{u}^{[i]}(t|t_k) + \bar{K}_i(\hat{x}^{[i]}(t|t_k) - \bar{x}^{[i]}(t|t_k))\right) + \\ &\quad + \textstyle\sum_{j\in\mathcal{N}_i}(A_{ij}\bar{x}^{[j]}(t|t_k) + B_{ij}\bar{u}^{[j]}(t|t_k)) \\ &= (A_{ii} + B_{ii}\bar{K}_i)\hat{x}^{[i]}(t|t_k) - B_{ii}\bar{K}_i\bar{x}^{[i]}(t|t_k) + \\ &\quad + \textstyle\sum_{j\in\mathcal{N}_i} A_{ij}\bar{x}^{[j]}(t|t_k) + \sum_{j=1}^{M} B_{ij}\bar{u}^{[j]}(t|t_k) \end{aligned} \tag{3.35}$$

On the other hand, the trajectory $\bar{x}^{[i]}(t|t_k)$, for all $t \in [t_{k+N}, t_{k+N+1}]$, is computed according to (3.8) with $\bar{u}^{[i]}(t|t_k) = K_i^d \bar{x}^{[i]}(t_{k+N}|t_k)$, and therefore $\bar{x}^{[i]}(t|t_k) = F_i^{zoh}(t - t_{k+N})\bar{x}^{[i]}(t_{k+N}|t_k)$.

From (3.12) $x^{[i]}(t_k) - \hat{x}^{[i]}(t_k) \in Z_i$ and, from (3.13)- (3.14), for $t \in [t_k, t_{k+1})$, it is guaranteed that $\hat{x}^{[j]}(t) - \tilde{x}^{[j]}(t) \in E_j$, $\hat{u}^{[j]}(t) - \tilde{u}^{[j]}(t) \in E_j$ for all $j \in \mathcal{N}_i$ and $w^{[i]}(t) \in \mathbb{W}_i$. Therefore, in view of the invariance of Z_i with respect to (3.7), it holds that $x^{[i]}(t_{k+1}) - \hat{x}^{[i]}(t_{k+1}|t_k) \in Z_i$.

For $t \in [t_{k+1}, t_{k+N-1})$, constraints (3.13), (3.14), (3.15) and (3.16) are verified in view of the feasibility of (3.11) at time t_k. For $t \in [t_{k+N-1}, t_{k+N})$, recalling (3.35) we have that

$$\begin{aligned} \dot{\hat{x}}^{[i]}(t|t_k) - \dot{\bar{x}}^{[i]}(t|t_k) &= (A_{ii} + B_{ii}\bar{K}_i)(\hat{x}^{[i]}(t|t_k) - \bar{x}^{[i]}(t|t_k)) \\ &\quad + \textstyle\sum_{j\in\mathcal{N}_i}\left(A_{ij}\bar{x}^{[j]}(t|t_k) + B_{ij}\bar{u}^{[j]}(t|t_k)\right) \end{aligned} \tag{3.36}$$

and recall also that $\tilde{x}^{[i]}(t) = \bar{x}^{[i]}(t|t_k)$ and $\tilde{u}^{[i]}(t) = \bar{u}^{[i]}(t|t_k)$ for all $i = 1, \dots, M$. In view of (3.17) $\hat{x}^{[i]}(t_{k+N-1}|t_k) - \bar{x}^{[i]}(t_{k+N-1}|t_k) \in S_i$ and from (3.18)- (3.19), it is guaranteed that $\sum_{j\in\mathcal{N}_i}(A_{ij}\bar{x}^{[j]}(t|t_k) + B_{ij}\bar{u}^{[j]}(t|t_k)) \in \bar{\mathbb{W}}_i$ for all $j \in \mathcal{N}_i$. In view of the invariance of S_i with respect to (3.10), it holds that $\hat{x}^{[i]}(t|t_k) - \bar{x}^{[i]}(t|t_k) = \hat{x}^{[i]}(t|t_k) - \tilde{x}^{[i]}(t) \in S_i$. Furthermore, since $\hat{u}^{[i]}(t|t_k) - \bar{u}^{[i]}(t|t_k) = \hat{u}^{[i]}(t|t_k) - \tilde{u}^{[i]}(t) \in \bar{K}_i S_i$ and $S_i \subseteq E_i$ and $\bar{K}_i S_i \subseteq U_i$, then (3.13) and (3.14) are also verified for $t \in [t_{k+N-1}, t_{k+N})$. This also proves that $\hat{x}^{[i]}(t_{k+N}|t_k) - \bar{x}^{[i]}(t_{k+N}|t_k) \in S_i$ and that (3.17) is satisfied. Moreover, being $\bar{\mathcal{E}}_i \oplus S_i \subseteq \hat{\mathbb{X}}_i$ and $\bar{\mathcal{U}}_i \oplus \bar{K}_i S_i \subseteq \hat{\mathbb{U}}_i$, constraints (3.15) and (3.16) are verified for $t \in [t_{k+N-1}, t_{k+N})$.

Finally, note that $\bar{x}^{[i]}(t_{k+N}) \in \bar{\mathbb{X}}_i^F$ in view of (3.20) and of the definition (3.29) of $\bar{\mathbb{X}}_i^F$ and (3.30b), the constraints (3.18), (3.19), (3.20) are also verified at time t_{k+1}. In view of this, the 4-uple $X_i(t_{k+1}|t_k)$ is a feasible solution to (3.11) at time t_{k+1}.

This implies that, given the optimal solution $X_i^*(t_{k+1})$ to the problem (3.11) at time t_{k+1} (which is proved to exist, provided that (3.11) is feasible at time t_k), for all $i = 1, \dots, M$ it holds that, by optimality

$$V_i^{N*}(x(t_{k+1})) = V_i^N(X_i^*(t_{k+1})) \le V_i^N(X_i(t_{k+1}|t_k)) \tag{3.37}$$

3.6.2 The collective problem

To prove the convergence to zero of the solution, we now define the collective problem, equivalent to the one considered in the previous sections. Define the vectors

$$\begin{aligned}
\hat{\mathbf{x}}(t) &= (\hat{x}^{[1]}(t), \dots, \hat{x}^{[M]}(t)), & \bar{\mathbf{x}}(t) &= (\bar{x}^{[1]}(t), \dots, \bar{x}^{[M]}(t))\\
\tilde{\mathbf{x}}(t) &= (\tilde{x}^{[1]}(t), \dots, \tilde{x}^{[M]}(t)), & \hat{\mathbf{u}}(t) &= (\hat{u}^{[1]}(t), \dots, \hat{u}^{[M]}(t))\\
\bar{\mathbf{u}}(t) &= (\bar{u}^{[1]}(t), \dots, \bar{u}^{[M]}(t)), & \tilde{\mathbf{u}}(t) &= (\tilde{u}^{[1]}(t), \dots, \tilde{u}^{[M]}(t))\\
\mathbf{w}(t) &= (w^{[1]}(t), \dots, w^{[M]}(t)), & \bar{\mathbf{w}}(t) &= (\bar{w}^{[1]}(t), \dots, \bar{w}^{[M]}(t))\\
\mathbf{z}(t) &= (z^{[1]}(t), \dots, z^{[M]}(t)), & \mathbf{s}(t) &= (s^{[1]}(t), \dots, s^{[M]}(t))
\end{aligned}$$

Then define the matrices $\mathbf{A}^* = \text{diag}(A_{11}, \dots, A_{MM})$, $\mathbf{B}^* = \text{diag}(B_{11}, \dots, B_{MM})$, $\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{A}^*$, $\tilde{\mathbf{B}} = \mathbf{B} - \mathbf{B}^*$. Collectively, we write equations (3.3), (3.5), and (3.8) as

$$\dot{\mathbf{x}}(t) = \mathbf{A}^*\mathbf{x}(t) + \mathbf{B}^*\mathbf{u}(t) + \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\tilde{\mathbf{u}}(t) + \mathbf{w}(t) \tag{3.38}$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}^*\hat{\mathbf{x}}(t) + \mathbf{B}^*\hat{\mathbf{u}}(t) + \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\tilde{\mathbf{u}}(t) \tag{3.39}$$

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{A}^*\bar{\mathbf{x}}(t) + \mathbf{B}^*\bar{\mathbf{u}}(t) \tag{3.40}$$

In view of (3.6) and (3.9), $\mathbf{u}(t) = \hat{\mathbf{u}}(t) + \mathbf{K}^c(\mathbf{x}(t) - \hat{\mathbf{x}}(t))$ and $\hat{\mathbf{u}}(t) = \bar{\mathbf{u}}(t) + \bar{\mathbf{K}}(\hat{\mathbf{x}}(t) - \bar{\mathbf{x}}(t))$. From this, and in view of (3.7) and (3.10),

$$\dot{\mathbf{z}}(t) = (\mathbf{A}^* + \mathbf{B}^*\mathbf{K}^c)\mathbf{z}(t) + \mathbf{w}(t) \tag{3.41}$$

$$\dot{\mathbf{s}}(t) = (\mathbf{A}^* + \mathbf{B}^*\bar{\mathbf{K}})\mathbf{s}(t) + \bar{\mathbf{w}}(t) \tag{3.42}$$

Minimizing (3.11) at time t_k for all $i = 1, \ldots, M$ is equivalent to solve the following collective minimization problem

$$\mathbf{V}^{N*}(\mathbf{x}(t_k)) = \min_{\mathbf{X}(t_k)} \mathbf{V}^N(\mathbf{X}(t_k)) \tag{3.43}$$

where $\mathbf{X}(t_k) = (X_1(t_k), \ldots, X_M(t_k))$, subject to the dynamic constraints (3.39), (3.40) and

$$\mathbf{x}(t_k) - \hat{\mathbf{x}}(t_k) \in \mathbb{Z} = \prod_{i=1}^{M} Z_i \tag{3.44a}$$

$$\hat{\mathbf{x}}(t) - \tilde{\mathbf{x}}(t) \in \mathbb{E} = \prod_{i=1}^{M} E_i \tag{3.44b}$$

$$\hat{\mathbf{u}}(t) - \tilde{\mathbf{u}}(t) \in \tilde{\mathbb{U}} = \prod_{i=1}^{M} U_i \tag{3.44c}$$

$$\hat{\mathbf{x}}(t) \in \hat{\mathbb{X}} = \prod_{i=1}^{M} \hat{\mathbb{X}}_i \tag{3.44d}$$

$$\hat{\mathbf{u}}(t) \in \hat{\mathbb{U}} = \prod_{i=1}^{M} \hat{\mathbb{U}}_i \tag{3.44e}$$

for all $t \in [t_k, t_{k+N-1})$, to

$$\hat{\mathbf{x}}(t_{k+N-1}) - \bar{\mathbf{x}}(t_{k+N-1}) \in \mathbb{S} = \prod_{i=1}^{M} S_i \tag{3.45}$$

$$\bar{\mathbf{x}}(t) \in \bar{\boldsymbol{\mathcal{E}}} = \prod_{i=1}^{M} \bar{\mathcal{E}}_i \tag{3.46}$$

$$\bar{\mathbf{x}}(t) \in \bar{\boldsymbol{\mathcal{U}}} = \prod_{i=1}^{M} \bar{\mathcal{U}}_i \tag{3.47}$$

and the terminal constraint

$$\bar{\mathbf{x}}(t_{k+N}) \in \bar{\mathbb{X}}^F = \prod_{i=1}^{M} \bar{\mathcal{X}}_i^F \tag{3.48}$$

The collective cost function $\mathbf{V}^N$ is

$$\mathbf{V}^N = \sum_{h=0}^{N-2} \hat{\mathbf{l}}(\hat{\mathbf{x}}(t_{k+h}), \hat{\mathbf{u}}(t_{k+h})) + \hat{\mathbf{V}}^F\left(\hat{\mathbf{x}}(t_{k+N-1})\right) + \bar{\mathbf{l}}(\bar{\mathbf{x}}(t_{k+N-1}), \bar{\mathbf{u}}(t_{k+N-1})) + \bar{\mathbf{V}}^F\left(\bar{\mathbf{x}}(t_{k+N})\right) \quad (3.49)$$

where, from (3.31):

$$\hat{\mathbf{l}}(\hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t)) = \frac{1}{2}\int_t^{t+T} (\|\hat{\mathbf{x}}(\eta)\|^2_{\hat{\mathbf{Q}}} + \|\hat{\mathbf{u}}(\eta)\|^2_{\hat{\mathbf{R}}}) d\eta \quad (3.50a)$$

$$\bar{\mathbf{l}}(\bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t)) = \frac{\lambda}{2}\int_t^{t+T} (\|\bar{\mathbf{x}}(\eta)\|^2_{\bar{\mathbf{Q}}} + \|\bar{\mathbf{u}}(\eta)\|^2_{\bar{\mathbf{R}}}) d\eta \quad (3.50b)$$

$$\hat{\mathbf{V}}^F(\hat{\mathbf{x}}(t)) = \frac{1}{2}\|\hat{\mathbf{x}}(t)\|^2_{\hat{\mathbf{P}}} \quad (3.50c)$$

$$\bar{\mathbf{V}}^F(\bar{\mathbf{x}}(t)) = \frac{\lambda}{2}\|\bar{\mathbf{x}}(t)\|^2_{\bar{\mathbf{P}}} \quad (3.50d)$$

and $\hat{\mathbf{Q}}$ =diag$(\hat{Q}_1, \ldots, \hat{Q}_M)$, $\hat{\mathbf{R}}$ =diag$(\hat{R}_1, \ldots, \hat{R}_M)$, $\hat{\mathbf{P}}$ =diag$(\hat{P}_1, \ldots, \hat{P}_M)$, $\bar{\mathbf{Q}}$ =diag$(\bar{Q}_1, \ldots, \bar{Q}_M)$, $\bar{\mathbf{R}}$ =diag$(\bar{R}_1, \ldots, \bar{R}_M)$, and $\bar{\mathbf{P}}$ =diag$(\bar{P}_1, \ldots, \bar{P}_M)$.

3.6.3 Proof of convergence

Denote with $\mathbf{X}(t_k|t_k) = (X_1(t_k|t_k), \ldots, X_M(t_k|t_k))$ the optimal solution to (3.43) at time t_k, and with $\mathbf{X}(t_{k+1}|t_k) = (X_1(t_{k+1}|t_k), \ldots, X_M(t_{k+1}|t_k))$ the feasible (non-optimal) solution to (3.43) at time t_{k+1}, where $X_i(t_{k+1}|t_k)$ is defined in (3.34), for all $i = 1, \ldots, M$. From (3.37) we have that

$$\begin{aligned} \mathbf{V}^{N*}(\mathbf{x}(t_{k+1})) - \mathbf{V}^{N*}(\mathbf{x}(t_k)) &\leq \mathbf{V}^N(\mathbf{X}(t_{k+1}|t_k)) - \mathbf{V}^N(\mathbf{X}(t_k|t_k)) \\ &\leq -\hat{\mathbf{l}}(\hat{\mathbf{x}}(t_k|t_k), \hat{\mathbf{u}}(t_k|t_k)) + (a) + (b) \end{aligned} \quad (3.51)$$

where

$$\begin{aligned} (a) &= \hat{\mathbf{l}}(\hat{\mathbf{x}}(t_{k+N-1}|t_k), \hat{\mathbf{u}}(t_{k+N-1}|t_k)) - \bar{\mathbf{l}}(\bar{\mathbf{x}}(t_{k+N-1}|t_k), \bar{\mathbf{u}}(t_{k+N-1}|t_k)) \\ &\quad + \hat{\mathbf{V}}^F(\hat{\mathbf{x}}(t_{k+N}|t_k)) - \hat{\mathbf{V}}^F(\hat{\mathbf{x}}(t_{k+N-1}|t_k)) \\ (b) &= \bar{\mathbf{l}}(\bar{\mathbf{x}}(t_{k+N}|t_k), \bar{\mathbf{u}}(t_{k+N}|t_k)) + \bar{\mathbf{V}}^F(\bar{\mathbf{x}}(t_{k+N+1}|t_k)) - \bar{\mathbf{V}}^F(\bar{\mathbf{x}}(t_{k+N}|t_k)) \end{aligned}$$

Consider first term (b). If matrices $\bar{P}_i$, $i = 1, \ldots, M$, are chosen as the solutions to the Lyapunov equations (3.28) then, from (3.32), for all $i = 1, \ldots, M$

$$\bar{V}_i^F(\bar{x}_i(t_{k+N+1}|t_k)) - \bar{V}_i^F(\bar{x}_i(t_{k+N}|t_k)) \leq -\bar{l}_i(\bar{x}_i(t_{k+N}|t_k), \bar{u}_i(t_{k+N}|t_k)) \quad (3.52)$$

which, collectively, implies that $(b) \leq 0$.

Considering now term (a), define the following collective quantities: $\mathbf{F}^* = \text{diag}(\bar{F}_{11}, \ldots, \bar{F}_{MM})$, $\mathbf{A}^{zoh}(\eta) = \text{diag}(A_{11}^{zoh}(\eta), \ldots, A_{MM}^{zoh}(\eta))$, $\mathbf{B}^{zoh}(\eta) = \text{diag}(B_{11}^{zoh}(\eta), \ldots, B_{MM}^{zoh}(\eta))$. Since $\bar{\mathbf{u}}(t|t_k)$ is constant for all $t \in [t_{k+N-1}, t_{k+N})$, and recalling (3.35), for $t \in [t_{k+N-1}, t_{k+N}]$ it results that

$$\bar{\mathbf{x}}(t|t_k) = \mathbf{A}^{zoh}(t - t_{k+N-1})\bar{\mathbf{x}}(t_{k+N-1}|t_k) + \mathbf{B}^{zoh}(t - t_{k+N-1})\bar{\mathbf{u}}(t_{k+N-1}|t_k) \tag{3.53a}$$

and, from (3.35)

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t|t_k) =& \bar{\mathbf{F}}^*\hat{\mathbf{x}}(t|t_k) + (\tilde{\mathbf{A}} - \mathbf{B}^*\bar{\mathbf{K}})\bar{\mathbf{x}}(t|t_k) + \mathbf{B}\bar{\mathbf{u}}(t|t_k)\\ =& \bar{\mathbf{F}}^*\hat{\mathbf{x}}(t|t_k) + (\tilde{\mathbf{A}} - \mathbf{B}^*\bar{\mathbf{K}})\mathbf{A}^{zoh}(t - t_{k+N-1})\bar{\mathbf{x}}(t_{k+N-1}|t_k) +\\ &+ \Big((\tilde{\mathbf{A}} - \mathbf{B}^*\bar{\mathbf{K}})\mathbf{B}^{zoh}(t - t_{k+N-1}) + \mathbf{B}\Big)\bar{\mathbf{u}}(t|t_k)\end{aligned} \tag{3.53b}$$

Therefore, solving (3.53b), we obtain

$$\begin{aligned}\hat{\mathbf{x}}(t|t_k) =& \bar{\mathbf{\Phi}}_x(t - t_{k+N-1})\hat{\mathbf{x}}(t_{k+N-1}|t_k) + \bar{\mathbf{\Gamma}}_{1x}(t - t_{k+N-1})\bar{\mathbf{x}}(t_{k+N-1}|t_k)\\ &+ \bar{\mathbf{\Gamma}}_{2x}(t - t_{k+N-1})\bar{\mathbf{u}}(t_{k+N-1}|t_k)\end{aligned} \tag{3.54}$$

$$\begin{aligned}\hat{\mathbf{u}}(t|t_k) =& \bar{\mathbf{\Phi}}_u(t - t_{k+N-1})\hat{\mathbf{x}}(t_{k+N-1}|t_k) + \bar{\mathbf{\Gamma}}_{1u}(t - t_{k+N-1})\bar{\mathbf{x}}(t_{k+N-1}|t_k)\\ &+ \bar{\mathbf{\Gamma}}_{2u}(t - t_{k+N-1})\bar{\mathbf{u}}(t_{k+N-1}|t_k)\end{aligned} \tag{3.55}$$

where $\bar{\mathbf{\Phi}}_x(\eta) = e^{\mathbf{F}^*\eta}$, $\bar{\mathbf{\Gamma}}_{1x}(\eta) = \int_0^\eta e^{\mathbf{F}^*(\eta-\nu)}(\tilde{\mathbf{A}} - \mathbf{B}^*\bar{\mathbf{K}})\mathbf{A}^{zoh}(\nu)d\nu$, $\bar{\mathbf{\Gamma}}_{2x}(\eta) = \int_0^\eta e^{\mathbf{F}^*(\eta-\nu)}((\tilde{\mathbf{A}} - \mathbf{B}^*\bar{\mathbf{K}})\mathbf{B}^{zoh}(\nu) + \mathbf{B})d\nu$, $\bar{\mathbf{\Phi}}_u(\eta) = \bar{\mathbf{K}}\bar{\mathbf{\Phi}}_x(\eta)$, $\bar{\mathbf{\Gamma}}_{1u}(\eta) = \bar{\mathbf{K}}(\bar{\mathbf{\Gamma}}_{1x}(\eta) - \mathbf{A}^{zoh}(\eta))$, $\bar{\mathbf{\Gamma}}_{2u}(\eta) = \mathbf{I} + \bar{\mathbf{K}}(\bar{\mathbf{\Gamma}}_{2x}(\eta) - \mathbf{B}^{zoh}(\eta))$.

Denote, for brevity, $\hat{\mathbf{x}}_{k+N-1} = \hat{\mathbf{x}}(t_{k+N-1}|t_k)$, $\mathbf{v}_{k+N-1} = (\bar{\mathbf{x}}(t_{k+N-1}|t_k), \bar{\mathbf{u}}(t_{k+N-1}|t_k))$, and

$$\begin{aligned}\bar{\mathbf{\Gamma}}_x(\eta) &= \begin{bmatrix}\bar{\mathbf{\Gamma}}_{1x}(\eta) & \bar{\mathbf{\Gamma}}_{2x}(\eta)\end{bmatrix}\\ \bar{\mathbf{\Gamma}}_u(\eta) &= \begin{bmatrix}\bar{\mathbf{\Gamma}}_{1u}(\eta) & \bar{\mathbf{\Gamma}}_{2u}(\eta)\end{bmatrix}\\ \bar{\mathbf{A}}_x(\eta) &= \begin{bmatrix}\mathbf{A}^{zoh}(\eta) & \mathbf{B}^{zoh}(\eta)\end{bmatrix}\\ \bar{\mathbf{A}}_u(\eta) &= \begin{bmatrix}\mathbf{0} & \mathbf{I}\end{bmatrix}\end{aligned}$$

Then, in view of (3.55) we compute the elements of term (a) as

$$\begin{aligned}\hat{\mathbf{l}}(\hat{\mathbf{x}}(t_{k+N-1}|t_k), \hat{\mathbf{u}}(t_{k+N-1}|t_k)) &= \frac{1}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2_{\int_0^T \bar{\mathbf{\Phi}}_x(\eta)^T\hat{\mathbf{Q}}\bar{\mathbf{\Phi}}_x(\eta)+\bar{\mathbf{\Phi}}_u(\eta)^T\hat{\mathbf{R}}\bar{\mathbf{\Phi}}_u(\eta)d\eta} \\ &+ \frac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\int_0^T \bar{\mathbf{\Gamma}}_x(\eta)^T\hat{\mathbf{Q}}\bar{\mathbf{\Gamma}}_x(\eta)+\bar{\mathbf{\Gamma}}_u(\eta)^T\hat{\mathbf{R}}\bar{\mathbf{\Gamma}}_u(\eta)d\eta} \\ &+ \hat{\mathbf{x}}_{k+N-1}^T(\int_0^T \bar{\mathbf{\Phi}}_x(\eta)^T\hat{\mathbf{Q}}\bar{\mathbf{\Gamma}}_x(\eta) \\ &+ \bar{\mathbf{\Phi}}_u(\eta)^T\hat{\mathbf{R}}\bar{\mathbf{\Gamma}}_u(\eta)d\eta)\mathbf{v}_{k+N-1} \qquad (3.56a)\\ \hat{\mathbf{V}}^F(\hat{\mathbf{x}}(t_{k+N}|t_k)) &= \frac{1}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2_{\bar{\mathbf{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\mathbf{\Phi}}_x(T)} \\ +\frac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\bar{\mathbf{\Gamma}}_x(T)^T\hat{\mathbf{P}}\bar{\mathbf{\Gamma}}_x(T)} \\ &+ \hat{\mathbf{x}}_{k+N-1}^T\bar{\mathbf{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\mathbf{\Gamma}}_x(T)\mathbf{v}_{k+N-1} \qquad (3.56b)\\ \bar{\mathbf{l}}(\bar{\mathbf{x}}(t_{k+N-1}|t_k), \bar{\mathbf{u}}(t_{k+N-1}|t_k)) &= \frac{\lambda}{2}\|\mathbf{v}_{k+N-1}\|^2_{\int_0^T \bar{\mathbf{A}}_x(\eta)^T\bar{\mathbf{Q}}\bar{\mathbf{A}}_x(\eta))+\bar{\mathbf{A}}_u(\eta)^T\bar{\mathbf{R}}\bar{\mathbf{A}}_u(\eta)d\eta} \qquad (3.56c)\end{aligned}$$

Define, for simplicity:

$$\mathcal{Q}_v = \int_0^T \left(\bar{\mathbf{\Gamma}}_x(\eta)^T\hat{\mathbf{Q}}\bar{\mathbf{\Gamma}}_x(\eta) + \bar{\mathbf{\Gamma}}_u(\eta)^T\hat{\mathbf{R}}\bar{\mathbf{\Gamma}}_u(\eta)\right) d\eta \qquad (3.57a)$$

$$\mathcal{S}_{xv} = \int_0^T \left(\bar{\mathbf{\Phi}}_x(\eta)^T\hat{\mathbf{Q}}\bar{\mathbf{\Gamma}}_x(\eta) + \bar{\mathbf{\Phi}}_u(\eta)^T\hat{\mathbf{R}}\bar{\mathbf{\Gamma}}_u(\eta)\right) d\eta \qquad (3.57b)$$

$$\mathcal{R}_v = \int_0^T \left(\bar{\mathbf{A}}_x(\eta)^T\bar{\mathbf{Q}}\bar{\mathbf{A}}_x(\eta)) + \bar{\mathbf{A}}_u(\eta)^T\bar{\mathbf{R}}\bar{\mathbf{A}}_u(\eta)\right) d\eta \qquad (3.57c)$$

Therefore

$$\begin{aligned}(a) \quad = \quad & \tfrac{1}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2_{\bar{\mathbf{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\mathbf{\Phi}}_x(T)-\hat{\mathbf{P}}+\mathcal{Q}_x} + \tfrac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\bar{\mathbf{\Gamma}}_x(T)^T\hat{\mathbf{P}}\bar{\mathbf{\Gamma}}_x(T)+\mathcal{Q}_v-\lambda\mathcal{R}_v} \\ & +\hat{\mathbf{x}}_{k+N-1}^T(\bar{\mathbf{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\mathbf{\Gamma}}_x(T) + \mathcal{S}_{xv})\mathbf{v}_{k+N-1}\end{aligned} \qquad (3.58)$$

Recall that $\hat{\mathbf{P}}$ is the block-diagonal matrix whose blocks $\hat{P}_i$ satisfy (3.33) for all $i = 1, \ldots, M$, i.e., such that $\hat{\mathbf{P}}$ satisfies

$$\bar{\mathbf{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\mathbf{\Phi}}_x(T) - \hat{\mathbf{P}} + \mathcal{Q}_x + \alpha\mathbf{I} = 0 \qquad (3.59)$$

where $\alpha > 1$ is an arbitrary scalar. The following procedure is proposed for defining a suitable scalar λ and matrix $\hat{\mathbf{P}}$.

1. Define $\mathcal{S}_{Pxv} = \bar{\mathbf{\Phi}}_x(T)^T \hat{\mathbf{P}} \bar{\mathbf{\Gamma}}_x(T) + \mathcal{S}_{xv}$ and an arbitrary scalar $\beta > 0$ such that

$$\beta \mathbf{I} \geq \mathcal{S}_{Pxv}^T \mathcal{S}_{Pxv} \quad (3.60)$$

   or equivalently

$$\beta \geq \|\mathcal{S}_{Pxv}\|_2^2 \quad (3.61)$$

2. Define $\bar{\lambda}$ as the smallest value of $\lambda > 0$ satisfying

$$\lambda \mathcal{R}_v - \bar{\mathbf{\Gamma}}_x(T)^T \hat{\mathbf{P}} \bar{\mathbf{\Gamma}}_x(T) - \mathcal{Q}_v \geq \beta \mathbf{I} \quad (3.62)$$

   Note that, given $\hat{\mathbf{P}}$ and $\beta > 0$, since $\mathcal{R}_v > 0$, it is always possible to define $\bar{\lambda} > 0$. finally, set $\lambda > \bar{\lambda}$.

According to the sketched procedure and in view of (3.59) and (3.62), from (3.58) we can write

$$(a) \quad \leq \quad -\tfrac{\alpha}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2 - \tfrac{\beta}{2}\|\mathbf{v}_{k+N-1}\|^2 + \hat{\mathbf{x}}_{k+N-1}^T \mathcal{S}_{Pxv} \mathbf{v}_{k+N-1} \quad (3.63)$$

Since

$$0 \leq \tfrac{1}{2}\|\hat{\mathbf{x}}_{k+N-1} - \mathcal{S}_{Pxv}\mathbf{v}_{k+N-1}\|^2 = \tfrac{1}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2 + \\ + \tfrac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\mathcal{S}_{Pxv}^T \mathcal{S}_{Pxv}} - \hat{\mathbf{x}}_{k+N-1}^T \mathcal{S}_{Pxv} \mathbf{v}_{k+N-1}$$

it follows that

$$\hat{\mathbf{x}}_{k+N-1}^T \mathcal{S}_{Pxv} \mathbf{v}_{k+N-1} \leq \tfrac{1}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2 + \tfrac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\mathcal{S}_{Pxv}^T \mathcal{S}_{Pxv}}$$

and, from (3.63)

$$(a) \quad \leq \quad \tfrac{1-\alpha}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2 + \tfrac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\mathcal{S}_{Pxv}^T \mathcal{S}_{Pxv} - \beta \mathbf{I}} \quad (3.64)$$

Therefore, since β satisfies (3.60) and $\alpha > 1$, then $(a) \leq 0$.

From (3.51), and having proved that both $(a) \leq 0$ and $(b) \leq 0$, we obtain that

$$\mathbf{V}^{N*}(\mathbf{x}(t_{k+1})) - \mathbf{V}^{N*}(\mathbf{x}(t_k)) \leq -\hat{\mathbf{l}}(\hat{\mathbf{x}}(t_k|t_k), \hat{\mathbf{u}}(t_k|t_k)) \quad (3.65)$$

Therefore $\hat{\mathbf{l}}(\hat{\mathbf{x}}(t_k|t_k), \hat{\mathbf{u}}(t_k|t_k)) \to 0$ as $k \to \infty$. Under suitable smoothness assumptions on $\hat{\mathbf{u}}(t|t_k)$ and $\hat{\mathbf{x}}(t|t_k)$ and since $\hat{\mathbf{Q}} > 0$ and $\hat{\mathbf{R}} > 0$, it follows that $\hat{\mathbf{x}}([t_k, t_{k+1})|t_k) \to 0$ and $\hat{\mathbf{u}}([t_k, t_{k+1})|t_k) \to 0$ as $k \to \infty$.

Recalling now system (3.1) where, for all $k \in \mathbb{N}$, $t \in [t_k, t_{k+1})$, $\mathbf{u}(t) = \hat{\mathbf{u}}(t|t_k) + \mathbf{K}^c(\mathbf{x}(t|t_k) - \hat{\mathbf{x}}(t))$. We can write

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{B}\mathbf{K}^c)\mathbf{x}(t) + \mathbf{B}(\hat{\mathbf{u}}(t|t_k) - \mathbf{K}^c \mathbf{x}(t|t_k))$$

Since $\mathbf{B}(\hat{\mathbf{u}}(t|t_k) - \mathbf{K}^c \mathbf{x}(t|t_k))$ is an asymptotically vanishing term, and since $\mathbf{A} + \mathbf{B}\mathbf{K}^c$ is Hurwitz in view of Assumption 3.2, we obtain that $\mathbf{x}(t) \to 0$ as $t \to \infty$.

Part III

Solutions to the tracking problem

Introduction to the tracking problem

Most of the contributions in the field of model predictive control are referred to the so-called regulation problem, i.e. asymptotically steering the state of the system to zero [105, 135, 136]. On the contrary, minor attention (see, e.g., [53, 88]) has been placed in the design of control schemes for the asymptotic tracking of constant reference output signals, that represent an important issue from the industrial point of view.

In the development of efficient industrial MPC algorithms, two main issues must be considered: *i)* the offset-free problem and *ii)* the unfeasible reference problem. The offset-free problem requires to develop methods which can guarantee asymptotic zero error regulation for piecewise constant and feasible reference signals, while the unfeasible reference problem aims at finding suitable solutions when the nominal constant reference signal cannot be reached due to the presence of state and/or control constraints.

Regarding the offset-free problem, many solutions have been proposed so far. The most popular one consists of augmenting the model of the plant under control with an artificial disturbance, which must be estimated together with the system state. This disturbance can account for possible model mismatch or for the presence of real unknown exogenous signals. Depending on its assumed dynamics, many algorithms have been developed, such as those described in [96, 111, 113, 118, 120, 123]. Another approach directly relies on the Internal Model Principle [39], where an internal model of the reference, i.e. an integrator, is directly included in the control scheme and fed by the output error. Then, the MPC algorithm is designed to stabilize the ensemble of the plant and the integrator. This strategy has been followed in [99, 104] where also more general exogenous signals and nonlinear, non square systems have been considered. A third solution to the offset-free problem consists of describing the system in the so-called "velocity-form", see [122, 168], where the enlarged state is composed by the state increments and the output error, while the manipulated variable is the control increment.

The unfeasible reference problem has been discussed in e.g. [29, 119, 135]. More recently, a bright solution has been proposed in [6, 87–89], see Chapter 1. In these papers, the MPC cost function is complemented by a term explicitly penalizing the distance between the required (possibly unfeasible) reference signal and an artificial, but feasible, reference, which turns out to be one of the optimization variables. Stability and convergence results are proven both in nominal conditions and for perturbed systems.

As for the contributions in the field of distributed control, only few papers on distributed MPC for tracking have been published (see, e.g., [53], where a cooperative distributed MPC scheme is proposed). Indeed, in a distributed setting, the standard approach, based on the reformulation of the tracking problem as a regulation one by computing at any set-point change of the output the corresponding state and control target values, cannot be followed due to the decentralization constraint.

Often, in industrial applications, hierarchical structures are used [149], e.g. including *i)* a Real Time Optimization (RTO) for computing the optimal operating conditions and *ii)* a centralized regulator with Model Predictive Control (MPC) for tracking purposes.

Two-layer structures, although very efficient in many practical applications, pose great difficulties when a distributed control is used at the lower layer. In fact, the references computed at the higher layer can ignore the presence of dynamic constraints among the subsystems and, as such, can lead to infeasible local optimization problems. This prevents one from directly applying the many decentralized and distributed MPC algorithms recently developed (see the examples reported in Chapter 1) for the regulation problem. In addition, in a distributed setting, the decentralization constraints do not allow to follow the standard approach, based on the reformulation of the tracking problem as a regulation one by computing at any set-point change of the output the corresponding state and control target values. An alternative approach is described in [23,162], where a distributed sequential reference-governor approach is proposed.

4

DPC for tracking

In this Chapter, a distributed MPC method based on Distributed Predictive Control (DPC) for the solution of the tracking problem is discussed. It is based on a hierarchical structure, depicted in Figure 4.1, that consists of three layers: *I)* a standard RTO optimization layer; *II)* an intermediate layer which transforms, for each local subsystem, the references $y^{[i]}_{set-point}$ computed with RTO into feasible trajectories $\tilde{x}^{[i]}$, $\tilde{u}^{[i]}$ and $\tilde{y}^{[i]}$ for the state, input and output variables, respectively; III) in the lower layer local MPC regulators $\mathbf{C}_i$ can communicate according to a prescribed information pattern and are designed with the state-feedback DPC algorithm originally developed for the solution of the regulation problem.

Notably, in the intermediate layer the reference trajectories are computed according to the same information pattern adopted at the lower layer, and the overall scheme guarantees that state and control constraints are fulfilled. Moreover, the controlled outputs reach the prescribed reference values computed with RTO whenever possible, or their nearest feasible value when feasibility problems arise due to the constraints.

4.1 Interacting subsystems

Consider the collective dynamical model

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \tag{4.1a}$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k \tag{4.1b}$$

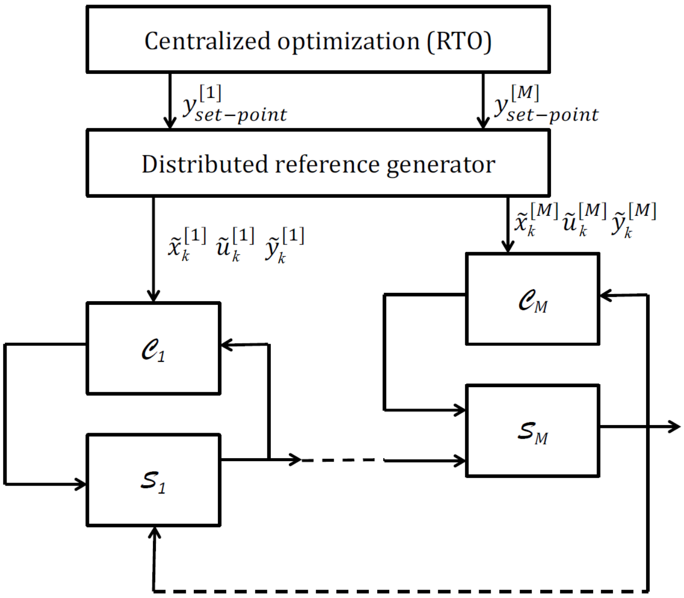

Figure 4.1: Overall control architecture for distributed tracking.

in which $\mathbf{x}_k \in \mathbb{R}^n$ is the collective state vector, $\mathbf{u}_k \in \mathbb{R}^m$ is the collective input vector and $\mathbf{y}_k \in \mathbb{R}^m$ is the collective output vector.

The dynamic system (4.1) can be decomposed in a set of M dynamically interacting non-overlapping subsystems which, according to the notation used in [94], are described by

$$x^{[i]}_{k+1} = A_{ii}x^{[i]}_k + B_{ii}u^{[i]}_k + E_i s^{[i]}_k \tag{4.2a}$$
$$y^{[i]}_k = C_i x^{[i]}_k \tag{4.2b}$$
$$z^{[i]}_k = C_{zi}x^{[i]}_k + D_{zi}u^{[i]}_k \tag{4.2c}$$

where $x^{[i]}_k \in \mathbb{R}^{n_i}$ and $u^{[i]}_k \in \mathbb{R}^{m_i}$ are the state and input vectors, respectively, of the i-th subsystem, while $y^{[i]}_k \in \mathbb{R}^{m_i}$ is its output vector. According to the non-overlapping decomposition, it holds that $\mathbf{x}_k = (x^{[1]}_k, \dots, x^{[M]}_k)$, $n = \sum_{i=1}^M n_i$, that $\mathbf{u}_k = (u^{[1]}_k, \dots, u^{[M]}_k)$, $m = \sum_{i=1}^M m_i$, and that $\mathbf{y}_k = (y^{[1]}_k, \dots, y^{[M]}_k)$. In line with the interaction-oriented models introduced in [94], the coupling input and output vectors $s^{[i]}_k$ and $z^{[i]}_k$, respectively, are defined to characterize the interconnections among the subsystems; in a collective form, they are defined as $\mathbf{s}_k = (s^{[1]}_k, \dots, s^{[M]}_k)$, $\mathbf{z}_k = (z^{[1]}_k, \dots, z^{[M]}_k)$, and the interconnections among subsystems are described by means of the algebraic equation

$$\mathbf{s}_k = \mathbf{L}\mathbf{z}_k \tag{4.3}$$

where $\mathbf{L}$ is called interconnection matrix. More specifically, the coupling input $s_k^{[i]}$ to subsystem i depends on the coupling output $z_k^{[j]}$ of the j-th subsystem according to

$$s_k^{[i]} = \sum_{j=1}^{M} L_{ij} z_k^{[j]} \tag{4.4}$$

We say that subsystem j is a dynamic neighbor of subsystem i if and only if $L_{ij} \neq 0$, and we denote as $\mathcal{N}_i$ the set of dynamic neighbors of subsystem i (which excludes i).

The input and state variables are subject to the "local" constraints $u_k^{[i]} \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ and $x_k^{[i]} \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$, respectively, where the sets $\mathbb{U}_i$ and $\mathbb{X}_i$ are convex.

The state transition matrices $A_{11} \in \mathbb{R}^{n_1 \times n_1}, \dots, A_{MM} \in \mathbb{R}^{n_M \times n_M}$ of the M subsystems are the diagonal blocks of $\mathbf{A}$, whereas the dynamic coupling terms between subsystems correspond to the non-diagonal blocks of $\mathbf{A}$, i.e., $A_{ij} = E_i L_{ij} C_{zj}$, with $j \neq i$. Correspondingly, B_{ii}, $i = 1, \dots, M$, are the diagonal blocks of $\mathbf{B}$, whereas the influence of the input of a subsystem upon the state of different subsystems is represented by the off-diagonal terms of $\mathbf{B}$, i.e., $B_{ij} = E_i L_{ij} D_{zj}$, with $j \neq i$. The collective output matrix is defined as $\mathbf{C} = \text{diag}(C_{11}, \dots, C_{MM})$.

Concerning system (4.1a) and its partition, the following main assumption on decentralized stabilizability is introduced:

Assumption 4.1 *There exists a block-diagonal matrix* $\mathbf{K}$*, i.e.* $\mathbf{K} = diag(K_1, \dots, K_M)$*, with* $K_i \in R^{m_i \times n_i}$*,* $i = 1, \dots, M$ *such that:*

i) $\mathbf{F} = \mathbf{A} + \mathbf{B}\mathbf{K}$ *is Schur.*

ii) $F_{ii} = (A_{ii} + B_{ii} K_i)$ *is Schur,* $i = 1, \dots, M$*.*

□

We recall that the design of the stabilizing matrix $\mathbf{K}$ can be performed according to the procedure proposed in Chapter 1.

Moreover, in order to solve the tracking problem for constant reference signals, the following standard assumption is made.

Assumption 4.2 *Defining*

$$\mathbf{S} = \begin{bmatrix} I_n - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix}$$

then rank$(\mathbf{S}) = n + m$. □

4.2 Control system architecture

We want to design a distributed state-feedback control law, based on MPC, for the tracking of a given constant set-point signal $y^{[i]}_{set-point} \in \mathbb{R}^{m_i}$, where $y^{[i]}_{set-point}$ can be obtained, e.g., by means of any RTO method, see again Figure 4.1. More specifically, our aim is to asymptotically steer the system output $y^{[i]}_t$ to a constant desired value $y^{[i]}_{set-point}$ for all $i = 1, \ldots, M$. The main idea behind the proposed algorithm is to consider a three-layer control architecture, see Figure 4.2.

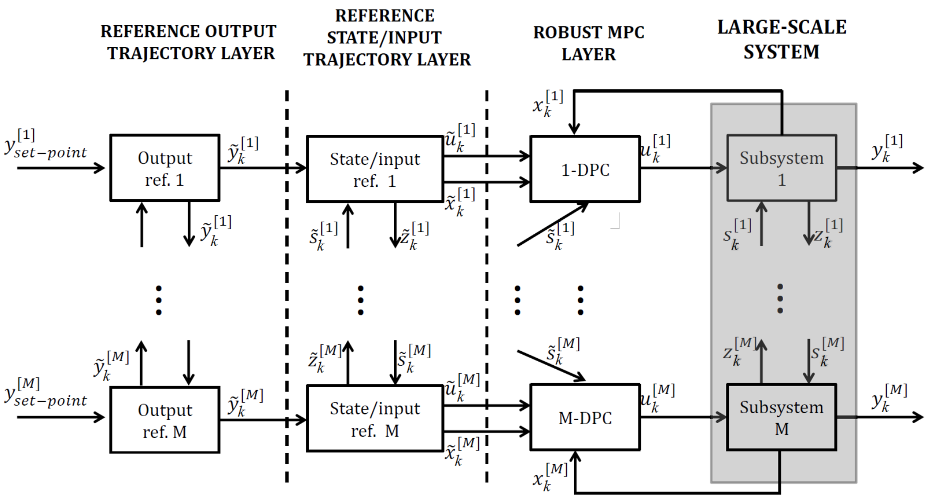

Figure 4.2: Control system architecture for distributed tracking.

1) The reference output trajectory management layer. For each subsystem $i = 1, \ldots, M$, a local reference trajectory management unit is required, which defines the reference trajectory $\tilde{y}^{[i]}_{k+\nu}$ of the output $y^{[i]}_{k+\nu}$. Although it would be natural to take $\tilde{y}^{[i]}_{k+\nu} = y^{[i]}_{set-point}$ for all $\nu \geq 0$, this choice could easily lead to infeasible standard MPC optimization problems, even in the centralized framework. Furthermore, in the distributed context this point is particularly critical, since too rapid changes in the output reference trajectory for a given subsystem could greatly affect the performance and the behavior of the other subsystems. Therefore $\tilde{y}^{[i]}$ will be regarded as an argument of an optimization problem rather than a fixed parameter, and constraints limiting the time variation of the local reference signals will be defined and computed. This layer is completely decentralized, i.e., the transmission of information between local reference trajectory management units is not needed.

2) The reference state and input trajectory layer. For each subsystem $i = 1, \dots, M$ assume that at any time instant k the future reference trajectories $\tilde{y}^{[i]}_{k+\nu}$, $\nu = 0, \dots, N-1$, are available for all $i = 1, \dots, M$. In order to define the reference trajectories $\tilde{x}^{[i]}_k$, $\tilde{u}^{[i]}_k$, and $\tilde{z}^{[i]}_k$ of the corresponding state, input, and coupling output variables, we design a suitable algorithm, using the output reference information as data.

3) The robust MPC layer. For each subsystem $i = 1, \dots, M$, a robust MPC unit is designed to drive the real state and input trajectories $x^{[i]}_k$ and $u^{[i]}_k$ as close as possible to the reference ones $\tilde{x}^{[i]}_k$, $\tilde{u}^{[i]}_k$, while respecting the constraints on the same variables. As in the case of the reference state and input trajectory layer, information is required to be transmitted from reference trajectory management units of neighboring regulators, in a neighbor-to-neighbor fashion.

4.2.1 The reference output trajectory management layer

For each subsystem $i = 1, \dots, M$, a local reference trajectory management unit defines, at any time k, the reference trajectory $\tilde{y}^{[i]}_{k+\nu}$ of the output $y^{[i]}_{k+\nu}$. Similarly to the approach taken in [88], the values $\tilde{y}^{[i]}$ will be regarded as an argument of an optimization problem itself, rather than a fixed parameter.

In the distributed context, too rapid changes of the output reference trajectory of a given subsystem could greatly affect the performance and the behavior of the other subsystems. Therefore, the main requirement for guaranteeing good performance and constraint satisfaction of our control scheme is to limit the rate of variation in time of the output reference signals. Therefore we will require that, for all $i = 1, \dots, M$, for all $k \geq 0$

$$\tilde{y}^{[i]}_{k+1} \in \tilde{y}^{[i]}_k \oplus \mathcal{B}^{(m_i)}_{p,\varepsilon}(0) \tag{4.5}$$

This layer is completely distributed, i.e., the reference trajectories $\tilde{y}_k^{[i]}$ of any subsystem i are computed only on the basis of local information and of the information provided by the reference trajectory management units of its constraint neighbors.

4.2.2 The reference state and input trajectory layer

Two different methods can be used to obtain the signals $(\tilde{x}_k^{[i]}, \tilde{u}_k^{[i]})$ based on the desired output trajectory $\tilde{y}_k^{[i]}$. The first one relies on the use of an integrator to expand the system (4.2a), while the second one makes use of an observer. In this Section, we provide detailed descriptions of both. As they are two alternative options, we will use the same notation for the two cases in order to improve the readability of the remainder of the Chapter. Anyway, it is important to recall that, depending on the technique chosen for this layer, the same symbols can take different meanings.

Computing the reference state and input trajectories using an integrator

For each subsystem $i = 1, \dots, M$ assume that at any time instant k the future reference trajectories $\tilde{y}_{k+\nu}^{[i]}$, $\nu = 0, \dots, N-1$, are available. In order to define the reference trajectories $(\tilde{x}_k^{[i]}, \tilde{u}_k^{[i]})$ based on the desired output trajectory $\tilde{y}_k^{[i]}$, we expand the system (4.2a), referred to the reference trajectories, with an integrator, i.e.,

$$\tilde{x}_{k+1}^{[i]} = A_{ii}\tilde{x}_k^{[i]} + B_{ii}\tilde{u}_k^{[i]} + E_i\tilde{s}_k^{[i]} \tag{4.6a}$$

$$\tilde{e}_{k+1}^{[i]} = \tilde{e}_k^{[i]} + \tilde{y}_{k+1}^{[i]} - C_i\tilde{x}_k^{[i]} \tag{4.6b}$$

where, similarly to (4.2c) and (4.4)

$$\tilde{z}_k^{[i]} = C_{zi}\tilde{x}_k^{[i]} + D_{zi}\tilde{u}_k^{[i]} \tag{4.6c}$$

$$\tilde{s}_k^{[i]} = \sum_{j\in\mathcal{N}_i} L_{ij}\tilde{z}_k^{[j]} \tag{4.6d}$$

Define $\chi_k^{[i]} = (\tilde{x}_k^{[i]}, \tilde{e}_k^{[i]})$,

$$\mathcal{A}_{ij} = \begin{cases} \begin{bmatrix} A_{ii} & 0 \\ -C_i & I_{m_i} \end{bmatrix} & \text{if } j = i \\ \\ \begin{bmatrix} A_{ij} & 0 \\ 0 & 0 \end{bmatrix} & \text{if } j \neq i \end{cases}, \mathcal{B}_{ij} = \begin{bmatrix} B_{ij} \\ 0 \end{bmatrix}, \mathcal{G}_i = \begin{bmatrix} 0 \\ I_{m_i} \end{bmatrix} \tag{4.6e}$$

and consider the control law

$$\tilde{u}_k^{[i]} = \mathcal{K}_i \chi_k^{[i]} \tag{4.6f}$$

where $\mathcal{K}_i = \begin{bmatrix} K_i^x & K_i^e \end{bmatrix}$. Letting $\mathcal{F}_{ij} = \mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_j$, the dynamics of the variable $\chi_k^{[i]}$ is therefore defined by the following dynamical system

$$\chi_{k+1}^{[i]} = \mathcal{F}_{ii}\chi_k^{[i]} + \sum_{j\in\mathcal{N}_i} \mathcal{F}_{ij}\chi_k^{[j]} + \mathcal{G}_i \tilde{y}_{k+1}^{[i]} \tag{4.7}$$

The gain matrix $\mathcal{K}_i$ is to be determined as follows: denoting by $\mathcal{A}$ and $\mathcal{B}$ the matrices whose block elements are $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$, respectively, and $\mathcal{K} = \mathrm{diag}(\mathcal{K}_1, \ldots, \mathcal{K}_M)$, the following assumption must be fulfilled

Assumption 4.3 *The matrix $\mathcal{F} = \mathcal{A} + \mathcal{B}\mathcal{K}$ is Schur.* □

Note that the synthesis of the $\mathcal{K}_i$'s can be performed according to the procedures proposed in Chapter 1.

Define, for all $i = 1, \ldots, M$ and for all $k \geq 0$, $\chi_k^{[i]ss} = (x_k^{[i]ss}, e_k^{[i]ss})$, i.e., the steady-state condition for (4.7) corresponding to the reference outputs $\tilde{y}_k^{[i]}$ assumed constant, i.e., $\tilde{y}_{k+1}^{[i]} = \tilde{y}_k^{[i]}$, and satisfying for all $i = 1, \ldots, M$

$$\chi_k^{[i]ss} = \mathcal{F}_{ii}\chi_k^{[i]ss} + \sum_{j\in\mathcal{N}_i} \mathcal{F}_{ij}\chi_k^{[j]ss} + \mathcal{G}_i \tilde{y}_k^{[i]} \tag{4.8}$$

In view of (4.6), $C_i x_k^{[i]ss} = \tilde{y}_{k+1}^{[i]}$ and $\mathcal{F}$ is Schur stable (see Assumption 4.3). Then a solution to the system (4.8) exists and is unique. Collectively define $\boldsymbol{\chi}_k^{ss} = (\chi_k^{[1]ss}, \ldots, \chi_k^{[M]ss})$, $\boldsymbol{\chi}_k = (\chi_k^{[1]}, \ldots, \chi_k^{[M]})$, and $\tilde{\mathbf{y}}_k = (\tilde{y}_k^{[1]}, \ldots, \tilde{y}_k^{[M]})$. From (4.6e)-(4.8) we can collectively write

$$\boldsymbol{\chi}_{k+1}^{ss} - \boldsymbol{\chi}_k^{ss} = (I_{n+p} - \mathcal{F})^{-1}\mathcal{G}(\tilde{\mathbf{y}}_{k+1} - \tilde{\mathbf{y}}_k) \tag{4.9}$$

where $\mathcal{G} = \mathrm{diag}(\mathcal{G}_1, \ldots, \mathcal{G}_M)$. Therefore

$$\begin{aligned} \boldsymbol{\chi}_{k+1} - \boldsymbol{\chi}_{k+1}^{ss} &= \mathcal{F}(\boldsymbol{\chi}_k - \boldsymbol{\chi}_k^{ss}) + \mathcal{G}(\tilde{\mathbf{y}}_{k+1} - \tilde{\mathbf{y}}_k) + \boldsymbol{\chi}_k^{ss} - \boldsymbol{\chi}_{k+1}^{ss} \\ &= \mathcal{F}(\boldsymbol{\chi}_k - \boldsymbol{\chi}_k^{ss}) + (I_{n+p} - (I_{n+p} - \mathcal{F})^{-1})\mathcal{G}(\tilde{\mathbf{y}}_{k+1} - \tilde{\mathbf{y}}_k) \\ &= \mathcal{F}(\boldsymbol{\chi}_k - \boldsymbol{\chi}_k^{ss}) - (I_{n+p} - \mathcal{F})^{-1}\mathcal{F}\mathcal{G}(\tilde{\mathbf{y}}_{k+1} - \tilde{\mathbf{y}}_k) \end{aligned} \tag{4.10}$$

We can rewrite (4.10) as

$$\boldsymbol{\chi}_{k+1} - \boldsymbol{\chi}_{k+1}^{ss} = \mathcal{F}(\boldsymbol{\chi}_k - \boldsymbol{\chi}_k^{ss}) + \tilde{\mathbf{w}}_k \tag{4.11}$$

where $\tilde{\mathbf{w}}_k$ can be seen as a bounded disturbance. In fact, in view of (4.5)

$$\tilde{\mathbf{y}}_{k+1} - \tilde{\mathbf{y}}_k \in \prod_{i=1}^{M} \mathcal{B}_{p,\varepsilon}^{(m_i)}(0) \tag{4.12}$$

and therefore $\tilde{\mathbf{w}}_k \in \tilde{\mathbb{W}} = -(I_{n+p} - \mathcal{F})^{-1}\mathcal{F}\mathcal{G}\prod_{i=1}^{M} \mathcal{B}_{p,\varepsilon}^{(m_i)}(0)$.

Under Assumption 4.3, for the system (4.11) there exists a possibly non-rectangular Robust Positive Invariant (RPI) set $\mathbf{\Delta}^\chi$ such that, if $\boldsymbol{\chi}_k - \boldsymbol{\chi}_k^{ss} \in \mathbf{\Delta}^\chi$, then it is guaranteed that $\boldsymbol{\chi}_{k+\nu} - \boldsymbol{\chi}_{k+\nu}^{ss} \in \mathbf{\Delta}^\chi$ for all $\nu \geq 0$. This, in turn, implies that there exist sets Δ_i^χ, $i = 1, \ldots, M$, defined in such a way that $\mathbf{\Delta}^\chi \subseteq \prod_{i=1}^{M} \Delta_i^\chi$, such that it is guaranteed that, for any initial condition $\boldsymbol{\chi}_0 - \boldsymbol{\chi}_0^{ss} \in \mathbf{\Delta}^\chi$, then

$$\chi_k^{[i]} - \chi_k^{[i]ss} \in \Delta_i^\chi \tag{4.13}$$

for all $k \geq 0$.

Computing the reference state and input trajectories using an observer

Denote

$$\mathcal{A}_{ij} = \begin{cases} \begin{bmatrix} A_{ii} & B_{ii} \\ 0 & I_{m_i} \end{bmatrix} & \text{if } j = i \\ \\ \begin{bmatrix} A_{ij} & B_{ij} \\ 0 & 0 \end{bmatrix} & \text{if } j \neq i \end{cases}, \mathcal{C}_i = \begin{bmatrix} C_{ii} & 0 \end{bmatrix} \tag{4.14}$$

Define, for all $i = 1, \ldots, M$ and for all $k \geq 0$, $\chi_k^{[i]ss} = (x_k^{[i]ss}, u_k^{[i]ss})$ where $x_k^{[i]ss}$ and $u_k^{[i]ss}$ are the steady-state state and input values corresponding to the reference outputs $\tilde{y}_k^{[i]}$ and satisfy the following steady-state equations

$$\begin{array}{rcl} \chi_k^{[i]ss} & = & \mathcal{A}_{ii}\chi_k^{[i]ss} + \sum_{j\in\mathcal{N}_i} \mathcal{A}_{ij}\chi_k^{[j]ss} \\ \tilde{y}_k^{[i]} & = & \mathcal{C}_i\chi_k^{[i]ss} \end{array} \tag{4.15}$$

It is easy to verify that Assumption 4.2 guarantees that a solution to the system (4.15) exists and is unique. In view of this, letting $\mathbf{x}_k^{ss} = (x_k^{[1]ss}, \ldots, x_k^{[M]ss})$, $\mathbf{u}_k^{ss} = (u_k^{[1]ss}, \ldots, u_k^{[M]ss})$, and $\tilde{\mathbf{y}}_k = (\tilde{y}_k^{[1]}, \ldots, \tilde{y}_k^{[M]})$, from (4.15) one has

$$\begin{bmatrix} \mathbf{x}_k^{ss} - \mathbf{x}_{k+1}^{ss} \\ \mathbf{u}_k^{ss} - \mathbf{u}_{k+1}^{ss} \end{bmatrix} \in -\mathbf{S}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \prod_{i=1}^{M} \mathcal{B}_{p,\varepsilon}^{(m_i)}(0) \tag{4.16}$$

from which it follows that, for all $i = 1, \dots, M$, there exists a set Δ_i^{ss} such that, for all $k \geq 0$

$$\chi_k^{[i]ss} - \chi_{k+1}^{[i]ss} \in \Delta_i^{ss} \tag{4.17}$$

An observer is now designed to provide an estimate $\chi_k^{[i]} = (\tilde{x}_k^{[i]}, \tilde{u}_k^{[i]})$ of the collective variable $\chi_k^{[i]ss}$ using the output reference information as data. We let

$$\tilde{s}_k^{[i]} = \sum_{j \in \mathcal{N}_i} L_{ij} \tilde{z}_k^{[j]} \tag{4.18}$$

The dynamics of the variable $\chi_k^{[i]}$ is defined by the following dynamical system

$$\chi_{k+1}^{[i]} = \mathcal{A}_{ii}\chi_k^{[i]} + \sum_{j \in \mathcal{N}_i} \mathcal{A}_{ij}\chi_k^{[j]} + \mathcal{G}_i(\tilde{y}_{k+1}^{[i]} - \mathcal{C}_i\chi_k^{[i]}) \tag{4.19}$$

where $\mathcal{G}_i = \begin{bmatrix} G_i^x \\ G_i^u \end{bmatrix}$ are gains to be determined as follows: denoting by $\mathcal{A}$ the matrix whose block elements are $\mathcal{A}_{ij}$, $\mathcal{C} = \text{diag}(\mathcal{C}_i)$, and $\mathcal{G} = \text{diag}(\mathcal{G}_i)$, the following assumption must be fulfilled

Assumption 4.4 *The matrix $\mathcal{A} - \mathcal{G}\mathcal{C}$ is Schur.* □

Note that the synthesis of the $\mathcal{G}_i$s can be performed according to the procedures proposed in Chapter 1.

From (4.14)-(4.15), it follows that

$$\begin{aligned}\chi_{k+1}^{[i]} - \chi_{k+1}^{[i]ss} &= (\mathcal{A}_{ii} - \mathcal{G}_i\mathcal{C}_i)(\chi_k^{[i]} - \chi_k^{[i]ss}) \\ &+ \sum_{j \in \mathcal{N}_i} \mathcal{A}_{ij}(\chi_k^{[j]} - \chi_k^{[j]ss}) + (\mathcal{A}_{ii} - \mathcal{G}_i\mathcal{C}_i)(\chi_k^{[i]ss} - \chi_{k+1}^{[i]ss}) \\ &+ \sum_{j \in \mathcal{N}_i} \mathcal{A}_{ij}(\chi_k^{[j]ss} - \chi_{k+1}^{[j]ss})\end{aligned} \tag{4.20}$$

In view of (4.17) we can rewrite (4.20) as

$$\begin{aligned}\chi_{k+1}^{[i]} - \chi_{k+1}^{[i]ss} = \; & (\mathcal{A}_{ii} - \mathcal{G}_i\mathcal{C}_i)(\chi_k^{[i]} - \chi_k^{[i]ss}) \\ & + \textstyle\sum_{j \in \mathcal{N}_i} \mathcal{A}_{ij}(\chi_k^{[j]} - \chi_k^{[j]ss}) + \tilde{w}_k^{[i]}\end{aligned} \tag{4.21}$$

where

$$\tilde{w}_k^{[i]} = (\mathcal{A}_{ii} - \mathcal{G}_i\mathcal{C}_i)(\chi_k^{[i]ss} - \chi_{k+1}^{[i]ss}) + \sum_{j \in \mathcal{N}_i} \mathcal{A}_{ij}(\chi_k^{[j]ss} - \chi_{k+1}^{[j]ss}) \in \tilde{\mathbb{W}}_i$$

can be regarded as a bounded disturbance in view of (4.17) and

$$\tilde{\mathbb{W}}_i = (\mathcal{A}_{ii} - \mathcal{G}_i\mathcal{C}_i)\Delta_i^{ss} \oplus (\bigoplus_{j\in\mathcal{N}_i} \mathcal{A}_{ij}\Delta_j^{ss}) \tag{4.22}$$

Under Assumption 4.4, for the system (4.21) there exists a possibly non-rectangular Robust Positive Invariant (RPI) set $\mathbf{\Delta}^{\chi}$ such that, if $(\tilde{\mathbf{x}}_k - \mathbf{x}_k^{ss}, \tilde{\mathbf{u}}_k - \mathbf{u}_k^{ss}) \in \mathbf{\Delta}^{\chi}$, then it is guaranteed that $(\tilde{\mathbf{x}}_{k+1} - \mathbf{x}_{k+1}^{ss}, \tilde{\mathbf{u}}_{k+1} - \mathbf{u}_{k+1}^{ss})) \in \mathbf{\Delta}^{\chi}$. This, in turn, implies that there exist sets Δ_i^{χ}, $i = 1, \ldots, M$, such that, for any initial condition $(\tilde{\mathbf{x}}_0 - \mathbf{x}_0^{ss}, \tilde{\mathbf{u}}_0 - \mathbf{u}_0^{ss}) \in \mathbf{\Delta}^{\chi}$, it is possible to guarantee that

$$(\chi_k^{[i]} - \chi_k^{[i]ss}) \in \Delta_i^{\chi} \tag{4.23}$$

for all $k \geq 0$.

4.2.3 The robust MPC layer

For each subsystem $i = 1, \ldots, M$, a robust MPC unit is designed to drive the real state and input trajectories $x_k^{[i]}$ and $u_k^{[i]}$ as close as possible to the reference ones $\tilde{x}_k^{[i]}$, $\tilde{u}_k^{[i]}$, while respecting the constraints on the same variables. As in the case of the reference state and input trajectory layer, information is required to be transmitted from reference trajectory management units of neighboring regulators, in a neighbor-to-neighbor fashion. Similarly to the regulation case, by adding suitable constraints to the MPC problem formulation, for each subsystem and for all $k \geq 0$ we will be able to guarantee that the actual coupling output trajectories lie in specified time-invariant neighborhoods of the reference trajectories. If $z_k^{[i]} \in \tilde{z}_k^{[i]} \oplus \mathcal{Z}_i$, where $0 \in \mathcal{Z}_i$, in view of (4.3) and (4.6d) (or (4.18)) we guarantee that $s_k^{[i]} \in \tilde{s}_k^{[i]} \oplus \mathcal{S}_i$, where $\mathcal{S}_i = \bigoplus_{j\in\mathcal{N}_i} L_{ij}\mathcal{Z}_j$. In this way, (4.2a) can be written as

$$x_{k+1}^{[i]} = A_{ii}x_k^{[i]} + B_{ii}u_k^{[i]} + E_i\tilde{s}_k^{[i]} + E_i(s_k^{[i]} - \tilde{s}_k^{[i]}) \tag{4.24}$$

where $E_i(s_k^{[i]} - \tilde{s}_k^{[i]})$ is a bounded disturbance and the term $E_i\tilde{s}_{k+\nu}^{[i]}$ can be interpreted as an input, known in advance over the prediction horizon $\nu = 0, \ldots, N-1$.

For the statement of the individual MPC sub-problems, henceforth called i-DPC problems, we define the i-th subsystem nominal model associated to equation (4.24)

$$\hat{x}_{k+1}^{[i]} = A_{ii}\hat{x}_k^{[i]} + B_{ii}\hat{u}_k^{[i]} + E_i\tilde{s}_k^{[i]} \tag{4.25}$$

or, if the observer is used

$$\hat{x}_{k+1}^{[i]} = A_{ii}\hat{x}_k^{[i]} + B_{ii}\hat{u}_k^{[i]} + E_i\tilde{s}_k^{[i]} + G_i^x(\tilde{y}_{k+1}^{[i]} - C_{ii}\tilde{x}_k^{[i]}) \tag{4.26}$$

Then we let

$$\hat{z}_k^{[i]} = C_{zi}\hat{x}_k^{[i]} + D_{zi}\hat{u}_k^{[i]} \tag{4.27}$$

The control law for the i-th subsystem (4.24), for all $k \geq 0$, is assumed to be given by

$$u_k^{[i]} = \hat{u}_k^{[i]} + K_i(x_k^{[i]} - \hat{x}_k^{[i]}) \tag{4.28}$$

where K_i satisfies Assumption 4.1. We also define $\varepsilon_k^{[i]} = x_k^{[i]} - \hat{x}_k^{[i]}$.

If the integrator is used for the reference state and inputs trajectory layer, from equation (4.24), (4.25) and (4.28) we obtain

$$\varepsilon_{k+1}^{[i]} = F_{ii}\varepsilon_k^{[i]} + w_k^{[i]} \tag{4.29}$$

where

$$w_k^{[i]} = E_i(s_k^{[i]} - \tilde{s}_k^{[i]}) \tag{4.30}$$

is a bounded disturbance since $s_k^{[i]} - \tilde{s}_k^{[i]} \in \mathcal{S}_i$. It follows that

$$w_k^{[i]} \in \mathbb{W}_i = E_i\mathcal{S}_i \tag{4.31}$$

On the other hand, if the observer is used for the reference state and inputs trajectory layer, from (4.24), (4.26) and (4.28) we obtain again the expression (4.29) where, in this case,

$$w_k^{[i]} = E_i(s_k^{[i]} - \tilde{s}_k^{[i]}) - G_i^x(\tilde{y}_{k+1}^{[i]} - C_{ii}\tilde{x}_k^{[i]}) \tag{4.32}$$

is a bounded disturbance since $s_k^{[i]} - \tilde{s}_k^{[i]} \in \mathcal{S}_i$ and, in view of (4.5) and (4.23)

$$\begin{aligned} \tilde{y}_{k+1}^{[i]} - C_{ii}\tilde{x}_k^{[i]} &= \tilde{y}_{k+1}^{[i]} - \tilde{y}_k^{[i]} + \tilde{y}_k^{[i]} - C_{ii}\tilde{x}_k^{[i]} \\ &= \tilde{y}_{k+1}^{[i]} - \tilde{y}_k^{[i]} + (-\mathcal{C}_i)(\chi_k^{[i]} - \chi_k^{[i]ss}) \in \mathcal{B}_{p,\varepsilon}^{(m_i)}(0) \oplus (-\mathcal{C}_i)\Delta_i^{\chi} \end{aligned} \tag{4.33}$$

It follows that

$$w_k^{[i]} \in \mathbb{W}_i = E_i\mathcal{S}_i \oplus (-G_i^x)(\mathcal{B}_{p,\varepsilon}^{(m_i)}(0) \oplus (-\mathcal{C}_i)\Delta_i^{\chi}) \tag{4.34}$$

In both cases, since $w_k^{[i]}$ is bounded and F_{ii} is Schur, there exists an RPI $\mathcal{E}_i$ for (4.29) such that, for all $\varepsilon_k^{[i]} \in \mathcal{E}_i$, then $\varepsilon_{k+1}^{[i]} \in \mathcal{E}_i$. Therefore at time $k+1$, in view of (4.2c) and (4.27), it holds that $z_{k+1}^{[i]} - \hat{z}_{k+1}^{[i]} = (C_{zi} + D_{zi}K_i)\varepsilon_{k+1}^{[i]} \in (C_{zi} + D_{zi}K_i)\mathcal{E}_i$.

In order to guarantee that, at time $k+1$, $z_{k+1}^{[i]} - \tilde{z}_{k+1}^{[i]} \in \mathcal{Z}_i$ can be still verified by adding suitable constraints to the optimization problems, the following assumption must be fulfilled.

Assumption 4.5 *For all $i = 1, \dots, M$, there exists a positive scalar ρ_i such that*

$$(C_{zi} + D_{zi}K_i)\mathcal{E}_i \oplus \mathcal{B}_{p,\rho_i}(0) \subseteq \mathcal{Z}_i \tag{4.35}$$

□

If Assumption 4.5 is fulfilled, we define, for all $i = 1, \dots, M$, the convex neighborhood of the origin Δ_i^z satisfying

$$\Delta_i^z \subseteq \mathcal{Z}_i \ominus (C_{zi} + D_{zi}K_i)\mathcal{E}_i \tag{4.36}$$

and we consider the constraint $\hat{z}_{k+1}^{[i]} - \tilde{z}_{k+1}^{[i]} \in \Delta_i^z$, in such a way that

$$z_{k+1}^{[i]} - \tilde{z}_{k+1}^{[i]} = z_{k+1}^{[i]} - \hat{z}_{k+1}^{[i]} + \hat{z}_{k+1}^{[i]} - \tilde{z}_{k+1}^{[i]} \in (C_{zi} + D_{zi}K_i)\mathcal{E}_i \oplus \Delta_i^z \subseteq \mathcal{Z}_i \tag{4.37}$$

as required at all time steps $k \geq 0$.

4.3 The distributed predictive control algorithm

It is now possible to state the i-DPC problem, to be solved at any time instant. The overall design problem is composed by a preliminary centralized off-line design and an on-line solution of the M i-DPC problems. These two steps are detailed in the following.

4.3.1 Off-line design

The off-line design consists of the following procedure:

1. if the integrator is used for the reference state and inputs trajectory layer: compute the matrices $\mathbf{K}$ and $\mathcal{K}$ satisfying Assumptions 4.1 and 4.3. If the observer is used for the reference state and inputs trajectory layer: compute the matrices $\mathbf{K}$ and $\mathcal{G}$ satisfying Assumptions 4.1 and 4.4.

2. If the integrator is used for the reference state and inputs trajectory layer: define $\mathcal{B}^{(m_i)}_{p,\varepsilon}(0)$, compute $\boldsymbol{\Delta}^{\chi}$ (a RPI for (4.11)) and Δ^{χ}_i. If the observer is used for the reference state and inputs trajectory layer: compute Δ^{ss}_i with (4.16), (4.17); compute $\tilde{\mathbb{W}}_i$ with (4.22) and $\tilde{\mathbb{W}} = \prod_{i=1}^{M} \tilde{\mathbb{W}}_i$; compute $\boldsymbol{\Delta}^{\chi}$, which represents a RPI for the collection of subsystems (4.21), and compute Δ^{χ}_i.

3. Compute the RPI sets $\mathcal{E}_i$ for the subsystems (4.29) and the sets Δ^z_i satisfying (4.36) and (4.37);

4. compute $\hat{\mathbb{X}}_i \subseteq \mathbb{X}_i \ominus \mathcal{E}_i$, $\hat{\mathbb{U}}_i \subseteq \mathbb{U}_i \ominus K_i\mathcal{E}_i$, the positively invariant set Σ_i for the equation

$$\delta x^{[i]}_{k+1} = F_{ii}\delta x^{[i]}_k \tag{4.38}$$

such that

$$(C_{zi} + D_{zi}K_i)\Sigma_i \subseteq \Delta^z_i \tag{4.39}$$

5. If the integrator is used for the reference state and inputs trajectory layer: compute the convex sets $\mathbb{Y}_i$ such that

$$\begin{aligned}\begin{bmatrix} I_{n_i} & 0 \\ K^x_i & K^e_i \end{bmatrix} \left(\Gamma_i (I_{n+p} - \mathcal{F})^{-1}\mathcal{G} \textstyle\prod_{j=1}^{M} \mathbb{Y}_j \oplus \Delta^{\chi}_i\right) \oplus \\ \oplus \begin{bmatrix} I_{n_i} \\ K_i \end{bmatrix} \Sigma_i \subseteq \hat{\mathbb{X}}_i \times \hat{\mathbb{U}}_i\end{aligned} \tag{4.40}$$

where Γ_i is the matrix, of suitable dimensions, that selects the subvector $\chi^{[i]}_k$ out of $\boldsymbol{\chi}_k$. Specifically, $\mathbb{Y}_i$ is the set associated to $\tilde{y}^{[i]}_k$ such that the corresponding steady-state state and input satisfy the control and state constraints defined by $\hat{\mathbb{X}}_i$ and $\hat{\mathbb{U}}_i$.

If the observer is used for the reference state and inputs trajectory layer: compute the convex sets $\mathbb{Y}_i$ such that

$$H_i \mathbf{S}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \prod_{j=1}^{M} \mathbb{Y}_j \oplus \Delta^{xu}_i \oplus \begin{bmatrix} I_{n_i} \\ K_i \end{bmatrix} \Sigma_i \subseteq \hat{\mathbb{X}}_i \times \hat{\mathbb{U}}_i \tag{4.41}$$

where H_i is the matrix, of suitable dimensions, that selects the vector $(x^{[i]}, u^{[i]})$ out of $(\mathbf{x}, \mathbf{u})$, $i = 1, \dots, M$.

4.3.2 On-line design

At any time instant the on-line design is performed in two steps, both based on the solution of a suitable distributed and independent optimization problem. First, the output reference trajectories $\tilde{y}_k^{[i]}$ are recursively updated. Then, the real control input $u_k^{[i]}$ is computed with MPC.

Computation of the reference outputs

The output reference trajectory $\tilde{y}_{t+N}^{[i]}$ is computed by the reference output trajectory management layer to minimize the distance with the set-points and, at the same time, to fulfill the constraints. The optimization problem to be solved at each time instant k is the following

$$\min_{\bar{y}_{k+N}^{[i]}} V_i^y(\bar{y}_{k+N}^{[i]}, k) \tag{4.42}$$

subject to

$$\bar{y}_{k+N}^{[i]} - \tilde{y}_{k+N-1}^{[i]} \in \mathcal{B}_{p,\varepsilon}^{(m_i)}(0) \tag{4.43}$$

$$\bar{y}_{k+N}^{[i]} \in \mathbb{Y}_i \tag{4.44}$$

where

$$V_i^y(\bar{y}_{k+N}^{[i]}) = \gamma\|\bar{y}_{k+N}^{[i]} - \tilde{y}_{k+N-1}^{[i]}\|^2 + \|\bar{y}_{k+N}^{[i]} - y_{set-point}^{[i]}\|_{T_i}^2$$

The weight T_i must verify the inequality

$$T_i > \gamma I_{m_i} \tag{4.45}$$

while γ is an arbitrarily small positive constant.

At time k, $\bar{y}_{k+N|k}^{[i]}$ is the solution to the optimization problem (4.42).

Remark 4.1 *In the implementation described in this Chapter, coupling constraints are not considered for simplicity. However, it is possible to include in the problem constraints involving the state of more than one subsystems. This, for instance, can be handled by imposing suitable constraints at the reference output trajectory management layer level. This implies that trasmission of information must be scheduled between local output trajectory management units. This issue has been explored in details in [60] and applied for controlling unicycle robots with collision avoidance capabilities.*

Computation of the control variables

The i-DPC problem solved by the i-th robust MPC layer unit is defined as follows:

$$\min_{\hat{x}_k^{[i]},\hat{u}_{[k:k+N-1]}^{[i]}} V_i^N(\hat{x}_k^{[i]}, \hat{u}_{[k:k+N-1]}^{[i]}) \tag{4.46}$$

where

$$\begin{aligned} V_i^N(\hat{x}_k^{[i]}, \hat{u}_{[k:k+N-1]}^{[i]}) = &\sum_{\nu=0}^{N-1} \|\hat{x}_{k+\nu}^{[i]} - \tilde{x}_{k+\nu}^{[i]}\|_{Q_i}^2 + \|\hat{u}_{k+\nu}^{[i]} - \tilde{u}_{k+\nu}^{[i]}\|_{R_i}^2 \\ &+ \|\hat{x}_{k+N}^{[i]} - \tilde{x}_{k+N}^{[i]}\|_{P_i}^2 \end{aligned} \tag{4.47}$$

subject to (4.25) (or to (4.26)) and, for $\nu = 0, \ldots, N-1$,

$$x_k^{[i]} - \hat{x}_k^{[i]} \in \mathcal{E}_i \tag{4.48a}$$

$$\hat{z}_{k+\nu}^{[i]} - \tilde{z}_{k+\nu}^{[i]} \in \Delta_i^z \tag{4.48b}$$

$$\hat{x}_{k+\nu}^{[i]} \in \hat{\mathbb{X}}_i \tag{4.48c}$$

$$\hat{u}_{k+\nu}^{[i]} \in \hat{\mathbb{U}}_i \tag{4.48d}$$

and to the terminal constraint

$$\hat{x}_{k+N}^{[i]} - \tilde{x}_{k+N}^{[i]} \in \Sigma_i \tag{4.49}$$

Note that, in case the observer is used for the reference state and input trajectory layer, the value $\tilde{x}_{k+N}^{[i]}$ has to be computed with

$$\begin{aligned} \bar{x}_{k+N}^{[i]}(\bar{y}_{k+N|k}^{[i]}) = &A_{ii}\tilde{x}_{k+N-1}^{[i]} + B_{ii}\tilde{u}_{k+N-1}^{[i]} + E_i\tilde{s}_{k+N-1}^{[i]} \\ &+ G_i^x(\bar{y}_{k+N|k}^{[i]} - C_{ii}\tilde{x}_{k+N-1}^{[i]}) \end{aligned} \tag{4.50}$$

The weights Q_i and R_i in the performance index (4.47) must be taken as positive definite matrices of appropriate dimensions while, in order to prove the convergence properties of the proposed approach, it is advisable to select the matrices P_i as the solutions of the (fully independent) Lyapunov equations

$$F_{ii}^T P_i F_{ii} - P_i = -(Q_i + K_i^T R_i K_i) \tag{4.51}$$

At time k, the tuple $(\hat{x}_{k|k}^{[i]}, \hat{u}_{[k:k+N-1]|k}^{[i]}, \bar{y}_{k+N|k}^{[i]})$ is the solution to the i-DPC problem and $\hat{u}_{k|k}^{[i]}$ is the input to the nominal system (4.25) (or to the nominal system (4.26)).

Remark 4.2 *It is important to note that the problems* (4.42) *and* (4.46) *are independent from each other. In fact, on the one hand, it is easy to see that* (4.42) *does not depend on* $\hat{x}_k^{[i]}$ *and* $\hat{u}_{[k:k+N-1]}^{[i]}$. *On the other hand note that, both the cost function* V_i^N *and the constraints* (4.48) *are independent of* $\bar{y}_{k+N|k}^{[i]}$.

Then, according to (4.28), the input to the system (4.2a) is

$$u_k^{[i]} = \hat{u}_{k|k}^{[i]} + K_i(x_k^{[i]} - \hat{x}_{k|k}^{[i]}) \tag{4.52}$$

Moreover, we set $\tilde{y}_{k+N}^{[i]} = \bar{y}_{k+N|k}^{[i]}$ and the references $\tilde{e}_{k+N}^{[i]}$ and $\tilde{x}_{k+N+1}^{[i]}$ are computed from (4.6b) and (4.6a), respectively. Finally $\tilde{u}_{k+N}^{[i]} = K_i^x \tilde{x}_{k+N}^{[i]} + K_i^e \tilde{e}_{k+N}^{[i]}$ from (4.6f) (in case the observer is used, $\tilde{x}_{k+N}^{[i]}$ and $\tilde{u}_{k+N}^{[i]}$ are computed with (4.19) once $\tilde{y}_{k+N}^{[i]}$ is given).

Denoting by $\hat{x}_{k+\nu|k}^{[i]}$ the state trajectory of system (4.25) (or of system (4.26)) stemming from $\hat{x}_{k|k}^{[i]}$ and $\hat{u}_{[k:k+N-1]|k}^{[i]}$, at time k it is also possible to compute $\hat{x}_{k+N|k}^{[i]}$.

The properties of the proposed distributed MPC algorithm for tracking can now be summarized in the following result (the proof is reported in the Appendix).

Theorem 4.1 *Let Assumptions 4.1-4.5 be verified and the tuning parameters be selected as previously described. If at time* $k = 0$ *a feasible solution to* (4.42), (4.46) *exists then, for all* $i = 1, \dots, M$

I) *Feasible solutions to* (4.42), (4.46) *exist for all* $k \geq 0$, *i.e., constraints* (4.43), (4.44), (4.48) *and* (4.49), *respectively, are verified. Furthermore, the constraints* $(x_k^{[i]}, u_k^{[i]}) \in \mathbb{X}_i \times \mathbb{U}_i$ *and for all* $i = 1, \dots, M$ *are fulfilled for all* $k \geq 0$.

II) *The resulting MPC controller asymptotically steers the i-th system to the admissible set-point* $y_{feas.set-point}^{[i]}$, *where* $y_{feas.set-point}^{[i]}$ *is the solution to*

$$y_{feas.set-point}^{[i]} = \underset{y^{[i]} \in \mathbb{Y}_i}{\operatorname{argmin}} \|y^{[i]} - y_{set-point}^{[i]}\|_{T_i}^2 \tag{4.53}$$

Note that when coupling static constraints (not considered in this Chapter) are present, the convergence to the nearest feasible solution to the prescribed set-point may be prevented for some initial conditions. These situations are denoted deadlock solutions in [162].

4.4 Simulation examples

We consider the problem of controlling the temperature of the apartment depicted in Figure 4.3 and constituted by two parts both with two rooms: rooms A and B belong to the first one, while rooms C and D to the second one. Each room is characterized by its own temperature (T_A, T_B, T_C and T_D) and is endowed with its own radiator (supplying heats q_A, q_B, q_C and q_D). For a detailed description of the model, of the used parameters and of the considered working point, we refer to Chapter 2. The discrete-time system of the form (4.1),

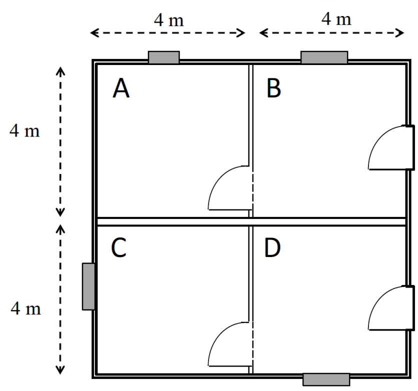

Figure 4.3: Schematic representation of a building with two apartments.

with $n = 4$, $m = 4$, is obtained by zero-order-hold discretization with sampling time $T = 30$ s, and it is described by the matrices

$$\mathbf{A} = \begin{bmatrix} 0.9731 & 0.0148 & 0.0059 & 0.0001 \\ 0.0148 & 0.9731 & 0.0001 & 0.0059 \\ 0.0059 & 0.0001 & 0.9731 & 0.0148 \\ 0.0001 & 0.0059 & 0.0148 & 0.9731 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 29.594 & 0.2243 & 0.0897 & 0.0009 \\ 0.2243 & 29.594 & 0.0009 & 0.0897 \\ 0.0897 & 0.0009 & 29.594 & 0.2243 \\ 0.0009 & 0.08973 & 0.2243 & 29.594 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The observer-based method has been used in the reference state and input trajectory layer. The matrices K_i and $\mathcal{G}_i$ fulfilling Assumption 4.1 and Assumption 4.4 have been computed as suggested in Chapter 2.

The weights used in the simulation are and $Q_1 = Q_2 = I_2$,

$R_1 = R_2 = I_2$, $T_1 = T_2 = I_2$, $\gamma = 10^{-6}$, and a prediction horizon of $N = 3$ was used.

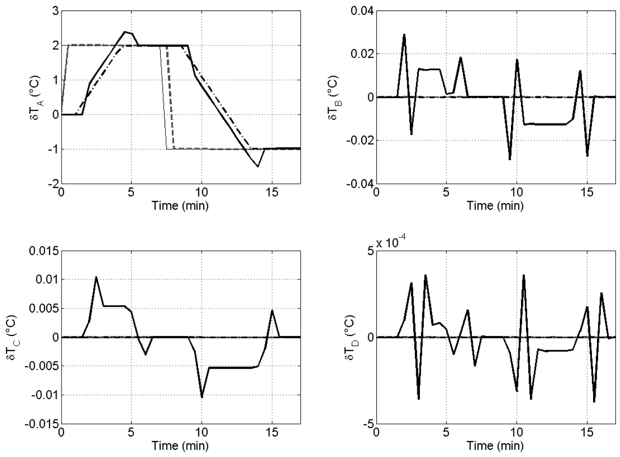

Figure 4.4: Trajectories of the output variables $y^{[1]}$ (above) and $y^{[2]}$ (below) obtained with DPC (black solid lines) and with cMPC (dashed gray lines). Think black lines: desired set-points $y^{[1,2]}_{set-point}$; black dash-dot lines: reference trajectories $\tilde{y}^{1,2}$.

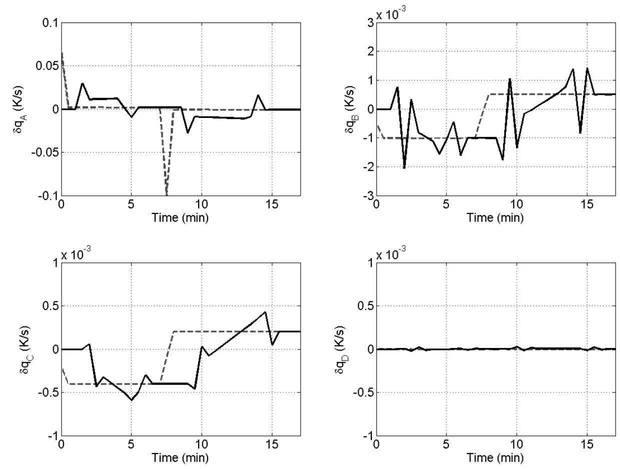

Figure 4.5: Trajectories of the inputs variables $u^{[1]}$ (above) and $u^{[2]}$ (below) obtained with DPC (black solid lines) and with cMPC (dashed gray lines).

In the simulations, the reference trajectories for $y^{[2]}_{set-point}$ are both always equal to zero, as well as the one related to T_B, while the first output of the first subsystem, T_A, should track a piece-wise constant reference trajectory, which values are 2 and -1. The results achieved are depicted in Figure 4.4, while the trajectories of the input variables are shown in Figure 4.5. In both these figures, a comparison between the outputs obtained with DPC and with centralized MPC is provided: it is possible to see that the transients obtained with DPC are slower, due to the limitation imposed to the set-point variations by constraint (4.43). Moreover, the distributed approach can lead to a reduction of the admissible set-point due to constraint (4.44), where the sets $\mathbb{Y}_i$ must satisfy (4.41).

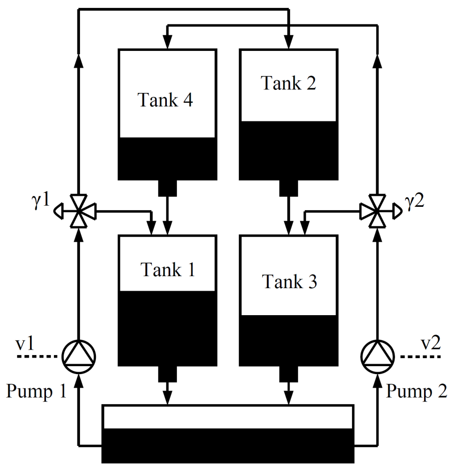

Figure 4.6: Schematic representation of the four-tanks system.

As second example, we consider now the four-tank system (see Figure 4.6) described in [74] and in Chapter 2, where we aim to control the water levels h_1 and h_3 of tanks 1 and 3 using the command voltages of the two pumps v_1 and v_2. In this case, we assume there are no external disturbances affecting the system.

The obtained linearized and discretized system with sampling time $T = 0.5$ s and zero-order-hold discretization has the form (4.1) with

$n = 4$, $m = 2$ where

$$\mathbf{A} = \begin{bmatrix} 0.9921 & 0 & 0 & 0.0206 \\ 0 & 0.9835 & 0 & 0 \\ 0 & 0.0165 & 0.9945 & 0 \\ 0 & 0 & 0 & 0.9793 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.0417 & 2.47 \cdot 10^{-4} \\ 0.0156 & 0 \\ 1.30 \cdot 10^{-4} & 0.0311 \\ 0 & 0.0235 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We first show the results obtained with a controller in which an observer has been used in the reference state and input trajectory layer. The matrices K_i and $\mathcal{G}_i$ fulfilling Assumption 4.1 and Assumption 4.4 have been computed by means of suitable LMIs (see Chapter 2).

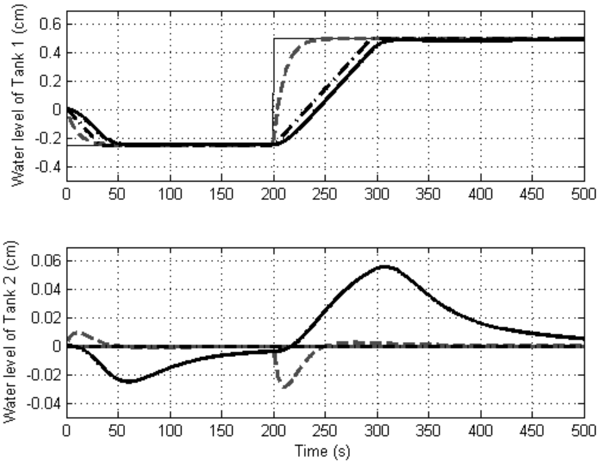

Figure 4.7: Trajectories of the output variables $y^{[1]}$ (above) and $y^{[2]}$ (below) obtained with DPC (black solid lines) and with cMPC (dashed gray lines). Think black lines: desired set-points $y^{[1,2]}_{set-point}$; black dash-dot lines: reference trajectories $\tilde{y}^{1,2}$.

The weights used in the simulation are $Q_1 = Q_2 = I_2$, $R_1 = R_2 = 1$, $T_1 = T_2 = 1$, $\gamma = 10^{-6}$, while the chosen prediction horizon is $N = 3$.

In the simulations, the reference trajectory for $y^{[2]}_{set-point}$ is always equal to zero, while the output of the first subsystem should track a piece-wise constant reference trajectory, which values are -0.25 and 0.5. The results achieved are depicted in Figure 4.7, while the trajectories of the input variables are shown in Figure 4.8.

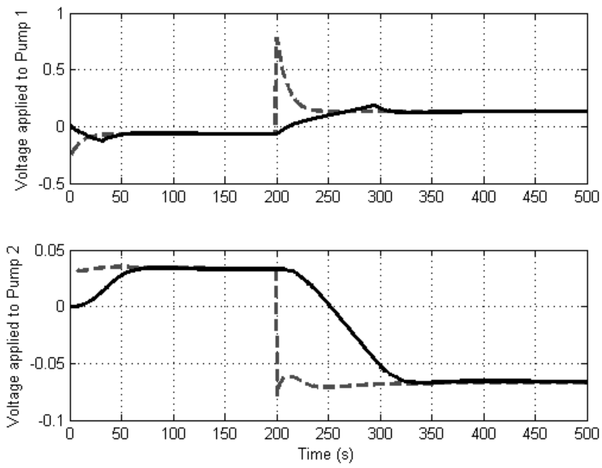

Figure 4.8: Trajectories of the inputs variables $u^{[1]}$ (above) and $u^{[2]}$ (below) obtained with DPC (black solid lines) and with cMPC (dashed gray lines).

In both these figures, a comparison between the outputs obtained with DPC and with centralized MPC is provided: it is possible to see that also in this case, the limitation imposed to the set-point variation leads to slower transients.

The same system has been used for testing the integrator-based method for the reference state and input trajectory layer. The matrices K_i and $\mathcal{K}_i$ fulfilling Assumption 4.1 and Assumption 4.2 have been computed as described in Chapter 2. The used weights and the prediction horizon are the ones shown for the previous case.

In the simulations, the reference trajectories $y^{[i]}_{set-point}$, $i = 1, 2$ are piece-wise constant (see Figure 4.9, grey dash-dotted lines). The results achieved are depicted in Figure 4.9, while the input trajectories are shown in Figure 4.10. Notably, the set-point $y^{[1]}_{set-point} = 2.5$ results infeasible to our algorithm, and hence the system output $y^{[1]}_t$ converges to the nearest feasible value. The system result to be faster than in the case where the observer was used, so important performance improvements can be obtained using the integrator for managing the reference state and input trajectory layer.

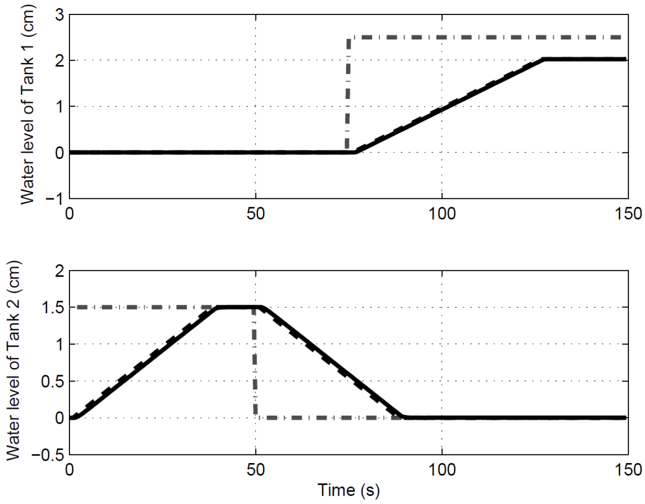

Figure 4.9: Trajectories of the output variables $y^{[1]}$ (above) and $y^{[2]}$ (below) (black solid lines) and reference outputs $\tilde{y}^{[1]}$ (above) and $\tilde{y}^{[2]}$ (below) (black dashed lines). Grey dash-dotted lines: desired set-points $y^{[1,2]}_{set-point}$.

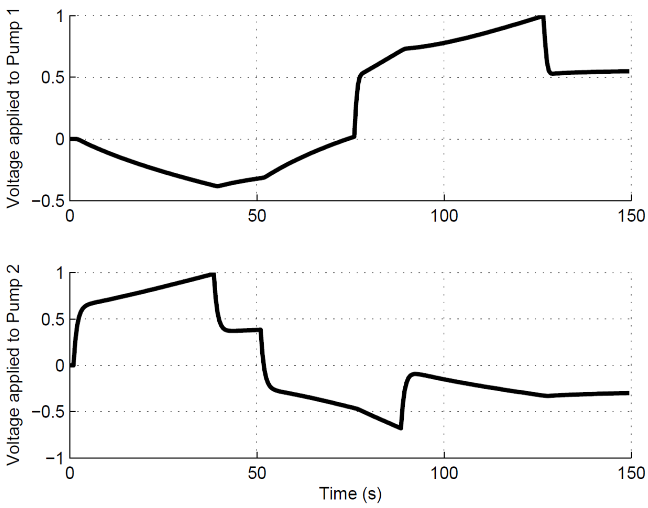

Figure 4.10: Trajectories of the inputs variables $u^{[1]}$ (above) and $u^{[2]}$ (below).

4.5 Conclusions

In this Chapter, a distributed MPC method for the solution of the tracking problem has been discussed, consisting in a hierarchical architecture based on Distributed Predictive Control. The overall scheme guarantees that state and control constraints are fulfilled and that the controlled outputs reach the prescribed reference values whenever possible, or their nearest feasible value when feasibility problems arise due to the constraints. The algorithm has been tested on two benchmark problems. In the next chapters, different solutions to the tracking problem including an integral action in the closed-loop will be presented.

4.6 Appendix

4.6.1 Proof of recursive feasibility and convergence of the reference management problem

Assume that, at step k, a solution $\bar{y}^{[i]}_{k+N|k}$ to (4.42) exists for all $i = 1, \ldots, M$. Then a solution to (4.42) exists at step $k+1$ for all $i = 1, \ldots, M$. In fact, taking $\bar{y}^{[i]}_{k+N+1} = \bar{y}^{[i]}_{k+N|k} = \tilde{y}^{[i]}_{k+N}$ one has $\bar{y}^{[i]}_{k+N|k} - \tilde{y}^{[i]}_{k+N} = 0 \in \mathcal{B}^{(m_i)}_{p,\varepsilon(0)}$ and $\bar{y}^{[i]}_{k+N|k} \in \mathbb{Y}_i$, hence verifying (4.43) and (4.44), respectively.

To prove convergence for the reference management layer note that, since at time $k+1$, $\bar{y}^{[i]}_{k+N+1} = \tilde{y}^{[i]}_{k+N}$ is a feasible solution, we have that, in view of the optimality of the solution $\bar{y}^{[i]}_{k+N+1|k+1}$

$$V^y_i(\bar{y}^{[i]}_{k+N+1|k+1}, k+1) \leq V^y_i(\bar{y}^{[i]}_{k+N|k}, k+1) \leq \|\bar{y}^{[i]}_{k+N|k} - y^{[i]}_{set-point}\|^2_{T_i} \tag{4.54}$$

In view of the fact that $\bar{y}^{[i]}_{k+N+1|k+1} = \tilde{y}^{[i]}_{k+N+1}$ for all k, we write $V^y_i(\bar{y}^{[i]}_{k+N+1|k+1}, k+1) = \gamma\|\tilde{y}^{[i]}_{k+N+1} - \tilde{y}^{[i]}_{k+N}\|^2 + \|\tilde{y}^{[i]}_{k+N+1} - y^{[i]}_{set-point}\|^2_{T_i}$, and we rewrite (4.54) as

$$\|\tilde{y}^{[i]}_{k+N+1} - y^{[i]}_{set-point}\|^2_{T_i} \leq \|\tilde{y}^{[i]}_{k+N} - y^{[i]}_{set-point}\|^2_{T_i} - \gamma\|\tilde{y}^{[i]}_{k+N+1} - \tilde{y}^{[i]}_{k+N}\|^2$$

From this we infer that $\tilde{y}^{[i]}_{k+N+1} - \tilde{y}^{[i]}_{k+N} \to 0$ as $k \to \infty$, and that

$$\|\tilde{y}^{[i]}_{k+N} - y^{[i]}_{set-point}\|^2_{T_i} \to \text{const} \tag{4.55}$$

as $k \to +\infty$.

Assume, by contradiction, that $\|\tilde{y}^{[i]}_{k+N} - y^{[i]}_{set-point}\|^2_{T_i} \to \bar{c}_i$, with $\bar{c}_i > c^o_i$, where

$$c^o_i = \|y^{[i]}_{feas.set-point} - y^{[i]}_{set-point}\|^2_{T_i} \tag{4.56}$$

Note that this implies that $\tilde{y}^{[i]}_{k+N} \neq y^{[i]}_{feas.set-point}$ for all $i = 1, \ldots, M$.

Assume that, given $\bar{k}$, for all $k \geq \bar{k}$ the optimal solution to (4.42) is $\bar{y}^{[i]}_{k+N|k} = \bar{y}^{[i]}$ where $\|\bar{y}^{[i]} - y^{[i]}_{set-point}\|^2_{T_i} = \bar{c}_i$. It results that

$$V^y_i(\bar{y}^{[i]}_{k+N|k}, k) = \bar{c}_i$$

On the other hand, an alternative solution is given by $\bar{\bar{y}}^{[i]}_{k+N}$, where

$$\bar{\bar{y}}^{[i]}_{k+N} = \lambda_i \bar{y}^{[i]} + (1 - \lambda_i) y^{[i]}_{feas.set-point}$$

with $\lambda_i \in [0, 1)$. This solution is feasible provided that *I)* $\bar{\bar{y}}^{[i]}_{k+N} - \bar{y}^{[i]} \in \mathcal{B}^{m_i}_{p,\varepsilon}(0)$ which can be verified if $(1 - \lambda_i)$ is sufficiently small, *II)* $\bar{\bar{y}}^{[i]}_{k+N} \in \mathbb{Y}_i$ which is also satisfied if $(1 - \lambda_i)$ is sufficiently small (since $\mathbb{Y}_i$ is convex and $\bar{y}^{[i]} \neq y^{[i]}_{feas.set-point}$).

According to this alternative solution

$$V^y_i(\bar{\bar{y}}^{[i]}_{k+N}, k) = \gamma \|\bar{\bar{y}}^{[i]}_{k+N} - \bar{y}^{[i]}\|^2 + \|\bar{\bar{y}}^{[i]}_{k+N} - y^{[i]}_{set-point}\|^2_{T_i}$$

Now if (4.45) is verified, then $V^y_i(\bar{\bar{y}}^{[i]}_{k+N}, k) < V^y_i(\bar{y}^{[i]}, k)$. This contradicts the assumption that $\|\tilde{y}^{[i]}_{k+N} - y^{[i]}_{set-point}\|^2_{T_i} \to \bar{c}_i$, with $\bar{c}_i > c^o_i$. Therefore, the only asymptotic solution compatible with (4.42), is that corresponding with $\tilde{y}^{[i]}_{k+N} = y^{[i]}_{feas.set-point}$ for all i.

It is now proved that $\tilde{y}^{[i]}_k \to y^{[i]}_{feas.set-point}$ for $k \to \infty$. In view of Assumption 4.2, this implies that $\mathcal{C}_i \chi^{[i]}_k = C_i \tilde{x}^{[i]}_k \to y^{[i]}_{feas.set-point}$ for all $i = 1, \ldots, M$.

4.6.2 Proof of recursive feasibility of the i-DPC problem

Assume that, at step k, a solution to (4.46) exists for all $i = 1, \ldots, M$, i.e., $(\hat{x}^{[i]}_{k|k}, \hat{u}^{[i]}_{[k:k+N-1]|k})$. Next we prove that, at step $k+1$, a solution to (4.46) exists for all $i = 1, \ldots, M$. To do so, we prove that the tuple $(\hat{x}^{[i]}_{k+1|k}, \hat{u}^{[i]}_{[k+1:k+N]|k})$ satisfies the constraints (4.25) (or (4.26)) and (4.48a)-(4.49) and is therefore a feasible (possibly suboptimal) solution to (4.46). Here $\hat{u}^{[i]}_{[t+1:t+N]|t}$ is obtained with

$$\hat{u}^{[i]}_{k+N|k} = \tilde{u}^{[i]}_{k+N} + K_i(\hat{x}^{[i]}_{k+N|k} - \tilde{x}^{[i]}_{k+N}) \tag{4.57}$$

First note that, in view of (4.6a), (4.25) and (4.57) (or, if the observer is used, in view of (4.26), and (4.57))

$$\hat{x}^{[i]}_{k+N+1|k} - \tilde{x}^{[i]}_{k+N+1} = F_{ii}(\hat{x}^{[i]}_{k+N|k} - \tilde{x}^{[i]}_{k+N}) \tag{4.58}$$

and therefore $\hat{x}^{[i]}_{k+N+1|k} - \tilde{x}^{[i]}_{k+N+1} \in \Sigma_i$ in view of the definition of Σ_i as a positively invariant set for (4.38), hence verifying (4.49). Therefore the constraint (4.49) is verified at step $k+1$.

Moreover, in view of the robust positive invariance of sets $\mathcal{E}_i$ with respect to equation (4.29), $i = 1, \ldots, M$, $x^{[i]}_{k+1} - \hat{x}^{[i]}_{k+1|k} \in \mathcal{E}_i$, and therefore (4.48a) is verified. Furthermore, in view of the feasibility of (4.48b)-(4.48d) at step k, it follows that constraints (4.48b)-(4.48d) are satisfied at step $k+\nu$ for $\nu = 1, \ldots, N-1$ and, from (4.49) and (4.39),

$$\hat{z}^{[i]}_{k+N|t} - \tilde{z}^{[i]}_{k+N} = (C_{zi} + D_{zi}K_i)(\hat{x}^{[i]}_{k+N|k} - \tilde{x}^{[i]}_{k+N}) \in (C_{zi} + D_{zi}K_i)\Sigma_i \subseteq \Delta^z_i \tag{4.59}$$

Hence constraint (4.48b) is verified at time $k+N$.

Suppose now that the integrator is used for the reference state and input trajectory layer. In this case we have that

$$\begin{bmatrix} \hat{x}^{[i]}_{k+N|k} \\ \hat{u}^{[i]}_{k+N|k} \end{bmatrix} \in \begin{bmatrix} \tilde{x}^{[i]}_{k+N} \\ \tilde{u}^{[i]}_{k+N} \end{bmatrix} \oplus \begin{bmatrix} I_{n_i} \\ K_i \end{bmatrix} \Sigma_i \tag{4.60}$$

where, from (4.13)

$$\begin{bmatrix} \tilde{x}^{[i]}_{k+N} \\ \tilde{u}^{[i]}_{k+N} \end{bmatrix} \in \begin{bmatrix} I_{n_i} & 0 \\ K^x_i & K^e_i \end{bmatrix} \left(\chi^{[i]ss}_{k+N} \oplus \Delta^\chi_i \right) \tag{4.61}$$

In turn, in view of (4.8) and similarly to (4.9),

$$\chi^{[i]ss}_{k+N} \in \Gamma_i (I_{n+p} - \mathcal{F})^{-1} \mathcal{G} \prod_{i=1}^{M} \mathbb{Y}_j \tag{4.62}$$

This eventually implies that, in view of (4.40)

$$\begin{bmatrix} \hat{x}^{[i]}_{k+N|k} \\ \hat{u}^{[i]}_{k+N|k} \end{bmatrix} \in \begin{bmatrix} I_{n_i} & 0 \\ K^x_i & K^e_i \end{bmatrix} \left(\Gamma_i (I_{n+p} - \mathcal{F})^{-1} \mathcal{G} \prod_{i=1}^{M} \mathbb{Y}_j \oplus \Delta^{\chi}_i \right) \oplus \oplus \begin{bmatrix} I_{n_i} \\ K_i \end{bmatrix} \Sigma_i \subseteq \hat{\mathbb{X}}_i \times \hat{\mathbb{U}}_i \tag{4.63}$$

which verifies constraints (4.48c) and (4.48d) at time $k + N$.

Assume instead that the observer is used for the reference state and input trajectory layer. In this case we have that

$$\begin{bmatrix} \hat{x}^{[i]}_{k+N|k} \\ \hat{u}^{[i]}_{k+N|k} \end{bmatrix} \in \begin{bmatrix} \tilde{x}^{[i]}_{k+N} \\ \tilde{u}^{[i]}_{k+N} \end{bmatrix} \oplus \begin{bmatrix} I_{n_i} \\ K_i \end{bmatrix} \Sigma_i$$

where, from (4.23)

$$\begin{bmatrix} \tilde{x}^{[i]}_{k+N} \\ \tilde{u}^{[i]}_{k+N} \end{bmatrix} \in \begin{bmatrix} x^{[i]ss}_{k+N} \\ u^{[i]ss}_{k+N} \end{bmatrix} \oplus \Delta^{xu}_i \tag{4.64}$$

In turn, in view of the definition of the sets $\mathbb{Y}_i$,

$$\begin{bmatrix} x^{[i]ss}_{k+N} \\ u^{[i]ss}_{k+N} \end{bmatrix} \in H_i \mathbf{S}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \prod_{j=1}^{M} \mathbb{Y}_j \tag{4.65}$$

This eventually implies that, in view of (4.41)

$$\begin{bmatrix} \hat{x}^{[i]}_{k+N|k} \\ \hat{u}^{[i]}_{k+N|k} \end{bmatrix} \in (H_i \mathbf{S}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \prod_{j=1}^{M} \mathbb{Y}_j) \oplus \Delta^{xu}_i \oplus \begin{bmatrix} I_{n_i} \\ K_i \end{bmatrix} \Sigma_i \subseteq \hat{\mathbb{X}}_i \times \hat{\mathbb{U}}_i$$

which verifies constraints (4.48c) and (4.48d) at time $k + N$.

4.6.3 Proof of convergence for the robust MPC layer

At time k the pair $(\hat{x}^{[i]}_{k|k}, \hat{u}^{[i]}_{[k:k+N-1]|k})$ is a solution to (4.46), leading to the optimal cost

$$V^{*N}_i(k) = \sum_{\nu=0}^{N-1} \|\hat{x}^{[i]}_{k+\nu|k} - \tilde{x}^{[i]}_{k+\nu}\|^2_{Q_i} + \|\hat{u}^{[i]}_{k+\nu|k} - \tilde{u}^{[i]}_{k+\nu}\|^2_{R_i} + \|\hat{x}^{[i]}_{k+N|k} - \tilde{x}^{[i]}_{k+N}\|^2_{P_i} \tag{4.66}$$

Since $(\hat{x}^{[i]}_{k+1|k}, \hat{u}^{[i]}_{[k+1:k+N]|k})$ is a feasible solution to (4.46) at time $k+1$, by optimality $V_i^{*N}(k+1) \leq V_i^N(\hat{x}^{[i]}_{k+1|k}, \hat{u}^{[i]}_{[k+1:k+N]|k})$, which is equal to

$$V_i^N(\hat{x}^{[i]}_{k+1|k}, \hat{u}^{[i]}_{[k+1:k+N]|k}) = \sum_{\nu=1}^{N} \|\hat{x}^{[i]}_{k+\nu|k} - \tilde{x}^{[i]}_{k+\nu}\|^2_{Q_i} + \|\hat{u}^{[i]}_{k+\nu|k} - \tilde{u}^{[i]}_{k+\nu}\|^2_{R_i} + \\ + \|\hat{x}^{[i]}_{k+N+1|k} - \tilde{x}^{[i]}_{k+N+1}\|^2_{P_i} \quad (4.67)$$

Adding and removing the terms $\|\hat{x}^{[i]}_{k|k} - \tilde{x}^{[i]}_{k}\|^2_{Q_i} + \|\hat{u}^{[i]}_{k|k} - \tilde{u}^{[i]}_{k}\|^2_{R_i}$, and $\|\hat{x}^{[i]}_{k+N|k} - \tilde{x}^{[i]}_{k+N}\|^2_{P_i}$ from the right hand side of (4.67), we obtain that $V_i^{*N}(k+1) \leq$

$$\sum_{\nu=0}^{N-1} \|\hat{x}^{[i]}_{k+\nu|k} - \tilde{x}^{[i]}_{k+\nu}\|^2_{Q_i} + \|\hat{u}^{[i]}_{k+\nu|k} - \tilde{u}^{[i]}_{k+\nu}\|^2_{R_i} + \|\hat{x}^{[i]}_{k+N|k} - \tilde{x}^{[i]}_{k+N}\|^2_{P_i} + \quad (4.68a)$$

$$+ \|\hat{x}^{[i]}_{k+N|k} - \tilde{x}^{[i]}_{k+N}\|^2_{Q_i} + \|\hat{u}^{[i]}_{k+N|t} - \tilde{u}^{[i]}_{k+N}\|^2_{R_i} + \quad (4.68b)$$

$$+ \|\hat{x}^{[i]}_{k+N+1|k} - \tilde{x}^{[i]}_{k+N+1}\|^2_{P_i} - \|\hat{x}^{[i]}_{k+N|k} - \tilde{x}^{[i]}_{k+N}\|^2_{P_i} + \quad (4.68c)$$

$$- (\|\hat{x}^{[i]}_{k|k} - \tilde{x}^{[i]}_{k}\|^2_{Q_i} + \|\hat{u}^{[i]}_{k|k} - \tilde{u}^{[i]}_{k}\|^2_{R_i}) \quad (4.68d)$$

Note that (4.68a) is equal to $V_i^{*N}(k)$ in (4.66). On the other hand, in view of (4.58), we can write (4.68b)-(4.68c) as

$$\text{(4.68b)- (4.68c)} = \|\hat{x}^{[i]}_{k+N|k} - \tilde{x}^{[i]}_{k+N}\|^2_{F_{ii}^T P_i F_{ii} - P_i + Q_i + K_i^T R_i K_i}$$

which, in view of (4.51), implies that

$$\text{(4.68b)- (4.68c)} = 0 \quad (4.69)$$

Therefore, for all $i = 1, \ldots, M$

$$V_i^{*N}(k+1) \leq V_i^{*N}(k) - (\|\hat{x}^{[i]}_{k|k} - \tilde{x}^{[i]}_{k}\|^2_{Q_i} + \|\hat{u}^{[i]}_{k|k} - \tilde{u}^{[i]}_{k}\|^2_{R_i}) \quad (4.70)$$

and, according to standard arguments in MPC [105], we prove that, for all $i = 1, \ldots, M$

$$\hat{x}^{[i]}_{k|k} \to \tilde{x}^{[i]}_{k} \quad (4.71a)$$

$$\hat{u}^{[i]}_{k|k} \to \tilde{u}^{[i]}_{k} \quad (4.71b)$$

as $k \to \infty$.

Suppose now the integrator is used for the reference state and input trajectory layer. In this case, consider the model (4.1a) and, collectively, the model (4.6a) and equation (4.28). We have that

$$\begin{array}{rcl}\mathbf{x}_{k+1} &=& \mathbf{A}\mathbf{x}_k + \mathbf{B}\left(\hat{\mathbf{u}}_{k|k} + \mathbf{K}(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})\right)\\ \tilde{\mathbf{x}}_{k+1} &=& \mathbf{A}\tilde{\mathbf{x}}_k + \mathbf{B}\tilde{\mathbf{u}}_k + \mathbf{B}\mathbf{K}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_k)\end{array} \tag{4.72}$$

for all $k \geq 0$. Denote $\mathbf{\Delta x}_k = \mathbf{x}_k - \tilde{\mathbf{x}}_k$, $\mathbf{\Delta}\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k|k} - \tilde{\mathbf{x}}_k$ and $\mathbf{\Delta}\hat{\mathbf{u}}_k = \hat{\mathbf{u}}_{k|k} - \tilde{\mathbf{u}}_k$. From (4.72)

$$\mathbf{\Delta x}_{k+1} = \mathbf{F}\mathbf{\Delta x}_k + \mathbf{B}\left(\mathbf{\Delta}\hat{\mathbf{u}}_k - \mathbf{K}\mathbf{\Delta}\hat{\mathbf{x}}_k\right) \tag{4.73}$$

Since, in view of (4.71), $\mathbf{B}\left(\mathbf{\Delta}\hat{\mathbf{u}}_k - \mathbf{K}\mathbf{\Delta}\hat{\mathbf{x}}_k\right) \to 0$ as $k \to \infty$, in view of Assumption 4.1 it holds that $\mathbf{\Delta x}_k \to 0$ as $k \to \infty$, which implies that asymptotically $C_i x_k^{[i]} \to C_i \tilde{x}_k^{[i]}$.

On the other hand, suppose we are in the case where the observer is used for the reference state and input trajectory layer. Consider the model (4.1a) and, collectively, the model (4.19) and equation (4.28). We have that

$$\begin{array}{rcl}\mathbf{x}_{k+1} &=& \mathbf{A}\mathbf{x}_k + \mathbf{B}\left(\hat{\mathbf{u}}_{k|k} + \mathbf{K}(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})\right)\\ \tilde{\mathbf{x}}_{k+1} &=& \mathbf{A}\tilde{\mathbf{x}}_k + \mathbf{B}\tilde{\mathbf{u}}_k + \mathbf{G}^x(\tilde{\mathbf{y}}_{k+1} - \mathbf{C}\tilde{\mathbf{x}}_k) + \mathbf{B}\mathbf{K}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_k)\end{array} \tag{4.74}$$

for all $k \geq 0$, where $\mathbf{G}^x = \mathrm{diag}(G_1^x, \ldots, G_M^x)$. From (4.74)

$$\mathbf{\Delta x}_{k+1} = \mathbf{F}\mathbf{\Delta x}_k + \mathbf{B}\left(\mathbf{\Delta}\hat{\mathbf{u}}_k - \mathbf{K}\mathbf{\Delta}\hat{\mathbf{x}}_k\right) + \mathbf{G}^x(\tilde{\mathbf{y}}_{k+1} - \mathbf{C}\tilde{\mathbf{x}}_k) \tag{4.75}$$

Since, in view of (4.71) and of the convergence of the reference trajectory layer, $\mathbf{B}\left(\mathbf{\Delta}\hat{\mathbf{u}}_k - \mathbf{K}\mathbf{\Delta}\hat{\mathbf{x}}_k\right) + \mathbf{G}^x(\tilde{\mathbf{y}}_{k+1} - \mathbf{C}\tilde{\mathbf{x}}_k) \to 0$ as $k \to \infty$, in view of Assumption 4.1 it holds that $\mathbf{\Delta x}_t \to 0$ as $k \to \infty$, which implies that asymptotically $C_{ii} x_k^{[i]} \to\to C_i \tilde{x}_k^{[i]}$.

5 DPC for systems in velocity-form

In this Chapter, DPC is extended to include an integral action in the closed-loop for the tracking of constant reference signals. Specifically, according to the approach proposed in e.g. [122,168], integrators are inserted into the loop and the corresponding enlarged system is described in velocity-form. If the process is affected by constant disturbances, this solution allows one to guarantee offset-free steady-state regulation for constant reference signals without the need to compute the corresponding steady-state values of the state and control vectors. Under standard assumptions in MPC, the closed-loop system enjoys stability properties, in the sense that the subsystems' state trajectories starting from given sets in the state space converge to the required equilibrium.

5.1 The system

5.1.1 System under control

Consider a discrete-time, linear system, which obeys to the linear dynamics

$$\begin{array}{rcl} \mathbf{x}_{k+1} & = & \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k & = & \mathbf{C}\mathbf{x}_k \end{array} \tag{5.1}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ is the state, $\mathbf{u}_k \in \mathbb{R}^m$ is the input, and $\mathbf{y}_k \in \mathbb{R}^m$ is the output. The constant output reference target value to be tracked is denoted by $\bar{\mathbf{y}} \in \mathbb{R}^m$. To guarantee the existence and the uniqueness of the steady-state pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R}^{m+n}$ such that the system output corresponds to $\bar{\mathbf{y}}$, the following standard assumption is made.

Assumption 5.1 *The input-output system* (5.1) *has no invariant zeros in* 1*, i.e.,*

$$\text{rank}\left(\begin{bmatrix} I_n - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix}\right) = n + m$$

□

The state and input vectors are constrained to lie in prescribed sets, i.e. $\mathbf{x} \in \mathbb{X}$, $\mathbf{u} \in \mathbb{U}$, where $\mathbb{X}$, $\mathbb{U}$ are compact and convex sets. As it will more formally specified in Assumption 5.4, constants $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ must lie in suitable subsets of $\mathbb{X}$ and $\mathbb{U}$, respectively, which in turn implicitly define a set of admissible constant output references $\bar{\mathbb{Y}}$. The sets $\mathbb{X}$ and $\mathbb{U}$ are here defined as the cartesian product of suitable subsets, according to the decomposition of the system into subsystems (see the next section).

5.1.2 Partitioned system

The system (5.1) is partitioned in M low order interconnected non overlapping subsystems, where a generic sub-model has $x_k^{[i]} \in \mathbb{R}^{n_i}$ as state vector, i.e., $\mathbf{x}_k = (x_k^{[1]}, \ldots, x_k^{[M]})$ and $\sum_{i=1}^{M} n_i = n$. Accordingly, the state transition matrices $A_{11} \in \mathbb{R}^{n_1 \times n_1}, \ldots, A_{MM} \in \mathbb{R}^{n_M \times n_M}$ of the M subsystems are the diagonal blocks of $\mathbf{A}$, whereas the non-diagonal blocks of $\mathbf{A}$ (i.e., A_{ij}, with $i \neq j$) define the dynamic coupling terms between subsystems. Also the input vector $\mathbf{u}_k$ is assumed to be partitioned in M non-overlapping sub-vectors $u_k^{[i]} \in \mathbb{R}^{m_i}$, where $\mathbf{u}_k = (u_k^{[1]}, \ldots, u_k^{[M]})$ and $u_k^{[i]}$ is denoted as the i-th subsystem input. Correspondingly, B_{ij}, $i, j = 1, \ldots, M$, are the blocks of $\mathbf{B}$ defining the direct influence of input $u_k^{[j]}$ upon the state $x_k^{[i]}$. Finally, the output $\mathbf{y}_k$ is partitioned into M non overlapping output vectors $y_k^{[i]} \in \mathbb{R}^{m_i}$, with $i = 1, \ldots, M$ and $\sum_{i=1}^{M} m_i = m$, where $y_k^{[i]}$ is assumed only to depend on $x_k^{[i]}$, for all $i = 1, \ldots, M$. This implies that $\mathbf{C}$ has a block diagonal structure $\mathbf{C} = \text{diag}(C_1, \ldots, C_M)$, where $C_i \in \mathbb{R}^{m_i \times n_i}$ for all $i = 1, \ldots, M$. In accordance with the output partition, the output reference target $\bar{\mathbf{y}}$ can be seen as decomposed into M local output reference targets $\bar{y}^{[i]}$, consistent with the definition of $y_t^{[i]}$. From now on, the subsystem j is said to be a dynamic neighbor of subsystem i if and only if the state and/or the input of j affect the dynamics of subsystem i i.e., if and only if $A_{ij} \neq 0$ and/or $B_{ij} \neq 0$. By $\mathcal{N}_i$ we denote the set of dynamic neighbors of subsystem i (which excludes i).

According to these partitions, the i-th subprocess obeys to the linear dynamics

$$\begin{array}{rcl} x_{k+1}^{[i]} &=& A_{ii}x_k^{[i]} + B_{ii}u_k^{[i]} + \sum_{j\in\mathcal{N}_i}\{A_{ij}x_k^{[j]} + B_{ij}u_k^{[j]}\} \\ y_k^{[i]} &=& C_i\, x_k^{[i]} \end{array} \tag{5.2}$$

where $x_k^{[i]} \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ and $u_k^{[i]} \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$, being $\mathbb{X}_i$ and $\mathbb{U}_i$ convex sets consistent with the adopted partition, i.e. $\mathbb{X} = \prod_{i=1}^{M} \mathbb{X}_i$, $\mathbb{U} = \prod_{i=1}^{M} \mathbb{U}_i$. We assume that each subsystem enjoys the following property.

Assumption 5.2 *For each subsystem* (5.2)*:*

i) the pair (A_{ii}, B_{ii}) *is reachable.*

ii) rank $\left(\begin{bmatrix} I_{n_i} - A_{ii} & -B_{ii} \\ C_{ii} & 0 \end{bmatrix}\right) = n_i + m_i$.

□

5.1.3 System with integrators

In order to solve the tracking problem, the partitioned system is now enlarged with m integrators and, according to [122,168], is described in "velocity form". Specifically, letting $\delta x_k^{[i]} = x_k^{[i]} - x_{k-1}^{[i]}$, $\varepsilon_k^{[i]} = y_k^{[i]} - \bar{y}^{[i]}$ and $\delta u_k^{[i]} = u_k^{[i]} - u_{k-1}^{[i]}$, system (5.2) is written as

$$\begin{array}{rcl} \delta x_{k+1}^{[i]} &=& A_{ii}\delta x_k^{[i]} + B_{ii}\delta u_k^{[i]} + \sum_{j\in\mathcal{N}_i}\{A_{ij}\delta x_k^{[j]} + B_{ij}\delta u_k^{[j]}\} \\ \varepsilon_{k+1}^{[i]} &=& C_iA_{ii}\delta x_k^{[i]} + \varepsilon_k^{[i]} + C_iB_{ii}\delta u_k^{[i]} + C_i\sum_{j\in\mathcal{N}_i}\{A_{ij}\delta x_k^{[j]} + B_{ij}\delta u_k^{[j]}\} \end{array} \tag{5.3}$$

Letting $\xi_k^{[i]} = (\delta x_k^{[i]}, \varepsilon_k^{[i]})$ and

$$\mathcal{A}_{ii} = \begin{bmatrix} A_{ii} & 0 \\ C_iA_{ii} & I_{m_i} \end{bmatrix},\ \mathcal{B}_{ii} = \begin{bmatrix} B_{ii} \\ C_iB_{ii} \end{bmatrix},\ \mathcal{A}_{ij} = \begin{bmatrix} A_{ij} & 0 \\ C_iA_{ij} & 0 \end{bmatrix}$$

$$\mathcal{B}_{ij} = \begin{bmatrix} B_{ij} \\ C_iB_{ij} \end{bmatrix},\ \mathcal{C}_i = \begin{bmatrix} 0 & I_{m_i} \end{bmatrix}$$

system (5.3) can be written in compact form as

$$\begin{array}{rcl} \xi_{k+1}^{[i]} &=& \mathcal{A}_{ii}\xi_k^{[i]} + \mathcal{B}_{ii}\delta u_k^{[i]} + \sum_{j\in\mathcal{N}_i}\{\mathcal{A}_{ij}\xi_k^{[j]} + \mathcal{B}_{ij}\delta u_k^{[j]}\} \\ \varepsilon_k^{[i]} &=& \mathcal{C}_i\xi_k^{[i]} \end{array} \tag{5.4}$$

For system (5.4) the following property holds (see the Appendix for the proof):

Proposition 5.1 *Under Assumption 5.2, the pair* $(\mathcal{A}_{ii}, \mathcal{B}_{ii})$ *is reachable.* ■

The set of M models (5.4) can be written in the collective form

$$\begin{aligned} \boldsymbol{\xi}_{k+1} &= \boldsymbol{\mathcal{A}}\boldsymbol{\xi}_k + \boldsymbol{\mathcal{B}}\boldsymbol{\delta}\mathbf{u}_k \\ \boldsymbol{\varepsilon}_k &= \boldsymbol{\mathcal{C}}\boldsymbol{\xi}_k \end{aligned} \tag{5.5}$$

where $\boldsymbol{\xi}_k = (\xi_k^{[1]}, \dots, \xi_k^{[M]})$, $\boldsymbol{\varepsilon}_k = (\varepsilon_k^{[1]}, \dots, \varepsilon_k^{[M]})$, $\boldsymbol{\delta}\mathbf{u}_k = (\delta u_k^{[1]}, \dots, \delta u_k^{[M]})$, while $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$ are the matrices whose block entries are $\mathcal{A}_{ij}$, $\mathcal{B}_{ij}$, and $\mathcal{C}_{ij}$, respectively.

Concerning system (5.5) and its partition, the following main assumption on decentralized stabilizability is introduced.

Assumption 5.3 *There exists a block-diagonal matrix* $\boldsymbol{\mathcal{K}}$, *i.e.* $\boldsymbol{\mathcal{K}} = diag(\mathcal{K}_1, \dots, \mathcal{K}_M)$ *with* $\mathcal{K}_i \in \mathbb{R}^{m_i \times (n_i + m_i)}$, $i = 1, \dots, M$, *such that*

i) The matrix $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}$ *is Schur.*

ii) The matrices $\mathcal{F}_{ii} = \mathcal{A}_{ii} + \mathcal{B}_{ii}\mathcal{K}_i$ *are Schur.*

□

5.2 The DPC algorithm for tracking

5.2.1 Nominal models and control law

DPC is developed under the assumption that, at any time instant k, each subsystem (5.4) transmits its state and input reference trajectories $\tilde{\xi}_{k+\nu}^{[i]}$ and $\delta\tilde{u}_{k+\nu}^{[i]}$, $\nu = 0, \dots, N-1$ to its neighbors. Moreover, by adding suitable constraints to its formulation, each subsystem is able to guarantee that, for all $k \geq 0$, its true state $\xi_k^{[i]}$ and input $\delta u_k^{[i]}$ trajectories lie in specified time-invariant neighborhoods of $\tilde{\xi}_k^{[i]}$ and $\delta\tilde{u}_k^{[i]}$ respectively, i.e, $\xi_k^{[i]} - \tilde{\xi}_k^{[i]} \in \mathbb{E}_i$ and $\delta u_k^{[i]} - \delta\tilde{u}_k^{[i]} \in \mathbb{E}_i^u$, where $0 \in \mathbb{E}_i$ and $0 \in \mathbb{E}_i^u$. In this way (5.4) can be written as

$$\xi_{k+1}^{[i]} = \mathcal{A}_{ii}\xi_k^{[i]} + \mathcal{B}_{ii}\delta u_k^{[i]} + \sum_{j \in \mathcal{N}_i} \{\mathcal{A}_{ij}\tilde{\xi}_k^{[j]} + \mathcal{B}_{ij}\delta\tilde{u}_k^{[j]}\} + w_k^{[i]} \tag{5.6}$$

where

$$w_k^{[i]} = \sum_{j \in \mathcal{N}_i} \{\mathcal{A}_{ij}(\xi_k^{[j]} - \tilde{\xi}_k^{[j]}) + \mathcal{B}_{ij}(\delta u_k^{[j]} - \delta\tilde{u}_k^{[j]})\} \in \mathbb{W}_i \tag{5.7}$$

is a bounded disturbance, specifically

$$\mathbb{W}_i = \bigoplus_{j\in\mathcal{N}_i}\{\mathcal{A}_{ij}\mathbb{E}_j \oplus \mathcal{B}_{ij}\mathbb{E}^u{}_j\} \tag{5.8}$$

and $\sum_{j\in\mathcal{N}_i}\{\mathcal{A}_{ij}\tilde{\xi}_k^{[j]} + \mathcal{B}_{ij}\delta\tilde{u}_k^{[j]}\}$ can be seen as an input, known in advance over the prediction horizon, which can be treated in the MPC formulation as a known disturbance.

With reference to the subsystems (5.6), it is possible to define the i-th subsystem nominal model as

$$\hat{\xi}_{k+1}^{[i]} = \mathcal{A}_{ii}\hat{\xi}_k^{[i]} + \mathcal{B}_{ii}\delta\hat{u}_k^{[i]} + \sum_{j\in\mathcal{N}_i}\{\mathcal{A}_{ij}\tilde{\xi}_k^{[j]} + \mathcal{B}_{ij}\delta\tilde{u}_k^{[j]}\} \tag{5.9}$$

According to the robust MPC approach based on tubes [107], the control law for the true i-th subsystem (5.6) is defined, for all $k \geq 0$, as

$$\delta u_k^{[i]} = \delta\hat{u}_k^{[i]} + \mathcal{K}_i(\xi_k^{[i]} - \hat{\xi}_k^{[i]}) \tag{5.10}$$

Letting $z_k^{[i]} = \xi_k^{[i]} - \hat{\xi}_k^{[i]}$, from (5.6)-(5.10) we obtain

$$z_{k+1}^{[i]} = (\mathcal{A}_{ii} + \mathcal{B}_{ii}\mathcal{K}_i)z_k^{[i]} + w_k^{[i]} \tag{5.11}$$

where $w_k^{[i]} \in \mathbb{W}_i$. Since $\mathbb{W}_i$ is bounded and $\mathcal{F}_{ii} = \mathcal{A}_{ii} + \mathcal{B}_{ii}\mathcal{K}_i$ is Schur, there exists a robust positively invariant (RPI) set $\mathbb{Z}_i$ (which is assumed to be symmetric with respect to the origin) for (5.11) such that, for all $z_k^{[i]} \in \mathbb{Z}_i$, then $z_{k+1}^{[i]} \in \mathbb{Z}_i$. Note that, correspondingly, $\delta u_k^{[i]} - \delta\hat{u}_k^{[i]} \in \mathcal{K}_i\mathbb{Z}_i$.

5.2.2 Input and state constraints

The reformulation (5.4) of (5.2) requires to transform the original constraints on the state and control variables $x^{[i]}$ and $u^{[i]}$ in terms of constraints on $\delta x^{[i]}$ (i.e., on $\xi^{[i]}$) and $\delta u^{[i]}$. To this end, first define the sets $\Delta\mathbb{E}_i$ and $\Delta\mathbb{U}_i$, for all $i = 1, \ldots, M$, containing the origin and satisfying $\Delta\mathbb{E}_i \oplus \mathbb{Z}_i \subseteq \mathbb{E}_i$ and $\Delta\mathbb{E}_i^u \oplus \mathcal{K}_i\mathbb{Z}_i \subseteq \mathbb{E}_i^u$, respectively. Then, note that, for $\nu = 1, \dots, N$, it holds that

$$x^{[i]}_{k+\nu} = x^{[i]}_k + \sum_{r=1}^{\nu} \delta\hat{x}^{[i]}_{k+r} + \sum_{r=1}^{\nu} (\delta x^{[i]}_{k+r\nu} - \delta\hat{x}^{[i]}_{k+\nu})$$

$$\in x^{[i]}_k + \sum_{r=1}^{\nu} \delta\hat{x}^{[i]}_{k+\nu} \bigoplus_{r=1}^{\nu} \left[I_{n_i} 0\right] \mathbb{Z}_i \tag{5.12a}$$

$$u^{[i]}_{k+\nu-1} = u^{[i]}_{k-1} + \sum_{r=1}^{\nu} \delta\hat{u}^{[i]}_{k+r-1} + \sum_{r=1}^{\nu} (\delta u^{[i]}_{k+r-1} - \delta\hat{u}^{[i]}_{k+r-1})$$

$$\in u^{[i]}_{k-1} + \sum_{r=1}^{\nu} \delta\hat{u}^{[i]}_{k+r-1} \bigoplus_{r=1}^{\nu} \mathcal{K}_i \mathbb{Z}_i \tag{5.12b}$$

Therefore, the constraints $x^{[i]}_{k+\nu} \in \mathbb{X}_i$ and $u^{[i]}_{k+\nu-1} \in \mathbb{U}_i$ for $\nu = 1, \dots, N$ translate in the following:

$$x^{[i]}_k + \sum_{r=1}^{\nu} \delta\hat{x}^{[i]}_{k+r} \in \hat{\mathbb{X}}_i(\nu) = \mathbb{X}_i \ominus \{\bigoplus_{r=1}^{\nu} \left[I_{n_i} \quad 0\right] \mathbb{Z}_i\} \tag{5.13a}$$

$$u^{[i]}_{k-1} + \sum_{r=1}^{\nu} \delta\hat{u}^{[i]}_{k+r-1} \in \hat{\mathbb{U}}_i(\nu) = \mathbb{U}_i \ominus \{\bigoplus_{r=1}^{\nu} \mathcal{K}_i \mathbb{Z}_i\} \tag{5.13b}$$

In the following, these constraints on $\delta\hat{x}^{[i]}$ will be transformed into equivalent constraints on $\hat{\xi}^{[i]}$ (see (5.18) and (5.19) below). Note that, in view of the definition (5.13), $\hat{\mathbb{U}}_i(\nu+1) \subseteq \hat{\mathbb{U}}_i(\nu)$ and $\hat{\mathbb{X}}_i(\nu+1) \subseteq \hat{\mathbb{X}}_i(\nu)$ for all ν.

5.2.3 i-DPC problems

The minimization problem for subsystem i (i-DPC) at instant k is now stated starting from the knowledge of the stabilizing gain matrix $\mathcal{K}_i$, of the sets $\mathbb{E}_i$, $\mathbb{E}^u_i$, $\mathbb{Z}_i$, $\Delta\mathbb{E}_i$, and $\Delta\mathbb{E}^u_i$ of the future input and state reference trajectories for i and its neighbors $\tilde{\xi}^{[j]}_{k+\nu}$ and $\delta\tilde{u}^{[j]}_{k+\nu}$, $\nu = 0, \dots, N-1$, $j \in \mathcal{N}_i$, and of the output reference target $\bar{y}^{[i]}$. With these ingredients, the i-DPC problem consists in the following:

$$\min_{\hat{\xi}^{[i]}_k, \delta\hat{u}^{[i]}_{[k:k+N-1]}} V^N_i(\hat{\xi}^{[i]}_k, \delta\hat{u}^{[i]}_{[k:k+N-1]}) = \sum_{\nu=0}^{N-1} \|\hat{\xi}^{[i]}_{k+\nu}\|^2_{Q_i} + \|\delta\hat{u}^{[i]}_{k+\nu}\|^2_{R_i} + \|\hat{\xi}^{[i]}_{k+N}\|^2_{P_i} \tag{5.14}$$

subject to the dynamic constraints (5.9), to the constraints

$$\xi_k^{[i]} - \hat{\xi}_k^{[i]} \in \mathbb{Z}_i \tag{5.15}$$

$$\hat{\xi}_{k+\nu}^{[i]} - \tilde{\xi}_{k+\nu}^{[i]} \in \Delta\mathbb{E}_i \tag{5.16}$$

$$\delta\hat{u}_{k+\nu}^{[i]} - \delta\tilde{u}_{k+\nu}^{[i]} \in \Delta\mathbb{E}_i^u \tag{5.17}$$

$$x_k^{[i]} + \begin{bmatrix} I_{n_i} & 0 \end{bmatrix} \sum_{r=1}^{\nu} \hat{\xi}_{k+r}^{[i]} \in \hat{\mathbb{X}}_i(\nu) \tag{5.18}$$

$$u_{k-1}^{[i]} + \sum_{r=1}^{\nu} \delta\hat{u}_{k+r-1}^{[i]} \in \hat{\mathbb{U}}_i(\nu) \tag{5.19}$$

with $\nu = 1, \ldots, N$, and to the terminal constraint

$$\hat{\xi}_{k+N}^{[i]} \in \Xi_i^F \tag{5.20}$$

where the sets Ξ_i^F (assumed to be symmetric with respect to the origin) are defined in the following.

Remark 5.1

I) *If* (5.15) *is satisfied and* (5.16)- (5.17) *are verified for* $\nu = 0$*, then* $\xi_k^{[i]} - \tilde{\xi}_k^{[i]} \in \Delta\mathbb{E}_i \oplus \mathbb{Z}_i \subseteq \mathbb{E}_i$ *and* $\delta u_k^{[i]} - \delta\tilde{u}_k^{[i]} \in \Delta\mathbb{E}_i^u \oplus \mathcal{K}_i\mathbb{Z}_i \subseteq \mathbb{E}_i^u$*, which implies that* $w_k^{[i]} \in \mathbb{W}_i$*. This, in view of the invariance property of* (5.11) *with respect to* $\mathbb{Z}_i$*, implies that* $\xi_{k+1}^{[i]} - \hat{\xi}_{k+1}^{[i]} \in \mathbb{Z}_i$*. Considering that constraints* (5.16) *and* (5.17) *are imposed over the whole prediction horizon, it follows by induction that* $w_{k+\nu}^{[i]} \in \mathbb{W}_i$ *and* $\xi_{k+\nu}^{[i]} - \hat{\xi}_{k+\nu}^{[i]} \in \mathbb{Z}_i$ *for all* $\nu = 1, \ldots, N$*. Therefore, the initially stated requirement on the boundedness of the disturbance* $w_k^{[i]}$ *is verified.*

II) *In the quadratic cost function* (5.14) *the positive definite matrices* Q_i*,* R_i *and* P_i *are suitable tuning parameters to be properly selected by the designer.*

III) *The final cost* $V_i^F = \|\hat{\xi}_{k+N}^{[i]}\|_{P_i}^2$ *must be selected (see the following Section) to guarantee suitable decreasing properties of the cost function.*

The solution of the i-DPC problem (5.14) at time k is the pair $(\hat{\xi}_{k/k}^{[i]}, \delta\hat{u}_{[k:k+N-1]/k}^{[i]})$; therefore, according to (5.10) and a receding horizon implementation, the input to the system (5.2), at instant k, is

$$\delta u_k^{[i]} = \delta\hat{u}_{k/k}^{[i]} + \mathcal{K}_i(\xi_k^{[i]} - \hat{\xi}_{k/k}^{[i]}) \tag{5.21}$$

Finally, letting $\hat{\xi}^{[i]}_{k+\nu/k}$ be the trajectory stemming from $\hat{\xi}^{[i]}_{k/k}$, $\delta\hat{u}^{[i]}_{[k:k+N-1]/k}$, and (5.9), the reference trajectories to be used in the next time instant $k+1$ are incrementally updated by appending the values

$$\tilde{\xi}^{[i]}_{k+N} = \hat{\xi}^{[i]}_{k+N/k} \tag{5.22a}$$

$$\delta\tilde{u}^{[i]}_{k+N} = \mathcal{K}_i\hat{\xi}^{[i]}_{k+N/k} \tag{5.22b}$$

to the reference trajectories previously defined for $k+\nu \leq k+N-1$.

5.3 Convergence results

The convergence properties of the proposed algorithm can be proved according to the results reported in Chapter 2 and in [50]. First define the set of admissible initial conditions $\boldsymbol{\xi}_0 = (\xi^{[1]}_0, \dots, \xi^{[M]}_0)$ and initial reference trajectories $\tilde{\xi}^{[j]}_{[0:N-1]}$, $\delta\tilde{u}^{[j]}_{[0:N-1]}$, for all $j = 1\dots, M$ as follows.

Definition 5.1 *Letting $\boldsymbol{\xi} = (\xi^{[1]}, \dots, \xi^{[M]})$, denote the feasibility region for all the i-DPC problems by*

$$\begin{aligned}\Xi^N := \; & \{\boldsymbol{\xi} : \text{ if } \xi^{[i]}_0 = \xi^{[i]} \text{ for all } i = 1,\dots,M \text{ then} \\ & \exists(\tilde{\xi}^{[1]}_{[0:N-1]},\dots,\tilde{\xi}^{[M]}_{[0:N-1]}), (\delta\tilde{u}^{[1]}_{[0:N-1]},\dots,\delta\tilde{u}^{[M]}_{[0:N-1]}), \\ & (\hat{\xi}^{[1]}_{0/0},\dots,\hat{\xi}^{[M]}_{0/0}), (\delta\hat{u}^{[1]}_{[0:N-1]},\dots,\delta\hat{u}^{[M]}_{[0:N-1]}) \text{ such} \\ & \text{that (5.2) and (5.15)- (5.20) are satisfied} \\ & \text{for all } i = 1,\dots,M\}\end{aligned}$$

Moreover, for each $\boldsymbol{\xi} \in \Xi^N$, let

$$\begin{aligned}\tilde{\Xi}^N_{\boldsymbol{\xi}} := \; & \{(\tilde{\xi}^{[1]}_{[0:N-1]},\dots,\tilde{\xi}^{[M]}_{[0:N-1]}), (\delta\tilde{u}^{[1]}_{[0:N-1]},\dots,\delta\tilde{u}^{[M]}_{[0:N-1]}) : \\ & \text{if } \xi^{[i]}_0 = \xi^{[i]} \text{ for all } i = 1,\dots,M \text{ then } \exists \\ & (\hat{\xi}^{[1]}_{0/0},\dots,\hat{\xi}^{[M]}_{0/0}), (\delta\hat{u}^{[1]}_{[0:N-1]},\dots,\delta\hat{u}^{[M]}_{[0:N-1]}) \text{ such} \\ & \text{that (5.2) and (5.15)- (5.20) are satisfied} \\ & \text{for all } i = 1,\dots,M\}\end{aligned}$$

be the region of feasible initial reference trajectories with respect to a given initial condition.

Assumption 5.4 *Letting $\mathbb{Z} = \prod_{i=1}^M \mathbb{Z}_i$, $\hat{\mathbb{U}} = \prod_{i=1}^M \hat{\mathbb{U}}_i$, and $\hat{\Xi}^F = \prod_{i=1}^M \hat{\Xi}^F_i$, it holds that:*

i) $\hat{\Xi}^F$ *is an invariant set for* $\hat{\boldsymbol{\xi}}^+ = \boldsymbol{\mathcal{F}}\hat{\boldsymbol{\xi}}$;

ii) *Defining for simplicity of notation* $\Delta\Xi = \bigoplus_{s=1}^{\infty} \boldsymbol{\mathcal{F}}^s(\hat{\Xi}^F \oplus \mathbb{Z})$, *and* $\mathbf{H} = diag(\begin{bmatrix} I_{n_1} & 0 \end{bmatrix}, \ldots, \begin{bmatrix} I_{n_M} & 0 \end{bmatrix})$ *the following properties must be satisfied:*

$$\bar{\mathbf{x}} \in \hat{\mathbb{X}}(N-1) \ominus \mathbf{H}\Delta\Xi \ominus \mathbf{H} \bigoplus_{r=1}^{N-1} \mathbb{Z} \tag{5.23a}$$

$$\bar{\mathbf{u}} \in \hat{\mathbb{U}}(N-1) \ominus \mathbf{K}\Delta\Xi \ominus \mathbf{K} \bigoplus_{r=1}^{N-1} \mathbb{Z} \tag{5.23b}$$

iii) *For all* $\hat{\boldsymbol{\xi}} \in \hat{\Xi}^F$

$$\mathbf{V}^F\left(\hat{\boldsymbol{\xi}}^+\right) - \mathbf{V}^F\left(\hat{\boldsymbol{\xi}}\right) \leq -\mathbf{l}(\hat{\boldsymbol{\xi}}, \boldsymbol{\delta}\hat{\mathbf{u}}) \tag{5.24}$$

where $\mathbf{V}^F(\hat{\boldsymbol{\xi}}) = \sum_{i=1}^{M} V_i^F(\hat{\xi}^{[i]})$ *and* $\mathbf{l}(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\delta}\mathbf{u}}) = \sum_{i=1}^{M} l_i(\hat{\xi}^{[i]}, \delta\hat{u}^{[i]})$, *being* $l_i(\hat{\xi}^{[i]}, \delta\hat{u}^{[i]}) = \|\hat{\xi}^{[i]}\|_{Q_i}^2 + \|\delta\hat{u}^{[i]}\|_{R_i}^2$.

□

Assumptions 5.4-*i)* and 5.4-*iii)* are standard in stabilizing MPC algorithms, while Assumption 5.4-*ii)* corresponds to require that the the state and control variables of the closed-loop system formed by (5.5) and the set of control laws (5.10) satisfy the constraints on **x** and **u**.

Assumption 5.5 *Given the sets* $\mathbb{E}_i$, $\mathbb{U}_i$, *and the RPI sets* $\mathbb{Z}_i$ *for equation* (5.11), *there exists a real positive constant* $\bar{\rho}_E > 0$ *such that* $\mathbb{Z}_i \oplus \mathbb{B}_{\bar{\rho}_E}^{(n_i)}(0) \subseteq \mathbb{E}_i$ *and* $\mathcal{K}_i\mathbb{Z}_i \oplus \mathbb{B}_{\bar{\rho}_E}^{(m_i)}(0) \subseteq \mathbb{U}_i$ *for all* $i = 1, \ldots, M$, *where* $\mathbb{B}_{\bar{\rho}_E}^{(dim)}(0)$ *is a ball of radius* $\bar{\rho}_E > 0$ *centered at the origin in the* $\mathbb{R}^{dim}$ *space.* □

The main convergence result is now stated; since the velocity-form allows one to transform a tracking problem in a regulation one, it can be proved along the lines reported in Chapter 2 and in [50].

Theorem 5.1 *Let Assumptions 5.1-5.5 be satisfied and let* $\Delta\mathbb{E}_i$ *and* $\Delta\mathbb{E}_i^u$ *be neighborhoods of the origin satisfying* $\Delta\mathbb{E}_i \oplus \mathbb{Z}_i \subseteq \mathbb{E}_i$ *and* $\Delta\mathbb{E}_i^u \oplus \mathcal{K}_i\mathbb{Z}_i \subseteq \mathbb{E}_i^u$. *Then, for any initial reference trajectories in* $\tilde{\Xi}_{\boldsymbol{\xi}_0}^N$, *the trajectory* $\boldsymbol{\xi}_k$, *starting from any initial condition* $\boldsymbol{\xi}_0 \in \Xi^N$, *asymptotically converges to the origin, so that the output* $\boldsymbol{y}$ *of system* (5.1) *converges to the reference* $\bar{\mathbf{y}}$. ■

5.4 Implementation issues

The DPC algorithm requires to compute the block diagonal matrix $\boldsymbol{\mathcal{K}} = \text{diag}(\mathcal{K}_1, \ldots, \mathcal{K}_M)$ satisfying Assumption 5.4. Moreover, condition (5.24) must be fulfilled by a proper selection of a matrix $\mathbf{P} = \text{diag}(P_1, \ldots, P_M)$. Eventually, the sets $\mathbb{E}_i$, $\mathbb{E}_i^u$, $\mathbb{Z}_i$, $\Delta\mathbb{E}_i$, $\Delta\mathbb{E}_i^u$ and Ξ_i^F must be characterized. Since the tracking problem has been transformed in a regulation one, the algorithms presented in Chapter 2 can be used.

5.5 Simulation example

Consider the four-tanks system of Figure 5.1, see [7,36,57,74,108,165], already introduced in Chapter 2. We recall that the goal is to control the levels h_1 and h_3 of Tanks 1 and 3. The manipulated inputs are the voltages of the two pumps v_1 and v_2. The parameters γ_1 and $\gamma_2 \in (0, 1)$ represent the fraction of water that flows inside the lower tanks, and are kept fixed during the simulations.

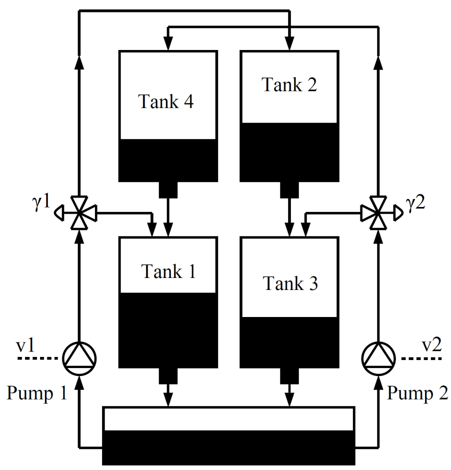

Figure 5.1: Schematic representation of the four-tanks system.

The dynamic model of the system and the values of the parameters have been described in Chapter 2. In this case, external disturbances are not considered.

The linearization of the dynamic system and its zero-order-hold discretization with sampling time $T = 0.5$ s, leads to the model (5.1) with $n = 4$, $m = 2$ and

$$\mathbf{A} = \begin{bmatrix} 0.9921 & 0 & 0 & 0.0206 \\ 0 & 0.9835 & 0 & 0 \\ 0 & 0.0165 & 0.9945 & 0 \\ 0 & 0 & 0 & 0.9793 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.0417 & 2.47 \cdot 10^{-4} \\ 0.0156 & 0 \\ 1.30 \cdot 10^{-4} & 0.0311 \\ 0 & 0.0235 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

In order to apply the DPC algorithm for tracking, this system has been partitioned into 2 subsystems: the first one constituted by Tank 1 and Tank 2, and the second one by Tank 3 and Tank 4. The partition of inputs, outputs and states is, therefore

$$\begin{array}{ccccccccc} x^{[1]} & = & \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T & , & u^{[1]} & = & u_1 & , & y^{[1]} = y_1 \\ x^{[2]} & = & \begin{bmatrix} x_3 & x_4 \end{bmatrix}^T & , & u^{[2]} & = & u_2 & , & y^{[2]} = y_2 \end{array}$$

The parameters used in the simulation are $Q_1 = Q_2 = 3.24 \cdot 10^{-9} \cdot I_3$, $R_1 = R_2 = 3.24 \cdot 10^{-15}$, $\overline{y}^{[1]} = -0.4$, $\overline{y}^{[2]} = 0$ and

$$\begin{array}{rcl} \mathcal{K}_1 & = & \begin{bmatrix} -10.87 & 0.25 & -3.21 \end{bmatrix} \\ \mathcal{K}_2 & = & \begin{bmatrix} -12.45 & 0.36 & -3.96 \end{bmatrix} \\ P_1 & = & \begin{bmatrix} 6.01 \cdot 10^{-6} & -4.98 \cdot 10^{-7} & 1.23 \cdot 10^{-6} \\ -4.98 \cdot 10^{-7} & 2.04 \cdot 10^{-6} & 2.47 \cdot 10^{-8} \\ 1.23 \cdot 10^{-6} & 2.47 \cdot 10^{-8} & 1.81 \cdot 10^{-6} \end{bmatrix} \\ P_2 & = & \begin{bmatrix} 7.37 \cdot 10^{-6} & -2.71 \cdot 10^{-7} & 1.59 \cdot 10^{-6} \\ -2.71 \cdot 10^{-7} & 1.46 \cdot 10^{-6} & 2.92 \cdot 10^{-7} \\ 1.59 \cdot 10^{-6} & 2.92 \cdot 10^{-7} & 2.73 \cdot 10^{-6} \end{bmatrix} \end{array}$$

In Figure 5.2, a comparison between the outputs obtained with DPC and with centralized MPC (cMPC) is provided, showing only a slight performance degradation of DPC with respect to cMPC.

5.6 Conclusions

The DPC algorithm for tracking presented in this Chapter has an important features which make it suited for industrial applications, specifically the integral action inserted in the closed-loop. On the other hand, the proposed method has guaranteed convergence properties only in

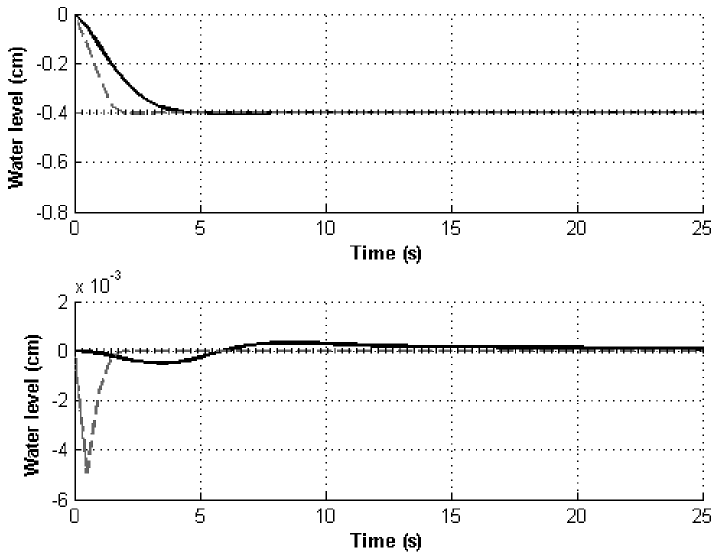

Figure 5.2: Trajectories of the output variables $y^{[1]}$ (above) and $y^{[2]}$ (below) obtained with DPC (solid lines) and with cMPC (dashed lines). Dotted lines: reference $\bar{y}^{[i]}$.

case the reference is kept constant: for time-varying reference signals, recursive feasibility can not be proved. In order to extend also to the cases of varying external setpoints the approach based on the velocity-form version of dynamic systems, its properties have been studied for the centralized case, as shown in Chapter 6.

5.7 Appendix

5.7.1 Proof of Proposition 5.1

The pair $(\mathcal{A}_{ii}, \mathcal{B}_{ii})$ is reachable iff, for all $\lambda \in \mathbb{C}$,

$$\begin{aligned} &\operatorname{rank}\left(\begin{bmatrix}\lambda I_{n_i+m_i} - \mathcal{A}_{ii} & -\mathcal{B}_{ii}\end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix}\lambda I_{n_i} - A_{ii} & -B_{ii} & 0 \\ -C_i A_{ii} & -C_i B_{ii} & (\lambda-1) I_{m_i}\end{bmatrix}\right) = n_i + m_i \end{aligned} \tag{5.25}$$

On the one hand, if $\lambda \neq 1$, since the term $(\lambda - 1)I_{m_i}$ has full rank m_i, and since the matrix

$$\begin{bmatrix}\lambda I_{n_i} - A_{ii} & -B_{ii} & 0 \\ -C_i A_{ii} & -C_i B_{ii} & (\lambda-1) I_{m_i}\end{bmatrix}$$

is block lower triangular, to guarantee (5.25) it is sufficient to guarantee that

$$\text{rank}(\begin{bmatrix}\lambda I_{n_i} - A_{ii} & -B_{ii}\end{bmatrix}) = n_i$$

which holds in view of the observability properties of the pair (A_{ii}, B_{ii}).

On the other hand, if $\lambda = 1$, to guarantee (5.25) it is sufficient to show that

$$\text{rank}\left(\begin{bmatrix} I_{n_i} - A_{ii} & -B_{ii} \\ -C_i A_{ii} & -C_i B_{ii} \end{bmatrix}\right) = n_i + m_i$$

which is equivalent to the condition *ii)* of Assumption 5.2.

6

Centralized MPC with integral action

The control technique presented in Chapter 5 has the important positive side of inserting an integral action in the closed-loop. This can be really useful in practice, since it leads to an offset-free tracking of constant reference points also in presence of constant disturbances (and/or errors in the model). The negative side of the proposed formulation is that feasibility is not guaranteed when setpoint changes are required. To overcome this issue, an in-depth study of centralized controllers for systems rewritten in velocity-form is discussed in this Chapter.

As already discussed, three different solutions to the offset-free problem have been proposed so far. The first one consists of augmenting the model of the plant under control with an artificial disturbance, which must be estimated together with the system state [96, 111, 113, 118, 120, 123]. Another approach consists of including an internal model of the reference fed by the output error in the control scheme [39, 99, 104]. A third solution consists of describing the system in velocity-form [122, 168].

Although all these solutions to the offset-free problem share the common idea to include, implicitly or explicitly, an integral action in the control loop, each one has its own advantages and drawbacks. In particular, the use of the velocity-form does not require the use of a state estimator even when the plant state is available and does not require to compute the steady state target for the plant state and control variables in order to properly formulate the optimization problem considered in the MPC formulation. On the other hand, in order to address the stability issue with MPC algorithms for systems in velocity form, in [168] the "zero terminal constraint" solution is suggested, while, to the best of the author knowledge, no stability results have been established for MPC algorithms based on the velocity-form and characterized by a terminal cost and a terminal constraint, terms usually considered to guarantee stability and convergence, see [105]. This is due to the difficulty to find an auxiliary control law and an associated terminal set where the constraints on the state and input of the system are fulfilled, see [122]. Note that the zero terminal constraint solution cannot be used for systems affected by bounded but non-constant disturbances.

In this Chapter, an MPC algorithm for linear systems described in velocity form is presented. It allows one to track piecewise constant signal rejecting constant disturbances, and relies on the approach described in [6,87–89] for the solution of the unfeasible reference problem. The proposed method guarantees stability properties and the convergence of the controlled output to the feasible reference or to the nearest artificial reference, i.e. the solution of the offset-free problem. The main point in the derivation of the method is related to the definition of an auxiliary stabilizing control law and of a region where its use can guarantee the fulfillment of the original state and control constraints. The proposed method is initially derived for nominal systems and is then extended for coping with disturbances and uncertainties. Specifically, the latter version is used in three cases, which differ from each other in view of the type of perturbation affecting the system, i.e., bounded disturbance, constant disturbance, and model uncertainty. All the proofs are reported in the Appendix to improve readability.

6.1 MPC for offset-free tracking: nominal systems

6.1.1 Statement of the problem

Consider a discrete-time, linear, time-invariant system, described by

$$\begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k \\ y_k & = & Cx_k \end{array} \tag{6.1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the input, and $y_k \in \mathbb{R}^m$ is the output. The inputs and state variables are subject to constraints, i.e. $x_k \in \mathbb{X}$ and $u_k \in \mathbb{U}$ for all instants k, where $\mathbb{X}$, $\mathbb{U}$ are compact and convex neighbors of the origin.

For system (6.1) we consider the problem of designing a state-feedback control system, based on MPC, for tracking a given constant reference signal $r^o \in \mathbb{R}^m$, i.e., that asymptotically steers the system output y_k to the desired value r^o. We underline that r^o is allowed to change without losing feasibility: in this sense, also if all the Chapter is developed referring to constant setpoints, the reader should remember that in practice the proposed method can be used to track piecewise constant references.

The following standard assumptions are made.

Assumption 6.1 i) *The state is measurable.*

ii) *The pair* (A, B) *is reachable.*

iii) *The input-output system* (6.1) *has no invariant zeros in* 1*, i.e.,* $rank(S) = n + m$*, where*

$$S = \begin{bmatrix} I_n - A & -B \\ -C & 0 \end{bmatrix}$$

□

Note that Assumption 6.1 *iii)* guarantees the existence and the uniqueness of the steady-state pair $(x^o, u^o) \in \mathbb{R}^{m+n}$ such that the system output corresponds to the desired value y^o.

6.1.2 The velocity form

In order to solve the tracking problem, the system is enlarged with m integrators and described in velocity-form. Specifically, denoting by $\hat{r}$ a generic tracking target (which could be different from r^o, for reasons which will become clear later on), the corresponding steady state condition is denoted $(\hat{x}, \hat{u})$. Letting $\delta x_k = x_k - x_{k-1}$, $\varepsilon_k = y_k - \hat{r}$, and $\delta u_k = u_k - u_{k-1}$, system (6.1) can be reformulated as follows

$$\begin{array}{rcl} \delta x_{k+1} & = & A\delta x_k + B\delta u_k \\ \varepsilon_{k+1} & = & CA\delta x_k + \varepsilon_k + CB\delta u_k \end{array} \tag{6.2}$$

Define $\xi_k = (\delta x_k, \varepsilon_k)$ and

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ CA & I_m \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} B \\ CB \end{bmatrix} \tag{6.3}$$

In this way system (6.2) can be written in compact form as

$$\xi_{k+1} = \mathcal{A}\xi_k + \mathcal{B}\delta u_k \tag{6.4}$$

The following proposition can be easily proved.

Proposition 6.1 *If Assumption 6.1 holds, then the pair* $(\mathcal{A}, \mathcal{B})$ *is reachable.* ■

In view of the reachability property of $(\mathcal{A}, \mathcal{B})$, it is possible to compute the gain $\mathcal{K}$ such that $\mathcal{F} = \mathcal{A} + \mathcal{B}\mathcal{K}$ is Schur.

The dynamics of system (6.4), under the state-feedback control law

$$\delta u_t = \mathcal{K}\xi_k \tag{6.5}$$

is given by

$$\xi_{k+1} = \mathcal{F}\xi_k \tag{6.6}$$

Lastly, knowing $\hat{r}$, it is possible to give a useful relationship between the states of the systems in velocity form and the states and inputs of the original system as shown with the following proposition.

Proposition 6.2 *Under Assumption 6.1 the following equation hold:*

$$\begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} = C^* \begin{bmatrix} \xi_k \\ \hat{r}_k \end{bmatrix} = \begin{bmatrix} C_\xi & C_y \end{bmatrix} \begin{bmatrix} \xi_k \\ \hat{r}_k \end{bmatrix} \tag{6.7}$$

where

$$C^* = \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix} \Sigma^{-1} \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & I_m \end{bmatrix} \tag{6.8}$$

and

$$\Sigma = \begin{bmatrix} A - I_n & B \\ CA & CB \end{bmatrix} \tag{6.9}$$

■

Where $C^* \in \mathbb{R}^{(n+m)\times(n+2m)}$, while $C_\xi \in \mathbb{R}^{(n+m)\times(n+m)}$ and $C_y \in \mathbb{R}^{(n+m)\times m}$ are two matrices such that $C^* = \begin{bmatrix} C_\xi & C_y \end{bmatrix}$. Note that the inverse of Σ always exists in view of Assumption 6.1 *iii)* (see the Appendix).

6.1.3 The maximal output admissible set

In the following, system (6.4) will be used to design an MPC algorithm with stability and tracking properties. To this end, (6.5) will be the auxiliary control law used to guarantee stability, see [105]. However, with respect to more standard formulations, the velocity form (6.4) with state and control increments δx_k and δu_k poses the problem of properly reformulating the constraints on the original variables x_k and u_k in terms of constraints on the state ξ_k of the closed-loop system (6.6).

To this regard, the main problem of this section is to define a suitable invariant set for the trajectory ξ_k and at the same time a set of output reference values $\hat{r}$, such that it is guaranteed that the original input and state variables u_k and x_k, respectively, lie in the feasibility sets $\mathbb{U}$ and $\mathbb{X}$.

In view of Proposition 6.2, the issue of computing an invariant set where $(\xi_k, \hat{r})$ must lie in order to guarantee that constraints $(x_k, u_k) \in \mathbb{X} \times \mathbb{U}$ are verified for all k can be cast as the problem of computing the maximal output admissible set (MOAS) for the following auxiliary system:

$$\begin{bmatrix} \xi_{k+1} \\ \hat{r} \end{bmatrix} = \begin{bmatrix} \mathcal{F} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \xi_k \\ \hat{r} \end{bmatrix} \tag{6.10a}$$

$$\begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} = \begin{bmatrix} C_\xi & C_y \end{bmatrix} \begin{bmatrix} \xi_k \\ \hat{r} \end{bmatrix} \tag{6.10b}$$

where (6.10a) and (6.10b) play the role of a state equation and of an output equation, respectively.

Letting

$$F^* = \begin{bmatrix} \mathcal{F} & 0 \\ 0 & I_m \end{bmatrix}$$

assume the following Assumption is fulfilled.

Assumption 6.2

i) The pair (F^*, C^*) is observable.

ii) $\mathbb{X} \times \mathbb{U}$ is a close polytope.

□

Then a polytopic inner approximation $\mathbb{O}_\epsilon$ to the MOAS can be computed in a finite number of steps [59], which is defined as follows:

$$\mathbb{O}_\epsilon = \{(\xi, \hat{r}) \in \mathbb{R}^{n+2m} : C_\xi \mathcal{F}^k \xi + C_y \hat{r} \in \mathbb{X} \times \mathbb{U} \;\; \forall\, k \geq 0 \text{ and } C_y \hat{r} \in \mathbb{X}_\epsilon \times \mathbb{U}_\epsilon\} \tag{6.11}$$

where $\mathbb{X}_\epsilon$ and $\mathbb{U}_\epsilon$ are close and compact sets satisfying $\mathbb{X}_\epsilon \oplus \mathcal{B}_\epsilon^n(0) \subseteq \mathbb{X}$ and $\mathbb{U}_\epsilon \oplus \mathcal{B}_\epsilon^m(0) \subseteq \mathbb{U}$, where $\mathcal{B}_\epsilon^{dim}(0)$ defines a ball of radius ϵ in the $\mathbb{R}^{dim}$ space, and ε can be arbitrarily small. The definition (6.11) enjoys a fundamental property: $\mathbb{O}_\epsilon$ results to be the set of initial conditions $(\xi, \hat{r})$ for the dynamic system (6.10a) such that, during the transient, it is guaranteed that $(x_k, u_k) = C_\xi \mathcal{F}^k \xi + C_y \hat{r} \in \mathbb{X} \times \mathbb{U}$, and the allowed steady state values $(\hat{x}, \hat{u}) = C_y \hat{r}$ belong to the set $\mathbb{X}_\epsilon \times \mathbb{U}_\epsilon \subset \text{int}(\mathbb{X} \times \mathbb{U})$ (in fact, note that when the nominal system is at an equilibrium point, $\hat{\xi}_k = 0$, thus $(\hat{x}, \hat{u}) = C^*(0, \hat{r}) = C_y \hat{r}$, i.e., $C_y = -S^{-1} \begin{bmatrix} 0 & I_m \end{bmatrix}'$. The latter is fundamental for the following results. For details on the computation of $\mathbb{O}_\epsilon$, please see [59] and Chapter 1.

6.1.4 The MPC problem

In this Section we introduce the MPC solution to the tracking problem stated in the previous section.

Similarly to [88], we remark that an arbitrary desired reference output r^o may easily lead to infeasible standard MPC optimization problems. In case this occurs, we assume (in line with the reference governor approach [58]) that the value $\hat{r}_k$ to be considered as the output reference trajectory in the MPC problem at time k is different from r^o and guarantees feasibility. As such, $\hat{r}_k$ will be regarded as an argument of the optimization problem itself, rather than a fixed parameter.

These considerations lead to state the following MPC optimization problem to be solved at each time instant k:

$$V_N^*(r^o, \delta x_k, y_k) = \min_{\hat{r}_k, \delta u_{[k:k+N-1]}} V_N(\hat{r}_k, \delta u_{[k:k+N-1]}; r^o, \delta x_k, y_k) \tag{6.12}$$

where

$$V_N = \sum_{\nu=0}^{N-1} \{ \|\xi_{k+\nu}\|_Q^2 + \|\delta u_{k+\nu}\|_R^2 \} + \|\xi_{k+N}\|_P^2 + \|\hat{r}_k - r^o\|_T^2 \tag{6.13}$$

subject to the dynamic constraint (6.4), to the constraints

$$C^* \begin{bmatrix} \hat{\xi}_{k+\nu} \\ \hat{y}_k \end{bmatrix} \in \mathbb{X} \times \mathbb{U} \tag{6.14}$$

for all $\nu = 1, \ldots, N-1$, and to the terminal constraint

$$\begin{bmatrix} \xi_{k+N} \\ \hat{r}_k \end{bmatrix} \in \mathbb{O}_\epsilon \tag{6.15}$$

As discussed, in the minimization problem (6.12), r^o and $\hat{r}_k$ are the fixed desired reference value and the "artificial" setpoint actually tracked at instant k, respectively. Furthermore, the input sequence $\delta u_{[k:k+N-1]}$ and $\hat{r}_k$ are the decision variables of the problem. The solution to (6.12) is denoted $\delta u_{[k:k+N-1]|k}, \hat{r}_{k|k}$.

Note that, in view of the definition of ξ and ε, the stated problem requires to optimize, at any time instant, not only the future control increments, but also the part of the current state ξ_k which depends on $\hat{r}$.

The weighting matrices $Q \in \mathbb{R}^{(n+m)\times(n+m)}$ and $R \in \mathbb{R}^{m\times m}$ are symmetric and positive definite, and $P \in \mathbb{R}^{(n+m)\times(n+m)}$ is the positive definite solution of the equation

$$\mathcal{F}^T P \mathcal{F} - P = -(Q + \mathcal{K}^T R \mathcal{K}) \tag{6.16}$$

In view of the dimensions of $\mathcal{F}$, $P \in \mathbb{R}^{(n+m)\times(n+m)}$. We can decompose it as follows

$$P = \begin{bmatrix} P_{xx} & P_{xy} \\ P_{xy}^T & P_{yy} \end{bmatrix}$$

where $P_{xx} \in \mathbb{R}^{n\times n}$, $P_{xy} \in \mathbb{R}^{n\times m}$, and $P_{yy} \in \mathbb{R}^{m\times m}$. Matrix $T \in \mathbb{R}^{m\times m}$ is a further tuning knob that must satisfy the constraint

$$T - P_{yy} \succ 0 \tag{6.17}$$

Some comments are in order. Constraints (6.14) are equivalent to require that $x_{k+\nu} \in \mathbb{X}$ for all $\nu = 1, \ldots, N-1$ and $u_{k+\nu} \in \mathbb{U}$ for all $\nu = 0, \ldots, N-2$. Furthermore, (6.15) implies that if, for all times $k+i$, $i \geq N$, the system (6.4) is controlled using the auxiliary state-feedback control law (6.5) then, in view of the invariance properties of $\mathbb{O}_\epsilon$, $x_{k+N+i} \in \mathbb{X}$ and $u_{k+N+i-1} \in \mathbb{U}$ are verified for all $i \geq 0$.

The main convergence result can now be stated.

Theorem 6.1 *Let Assumption 6.1 be verified and the design parameters Q, R, P, T, $\mathbb{X}\times\mathbb{U}$, and $\mathbb{O}_\epsilon$ be chosen as specified. Then, if at time $k=0$ a feasible solution to the optimization problem* (6.12)-(6.15) *exists, the resulting MPC control law asymptotically steers the nominal system output $\hat{y}_k$ to the admissible set-point r_{ad}, where*

$$r_{ad} = \underset{C_y y \in \mathbb{X}_\epsilon \times \mathbb{U}_\epsilon}{\operatorname{argmin}} \|y - r^o\|_T^2 \tag{6.18}$$

Moreover, $\delta u_{k|k} \to 0$ as $k \to \infty$, and the constraints $(x_k, u_k) \in \mathbb{X}\times\mathbb{U}$ are fulfilled for all $k \geq 0$. ■

6.2 MPC for offset-free tracking: disturbed systems

6.2.1 Statement of the problem and transformation in velocity-form

Assume system (6.1) is affected by a disturbance

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Cx_k \end{aligned} \tag{6.19}$$

where the disturbance $w_k \in \mathbb{W}$ is unknown, but bounded, and $\mathbb{W}$ includes the origin. Concerning system (6.19), we suppose that Assumption 6.1 is again fulfilled.

If, in addition to $\delta x_k = x_k - x_{k-1}$, $\varepsilon_k = y_k - \hat{r}$, $\delta u_k = u_k - u_{k-1}$, $\xi_k = (\delta x_k, \varepsilon_k)$, $\mathcal{A}$ and $\mathcal{B}$ (see (6.3)), we introduce also

$$\mathcal{B}_w = \begin{bmatrix} I_n \\ C \end{bmatrix}, \, \delta w_k = w_k - w_{k-1}, \, d_k = \mathcal{B}_w \delta w_k \tag{6.20}$$

system (6.19) can be reformulated as

$$\xi_{k+1} = \mathcal{A}\xi_k + \mathcal{B}\delta u_k + d_k \tag{6.21}$$

Associated with system (6.19), the following nominal system is defined.

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + B\hat{u}_k \\ \hat{y}_k &= C\hat{x}_k \end{aligned} \tag{6.22}$$

Letting $\delta\hat{x}_k = \hat{x}_k - \hat{x}_{k-1}$, $\hat{\varepsilon}_k = \hat{y}_k - \hat{r}$, $\delta\hat{u}_k = \hat{u}_k - \hat{u}_{k-1}$, $\hat{\xi}_k = (\delta\hat{x}_k, \hat{\varepsilon}_k)$, the nominal counterpart of system (6.21) is

$$\hat{\xi}_{k+1} = \mathcal{A}\hat{\xi}_k + \mathcal{B}\delta\hat{u}_k \tag{6.23}$$

According to [107] assume that, for all k, for the real system (6.21) the following control law is considered

$$\delta u_k = \delta\hat{u}_k + \mathcal{K}(\xi_k - \hat{\xi}_k) \tag{6.24}$$

where the gain $\mathcal{K}$ is defined a priori such that $\mathcal{F} = \mathcal{A} + \mathcal{B}\mathcal{K}$ is Schur (see Proposition 6.1). From (6.21), (6.23), and (6.24) it directly follows that

$$(\xi_{k+1} - \hat{\xi}_{k+1}) = \mathcal{F}(\xi_k - \hat{\xi}_k) + d_k \tag{6.25}$$

which is independent of the control input $\delta\hat{u}_k$ to the nominal system.

6.2.2 The maximal output admissible set computed using tightened constraints

One of the main obstacles for the use of the velocity form is due to the difficulty to transform the state and control constraints for the real disturbed system (6.19) in terms of equivalent constraints on the enlarged nominal state $\hat{\xi}_k$, of the reference signal $\hat{r}$. To this end, the approach outlined in the following consists in three logical steps:

I) we write $(x_k - \hat{x}_k, u_{k-1} - \hat{u}_{k-1})$ in terms of $(\xi_k - \hat{\xi}_k, w_{k-1})$;

II) we show that there exists a robust positively invariant (RPI) set (denoted $\tilde{\mathbb{Z}}$) where $(\xi_k - \hat{\xi}_k, w_{k-1})$ is guaranteed to lie;

III) we compute, from $\tilde{\mathbb{Z}}$, the tightened set $\hat{\mathbb{X}}_{\mathbb{U}}$ where $(\hat{x}_k, \hat{u}_{k-1})$ must lie to verify the constraints on x_k and u_k.

As for *I)*, the following result will be used.

Proposition 6.3 *Under Assumption 6.1, the following equations hold*

$$\begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} = C^* \begin{bmatrix} \xi_k \\ \hat{r}_k \end{bmatrix} + C_w w_{k-1} \tag{6.26}$$

$$\begin{bmatrix} \hat{x}_k \\ \hat{u}_{k-1} \end{bmatrix} = C^* \begin{bmatrix} \hat{\xi}_k \\ \hat{r}_k \end{bmatrix} \tag{6.27}$$

where $C^* = \begin{bmatrix} C_\xi & C_y \end{bmatrix}$ *is defined in equation* (6.8) *and*

$$C_w = \begin{bmatrix} I_n \\ 0 \end{bmatrix} - \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix} \Sigma^{-1} \begin{bmatrix} I_n \\ C \end{bmatrix} \tag{6.28}$$

■

From (6.26) and (6.27), the difference between real and nominal states and inputs is

$$\begin{bmatrix} x_k - \hat{x}_k \\ u_{k-1} - \hat{u}_{k-1} \end{bmatrix} = \begin{bmatrix} C_\xi & C_w \end{bmatrix} \begin{bmatrix} \xi_k - \hat{\xi}_k \\ w_{k-1} \end{bmatrix} \tag{6.29}$$

Concerning *II)*, defining $\omega_k = w_{k-1}$ one can write, from (6.20) and (6.25)

$$\begin{bmatrix} \xi_{k+1} - \hat{\xi}_{k+1} \\ \omega_{k+1} \end{bmatrix} = \tilde{\mathcal{F}} \begin{bmatrix} \xi_k - \hat{\xi}_k \\ \omega_k \end{bmatrix} + \tilde{\mathcal{B}}_w w_k \tag{6.30}$$

where

$$\tilde{\mathcal{F}} = \begin{bmatrix} \mathcal{F} & -\mathcal{B}_w \\ 0 & 0 \end{bmatrix}, \tilde{\mathcal{B}}_w = \begin{bmatrix} \mathcal{B}_w \\ I_n \end{bmatrix}$$

From (6.30), since the Schureness of $\tilde{\mathcal{F}}$ follows from the Schureness of $\mathcal{F}$, in view of [132] it is possible to compute the minimal RPI set $\tilde{\mathbb{Z}}$ for $(\xi_k - \hat{\xi}_k, \omega_k)$ as

$$\tilde{\mathbb{Z}} = \bigoplus_{\nu=0}^{\infty} \tilde{\mathcal{F}}^{\nu} \tilde{\mathcal{B}}_w \mathbb{W} \tag{6.31}$$

Therefore, in view of (6.31), if $(\xi_k - \hat{\xi}_k, \omega_k) \in \tilde{\mathbb{Z}}$ and $w_{k+j} \in \mathbb{W}$ for all $j \geq 0$, it holds that $(\xi_{k+j}, \omega_{k+j}) \in (\hat{\xi}_{k+j}, 0) \oplus \tilde{\mathbb{Z}}$ for all $j \geq 1$. An algorithm for the computation of a polytopic robustly invariant outer approximation of the latter set can be devised in line with [132], as shown in Chapter 1.

Remark 6.1 *An alternative choice is to set* $d_k \in \mathbb{D} = \mathcal{B}_w (\mathbb{W} \oplus (-\mathbb{W}))$ *in* (6.21) *and to compute a RPI set for* (6.25) *as* $\mathbb{Z} = \bigoplus_{\nu=0}^{\infty} \mathcal{F}^{\nu}\mathbb{D}$, *such that if* $(\xi_k - \hat{\xi}_k) \in \mathbb{Z}$ *then* $\xi_{k+j} \in \hat{\xi}_{k+j} \oplus \mathbb{Z}$ *for all* $j \geq 1$ *and for all possible disturbance realizations. Such a simpler but more conservative choice corresponds to replace* $\tilde{\mathbb{Z}}$ *in* (6.31) *with* $\mathbb{Z} \times \mathbb{W}$ *in the following developments.*

Point *III)*. Since $\omega_k = w_{k-1}$ and in view of (6.31) one has that, if $(\xi_k - \hat{\xi}, \omega_k) \in \tilde{\mathbb{Z}}$, then

$$\begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} = \begin{bmatrix} \hat{x}_k \\ \hat{u}_{k-1} \end{bmatrix} + \begin{bmatrix} C_\xi & C_w \end{bmatrix} \begin{bmatrix} \xi_k - \hat{\xi} \\ \omega_k \end{bmatrix} \in \begin{bmatrix} \hat{x}_k \\ \hat{u}_{k-1} \end{bmatrix} \oplus \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathbb{Z}} \tag{6.32}$$

Therefore, in order to guarantee that $(x_k, u_{k-1}) \in \mathbb{X} \times \mathbb{U}$, the following tightened constraints must be satisfied for $(\hat{x}_k, \hat{u}_{k-1})$

$$(\hat{x}_k, \hat{u}_{k-1}) \in \hat{\mathbb{X}}_{\mathbb{U}} \tag{6.33}$$

where $\hat{\mathbb{X}}_{\mathbb{U}}$ is the set, with non-empty interior, that verifies

$$\hat{\mathbb{X}}_{\mathbb{U}} \subseteq (\mathbb{X} \times \mathbb{U}) \ominus \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathbb{Z}} \tag{6.34}$$

In terms of ξ, $\hat{r}$ and in view of (6.27), equation (6.33) can be written as

$$C^* \begin{bmatrix} \hat{\xi}_k \\ \hat{r}_k \end{bmatrix} \in \hat{\mathbb{X}}_{\mathbb{U}} \tag{6.35}$$

In order to design an MPC stabilizing feedback control law, it is advisable to define the auxiliary control law

$$\delta \hat{u}_k = \mathcal{K}\hat{\xi}_k \tag{6.36}$$

to compute an invariant set for the nominal closed-loop system (6.23), (6.36), together with a set of output reference values $\hat{r}$, where $(\hat{\xi}_k, \hat{r})$ must lie in order to guarantee that constraints $(\hat{x}_k, \hat{u}_{k-1}) \in \hat{\mathbb{X}}_{\mathbb{U}}$ are verified for all k. This requires to compute the maximal output admissible set (MOAS) $\hat{\mathbb{O}}$ for the auxiliary nominal system with the dynamic equation

$$\begin{bmatrix} \hat{\xi}_{k+1} \\ \hat{r} \end{bmatrix} = \begin{bmatrix} \mathcal{F} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \hat{\xi}_k \\ \hat{r} \end{bmatrix} \tag{6.37}$$

and the output equation (6.27).

Provided that Assumption 6.2 is verified, an invariant, polytopic inner approximation $\hat{\mathbb{O}}_\epsilon$ of the MOAS can be computed in a finite number of steps [59]. Specifically, $\hat{\mathbb{O}}_\epsilon$ is defined as follows

$$\hat{\mathbb{O}}_\epsilon = \{(\hat{\xi}, \hat{r}) \in \mathbb{R}^{n+2m} : C_\xi \mathcal{F}^k \hat{\xi} + C_y \hat{r} \in \hat{\mathbb{X}}_{\mathbb{U}} \ \ \forall\, k \geq 0 \text{ and } C_y \hat{r} \in \hat{\mathbb{X}}_{\mathbb{U}}(\epsilon)\} \tag{6.38}$$

where $\hat{\mathbb{X}}_{\mathbb{U}}(\epsilon)$ is a close and compact set satisfying $\hat{\mathbb{X}}_{\mathbb{U}}(\epsilon) \oplus \mathcal{B}^n_\epsilon(0) \subseteq \hat{\mathbb{X}}_{\mathbb{U}}$, and ϵ can be arbitrarily small. Note that $\hat{\mathbb{O}}_\epsilon \subset \hat{\mathbb{O}}$ [59]. It is finally worth remarking that, in view of (6.38), a reference output $\hat{r}$ is feasible if $C_y \hat{r} \in \hat{\mathbb{X}}_{\mathbb{U}}(\epsilon)$.

6.2.3 The MPC problem

The case of bounded variable disturbance

In the formulation of the robust MPC problem and according to [107], at any time k the optimization problem is stated with reference to the nominal system (6.23) and its initial condition $\hat{\xi}_k$ is considered as an optimization variable, besides the future nominal input sequence $\delta\hat{u}_{[k:k+N-1]}$. Being $\hat{\xi}_k = (\delta\hat{x}_k, (\hat{y}_k - \hat{r}_k))$, this means that both $\delta\hat{x}_k$ and $\hat{y}_k$ are arguments of the optimization problem. As for possibly infeasible reference signals r^o, and following [88] also in this case, the value $\hat{r}_k$ to be considered as the output reference trajectory in the MPC problem at time k is regarded as an argument of the optimization problem itself and is computed in such a way that feasibility is guaranteed.

The MPC problem to be solved at each time instant k is:

$$V_N^*(r^o, \delta x_k, y_k) = \min_{\delta \hat{x}_k, \hat{y}_k, \hat{r}_k, \delta \hat{u}_{[k:k+N-1]}} V_N(\delta \hat{x}_k, \hat{y}_k, \hat{r}_k, \delta \hat{u}_{[k:k+N-1]}; r^o, \delta x_k, y_k) \tag{6.39}$$

where

$$V_N = \sum_{\nu=0}^{N-1} \{ \|\hat{\xi}_{k+\nu}\|_Q^2 + \|\delta \hat{u}_{k+\nu}\|_R^2 \} + \|\hat{\xi}_{k+N}\|_P^2 + \|\hat{r}_k - r^o\|_T^2$$

subject to the dynamic constraint (6.23), to the constraints

$$\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} (\xi_k - \hat{\xi}_k) \in \begin{bmatrix} 0 \\ -w_{k-1} \end{bmatrix} \oplus \tilde{\mathbb{Z}} \tag{6.40}$$

where, at time k, w_{k-1} can be explicitly computed from x_{k-1}, u_{k-1}, and x_k using (6.19), to

$$C^* \begin{bmatrix} \hat{\xi}_{k+\nu} \\ \hat{r}_k \end{bmatrix} \in \hat{\mathbb{X}}_{\mathbb{U}} \tag{6.41}$$

for all $\nu = 1, \ldots, N-1$, and to the terminal constraint

$$\begin{bmatrix} \hat{\xi}_{k+N} \\ \hat{r}_k \end{bmatrix} \in \hat{\mathbb{O}}_\epsilon \tag{6.42}$$

According to the receding horizon approach, at any time instant the optimal solution $\delta \hat{x}_{k|k}, \hat{y}_{k|k}, \hat{r}_{k|k}, \delta \hat{u}_{[k:k+N-1]|k}$ is found and the control law (6.24) is applied with $\delta \hat{u}_k = \delta \hat{u}_{k|k}$ and $\hat{\xi}_k = (\delta \hat{x}_{k|k}, (\hat{y}_{k|k} - \hat{r}_{k|k}))$.

A proper selection of the tuning parameters allows one to prove the convergence properties of the proposed control algorithm. In particular, the weighting matrices Q and R are assumed to be symmetric and positive definite, while P and T are defined as in equations 6.16 and 6.17.

Theorem 6.2 *Let Assumption 6.1 be verified and the design parameters Q, R, P, T, $\tilde{\mathbb{Z}}$, $\hat{\mathbb{X}}_{\mathbb{U}}$, and $\hat{\mathbb{O}}_\epsilon$ be chosen as specified. Then, if at time $k = 0$ a feasible solution to the optimization problem* (6.39)- (6.42) *exists, the resulting MPC control law asymptotically steers the nominal system output $\hat{y}_k$ to the admissible set-point r_{ad}, where*

$$r_{ad} = \underset{C_y y \in \hat{\mathbb{X}}_{\mathbb{U}}(\epsilon)}{\operatorname{argmin}} \|y - r^o\|_T^2 \tag{6.43}$$

Moreover, $\delta \hat{u}_{k|k} \to 0$ as $k \to \infty$, and the constraints $(x_k, u_k) \in \mathbb{X} \times \mathbb{U}$ are fulfilled for all $k \geq 0$. ■

This result guarantees that, for any feasible initial state, the proposed method asymptotically steers the nominal system output to the desired reference value r^o if r^o is admissible, or to the admissible output r_{ad} if r^o is not admissible. Since the controller always keeps $(\xi_k - \hat{\xi}_k)$ bounded for all $k \geq 0$, in view of (6.30), the output of the real system is driven in a neighborhood of r^o (or of r_{ad}). Moreover, the following corollary accounts for the case the disturbance converges to a constant value.

Corollary 6.1 *Let Assumption 6.1 be verified, the design parameters Q, R, P, T, $\tilde{\mathbb{Z}}$, $\hat{\mathbb{X}}_{\mathbb{U}}$, and $\hat{\mathbb{O}}_\epsilon$ be chosen as specified. If $w_k \to \bar{w}$ as $k \to \infty$ and if, at time $k = 0$, a feasible solution to the optimization problem* (6.39)- (6.42) *exists, then the resulting MPC control law asymptotically steers the system* (6.19) *output y_k to the admissible set-point r_{ad} given by* (6.43) *and the constraints $(x_k, u_k) \in \mathbb{X} \times \mathbb{U}$ are fulfilled for all $k \geq 0$.* ■

The case of constant unknown disturbance

When the disturbance $w_k \in \mathbb{W}$ is unknown but constant, i.e., $w_k = \bar{w}$, a standard and much simpler formulation of the MPC problem can be used. In fact, being $\delta w_k = 0$, the velocity forms of the system (6.21) and of its nominal model (6.23) coincide. Therefore $\tilde{\mathbb{Z}} = \{0\} \times \mathbb{W}$ which means that, provided that $\xi_k = \hat{\xi}_k$, $\xi_{k+j} = \hat{\xi}_{k+j}$ for all $\bar{w} \in \mathbb{W}$. In view of this, the robust tube-based MPC formulation is no longer needed and the optimization problem can be directly formulated on the plant variables ξ_k and δu_k. Moreover, to guarantee $(x_k, u_{k-1}) \in \mathbb{X} \times \mathbb{U}$, the set $\hat{\mathbb{X}}_{\mathbb{U}}$ in (6.33), (6.35), and (6.38) must verify

$$\hat{\mathbb{X}}_{\mathbb{U}} \subseteq (\mathbb{X} \times \mathbb{U}) \ominus C_w \mathbb{W} \tag{6.44}$$

which is the reformulation of (6.34) in this simpler case.

In summary, the optimization problem to be solved on-line has the following standard formulation of (non-robust) MPC.

$$V_N^*(r^o, \delta x_k, y_k) = \min_{\hat{r}_k, \delta u_{[k:k+N-1]}} V_N(\hat{r}_k, \delta u_{[k:k+N-1]}; r^o, \delta x_k, y_k) \tag{6.45}$$

where

$$V_N = \sum_{\nu=0}^{N-1} \{\|\xi_{k+\nu}\|_Q^2 + \|\delta u_{k+\nu}\|_R^2\} + \|\xi_{k+N}\|_P^2 + \|\hat{r}_k - r^o\|_T^2$$

subject to the dynamic constraint (6.21) with $d_k = 0$, to the constraints (6.41), for all $\nu = 1, \dots, N-1$, and to the terminal constraint (6.42).

Once the optimization problem has been solved at any time k and its optimal solution $\hat{r}_{k|k}, \delta u_{[k:k+N-1]|k}$ has been computed, the input applied to the real system is $u_k = u_{k-1} + \delta u_{k|k}$.

With arguments similar to those used for the general case, the following corollary of Theorem 6.2 can be stated.

Corollary 6.2 *Let Assumption 6.1 be verified, the design parameters Q, R, P, T, $\tilde{\mathbb{Z}}$, $\hat{\mathbb{X}}_{\mathbb{U}}$, and $\hat{\mathbb{O}}_\epsilon$ be chosen as specified. Consider the case of constant disturbances, i.e., $w_k = \bar{w}$ for all $k \geq 0$. Then, if at time $k = 0$ a feasible solution to the optimization problem* (6.45) *exists, the resulting MPC control law asymptotically steers the system* (6.19) *output y_k to the admissible set-point r_{ad} given by* (6.43) *and the constraints $(x_k, u_k) \in \mathbb{X} \times \mathbb{U}$ are fulfilled for all $k \geq 0$.* ■

It is worth recalling that, in the definition (6.43) of r_{ad}, the set $\hat{\mathbb{X}}_{\mathbb{U}}(\varepsilon)$ appears, which is defined on the basis of $\hat{\mathbb{X}}_{\mathbb{U}}$, which now must verify the condition (6.44).

The case of model uncertainties

A possible application of the results presented in this paper is the case where the disturbance w_k stems from model perturbations. Specifically, assume that the real system matrices are $A_r = A + A_\delta$ and $B_r = B + B_\delta$, with A_δ and B_δ unknown, but bounded. Since $x_k \in \mathbb{X}$ and $u_k \in \mathbb{U}$, the control problem can be stated as in (6.19), where the bounded disturbance is

$$w_k = A_\delta x_k + B_\delta u_k \tag{6.46}$$

We assume, according to [79], that

$$\begin{bmatrix} A_\delta & B_\delta \end{bmatrix} = \sum_{i=1}^{n_\alpha} \alpha_i \begin{bmatrix} A_i & B_i \end{bmatrix} \tag{6.47}$$

where $\alpha_i > 0$ are unknown time-invariant parameters satisfying $\sum_{i=1}^{n_A} \alpha_i = 1$. Define also

$$\mathcal{A}_\delta = \begin{bmatrix} A_\delta & 0 \\ CA_\delta & 0 \end{bmatrix} \quad \mathcal{B}_\delta = \begin{bmatrix} B_\delta \\ CB_\delta \end{bmatrix}$$

The following corollary of Theorem 6.2 can be stated, provided that all the ingredients of the MPC problem (6.39) can be properly defined, including $\hat{\mathbb{X}}_{\mathbb{U}}$.

Corollary 6.3 *Let Assumption 6.1 be verified, the design parameters* Q, R, P, T, $\tilde{\mathbb{Z}}$, $\hat{\mathbb{X}}_{\mathbb{U}}$, *and* $\hat{\mathbb{O}}_\epsilon$ *be chosen as specified, and model uncertainties be of the type* (6.46). *Furthermore, assume that the matrix* $\mathcal{F}_r = \mathcal{A} + \mathcal{A}_\delta + (\mathcal{B} + \mathcal{B}_\delta)\mathcal{K}$ *is Schur for all pairs* $\begin{bmatrix} A_\delta & B_\delta \end{bmatrix}$ *satisfying* (6.47). *Then, if at time* $k = 0$ *a feasible solution to the optimization problem* (6.39)- (6.42) *exists, the resulting MPC control law asymptotically steers the system* (6.19) *output* y_k *to the admissible set-point* r_{ad} *given by* (6.43) *and the constraints* $(x_k, u_k) \in \mathbb{X} \times \mathbb{U}$ *are fulfilled for all* $k \geq 0$. ■

Methods for designing the gain matrix $\mathcal{K}$ satisfying the assumptions of the corollary have been thoroughly studied in the past, also in the context of MPC, see, e.g., [35, 79].

One of the key points of the proposed approach lies in the existence of a proper set $\hat{\mathbb{X}}_{\mathbb{U}}$. According to its definition, it is computed as follows:

I) compute $\mathbb{W} = \text{convH}\left(\bigcup_{i=1}^{n_\alpha}\{\begin{bmatrix} A_i & B_i \end{bmatrix} \mathbb{X} \times \mathbb{U}\}\right)$, where convH denotes the convex hull.

II) Compute $\tilde{\mathbb{Z}}$ using (6.31).

III) Compute $\hat{\mathbb{X}}_{\mathbb{U}}$ using (6.34).

Note that step *III)* can be carried out provided that

$$\begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathbb{Z}} \subset \mathbb{X} \times \mathbb{U} \tag{6.48}$$

which, in view of the fact that $\tilde{\mathbb{Z}}$ is in turn computed from $\mathbb{X} \times \mathbb{U}$, is indeed a small gain requirement. In fact, small perturbations (i.e., small A_δ and B_δ) result in a small $\tilde{\mathbb{Z}}$. A simple sufficient condition guaranteeing (6.48) can be stated in case $\mathbb{X} \times \mathbb{U}$ is a zonotope i.e., a centrally symmetric convex polytope, defined in the following alternative ways:

$$\begin{aligned} \mathbb{X} \times \mathbb{U} &= \left\{(x, u) \in \mathbb{R}^{n+m} : \begin{bmatrix} x \\ u \end{bmatrix} = \Xi_{x,u} v \text{ where } \|v\|_\infty \leq 1\right\} \\ &= \left\{(x, u) \in \mathbb{R}^{n+m} : h_i^T \begin{bmatrix} x \\ u \end{bmatrix} \leq 1 \text{ for all } i = 1, \ldots, r\right\} \end{aligned} \tag{6.49}$$

for suitably-specified matrices $\Xi_{x,u}$ and vectors h_i, $i = 1, \dots, r$. Also define $H_{x,u}$ as

$$H_{x,u} = \begin{bmatrix} h_1 & \dots & h_r \end{bmatrix} \tag{6.50}$$

The following proposition provides a sufficient (but conservative) condition for the small gain one (6.48).

Proposition 6.4 *Assume that*

$$\sum_{\nu=0}^{\infty} \|\tilde{\mathcal{F}}^{\nu}\|_{\infty} \|H_{x,u}^T \begin{bmatrix} C_\xi & C_w \end{bmatrix}\|_{\infty} \max_{j=1,\dots,n_\alpha} \|\tilde{\mathcal{B}}_w \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u}\|_{\infty} < 1 \tag{6.51}$$

then $\hat{\mathbb{X}}_{\mathbb{U}}$ can be properly defined. ■

Checking (6.51) is straightforward: $H_{x,u}$ and $\Xi_{x,u}$ are given once sets $\mathbb{X}$ and $\mathbb{U}$ are defined, and all the matrices appearing in (6.51) are defined in the problem setup. Furthermore, since $\tilde{\mathcal{F}}$ is Schur, there exist constants $\alpha \geq 1$, $\beta \in [0, 1)$ such that $\|\tilde{\mathcal{F}}^{\nu}\|_{\infty} \leq \alpha\beta^{\nu}$. Therefore, by replacing $\sum_{\nu=0}^{\infty} \|\tilde{\mathcal{F}}^{\nu}\|_{\infty}$ with $\alpha/(1-\beta)$, we obtain a sufficient condition for (6.51).

6.3 Simulation examples

For assessing the performances of the nominal algorithm presented in Section 6.1, consider the following benchmark problem, originally proposed in [87] and [88]. The nominal dynamic model of the plant is

$$\begin{array}{lcl} x_{k+1} & = & Ax_k + Bu_k \\ y_k & = & Cx_k \end{array} \tag{6.52}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that, differently from [87] and [88], here both states correspond to outputs. The constraints on the states and inputs are $\|x\|_{\infty} \leq 5$ and $\|u\|_{\infty} \leq 3$, respectively. System (6.52) is used for the synthesis of the controller. Assume also that the plant is affected by both model perturbations and disturbances: the real transition and output matrices are A_{real} and B_{real}, respectively, and an additional constant unknown disturbance $\bar{w}$ is present. Therefore the evolution of the system obeys to

$$\begin{array}{rcl} x_{k+1} & = & A_{real}x_k + B_{real}u_k + \bar{w} \\ y_k & = & Cx_k \end{array} \tag{6.53}$$

where

$$A_{real} = \begin{bmatrix} 0.8 & 1.1 \\ 0.05 & 0.9 \end{bmatrix}, \; B_{real} = \begin{bmatrix} 0.55 & 0.05 \\ -0.05 & 0.9 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \bar{w} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

We set $Q = \text{diag}(0.1, 0.1, 1, 1)$, $R = \text{diag}(0.1, 0.1)$, and $N = 10$. Matrices $\mathcal{K}$ and P have been computed according to the LQ synthesis criterion:

$$\begin{array}{rcl} \mathcal{K} & = & \begin{bmatrix} -1.69 & -1.71 & -1.22 & 0.30 \\ -0.05 & -0.10 & 0.02 & -0.73 \end{bmatrix} \\ P & = & \begin{bmatrix} 0.44 & 0.34 & 0.24 & -0.06 \\ 0.34 & 0.54 & 0.24 & 0.01 \\ 0.24 & 0.24 & 1.41 & 0.10 \\ -0.06 & 0.01 & 0.10 & 1.37 \end{bmatrix} \end{array}$$

Matrix P_{yy} is therefore

$$P_{yy} = \begin{bmatrix} 1.41 & 0.10 \\ 0.10 & 1.37 \end{bmatrix}$$

Matrix T has been set to $T = 10P_{yy}$.

In the simulations, the reference trajectory for y_2 is constant and equal to zero. The first output y_1, instead, has a piece-wise constant reference trajectory, taking values 5, -5.5 and 3. The results are shown in Figure 6.1. In Figure 6.2 we show the trajectories of the input variables.

Importantly, note that every admissible reference signal is tracked without error and that the effects of the model perturbation and of the unknown disturbance have been rejected, thanks to the particular offset-free MPC formulation proposed in this paper. On the other hand, the setpoint -5.5 it is not admissible and the output is steered to the closest admissible value.

Although constraint satisfaction is not guaranteed for perturbed systems by the results presented in Section 6.1, they are accidentally satisfied in this example.

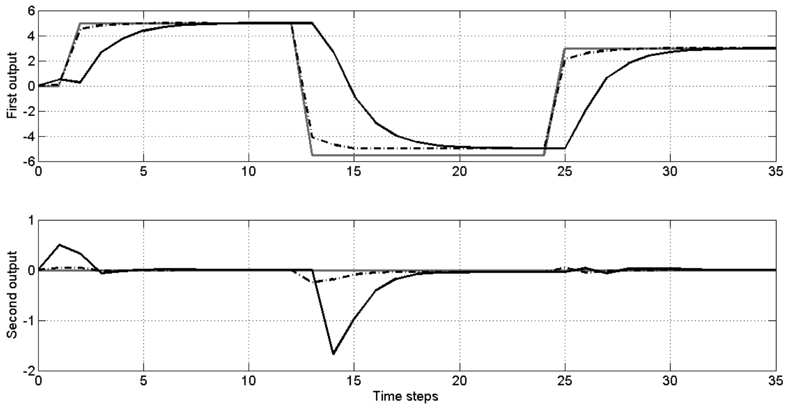

Figure 6.1: Trajectories of the real plant output variables y_1 (above) and y_2 (below). Grey solid line: real desired references ($\hat{y}_{1,2}$); black dash-dotted line: artificial references ($\bar{y}_{1,2}$); black solid line: outputs of the system ($y_{1,2}$).

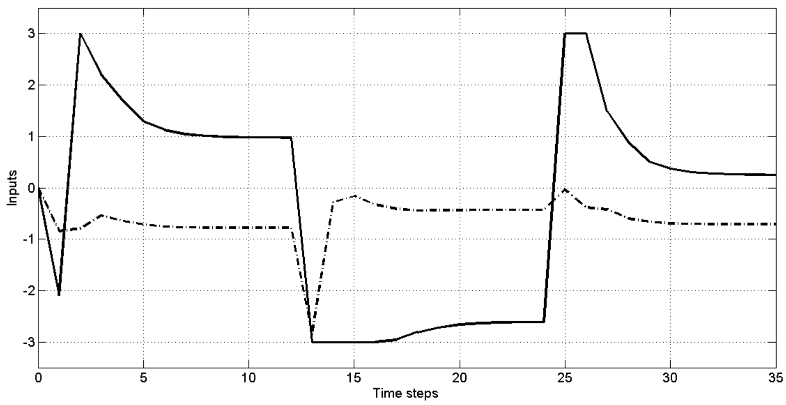

Figure 6.2: Trajectories of the inputs applied to the real system. Blak solid line: first input; black dash-dotted line: second input.

To test the robust version of the proposed algorithm, consider again this first benchmark problem. The dynamic model is given by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Cx_k \end{aligned} \tag{6.54}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The constraints on the states and inputs are again $\|x\|_\infty \leq 5$ and $\|u\|_\infty \leq 3$, while the variable disturbance w is such that $\|w\|_\infty \leq 0.1$. For the synthesis of the controller it has been set $N = 5$, $Q = I_4$, $R = 0.01 I_2$, while the matrices $\mathcal{K}$ and P have been computed with the LQ_∞ synthesis criterion. The weighting matrix T has been set to $T = 10P_{yy}$.

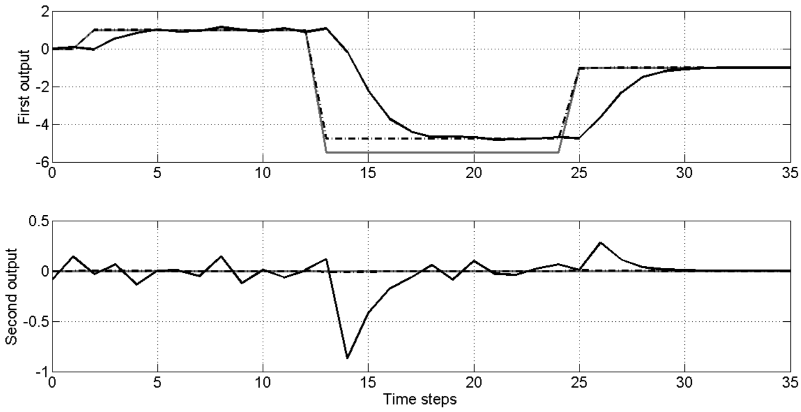

Figure 6.3: Trajectories of the real plant output variables y_1 (above) and y_2 (below). Grey solid line: real desired references ($r^o_{1,2}$); black dash-dotted line: artificial references ($\hat{r}_{1,2}$); black solid line: outputs of the system ($y_{1,2}$). The disturbance w_k is constant for $k \geq 24$.

In the reported simulations, the reference trajectory for y_2 is constant and equal to zero. The first output y_1, instead, has a piecewise constant reference trajectory, taking values 5, -5.5 and 3. The disturbance is randomly varying in the set $\mathbb{W}$ for $k \in [0, 23]$, while it is kept constant for $k \in [24, 35]$. The results are shown in Figure 6.3, while Figure 6.4 shows the trajectories of the input variables.

Note that every admissible reference signal is tracked, without error when $w_k = \bar{w}$, and that the effects of the unknown disturbance have

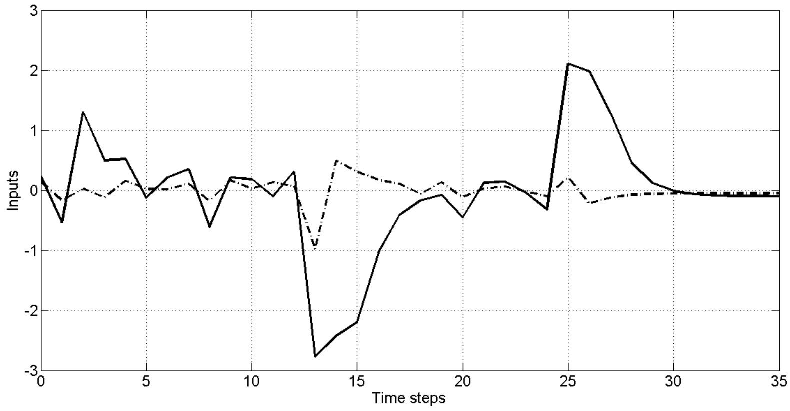

Figure 6.4: Trajectories of the inputs applied to the real system. Black solid line: first input; black dash-dotted line: second input.

been rejected. On the other hand, the setpoint −5.5 is not admissible and the output is steered to the closest admissible value. Constraints on states and inputs are always fulfilled thanks to the robust approach.

Let's now consider a second example, proposed in [69, 123], to evaluate the application of the nominal formulation of Section 6.1: its simple implementation and useful characteristics could in fact be of great interest from an industrial point of view. A continuous stirred-tank

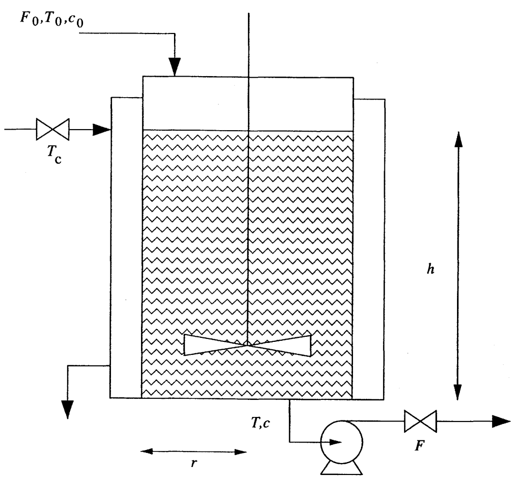

Figure 6.5: Sketch of a continuous stirred-tank reactor.

reactor (Figure 6.5) is considered: in the liquid phase, an irreversible reaction occurs. The temperature inside the reactor is regulated with external cooling. The nonlinear equations which give the dynamics of the CSTR are

$$\frac{dc}{dt} = \frac{F_0(c_0 - c)}{\pi r^2 h} - k_0 c\mathtt{e}(-\frac{E}{RT}) \tag{6.55a}$$

$$\frac{dT}{dt} = \frac{F_0(T_0 - T)}{\pi r^2 h} - \frac{\Delta H}{\rho C_p} k_0 c\mathtt{e}(-\frac{E}{RT}) + \frac{2Uh}{r\rho C_p}(T_c - T) \tag{6.55b}$$

$$\frac{dh}{dt} = \frac{F_0 - F}{\pi r^2} \tag{6.55c}$$

The three state variables, all measurable, are the level of the tank, h, the molar concentration of the reagent, c and the reactor temperature T. The first two are taken as the controlled outputs. The manipulated inputs to the system are the outlet flow rate, F, and the coolant liquid temperature, T_c. The inlet flow rate is assumed to be an unknown disturbance. In Table 6.1 the model parameters are reported. The system is linearized in correspondence of the stable steady state where $h^s = 0.659$ m, $c^s = 0.877$ mol l^{-1}, $T^s = 324.5$ K, $F^s = 100$ l min^{-1} and $T_c^s = 300$ l, having denoted with apex s the steady state. As shown in [123], using a sampling time of 1 min, the following linear discrete-time model is obtained

Table 6.1: CSTR parameters

Parameter	Nominal value
F_0	100 l min^{-1}
T_0	350 K
c_0	1 mol l^{-1}
r	0.219 m
k_0	7.2 × 10^{10} min^{-1}
E/R	8.750 K
U	915.6 W K m^{-2}
ρ	1 kg l^{-1}
C_p	0.239 J K g^{-1}
ΔH	-5 × 10^4 J mol^{-1}

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Cx_k \end{aligned} \tag{6.56}$$

with

$$A = \begin{bmatrix} 0.2511 & -3.368e^{-3} & -7.056e^{-4} \\ 11.06 & 0.3296 & -2.545 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} -5.426e^{-3} & 1.530e^{-5} \\ 1.297 & 0.1218 \\ 0 & -6.592e^{-2} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$w_k = \begin{bmatrix} -1.762e^{-5} \\ 7.784e^{-2} \\ 6.592e^{-2} \end{bmatrix} p_k$$

where $x = ((c - c^s), (T - T^s), (h - h^s))$, $u = ((T_c - T_c^s), (F - F^s))$, $y = ((c - c^s), (h - h^s))$ and $p = F_0 - F_0^s$.

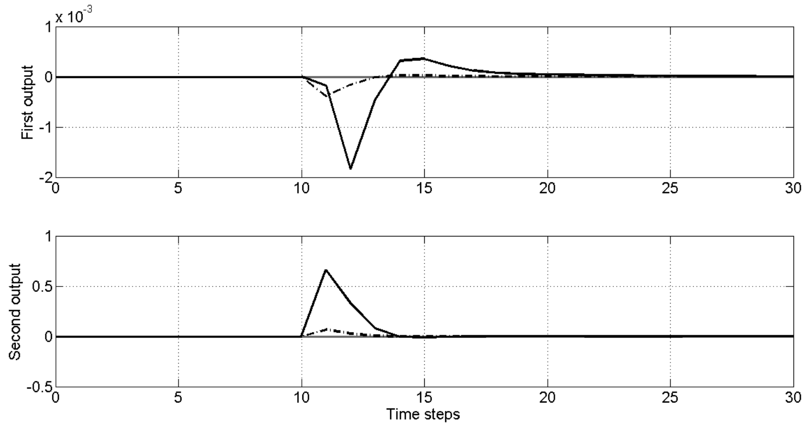

Figure 6.6: Trajectories of the plant output variables y_1 (above) and y_2 (below). Grey solid line: real desired references ($y^r_{1,2}$); black dash-dotted line: artificial references ($\bar{y}_{1,2}$); black solid line: outputs of the system ($y_{1,2}$).

At time $k = 10$ (i.e., $t = 10$ min), a disturbance $p = 10$ enters the plant, and the goal of the controller is to track the steady state values of the outputs (i.e., to regulate the outputs of the linearized system to zero).

The length of the prediction horizon is $N = 5$, while we put $Q = \text{diag}(0.01, 0.01, 0.01, 10, 10)$ and $R = \text{diag}(0.01, 0.01)$. Matrix $\mathcal{K}$ has been generated according to LQ criterion and we chose $T = 10P_{yy}$. The results are reported in Figure 6.6, from which it is clear that the disturbance is completely rejected. In Figure 6.7 the applied inputs are shown.

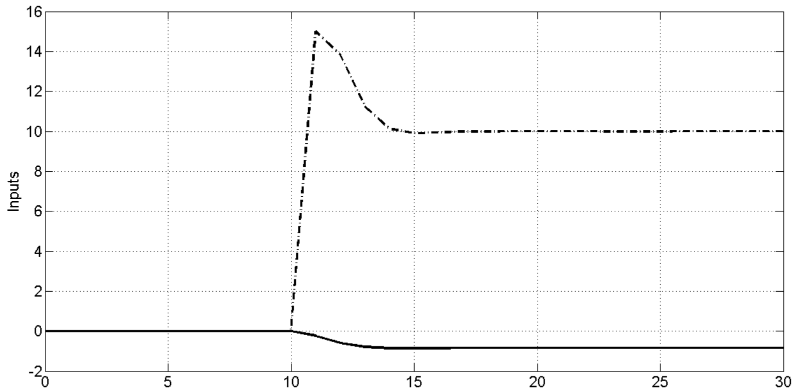

Figure 6.7: Trajectories of the inputs applied to the system. Blak solid line: first input; black dash-dotted line: second input.

6.4 Conclusions

In this paper, a robust MPC algorithm solving the offset-free tracking and the infeasible reference problems has been developed for disturbed systems expressed in velocity form. Convergence results have been obtained by suitably defining the auxiliary control law and the terminal set used in the problem formulation. In the next Chapter, the obtained results will be used to derive a decentralized control algorithm for tracking varying (piacewise constant) references.

6.5 Appendix

6.5.1 Proof of Proposition 6.2 and of Proposition 6.3

Being $\delta x_k = x_k - x_{k-1}$, system (6.19) can be written as

$$\begin{array}{rcl} \delta x_k & = & (A - I_n)x_{k-1} + Bu_{k-1} + w_{k-1} \\ y_k & = & Cx_k = C(Ax_{k-1} + Bu_{k-1} + w_{k-1}) \end{array} \tag{6.57}$$

which gives

$$\begin{bmatrix} x_{k-1} \\ u_{k-1} \end{bmatrix} = \begin{bmatrix} A - I_n & B \\ CA & CB \end{bmatrix}^{-1} \left(\begin{bmatrix} \delta x_k \\ y_k \end{bmatrix} - \begin{bmatrix} I_n \\ C \end{bmatrix} w_{k-1} \right) \tag{6.58}$$

The dynamics of the state of system (6.19) allows one to write also

$$\begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x_{k-1} \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} I_n \\ 0 \end{bmatrix} w_{k-1} \tag{6.59}$$

Inserting (6.58) inside (6.59) and recalling that

$$\begin{bmatrix} \delta x_k \\ y_k \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & I_m \end{bmatrix} \begin{bmatrix} \xi_k \\ \hat{r}_k \end{bmatrix}$$

(6.26) clearly follows. Expression (6.27) and (6.7) are obtained in the same way simply considering $w_{k-1} = 0$.

In order to find matrices C^* and C_w we need to prove that the inverse of Σ can be computed. It is easy to see that Σ is invertible if and only if S is invertible, which holds in view of Assumption 6.1. This can be proved by contradiction as follows. Let's suppose that Σ^{-1} can not be computed: this mean that there exists a vector $\mu = (\mu_x, \mu_u)$, with $\mu_x \in \mathbb{R}^n$ and $\mu_u \in \mathbb{R}^m$ such that $\Sigma\mu = 0$ with $\mu \neq 0$, i.e.,

$$\begin{array}{ll} A\mu_x + B\mu_u & = \mu_x \\ C(A\mu_x + B\mu_u) = C\mu_x & = 0 \end{array} \tag{6.60}$$

that can be written as $S\mu = 0$. Since $\mu \neq 0$, this contradicts Assumption 6.1 *iii)* and therefore proves the existence of Σ^{-1}.

6.5.2 Proof of Theorem 6.1 and of Theorem 6.2

The proof of Theorem 6.1 is a simplified version of the proof of Theorem 6.2, see [13], and therefore is here skipped. Moreover, once Theorem 6.2 and Corollary 6.2 have been proved, a very simple proof of Theorem 6.1 can be stated. We refer the reader to the proof of Corollary 6.2 for additional details.

As for the proof of Theorem 6.2, it follows the same rationale as the proof of Theorem 1 in [88]. Specifically, it is based on the following steps:

I) we prove that the only closed-loop stable equilibrium point such that $\hat{\xi}_k = 0$, compatible with the minimization problem (6.39), is the one corresponding to $\hat{r}_k = r_{ad}$.

II) We prove recursive feasibility and that the system converges asymptotically to an equilibrium point, i.e., that $\hat{\xi}_k \to 0$ as $k \to \infty$.

III) From the previous steps we infer that the nominal system asymptotically converges to the unique equilibrium point, which is compatible with (6.39).

Step I
First note that, among all admissible points $(\hat{\xi}_k, \hat{r}_k) = (\delta\hat{x}_k, \hat{y}_k - \hat{r}_k, \hat{r}_k)$, the point $(0, 0, r_{ad})$, where r_{ad} fulfills (6.43), is the equilibrium condition to (6.23) (corresponding to $\hat{y}_k = \hat{r}_k = r_{ad}$ and $\delta\hat{u}_{k+\nu} = 0$ for all ν) such that the cost function V_N is globally minimized. Recall that it is admissible since $C_y r_{ad} \in \hat{\mathbb{X}}_{\mathbb{U}}(\epsilon)$, see (6.38).

Consider a generic admissible reference $\hat{r} \neq r_{ad}$, such that $(\hat{x}_r, \hat{u}_r) = C_y\hat{r} \in \hat{\mathbb{X}}_{\mathbb{U}}(\epsilon)$, corresponding to the equilibrium $(\delta\hat{x}_k, \hat{y}_k - \hat{r}_k, \hat{r}_k) = (0, 0, \hat{r})$ for the system (6.37) (i.e., corresponding to an equilibrium to (6.23) in case $\delta\hat{u}_{k+\nu} = 0$ for all $\nu \geq 0$). The cost, associated to such equilibrium condition is

$$V_N = \|\hat{r} - r^o\|_T^2 \tag{6.61}$$

Instead, consider the solution corresponding to the trajectory starting from $(\delta\hat{x}_k, \hat{y}_k - \hat{r}_k, \hat{r}_k) = (0, \hat{r} - \tilde{r}, \tilde{r})$ (i.e., from the point $(\hat{x}_r, \hat{u}_r)$, which does not correspond now to an equilibrium point for (6.37), in view of the fact that the reference output is now $\tilde{r} \neq \hat{r}$), and evolving according to the auxiliary control law (6.36). If $\tilde{r}$ is defined according to $\tilde{r} = \lambda\hat{r} + (1 - \lambda)r_{ad}$, $\lambda \in [0, 1)$, then:

i) the corresponding equilibrium is admissible, i.e., $C_y\tilde{r} \in \hat{\mathbb{X}}_{\mathbb{U}}(\epsilon)$, in view of the convexity of $\hat{\mathbb{X}}_{\mathbb{U}}(\epsilon)$.

ii) since $C_y\tilde{r} \in \hat{\mathbb{X}}_{\mathbb{U}}(\epsilon)$, and being $\hat{\mathbb{O}}_\epsilon \subset \hat{\mathbb{O}}$, provided $(1 - \lambda)$ (and hence $\|\hat{r} - \tilde{r}\|$) is sufficiently small, we get that the initial condition $(0, \hat{r} - \tilde{r}, \tilde{r}) \in \hat{\mathbb{O}}$. In this way, it is possible to control the system with the auxiliary law (6.36).

The cost associated to such auxiliary control law is

$$\begin{aligned}\tilde{V}_N = \|\tilde{r} - r^o\|_T^2 + \sum_{\nu=0}^{N-1} \{ \|\mathcal{F}^\nu(0, \hat{r} - \tilde{r})\|_Q^2 + \|\mathcal{K}\mathcal{F}^\nu(0, \hat{r} - \tilde{r})\|_R^2 \} + \\ + \|\mathcal{F}^N(0, \hat{r} - \tilde{r})\|_P^2\end{aligned} \tag{6.62}$$

In view of (6.16), we obtain that the latter is equal to

$$\tilde{V}_N = \|\tilde{r} - r^o\|_T^2 + \|(0, \hat{r} - \tilde{r})\|_P^2 = \|\tilde{r} - r^o\|_T^2 + \|\hat{r} - \tilde{r}\|_{P_{yy}}^2 \tag{6.63}$$

since $P_{yy} \prec T$

$$\tilde{V}_N < \|\tilde{r} - r^o\|_T^2 + \|\hat{r} - \tilde{r}\|_T^2 \tag{6.64}$$

Note that

$$\begin{aligned} a &= \|\tilde{r} - r^o\|_T = \lambda\|\hat{r} - r^o\|_T \\ b &= \|\hat{r} - \tilde{r}\|_T = (1-\lambda)\|\hat{r} - r^o\|_T \\ c &= a + b = \|\hat{r} - r^o\|_T \end{aligned}$$

Since $a, b, c > 0$, if $a + b = c$ then $a^2 + b^2 < c^2$. From this, (6.64), and (6.61)

$$\tilde{V}_N < V_N$$

from which it follows that, for all $\hat{r} \neq r_{ad}$, $(0, 0, \hat{r})$ is not an equilibrium point for the closed loop system.

Step II

Recursive feasibility and convergence of $\hat{\xi}_k$ to zero, once $\hat{r}$ is kept constant, can be proved resorting to standard robust tube-based MPC arguments, see [107].

Feasibility: Consider that, at time k, a solution to (6.39) is $(\delta\hat{x}_{k|k}, \hat{y}_{k|k}, \hat{r}_{k|k}, \delta\hat{u}_{[k:k+N-1]|k})$. Then it is easy to see that a feasible solution to (6.39) at time $k+1$ is $(\delta\hat{x}_{k+1|k}, \hat{y}_{k+1|k}, \hat{r}_{k|k}, \delta\hat{u}_{[k+1:k+N]|k})$, where it is defined $\delta\hat{u}_{k+N|k} = \mathcal{K}\hat{\xi}_{k+N|k}$, where $\hat{\xi}_{k+N|k}$ stems from (6.23) with inputs $\delta\hat{u}_{[k:k+N-1]|k}$ and initial condition $\hat{\xi}_{k|k} = (\delta\hat{x}_{k|k}, (\hat{y}_{k|k} - \hat{r}_{k|k}))$.

Convergence: Consider the above-mentioned feasible solution at time $k + 1$. Following standard arguments for the proof of the convergence of robust tube-based MPC [107] we obtain that

$$V_N^*(r^o, \delta x_{k+1}, y_{k+1}) - V_N^*(r^o, \delta x_k, y_k) \leq -\|\hat{\xi}_{k|k}\|_Q^2 - \|\delta\hat{u}_{k|k}\|_R^2$$

In view of the positive definiteness of V_N^* and of Q and R, it holds that $\hat{\xi}_{k|k} \to 0$ and $\delta\hat{u}_{k|k} \to 0$ as $k \to \infty$.

Note that, in the case the disturbance is constant, the evolution of the real system in velocity form is indeed $\xi_{k+1} = \mathcal{A}\xi_k + \mathcal{B}\delta u_k$. Since the input increment to the real system is $\delta u_k = \delta\hat{u}_{k|k} + \mathcal{K}(\xi_k - \hat{\xi}_{k|k})$, the system dynamics is described by $\xi_{k+1} = \mathcal{F}\xi_k + \delta\hat{u}_{k|k} - \mathcal{K}\hat{\xi}_{k|k}$ where $\delta\hat{u}_{k|k} - \mathcal{K}\hat{\xi}_{k|k}$ is an asymptotically vanishing term. Being $\mathcal{F}$ Schur then $\xi_k \to 0$ as $k \to \infty$.

Step III

In view of Step II, convergence of $\hat{\xi}_k$ is guaranteed. Since, in view of Step I, the only equilibrium to (6.23) compatible with (6.39) is the one corresponding to $\hat{r}_k = r_{ad}$, then it also holds that $\hat{y}_k \to r_{ad}$

as $k \to \infty$. Since the control law (6.24) always keeps ξ_k in a neighborhood of $\hat{\xi}_k$, we have that y_k tends to a neighborhood of r_{ad}, i.e., $y_k \to r_{ad} \oplus \begin{bmatrix} 0 & I_m & 0 \end{bmatrix} \tilde{\mathbb{Z}}$ as $k \to \infty$.

6.5.3 Proof of Corollary 6.1

In view of Theorem 6.2, $\hat{y}_k \to r_{ad}$ as $k \to \infty$. Since the input increment to the real system is $\delta u_k = \delta \hat{u}_{k/k} + \mathcal{K}(\xi_k - \hat{\xi}_{k/k})$, the system dynamics is described by $\xi_{k+1} = \mathcal{F}\xi_k + \delta \hat{u}_{k/k} - \mathcal{K}\hat{\xi}_{k/k} + d_k$ where both $\delta \hat{u}_{k/k} - \mathcal{K}\hat{\xi}_{k/k}$ and $d_k = \mathcal{B}_w(w_k - w_{k-1})$ are asymptotically vanishing terms. Being $\mathcal{F}$ Schur stable, then $\xi_k \to 0$ as $k \to \infty$.

6.5.4 Proof of Corollary 6.2

In view of Theorem 6.2, $\hat{y}_k \to r_{ad}$ as $k \to \infty$. This, in general, means that y_k tends to a neighborhood of r_{ad}. Recalling that, in this particular case, $\tilde{\mathbb{Z}} = \{0\} \times \mathbb{W}$, we have that

$$y_k \to r_{ad} \oplus \begin{bmatrix} 0 & I_m & 0 \end{bmatrix} \tilde{\mathbb{Z}} = r_{ad} \oplus \{0\} = r_{ad}$$

as $k \to \infty$.

Note that this result allows one to derive a very simple proof of Theorem 6.1. In fact, the nominal algorithm can be seen as a particular case of constant disturbance, where $\bar{w} = 0$, $\mathbb{W} = \{0\}$, $\tilde{\mathbb{Z}} = \{0\} \times \{0\} = \{0\}$, $\hat{\mathbb{X}}_{\mathbb{U}} = \hat{\mathbb{X}} \times \hat{\mathbb{U}}$ and $\hat{\mathbb{O}}_\epsilon = \mathbb{O}_\epsilon$.

6.5.5 Proof of Corollary 6.3

In view of Theorem 6.2, it is proved that $\hat{y}_k \to r_{ad}$ as $k \to \infty$. We define $\varepsilon^a_k = y_k - r_{ad}$ and $\hat{\varepsilon}^a_k = \hat{y}_k - r_{ad}$. Noting that $\delta w_k = w_k - w_{k-1} = A_\delta \delta x_k + B_\delta \delta u_k$ we have that, similarly to (6.2)

$$\begin{array}{rcl} \delta x_{k+1} & = & (A + A_\delta)\delta x_k + (B + B_\delta)\delta u_k \\ \varepsilon^a_{k+1} & = & C(A + A_\delta)\delta x_k + \varepsilon^a_k + C(B + B_\delta)\delta u_k \end{array} \quad (6.65)$$

where, for all k, $\delta u_k = \delta \hat{u}_{k|k} + \mathcal{K} \begin{bmatrix} \delta x_k - \delta \hat{x}_{k|k} \\ \varepsilon^a_k - \hat{\varepsilon}^a_k \end{bmatrix}$. We can therefore write equation (6.65) as

$$\begin{bmatrix} \delta x_{k+1} \\ \varepsilon^a_{k+1} \end{bmatrix} = \mathcal{F}_r \begin{bmatrix} \delta x_k \\ \varepsilon^a_k \end{bmatrix} + \eta_k \quad (6.66)$$

where $\eta_k = \begin{bmatrix} B + B_\delta \\ C(B + B_\delta) \end{bmatrix} \left(\delta\hat{u}_{k|k} - \mathcal{K} \begin{bmatrix} \delta\hat{x}_{k|k} \\ \hat{\varepsilon}^a_k \end{bmatrix} \right)$ is an asymptotically vanishing term. Since $\mathcal{F}_r$ is Schur, it implies that $\delta x_k \to 0$ and $\varepsilon^a_k \to 0$ as $k \to \infty$.

6.5.6 Proof of Proposition 6.4

Recalling (6.49) and (6.34), the set $\hat{\mathbb{X}}_{\mathbb{U}}$ exists if there exists $\rho > 0$ such that

$$\begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathbb{Z}} \oplus \mathcal{B}^{(2n+m)}_\rho(0) \subseteq \mathbb{X} \times \mathbb{U} \tag{6.67}$$

which is verified if, for all $i = 1, \ldots, r$

$$\max_{\tilde{z} \in \tilde{\mathbb{Z}}} h_i^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{z} < 1 \tag{6.68}$$

In view of (6.31), (6.68) is verified if, for all $i = 1, \ldots, r$

$$\sum_{\nu=0}^{\infty} \max_{w \in \mathbb{W}} h_i^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathcal{F}}^\nu \tilde{\mathcal{B}}_w w < 1 \tag{6.69}$$

where, in view of (6.49), $w = \sum_{j=1}^{n_\alpha} \alpha_j \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u} v$, where $\|v\|_\infty \leq 1$. Therefore (6.69) is verified if, for all $i = 1, \ldots, r$ and for all α_j, $j = 1, \ldots, n_{alpha}$

$$\sum_{\nu=0}^{\infty} \max_{\|v\|_\infty \leq 1} h_i^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathcal{F}}^\nu \tilde{\mathcal{B}}_w \sum_{j=1}^{n_\alpha} \alpha_j \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u} v < 1 \tag{6.70}$$

Note that, for all $i = 1, \ldots, r$

$$\begin{aligned} &\textstyle\sum_{\nu=0}^{\infty} \max_{\|v\|_\infty \leq 1} h_i^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathcal{F}}^\nu \tilde{\mathcal{B}}_w \sum_{j=1}^{n_\alpha} \alpha_j \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u} v \\ &= \textstyle\sum_{\nu=0}^{\infty} \|h_i^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathcal{F}}^\nu \tilde{\mathcal{B}}_w \sum_{j=1}^{n_\alpha} \alpha_j \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u}\|_\infty \end{aligned}$$

Since, in view of (6.50) and for all $\nu = 0, \ldots, \infty$,

$$\begin{aligned} &\|h_i^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathcal{F}}^k \tilde{\mathcal{B}}_w \textstyle\sum_{j=1}^{n_\alpha} \alpha_j \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u}\|_\infty \\ &\leq \|H_{x,u}^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathcal{F}}^k \tilde{\mathcal{B}}_w \textstyle\sum_{j=1}^{n_\alpha} \alpha_j \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u}\|_\infty \end{aligned}$$

for all $i = 1, \ldots, r$, then

$$\begin{aligned}
&\textstyle\sum_{\nu=0}^{\infty} \|h_i^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathcal{F}}^\nu \tilde{\mathcal{B}}_w \sum_{j=1}^{n_\alpha} \alpha_j \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u}\|_\infty \\
&\le \textstyle\sum_{\nu=0}^{\infty} \|H_{x,u}^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathcal{F}}^\nu \tilde{\mathcal{B}}_w \sum_{j=1}^{n_\alpha} \alpha_j \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u}\|_\infty \\
&\le \textstyle\sum_{j=1}^{n_\alpha} \alpha_j \sum_{\nu=0}^{\infty} \|H_{x,u}^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathcal{F}}^\nu \tilde{\mathcal{B}}_w \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u}\|_\infty \\
&\le \max_{j=1,\ldots,n_\alpha} \left(\textstyle\sum_{k=0}^{\infty} \|H_{x,u}^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \tilde{\mathcal{F}}^\nu \tilde{\mathcal{B}}_w \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u}\|_\infty \right) \\
&\le \textstyle\sum_{\nu=0}^{\infty} \|\tilde{\mathcal{F}}^\nu\|_\infty \|H_{x,u}^T \begin{bmatrix} C_\xi & C_w \end{bmatrix} \|_\infty \cdot \\
&\qquad \cdot \max_{j=1,\ldots,n_\alpha} \|\tilde{\mathcal{B}}_w \begin{bmatrix} A_j & B_j \end{bmatrix} \Xi_{x,u}\|_\infty
\end{aligned}$$

Eventually, if (6.51) is verified, it follows that (6.68) is also verified.

7 DePC for tracking

Using the results shown in Chapter 6 for centralized controllers based on the velocity-form version of the dynamic systems, we present a decentralized algorithms for tracking piecewise constant references with integral action in the closed-loop.

The problem of designing decentralized regulators is of paramount importance in the industrial world, where most of the controllers are simple single-input-single-output PI (or PID) regulators. For this reason, synthesis methods of decentralized regulators have first been proposed since the 70's for control of large-scale systems mainly in the continuous-time framework [94,151,152,169]. The problem of designing decentralized regulators with offset-free tracking properties has been tackled in the 80's both in the continuous-time [40, 64] and in the discrete-time [145] contexts.

In this Chapter, we address the decentralized offset-free tracking problem of piecewise constant reference signals for systems subject to constraints on inputs and states. The systems are supposed to be constituted by a set of non-overlapping subsystems coupled by states and inputs. To reach our aim, first we recast the problem as a regulation one by reformulating the plant model in the velocity-form [14, 122, 126, 168]. Secondly we state the decentralized control problem as a tube-based robust one [107], where dynamic interactions among subsystems are interpreted as perturbations to be rejected, similarly to [50, 139]. This approach implies the definition of tightened state and input constraints for accounting for uncertainties. Convergence results are reported and a simulation example is provided to evaluate the performances of the control algorithm. The proposed decentralized predictive control (DePC) scheme has the great advantage that no transmission of information among the local controllers is required, at the price of a reduction of the size of feasibility sets.

7.1 The dynamical system

7.1.1 System under control

The process to be controlled is described by a discrete-time, linear, time invariant system with dynamic equations

$$\begin{array}{rcl} \mathbf{x}_{k+1} & = & \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{v}_k \\ \mathbf{y}_k & = & \mathbf{C}\mathbf{x}_k \end{array} \tag{7.1}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ is the state, $\mathbf{u}_k \in \mathbb{R}^m$ is the input, $\mathbf{v}_k \in \mathbb{V} \subset \mathbb{R}^n$ is an unknown bounded external disturbance and $\mathbf{y}_k \in \mathbb{R}^m$ is the output. The set $\mathbb{V}$ is convex, compact and includes the origin. The state and input vectors are constrained to lie within given convex and compact sets, i.e. $\mathbf{x}_k \in \mathbb{X} \subset \mathbb{R}^n$, $\mathbf{u}_k \in \mathbb{U} \subset \mathbb{R}^m$. The constant reference to be tracked is denoted by $\mathbf{r}^o \in \mathbb{R}^m$. To guarantee the existence and the uniqueness of the steady-state pair such that the system output corresponds to $\mathbf{r}^o$, the following standard assumption is made.

Assumption 7.1 *The input-output system* (7.1) *has no invariant zeros in* 1, *i.e.,*

$$\text{rank}\left(\begin{bmatrix} I_n - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix}\right) = n + m$$

□

7.1.2 Partitioned system

As shown in Chapter 5, system (7.1) is divided in M interacting and non overlapping subsystems, each one having $x_k^{[i]} \in \mathbb{R}^{n_i}$ as state vector, $u_k^{[i]} \in \mathbb{R}^{m_i}$ as input vector, $y_k^{[i]} \in \mathbb{R}^{m_i}$ as output vector and $v_k^{[i]} \in \mathbb{V}_i \subset \mathbb{R}^{n_i}$ as external disturbance, i.e., $\mathbf{x}_k = (x_k^{[1]}, \ldots, x_k^{[M]})$, $\mathbf{u}_k = (u_k^{[1]}, \ldots, u_k^{[M]})$, $\mathbf{y}_k = (y_k^{[1]}, \ldots, y_k^{[M]})$ and $\mathbf{v}_k = (v_k^{[1]}, \ldots, v_k^{[M]})$ (with $\sum_{i=1}^M n_i = n$ and $\sum_{i=1}^M m_i = m$). The sets $\mathbb{V}_i$ are convex, compact, and include the origin and $\mathbb{V}$ is defined as $\mathbb{V} = \prod_{i=1}^M \mathbb{V}_i$. We assume that, for all subsystems, $y_k^{[i]}$ depends only on $x_k^{[i]}$ (this means that $\mathbf{C}$ has a block diagonal structure). Eventually, The reference vector $\mathbf{r}^o$ is decomposed into M local output set-points $r^{o,[i]}$, consistent with the definition of $y_k^{[i]}$. According to this partition of system (7.1), as already seen in the previous Chapters, the matrices A_{ij} and B_{ij}, $i, j = 1, \dots, M$, whose dimensions are consistent with the partition of states and inputs, are the block entries of the matrices of $\mathbf{A}$ and $\mathbf{B}$. Given the i-th subsystem, by $\mathcal{N}_i$ we denote the set of its neighbors (which excludes i), i.e. the set of subsystems for which $A_{ij} \neq 0$ and/or $B_{ij} \neq 0$.

The dynamics of the i-th subsystem is given by

$$\begin{array}{rcl} x_{k+1}^{[i]} & = & A_{ii}\, x_k^{[i]} + B_{ii} u_k^{[i]} + \sum_{j \in \mathcal{N}_i} \{A_{ij} x_k^{[j]} + B_{ij} u_k^{[j]}\} + v_k^{[i]} \\ y_k^{[i]} & = & C_i\, x_k^{[i]} \end{array} \tag{7.2}$$

where $x_k^{[i]} \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ and $u_k^{[i]} \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$, being $\mathbb{X}_i$ and $\mathbb{U}_i$ convex, compact and neighbors of the origin, for all $i = 1, \dots, M$. We define $\mathbb{X} = \prod_{i=1}^M \mathbb{X}_i$, $\mathbb{U} = \prod_{i=1}^M \mathbb{U}_i$. A second assumption, on the properties of the subsystems, is introduced.

Assumption 7.2 *For each subsystem* (7.2)*:*

i) the pair (A_{ii}, B_{ii}) *is reachable.*

ii) $\operatorname{rank}\left(\begin{bmatrix} I_{n_i} - A_{ii} & -B_{ii} \\ C_{ii} & 0 \end{bmatrix}\right) = n_i + m_i.$

□

In order to design a decentralized controller, all the subsystems (7.2) are seen as independent processes affected by an unknown, bounded, external disturbance, i.e.

$$\begin{array}{rcl} x_{k+1}^{[i]} & = & A_{ii}\, x_k^{[i]} + B_{ii} u_k^{[i]} + w_k^{[i]} \\ y_k^{[i]} & = & C_i\, x_k^{[i]} \end{array} \tag{7.3}$$

where $w_k^{[i]} \in \mathbb{W}_i$ is defined as $w_k^{[i]} = \sum_{j \in \mathcal{N}_i} \{A_{ij} x_k^{[j]} + B_{ij} u_k^{[j]}\} + v_k^{[i]}$, thus having $\mathbb{W}_i = \bigoplus_{j \in \mathcal{N}_i} \{A_{ij} \mathbb{X}_j \oplus B_{ij} \mathbb{U}_j\} \oplus \mathbb{V}_i$.

7.1.3 Subsystems with integrators

To track the reference signal, m integrators are inserted in the partitioned subsystems, according to their description in velocity-form (see, e.g., [122, 168] and Chapters 5 and 6). Let $\hat{r}^{[i]}$ be a generic tracking target, which could be different from $r^{o,[i]}$, because using the approach described in [88] the setpoint is considered as one of the optimization variables. Then, defining $\delta x_k^{[i]} = x_k^{[i]} - x_{k-1}^{[i]}$, $\varepsilon_t^{[i]} = y_k^{[i]} - \hat{r}^{[i]}$, $\delta u_k^{[i]} = u_k^{[i]} - u_{k-1}^{[i]}$ and $\delta w_k^{[i]} = w_k^{[i]} - w_{k-1}^{[i]}$ system (7.3) can be reformulated as

$$\begin{array}{rcl} \delta x_{k+1}^{[i]} & = & A_{ii}\delta x_k^{[i]} + B_{ii}\delta u_k^{[i]} + \delta w_k^{[i]} \\ \varepsilon_{k+1}^{[i]} & = & C_i A_{ii}\delta x_k^{[i]} + \varepsilon_k^{[i]} + C_i B_{ii}\delta u_k^{[i]} + C_i\delta w_k^{[i]} \end{array} \tag{7.4}$$

Defining $\xi_k^{[i]} = (\delta x_k^{[i]}, \varepsilon_k^{[i]})$, $\delta v_k^{[i]} = v_k^{[i]} - v_{k-1}^{[i]}$ and

$$\mathcal{A}_{ii} = \begin{bmatrix} A_{ii} & 0 \\ C_i A_{ii} & I_{m_i} \end{bmatrix},\ \mathcal{B}_{ii} = \begin{bmatrix} B_{ii} \\ C_i B_{ii} \end{bmatrix},\ \mathcal{B}_i^w = \begin{bmatrix} I_{n_i} \\ C_i \end{bmatrix}$$

$$\mathcal{A}_{ij} = \begin{bmatrix} A_{ij} & 0 \\ C_i A_{ij} & 0 \end{bmatrix},\ \mathcal{B}_{ij} = \begin{bmatrix} B_{ij} \\ C_i B_{ij} \end{bmatrix},\ d_k^{[i]} = \mathcal{B}_i^w \delta w_k^{[i]} \tag{7.5}$$

system (7.2) can be reformulated in compact form as

$$\xi_{k+1}^{[i]} = \mathcal{A}_{ii}\xi_k^{[i]} + \mathcal{B}_{ii}\delta u_k^{[i]} + d_k^{[i]} \tag{7.6}$$

which is equivalent to

$$\xi_{k+1}^{[i]} = \mathcal{A}_{ii}\xi_k^{[i]} + \mathcal{B}_{ii}\delta u_k^{[i]} + \sum_{j\in\mathcal{N}_i}\{\mathcal{A}_{ij}\xi_k^{[j]} + \mathcal{B}_{ij}\delta u_k^{[j]}\} + \mathcal{B}_i^w\delta v_k^{[i]} \tag{7.7}$$

As shown in Chapter 4 [12]:

Proposition 7.1 *Under Assumption 7.2, the pair $(\mathcal{A}_{ii}, \mathcal{B}_{ii})$ is reachable.* ■

The set of M models (7.7) can be collectively written as

$$\boldsymbol{\xi}_{k+1} = \boldsymbol{\mathcal{A}}\boldsymbol{\xi}_k + \boldsymbol{\mathcal{B}}\boldsymbol{\delta}\mathbf{u}_k + \boldsymbol{\mathcal{B}}^w\boldsymbol{\delta}\mathbf{v}_k \tag{7.8}$$

where $\boldsymbol{\xi}_k = (\xi_k^{[1]}, \dots, \xi_k^{[M]})$, $\boldsymbol{\delta}\mathbf{u}_k = (\delta u_k^{[1]}, \dots, \delta u_k^{[M]})$, $\boldsymbol{\delta}\mathbf{v}_k = (\delta v_k^{[1]}, \dots, \delta v_k^{[M]})$, $\boldsymbol{\mathcal{B}}^w = \mathrm{diag}(\mathcal{B}_1^w, \dots, \mathcal{B}_M^w)$ while $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ are the matrices whose block entries are $\mathcal{A}_{ij}$, $\mathcal{B}_{ij}$, respectively.

7.2 The DePC algorithm for tracking

7.2.1 Nominal models and control law

The following nominal subsystem is introduced

$$\begin{array}{rcl} \hat{x}_{k+1}^{[i]} & = & A_{ii}\hat{x}_k^{[i]} + B_{ii}\hat{u}_k^{[i]} \\ \hat{y}_k^{[i]} & = & C_i\hat{x}_k^{[i]} \end{array} \tag{7.9}$$

consisting in (7.3) where coupling terms and external disturbance are neglected. Define $\delta\hat{x}_k^{[i]} = \hat{x}_k^{[i]} - \hat{x}_{k-1}^{[i]}$, $\hat{\varepsilon}_k^{[i]} = \hat{y}_k^{[i]} - \hat{r}^{[i]}$, $\delta\hat{u}_k^{[i]} = \hat{u}_k^{[i]} - \hat{u}_{k-1}^{[i]}$, and $\hat{\xi}_k^{[i]} = (\delta\hat{x}_k^{[i]}, \hat{\varepsilon}_k^{[i]})$. The velocity-form model corresponding to (7.9) is

$$\hat{\xi}_{k+1}^{[i]} = \mathcal{A}_{ii}\hat{\xi}_k^{[i]} + \mathcal{B}_{ii}\delta\hat{u}_k^{[i]} \tag{7.10}$$

To compute a decentralized stabilizing state feedback auxiliary control law for all the M nominal subsystems (7.10), a new assumption is introduced.

Assumption 7.3 *There exists a block-diagonal matrix* $\boldsymbol{\mathcal{K}} = diag(\mathcal{K}_1, \ldots, \mathcal{K}_M)$, *with* $\mathcal{K}_i \in \mathbb{R}^{m_i \times (n_i+m_i)}$, $i = 1, ..., M$, *such that:*

i) the matrix $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}$ *is Schur.*

ii) The matrices $\mathcal{F}_{ii} = \mathcal{A}_{ii} + \mathcal{B}_{ii}\mathcal{K}_i$ *are Schur.*

□

We will compute the control action of the real subsystems (7.6) as

$$\delta u_k^{[i]} = \delta\hat{u}_k^{[i]} + \mathcal{K}_i(\xi_k^{[i]} - \hat{\xi}_k^{[i]}) \tag{7.11}$$

For the nominal subsystems (7.10), consider the control laws

$$\delta\hat{u}_k^{[i]} = \mathcal{K}_i\hat{\xi}_k^{[i]} \tag{7.12}$$

From (7.6), (7.10), (7.12), and (7.11) we have

$$(\xi_{k+1}^{[i]} - \hat{\xi}_{k+1}^{[i]}) = \mathcal{F}_{ii}(\xi_k^{[i]} - \hat{\xi}_k^{[i]}) + d_k^{[i]} \tag{7.13}$$

7.2.2 Constraints for inputs and states

One of the main problems related to the formulation of MPC problems when applied to systems formulated in velocity form is how to properly define the input, state, and terminal constraint sets. For this purpose, we need the following result, proved in Chapter 6 for centralized systems [14].

Proposition 7.2 *For all $i = 1, \dots, M$, under Assumption 7.2 the following equations hold*

$$\begin{bmatrix} x_k^{[i]} \\ u_{k-1}^{[i]} \end{bmatrix} = C_i^* \begin{bmatrix} \xi_k^{[i]} \\ \hat{r}^{[i]} \end{bmatrix} + C_i^w w_{k-1}^{[i]} \tag{7.14}$$

$$\begin{bmatrix} \hat{x}_k^{[i]} \\ \hat{u}_{k-1}^{[i]} \end{bmatrix} = C_i^* \begin{bmatrix} \hat{\xi}_k^{[i]} \\ \hat{r}^{[i]} \end{bmatrix} \tag{7.15}$$

where

$$\begin{aligned} C_i^* &= \begin{bmatrix} A_{ii} & B_{ii} \\ 0 & I_{m_i} \end{bmatrix} \begin{bmatrix} A_{ii} - I_{n_i} & B_{ii} \\ C_i A_{ii} & C_i B_{ii} \end{bmatrix}^{-1} \begin{bmatrix} I_{n_i} & 0 & 0 \\ 0 & I_{m_i} & I_{m_i} \end{bmatrix} \\ C_i^w &= \begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix} - \begin{bmatrix} A_{ii} & B_{ii} \\ 0 & I_{m_i} \end{bmatrix} \begin{bmatrix} A_{ii} - I_{n_i} & B_{ii} \\ C_i A_{ii} & C_i B_{ii} \end{bmatrix}^{-1} \begin{bmatrix} I_{n_i} \\ C_i \end{bmatrix} \end{aligned}$$

■

It is worth underling that the inverse appearing in the definition of C_i^* and C_i^w always exists since Assumption 7.2 is verified, see Chapter 6.

Let now partition matrix C_i^* as follows: $C_i^* = \begin{bmatrix} C_i^\xi & C_i^y \end{bmatrix}$, where $C_i^\xi \in \mathbb{R}^{(n_i+m_i)\times(n_i+m_i)}$ and $C_i^y \in \mathbb{R}^{(n_i+m_i)\times m_i}$. From equations (7.14) and (7.15) we obtain

$$\begin{bmatrix} x_k^{[i]} - \hat{x}_k^{[i]} \\ u_{k-1}^{[i]} - \hat{u}_{k-1}^{[i]} \end{bmatrix} = \begin{bmatrix} C_i^\xi & C_i^w \end{bmatrix} \begin{bmatrix} \xi_k^{[i]} - \hat{\xi}_k^{[i]} \\ w_{k-1}^{[i]} \end{bmatrix} \tag{7.16}$$

Defining $\omega_k^{[i]} = w_{k-1}^{[i]}$ one can write, from (7.5) and (7.13)

$$\begin{bmatrix} \xi_{k+1}^{[i]} - \hat{\xi}_{k+1}^{[i]} \\ \omega_{k+1}^{[i]} \end{bmatrix} = \tilde{\mathcal{F}}_{ii} \begin{bmatrix} \xi_k^{[i]} - \hat{\xi}_k^{[i]} \\ \omega_k^{[i]} \end{bmatrix} + \tilde{\mathcal{B}}_i^w w_k^{[i]} \tag{7.17}$$

where

$$\tilde{\mathcal{F}}_{ii} = \begin{bmatrix} \mathcal{F}_{ii} & -\mathcal{B}_i^w \\ 0 & 0 \end{bmatrix}, \; \tilde{\mathcal{B}}_i^w = \begin{bmatrix} \mathcal{B}_i^w \\ I_{n_i} \end{bmatrix}$$

Since $\mathcal{F}_{ii}$ is Schur, $\tilde{\mathcal{F}}_{ii}$ is Schur as well. In view of this, it is possible (see [131]) to compute the minimal robust positively invariant (RPI) set for (7.17) as $\tilde{\mathbb{Z}}_i = \bigoplus_{k=0}^{+\infty} \tilde{\mathcal{F}}_{ii}^k \tilde{\mathcal{B}}_i^w \mathbb{W}_i$.

Moreover, considering (7.16), if $(\xi_k^{[i]} - \hat{\xi}_k^{[i]}, \omega_k^{[i]}) \in \tilde{\mathbb{Z}}_i$, then to verify the constraints $(x_k^{[i]}, u_{k-1}^{[i]}) \in \mathbb{X} \times \mathbb{U}$ it has to be guaranteed that

$$(\hat{x}_k^{[i]}, \hat{u}_{k-1}^{[i]}) \in \hat{\mathbb{X}}_i^{\mathbb{U}} \tag{7.18}$$

where $\hat{\mathbb{X}}_i^{\mathbb{U}}$ must satisfy

$$\hat{\mathbb{X}}_i^{\mathbb{U}} \subseteq (\mathbb{X}_i \times \mathbb{U}_i) \ominus \begin{bmatrix} C_i^{\xi} & C_i^{w} \end{bmatrix} \tilde{\mathbb{Z}}_i \tag{7.19}$$

An assumption on the existence of sets $\hat{\mathbb{X}}_i^{\mathbb{U}}$ is required.

Assumption 7.4 *For each subsystem* (7.3) *there exists* $\mathcal{B}_{p,\varepsilon}^{(n_i+m_i)}(0)$ *such that*

$$\begin{bmatrix} C_i^{\xi} & C_i^{w} \end{bmatrix} \tilde{\mathbb{Z}}_i \oplus \mathcal{B}_{p,\varepsilon}^{(n_i+m_i)}(0) \subseteq (\mathbb{X}_i \times \mathbb{U}_i) \tag{7.20}$$

□

In terms of $\hat{\xi}_k^{[i]}$, $\hat{r}^{[i]}$ and in view of (7.15), Equation (7.18) can be written as

$$C_i^* \begin{bmatrix} \hat{\xi}_k^{[i]} \\ \hat{r}^{[i]} \end{bmatrix} \in \hat{\mathbb{X}}_i^{\mathbb{U}} \tag{7.21}$$

7.2.3 Computation of the terminal set

An invariant set for the nominal closed-loop subsystem (7.10)- (7.12) must be computed, together with a set of output reference values $\hat{r}^{[i]}$, where $(\hat{\xi}_k^{[i]}, \hat{r}^{[i]})$ must lie in order to guarantee that constraints (7.18) are verified for all k when the control law (7.12) is used. To this end, as explained in Chapter 1, it is sufficient to compute the maximal output admissible set (MOAS) [59] $\hat{\mathbb{O}}_i$ for the following system:

$$\begin{bmatrix} \hat{\xi}_{k+1}^{[i]} \\ \hat{r}^{[i]} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_{ii} & 0 \\ 0 & I_{m_i} \end{bmatrix} \begin{bmatrix} \hat{\xi}_k^{[i]} \\ \hat{r}^{[i]} \end{bmatrix} \tag{7.22a}$$

$$\begin{bmatrix} \hat{x}_k^{[i]} \\ \hat{u}_{k-1}^{[i]} \end{bmatrix} = C_i^* \begin{bmatrix} \hat{\xi}_k^{[i]} \\ \hat{r}^{[i]} \end{bmatrix} \tag{7.22b}$$

where (7.22a) and (7.22b) act as a state equation and as an output equation, respectively.

Assumption 7.5 *For each subsystem:*

i) the pair $\left(\begin{bmatrix} \mathcal{F}_{ii} & 0 \\ 0 & I_{m_i} \end{bmatrix}, C_i^*\right)$ *is observable.*

ii) $\hat{\mathbb{X}}_i^{\mathbb{U}}$ *is a closed polytope.*

□

Under this assumption, see Chapter 1, an invariant, polytopic inner approximation $\hat{\mathbb{O}}_i^\epsilon$ to the MOAS can be computed in a finite number of steps. Specifically, $\hat{\mathbb{O}}_i^\epsilon$ is defined as follows

$$\hat{\mathbb{O}}_i^\epsilon = \{(\hat{\xi}^{[i]}, \hat{r}^{[i]}) \in \mathbb{R}^{n_i+2m_i} : C_i^\xi \mathcal{F}_{ii}^k \hat{\xi}^{[i]} + C_i^y \hat{r}^{[i]} \in \hat{\mathbb{X}}_i^{\mathbb{U}} \,\forall\, k \geq 0, \\ C_i^y \hat{r}^{[i]} \in \hat{\mathbb{X}}_i^{\mathbb{U}}(\epsilon)\} \quad (7.23)$$

where $\hat{\mathbb{X}}_i^{\mathbb{U}}(\epsilon)$ is a close and compact set satisfying $\hat{\mathbb{X}}_i^{\mathbb{U}}(\epsilon) \oplus \mathcal{B}_\epsilon^{n_i+m_i}(0) \subseteq \hat{\mathbb{X}}_i^{\mathbb{U}}$, and ϵ can be arbitrarily small.

7.2.4 *i*-DePC problems

At any time step k an MPC optimization problem is solved where, similarly to [107], the predictions of the state variables are computed using the nominal model (7.10), while its initial condition $\hat{\xi}_k^{[i]}$ is regarded to as an optimization variable, as well as the nominal input sequence $\delta\hat{u}_{[k:k+N-1]}^{[i]}$.

In the proposed control scheme, infeasible reference set-points are handled similarly to [88]. Specifically, the value $\hat{r}_k^{[i]}$, which is the value of $\hat{r}^{[i]}$ decided at time k, is regarded to as an argument of the optimization problem itself, rather than a fixed parameter. Being $\hat{\xi}_k^{[i]} = (\delta\hat{x}_k^{[i]}, (\hat{y}_k^{[i]} - \hat{r}_k^{[i]}))$, this means that $\delta\hat{x}_k^{[i]}$, $\hat{y}_k^{[i]}$ and $\hat{r}_k^{[i]}$ are all different arguments of the optimization problem.

The decentralized predictive control problem solved on-line by the i-th subsystem (denoted i-DePC) at each discrete time instant k is:

$$V_i^{N*}(r^{o,[i]}, \delta x_k^{[i]}, y_k^{[i]}) = \\ \min_{\delta\hat{x}_k^{[i]}, \hat{y}_k^{[i]}, \hat{r}_k^{[i]}, \delta\hat{u}_{[k:k+N-1]}^{[i]}} V_i^N(\delta\hat{x}_k^{[i]}, \hat{y}_k^{[i]}, \hat{r}_k^{[i]}, \delta\hat{u}_{[k:k+N-1]}^{[i]}; r^{o,[i]}, \delta x_k^{[i]}, y_k^{[i]}) \quad (7.24)$$

where

$$V_i^N = \|\hat{r}_k^{[i]} - r^{o,[i]}\|_T^2 + \sum_{\nu=k}^{k+N-1} \{\|\hat{\xi}_\nu^{[i]}\|_{Q_i}^2 + \|\delta\hat{u}_\nu^{[i]}\|_{R_i}^2\} + \|\hat{\xi}_{k+N}^{[i]}\|_{P_i}^2$$

subject to the dynamic constraint (7.10), to the constraints

$$\begin{bmatrix} I_{n_i+m_i} \\ 0 \end{bmatrix} (\xi_k^{[i]} - \hat{\xi}[i]_k) \in \begin{bmatrix} 0 \\ -w_{k-1}^{[i]} \end{bmatrix} \oplus \tilde{\mathbb{Z}}_i \quad (7.25)$$

$$C_i^* \begin{bmatrix} \hat{\xi}^{[i]}_{k+k} \\ \hat{r}^{[i]}_{k} \end{bmatrix} \in \hat{\mathbb{X}}_i^{\mathbb{U}} \tag{7.26}$$

for all $\nu = 1, \dots, N-1$, and to the terminal constraint

$$\begin{bmatrix} \hat{\xi}^{[i]}_{k+N} \\ \hat{r}^{[i]}_{k} \end{bmatrix} \in \hat{\mathbb{O}}_i^{\epsilon} \tag{7.27}$$

It is worth remarking that the term $w^{[i]}_{k-1}$ used in (7.25) can be computed, at time k, by subsystem i using (7.3). At any time instant the optimal solution $\delta\hat{x}^{[i]}_{k|k}, \hat{y}^{[i]}_{k|k}, \hat{r}^{[i]}_{k|k}, \delta\hat{u}^{[i]}_{[k:k+N-1]|k}$ is found and the control law (7.11) is applied with $\delta\hat{u}^{[i]}_{k} = \delta\hat{u}^{[i]}_{k|k}$ and $\hat{\xi}^{[i]}_{k} = (\delta\hat{x}^{[i]}_{k|k}, (\hat{y}^{[i]}_{k|k} - \hat{r}_{k|k})^{[i]})$.

The convergence properties of the proposed control algorithm can be proved by properly selecting the tuning parameters: $Q_i \succ 0$ and $R_i \succ 0$, while $P_i \succ 0$ is assumed to be chosen as the solution of the Lyapunov equation

$$\mathcal{F}_{ii}^T P_i \mathcal{F}_{ii} - P_i = -(Q_i + \mathcal{K}_i^T R_i \mathcal{K}_i) \tag{7.28}$$

As for the matrix T_i, decompose $P_i \in \mathbb{R}^{(n_i+m_i)\times(n_i+m_i)}$ as follows

$$P_i = \begin{bmatrix} P_i^{xx} & P_i^{xy} \\ P_i^{yx} & P_i^{yy} \end{bmatrix}$$

where $P_i^{xx} \in \mathbb{R}^{n_i \times n_i}$, and select T_i to satisfy the inequality

$$T_i - P_i^{yy} \succ 0 \tag{7.29}$$

7.3 Convergence results

In this section the main convergence properties of the proposed method are given. In particular, we first resort to the results provided in Chapter 6 as a preliminary step. Importantly, if Assumption 7.4 is verified for all i, the decentralized control problem is, as a matter of fact, decomposed into independent "decoupled" robust MPC problems. The recursive feasibility properties and the convergence of the nominal subsystems' outputs $\hat{y}^{[i]}_t$ is therefore guaranteed by the theory described in Chapter 6 reported here for completeness.

Theorem 7.1 ([14]) *For each $i = 1, \dots, M$, let Assumption 7.2 be verified and the design parameters Q_i, R_i, P_i, T_i, $\tilde{\mathbb{Z}}_i$, $\hat{\mathbb{X}}_i^{\mathbb{U}}$, and $\hat{\mathbb{O}}_i^{\epsilon}$*

be chosen as specified. Then, if at time $k = 0$ a feasible solution to the i-DePC optimization problem (7.24) - (7.27) exists, then the resulting control law asymptotically steers the nominal output $\hat{y}_k^{[i]}$ of subsystem (7.9) - (7.10) to the admissible set-points $r_{ad}^{[i]}$, where

$$r_{ad}^{[i]} = \underset{y^{[i]} \in \begin{bmatrix} 0 & I_{m_i} \end{bmatrix} \mathbb{O}_i^{\rho}}{\operatorname{argmin}} \|y^{[i]} - r^{o,[i]}\|_T^2 \tag{7.30}$$

Moreover $\delta\hat{x}_{k|k}^{[i]} \to 0$, $\delta\hat{u}_{k:k+N-1|k}^{[i]} \to 0$ as $k \to \infty$ and the constraints $(x_k^{[i]}, u_k^{[i]}) \in \mathbb{X}_i \times \mathbb{U}_i$ are fulfilled for all $k \geq 0$. ■

The result stated in Theorem 7.1 is used to prove convergence of the real outputs $y_k^{[i]}$ to a neighborhood of the desired (feasible) reference set-points. The proof is provided by considering the collective system.

Corollary 7.1 *For all $i = 1, \dots, M$, let Assumptions 7.2 and 7.3 be verified and the design parameters Q_i, R_i, P_i, T_i, $\tilde{\mathbb{Z}}_i$, $\hat{\mathbb{X}}_i^{\mathbb{U}}$, and $\hat{\mathbb{O}}_i^{\epsilon}$ be chosen as specified. Then, if at time $k = 0$ a feasible solution to the i-DePC optimization problem (7.24) - (7.27) exists, then the constraints $(x_k^{[i]}, u_k^{[i]}) \in \mathbb{X}_i \times \mathbb{U}_i$ are fulfilled for all $k \geq 0$ and for all $i = 1, \dots, M$. Moreover, the resulting control law asymptotically steers the output $\mathbf{y}_k$ of system (7.1) to:*

I) the value $\mathbf{r}_{ad}$ in case the exogenous unknown disturbance $\mathbf{v}$ is constant;

II) the set $\mathbf{r}_{ad} \oplus \Delta_r$, in case the unknown exogenous unknown disturbance $\mathbf{v}$ is time-varying. Here $\mathbf{r}_{ad} = (r_{ad}^{[1]}, \dots, r_{ad}^{[M]})$ and $\Delta_r = \mathbf{H}_y \mathbb{Z}_v$, where $\mathbb{Z}_v$ is the minimal RPI set for the collective system

$$\begin{bmatrix} \boldsymbol{\xi}_{k+1} \\ \boldsymbol{\mu}_{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{F}} & -\boldsymbol{\mathcal{B}}^w \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_t \\ \boldsymbol{\mu}_k \end{bmatrix} + \begin{bmatrix} \boldsymbol{\mathcal{B}}^w \\ I_n \end{bmatrix} \mathbf{v}_k \tag{7.31}$$

having defined $\boldsymbol{\mu}_k = (\mu_k^{[1]}, \dots, \mu_k^{[M]}) = \mathbf{v}_{k-1}$, while $\mathbf{H}_y \in \mathbb{R}^{m \times (2n+m)}$ is the matrix that selects vector $\boldsymbol{\varepsilon}_k$ out of $(\boldsymbol{\xi}_k, \boldsymbol{\mu}_k)$.

■

Proof 7.1 *First, using Theorem 7.1 we can guarantee that for all $i = 1, \dots, M$ $\hat{y}_k^{[i]} \to r_{ad}^{[i]}$, $\delta\hat{x}_{k|k}^{[i]} \to 0$, $\delta\hat{u}_{k:k+N-1|k}^{[i]} \to 0$ as $k \to \infty$ and that* *the constraints* $(x_k^{[i]}, u_k^{[i]}) \in \mathbb{X}_i \times \mathbb{U}_i$ *are fulfilled for all* $k \geq 0$. *Secondly, we define* $\varepsilon_k^{a,[i]} = y_k^{[i]} - r_{ad}^{[i]}$ *and* $\hat{\varepsilon}_k^{a,[i]} = \hat{y}_k^{[i]} - r_{ad}^{[i]}$. *Similarly to* (7.7), *we write*

$$\begin{bmatrix} \delta x_{k+1}^{[i]} \\ \varepsilon_{k+1}^{a,[i]} \end{bmatrix} = \sum_{j \in \mathcal{N}_i \cup \{i\}} \left\{ \mathcal{A}_{ij} \begin{bmatrix} \delta x_k^{[j]} \\ \varepsilon_k^{a,[j]} \end{bmatrix} + \mathcal{B}_{ij} \delta u_k^{[j]} \right\} + \mathcal{B}_i^w \delta v_k^{[i]} \tag{7.32}$$

Under (7.11) *and recalling that* $\varepsilon_k^{a,[i]} - \hat{\varepsilon}_k^{a,[i]} = y_k^{[i]} - \hat{y}_k^{[i]} = \varepsilon_k^{[i]} - \hat{\varepsilon}_k^{[i]}$, *equation* (7.32) *is equivalent to*

$$\begin{bmatrix} \delta x_{k+1}^{[i]} \\ \varepsilon_{k+1}^{a,[i]} \end{bmatrix} = \sum_{j \in \mathcal{N}_i \cup \{i\}} (\mathcal{A}_{ij} + \mathcal{B}_{ij} \mathcal{K}_j) \begin{bmatrix} \delta x_k^{[j]} \\ \varepsilon_k^{a,[j]} \end{bmatrix} + \eta_k^{[i]} + \mathcal{B}_i^w \delta v_k^{[i]} \tag{7.33}$$

where

$$\eta_k^{[i]} = \sum_{j \in \mathcal{N}_i \cup \{i\}} \mathcal{B}_{ij} (\delta \hat{u}_k^{[j]} - \mathcal{K}_j \begin{bmatrix} \delta \hat{x}_k^{[j]} \\ \hat{\varepsilon}_k^{a,[j]} \end{bmatrix})$$

is an asymptotically vanishing term. Collectively, defining $\boldsymbol{\xi}_k^a = (\delta x_{k+1}^{[1]}, \varepsilon_{k+1}^{a,[1]}, \delta x_{k+1}^{[2]}, \ldots, \varepsilon_{k+1}^{a,[M]})$, $\boldsymbol{\mu}_k = (\mu_k^{[1]}, \ldots, \mu_k^{[M]}) = \mathbf{v}_{k-1}$, *and* $\boldsymbol{\eta}_t = (\eta_k^{[1]}, \ldots, \eta_k^{[M]})$, *we can write equation* (7.33) *as*

$$\begin{bmatrix} \boldsymbol{\xi}_{k+1}^a \\ \boldsymbol{\mu}_{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{F}} & -\boldsymbol{\mathcal{B}}^w \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_k^a \\ \boldsymbol{\mu}_k \end{bmatrix} + \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \boldsymbol{\eta}_k + \begin{bmatrix} \boldsymbol{\mathcal{B}}^w \\ I_n \end{bmatrix} \mathbf{v}_k \tag{7.34}$$

Since $\boldsymbol{\mathcal{F}}$ *is Schur, in view of Assumption 7.3* i), *then:*

I) in case $\mathbf{v}_k$ *is constant, i.e.,* $\mathbf{v}_k = \bar{\mathbf{v}}$ *for all* $k \geq 0$, *the limit set for the trajectories of* $\boldsymbol{\xi}_k^a$ *in equation* (7.34) *is the origin, which implies that* $y_k^{[i]} \to \hat{y}_k^{[i]}$ *as* $k \to +\infty$ *for all* $i = 1, \ldots, M$.

II) In general, see [133], the limit set of all trajectories $(\boldsymbol{\xi}_k^a, \boldsymbol{\mu}_k)$ *is* $\mathbb{Z}_v$, *from which it follows that* $\mathbf{y}_k \to \hat{\mathbf{y}}_k \oplus \mathbf{H}_y \mathbb{Z}_v$.

7.4 Example

Consider the problem of regulating the temperatures T_A, T_B, T_C and T_D of the four rooms of the building sketched in Figure 7.1. Rooms A and B belong to the first apartment while rooms C and D to the second one. Each room has a radiator supplying heats q_A, q_B, q_C and q_D. We refer the reader to Chapter 2 for details about the dynamic

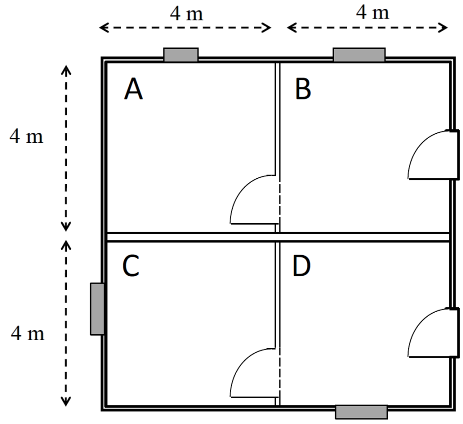

Figure 7.1: Schematic representation of a building with two apartments

model, the values of the parameters and the considered equilibrium point.

The discrete-time system of the form (7.1) (with $n = 4$ and $m = 4$) is obtained by zero-order-hold discretization with sampling time $T = 10$ s. The partition of inputs, outputs and states is:

$$x^{[1]} = \begin{bmatrix} \delta T_A & \delta T_B \end{bmatrix}^T, \; u^{[1]} = \begin{bmatrix} \delta q_A & \delta q_B \end{bmatrix}^T, \; y^{[1]} = \begin{bmatrix} \delta T_A & \delta T_B \end{bmatrix}^T$$

$$x^{[2]} = \begin{bmatrix} \delta T_C & \delta T_D \end{bmatrix}^T, \; u^{[2]} = \begin{bmatrix} \delta q_C & \delta q_D \end{bmatrix}^T, \; y^{[2]} = \begin{bmatrix} \delta T_C & \delta T_D \end{bmatrix}^T$$

The considered constraints on the inputs and the states of the linearized system have been chosen as:

$$x^{[1]}_{min} = x^{[2]}_{min} = (-5, -5), \; x^{[1]}_{max} = x^{[2]}_{max} = (10, 10)$$

$$u^{[1]}_{min} = u^{[2]}_{min} = (-0.26, -0.26), \; u^{[1]}_{max} = u^{[2]}_{max} = (0.25, 0.25)$$

The matrices $\mathcal{K}_i$ fulfilling Assumption 7.3 are obtained by solving suitable linear matrix inequalities [18].The weighting matrices used in the simulation are $Q_1 = Q_2 = 2I_2$, $R_1 = R_2 = I_2$. We finally set $T_i = 10P_i^{yy}$ and $N = 3$.

In the simulations, the reference trajectories for $y^{[2]}_{set-point}$ are both always equal to zero, as well as the one related to T_B, while T_A must track a piece-wise constant reference trajectory, whose values are 7, -7 and 3. The results are shown in Figure 7.2, while the values of the input variables are depicted in Figure 7.3. To show the capability of our algorithm to reject constant external disturbances, a step of the external temperature is added, i.e., $T_E(t) = 10$ °C for $t \geq 4.5$ min.

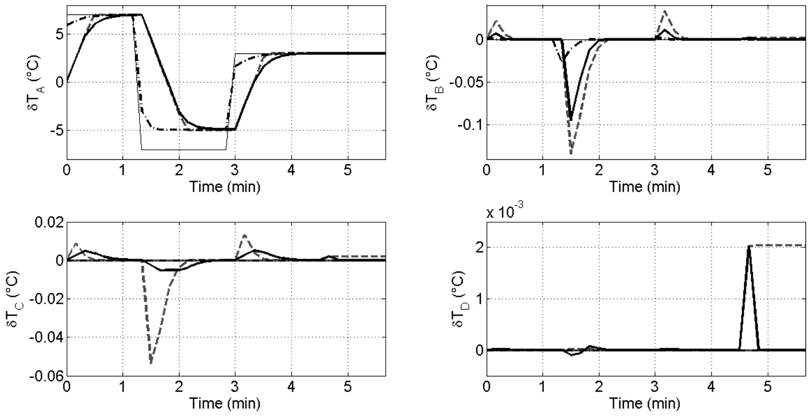

Figure 7.2: Trajectories of the output variables $y^{[1]}$ (above) and $y^{[2]}$ (below) obtained with DPC (black solid lines) and with cMPC (dashed gray lines). Think black lines: desired set-points $y^{[1,2]}_{set-point}$; black dash-dot lines: reference trajectories $\tilde{y}^{1,2}$.

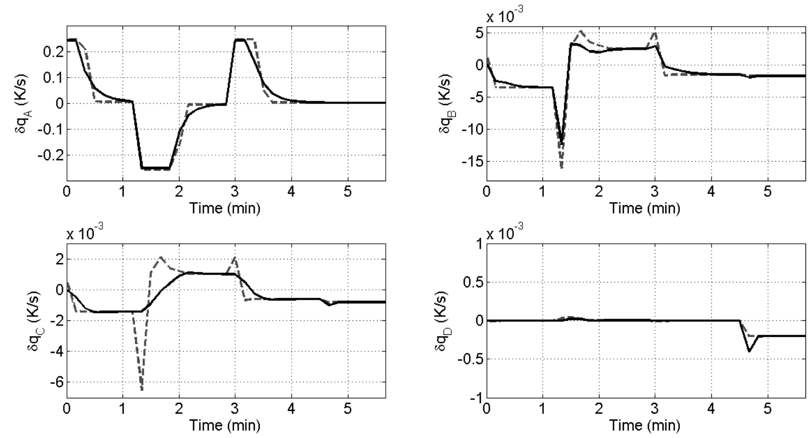

Figure 7.3: Trajectories of the inputs variables $u^{[1]}$ (above) and $u^{[2]}$ (below) obtained with DPC (black solid lines) and with cMPC (dashed gray lines).

In both these figures a comparison between the outputs obtained with DPC and with centralized MPC is provided, showing the effectiveness of our approach. Note that the infeasible target -7 is managed reaching the closest admissible output.

7.5 Conclusions

In this Chapter we presented a decentralized control algorithm based on MPC for offset-free tracking of varying (piecewise constant) references. The integral action has been inserted in the closed-loop rewriting the dynamic system in velocity-form. The proposed controller can handle infeasible references and is fully decentralized. As no information is required to be transmitted among subsystems, this technique can be useful in cases where communication is not possible or is affected by frequent disruptions.

Part IV

Conclusions

8

Conclusions

In this thesis several algorithm based on MPC for distributed and decentralized control of linear systems subject to constraints on inputs and states have been presented. All the control techniques rely on the idea of considering the coupling terms among the subsystems as disturbances to be rejected. Specifically, each control system is required to reject a know disturbance, constituted by the nominal state and inputs trajectories of the other subsystems, and an unknown (but bounded) disturbance, which is the differences between the nominal trajectories and the real values of the inputs and of the states of the neighbors. To reject this second term, the robust tube-based MPC approach has been adopted.

Initially, a regulation problem for distributed control has been solved, and some practical implementation issues have been presented and solved. A continuous-time version of the proposed approach has been also provided. Then, the tracking problem has been considered and a number of solutions has been proposed. The first one can be used to track a piecewise constant target, and is based on the inclusion among the optimization variables of the MPC problem of the value of the reference point that each subsystem really tracks. A penalization of its distance from the desired external set point guarantees feasibility at each time instant. To reject constant disturbances, a second solution has been developed by inserting an integral action in the closed-loop. To this end, the system under control has been rewritten in velocity-form assuming that the desired setpoint is constant. In order to obtain an algorithm exploiting the velocity-form capabilities to be used also in case of variable references, an in-depth study of the properties of systems rewritten in velocity-form for the centralized case has been presented. Eventually, the results derived for this latter case have been used to develop a completely decentralized approach with integral action for tracking piecewise constant references.

All the presented algorithms have been tested in simulation, both to understand their performances and to evaluate the complexity of the controller design and the required online computational load.

Several properties make the control methods shown in this thesis suitable also for industrial applications. First of all, they all rely on the robust tube-based MPC algorithm, therefore it is straightforward to extend them in order to include the possibility of rejecting additive external disturbances. In particular, huge efforts have been put in studying how to insert an integral action in the closed-loop system. Such characteristic is extremely important for the industry, since it allows one to perfectly reject constant unknown disturbances and to delete steady-state offsets caused by errors in the model. Moreover, all the algorithms have been designed for controlling a wide class of large scale systems, in fact they can be used for dynamically coupled subsystems, as well as for subsystems with coupled constraints (not considered in this thesis).

Secondly, concerning the distributed techniques, the transmission of information is extremely limited. Each subsystem is required to know only the model governing its dynamics and how it is influenced by the inputs and outputs of the neighboring subsystems. Only neighbor-to-neighbor communication among subsystems is needed and the data transmitted, at each time instant, by each subsystem correspond only to the value of the reference trajectories of states and inputs at the end of the prediction horizon. Notably, information must be transmitted and received with a non-iterative communication protocol i.e, once within a sampling time. The decentralized control scheme, by definition adopted in this thesis, does not require any online transmission of information at all. We also remark that as the number of subsystems grows (while, for instance, the average number of neighbors for each subsystem remains constant), the information required to be stored, processed and transmitted by each subsystem not linked to the new subsystems remains constant.

Extensions to the output-feedback control case have already been developed and do not represent a critical issue. Basically, this improvement only needs to apply the same approach previously described for coping with unknown exogenous additive disturbances.

An important result is also that the online computational load required to each local controller is the same of a standard robust tube-based MPC algorithm. Thus, since the large-scale constrained optimal control problem is partitioned in many independent subproblems, the overall computational complexity is extremely reduced. Note also that the number of optimization variables in the local optimization problems remains constant regardless of the topology (and complexity) of the interconnection network.

Eventually, as shown in the reported simulation examples, even if the solution computed at each sampling time by the local controllers is in general globally suboptimal considering the whole large-scale system, the performances obtained using the distributed and decentralized algorithms have always been close to those of centralized controllers.

On the other hand, the proposed distributed techniques are affected by some non-negligible issues. First of all, the offline design can be extremely difficult. The problem of computing the RPI sets for all the interconnected subsystems has been solved only through methods based on sufficient conditions and that depend on some arbitrary parameter choices. It is still not possible to evaluate if the proposed methods can be applied to a given system without resorting to a trial-and-error (possibly infinite) procedure.

Secondly, also when one of the described distributed algorithm can be used for controlling a given system, the offline computational load due to the required sets manipulations can be extremely heavy. In practice, it turns out to be difficult to apply such techniques to systems constituted by subsystems with more than 6 or 7 states.

Finally, the fact that the interactions among subsystems are seen as disturbances to be rejected implies that the controllers described in this thesis are suited for systems characterized by weak interactions.

To overcome these problems, distributed algorithms based on probabilistic approaches are currently being studied. Also possible variations where robust approaches are (partially) substituted by an increased communication load could help in enlarging the range of systems where successful applications can be obtained.

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