POLITECNICO DI MILANO
DEPARTMENT OF MATHEMATICS
DOCTORAL PROGRAMME IN
MATHEMATICAL MODELS AND METHODS IN ENGINEERING

ASYMPTOTIC ANALYSIS OF EVOLUTION EQUATIONS WITH NONCLASSICAL HEAT CONDUCTION

Doctoral Dissertation of: FILIPPO DELL'ORO

Advisor: Prof. VITTORINO PATA

Tutor: Prof. DARIO PIEROTTI
Chair of the PhD Program: Prof. ROBERTO LUCCHETTI

CYCLE XXVI

Summary

The present doctoral thesis deals with asymptotic behavior of evolution equations with nonclassical heat conduction. First, we consider the strongly damped nonlinear wave equation on a bounded smooth domain $\Omega \subset \mathbb{R}^3$

$$u_{tt} - \Delta u_t - \Delta u + f(u_t) + g(u) = h$$

which serves as a model in the description of type III thermal evolution within the theory of Green and Naghdi. In particular, the nonlinearity f acting on u_t is allowed to be nonmonotone and to exhibit a critical growth of polynomial order 5. The main focus is the longterm analysis of the related solution semigroup, which is shown to possess global and exponential attractors of optimal regularity in the natural weak energy space. Then, we analyze two evolution systems ruling the dynamics of type III thermoelastic extensible beams or Berger plates with memory. Specifically, we study the decay properties of the solution semigroup generated by an abstract version of the linear system

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta \alpha_t = 0 \\ \alpha_{tt} - \Delta \alpha - \displaystyle\int_0^\infty \mu(s) \Delta[\alpha(t) - \alpha(t-s)]\,\mathrm{d}s - \Delta u_t = 0 \end{cases}$$

along with the limit situation without memory

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta \alpha_t = 0 \\ \alpha_{tt} - \Delta \alpha - \Delta \alpha_t - \Delta u_t = 0 \end{cases}$$

and the existence of regular global attractors for an abstract version of the nonlinear model

$$\begin{cases} u_{tt} - \omega \Delta u_{tt} + \Delta^2 u - \big[b + \|\nabla u\|^2_{L^2(\Omega)}\big] \Delta u + \Delta \alpha_t = g \\ \alpha_{tt} - \Delta \alpha - \displaystyle\int_0^\infty \mu(s) \Delta[\alpha(t) - \alpha(t-s)]\,\mathrm{d}s - \Delta u_t = 0. \end{cases}$$

Moreover, we discuss the asymptotic behavior of the nonlinear type III Caginalp phase-field system

$$\begin{cases} u_t - \Delta u + \phi(u) = \alpha_t \\ \alpha_{tt} - \Delta\alpha_t - \Delta\alpha + g(\alpha) = -u_t \end{cases}$$

on a bounded smooth domain $\Omega \subset \mathbb{R}^3$, with nonlinearities ϕ and g of polynomial critical growth 5, proving the existence of the regular global attractor. Finally, we analyze the linear differential system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0 \\ \rho_3\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s + \delta\psi_{tx} = 0 \end{cases}$$

describing a Timoshenko beam coupled with a temperature evolution of Gurtin-Pipkin type. A necessary and sufficient condition for exponential stability is established in terms of the structural parameters of the equations. In particular, we generalize previously known results on the Fourier-Timoshenko and the Cattaneo-Timoshenko beam models.

In the first chapter of the thesis we introduce some preliminary results about infinite-dimensional dynamical systems and linear semigroups needed in the course of the investigation. The remaining chapters correspond to the following papers, written during the three years of PhD.

- F. Dell'Oro and V. Pata, *Long-term analysis of strongly damped nonlinear wave equations,* Nonlinearity **24** (2011), 3413–3435, (Chapter 2 and Chapter 3).
- F. Dell'Oro and V. Pata, *Strongly damped wave equations with critical nonlinearities,* Nonlinear Anal. **75** (2012), 5723–5735, (Chapter 4).
- F. Dell'Oro, *Global attractors for strongly damped wave equations with subcritical-critical nonlinearities,* Commun. Pure Appl. Anal. **12** (2013), 1015–1027, (Chapter 5).
- M. Coti Zelati, F. Dell'Oro and V. Pata, *Energy decay of type III linear thermoelastic plates with memory,* J. Math. Anal. Appl. **401** (2013), 357–366, (Chapter 6).
- F. Dell'Oro and V. Pata, *Memory relaxation of type III thermoelastic extensible beams and Berger plates,* Evol. Equ. Control Theory **1** (2012), 251–270, (Chapter 6).
- M. Conti, F. Dell'Oro and A. Miranville, *Asymptotic behavior of a generalization of the Caginalp phase-field system,* Asymptot. Anal. **81** (2013), 297–314, (Chapter 7).
- F. Dell'Oro and V. Pata, *On the stability of Timoshenko systems with Gurtin-Pipkin thermal law,* submitted, (Chapter 8).

Acknowledgments

I thank my advisor Prof. Vittorino Pata for introducing me to the fascinating field of infinite-dimensional dynamical systems, and for the competence, enthusiasm, professionality and constant guidance during these three years of PhD.

I also thank Prof. Monica Conti for the interest, help and several remarkable teachings, and Prof. Jaime Muñoz Rivera for the kind hospitality in Petrópolis and many stimulating discussions.

I express my sincere gratitude to my family for the support and the quotidian encouragement. I heartily thank Silvia for her love and her constant precious presence in my life.

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Introduction

The aim of the research contained in the present doctoral thesis is the mathematical analysis of well-posedness and asymptotic behavior of linear and nonlinear dissipative partial differential equations with nonclassical heat conduction, that is, thermal evolutions where the temperature may travel with finite speed propagation. In the linear case, we mainly focus on the stability properties of the associated semigroups, analyzing the decay to zero of the solutions. In the nonlinear situation, we dwell on existence and regularity of finite-dimensional global and exponential attractors, providing a complete description of the asymptotic dynamics by means of suitable "small" regions of the phase space.

Hereafter is a detailed discussion of the models considered in the thesis. In particular, the nonclassical character of the temperature is stressed and the physical meaning and relevance are explained.

Nonlinear Heat Conduction of Type III

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. The thermal evolution in a homogenous isotropic (rigid) heat conductor occupying the space-time cylinder $\Omega_T = \Omega \times (0, T)$ is governed by the balance equation

$$e_t + \operatorname{div} \mathbf{q} = F.$$

Here, the internal energy e is a function of the *relative temperature field*

$$\vartheta = \vartheta(\boldsymbol{x}, t) : \Omega_T \to \mathbb{R},$$

that is, the temperature variation from an equilibrium reference value, while

$$\mathbf{q} = \mathbf{q}(\boldsymbol{x}, t) : \Omega_T \to \mathbb{R}^3$$

is the *heat flux vector*. Finally, F represents a source term. We also assume the Dirichlet boundary condition

$$\vartheta(\boldsymbol{x}, t)_{|\boldsymbol{x} \in \partial\Omega} = 0,$$

expressing the fact that the boundary $\partial\Omega$ of the conductor is kept at null (i.e. equilibrium) temperature for all times. Considering only small variations of ϑ and $\nabla\vartheta$, the internal energy fulfills with good approximation the equality

$$e(\boldsymbol{x}, t) = e_0(\boldsymbol{x}) + c\vartheta(\boldsymbol{x}, t),$$

where e_0 is the internal energy at equilibrium and $c > 0$ is the specific heat. Accordingly, the balance equation becomes

$$c\vartheta_t + \operatorname{div} \mathbf{q} = F. \tag{1}$$

For a general F of the form

$$F(\boldsymbol{x}, t) = -f(\vartheta(\boldsymbol{x}, t)) + h(\boldsymbol{x}), \tag{2}$$

accounting for the simultaneous presence of a time-independent external heat supply and a nonlinearly temperature-dependent internal source, equation (1) reads

$$c\vartheta_t + \operatorname{div} \mathbf{q} + f(\vartheta) = h. \tag{3}$$

To complete the picture, a further relation is needed: the so-called constitutive law for the heat flux, establishing a link between $\mathbf{q}$ and ϑ. In fact, the choice of the constitutive law is what really determines the model. At the same time, being a purely heuristic interpretation of the physical phenomenon, it may reflect different individual perceptions of reality, or even philosophical beliefs. For instance, for the classical Fourier constitutive law

$$\mathbf{q} + \kappa\nabla\vartheta = 0, \quad \kappa > 0, \tag{4}$$

we deduce from (3) the familiar reaction-diffusion equation

$$c\vartheta_t - \kappa\Delta\vartheta + f(\vartheta) = h.$$

Nevertheless, such an equation predicts instantaneous propagation of (thermal) signals, a typical side-effect of parabolicity. This feature, sometimes called the *paradox of heat conduction* (see e.g. [14, 32]), has often encountered strong criticism in the scientific community, up to be perceived as "physically unrealistic" by some authors. Therefore, several attempts have been made through the years in order to introduce some hyperbolicity in the mathematical modeling of heat conduction (see e.g. [8, 48]). A possible choice is adopting the Maxwell-Cattaneo law [8], namely, the differential perturbation of (4)

$$\mathbf{q} + \varepsilon\mathbf{q}_t + \kappa\nabla\vartheta = 0, \quad \kappa \gg \varepsilon > 0. \tag{5}$$

In which case, the sum (3)+$\varepsilon\partial_t$(3) entails the hyperbolic reaction-diffusion equation

$$\varepsilon c\vartheta_{tt} - \kappa\Delta\vartheta + [c + \varepsilon f'(\vartheta)]\vartheta_t + f(\vartheta) = h,$$

widely employed in the description of many interesting phenomena, such as chemical reacting systems, gene selection, population dynamics or forest fire propagation, to name a few (cf. [33, 59, 60]). Another strategy is relaxing (4) by means of a time-convolution against a suitable (e.g. convex, decreasing and summable) kernel μ. Precisely, omitting the dependence on $\boldsymbol{x}$,

$$\mathbf{q}(t) = -\kappa_0\nabla\vartheta(t) - \int_{-\infty}^{t} \mu(t-s)\nabla\vartheta(s)\,\mathrm{d}s. \tag{6}$$

The constant κ_0 can be either strictly positive or zero, according to the models of Coleman-Gurtin [18] or Gurtin-Pipkin [48], respectively. Plugging (6) into (3) we end up with the integrodifferential equation

$$c\vartheta_t - \kappa_0 \Delta\vartheta - \int_0^\infty \mu(s)\Delta\vartheta(t-s)\,\mathrm{d}s + f(\vartheta) = h.$$

Quite interestingly, in the (fully hyperbolic) case $\kappa_0 = 0$, we recover (5) as the particular instance of (6) corresponding to the kernel

$$\mu(s) = \frac{\kappa}{\varepsilon}\,\mathrm{e}^{-s/\varepsilon}.$$

In a different fashion, the theory of *heat conduction of type III* devised by Green and Naghdi [43–47, 90] considers instead a perturbation of the classical law (4) of integral kind. Indeed, the Fourier law is modified in the following manner:

$$\mathbf{q} + \kappa\nabla\vartheta + \omega\nabla u = 0, \quad \kappa, \omega > 0, \tag{7}$$

where an additional independent variable appears: the *thermal displacement* $u : \Omega_T \to \mathbb{R}$, defined as

$$u(\boldsymbol{x}, t) = u(\boldsymbol{x}, 0) + \int_0^t \vartheta(\boldsymbol{x}, s)\,\mathrm{d}s,$$

hence satisfying the equality

$$u_t(\boldsymbol{x}, t) = \vartheta(\boldsymbol{x}, t), \quad \forall\,(\boldsymbol{x}, t) \in \Omega_T,$$

and (for consistency) complying with the Dirichlet boundary condition

$$u(\boldsymbol{x}, t)_{|\boldsymbol{x}\in\partial\Omega} = 0.$$

Using (7) the balance equation (1) turns into

$$cu_{tt} - \kappa\Delta u_t - \omega\Delta u = F.$$

Replacing for more generality (2) with

$$F(\boldsymbol{x}, t) = -f(\vartheta(\boldsymbol{x}, t)) - g(u(\boldsymbol{x}, t)) + h(\boldsymbol{x}),$$

allowing the source term to contain a further contribution depending nonlinearly on the thermal displacement, we finally arrive at the boundary-value problem

$$\begin{cases} cu_{tt} - \kappa\Delta u_t - \omega\Delta u + f(u_t) + g(u) = h, \\ u_{|\partial\Omega} = 0, \\ u_{t|\partial\Omega} = 0, \end{cases}$$

which will be analyzed in Chapters 2-5. We refer the reader to [80–87] for discussions and other developments related to type III heat conduction.

Type III Thermoelastic Extensible Beams and Berger Plates

For $n = 1, 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Given the parameters $\omega > 0$ and $b \in \mathbb{R}$, we consider the evolution system of coupled equations on the space-time cylinder $\Omega_+ = \Omega \times \mathbb{R}^+$

$$u_{tt} - \omega\Delta u_{tt} + \Delta^2 u - \big[b + \|\nabla u\|^2_{L^2(\Omega)}\big]\Delta u + \Delta\vartheta = g, \tag{8}$$

$$\vartheta_t + \operatorname{div}\mathbf{q} - \Delta u_t = 0, \tag{9}$$

in the unknown variables

$$u = u(\boldsymbol{x}, t) : \Omega_+ \to \mathbb{R}, \qquad \vartheta = \vartheta(\boldsymbol{x}, t) : \Omega_+ \to \mathbb{R}, \qquad \mathbf{q} = \mathbf{q}(\boldsymbol{x}, t) : \Omega_+ \to \mathbb{R}^n.$$

Such a system, written here in normalized dimensionless form, rules the dynamics of a thermoelastic extensible beam (for $n = 1$) or Berger plate (for $n = 2$) of shape Ω at rest (see [4, 93]). Accordingly, the variable u stands for the vertical displacement from equilibrium, ϑ is the (relative) temperature and $\mathbf{q}$ is the heat flux vector obeying some constitutive law, depending on one's favorite choice of heat conduction model. The term $-\omega\Delta u_{tt}$ appearing in the first equation witnesses the presence of rotational inertia, whereas the real parameter b accounts for the axial force acting in the reference configuration: $b > 0$ if the beam (or plate) is stretched, $b < 0$ if compressed. Finally, the function $g : \Omega \to \mathbb{R}$ describes a lateral load distribution. We also assume that the ends of the beam (or plate) are hinged, which translates into the hinged boundary conditions for u

$$u(\boldsymbol{x}, t)_{|\boldsymbol{x}\in\partial\Omega} = \Delta u(\boldsymbol{x}, t)_{|\boldsymbol{x}\in\partial\Omega} = 0,$$

and we take the Dirichlet boundary condition for ϑ

$$\vartheta(\boldsymbol{x}, t)_{|\boldsymbol{x}\in\partial\Omega} = 0,$$

expressing the fact that the boundary $\partial\Omega$ is kept at null (i.e. equilibrium) temperature for all times. It is worth noting that different boundary conditions for u are physically significant as well, such as the clamped boundary conditions

$$u_{|\partial\Omega} = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0,$$

where ν is the outer normal vector. However, the mathematical analysis carried out in this thesis (see Chapter 6) depends on the specific structure of the hinged boundary conditions (the so called "commutative case"). In the clamped case major modifications on the needed tools are required and the proofs become much more technical.

We are left to specify the constitutive relation for the heat flux, establishing a link between $\mathbf{q}$ and ϑ. Adopting for instance the classical Fourier law (the physical constants are set to 1)

$$\mathbf{q} = -\nabla\vartheta,$$

equation (9) turns into

$$\vartheta_t - \Delta\vartheta - \Delta u_t = 0.$$

In a different fashion, as already said, the theory of heat conduction of type III devised by Green and Naghdi considers a perturbation of the classical law of integral kind, by means of the so-called thermal displacement

$$\alpha(\boldsymbol{x}, t) = \alpha_0(\boldsymbol{x}) + \int_0^t \vartheta(\boldsymbol{x}, s)\,\mathrm{d}s,$$

satisfying the equality $\alpha_t = \vartheta$ and (for consistency) complying with the Dirichlet boundary condition

$$\alpha(\boldsymbol{x}, t)_{|\boldsymbol{x}\in\partial\Omega} = 0.$$

Then, the Fourier law is modified as

$$\mathbf{q} = -\nabla\alpha_t - \nabla\alpha,$$

so that (9) takes the form

$$\alpha_{tt} - \Delta\alpha - \Delta\alpha_t - \Delta u_t = 0. \tag{10}$$

Still, the equation predicts infinite speed propagation of (thermal) signals, due to its partially parabolic character which provides an instantaneous regularization of α_t. Such an effect is not expected (nor observed) in real conductors. Similarly to what done in [30] for the Fourier case, a possible answer is considering a memory relaxations of the above constitutive law of the form

$$\mathbf{q}(t) = -\int_0^\infty \kappa(s)\nabla\alpha_t(t-s)\,\mathrm{d}s - \nabla\alpha(t),$$

for some bounded convex summable function κ (the memory kernel) of total mass

$$\int_0^\infty \kappa(s)\,\mathrm{d}s = 1.$$

Up to a rescaling, we may also suppose $\kappa(0) = 1$. Accordingly, (9) becomes

$$\alpha_{tt} - \Delta\alpha - \int_0^\infty \kappa(s)\Delta\alpha_t(t-s)\,\mathrm{d}s - \Delta u_t = 0, \tag{11}$$

where the past history of the temperature is supposed to be known and regarded as an initial datum of the problem. It is readily seen that, when the function κ converges in the distributional sense to the Dirac mass at zero, equation (10) is formally recovered from (11). From the physical viewpoint, this means that (10) is close to (11) when the memory kernel is concentrated, i.e. when the system keeps a very short memory of the past effects. As a matter of fact, (11) can be given a more convenient form. Indeed, defining the differentiated kernel

$$\mu(s) = -\kappa'(s),$$

a formal integration by parts yields

$$\int_0^\infty \kappa(s)\Delta\alpha_t(t-s)\,\mathrm{d}s = \int_0^\infty \mu(s)\Delta[\alpha(t) - \alpha(t-s)]\,\mathrm{d}s.$$

In summary, equation (8) and the particular concrete realization (11) of (9) give rise to the system

$$\begin{cases} u_{tt} - \omega\Delta u_{tt} + \Delta^2 u - \big[b + \|\nabla u\|^2_{L^2(\Omega)}\big]\Delta u + \Delta\alpha_t = g, \\ \alpha_{tt} - \Delta\alpha - \displaystyle\int_0^\infty \mu(s)\Delta[\alpha(t) - \alpha(t-s)]\,\mathrm{d}s - \Delta u_t = 0, \end{cases}$$

which will be analyzed in Chapter 6 (actually, in a more general abstract form). As a matter of fact, from the physical viewpoint, it is also relevant to neglect the effect of the rotational inertia on the plate (see e.g. [40, 51, 55, 57]). This corresponds to the limit situation when $\omega = 0$. Hence, we will also consider the following linear and homogeneous version of the above model

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta\alpha_t = 0, \\ \alpha_{tt} - \Delta\alpha - \displaystyle\int_0^\infty \mu(s)\Delta[\alpha(t) - \alpha(t-s)]\,\mathrm{d}s - \Delta u_t = 0, \end{cases}$$

along with the system

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta\alpha_t = 0, \\ \alpha_{tt} - \Delta\alpha - \Delta\alpha_t - \Delta u_t = 0, \end{cases}$$

formally obtained when the memory kernel collapses into the Dirac mass at zero. As we will see, the presence of the memory produces a lack of exponential stability of the associated linear semigroups, preventing the analysis of the asymptotic properties in the nonlinear case.

Type III Nonlinear Caginalp Phase-Field Systems

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. The thermal evolution of a material occupying a volume Ω, with order parameter u and (relative) temperature ϑ, is governed by the equations

$$\frac{\partial u}{\partial t} = -\frac{\partial \Psi}{\partial u}, \tag{12}$$

$$\frac{\partial H}{\partial t} + \operatorname{div}\mathbf{q} = F. \tag{13}$$

Here, Ψ denotes the *total free energy* of the system defined as

$$\Psi(u,\vartheta) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + \Phi_0(u) - u\vartheta - \frac{1}{2}\vartheta^2\Big)\,\mathrm{d}\boldsymbol{x},$$

where the potential

$$\Phi_0(s) = \int_0^s \phi(y)\,\mathrm{d}y \quad \text{for some real } \phi : \mathbb{R} \to \mathbb{R},$$

has a typical double-well shape (e.g. $\Phi_0(s) = (s^2 - 1)^2$). Besides, H stands for the *enthalpy* of the material

$$H = -\frac{\partial \Psi}{\partial \vartheta} = \vartheta + u,$$

$\mathbf{q}$ is the heat flux vector and F represents a source term. Finally, we assume the Dirichlet boundary condition for u and ϑ

$$u(t)_{|\partial\Omega} = \vartheta(t)_{|\partial\Omega} = 0.$$

Accordingly, equation (12) reads

$$u_t - \Delta u + \phi(u) = \vartheta, \tag{14}$$

while the concrete form of (13) depends on the choice of the constitutive law for the heat flux. For example, within the classical Fourier law

$$\mathbf{q} = -k\nabla\vartheta, \quad k > 0,$$

we deduce from (13) the reaction-diffusion equation

$$\vartheta_t - k\Delta\vartheta = -u_t + F$$

which, coupled with (14), constitutes the original Caginalp phase-field system [6]. Instead, adopting the theory of heat conduction of type III, the heat flux takes the form

$$\mathbf{q} = -k^\star \nabla\alpha - k\nabla\vartheta, \quad k^\star > 0,$$

where the variable

$$\alpha(t) = \int_0^t \vartheta(\tau)\,\mathrm{d}\tau + \alpha(0)$$

represents the thermal displacement. In turn, the balance equation (13) translates into

$$\alpha_{tt} - k\Delta\alpha_t - k^\star\Delta\alpha = u_t + F.$$

In conclusion, for a general term F of the form

$$F = -g(\alpha)$$

accounting for the presence of a nonlinear internal source depending on the displacement, and setting the physical constants to 1, we end up with the following nonlinear phase-field system of Caginalp type

$$\begin{cases} u_t - \Delta u + \phi(u) = \alpha_t, \\ \alpha_{tt} - \Delta\alpha_t - \Delta\alpha + g(\alpha) = -u_t. \end{cases}$$

This model will be studied in Chapter 7. We refer the reader to [62–65] for further discussions related to phase-field systems with nonclassical heat conduction.

Timoshenko Systems with Gurtin-Pipkin Thermal Law

Given $\ell > 0$, we consider the thermoelastic beam model of Timoshenko type [92]

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\vartheta_x = 0, \\ \rho_3\vartheta_t + q_x + \delta\psi_{tx} = 0, \end{cases} \tag{15}$$

where the unknown variables

$$\varphi, \psi, \vartheta, q : \ (x,t) \in [0,\ell] \times [0,\infty) \mapsto \mathbb{R}$$

represent the transverse displacement of a beam with reference configuration $[0,\ell]$, the rotation angle of a filament, the relative temperature and the heat flux vector, respectively. Here, ρ_1, ρ_2, ρ_3 as well as κ, b, δ are strictly positive fixed constants. The system is complemented with the Dirichlet boundary conditions for φ and ϑ

$$\varphi(0,t) = \varphi(\ell,t) = \vartheta(0,t) = \vartheta(\ell,t) = 0,$$

and the Neumann one for ψ

$$\psi_x(0,t) = \psi_x(\ell,t) = 0.$$

Such conditions, commonly adopted in the literature, seem to be the most feasible from a physical viewpoint. To complete the picture, we need to establish a link between q and ϑ, through the constitutive law for the heat flux. We assume the Gurtin-Pipkin heat conduction law

$$\beta q(t) + \int_0^\infty g(s)\vartheta_x(t-s)\,\mathrm{d}s = 0, \quad \beta > 0, \tag{16}$$

where the memory kernel g is a (bounded) convex summable function on $[0,\infty)$ of total mass

$$\int_0^\infty g(s)\,\mathrm{d}s = 1.$$

As already said, equation (16) can be viewed as a memory relaxation of the Fourier law, inducing (similarly to the Cattaneo law) a fully hyperbolic mechanism of heat transfer. In this perspective, it may be considered a more realistic description of physical reality. Accordingly, system (15) turns into

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\vartheta_x = 0, \\ \rho_3\vartheta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g(s)\vartheta_{xx}(t-s)\,\mathrm{d}s + \delta\psi_{tx} = 0, \end{cases}$$

and this model will be studied in the final Chapter 8.

CHAPTER 1

Preliminaries

In this first chapter, we recall some basic tools from the theory of infinite-dimensional dynamical systems and linear semigroups. A more detailed presentation can be found in the classical books [2, 13, 24, 49, 50, 53, 77, 88, 91].

1.1 Infinite-Dimensional Dynamical Systems

Nonlinear dynamical systems play a crucial role in the modern study of several physical phenomena where some kind of evolution is taken into account. In particular, many dynamics are characterized by the presence of some dissipation mechanisms (e.g. friction or viscosity) which produce a loss of energy in the system. Roughly speaking, from the mathematical viewpoint dissipation is represented by the existence of a set in the phase space called *absorbing set* (see Definition 1.1.3). Nevertheless, in order to have a better understanding of the asymptotic behavior of the system, some additional "good" geometrical and topological properties (e.g. compactness or finite fractal/Hausdorff dimension) are necessary. This leads to the modern concept of *attractor* (see Definition 1.1.7), that is, the minimal compact set which attracts uniformly all the bounded sets of the phase space.

1.1.1 Dissipative dynamical systems

We begin with some definitions.

Definition 1.1.1. *Let X be a real Banach space. A dynamical system (otherwise called $\mathcal{C}_0$-semigroup of operators) on X is a one-parameter family of functions $S(t) : X \to X$ depending on $t \geq 0$ satisfying the following properties:*

(S.1) $S(0) = \mathbb{I}$;[1]

(S.2) $S(t+\tau) = S(t)S(\tau)$ *for all* $t, \tau \geq 0$*;*

(S.3) $t \mapsto S(t)x \in C([0,\infty), X)$ *for all* $x \in X$*;*

(S.4) $S(t) \in C(X, X)$ *for all* $t \geq 0$.

Remark 1.1.1. *In light of some recent developments (see [9] and [13, Chapter XI]), the notion of dynamical system can be actually given in a more general form, removing the continuity assumptions (S.3) and (S.4) from Definition 1.1.1 (see the forthcoming Remark 1.1.2 for a more detailed discussion).*

Along this section, $S(t)$ will always denote a dynamical system acting on a Banach space X.

Definition 1.1.2. *A nonempty set* $\mathcal{B} \subset X$ *is called invariant for* $S(t)$ *if*

$$S(t)\mathcal{B} \subset \mathcal{B}, \quad \forall t \geq 0.$$

Definition 1.1.3. *A subset* $\mathbb{B} \subset X$ *is called absorbing set if it is bounded*[2] *and for any bounded set* $\mathcal{B} \subset X$ *there exists an entering time* $t_{\rm e} = t_{\rm e}(\mathcal{B}) \geq 0$ *such that*

$$S(t)\mathcal{B} \subset \mathbb{B}, \quad \forall t \geq t_{\rm e}.$$

It is worth noting that, once we have proved the existence of an absorbing set $\mathbb{B}$, an invariant absorbing set can be easily constructed through the formula

$$\bigcup_{t \geq t_{\rm e}} S(t)\mathbb{B} \subset \mathbb{B}, \quad t_{\rm e} = t_{\rm e}(\mathbb{B}).$$

As already mentioned, a dynamical systems is called *dissipative* if it possesses an absorbing set. We also need the notion of ω-limit set.

Definition 1.1.4. *The* ω*-limit set of a nonempty set* $\mathcal{B} \subset X$ *is defined as*

$$\omega(\mathcal{B}) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} S(\tau)\mathcal{B}}.$$

Thus, since $\omega(\mathcal{B})$ in some sense captures the dynamics of the orbits of $\mathcal{B}$, if the dynamical system possesses an absorbing set $\mathbb{B}$ one might try to describe the asymptotic behavior of the whole system through union

$$\bigcup_{x \in \mathbb{B}} \omega(x),$$

since any trajectory eventually enters into $\mathbb{B}$. However, this set turns out to be too "small" in order to provide the necessary information, as will be clear in the sequel.

[1] Here, $\mathbb{I}$ denotes the identity on X.

[2] Some authors do not require boundedness in the definition of absorbing set.

1.1.2 Global and exponential attractors

Due to the fact that the phase space X can be infinite-dimensional, existence of absorbing set usually gives poor information on the longterm dynamics. Indeed, for instance, balls are not compact in the infinite-dimensional case. Therefore, one might think to investigate, for example, existence of compact absorbing set. However, when dealing with concrete dynamical systems generated by partial differential equations arising in Mathematical Physics, compact absorbing sets pop up when the equation exhibits regularizing effects on the solution, that is, when the dynamics is parabolic. Thus, in the hyperbolic case, compact absorbing set are out of reach. The central idea is then to consider compact sets that "attract" (rather than absorb) the orbits originating from bounded sets.

Definition 1.1.5. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are nonempty subsets of X, the Hausdorff semidistance between $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined as*

$$\boldsymbol{\delta}_X(\mathcal{B}_1,\mathcal{B}_2) = \sup_{z_1\in\mathcal{B}_1}\inf_{z_2\in\mathcal{B}_2}\|z_1-z_2\|_X.$$

Observe that the Hausdorff semidistance is not symmetric. Moreover, it is easy to see that

$$\boldsymbol{\delta}_X(\mathcal{B}_1,\mathcal{B}_2)=0 \quad \text{if and only if} \quad \mathcal{B}_1\subset\overline{\mathcal{B}}_2,$$

where $\overline{\mathcal{B}}_2$ denotes the closure in X of the set $\mathcal{B}_2$.

Definition 1.1.6. *A set $\mathbb{K}\subset X$ is called attracting for $S(t)$ if*

$$\lim_{t\to\infty}\boldsymbol{\delta}_X(S(t)\mathcal{B},\mathbb{K})=0,$$

for any bounded set $\mathcal{B}\subset X$. The dynamical system $S(t)$ is called asymptotically compact if has a compact attracting set.

Definition 1.1.7. *A compact set $\mathbb{A}\subset X$ which is at the same time attracting and fully invariant (i.e. $S(t)\mathbb{A}=\mathbb{A}$ for every $t\geq 0$) is called the global attractor of $S(t)$.*

It is well-known that the global attractor of a dynamical system, provided it exists, is unique and connected (see e.g. [2, 49, 50, 91]). Besides, in several concrete situations arising in dynamical systems generated by partial differential equations, the attractor $\mathbb{A}$ has finite fractal dimension

$$\dim_{\mathrm{f}}(\mathbb{A})=\limsup_{\varepsilon\to 0}\frac{\ln N_\varepsilon}{\ln\frac{1}{\varepsilon}},$$

where N_ε is the smallest number of ε-balls of X covering $\mathbb{A}$. In this situation, roughly speaking, the long-term dynamics becomes finite-dimensional (see e.g. [91]).

We now state one of the main abstract results concerning existence of global attractors. To this aim, we will lean on the notion of *Kuratowski measure of noncompactness* of a bounded set $\mathcal{B}\subset X$. This is by definition

$$\alpha(\mathcal{B})=\inf\big\{d:\mathcal{B}\text{ is covered by finitely many sets of diameter less than }d\big\}.$$

Accordingly, $\alpha(\mathcal{B})=0$ if and only if $\mathcal{B}$ is totally bounded, i.e. precompact in a Banach space framework. Further straightforward properties are listed below (cf. [49]):

- $\alpha(\mathcal{B}) = \alpha(\overline{\mathcal{B}})$;
- $\alpha(\mathcal{B}) \leq \operatorname{diam}(\mathcal{B})$;
- $\alpha(\mathcal{B}_1 \cup \mathcal{B}_2) = \max\{\alpha(\mathcal{B}_1), \alpha(\mathcal{B}_2)\}$;
- $\mathcal{B}_1 \subset \mathcal{B}_2 \;\; \Rightarrow \;\; \alpha(\mathcal{B}_1) \leq \alpha(\mathcal{B}_2)$.

The result reads as follows.

Theorem 1.1.1. *Let $S(t) : X \to X$ be a dissipative dynamical system acting on a Banach space X, and let $\mathbb{B}$ an absorbing set. If there exists a sequence $t_n \geq 0$ such that*

$$\lim_{n\to\infty} \alpha(S(t_n)\mathbb{B}) = 0,$$

then $\omega(\mathbb{B})$ is the global attractor of $S(t)$.

We address the reader to the classical books [2,13,49,91] for a proof (but see also [75] and Theorem A.2 in the final appendix).

Remark 1.1.2. *The basic objects of the theory introduced so far (absorbing and attracting sets, global attractors) can be in fact revisited only in terms of their attraction properties, without any continuity assumption on $S(t)$. Within this approach, a slight different notion of global attractor is necessary, the minimality with respect to attraction being the sole characterizing property. The invariance is discussed only in a second moment, as a consequence of some kind of continuity. A detailed discussion can be found in [9].*

Nonetheless, the global attractor is usually affected by an essential drawback. Indeed, the attraction rate can be arbitrarily slow and, in general, cannot be explicitly estimated. As a consequence, the global attractor may be very sensitive to small perturbations. Although not crucial from the theoretical side, this problem becomes significant for practical purposes (e.g. numerical simulations). In order to overcome these difficulties, a new object has been introduced in [26], namely, the so-called exponential attractor.

Definition 1.1.8. *An exponential attractor is a compact invariant set $\mathbb{E} \subset X$ of finite fractal dimension satisfying for all bounded set $\mathcal{B} \subset X$ the exponential attraction property*

$$\boldsymbol{\delta}_X(S(t)\mathcal{B}, \mathbb{E}) \leq \mathfrak{I}(\|\mathcal{B}\|_X)\mathrm{e}^{-\omega t}$$

for some $\omega > 0$ and some increasing function $\mathfrak{I} : \mathbb{R}^+ \to \mathbb{R}^+$.

Contrary to the global attractor, an exponential attractor is not unique. With regard to sufficient conditions for the existence of exponential attractors in Hilbert spaces we refer to [1, 26]. In a Banach space setting, the first result was devised in [27] (see also [17] and the final appendix of this thesis).

1.1.3 Gradient systems

In this section we analyze a special class of dynamical systems, the so-called gradient systems, characterized by the existence of a *Lyapunov functional*. We begin with some definitions.

Definition 1.1.9. *A function $Z \in C(\mathbb{R}, X)$ is called a complete bounded trajectory (*CBT*) of $S(t)$ if*

$$\sup_{\tau \in \mathbb{R}} \|Z(\tau)\|_X < \infty$$

and

$$S(t)Z(\tau) = Z(t+\tau), \quad \forall t \geq 0, \forall \tau \in \mathbb{R}.$$

We also introduce the set of stationary points of $S(t)$

$$\mathbb{S} = \{z \in X : S(t)z = z, \forall t \geq 0\},$$

and the unstable set of $\mathbb{S}$, that is,

$$W(\mathbb{S}) = \Big\{ Z(0) : Z \text{ CBT and } \lim_{\tau \to -\infty} \|Z(\tau) - \mathbb{S}\|_X = 0 \Big\}.$$

Definition 1.1.10. *A Lyapunov functional for the dynamical system $S(t)$ is a function $\Lambda \in C(X, \mathbb{R})$ such that*

(i) $\Lambda(z) \to \infty$ *if and only if* $\|z\|_X \to \infty$*;*

(ii) $\Lambda(S(t)z) \leq \Lambda(z)$ *for every $z \in X$ and every $t \geq 0$;*

(iii) $\Lambda(S(t)z) = \Lambda(z)$ *for all $t \geq 0$ implies that $z \in \mathbb{S}$.*

If there exists a Lyapunov functional, then $S(t)$ is called a gradient system.

We report the following standard abstract result on existence of global attractors for gradient systems (see [49, 56]).

Lemma 1.1.1. *Let $S(t) : X \to X$ be a gradient system acting on a Banach space X. Assume that*

(i) the set $\mathbb{S}$ of the stationary points of $S(t)$ is bounded in X;

(ii) for every $R \geq 0$ there exist a positive function $\mathfrak{I}_R$ vanishing at infinity and a compact set $\mathbb{K}_R \subset X$ such that $S(t)$ can be split into the sum $S_0(t) + S_1(t)$, where the one-parameter operators $S_0(t)$ and $S_1(t)$ fulfill

$$\|S_0(t)z\|_X \leq \mathfrak{I}_R(t) \qquad \textit{and} \qquad S_1(t)z \subset \mathbb{K}_R,$$

whenever $\|z\|_X \leq R$ and $t \geq 0$.

Then, $S(t)$ possesses a connected global attractor $\mathbb{A}$, which consists of the unstable set $W(\mathbb{S})$. Moreover, $\mathbb{A}$ is a subset of $\mathbb{K}_R$ for some $R > 0$.

In conclusion, roughly speaking, the asymptotic dynamics of gradient systems can be fully described by means of complete bounded trajectories departing (at $-\infty$) from the set of stationary points of $S(t)$.

1.2 Linear Semigroups

We now consider the particular situation where $S(t)$ is a linear operator for every $t \geq 0$. With standard notation, we will denote by $L(X)$ the space of bounded linear operators from X into X.

Definition 1.2.1. *Let X be a real Banach space. A linear dynamical system (otherwise called $\mathcal{C}_0$-semigroup of bounded linear operators) acting on X is a family of maps*

$$S(t) \in L(X)$$

depending on $t \geq 0$ satisfying the semigroup properties (S.1)-(S.2) together with

(S.3') $\lim_{t \to 0} S(t)x = x$ *for all $x \in X$.*

Notice that assumption (S.3') and the semigroup properties imply the continuity

$$t \mapsto S(t)x \in \mathcal{C}([0,\infty), X) \text{ for every fixed } x \in X.$$

When dealing with linear dynamical systems, an important concept is the one of infinitesimal generator.

Definition 1.2.2. *The linear operator A with domain*

$$\mathfrak{D}(A) = \left\{ x \in X : \lim_{t \to 0} \frac{S(t)x - x}{t} \text{ exists} \right\}$$

defined as

$$Ax = \lim_{t \to 0} \frac{S(t)x - x}{t}$$

is the called the infinitesimal generator of the linear dynamical system $S(t)$.

It is possible to show that A is a closed, densely defined operator which uniquely determines the linear dynamical system (see e.g. [77]). Formally, one writes

$$S(t) = \mathrm{e}^{tA}$$

to indicate that the operator A is the infinitesimal generator of the semigroup $S(t)$.

One important and natural question is then to determine whenever a closed linear operator with dense domain in X is the infinitesimal generator of a linear dynamical system $S(t)$. As a matter of fact, a necessary and sufficient condition is given by the Hille-Yosida Theorem (see e.g. [77]), so that the problem is nowadays completely solved from the theoretical viewpoint. However, the Hille-Yosida Theorem is somehow difficult to apply in several concrete situations, as it involves the knowledge of the spectrum of the operator A (usually, not an easy task to achieve). Nevertheless, in a Hilbert space setting, there exists a more effective criterion. We need two preliminary definitions.

Definition 1.2.3. *$S(t)$ is called a contraction semigroup if*

$$\|S(t)\|_{L(X)} \leq 1, \quad \forall t \geq 0.$$

Definition 1.2.4. *A linear operator A on a real Hilbert space X is dissipative if*

$$\langle Ax, x\rangle \le 0, \quad \forall x \in \mathfrak{D}(A).$$

The result reads as follows.

Lemma 1.2.1 (Lumer-Phillips)**.** *Let A be a densely defined linear operator on a real Hilbert space X. Then A is the infinitesimal generator of a contraction semigroup $S(t)$ if and only if*

(i) A is dissipative; and

(ii) Range$(I - A) = X$.

We address the reader to [77] for the proof. The Lumer-Phillips Theorem turns out to be a very useful tool in the study of linear dynamical systems generated by partial differential equations, as we will see in Chapter 6.

Another fundamental problem is the one of asymptotic stability, that is, the study of the decay properties of the trajectories.

Definition 1.2.5. *The linear dynamical system $S(t)$ is said to be*

- *stable if*
$$\lim_{t\to\infty} \|S(t)x\|_X = 0, \quad \forall x \in X;$$
- *exponentially stable if there are $M \ge 1$ and $\varepsilon > 0$ such that*
$$\|S(t)\|_{L(X)} \le M\mathrm{e}^{-\varepsilon t}, \quad \forall t \ge 0.$$

Exploiting the semigroup properties, when lack of exponential stability occurs we can say that there is no decay pattern valid for all $x \in X$ (see [11, 77]).

In order to deal with concrete cases of linear dynamical systems generated by partial differential equations, we will also exploit an operative abstract criterion developed in [79] (but see also [37] for the statement used here). First, we need a definition.

Definition 1.2.6. *The complexification of a real Banach space X is the complex Banach space $X_{\mathbb{C}}$ defined as*

$$X_{\mathbb{C}} = X \oplus \mathrm{i}X = \{z = x + \mathrm{i}y \quad \textit{with} \;\; x, y \in X\}$$

and endowed with the norm

$$\|x + \mathrm{i}y\|_{X_{\mathbb{C}}} = \sqrt{\|x\|^2 + \|y\|^2}\,.$$

Analogously, the complexification $A_{\mathbb{C}}$ of a linear operator of A on X is the (linear) operator on $X_{\mathbb{C}}$ with domain

$$\mathfrak{D}(A_{\mathbb{C}}) = \{z = x + \mathrm{i}y \quad \textit{with} \;\; x, y \in \mathfrak{D}(A)\}$$

defined by

$$A_{\mathbb{C}}(x + \mathrm{i}y) = Ax + \mathrm{i}Ay.$$

The result is the following.

Lemma 1.2.2. *A linear dynamical system $S(t) = \mathrm{e}^{tA}$ acting on a real Hilbert space X is exponentially stable if and only if there exists $\varepsilon > 0$ such that*

$$\inf_{\lambda \in \mathbb{R}} \|\mathrm{i}\lambda z - A_{\mathbb{C}} z\|_{X_{\mathbb{C}}} \geq \varepsilon \|z\|_{X_{\mathbb{C}}}, \quad \forall z \in \mathfrak{D}(A_{\mathbb{C}}), \tag{1.2.1}$$

where $A_{\mathbb{C}}$ and $X_{\mathbb{C}}$ are understood to be the complexifications of the original infinitesimal generator A and space X, respectively.

CHAPTER 2

Strongly Damped Nonlinear Wave Equations

2.1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Calling $\mathrm{H} = L^2(\Omega)$, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and introducing the strictly positive Dirichlet operator

$$A = -\Delta \qquad \text{with domain} \qquad \mathfrak{D}(A) = H^2(\Omega) \cap H^1_0(\Omega) \Subset \mathrm{H},$$

we consider the evolution equation in the unknown $u = u(\boldsymbol{x}, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$

$$u_{tt} + Au_t + Au + f(u_t) + g(u) = h \tag{2.1.1}$$

subject to the initial conditions

$$u(\boldsymbol{x}, 0) = a(\boldsymbol{x}) \qquad \text{and} \qquad u_t(\boldsymbol{x}, 0) = b(\boldsymbol{x}),$$

where $a, b : \Omega \to \mathbb{R}$ are assigned data.

The time-independent external source $h = h(\boldsymbol{x})$ is taken in H, while the nonlinearities comply with the following assumptions, $\lambda_1 > 0$ being the first eigenvalue of A.

Assumptions on $\boldsymbol{f}$. *Let $f \in \mathcal{C}^1(\mathbb{R})$, with $f(0) = 0$, satisfy for every $s \in \mathbb{R}$ and some $c \geq 0$ the growth bound*

$$|f'(s)| \leq c + c|s|^4, \tag{2.1.2}$$

along with the dissipativity condition

$$\liminf_{|s|\to\infty} f'(s) > -\lambda_1. \tag{2.1.3}$$

Assumptions on g. *Let $g \in \mathcal{C}^1(\mathbb{R})$, with $g(0) = 0$, satisfy for every $s \in \mathbb{R}$ and some $c \geq 0$ the growth bound*

$$|g'(s)| \leq c + c|s|^{p-1} \quad \textit{with} \qquad p \in [1,5], \tag{2.1.4}$$

along with the dissipativity conditions

$$\liminf_{|s|\to\infty} \frac{g(s)}{s} > -\lambda_1, \tag{2.1.5}$$

$$\liminf_{|s|\to\infty} \frac{sg(s) - c_1\int_0^s g(y)\mathrm{d}y}{s^2} > -\frac{\lambda_1}{2}, \tag{2.1.6}$$

for some $c_1 > 0$. In fact (2.1.5)-(2.1.6) *are automatically verified (with $c_1 = 1$) if g fulfills the same dissipation condition* (2.1.3) *of f, slightly less general but still widely used in the literature.*

As explained in the introduction of the thesis, equation (2.1.1), here written in dimensionless form, rules the thermal evolution in a rigid body of shape Ω within the theory of heat conduction of type III devised by Green and Nagdhi. However, other physical interpretations are possible, for example viscoelasticity of Kelvin-Voigt type.

After introducing the notation and the functional setting (see Section 2.2), in the successive Section 2.3 we consider an equivalent formulation of the problem, more suitable for our purposes. Well-posedness is proved in Section 2.4, yielding a solution semigroup $S(t)$ (dynamical system) acting on the natural weak energy space. Finally, in Sections 2.5-2.6, we dwell on the dissipative character of the semigroup, witnessed by the existence of (bounded) regular absorbing sets.

2.2 Preliminaries

2.2.1 Notation

For $\sigma \in \mathbb{R}$, we define the hierarchy of (compactly) nested Hilbert spaces

$$\mathrm{H}_\sigma = \mathfrak{D}(A^{\frac{\sigma}{2}}), \qquad \langle w, v\rangle_\sigma = \langle A^{\frac{\sigma}{2}}w, A^{\frac{\sigma}{2}}v\rangle, \qquad \|w\|_\sigma = \|A^{\frac{\sigma}{2}}w\|.$$

For $\sigma > 0$, it is understood that $\mathrm{H}_{-\sigma}$ denotes the completion of the domain, so that $\mathrm{H}_{-\sigma}$ is the dual space of H_σ. Moreover, the subscript σ is always omitted whenever zero. The symbol $\langle\cdot,\cdot\rangle$ also stands for duality product between H_σ and its dual space $\mathrm{H}_{-\sigma}$. In particular,

$$\mathrm{H}_2 = H^2(\Omega)\cap H_0^1(\Omega) \quad \Subset \quad \mathrm{H}_1 = H_0^1(\Omega) \quad \Subset \quad \mathrm{H} = L^2(\Omega) \quad \Subset \quad \mathrm{H}_{-1} = H^{-1}(\Omega),$$

and we have the Poincaré inequality

$$\lambda_1\|w\|^2 \leq \|w\|_1^2, \quad \forall w \in \mathrm{H}_1.$$

Then we define the natural energy spaces

$$\mathcal{H}_\sigma = \mathrm{H}_{\sigma+1}\times\mathrm{H}_\sigma$$

endowed with the standard product norms

$$\|\{w_1, w_2\}\|^2_{\mathcal{H}_\sigma} = \|w_1\|^2_{\sigma+1} + \|w_2\|^2_\sigma.$$

We will also encounter the "asymmetric" energy spaces

$$\mathcal{V}_\sigma = \mathrm{H}_\sigma \times \mathrm{H}_\sigma.$$

2.2.2 General agreements

Without loss of generality, we may (and do) suppose $c_1 = 1$ in (2.1.6). Along the chapter, we will perform a number of formal energy-type estimates, which are rigorously justified in a Galerkin approximation scheme. In the proofs, we will always adopt the symbol Λ (or Λ_ε) to denote some energy functional, specifying its particular structure from case to case. Moreover, the Hölder, Young and Poincaré inequalities will be tacitly used in several occasions, as well as the Sobolev embedding

$$\mathrm{H}_1 \subset L^6(\Omega).$$

2.2.3 A technical lemma

We report a Gronwall-type lemma from the very recent paper [70].

Lemma 2.2.1. *Given $k \geq 1$ and $C \geq 0$, let $\Lambda_\varepsilon : [0, \infty) \to [0, \infty)$ be a family of absolutely continuous functions satisfying for every $\varepsilon > 0$ small the inequalities*

$$\frac{1}{k}\Lambda_0 \leq \Lambda_\varepsilon \leq k\Lambda_0 \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon \leq C\varepsilon^6\Lambda_\varepsilon^3 + C$$

for some continuous $\Lambda_0 : [0, \infty) \to [0, \infty)$. Then there are constants $\delta > 0$, $R \geq 0$ and an increasing function $\mathfrak{I} \geq 0$ such that

$$\Lambda_0(t) \leq \mathfrak{I}(\Lambda_0(0))\mathrm{e}^{-\delta t} + R.$$

2.3 An Equivalent Formulation

Equation (2.1.1) can be given an equivalent formulation, allowing to render the calculations much simpler.

2.3.1 Decompositions of the nonlinear terms

The first step is splitting f and g into the sums of suitable functions.

Lemma 2.3.1. *For every fixed $\lambda < \lambda_1$ sufficiently close to λ_1, the decomposition*

$$f(s) = \phi(s) - \lambda s + \phi_{\mathrm{c}}(s)$$

holds for some $\phi, \phi_{\mathrm{c}} \in \mathcal{C}^1(\mathbb{R})$ with the following properties:

- *ϕ_{c} is compactly supported with $\phi_{\mathrm{c}}(0) = 0$;*

- *ϕ vanish inside $[-1,1]$ and fulfills for some $c \ge 0$ and every $s \in \mathbb{R}$ the bounds*

$$0 \le \phi'(s) \le c|s|^4.$$

Proof. In light of (2.1.3), fix any λ subject to the bounds

$$\liminf_{|s|\to\infty} f'(s) > -\lambda > -\lambda_1.$$

Hence,

$$f'(s) \ge -\lambda, \quad \forall\, |s| \ge k,$$

for $k \ge 1$ large enough. Choosing then any smooth function $\varrho : \mathbb{R} \to [0,1]$ satisfying

$$s\varrho'(s) \ge 0 \qquad \text{and} \qquad \varrho(s) = \begin{cases} 0 & \text{if } |s| \le k, \\ 1 & \text{if } |s| \ge k+1, \end{cases}$$

it is immediate to check that

$$\phi(s) = \varrho(s)[f(s) + \lambda s] \qquad \text{and} \qquad \phi_{\rm c}(s) = [1 - \varrho(s)][f(s) + \lambda s]$$

comply with the requirements. □

Lemma 2.3.2. *For every fixed $\lambda < \lambda_1$ sufficiently close to λ_1, the decomposition*

$$g(s) = \gamma(s) - \lambda s + \gamma_{\rm c}(s)$$

holds for some $\gamma, \gamma_{\rm c} \in \mathcal{C}^1(\mathbb{R})$ with the following properties:

- *$\gamma_{\rm c}$ is compactly supported with $\gamma_{\rm c}(0) = 0$;*
- *γ vanish inside $[-1,1]$ and fulfills for some $c \ge 0$ and every $s \in \mathbb{R}$ the bounds*

$$0 \le \int_0^s \gamma(y)\,{\rm d}y \le s\gamma(s) \qquad \textit{and} \qquad |\gamma'(s)| \le c|s|^{p-1}.$$

Proof. Using this time (2.1.5)-(2.1.6), where we put $c_1 = 1$, for any fixed $\lambda < \lambda_1$ close to λ_1 and every $|s| \ge k \ge 1$ large enough we get

$$s[g(s) + \lambda s] \ge 0 \qquad \text{and} \qquad \int_0^s g(y){\rm d}y \le sg(s) + \frac{1}{2}\lambda s^2.$$

Similarly to the previous proof, we define

$$\gamma(s) = \varrho(s)[g(s) + \lambda s] \qquad \text{and} \qquad \gamma_{\rm c}(s) = [1 - \varrho(s)][g(s) + \lambda s].$$

Then, the first inequality tells that

$$0 \le \int_0^s \gamma(y)\,{\rm d}y \le \varrho(s)\int_0^s [g(y) + \lambda y]\,{\rm d}y = \varrho(s)\left[\int_0^s g(y)\,{\rm d}y + \frac{1}{2}\lambda s^2\right],$$

and by applying the second one we establish the desired integral estimate, whereas the growth bound on γ' is straightforward. □

Due to Lemma 2.3.1 and Lemma 2.3.2, the functionals on H_1 given by

$$\Phi_0(w) = 2\int_\Omega \int_0^{w(\boldsymbol{x})} \phi(y)\,\mathrm{d}y\mathrm{d}\boldsymbol{x}, \qquad \Gamma_0(w) = 2\int_\Omega \int_0^{w(\boldsymbol{x})} \gamma(y)\,\mathrm{d}y\mathrm{d}\boldsymbol{x},$$

and

$$\Phi_1(w) = \langle \phi(w), w\rangle, \qquad \Gamma_1(w) = \langle \gamma(w), w\rangle,$$

fulfill for every $w \in \mathrm{H}_1$ the inequalities

$$0 \le \Phi_0(w) \le 2\Phi_1(w), \tag{2.3.1}$$

$$0 \le \Gamma_0(w) \le 2\Gamma_1(w). \tag{2.3.2}$$

Moreover, since

$$|\phi(s)|^{\frac{6}{5}} = |\phi(s)|^{\frac{1}{5}}|\phi(s)| \le c|s||\phi(s)| = cs\phi(s),$$

we deduce that for all $C > 0$ sufficiently large

$$\|\phi(w)\|_{L^{6/5}} \le C[\Phi_1(w)]^{\frac{5}{6}}, \quad \forall w \in \mathrm{H}_1. \tag{2.3.3}$$

2.3.2 The equation revisited

For a fixed $\lambda < \lambda_1$ complying with Lemma 2.3.1 and Lemma 2.3.2, we rewrite (2.1.1) in the equivalent form

$$u_{tt} + Bu_t + Bu + \phi(u_t) + \gamma(u) = q, \tag{2.3.4}$$

where

$$q = h - \phi_{\mathrm{c}}(u_t) - \gamma_{\mathrm{c}}(u) \in L^\infty(\mathbb{R}^+; \mathrm{H})$$

and

$$B = A - \lambda I \quad \text{with domain} \quad \mathfrak{D}(B) = \mathfrak{D}(A)$$

is a positive operator commuting with A. In particular, the bilinear form

$$(w, v)_\sigma = \langle w, A^{-1}Bv\rangle_\sigma = \langle w, v\rangle_\sigma - \lambda\langle w, v\rangle_{\sigma-1}$$

defines an equivalent inner product on the space H_σ whose induced norm $|\cdot|_\sigma$, in light of the Poincaré inequality, satisfies

$$\frac{\lambda_1 - \lambda}{\lambda_1}\|w\|_\sigma^2 \le |w|_\sigma^2 = \|w\|_\sigma^2 - \lambda\|w\|_{\sigma-1}^2 \le \|w\|_\sigma^2. \tag{2.3.5}$$

2.3.3 The renormed spaces

Aiming to deal with the reformulated version (2.3.4) of the original equation, it is convenient to redefine the norms of the energy spaces. Accordingly, we agree to consider $\mathcal{H}_\sigma$ and $\mathcal{V}_\sigma$ endowed with the equivalent norms

$$|\{w, v\}|_{\mathcal{H}_\sigma}^2 = |w|_{\sigma+1}^2 + \|v\|_\sigma^2 \qquad \text{and} \qquad |\{w, v\}|_{\mathcal{V}_\sigma}^2 = |w|_\sigma^2 + |v|_\sigma^2.$$

Whenever needed, the norm inequalities (2.3.5) will be applied without explicit mention.

2.3.4 Formal estimates

We conclude by establishing some general differential relations, widely used in the forthcoming proofs. Given a vector-valued function w, as regular as required, we set

$$\Box w = 2w_{tt} + 2Bw_t + 2Bw.$$

The following identities are verified by direct calculations:

$$\langle \Box w, w\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\big[|w|_1^2 + 2\langle w, w_t\rangle\big] + 2|w|_1^2 - 2\|w_t\|^2, \tag{2.3.6}$$

$$\langle \Box w, w_t\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\big[|w|_1^2 + \|w_t\|^2\big] + 2|w_t|_1^2, \tag{2.3.7}$$

$$\langle \Box w, w_{tt}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\big[|w_t|_1^2 + 2(w, w_t)_1\big] + 2\|w_{tt}\|^2 - 2|w_t|_1^2. \tag{2.3.8}$$

Next, we define the family of energy functionals depending on $\varepsilon \geq 0$

$$\Pi_\varepsilon(w) = (1+\varepsilon)|w|_1^2 + \|w_t\|^2 + 2\varepsilon\langle w_t, w\rangle. \tag{2.3.9}$$

Exploiting (2.3.6)-(2.3.7), the inequalities

$$\Pi_0(w) \leq 2\Pi_\varepsilon(w) \leq 4\Pi_0(w) \tag{2.3.10}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_\varepsilon(w) + \varepsilon\Pi_\varepsilon(w) + \frac{1}{2}\big[\varepsilon|w|_1^2 + 3|w_t|_1^2\big] \leq \langle \Box w, w_t + \varepsilon w\rangle \tag{2.3.11}$$

are easily seen to hold for every $\varepsilon > 0$ sufficiently small.

2.4 The Solution Semigroup

2.4.1 Well-posedness

First, we stipulate the definition of solution.

Definition 2.4.1. *Given $T > 0$, we call weak solution to* (2.3.4) *on* $[0,T]$ *a function*

$$u \in \mathcal{C}([0,T], \mathrm{H}_1) \cap \mathcal{C}^1([0,T], \mathrm{H}) \cap W^{1,2}(0,T; \mathrm{H}_1)$$

satisfying for almost every $t \in [0,T]$ and every test $\theta \in \mathrm{H}_1$ the equality

$$\langle u_{tt}, \theta\rangle + (u_t, \theta)_1 + (u, \theta)_1 + \langle \phi(u_t), \theta\rangle + \langle \gamma(u), \theta\rangle = \langle q, \theta\rangle.$$

Theorem 2.4.1. *For every $T > 0$ and every $z = \{a, b\} \in \mathcal{H}$ there is a unique weak solution u to* (2.3.4) *on* $[0,T]$ *subject to the initial conditions*

$$\{u(0), u_t(0)\} = z.$$

Moreover, given any pair of initial data $z_1, z_2 \in \mathcal{H}$, there exists a constant $C \geq 0$ depending (increasingly) on the norms of $z_\imath$ such that the difference $\bar{u} = u_1 - u_2$ of the corresponding solutions satisfies the continuous dependence estimate

$$\|\{\bar{u}, \bar{u}_t\}\|_{L^\infty(0,T;\mathcal{H})} + \|\bar{u}_t\|_{L^2(0,T;\mathrm{H}_1)} \leq C\mathrm{e}^{CT}|z_1 - z_2|_{\mathcal{H}}.$$

2.4.2 Sketch of the proof

The continuous dependence is essentially contained in the proof of Theorem 3.2.1 in the next chapter (by setting $\varepsilon = 0$). Concerning existence, we follow the usual Galerkin procedure, considering the solutions u_n to the corresponding n-dimensional approximating problems. Arguing as in the forthcoming Theorem 2.5.1 and Corollary 2.5.1, with the aid of (2.3.3), we deduce the uniform boundedness of

$$\begin{aligned} u_n \ &\text{ in } \ L^\infty(0,T;\mathrm{H}_1), \\ \partial_t u_n \ &\text{ in } \ L^\infty(0,T;\mathrm{H}) \cap L^2(0,T;\mathrm{H}_1), \end{aligned}$$

and, calling $\Omega_T = \Omega \times (0,T)$, those of

$$\gamma(u_n) \ \text{ and } \ \phi(\partial_t u_n) \ \text{ in } \ L^{\frac{6}{5}}(\Omega_T).$$

Hence, we can extract weakly or weakly-$*$ convergent subsequences

$$u_n \rightharpoonup u, \qquad \partial_t u_n \rightharpoonup u_t, \qquad \gamma(u_n) \rightharpoonup \gamma_\star, \qquad \phi(\partial_t u_n) \rightharpoonup \phi_\star,$$

in the respective spaces. Proving the claimed continuity in time of u is standard matter. The only difficulty is identifying the limits of the nonlinearities, i.e. showing the equalities

$$\gamma_\star = \gamma(u) \qquad \text{and} \qquad \phi_\star = \phi(u_t).$$

The first is a consequence of the so-called weak dominated convergence theorem. Indeed, from the Sobolev compact embedding

$$W^{1,2}(0,T;\mathrm{H}_1) \Subset \mathcal{C}([0,T],\mathrm{H}),$$

we learn that, up to a subsequence,

$$u_n \to u \ \text{ a.e. in } \Omega_T \quad \Rightarrow \quad \gamma(u_n) \to \gamma(u) \ \text{ a.e. in } \Omega_T.$$

For every fixed $\beta \in L^6(\Omega_T)$, the latter convergence together with the $L^{\frac{6}{5}}$-bound of $\gamma(u_n)$ entail the limit

$$\int_0^T \langle \gamma(u_n(t)), \beta(t)\rangle \,\mathrm{d}t \to \int_0^T \langle \gamma(u(t)), \beta(t)\rangle \,\mathrm{d}t.$$

This provides the equality

$$\int_0^T \langle \gamma_\star(t), \beta(t)\rangle \,\mathrm{d}t = \int_0^T \langle \gamma(u(t)), \beta(t)\rangle \,\mathrm{d}t,$$

in turn implying

$$\gamma_\star = \gamma(u) \ \text{ a.e. in } \Omega_T.$$

Instead, the identification of $\phi_\star$ requires an additional argument. For $\tau < T$ arbitrarily chosen, proceeding as in the proof of the forthcoming Theorem 2.6.1, we obtain the uniform boundedness of

$$\partial_{tt} u_n \ \text{ in } \ L^2(\tau,T;\mathrm{H}_1),$$

with a bound depending on τ (and blowing up when $\tau \to 0$). Still, this is enough to infer the uniform boundedness of

$$\partial_t u_n \quad \text{in} \quad W^{1,2}(\tau, T; \mathrm{H}_1) \Subset \mathcal{C}([\tau, T], \mathrm{H}),$$

and conclude that, calling $\Omega_{\tau,T} = \Omega \times (\tau, T)$,

$$\partial_t u_n \to u_t \quad \text{a.e. in } \Omega_{\tau,T} \quad \Rightarrow \quad \phi(\partial_t u_n) \to \phi(u_t) \quad \text{a.e. in } \Omega_{\tau,T}.$$

Then, repeating the previous argument with $\Omega_{\tau,T}$ in place of Ω_T, we establish the equality

$$\phi_\star = \phi(u_t) \quad \text{a.e. in } \Omega_{\tau,T},$$

which extends on the whole cylinder Ω_T by letting $\tau \to 0$. □

2.4.3 The semigroup

The main consequence of Theorem 2.4.1 is that the family of maps

$$S(t) : \mathcal{H} \to \mathcal{H} \qquad \text{acting as} \qquad S(t)z = \{u(t), u_t(t)\}, \tag{2.4.1}$$

where u is the solution on any interval $[0, T]$ containing t with initial data $z = \{a, b\} \in \mathcal{H}$, defines a dynamical system on $\mathcal{H}$ (normed by $|\cdot|_{\mathcal{H}}$).

In what follows, we will often refer more correctly to (2.4.1) when speaking of solution with initial data z, whose corresponding (doubled) energy is by definition

$$E(t) = |S(t)z|^2_{\mathcal{H}} = |u(t)|_1^2 + \|u_t(t)\|^2.$$

2.5 Dissipativity

In this section, we consider a nonlinearity $g(u)$ of critical order $p = 5$. The dissipative character of $S(t)$ is witnesses by the existence of absorbing sets, capturing the trajectories originating from bounded sets of initial data uniformly in time. The existence of an invariant absorbing set for $S(t)$ is an immediate corollary of the next result.

Theorem 2.5.1. *The dissipative estimate*

$$E(t) \leq \mathfrak{I}(E(0))\mathrm{e}^{-\delta t} + R$$

holds for some structural quantities $\delta > 0$, $R \geq 0$ and $\mathfrak{I} : [0, \infty) \to [0, \infty)$ increasing.

Proof. Along the proof, $C \geq 0$ will denote a *generic* constant, possibly depending on ϕ, γ, q, but independent of the initial energy $E(0)$. Due to (2.3.9)-(2.3.10) and the positivity of Γ_0, the family of functionals

$$\Lambda_\varepsilon = \Pi_\varepsilon(u) + \Gamma_0(u)$$

satisfies for every $\varepsilon > 0$ small

$$E = \Pi_0(u) \leq \Lambda_0 \leq 2\Lambda_\varepsilon \leq 4\Lambda_0. \tag{2.5.1}$$

The product in H of (2.3.4) and $2u_t + 2\varepsilon u$ reads

$$\langle \Box u, u_t + \varepsilon u\rangle + 2\Phi_1(u_t) = -2\langle \gamma(u), u_t + \varepsilon u\rangle - 2\varepsilon\langle \phi(u_t), u\rangle + 2\langle q, u_t + \varepsilon u\rangle,$$

and an application of (2.3.11) entails

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Pi_\varepsilon(u) + \varepsilon\Pi_\varepsilon(u) + \frac{1}{2}\big[\varepsilon|u|_1^2 + 3|u_t|_1^2\big] + 2\Phi_1(u_t) \\ &\quad \le -2\langle \gamma(u), u_t + \varepsilon u\rangle - 2\varepsilon\langle \phi(u_t), u\rangle + 2\langle q, u_t + \varepsilon u\rangle. \end{aligned} \tag{2.5.2}$$

Recalling (2.3.2), we have

$$-2\langle \gamma(u), u_t + \varepsilon u\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_0(u) - 2\varepsilon\Gamma_1(u) \le -\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_0(u) - \varepsilon\Gamma_0(u),$$

whereas (2.3.3) and (2.5.1) yield

$$-2\varepsilon\langle \phi(u_t), u\rangle \le 2\varepsilon\|\phi(u_t)\|_{L^{6/5}}\|u\|_{L^6} \le C\varepsilon[\Phi_1(u_t)]^{\frac{5}{6}}|u|_1 \le \Phi_1(u_t) + C\varepsilon^6\Lambda_\varepsilon^3.$$

Finally,

$$2\langle q, u_t + \varepsilon u\rangle \le 2\varepsilon\|q\|\|u\| + 2\|q\|\|u_t\| \le \frac{1}{2}\big[\varepsilon|u|_1^2 + |u_t|_1^2\big] + C.$$

Plugging the three inequalities in (2.5.2), we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon + |u_t|_1^2 + \Phi_1(u_t) \le C\varepsilon^6\Lambda_\varepsilon^3 + C. \tag{2.5.3}$$

Within (2.5.1) and (2.5.3), we meet the hypotheses of Lemma 2.2.1. Thus,

$$E(t) \le \Lambda_0(t) \le \mathfrak{I}(\Lambda_0(0))\mathrm{e}^{-\delta t} + R,$$

for some $\delta > 0$, $R \ge 0$ and $\mathfrak{I} \ge 0$ increasing. On the other hand, from the growth bound on γ we infer the control

$$\Lambda_0 = E + \Gamma_0(u) \le cE\big[1 + E^2\big], \quad c \ge 1.$$

Accordingly, upon redefining $\mathfrak{I}$ in the obvious way, the theorem is proven. □

For initial data $z \in \mathbb{B}$ invariant absorbing, an integration in time of (2.5.3) with $\varepsilon = 0$ provides a corollary.

Corollary 2.5.1. *For any invariant absorbing set $\mathbb{B}$ there is $C = C(\mathbb{B}) \ge 0$ such that*

$$\sup_{z\in\mathbb{B}} \int_t^T \big[|u_t(\tau)|_1^2 + \Phi_1(u_t(\tau))\big]\,\mathrm{d}\tau \le C + C(T-t), \quad \forall\, T > t \ge 0.$$

2.6 Partial Regularization

In the same spirit of [73], a full exploitation of the partially parabolic features of the equation allows us to gain additional regularity on the velocity component of the solution.

Theorem 2.6.1. *There exists an invariant absorbing set* $\mathbb{B}$ *satisfying*

$$\sup_{t\geq 0}\sup_{z\in\mathbb{B}}\left[|u_t(t)|_1 + \|u_{tt}(t)\| + \int_t^{t+1} |u_{tt}(\tau)|_1^2\,\mathrm{d}\tau\right] < \infty.$$

In particular, $\mathbb{B}$ *is a bounded subset of* $\mathcal{V}_1$.

The proof will require some passages. We begin with a simple observation, stated as a lemma.

Lemma 2.6.1. *If* $\mathbb{B}_0$ *is an invariant absorbing set, then*

$$\mathbb{B}_1 = S(1)\mathbb{B}_0 \subset \mathbb{B}_0$$

remains invariant and absorbing, and any (bounded) function $\Lambda : \mathbb{B}_1 \to \mathbb{R}$ *satisfies*

$$\sup_{t\geq 0}\sup_{z\in\mathbb{B}_1}\Lambda(S(t)z) = \sup_{t\geq 0}\sup_{z\in\mathbb{B}_0}\Lambda(S(t+1)z) \leq \sup_{z\in\mathbb{B}_0}\Lambda(S(1)z).$$

Lemma 2.6.2. *There exists an invariant absorbing set* $\mathbb{B}_1$*, together with a constant* $C = C(\mathbb{B}_1) \geq 0$*, such that for all initial data in* $\mathbb{B}_1$

$$\sup_{t\geq 0}|u_t(t)|_1 \leq C \qquad \text{and} \qquad \int_0^1 \|u_{tt}(t)\|^2\,\mathrm{d}t \leq C.$$

Proof. Fix an arbitrary invariant absorbing set $\mathbb{B}_0$, and consider initial data $z \in \mathbb{B}_0$. In what follows, $C \geq 0$ is a *generic* constant depending only on $\mathbb{B}_0$. By (2.3.1) and the growth bound on γ, the functional

$$\Lambda = \Lambda(S(t)z) = |u_t|_1^2 + \Phi_0(u_t) + 2(u, u_t)_1 + 2\langle\gamma(u), u_t\rangle + K$$

fulfills for $K = K(\mathbb{B}_0) > 0$ large enough the uniform controls

$$|u_t|_1^2 \leq 2\Lambda \leq C\big[1 + |u_t|_1^2 + \Phi_1(u_t)\big].$$

In particular, we see from Corollary 2.5.1 that

$$\int_0^1 \Lambda(S(t)z)\,\mathrm{d}t \leq C.$$

Recalling (2.3.8), the product in H of (2.3.4) and $2u_{tt}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\|u_{tt}\|^2 = 2|u_t|_1^2 + 2\langle\gamma'(u)u_t, u_t\rangle + 2\langle q, u_{tt}\rangle.$$

Estimating the right-hand side as

$$2|u_t|_1^2 + 2\langle\gamma'(u)u_t, u_t\rangle \leq 2|u_t|_1^2 + 2\|\gamma'(u)\|_{L^{3/2}}\|u_t\|_{L^6}^2 \leq C\big(1 + \|u\|_1^4\big)|u_t|_1^2 \leq C\Lambda,$$

and

$$2\langle q, u_{tt}\rangle \leq 2\|q\|\|u_{tt}\| \leq \|u_{tt}\|^2 + C,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \|u_{tt}\|^2 \leq C\Lambda + C. \tag{2.6.1}$$

Hence, multiplying at every fixed time $t \in [0,1]$ both terms of (2.6.1) by t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[t\Lambda(S(t)z)\big] \leq C\Lambda(S(t)z) + C,$$

and a subsequent integration on $[0,1]$ gives

$$\Lambda(S(1)z) \leq C\int_0^1 \Lambda(S(t)z)\,\mathrm{d}t + C \leq C.$$

Choosing

$$\mathbb{B}_1 = S(1)\mathbb{B}_0 \subset \mathbb{B}_0$$

and applying Lemma 2.6.1, we draw the uniform estimate

$$\sup_{t\geq 0}\sup_{z\in\mathbb{B}_1} \Lambda(S(t)z) \leq \sup_{z\in\mathbb{B}_0} \Lambda(S(1)z) \leq C,$$

establishing the desired bound

$$\sup_{t\geq 0}\sup_{z\in\mathbb{B}_1} |u_t(t)|_1 \leq C.$$

In turn, for initial data $z \in \mathbb{B}_1$, the differential inequality (2.6.1) improves to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \|u_{tt}\|^2 \leq C,$$

and an integration over $[0,1]$ provides the remaining integral control. □

Proof of Theorem 2.6.1. We now take initial data z from the invariant absorbing set $\mathbb{B}_1$ of the previous lemma. Accordingly, $C \geq 0$ will stand for a *generic* constant depending only on $\mathbb{B}_1$. Differentiating (2.3.4) with respect to time, we get

$$\Box u_t = -2\phi'(u_t)u_{tt} - 2\gamma'(u)u_t + 2q_t.$$

Multiplying both terms by u_{tt}, we obtain from (2.3.7)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2|u_{tt}|_1^2 = -2\langle\phi'(u_t)u_{tt}, u_{tt}\rangle + 2\langle q_t, u_{tt}\rangle - 2\langle\gamma'(u)u_t, u_{tt}\rangle,$$

where

$$\Lambda = \Lambda(S(t)z) = |u_t|_1^2 + \|u_{tt}\|^2.$$

Since $\phi' \geq 0$,

$$-2\langle\phi'(u_t)u_{tt}, u_{tt}\rangle \leq 0.$$

Concerning the other two terms, we have

$$2\langle q_t, u_{tt}\rangle = -2\langle\phi_c'(u_t)u_{tt} + \gamma_c'(u)u_t, u_{tt}\rangle \leq C\|u_{tt}\|^2 + C,$$

and, after Lemma 2.6.2,

$$-2\langle\gamma'(u)u_t, u_{tt}\rangle \leq 2\|\gamma'(u)\|_{L^{3/2}}\|u_t\|_{L^6}\|u_{tt}\|_{L^6} \leq C|u_{tt}|_1 \leq |u_{tt}|_1^2 + C.$$

Summarizing, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + |u_{tt}|_1^2 \leq C\Lambda + C. \tag{2.6.2}$$

By Lemma 2.6.2 we infer also the integral control

$$\int_0^1 \Lambda(S(t)z)\,\mathrm{d}t \leq C.$$

Hence, multiplying by t and integrating on $[0,1]$, we get

$$\Lambda(S(1)z) \leq C.$$

Defining

$$\mathbb{B} = S(1)\mathbb{B}_1 \subset \mathbb{B}_1,$$

we conclude from Lemma 2.6.1 that

$$\sup_{t\geq 0}\sup_{z\in\mathbb{B}}\left[|u_t(t)|_1^2 + \|u_{tt}(t)\|^2\right] = \sup_{t\geq 0}\sup_{z\in\mathbb{B}}\Lambda(S(t)z) \leq \sup_{z\in\mathbb{B}_1}\Lambda(S(1)z) \leq C.$$

At this point, choosing initial data $z \in \mathbb{B}$, we can rewrite (2.6.2) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + |u_{tt}|_1^2 \leq C,$$

and an integration over $[t, t+1]$ completes the argument. □

CHAPTER 3

Attractor for the SDNWE. The Critical-Subcritical Case

3.1 Introduction

The longterm properties of the strongly damped wave equation (2.1.1) considered in the previous chapter have been widely investigated by several authors. The existence of a regular global attractor in the case $f \equiv 0$ and within a critical growth condition on g is well-known (see e.g. [7, 21, 72, 73]). Moreover, in presence of a linear (or at most sublinear) f, many results are available in the literature. To the best of our knowledge the more challenging case of a superlinear f has been tackled in [17, 54]. There, with reference to assumption (2.1.4), the existence of the global attractor (without additional regularity) is attained provided that $p < 3$ or $p \leq 3$, respectively, and f has a subcritical growth

$$|f'(s)| \leq c + c|s|^{r-1}, \quad r < 5.$$

The focus of this chapters is the asymptotic behavior of the semigroup generated by equation (2.1.1) in the critical-subcritical case. More precisely, for a nonlinearity f of critical polynomial order $r = 5$, we obtain the global attractor whenever $p < 5$, establishing its optimal regularity within the further restriction $p \leq 4$.

In the next Section 3.2 we establish the existence of the global attractor $\mathbb{A}$ for $S(t)$. In Section 3.3 we give a conditional regularity result for $\mathbb{A}$, whose optimal regularity is demonstrated in the final Section 3.4.

Remark 3.1.1. *Analogously to what done in [52] for the case $f \equiv 0$, it is also possible to study the asymptotic properties of equation* (2.1.1) *in the "supercritical" situation (i.e. beyond the quintic growth) assuming a further polynomial control from below and changing the phase space.*

3.2 The Global Attractor

According to Theorem 2.4.1, along the chapter $S(t)$ will denote the solution semigroup generated by equation (2.1.1), acting on the phase space $\mathcal{H}$. The main result reads as follows.

Theorem 3.2.1. *For any fixed $p < 5$, the semigroup $S(t) : \mathcal{H} \to \mathcal{H}$ possesses a connected global attractor $\mathbb{A}$.*

Since a fully invariant bounded set is contained in every absorbing set, Theorem 2.6.1 proved in the previous chapter provides an immediate corollary.

Corollary 3.2.1. *Any solution $S(t)z = \{u(t), u_t(t)\}$ on $\mathbb{A}$ fulfills*

$$\sup_{t\geq 0}\sup_{z\in\mathbb{A}}\left[|u_t(t)|_1 + \|u_{tt}(t)\| + \int_t^{t+1}|u_{tt}(\tau)|_1^2\,\mathrm{d}\tau\right] < \infty.$$

As a byproduct, $\mathbb{A}$ is bounded in $\mathcal{V}_1$.

According to the abstract Theorem A.1 stated in the final appendix, the existence of $\mathbb{A}$ is attained once we find a constant $\nu < 1$, a time $T > 0$ and a precompact pseudometric $\boldsymbol{\rho}$ on $\mathbb{B}$ such that the map $S = S(T)$, known to be continuous on the whole space $\mathcal{H}$ from Theorem 2.4.1, satisfies the inequality

$$|Sz_1 - Sz_2|_{\mathcal{H}} \leq \nu|z_1 - z_2|_{\mathcal{H}} + \boldsymbol{\rho}(z_1, z_2), \quad \forall\, z_1, z_2 \in \mathbb{B}. \tag{3.2.1}$$

Up to an inessential multiplicative constant, the pseudometric that will do reads

$$\boldsymbol{\rho}_{T,m}(z_1, z_2) = \sup_{t\in[0,T]}\left[\|u_1(t) - u_2(t)\|_m^2 + \|\partial_t u_1(t) - \partial_t u_2(t)\|^2\right]^{\frac{1}{2}},$$

with $T > 0$ and $m < 1$ to be properly chosen, where

$$\{u_\imath(t), \partial_t u_\imath(t)\} = S(t)z_\imath, \quad \imath = 1, 2.$$

The precompactness of $\boldsymbol{\rho}_{T,m}$ follows from standard Sobolev compact embeddings. Indeed, if z_n is a sequence in $\mathbb{B}$, the corresponding solutions u_n are known from Theorem 2.6.1 to be uniformly bounded in

$$W^{2,2}(0, T; \mathrm{H}_1) \Subset \mathcal{C}([0, T], \mathrm{H}_m) \cap \mathcal{C}^1([0, T], \mathrm{H}).$$

Proof of Theorem 3.2.1. Along the proof, we will exploit several times the uniform bounds provided by Theorem 2.6.1. Accordingly, $C \geq 0$ will stand for a *generic* constant depending only on $\mathbb{B}$. Without loss of generality, we assume $p \in [3, 5)$ and we fix

$$m = \frac{p-3}{2} \in [0, 1).$$

The time $T > 0$ is also understood to be fixed, albeit its precise value will be specified in a later moment. Let then $z_\imath \in \mathbb{B}$ be arbitrarily given, and call for short

$$D = \boldsymbol{\rho}_{T,m}(z_1, z_2).$$

The difference of the solutions

$$\{\bar{u}(t), \bar{u}_t(t)\} = S(t)z_1 - S(t)z_2$$

fulfills the equation

$$\Box \bar{u} = 2\phi_{\rm d} + 2\gamma_{\rm d},$$

where we put

$$\begin{aligned} \phi_{\rm d} &= \phi(\partial_t u_2) - \phi(\partial_t u_1) + \phi_{\rm c}(\partial_t u_2) - \phi_{\rm c}(\partial_t u_1), \\ \gamma_{\rm d} &= \gamma(u_2) - \gamma(u_1) + \gamma_{\rm c}(u_2) - \gamma_{\rm c}(u_1). \end{aligned}$$

Thus, by (2.3.9) and (2.3.11), a multiplication in H by $\bar{u}_t + \varepsilon\bar{u}$ with $\varepsilon > 0$ small leads to

$$\frac{\rm d}{{\rm d}t}\Pi_\varepsilon(\bar{u}) + \varepsilon\Pi_\varepsilon(\bar{u}) + \frac{1}{2}\big[\varepsilon|\bar{u}|_1^2 + 3|\bar{u}_t|_1^2\big] \leq 2\langle\phi_{\rm d}, \bar{u}_t\rangle + 2\varepsilon\langle\phi_{\rm d}, \bar{u}\rangle + 2\langle\gamma_{\rm d}, \bar{u}_t + \varepsilon\bar{u}\rangle.$$

We now estimate the right-hand side uniformly with respect to $t \in [0, T]$, being D the only quantity dependent on T. Firstly, we learn from (2.3.1) that

$$2\langle\phi_{\rm d}, \bar{u}_t\rangle \leq C\|\bar{u}_t\|^2 \leq CD^2,$$

whereas the growth bound on ϕ entails

$$2\varepsilon\langle\phi_{\rm d}, \bar{u}\rangle \leq C\varepsilon\big[1 + \|\partial_t u_1\|_{L^6} + \|\partial_t u_2\|_{L^6}\big]^4\|\bar{u}\|_{L^6}\|\bar{u}_t\|_{L^6} \leq \frac{1}{2}|\bar{u}_t|_1^2 + C\varepsilon^2|\bar{u}|_1^2.$$

Concerning the last term, from the Sobolev embedding

$$\mathrm{H}_m \subset L^{\frac{6}{3-2m}}(\Omega)$$

we obtain

$$\begin{aligned} 2\langle\gamma_{\rm d}, \bar{u}_t + \varepsilon\bar{u}\rangle &\leq C\big[1 + \|u_1\|_{L^6} + \|u_2\|_{L^6}\big]^{2+2m}\|\bar{u}\|_{L^{6/(3-2m)}}\big[\|\bar{u}_t\|_{L^6} + \varepsilon\|\bar{u}\|_{L^6}\big] \\ &\leq C\|\bar{u}\|_m\big[|\bar{u}_t|_1 + \varepsilon|\bar{u}|_1\big] \\ &\leq |\bar{u}_t|_1^2 + C\varepsilon^2|\bar{u}|_1^2 + CD^2. \end{aligned}$$

In summary, up to fixing ε small enough, the right-hand side is controlled by

$$C\varepsilon^2|\bar{u}|_1^2 + \frac{3}{2}|\bar{u}_t|_1^2 + CD^2 \leq \frac{1}{2}\big[\varepsilon|\bar{u}|_1^2 + 3|\bar{u}_t|_1^2\big] + CD^2.$$

Accordingly,

$$\frac{\rm d}{{\rm d}t}\Pi_\varepsilon(\bar{u}) + \varepsilon\Pi_\varepsilon(\bar{u}) \leq CD^2,$$

and the standard Gronwall lemma on $[0, T]$ yields

$$\Pi_\varepsilon(\bar{u}(T)) \leq \Pi_\varepsilon(\bar{u}(0))\mathrm{e}^{-\varepsilon T} + CD^2.$$

Therefore, invoking (2.3.9)-(2.3.10),

$$|Sz_1 - Sz_2|_{\mathcal{H}}^2 \leq 2\Pi_\varepsilon(\bar{u}(T)) \leq \nu^2|z_1 - z_2|_{\mathcal{H}}^2 + CD^2,$$

Choosing $T > 0$ large enough such that $\nu < 1$, we recover (3.2.1). $\square$

3.3 A Conditional Result

The next issue is the regularity of the attractor. To this end, let $\mathbb{A}$ be the global attractor of $S(t)$, whose existence is actually unproven for the critical case $p = 5$. Introducing the two-component vector

$$h_\star = \{-B^{-1}h, 0\} \in \mathcal{H}_1,$$

we define the translate of the attractor

$$\mathbb{A}_\star = \mathbb{A} + h_\star = \{z + h_\star :\ z \in \mathbb{A}\}.$$

We also need the following standard result (cf. Lemma 2.1 in [12]).

Lemma 3.3.1. *Given a nonnegative locally summable function φ on $\mathbb{R}^+$, the inequality*

$$\int_0^t \mathrm{e}^{-\varepsilon(t-s)}\varphi(\tau)\,\mathrm{d}\tau \le \frac{1}{1-\mathrm{e}^{-\varepsilon}} \sup_{T\ge 0} \int_T^{T+1} \varphi(\tau)\,\mathrm{d}\tau$$

holds for every $t \ge 0$ and every $\varepsilon > 0$.

A conditional regularity result holds.

Proposition 3.3.1. *Assume to know that $\mathbb{A}$ is bounded in $\mathcal{H}_1$. Then $\mathbb{A}_\star$ is bounded in $\mathcal{H}_2$. In turn, $\mathbb{A}$ is bounded in $\mathcal{V}_2$, and is bounded in $\mathcal{H}_2$ whenever $h \in \mathrm{H}_1$.*

Proof. Let $C \ge 0$ be a *generic* constant depending only on $\mathbb{A}$, and consider an arbitrary solution

$$S(t)z = \{u(t), u_t(t)\} \qquad \text{with} \qquad z = \{a, b\} \in \mathbb{A}.$$

We first show the boundedness of $\mathbb{A}$ in $\mathcal{V}_2$. To this end, on account of the full invariance of the attractor, it is enough proving the uniform bound

$$|u_t(1)|_2 \le C.$$

Indeed, recalling also Corollary 3.2.1, the vector $\xi = u_t(1)$ fulfills

$$B\xi + \phi(\xi) = q(1) - u_{tt}(1) - Bu(1) - \gamma(u(1)) \in \mathrm{H},$$

and from the monotonicity of ϕ we obtain

$$2|\xi|_2^2 \le 2|\xi|_2^2 + 2\langle \phi'(\xi)\nabla\xi, \nabla\xi\rangle = 2\langle A\xi, B\xi + \phi(\xi)\rangle \le |\xi|_2^2 + C.$$

Next, defining the function

$$\zeta = B^{-1}[q - h - u_{tt} - \phi(u_t) - \gamma(u)],$$

we verify by direct calculations the identity

$$u(t) - B^{-1}h - \mathrm{e}^{-t}(a - B^{-1}h) = \int_0^t \mathrm{e}^{-(t-\tau)}\zeta(\tau)\,\mathrm{d}\tau.$$

Exploiting the $\mathcal{V}_2$-boundedness of the attractor proved earlier, together with the integral estimate in Corollary 3.2.1, an application of Lemma 3.3.1 provides the control

$$\int_0^t \mathrm{e}^{-(t-\tau)} |\zeta(\tau)|_3 \, \mathrm{d}\tau \leq \frac{1}{1-\mathrm{e}^{-1}} \sup_{T \geq 0} \int_T^{T+1} |\zeta(\tau)|_3 \, \mathrm{d}\tau \leq C.$$

Accordingly,

$$|u(t) - B^{-1}h - \mathrm{e}^{-t}(a - B^{-1}h)|_3 \leq C,$$

yielding in turn, as $\mathbb{A}$ is bounded in $\mathcal{V}_2$,

$$|S(t)z + h_\star - \mathrm{e}^{-t}(z + h_\star)|_{\mathcal{H}_2} \leq C.$$

Letting $t \to \infty$ and appealing once more to the full invariance of the attractor, we conclude that $\mathbb{A}_\star$ is bounded in $\mathcal{H}_2$. □

3.4 Regularity of the Attractor

The (optimal) regularity of the global attractor $\mathbb{A}$ is attained within a restriction on the growth of g.

Theorem 3.4.1. *Assume that $p \leq 4$. Then the translate $\mathbb{A}_\star$ of the global attractor is bounded in $\mathcal{H}_2$.*

The proof will be carried out in a number of lemmas. We preliminarily observe that, due to Proposition 3.3.1, it suffices showing the boundedness of $\mathbb{A}$ in $\mathcal{H}_1$. To this aim, calling as usual $C \geq 0$ a *generic* constant depending only on $\mathbb{A}$, let us consider an arbitrary solution

$$S(t)z = \{u(t), u_t(t)\} \qquad \text{with} \qquad z = \{a, b\} \in \mathbb{A},$$

which is known by Corollary 3.2.1 to satisfy uniformly in time

$$|u|_1 + |u_t|_1 \leq C. \tag{3.4.1}$$

Then, we split u into the sum

$$u(t) = v(t) + w(t),$$

where v and w solve the Cauchy problems

$$\begin{cases} v_{tt} + Bv + Bv_t + \phi(u_t) - \phi(w_t) + \gamma(v) = 0, \\ v(0) = a, \\ v_t(0) = b, \end{cases} \tag{3.4.2}$$

and

$$\begin{cases} w_{tt} + Bw + Bw_t + \phi(w_t) + \gamma(u) - \gamma(v) = q, \\ w(0) = 0, \\ w_t(0) = 0. \end{cases} \tag{3.4.3}$$

Lemma 3.4.1. *The uniform bound* $|v|_1 + \|v_t\| \le C$ *holds, along with the integral estimate*

$$\int_0^\infty |v_t(t)|_1^2 \, \mathrm{d}t \le C. \tag{3.4.4}$$

Proof. Multiplying (3.4.2) by $2v_t$, and using the monotonicity of ϕ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[|v|_1^2 + \|v_t\|^2 + \Gamma_0(v)\big] + 2|v_t|_1^2 \le 0.$$

Recalling that $\Gamma_0(v)$ is positive, an integration over $[0,t]$ gives

$$|v(t)|_1^2 + \|v_t(t)\|^2 + 2\int_0^t |v_t(\tau)|_1^2 \, \mathrm{d}\tau \le C.$$

Since $t > 0$ is arbitrary, we are finished. □

Collecting (3.4.1) and (3.4.4) we draw an immediate corollary.

Corollary 3.4.1. *There is* $M = M(\mathbb{A}) \ge 0$ *such that, for any time* $T \ge 1$*, the estimate*

$$|w_t(t_T)|_1 \le M$$

occurs for some $t_T = t_T(z) \in [T-1, T]$.

Lemma 3.4.2. *The uniform bound* $|w_t|_1 \le C$ *holds.*

Proof. Arguing as in Lemma 2.6.2, we introduce the functional

$$\Lambda = |w_t|_1^2 + \Phi_0(w_t) + 2(w, w_t)_1 + 2\langle \gamma(u) - \gamma(v), w_t\rangle + K$$

with $K = K(\mathbb{A}) > 0$ large enough in order to have

$$|w_t|_1^2 \le 2\Lambda \le C\big[1 + |w_t|_1^6\big].$$

Indeed, thanks to (3.4.1) and Lemma 3.4.1,

$$2\big|\langle \gamma(u) - \gamma(v), w_t\rangle\big| \le 2\|\gamma(u) - \gamma(v)\|_{L^{6/5}} \|w_t\|_{L^6} \le \frac{1}{4}|w_t|_1^2 + C.$$

The product in H of (3.4.3) and $2w_{tt}$ (cf. (2.3.8)) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\|w_{tt}\|^2 = 2|w_t|_1^2 + 2\langle \gamma'(u)u_t - \gamma'(v)v_t, w_t\rangle + 2\langle q, w_{tt}\rangle.$$

Appealing again to (3.4.1) and Lemma 3.4.1, and using the straightforward inequality

$$|w_t|_1 \le |v_t|_1 + |u_t|_1 \le |v_t|_1 + C,$$

the right-hand side is controlled by

$$\begin{aligned} 2|w_t|_1^2 &+ 2\big[\|\gamma'(u)\|_{L^{3/2}}\|u_t\|_{L^6} + \|\gamma'(v)\|_{L^{3/2}}\|v_t\|_{L^6}\big]\|w_t\|_{L^6} + 2\|q\|\|w_{tt}\| \\ &\le 2\|w_{tt}\|^2 + C|w_t|_1^2 + C|v_t|_1|w_t|_1 + C \\ &\le 2\|w_{tt}\|^2 + C|v_t|_1^2 + C. \end{aligned}$$

Therefore, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda \leq C|v_t|_1^2 + C.$$

At this point, for every fixed $T > 0$, we integrate over $[t,T]$, for some positive $t \geq T-1$. By virtue of (3.4.4), this gives

$$|w_t(T)|_1^2 \leq 2\Lambda(T) \leq C + 2\Lambda(t) \leq C\big[1 + |w_t(t)|_1^6\big].$$

If $T \leq 1$ we choose $t = 0$, otherwise we choose $t = t_T$ as in Corollary 3.4.1. In either case, the desired bound follows. □

We subsume (3.4.1), Lemma 3.4.1 and Lemma 3.4.2 into the uniform estimate

$$|u|_1 + |v|_1 + |w|_1 + |u_t|_1 + |v_t|_1 + |w_t|_1 \leq C. \tag{3.4.5}$$

We are now able to prove the (exponential) decay of the solutions to (3.4.2).

Lemma 3.4.3. *There exists $\delta = \delta(\mathbb{A}) > 0$ such that*

$$|\{v(t), v_t(t)\}|_{\mathcal{H}} \leq C\mathrm{e}^{-\delta t}.$$

Proof. With reference to equation (3.4.2), we introduce the family of functionals

$$\Lambda_\varepsilon = \Pi_\varepsilon(v) + \Gamma_0(v),$$

and we recast word by word the proof of Theorem 2.5.1 (with v in place of u), the only differences here being that $q = 0$ and the term $\phi(u_t)$ is replaced by $\phi(u_t) - \phi(w_t)$. Then we find the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon + \frac{1}{2}\big[\varepsilon|v|_1^2 + 3|v_t|_1^2\big] + 2\langle\phi(u_t) - \phi(w_t), v_t\rangle \leq -2\varepsilon\langle\phi(u_t) - \phi(w_t), v\rangle.$$

The monotonicity of ϕ ensures that

$$2\langle\phi(u_t) - \phi(w_t), v_t\rangle \geq 0,$$

while using (3.4.5) we draw the estimate

$$-2\varepsilon\langle\phi(u_t) - \phi(w_t), v\rangle \leq 2\varepsilon\|\phi(u_t) - \phi(w_t)\|_{L^{6/5}}\|v\|_{L^6} \leq C\varepsilon|v_t|_1|v|_1.$$

It is readily seen that, up to fixing $\varepsilon = \varepsilon(\mathbb{A}) > 0$ small enough, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon \leq 0,$$

and the Gronwall lemma together with (2.5.1) give the desired decay. □

The growth bound $p \leq 4$, not used so far, will play a role in the next lemma, where we will need the inequality

$$\|\gamma(u) - \gamma(v)\| \leq C|w|_2^{\frac{1}{2}}. \tag{3.4.6}$$

Indeed, if $p \leq 4$, we can write

$$\|\gamma(u) - \gamma(v)\| \leq C\big[1 + \|u\|_{L^6}^3 + \|v\|_{L^6}^3\big]\|w\|_{L^\infty},$$

and the assertion follows from (3.4.5) and the Agmon inequality

$$\|w\|_{L^\infty}^2 \leq c_\Omega \|w\|_1 \|w\|_2,$$

where $c_\Omega > 0$ depending only on the three-dimensional domain Ω.

Lemma 3.4.4. *We have the uniform bound*

$$|\{w(t), w_t(t)\}|_{\mathcal{H}_1} \leq C.$$

Proof. On account of (2.3.9)-(2.3.10), for every $\varepsilon > 0$ small the energy functional

$$\Lambda_\varepsilon = \Pi_\varepsilon(A^{\frac{1}{2}}w)$$

fulfills

$$|\{w, w_t\}|_{\mathcal{H}_1}^2 \leq 2\Lambda_\varepsilon.$$

We take the product in H of (3.4.3) and $2Aw_t + 2\varepsilon Aw$. Since the operators A and B commute, by (2.3.11) and the monotonicity of ϕ, this gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon + \frac{1}{2}\big[\varepsilon|w|_2^2 + 3|w_t|_2^2\big] &\leq \langle \Box w, Aw_t + \varepsilon Aw\rangle + 2\langle \phi'(w_t)\nabla w_t, \nabla w_t\rangle \\ &= 2\langle Q, Aw_t + \varepsilon Aw\rangle - 2\varepsilon\langle \phi'(w_t)\nabla w_t, \nabla w\rangle,\end{aligned}$$

having set

$$Q = q - \gamma(u) + \gamma(v).$$

Exploiting (3.4.5) and (3.4.6), we derive the controls

$$2\langle Q, Aw_t + \varepsilon Aw\rangle \leq C\big[1 + |w|_2^{\frac{1}{2}}\big]\big[|w_t|_2 + \varepsilon|w|_2\big] \leq C\varepsilon^{-\frac{4}{3}} + C\varepsilon^{\frac{4}{3}}|w|_2^2 + \frac{1}{2}|w_t|_2^2,$$

and

$$-2\varepsilon\langle \phi'(w_t)\nabla w_t, \nabla w\rangle \leq C\varepsilon\|\phi'(w_t)\|_{L^{3/2}}\|\nabla w_t\|_{L^6}\|\nabla w\|_{L^6} \leq C\varepsilon^2|w|_2^2 + |w_t|_2^2.$$

Therefore, once the parameter $\varepsilon > 0$ is fixed small enough, the right-hand side of the differential inequality becomes less than

$$\frac{1}{2}\big[\varepsilon|w|_2^2 + 3|w_t|_2^2\big] + C,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon \leq C.$$

Since $\Lambda_\varepsilon(0) = 0$, applying the standard Gronwall lemma we are led to

$$|\{w(t), w_t(t)\}|_{\mathcal{H}_1}^2 \leq 2\Lambda_\varepsilon(t) \leq C. \qquad \square$$

Conclusion of the Proof of Theorem 3.4.1. Lemma 3.4.3 and Lemma 3.4.4 tell that the solutions originating from $\mathbb{A}$ are uniformly attracted by a (proper) closed ball $\mathcal{B}$ of $\mathcal{H}_1$ centered at zero. Since $\mathbb{A}$ is fully invariant, this means that $\mathbb{A}$ is contained in the $\mathcal{H}$-closure of $\mathcal{B}$. But closed balls of $\mathcal{H}_1$ are closed in $\mathcal{H}$ as well, for $\mathcal{H}_1$ is reflexive. $\square$

CHAPTER 4

Attractor for the SDNWE. The Fully-Critical Case

4.1 Introduction

The aim of the present chapter is the longterm analysis of the strongly damped nonlinear wave equation (2.1.1) where both the nonlinearities f and g exhibit a critical growth of polynomial order 5. More precisely, we stipulate the following assumptions.

Hypotheses on f and g. *Let $f \in \mathcal{C}^1(\mathbb{R})$ and $g \in \mathcal{C}^2(\mathbb{R})$, with $f(0) = g(0) = 0$, satisfy for every $s \in \mathbb{R}$ and some $c \geq 0$ the growth bounds*

$$|f'(s)| \leq c + c|s|^4, \tag{4.1.1}$$

$$|g''(s)| \leq c + c|s|^3, \tag{4.1.2}$$

along with the dissipation conditions

$$\inf_{s\in\mathbb{R}} f'(s) = -\lambda > -\lambda_1, \tag{4.1.3}$$

$$\liminf_{|s|\to\infty} g'(s) > -\lambda_1. \tag{4.1.4}$$

The dissipation condition for g is standard: roughly speaking, (4.1.4) tells that g is essentially monotone at infinity. On the contrary, the more restrictive (4.1.3) implies the essential monotonicity of f on the whole real line. In this chapter, assuming the (strong) monotonicity of the damping operator

$$-\Delta u_t + f(u_t)$$

ensured by (4.1.3), we prove the existence of an exponential attractor of optimal regularity and, in turn, the one of the regular global attractor of finite fractal dimension.

Still, the existence of exponential (or even global) attractors in the fully critical case when (4.1.3) is replaced by weaker the dissipativity condition

$$\liminf_{|s|\to\infty} f'(s) > -\lambda_1$$

remains an open (and possibly quite challenging) question.

In Section 4.2 we focus on the dynamical system and its dissipative properties, recalling some results proved in Chapter 2. Section 4.3 is devoted to the main theorem on the existence of global and exponential attractors whose proof is carried out in the last Sections 4.4-4.6.

4.2 The Dissipative Dynamical System

We begin by reporting a generalized version of the Gronwall lemma (see [20, Lemma 2.1]).

Lemma 4.2.1. *Setting $\mathbb{R}^+_0 = [0,\infty)$, let $\Lambda : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ be an absolutely continuous function satisfying, for some $\nu > 0$ and $k \geq 0$, the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + 2\nu\Lambda(t) \leq \mu(t)\Lambda(t) + k,$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ fulfills

$$\int_t^T \mu(\tau)\,\mathrm{d}\tau \leq \nu(T-t) + m$$

for every $T > t \geq 0$ and some $m \geq 0$. Then

$$\Lambda(t) \leq \Lambda(0)\mathrm{e}^m\mathrm{e}^{-\nu t} + \frac{k\mathrm{e}^m}{\nu}.$$

For future use, we also recall the existence and uniqueness result.

Theorem 4.2.1. *For every $T > 0$ and every initial datum $z = \{a, b\} \in \mathcal{H}$, problem* (2.1.1) *admits a unique weak solution*

$$u \in \mathcal{C}([0,T],\mathrm{H}_1) \cap \mathcal{C}^1([0,T],\mathrm{H}) \cap W^{1,2}(0,T;\mathrm{H}_1).$$

In addition, for any pair of initial data $z_1, z_2 \in \mathcal{H}$, the difference of the corresponding solutions $\bar{u}(t)$ satisfies

$$\|\{\bar{u}(t),\bar{u}_t(t)\}\|_{\mathcal{H}} \leq C\mathrm{e}^{Ct}\|z_1 - z_2\|_{\mathcal{H}}, \tag{4.2.1}$$

for some constant $C \geq 0$ depending (increasingly) only on the norms of z_1, z_2.

As a byproduct, (2.1.1) generates a dynamical system $S(t)$ acting on the phase space

$$\mathcal{H} = \mathrm{H}_1 \times \mathrm{H},$$

defined by the rule

$$S(t)z = \{u(t), u_t(t)\},$$

where $u(t)$ is the solution at time t with initial data $\{u(0), u_t(0)\} = z$.

4.2.1 Dissipative estimates

In light of our purposes, it is convenient to consider the solutions to the more general equation

$$u_{tt} + Au_t + Au + f(u_t) + g(u) = q, \tag{4.2.2}$$

with a time-dependent external source

$$q = q(\boldsymbol{x}, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}.$$

One of the main theorems of Chapter 2 is the following.[1]

Theorem 4.2.2. *Suppose $q \in L^\infty(\mathbb{R}^+; \mathrm{H})$. Then the solution u to* (4.2.2) *with initial data $z \in \mathcal{H}$ fulfills the dissipative estimate*

$$\|\{u(t), u_t(t)\}\|_{\mathcal{H}} \leq \mathfrak{I}(\|z\|_{\mathcal{H}})\mathrm{e}^{-\delta t} + R,$$

for some constants $\delta > 0$, $R \geq 0$ and some increasing function $\mathfrak{I} : \mathbb{R}^+ \to \mathbb{R}^+$.

For the particular case $q = h$, Theorem 4.2.2 provides the existence of an (invariant) absorbing set for $S(t)$. Actually, an exploitation of the partially parabolic features of the dynamical system provides the following more general result (Theorem 2.6.1).

Theorem 4.2.3. *There exists an invariant absorbing set $\mathbb{B}_0$ such that*

$$\sup_{t\geq 0} \sup_{z\in\mathbb{B}_0} \left[\|u_t(t)\|_1 + \|u_{tt}(t)\| + \int_t^{t+1} \|u_{tt}(\tau)\|_1^2 \, \mathrm{d}\tau \right] < \infty.$$

In particular, $\mathbb{B}_0$ is bounded in $\mathcal{V}_1$.

Remark 4.2.1. *Along the chapter, we will often make use without explicit mention of the Sobolev embedding*

$$\mathrm{H}_s \subset L^{\frac{6}{3-2s}}(\Omega), \quad s \in [0, \tfrac{3}{2}),$$

as well as of the Hölder, Young and Poincaré inequalities. Besides, we will perform formal energy-type estimates, justified in a proper Galerkin approximation scheme.

4.2.2 The dissipation integral

In what follows, $C \geq 0$ will denote a *generic* constant depending only on the invariant absorbing set $\mathbb{B}_0$ of Theorem 4.2.3. We define the functionals on H_1

$$G(w) = 2\int_\Omega \int_0^{w(\boldsymbol{x})} g(y)\, \mathrm{d}y\mathrm{d}\boldsymbol{x} \qquad \text{and} \qquad Q(w) = 2\langle h, w\rangle.$$

An exploitation of (4.1.3) allows us to obtain further information on the integrability of the velocity field.

Lemma 4.2.2. *We have the dissipation integral*

$$\sup_{z\in\mathbb{B}_0} \int_0^\infty \|u_t(\tau)\|_1^2 \, \mathrm{d}\tau \leq C.$$

[1] It is immediate to see that the proof of Theorem 2.5.1 is still valid replacing the time-independent forcing term h with $q \in L^\infty(\mathbb{R}^+; \mathrm{H})$.

Proof. By (4.1.2), the functional

$$\Lambda = \|u_t\|^2 + \|u\|_1^2 + G(u) + Q(u) + K$$

fulfills for $K = K(\mathbb{B}_0) > 0$ large enough the uniform controls

$$0 \leq \Lambda \leq C.$$

Taking product in H of (2.1.1) and $2u_t$ and exploiting (4.1.3), we easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \nu\|u_t\|_1^2 \leq 0$$

for some $\nu > 0$. Integrating the last inequality over $[0, t]$, we get

$$\int_0^t \|u_t(\tau)\|_1^2 \mathrm{d}\tau \leq C,$$

and since $t > 0$ is arbitrary we are finished. □

4.3 Global and Exponential Attractors

4.3.1 Statement of the result

The main result of the chapter reads as follows.

Theorem 4.3.1. *The dynamical system $S(t)$ on $\mathcal{H}$ admits an exponential attractor $\mathbb{E}$ contained and bounded in $\mathcal{V}_2$.*

In particular, $S(t)$ is asymptotically compact, and an immediate corollary can be drawn.

Corollary 4.3.1. *There exists the global attractor $\mathbb{A} \subset \mathbb{E}$ of the dynamical system $S(t)$. Accordingly,*

$$\dim_{\mathrm{f}}(\mathbb{A}) \leq \dim_{\mathrm{f}}(\mathbb{E}) < \infty.$$

Remark 4.3.1. *Since $\mathbb{E}$ is bounded in $\mathcal{V}_2$, arguing as in the previous chapter we can actually deduce the boundedness in $\mathcal{H}_2$ of the translate*

$$\mathbb{E}_\star = \mathbb{E} + h_\star,$$

where we set

$$h_\star = \{-A^{-1}h, 0\} \in \mathcal{H}_1.$$

4.3.2 Proof of Theorem 4.3.1

First, we establish three lemmas, whose proof will be given in the next sections.

Lemma 4.3.1. *There exists a closed set $\mathbb{B}_1 \subset \mathcal{H}$ bounded in $\mathcal{H}_1$ such that*

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathbb{B}_0, \mathbb{B}_1) \leq C_0 \mathrm{e}^{-\omega_0 t}$$

for some $C_0 \geq 0$ and $\omega_0 > 0$, where $\mathbb{B}_0$ is the invariant absorbing set of Theorem 4.2.3.

Lemma 4.3.2. *There exists a closed invariant set $\mathbb{B}_2 \subset \mathcal{H}$ such that*

$$\sup_{t\geq 0}\sup_{z\in\mathbb{B}_2}\big[\|u(t)\|_2 + \|u_t(t)\|_2 + \|u_{tt}(t)\|\big] < \infty \tag{4.3.1}$$

and

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathbb{B}_1, \mathbb{B}_2) \leq C_1 \mathrm{e}^{-\omega_1 t}$$

for some $C_1 \geq 0$ and $\omega_1 > 0$. In particular, $\mathbb{B}_2$ is bounded in $\mathcal{V}_2$.

Lemma 4.3.2 along with (4.2.1) provide an immediate corollary.

Corollary 4.3.2. *For every $T > 0$, the map*

$$(t, z) \mapsto S(t)z$$

is Lipschitz continuous on $[0, T] \times \mathbb{B}_2$ into $\mathcal{H}$, with a Lipschitz constant $\ell = \ell(T) > 0$.

Lemma 4.3.3. *There exists a compact invariant set $\mathbb{E} \subset \mathcal{H}$, with $\dim_{\mathrm{f}}(\mathbb{E}) < \infty$, bounded in $\mathcal{V}_2$ such that*

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathbb{B}_2, \mathbb{E}) \leq C_2 \mathrm{e}^{-\omega_2 t}$$

for some $C_2 \geq 0$ and $\omega_2 > 0$.

We now appeal to the transitivity property of exponential attraction devised in [28].

Lemma 4.3.4. *Let $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{H}$ be such that*

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathcal{B}_0, \mathcal{B}_1) \leq C_0 \mathrm{e}^{-\omega_0 t} \qquad \textit{and} \qquad \boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathcal{B}_1, \mathcal{B}_2) \leq C_1 \mathrm{e}^{-\omega_1 t}$$

for some $C_0, C_1 \geq 0$ and $\omega_0, \omega_1 > 0$. Assume also that for all $z, \zeta \in \bigcup_{t\geq 0} S(t)\mathcal{B}_i$

$$\|S(t)z - S(t)\zeta\|_{\mathcal{H}} \leq K\mathrm{e}^{\kappa t}\|z - \zeta\|_{\mathcal{H}},$$

for some $K \geq 0$ and $\kappa > 0$. Then

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathcal{B}_0, \mathcal{B}_2) \leq C\mathrm{e}^{-\omega t},$$

where $C = KC_0 + C_1$ and $\omega = \frac{\omega_0\omega_1}{\kappa+\omega_0+\omega_1}$.

In light of the above results and recalling (4.2.1), by applying twice Lemma 4.3.4 we conclude that

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathbb{B}_0, \mathbb{E}) \leq C\mathrm{e}^{-\omega t}$$

for some $C \geq 0$ and $\omega > 0$. At this point, since $\mathbb{B}_0$ is absorbing, it is standard matter verifying that

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathcal{B}, \mathbb{E}) \leq \mathfrak{I}(\|\mathcal{B}\|_{\mathcal{H}})\mathrm{e}^{-\omega t}$$

for every bounded set $\mathcal{B} \subset \mathcal{H}$ and some increasing function $\mathfrak{I} : \mathbb{R}^+ \to \mathbb{R}^+$. □

4.4 Proof of Lemma 4.3.1

We proceed through a number of steps. Exploiting an idea from [74], fixing $\theta > 0$ suitably large we decompose the functions f and g as

$$f(s) = \phi(s) - \lambda s \qquad \text{and} \qquad g(s) = \gamma(s) - \theta s.$$

Due to (4.1.1)-(4.1.4), it is apparent that

$$0 \leq \phi'(s) \leq c + c|s|^4, \qquad \gamma'(s) \geq 0, \qquad |\gamma''(s)| \leq c + c|s|^3.$$

Introducing the positive functionals

$$\Phi(u) = 2 \int_\Omega \int_0^{u(\boldsymbol{x})} \phi(y)\,\mathrm{d}y\mathrm{d}\boldsymbol{x},$$

and

$$\Gamma(u) = 2 \int_\Omega \int_0^{u(\boldsymbol{x})} \gamma(y)\,\mathrm{d}y\mathrm{d}\boldsymbol{x},$$

we consider an arbitrary solution $S(t)z = \{u(t), u_t(t)\}$ with initial data $z = \{a, b\} \in \mathbb{B}_0$. Then, we write

$$u(t) = v(t) + w(t),$$

where v and w solve

$$\begin{cases} v_{tt} + Av_t + Av + \phi(u_t) - \phi(w_t) + \gamma(u) - \gamma(w) = 0, \\ v(0) = a, \\ v_t(0) = b, \end{cases} \tag{4.4.1}$$

and

$$\begin{cases} w_{tt} + Aw_t + Aw + \phi(w_t) + \gamma(w) = q, \\ w(0) = 0, \\ w_t(0) = 0, \end{cases} \tag{4.4.2}$$

with

$$q = \lambda u_t + \theta u + h.$$

Throughout this section, $C \geq 0$ will denote a *generic* constant depending only on $\mathbb{B}_0$. Following a standard approach (see e.g. [2, 49, 50, 91]), we will show that system (4.4.1) is exponentially stable, whereas the solutions to (4.4.2) are uniformly bounded in a more regular space.

Lemma 4.4.1. *We have*

$$\sup_{t \geq 0} \big[\|w(t)\|_1 + \|w_t(t)\| \big] \leq C.$$

Proof. We learn from Theorem 4.2.2 that $q \in L^\infty(\mathbb{R}^+; \mathrm{H})$. Thus Theorem 4.2.2 applies to system (4.4.2). $\square$

Lemma 4.4.2. *For every $\nu > 0$ and every $T > t \geq 0$,*

$$\int_t^T \|w_t(\tau)\|_1^2 \, \mathrm{d}\tau \leq \nu(T-t) + \frac{C}{\nu}.$$

Proof. Arguing as in Lemma 4.2.2, we see that the functional

$$\Lambda = \|w_t\|^2 + \|w\|_1^2 + \Gamma(w) - 2\theta\langle u, w\rangle - 2\langle h, w\rangle + K$$

satisfies for $K = K(\mathbb{B}_0) > 0$ large enough the uniform controls

$$0 \leq \Lambda \leq C.$$

Multiplying (4.4.2) by $2w_t$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\|w_t\|_1^2 + 2\langle \phi(w_t), w_t\rangle = 2\lambda\langle u_t, w_t\rangle - 2\theta\langle u_t, w\rangle.$$

The monotonicity of ϕ gives

$$2\langle \phi(w_t), w_t\rangle \geq 0,$$

while, exploiting Lemma 4.4.1,

$$2\lambda\langle u_t, w_t\rangle - 2\theta\langle u_t, w\rangle \leq C\|u_t\|_1.$$

Thus, for every $\nu > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\|w_t\|_1^2 \leq \nu + \frac{C}{\nu}\|u_t\|_1^2,$$

and the claim is proven by integrating on $[t, T]$, on account of Lemma 4.2.2. □

We subsume Lemma 4.2.2 and Lemma 4.4.2 into the integral bound

$$\int_t^T \big[\|u_t(\tau)\|_1^2 + \|w_t(\tau)\|_1^2\big] \, \mathrm{d}\tau \leq \nu(T-t) + \frac{C}{\nu} \tag{4.4.3}$$

for any $T > t \geq 0$ and any $\nu > 0$ small. The next lemma provides a regularization estimate for w_t.

Lemma 4.4.3. *We have*

$$\sup_{t\geq 0} \|w_t(t)\|_1 \leq C.$$

Proof. By the positivity of Φ and Γ and the growth bounds of ϕ and γ, we infer that the functional

$$\Lambda = \|w_t\|_1^2 + 2\langle w_t, w\rangle_1 + \Phi(w_t) + \Gamma(w) + K$$

satisfies, for $K = K(\mathbb{B}_0) > 0$ sufficiently large

$$\|w_t\|_1^2 \leq 2\Lambda \leq C\big[1 + \|w_t\|_1^6\big].$$

The product in H of (4.4.2) and $2w_{tt}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\|w_{tt}\|^2 = 2\|w_t\|_1^2 + \langle \gamma'(w)w_t, w_t\rangle + 2\langle q, w_{tt}\rangle.$$

The estimate

$$\langle\gamma'(w)w_t, w_t\rangle + 2\langle q, w_{tt}\rangle \le C\|w_t\|_1^2 + \|w_{tt}\|^2 + C$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda \le C\|w_t\|_1^2 + C.$$

Thus, for every fixed $T > 0$, integrating over $[t, T]$ for some positive $t \ge T - 1$ and using Lemma 4.4.2, we arrive at

$$\|w_t(T)\|_1^2 \le 2\Lambda(T) \le C + 2\Lambda(t) \le C\big[1 + \|w_t(t)\|_1^6\big].$$

If $T \le 1$ we choose $t = 0$. Otherwise we observe that, in view of Lemma 4.4.2, there exists $M = M(\mathbb{B}_0) \ge 0$ such that, for some $t_T \in [T-1, T]$,

$$\|w_t(t_T)\|_1 \le M.$$

Choosing $t = t_T$ the proof is over. □

Lemma 4.4.4. *There exists* $\omega_0 = \omega_0(\mathbb{B}_0) > 0$ *such that*

$$\|\{v(t), v_t(t)\}\|_{\mathcal{H}} \le C\mathrm{e}^{-\omega_0 t}.$$

Proof. Introduce the positive function

$$p = \int_0^1 \gamma'(su + (1-s)w)\,\mathrm{d}s$$

and observe that

$$\gamma(u) - \gamma(w) = vp.$$

Defining

$$P = \langle vp, v\rangle,$$

it follows from (2.3.10) and the positivity of P that the family of functionals

$$\Lambda_\varepsilon = \Pi_\varepsilon(v) + P$$

satisfies, for $\varepsilon > 0$ sufficiently small,

$$\|v(t)\|_1^2 + \|v_t(t)\|^2 \le 2\Lambda_\varepsilon \le 4\Lambda_0. \tag{4.4.4}$$

On the other hand, from the growth bound on γ, we obtain

$$\Lambda_0 \le C\big[\|v(t)\|_1^2 + \|v_t(t)\|^2\big]. \tag{4.4.5}$$

Multiplying (4.4.1) by $2v_t + 2\varepsilon v$ we infer that

$$\langle \Box v, v_t + \varepsilon v\rangle + 2\langle\phi(u_t) - \phi(w_t), v_t\rangle + 2\langle vp, v_t + \varepsilon v\rangle = -2\varepsilon\langle\phi(u_t) - \phi(w_t), v\rangle.$$

Exploiting the monotonicity of ϕ and (2.3.11) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon + \frac{1}{2}\big[\varepsilon\|v\|_1^2 + 3\|v_t\|_1^2\big] \le \langle p_t, v^2\rangle - 2\varepsilon\langle\phi(u_t) - \phi(w_t), v\rangle.$$

By the growth bound on γ, the function

$$p_t = \int_0^1 \gamma''(su + (1-s)w)\,[su_t + (1-s)w_t]\,\mathrm{d}s$$

fulfills

$$\|p_t\|_{L^{3/2}} \le C\big[\|u_t\|_1 + \|w_t\|_1\big].$$

Thus

$$\langle p_t, v^2\rangle \le C\|p_t\|_{L^{3/2}}\|v\|_1^2 \le C\big[\|u_t\|_1 + \|w_t\|_1\big]\|v\|_1^2 \le \frac{\varepsilon}{4}\|v\|_1^2 + \frac{C}{\varepsilon}\big[\|u_t\|_1^2 + \|w_t\|_1^2\big]\Lambda_\varepsilon.$$

Using Lemma 4.4.3 and the growth bound on ϕ, we deduce the control

$$-2\varepsilon\langle\phi(u_t) - \phi(w_t), v\rangle \le C\varepsilon\|v_t\|_1\|v\|_1 \le \frac{\varepsilon}{4}\|v\|_1^2 + \frac{3}{2}\|v_t\|_1^2.$$

Therefore, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon \le C\big[\|u_t\|_1^2 + \|w_t\|_1^2\big]\Lambda_\varepsilon.$$

In view of (4.4.3), an application of Lemma 4.2.1 together with (4.4.4)-(4.4.5) gives the desired decay. □

Lemma 4.4.5. *We have the uniform bound*

$$\|\{w(t), w_t(t)\}\|_{\mathcal{H}_1} \le C.$$

Proof. By (2.3.10) and the monotonicity of γ, we see that the functional

$$\Lambda_\varepsilon = \Pi_\varepsilon(A^{\frac{1}{2}}w) + 2\langle\gamma(w), Aw\rangle$$

fulfills

$$\|\{w(t), w_t(t)\}\|_{\mathcal{H}_1}^2 \le 2\Lambda_\varepsilon.$$

Taking the product in H of (4.4.2) and $2Aw_t + 2\varepsilon Aw$, exploiting (2.3.11) and the monotonicity of ϕ, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon + \frac{1}{2}\big[\varepsilon\|w\|_2^2 + 3\|w_t\|_2^2\big] \\
&\le \langle\Box w, Aw_t + \varepsilon Aw\rangle + 2\langle\gamma(w), Aw_t + \varepsilon Aw\rangle + 2\langle\gamma'(w)w_t, Aw\rangle + 2\langle\phi'(w_t)\nabla w_t, \nabla w_t\rangle \\
&= 2\langle q, Aw_t + \varepsilon Aw\rangle + 2\langle\gamma'(w)w_t, Aw\rangle - 2\varepsilon\langle\phi'(w_t)\nabla w_t, \nabla w\rangle.
\end{aligned}$$

We have

$$2\langle q, Aw_t + \varepsilon Aw\rangle \le C + \|w_t\|_2^2 + \varepsilon^2\|w\|_2^2,$$

and

$$-2\varepsilon\langle\phi'(w_t)\nabla w_t, \nabla w\rangle \le C\varepsilon^2\|w\|_2^2 + \frac{1}{2}\|w_t\|_2^2.$$

Furthermore, by the Agmon inequality,

$$2\langle\gamma'(w)w_t, Aw\rangle \le C\|w\|_2^2\|w_t\|_1.$$

Thus, for a fixed $\varepsilon > 0$ sufficiently small,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon \leq C\|w_t\|_1\Lambda_\varepsilon + C. \tag{4.4.6}$$

For every $T > t \geq 0$, using Lemma 4.4.2 we estimate

$$\begin{aligned}\int_t^T \|w_t(\tau)\|_1 \mathrm{d}\tau &\leq \sqrt{T-t}\left(\int_t^T \|w_t(\tau)\|_1^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \\ &\leq \sqrt{T-t}\left(\nu(T-t) + \frac{C}{\nu}\right)^{\frac{1}{2}} \leq \sqrt{\nu}(T-t) + \frac{C}{\nu\sqrt{\nu}}.\end{aligned}$$

In summary, for $\nu > 0$ small enough,

$$\int_t^T \|w_t(\tau)\|_1 \mathrm{d}\tau \leq \nu(T-t) + \frac{C}{\nu^3}. \tag{4.4.7}$$

Hence, choosing $\nu = \frac{\varepsilon}{2}$, applying Lemma 4.2.1 to (4.4.6) and noting that $\Lambda_\varepsilon(0) = 0$, we are finished. □

Finally, we take a closed ball $\mathbb{B}_1$ of $\mathcal{H}_1$ centered at zero with radius sufficiently large (i.e. larger than the C in the statement of Lemma 4.4.5). Then, collecting Lemma 4.4.4 and Lemma 4.4.5, the proof of Lemma 4.3.1 is completed. □

4.5 Proof of Lemma 4.3.2

In this section, $C \geq 0$ will stand for a *generic* constant depending only on $\mathbb{B}_1$.

Lemma 4.5.1. *The following estimate holds:*

$$\sup_{t\geq 0}\sup_{z\in\mathbb{B}_1} \|\{u(t), u_t(t)\}\|_{\mathcal{H}_1} \leq C.$$

Proof. Let $u(t)$ be a solution of (2.1.1) with initial data $z = \{a, b\} \in \mathbb{B}_1$, and observe that Lemma 4.4.3 and equation (4.4.7) remain true with u in place of w. Recasting the proof of Lemma 4.4.5, we end up with an inequality analogous to (4.4.6). Since the initial data belong to $\mathbb{B}_1$, we apply Lemma 4.2.1 and the desired estimate is drawn. □

Next, calling $t_\mathrm{e} \geq 0$ the entering time of $\mathbb{B}_1$ into $\mathbb{B}_0$, we consider the invariant set

$$\mathbb{K} = \bigcup_{t\geq t_\mathrm{e}} S(t)\mathbb{B}_1 \subset \mathbb{B}_0.$$

The properties of the set $\mathbb{K}$ are summarized in the following lemma.

Lemma 4.5.2. *We have*

$$\sup_{t\geq 0}\sup_{z\in\mathbb{K}} \left[\|u(t)\|_2 + \|u_t(t)\|_2 + \|u_{tt}(t)\|\right] \leq C.$$

Proof. Take initial data $z \in \mathbb{K}$. We know from Theorem 4.2.3 that $u_{tt} \in L^\infty(\mathbb{R}^+; \mathrm{H})$. Moreover, Lemma 4.5.1 tells that $u \in L^\infty(\mathbb{R}^+; \mathrm{H}_2)$. Hence, for all $t \geq 0$,

$$Au_t(t) + \phi(u_t(t)) = h + \lambda u_t(t) - u_{tt}(t) - Au(t) - g(u(t)) \in \mathrm{H}.$$

Exploiting the monotonicity of ϕ, we deduce that

$$\begin{aligned} 2\|u_t(t)\|_2^2 &\leq 2\|u_t(t)\|_2^2 + 2\langle \phi'(u_t(t))\nabla u_t(t), \nabla u_t(t)\rangle \\ &= 2\langle Au_t(t), Au_t(t) + \phi(u_t(t))\rangle \leq \|u_t(t)\|_2^2 + C, \end{aligned}$$

which yields the desired conclusion. □

Finally, we define

$$\mathbb{B}_2 = \overline{\mathbb{K}}^{\mathcal{H}} \quad (\text{closure in } \mathcal{H}).$$

By the continuity of $S(t)$ in $\mathcal{H}$ and the invariance of $\mathbb{K}$, we immediately get

$$S(t)\mathbb{B}_2 = S(t)\overline{\mathbb{K}}^{\mathcal{H}} \subset \overline{S(t)\mathbb{K}}^{\mathcal{H}} \subset \overline{\mathbb{K}}^{\mathcal{H}} = \mathbb{B}_2.$$

Thus, $\mathbb{B}_2$ is invariant. Moreover, $\mathbb{B}_2$ absorbs $\mathbb{B}_1$ by construction, and hence attracts $\mathbb{B}_1$ at an arbitrary rate $\omega_1 > 0$, up to choosing the constant C_1 sufficiently large. We are left to show (4.3.1). To this end, fixed any time $t \geq 0$, take $u(t)$ originating from initial data $z \in \mathbb{B}_2$. Select a sequence

$$z_n \to z \qquad \text{with} \qquad z_n \in \mathbb{K},$$

whose corresponding solution fulfills (up to a subsequence)

$$u_n \to u \quad \text{weakly in } \mathrm{H}_2,$$

where the identification of the limit is attained by exploiting the semigroup continuity. Thus

$$\|u(t)\|_2 \leq \liminf_{n\to\infty} \|u_n(t)\|_2 \leq C.$$

The remaining estimates for u_t in $L^\infty(\mathbb{R}^+; \mathrm{H}_2)$ and u_{tt} in $L^\infty(\mathbb{R}^+; \mathrm{H})$ are similar. Finally, from (4.3.1) we learn that $\mathbb{B}_2$ is bounded in $\mathcal{V}_2$ and Lemma 4.3.2 is proven. □

4.6 Proof of Lemma 4.3.3

In order to prove Lemma 4.3.3, we follow the standard abstract scheme described in the final appendix (see Lemma A.1 and Theorem A.3). For every initial data $z \in \mathbb{B}_2$, denote by $L(t)z$ the solution at time t to the linear homogeneous problem

$$\begin{cases} u_{tt} + Au_t + Au = 0, \\ (u(0), u_t(0)) = z, \end{cases}$$

and let

$$K(t)z = S(t)z - L(t)z.$$

Given two solutions

$$S(t)z_1 = (u_1(t), \partial_t u_1(t)) \qquad \text{and} \qquad S(t)z_2 = (u_2(t), \partial_t u_2(t))$$

originating from $z_1, z_2 \in \mathbb{B}_2$, we make the decomposition

$$S(t)z_1 - S(t)z_2 = (\bar{u}(t), \bar{u}_t(t)) = (\bar{v}(t), \bar{v}_t(t)) + (\bar{w}(t), \bar{w}_t(t)),$$

where

$$\begin{cases} \bar{v}_{tt} + A\bar{v}_t + A\bar{v} = 0, \\ (\bar{v}(0), \bar{v}_t(0)) = z_1 - z_2, \end{cases} \tag{4.6.1}$$

and

$$\begin{cases} \bar{w}_{tt} + A\bar{w}_t + A\bar{w} + f(\partial_t u_1) - f(\partial_t u_2) + g(u_1) - g(u_2) = 0, \\ (\bar{w}(0), \bar{w}_t(0)) = 0. \end{cases} \tag{4.6.2}$$

Lemma 4.6.1. *For every $t > 0$ sufficiently large, we have the estimate*

$$\|\{\bar{v}(t), \bar{v}_t(t)\}\|_{\mathcal{H}} \leq \vartheta \|z_1 - z_2\|_{\mathcal{H}},$$

for some $\vartheta < 1$.

Proof. Multiplying (4.6.1) by $2\bar{v}_t + 2\varepsilon\bar{v}$ and exploiting (2.3.11), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_\varepsilon(\bar{v}) + \varepsilon\Pi_\varepsilon(\bar{v}) \leq \langle \Box\bar{v}, \bar{v}_t + \varepsilon\bar{v}\rangle = 0.$$

Taking into account (2.3.10), the Gronwall lemma yields the exponential decay of the linear semigroup generated by (4.6.1), which readily implies the claim. $\square$

In the sequel the *generic* constant $C \geq 0$ will depend only on $\mathbb{B}_2$.

Lemma 4.6.2. *The following inequality holds*

$$\|\{\bar{w}(t), \bar{w}_t(t)\}\|_{\mathcal{H}_1} \leq C\mathrm{e}^{Ct}\|z_1 - z_2\|_{\mathcal{H}}.$$

Proof. We consider, for $\varepsilon > 0$ small, the functional

$$\Lambda_\varepsilon = \Pi_\varepsilon(A^{\frac{1}{2}}\bar{w}).$$

We take the product in H of (4.6.2) and $2A\bar{w}_t + 2\varepsilon A\bar{w}$. Using (2.3.11), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon + \frac{1}{2}\big[\varepsilon\|\bar{w}\|_2^2 + 3\|\bar{w}_t\|_2^2\big] &\leq \langle \Box\bar{w}, A\bar{w}_t + \varepsilon A\bar{w}\rangle \\ &= -2\langle f(\partial_t u_1) - f(\partial_t u_2), A\bar{w}_t + \varepsilon A\bar{w}\rangle \\ &\quad - 2\langle g(u_1) - g(u_2), A\bar{w}_t + \varepsilon A\bar{w}\rangle. \end{aligned}$$

Due to (4.1.1) and the Agmon inequality,

$$\|f(\partial_t u_1) - f(\partial_t u_2)\| \leq C\|\partial_t u_1 - \partial_t u_2\|.$$

Thus

$$-2\langle f(\partial_t u_1) - f(\partial_t u_2), A\bar{w}_t + \varepsilon A\bar{w}\rangle \le C\|\bar{u}_t\|\|\bar{w}_t + \varepsilon\bar{w}\|_2.$$

Moreover, by (4.1.2),

$$-2\langle g(u_1) - g(u_2), A\bar{w}_t + \varepsilon A\bar{w}\rangle \le C\|\bar{u}\|_1\|\bar{w}_t + \varepsilon\bar{w}\|_2.$$

A final application of the Hölder inequality entails

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon \le C\big[\|\bar{u}\|_1^2 + \|\bar{u}_t\|^2\big].$$

Since $\Lambda_\varepsilon(0) = 0$, applying (4.2.1) the Gronwall lemma and (2.3.10) we are finished. □

At this point, for $t_\star > 0$ sufficiently large, we set

$$L = L(t_\star) \qquad \text{and} \qquad K = K(t_\star).$$

In light of Lemmas 4.6.1-4.6.2, the assumptions (A.1) and (A.2) of Lemma A.1 in the appendix are verified. This ensures the existence of a discrete exponential attractor $\mathbb{E}_{\mathrm{d}} \subset \mathbb{B}_2$ for the discrete semigroup

$$S = S(t_\star).$$

Recalling Corollary 4.3.2, we learn from Theorem A.3 that

$$\mathbb{E} = \bigcup_{\tau\in[0,t_\star]} S(\tau)\mathbb{E}_{\mathrm{d}} \subset \mathbb{B}_2$$

is an exponential attractor for the semigroup $S(t)$ on $\mathbb{B}_2$. The proof of Lemma 4.3.3 is over. □

CHAPTER 5

Attractor for the SDNWE. The Subcritical-Critical Case

5.1 Introduction

We begin by summarizing the results obtained in Chapters 3 and 4. We have analyzed the nonlinear strongly damped wave equation

$$u_{tt} - \Delta u_t - \Delta u + f(u_t) + g(u) = h \tag{5.1.1}$$

on a bounded smooth domain $\Omega \subset \mathbb{R}^3$. The asymptotic properties of the semigroup generated by (5.1.1) in the presence of a linear (or at most sublinear) f have been studied by many authors. On the contrary, the analysis in the case of a superlinear f, which introduces highly non trivial technical difficulties due to the simultaneous interaction between two nonlinearities, is far from being completely exhaustive. We focused on two challenging situations: the critical-subcritical case

$$\begin{cases} |f'(s)| \leq c + c|s|^4, \\ |g'(s)| \leq c + c|s|^{p-1}, \qquad p < 5, \end{cases} \tag{5.1.2}$$

and the fully critical case

$$\begin{cases} |f'(s)| \leq c + c|s|^4, \\ |g'(s)| \leq c + c|s|^4. \end{cases} \tag{5.1.3}$$

Case (5.1.2) has been investigated in Chapter 3, where the existence of the global attractor is proved. Moreover, exploiting a suitable decomposition of the semigroup and under the additional hypothesis $p \leq 4$, the (optimal) regularity of the attractor is reached.

The fully critical situation (5.1.3) has been studied in Chapter 4 within the stronger dissipativity assumption

$$\inf_{s\in\mathbb{R}} f'(s) = -\lambda > -\lambda_1.$$

This hypothesis furnishes the dissipation integral

$$\int_0^\infty \|\nabla u_t(\tau)\|^2\,\mathrm{d}\tau < \infty,$$

and the existence of an exponential attractor of optimal regularity (and then the one of a regular global attractor of finite fractal dimension) can be proved. The aim of this chapter is to analyze the subcritical-critical situation. More precisely, we stipulate the following hypothesis.

Assumptions on f. *We suppose that $f \in \mathcal{C}^1(\mathbb{R})$, with $f(0) = 0$, satisfies for every $s \in \mathbb{R}$ and some $c \geq 0$ the subcritical growth bound*

$$|f'(s)| \leq c + c|s|^{r-1}, \qquad r \in [1,5), \tag{5.1.4}$$

along with the dissipation condition

$$\liminf_{|s|\to\infty} f'(s) > -\lambda_1. \tag{5.1.5}$$

Assumptions on g. *We suppose that $g \in \mathcal{C}^1(\mathbb{R})$, with $g(0) = 0$, satisfies for every $s \in \mathbb{R}$ and some $c \geq 0$ the critical growth bound*

$$|g'(s)| \leq c + c|s|^4, \tag{5.1.6}$$

along with the dissipation conditions

$$\liminf_{|s|\to\infty} \frac{g(s)}{s} > -\lambda_1, \tag{5.1.7}$$

$$\liminf_{|s|\to\infty} \frac{sg(s) - c_1\int_0^s g(y)\mathrm{d}y}{s^2} > -\frac{\lambda_1}{2}, \tag{5.1.8}$$

for some $c_1 > 0$.

Hypotheses (5.1.7)-(5.1.8) are verified in one shot if g satisfies the stronger dissipativity assumption

$$\liminf_{|s|\to\infty} g'(s) > -\lambda_1.$$

Moreover, without loss of generality, we may suppose $c_1 = 1$ in (5.1.8). Although, in principle, the critical growth on f is harder to treat than the one of g, the very same method used in Chapter 3 does not seem to work. In particular, the study of the regularity of the attractor requires a bootstrap procedure, contrary to Chapter 3 where the regularity is attained in one step. Thus, the techniques of the present chapter are completely different and specifically tailored for this subcritical-critical situation.

The remaining part of the chapter is organized as follows. In the next Section 5.2 we state and demonstrate the existence of the global attractor, whose optimal regularity is proved in the subsequent Section 5.3.

5.2 The Global Attractor

Let us introduce some notation. For an arbitrary Banach space $\mathcal{X}$ and $r \geq 0$, we define the (closed) ball

$$\mathsf{B}_{\mathcal{X}}(r) = \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \leq r\}.$$

We also denote by $\mathfrak{I}$ and $\mathfrak{D}$ the set of continuous increasing functions $J : \mathbb{R}^+ \to \mathbb{R}^+$ and the set of continuous decreasing functions $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta(\infty) < 1$, respectively. We also recall that, in light of Lemmas 2.3.1 and 2.3.2, for every fixed $\lambda < \lambda_1$ we can decompose

$$f(s) = \phi(s) - \lambda s + \phi_{\rm c}(s)$$

and

$$g(s) = \gamma(s) - \lambda s + \gamma_{\rm c}(s)$$

for some $\phi, \phi_{\rm c} \in \mathcal{C}^1(\mathbb{R})$ and $\gamma, \gamma_{\rm c} \in \mathcal{C}^1(\mathbb{R})$, satisfying the following properties:

- $\phi_{\rm c}$ and $\gamma_{\rm c}$ are compactly supported, with $\phi_{\rm c}(0) = \gamma_{\rm c}(0) = 0$;
- ϕ vanish inside $[-1, 1]$ and fulfills for some $c \geq 0$ and every $s \in \mathbb{R}$ the bounds

  $$0 \leq \phi'(s) \leq c|s|^{r-1};$$

- γ vanish inside $[-1, 1]$ and fulfills for some $c \geq 0$ and every $s \in \mathbb{R}$ the bounds

  $$|\gamma'(s)| \leq c|s|^4,$$

  and

  $$0 \leq \Gamma(s) \leq s\gamma(s), \tag{5.2.1}$$

  where $\Gamma(s) = \int_0^s \gamma(y)\,\mathrm{d}y$.

Accordingly, we rewrite (5.1.1) as

$$u_{tt} + Bu_t + Bu + \phi(u_t) + \gamma(u) = q, \tag{5.2.2}$$

where

$$q = h - \phi_{\rm c}(u_t) - \gamma_{\rm c}(u) \in L^\infty(\mathbb{R}^+; \mathrm{H})$$

and

$$B = A - \lambda I \qquad \text{with} \qquad \mathrm{dom}(B) = \mathrm{dom}(A)$$

is a positive operator commuting with A. In particular, the bilinear form

$$(w, v)_\sigma = \langle w, A^{-1}Bv\rangle_\sigma = \langle w, v\rangle_\sigma - \lambda\langle w, v\rangle_{\sigma-1}$$

defines an equivalent inner product on the space H_σ, with induced norm $|\cdot|_\sigma$. Finally, for further use, we report some estimates involving the linear part of (5.2.2) (cf. (2.3.10) and (2.3.11)). Given a vector-valued function $w = w(t)$ as regular as needed, we define

$$\Box w = 2w_{tt} + 2Bw_t + 2Bw,$$

and we introduce the family of functionals depending on $\varepsilon \geq 0$

$$\Pi_\varepsilon(w) = (1+\varepsilon)|w|_1^2 + \|w_t\|^2 + 2\varepsilon\langle w_t, w\rangle.$$

By direct calculations, the inequalities

$$\Pi_0(w) \leq 2\Pi_\varepsilon(w) \leq 4\Pi_0(w) \tag{5.2.3}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_\varepsilon(A^{\frac{s}{2}}w) + \varepsilon\Pi_\varepsilon(A^{\frac{s}{2}}w) + \frac{\varepsilon}{2}|w|_{1+s}^2 + \frac{3}{2}\|w_t\|_{1+s}^2 \leq \langle \Box w, A^s w_t + \varepsilon A^s w\rangle \tag{5.2.4}$$

hold for every $\varepsilon > 0$ sufficiently small and every $s \in \mathbb{R}$.

We now state and prove one of the main results of the chapter.

Theorem 5.2.1. *The dynamical system $S(t) : \mathcal{H} \to \mathcal{H}$*[1] *possesses a (connected) global attractor $\mathbb{A}$ such that*

$$\sup_{t\geq 0}\sup_{z\in\mathbb{A}} \big[|u_t(t)|_1 + \|u_{tt}(t)\|\big] < \infty. \tag{5.2.5}$$

In particular, $\mathbb{A}$ is bounded in $\mathcal{V}_1$.

With reference to Theorem 4.2.3, along the section $C \geq 0$ stands for a *generic* constant depending only on the invariant absorbing set $\mathbb{B}_0$. For every initial data $z = \{a, b\} \in \mathbb{B}_0$, we consider an arbitrary solution

$$S(t)z = \{u(t), u_t(t)\}$$

and we split

$$u(t) = v(t) + w(t),$$

where v and w solve the Cauchy problems

$$\begin{cases} v_{tt} + Bv_t + Bv + \gamma(v) = 0, \\ v(0) = a, \\ v_t(0) = b, \end{cases} \tag{5.2.6}$$

and

$$\begin{cases} w_{tt} + Bw_t + Bw + \phi(u_t) + \gamma(u) - \gamma(v) = q, \\ w(0) = 0, \\ w_t(0) = 0. \end{cases} \tag{5.2.7}$$

We first show the (uniform) exponential decay of the solutions to (5.2.6).

Lemma 5.2.1. *There exists $\delta = \delta(\mathbb{B}_0) > 0$ such that*

$$|\{v(t), v_t(t)\}|_{\mathcal{H}} \leq C\mathrm{e}^{-\delta t}.$$

[1] We recall that, according to Theorem 2.4.1, equation (5.2.2) generates a dynamical system $S(t)$ acting on the phase space $\mathcal{H}$.

Proof. Denoting
$$\Gamma_0(v) = 2\langle \Gamma(v), 1\rangle,$$
due to (5.2.1) and (5.2.3) the family of functionals
$$\Lambda_\varepsilon = \Pi_\varepsilon(v) + \Gamma_0(v)$$
fulfills, for every $\varepsilon > 0$ small,
$$\Pi_0(v) \le \Lambda_0 \le 2\Lambda_\varepsilon \le 4\Lambda_0. \tag{5.2.8}$$
On the other hand, from the growth bound on γ we infer the control
$$\Lambda_0 \le c\Pi_0(v)\big[1 + \Pi_0(v)^2\big], \quad c \ge 1. \tag{5.2.9}$$
Taking the product in H of (5.2.6) and $2v_t + 2\varepsilon v$ we obtain
$$\langle \Box v, v_t + \varepsilon v\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\Gamma_0(v) + 2\varepsilon\langle\gamma(v), v\rangle = 0.$$
Thus, applying (5.2.4) with $s = 0$ and (5.2.1)
$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Lambda_\varepsilon \le 0,$$
and the Gronwall lemma together with (5.2.8) and (5.2.9) give the desired decay. □

Next, we define the number
$$\varkappa = \min\{\tfrac{1}{4}, \tfrac{5-r}{2}\} > 0, \tag{5.2.10}$$
and we prove a time-dependent bound of the solutions to (5.2.7).

Lemma 5.2.2. *There exists a function $J \in \mathfrak{J}$ such that*
$$|\{w(t), w_t(t)\}|_{\mathcal{H}_\varkappa} \le J(t).$$

Proof. Multiplying (5.2.7) by $2A^\varkappa w_t$ we obtain
$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_0(A^{\frac{\varkappa}{2}}w) + 2|w_t|^2_{1+\varkappa} = -2\langle\phi(u_t), A^\varkappa w_t\rangle + 2\langle\gamma(v) - \gamma(u), A^\varkappa w_t\rangle + 2\langle q, A^\varkappa w_t\rangle.$$
Exploiting Theorem 4.2.3, Lemma 5.2.1 and the growth bounds on ϕ and γ we derive the controls
$$-2\langle\phi(u_t), A^\varkappa w_t\rangle \le C\|u_t\|^r_{L^{6r/(5-2\varkappa)}}\|w_t\|_{1+\varkappa} \le C\|u_t\|^r_{L^6}\|w_t\|_{1+\varkappa} \le C + \frac{1}{2}|w_t|^2_{1+\varkappa}$$
and
$$\begin{aligned}2\langle\gamma(v) - \gamma(u), A^\varkappa w_t\rangle &\le C\big[\|v\|^4_{L^6} + \|u\|^4_{L^6}\big]\|w\|_{L^{6/(1-2\varkappa)}}\|w_t\|_{1+\varkappa}\\ &\le C|w|_{1+\varkappa}|w_t|_{1+\varkappa} \le C\,\Pi_0(A^{\frac{\varkappa}{2}}w) + \frac{1}{2}|w_t|^2_{1+\varkappa}.\end{aligned}$$

Moreover,

$$2\langle q, A^{\varkappa} w_t\rangle \leq C\|w_t\|_{1+\varkappa} \leq C + \frac{1}{2}|w_t|^2_{1+\varkappa}.$$

Therefore, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_0(A^{\frac{\varkappa}{2}}w) \leq C\,\Pi_0(A^{\frac{\varkappa}{2}}w) + C,$$

and the Gronwall lemma completes the proof. □

Proof of Theorem 5.2.1. In light of Lemma 5.2.1 and Lemma 5.2.2 we infer that

$$\lim_{t\to\infty}\big[\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathbb{B}, \mathcal{C}(t))\big] = 0,$$

where, for every fixed t, $\mathcal{C}(t)$ is a bounded subset of $\mathcal{H}_\varkappa \Subset \mathcal{H}$. Thus, the existence of the global attractor $\mathbb{A}$ is proved. Moreover, since $\mathbb{A}$ is fully invariant, it is contained in every absorbing set and then the uniform bound (5.2.5) follows immediately from Theorem 4.2.3. □

5.3 Regularity of the Attractor

In this section, we prove the optimal regularity of the global attractor $\mathbb{A}$.

Theorem 5.3.1. *The global attractor $\mathbb{A}$ of the dynamical system $S(t)$ is bounded in $\mathcal{V}_2$.*

The proof of Theorem 5.3.1, carried out with a bootstrap procedure, relies on two lemmas. The first Lemma 5.3.1 is a technical result necessary to apply the bootstrapping argument contained in the subsequent Lemma 5.3.2.

Throughout the section, we set $\varkappa$ as in (5.2.10).

Lemma 5.3.1. *The global attractor $\mathbb{A}$ is bounded in $\mathcal{V}_{1+\varkappa}$.*

Proof. In what follows, $C, \delta > 0$ and $J \in \mathfrak{I}$ will denote *generic* constants and a *generic* function, respectively, depending only on the invariant absorbing set $\mathbb{B}_0$ of Theorem 4.2.3. We divide the proof in two steps. First, we prove that $\mathbb{A}$ is bounded in $\mathcal{H}_\varkappa$; subsequently, we show the boundedness of $\mathbb{A}$ in $\mathcal{V}_{1+\varkappa}$.

Step 1. Our aim is to apply the abstract Theorem A.4 reported in the final appendix with $\mathcal{V} = \mathcal{H}_\varkappa$. Denote $r = \|\mathbb{B}_0\|_{\mathcal{H}}$ and take $z \in \mathbb{B}_0$. For $x \in \mathsf{B}_{\mathcal{H}}(r)$ and $y \in \mathcal{H}_\varkappa$ such that $x + y = z$, we define

$$V_z(t)x = (\hat{v}(t), \hat{v}_t(t)) \qquad \text{and} \qquad U_z(t)y = (\hat{w}(t), \hat{w}_t(t)),$$

where $\hat{v}(t)$ and $\hat{w}(t)$ solve

$$\begin{cases} \hat{v}_{tt} + B\hat{v}_t + B\hat{v} + \gamma(\hat{v}) = 0, \\ \hat{v}(0) = x, \end{cases}$$

and

$$\begin{cases} \hat{w}_{tt} + B\hat{w}_t + B\hat{w} + \phi(u_t) + \gamma(u) - \gamma(\hat{v}) = q, \\ \hat{w}(0) = y. \end{cases} \tag{5.3.1}$$

Assumption (i) of Theorem A.4 holds by construction, while (ii) follows easily recasting the proof of Lemma 5.2.1. It remains to verify (iii). Taking the product in H of (5.3.1) and $2A^{\varkappa}\hat{w}_t + 2\varepsilon A^{\varkappa}\hat{w}$ and using (5.2.4) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Pi_\varepsilon(A^{\frac{\varkappa}{2}}\hat{w}) + \varepsilon\Pi_\varepsilon(A^{\frac{\varkappa}{2}}\hat{w}) + \frac{\varepsilon}{2}|\hat{w}|^2_{1+\varkappa} + \frac{3}{2}\|\hat{w}_t\|^2_{1+\varkappa}& \\ \le \langle \Box\hat{w}, A^{\varkappa}\hat{w}_t + \varepsilon A^{\varkappa}\hat{w}\rangle& \\ = -2\langle\phi(u_t), A^{\varkappa}\hat{w}_t + \varepsilon A^{\varkappa}\hat{w}\rangle + 2\langle Q, A^{\varkappa}\hat{w}_t + \varepsilon A^{\varkappa}\hat{w}\rangle&,\end{aligned}$$

having set $Q = q + \gamma(\hat{v}) - \gamma(u)$. In light of Theorem 4.2.3 and the growth bound on ϕ, we estimate

$$\begin{aligned}-2\langle\phi(u_t), A^{\varkappa}\hat{w}_t + \varepsilon A^{\varkappa}\hat{w}\rangle &\le C\|u_t\|^r_{L^{6r/(5-2\varkappa)}}\big[\|A^{\varkappa}\hat{w}_t\|_{L^{6/(1+2\varkappa)}} + \varepsilon\|A^{\varkappa}\hat{w}\|_{L^{6/(1+2\varkappa)}}\big] \\ &\le C\|u_t\|^r_{L^6}\big[\|\hat{w}_t\|_{1+\varkappa} + \varepsilon\|\hat{w}\|_{1+\varkappa}\big] \\ &\le \frac{1}{2}\|\hat{w}_t\|^2_{1+\varkappa} + \frac{\varepsilon}{4}|\hat{w}|^2_{1+\varkappa} + C.\end{aligned}$$

Next, we compute

$$\begin{aligned}|Q| \le |q| + |\gamma(\hat{v}) - \gamma(u)| &\le |q| + C|\hat{w}|\big[|\hat{v}| + |u|\big]^4 \\ &\le |q| + C|\hat{w}|\big[|\hat{v}| + |v|\big]^4 + C|w|^4\big[|u| + |\hat{v}|\big].\end{aligned}$$

Thus, exploiting the growth bound on γ

$$\begin{aligned}&2\langle Q, A^{\varkappa}\hat{w}_t + \varepsilon A^{\varkappa}\hat{w}\rangle \\ &\quad\le C\Big[1 + \|\hat{w}\|_{L^{6/(1-2\varkappa)}}\big[\|\hat{v}\|_{L^6} + \|v\|_{L^6}\big]^4 + \|w\|^4_{L^{24/(4-2\varkappa)}}\big[\|u\|_{L^6} + \|\hat{v}\|_{L^6}\big]\Big] \\ &\qquad\cdot\big[\|\hat{w}_t\|_{1+\varkappa} + \varepsilon\|\hat{w}\|_{1+\varkappa}\big].\end{aligned}$$

Appealing to Lemma 5.2.1, we infer

$$\|\hat{w}\|_{L^{6/(1-2\varkappa)}}\big[\|\hat{v}\|_{L^6} + \|v\|_{L^6}\big]^4 \le C\mathrm{e}^{-\delta t}|\hat{w}|_{1+\varkappa}$$

and

$$\|w\|^4_{L^{24/(4-2\varkappa)}}\big[\|u\|_{L^6} + \|\hat{v}\|_{L^6}\big] \le C|w|^4_{1+\varkappa}.$$

In summary, applying also Lemma 5.2.2, the right-hand side is controlled by

$$C\mathrm{e}^{-\delta t}|\hat{w}|^2_{1+\varkappa} + J(t) + \frac{1}{2}\|\hat{w}_t\|^2_{1+\varkappa} + \frac{\varepsilon}{4}|\hat{w}|^2_{1+\varkappa},$$

and we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_\varepsilon(A^{\frac{\varkappa}{2}}\hat{w}) + \varepsilon\Pi_\varepsilon(A^{\frac{\varkappa}{2}}\hat{w}) \le C\mathrm{e}^{-\delta t}\,\Pi_\varepsilon(A^{\frac{\varkappa}{2}}\hat{w}) + J(t).$$

An application of the standard Gronwall lemma together with (5.2.3) provides the estimate (iii). Therefore, invoking Theorem A.4, we conclude that the global attractor $\mathbb{A}$ is bounded in $\mathcal{H}_\varkappa$.

Step 2. Once the regularity in $\mathcal{H}_\varkappa$ is established, choose an arbitrary initial data $z = \{a, b\} \in \mathbb{A}$. Theorem 5.2.1 yields

$$\sup_{t\geq 0} \big[|u(t)|_{1+\varkappa} + |u_t(t)|_1 + \|u_{tt}(t)\|\big] \leq C, \tag{5.3.2}$$

where $C \geq 0$ this time depends only on $\mathbb{A}$. Then, we write equation (5.2.2) as

$$Bu_t = -u_{tt} - Bu - \phi(u_t) - \gamma(u) + q,$$

and we multiply by $A^\varkappa u_t$ to get

$$|u_t|^2_{1+\varkappa} = -\langle u_{tt} + Bu, A^\varkappa u_t\rangle - \langle \phi(u_t) + \gamma(u) - q, A^\varkappa u_t\rangle.$$

From (5.3.2) and the growth bounds on ϕ and γ we derive the controls

$$-\langle u_{tt} + Bu, A^\varkappa u_t\rangle \leq C|u_t|_{1+\varkappa} \leq \frac{1}{4}|u_t|^2_{1+\varkappa} + C,$$

and

$$\begin{aligned} -\langle \phi(u_t) + \gamma(u) - q, A^\varkappa u_t\rangle \leq & C\big[1 + \|u_t\|^r_{L^{6r/(5-2\varkappa)}} + \|u\|^5_{L^{30/(5-2\varkappa)}}\big]\|u_t\|_{1+\varkappa} \\ \leq & C|u_t|_{1+\varkappa} \leq \frac{1}{4}|u_t|^2_{1+\varkappa} + C, \end{aligned}$$

so that

$$\sup_{t\geq 0} |u_t(t)|_{1+\varkappa} \leq C. \tag{5.3.3}$$

Collecting estimates (5.3.2)-(5.3.3) and exploiting the fully invariance of $\mathbb{A}$, the proof of Lemma 5.3.1 is finished. ☐

Lemma 5.3.2. *Let $s \in [\varkappa, 1]$ be given and set*

$$\ell = \min\{\varkappa, 1 - s\}.$$

The following implication holds

$$\mathbb{A} \text{ bounded in } \mathcal{V}_{1+s} \quad \Rightarrow \quad \mathbb{A} \text{ bounded in } \mathcal{V}_{1+s+\ell}.$$

Proof. We consider an arbitrary solution

$$S(t)z = u(t) \qquad \text{with} \qquad z = \{a, b\} \in \mathbb{A},$$

and we make the decomposition

$$u(t) = v(t) + w(t)$$

where

$$\begin{cases} v_{tt} + Bv_t + Bv = 0, \\ v(0) = a, \\ v_t(0) = b, \end{cases} \tag{5.3.4}$$

and

$$\begin{cases} w_{tt} + Bw_t + Bw + \phi(u_t) + \gamma(u) = q, \\ w(0) = 0, \\ w_t(0) = 0. \end{cases} \tag{5.3.5}$$

Along the proof, the *generic* constant $C \geq 0$ depends only on $\mathbb{A}$. Multiplying (5.3.4) by $2v_t + 2\varepsilon v$ and using (5.2.4) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_\varepsilon(v) + \varepsilon\Pi_\varepsilon(v) \leq \langle \Box v, v_t + \varepsilon v\rangle = 0.$$

Thus, taking into account (5.2.3), an application of the Gronwall lemma gives

$$|\{v(t), v_t(t)\}|_{\mathcal{H}} \leq C\mathrm{e}^{-\nu t}, \tag{5.3.6}$$

for some $\nu = \nu(\mathbb{A}) > 0$. Then, we multiply (5.3.5) by $2A^{s+\ell}w_t + 2\varepsilon A^{s+\ell}w$. Appealing to (5.2.4) we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Pi_\varepsilon(A^{\frac{s+\ell}{2}}w) + \varepsilon\Pi_\varepsilon(A^{\frac{s+\ell}{2}}w) + \frac{\varepsilon}{2}|w|^2_{1+s+\ell} + \frac{3}{2}\|w_t\|^2_{1+s+\ell} \\ \leq \langle \Box w, A^{s+\ell}w_t + \varepsilon A^{s+\ell}w\rangle \\ = -2\langle \phi(u_t) + \gamma(u), A^{s+\ell}w_t + \varepsilon A^{s+\ell}w\rangle + 2\langle q, A^{s+\ell}w_t + \varepsilon A^{s+\ell}w\rangle. \end{aligned}$$

Exploiting the growth bounds on ϕ and γ we estimate

$$\begin{aligned} -2\langle \phi(u_t) + \gamma(u), A^{s+\ell}w_t + \varepsilon A^{s+\ell}w\rangle \leq\; & C\big[\|u_t\|^r_{L^{6r/(5-2(s+\ell))}} + \|u\|^5_{L^{30/(5-2(s+\ell))}}\big] \\ & \cdot \big[\|A^{s+\ell}w_t\|_{L^{6/(1+2(s+\ell))}} + \varepsilon\|A^{s+\ell}w\|_{L^{6/(1+2(s+\ell))}}\big]. \end{aligned}$$

If $s < \frac{1}{2}$, using the Sobolev embedding

$$\mathrm{H}_{1+s} \subset L^{\frac{6}{1-2s}}(\Omega), \quad s \in [0, \tfrac{1}{2})$$

we have

$$\begin{aligned} \big[\|u_t\|^r_{L^{6r/(5-2(s+\ell))}} + \|u\|^5_{L^{30/(5-2(s+\ell))}}\big] &\leq C\big[\|u_t\|^r_{L^{6/(1-2s)}} + \|u\|^5_{L^{6/(1-2s)}}\big] \\ &\leq C\big[\|u_t\|^r_{1+s} + \|u\|^5_{1+s}\big] \leq C. \end{aligned}$$

If $s \geq \frac{1}{2}$, we still have

$$\big[\|u_t\|^r_{L^{6r/(5-2(s+\ell))}} + \|u\|^5_{L^{30/(5-2(s+\ell))}}\big] \leq C,$$

by means of the continuous embedding

$$\mathrm{H}_{1+s} \subset L^p(\Omega), \qquad 1 \leq p < \infty.$$

In both cases we infer

$$\begin{aligned} -2\langle \phi(u_t) + \gamma(u), A^{s+\ell}w_t + \varepsilon A^{s+\ell}w\rangle &\leq C\big[\|w_t\|_{1+s+\ell} + \varepsilon\|w\|_{1+s+\ell}\big] \\ &\leq \frac{1}{2}\|w_t\|^2_{1+s+\ell} + \frac{\varepsilon}{4}|w|^2_{1+s+\ell} + C. \end{aligned}$$

Moreover,

$$\begin{aligned} 2\langle q, A^{s+\ell} w_t + \varepsilon A^{s+\ell} w\rangle &\le C\big[\|w_t\|_{1+s+\ell} + \varepsilon \|w\|_{1+s+\ell}\big] \\ &\le \frac{1}{2}\|w_t\|^2_{1+s+\ell} + \frac{\varepsilon}{4}|w|^2_{1+s+\ell} + C, \end{aligned}$$

and we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_\varepsilon(A^{\frac{s+\ell}{2}} w) + \varepsilon \Pi_\varepsilon(A^{\frac{s+\ell}{2}} w) \le C.$$

Therefore, applying the Gronwall lemma and (5.2.3) we obtain

$$|\{w(t), w_t(t)\}|_{\mathcal{H}_{s+\ell}} \le C. \tag{5.3.7}$$

From (5.3.6) and (5.3.7) we learn that the solutions originating from $\mathbb{A}$ are attracted by a proper closed ball B of $\mathcal{H}_{s+\ell}$ centered at zero. Since $\mathbb{A}$ is fully invariant this implies that $\mathbb{A}$ is contained in the $\mathcal{H}$-closure of B and thus it is bounded in $\mathcal{H}_{s+\ell}$. Besides, from (5.2.5), we establish the uniform control

$$\sup_{t\ge 0}\sup_{z\in\mathbb{A}} \big[|u(t)|_{1+s+\ell} + |u_t(t)|_{1+s} + \|u_{tt}(t)\|\big] < \infty. \tag{5.3.8}$$

At this point, for initial data $z = \{a, b\} \in \mathbb{A}$, we rewrite equation (5.2.2) as

$$Bu_t = -u_{tt} - Bu - \phi(u_t) - \gamma(u) + q.$$

Multiplying by $A^{s+\ell} u_t$ we infer

$$|u_t|^2_{1+s+\ell} = -\langle u_{tt} + Bu, A^{s+\ell} u_t\rangle - \langle \phi(u_t) + \gamma(u) - q, A^{s+\ell} u_t\rangle.$$

Exploiting (5.3.8) and the growth bounds on ϕ and γ we have

$$-\langle u_{tt} + Bu, A^{s+\ell} u_t\rangle \le C|u_t|_{1+s+\ell} \le \frac{1}{4}|u_t|^2_{1+s+\ell} + C,$$

and

$$-\langle \phi(u_t) + \gamma(u) - q, A^{s+\ell} u_t\rangle \le C\big[1 + \|u_t\|^r_{L^{6r/(5-2(s+\ell))}} + \|u\|^5_{L^{30/(5-2(s+\ell))}}\big]\|u_t\|_{1+s+\ell}.$$

As before, if $s < \frac{1}{2}$ we estimate

$$\big[\|u_t\|^r_{L^{6r/(5-2(s+\ell))}} + \|u\|^5_{L^{30/(5-2(s+\ell))}}\big] \le C\big[\|u_t\|^r_{1+s} + \|u\|^5_{1+s}\big] \le C,$$

whereas if $s \ge \frac{1}{2}$ we still estimate

$$\big[\|u_t\|^r_{L^{6r/(5-2(s+\ell))}} + \|u\|^5_{L^{30/(5-2(s+\ell))}}\big] \le C,$$

and we are led to

$$-\langle \phi(u_t) + \gamma(u) - q, A^{s+\ell} u_t\rangle \le C|u_t|_{1+s+\ell} \le \frac{1}{4}|u_t|^2_{1+s+\ell} + C.$$

In conclusion

$$\sup_{t\ge 0}|u_t(t)|_{1+s+\ell} \le C,$$

and we are finished. □

Conclusion of the Proof of Theorem 5.3.1. From Lemma 5.3.1 we readily infer that $\mathbb{A}$ is bounded in $\mathcal{V}_{1+\varkappa}$. Thus, applying Lemma 5.3.2 starting from $s = \varkappa$, it is apparent that after a finite number of steps we get that $\mathbb{A}$ is bounded in $\mathcal{V}_2$. This completes the argument. □

CHAPTER 6

Thermoelastic Extensible Beams and Berger Plates

6.1 Introduction

Let H be a real Hilbert space with inner product and norm $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, and let

$$A : \mathfrak{D}(A) \Subset H \to H$$

be a strictly positive linear operator with domain $\mathfrak{D}(A)$ compactly embedded into H. We consider for $t > 0$ the evolution system in the unknowns $u = u(t)$ and $\alpha = \alpha(t)$

$$\begin{cases} \ddot{u} + \omega A \ddot{u} + A^2 u + f(\|A^{\frac{1}{2}} u\|^2) A u - A \dot{\alpha} = g, \\ \ddot{\alpha} + A\alpha + \displaystyle\int_0^\infty \mu(s) A[\alpha(t) - \alpha(t-s)] \, \mathrm{d}s + A\dot{u} = 0, \end{cases} \tag{6.1.1}$$

where $u(0)$, $\dot{u}(0)$, $\alpha(0)$ and $\dot{\alpha}(0)$, as well as the past history $\alpha(-s)$ of the variable α appearing in the convolution integral are understood to be assigned data of the problem, whereas $\omega > 0$ is a fixed parameter.

The following general assumptions on the constitutive terms are made:

- The function $g \in H$ is independent of time.
- The convolution kernel μ is a summable nonincreasing piecewise absolutely continuous function on $\mathbb{R}^+ = (0, \infty)$ subject to the normalization conditions

$$\int_0^\infty \mu(s) \, \mathrm{d}s = \int_0^\infty s\mu(s) \, \mathrm{d}s = 1 \tag{6.1.2}$$

and whose discontinuity points (if any) form an increasing sequence $\{s_n\}_{n \geq 1}$.

• The nonlinear map $f \in \mathcal{C}^1(\mathbb{R}_0^+)$ fulfills for every $x \geq 0$

$$f(x)x \geq -a_0 x + b_0 \tag{6.1.3}$$

for some $b_0 \in \mathbb{R}$ and $a_0 < \lambda_1$, where

$$\lambda_1 = \min_{0 \neq u \in \mathfrak{D}(A)} \frac{\|Au\|}{\|u\|} > 0$$

is the first eigenvalue of the operator A. Accordingly, the primitive

$$F(x) = \int_0^x f(y)\,\mathrm{d}y$$

is readily seen to satisfy the inequality

$$F(x) \geq -ax + b \tag{6.1.4}$$

for some $b \in \mathbb{R}$ and $a_0 < a < \lambda_1$.

Remark 6.1.1. *As detailed in the introduction of the thesis, system* (6.1.1) *can be viewed as an abstract version of an evolution model describing the vibrations of thermoelastic beams and plates, corresponding to the choice $H = L^2(\Omega)$, $f(x) = x + b$ and*

$$A = -\Delta, \qquad \mathfrak{D}(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

for some bounded smooth domain $\Omega \subset \mathbb{R}^n$ ($n = 1, 2$).

Since the seminal work of Woinowsky-Krieger [93], nonlinear evolution systems modeling thermoelastic beams or plates, within different types of heat conduction laws, have been widely investigated (see e.g. [5,16,23,38,78] and references therein). In [23], in place of the classical Fourier law, the authors consider the type III heat conduction law proposed by Green and Naghdi. More specifically, they study the system

$$\begin{cases} \ddot{u} + A^2 u + (b + \|A^{\frac{1}{2}}u\|^2)Au - A\dot{\alpha} = g, \\ \ddot{\alpha} + A\alpha + A\dot{\alpha} + A\dot{u} = 0, \end{cases} \tag{6.1.5}$$

where the dissipation is entirely contributed by the second equation ruling the evolution of the thermal displacement α. When dealing with models of this kind, one difficulty lies in the fact that the mechanical component does not cause any loss of energy, which renders the asymptotic analysis nontrivial.

The main focus of the first part of this chapter (from Section 6.2 to Section 6.10) is the longtime behavior of the solution semigroup generated by (6.1.1), which can be viewed as a memory relaxation of system (6.1.5). In particular, the dissipation mechanism is only thermal and merely due to the convolution (or memory) integral, without any additional instantaneous dissipative term. We prove the existence of the regular global attractor for the associated semigroup in the natural weak energy space, provided that μ satisfies a suitable decay assumption (see (6.4.1) below). The strategy is based on the existence of a Lyapunov functional, which reflects the gradient system structure of the problem, along with the exploitation of certain dissipation integrals and sharp energy estimates.

As already said in the introduction of the thesis, from the physical viewpoint it is also relevant to neglect the rotational inertia term $\omega A\ddot{u}$ appearing in the first equation of (6.1.1). Accordingly, in the second part of the chapter (from Section 6.11 to Section 6.14) we study the corresponding linear homogeneous system with $\omega = 0$, i.e.

$$\begin{cases} \ddot{u} + A^2 u - A\dot{\alpha} = 0, \\ \ddot{\alpha} + A\alpha + \displaystyle\int_0^\infty \mu(s) A[\alpha(t) - \alpha(t-s)]\,\mathrm{d}s + A\dot{u} = 0, \end{cases} \tag{6.1.6}$$

along with the limit situation without memory

$$\begin{cases} \ddot{u} + A^2 u - A\dot{\alpha} = 0, \\ \ddot{\alpha} + A\alpha + A\dot{\alpha} + A\dot{u} = 0. \end{cases} \tag{6.1.7}$$

Our main results are the lack of exponential stability of the solution semigroup generated by system (6.1.6), and the exponential stability of the one associated with (6.1.7) (see Theorems 6.13.1 and 6.14.1). This witnesses that the dissipation mechanism of system (6.1.1), entirely contributed by the memory, is extremely weak and hence it is hard to obtain stabilization properties. Actually, the two liner systems above will be studied in a more general form, with a coupling term depending on a real parameter $\sigma \leq \frac{3}{2}$.

6.2 Functional Setting and Notation

Analogously to Chapter 2, for $r \in \mathbb{R}$, we define the compactly nested family of Hilbert spaces (r will be always omitted whenever zero)

$$H^r = \mathfrak{D}(A^{\frac{r}{2}}), \qquad \langle u, v\rangle_r = \langle A^{\frac{r}{2}} u, A^{\frac{r}{2}} v\rangle, \qquad \|u\|_r = \|A^{\frac{r}{2}} u\|.$$

For $r > 0$, it is understood that H^{-r} denotes the completion of the domain, so that H^{-r} is the dual space of H^r. Accordingly, the symbol $\langle\cdot,\cdot\rangle$ also stands for duality product between H^r and H^{-r}, and we have the generalized Poincaré inequalities

$$\lambda_1 \|u\|_r^2 \leq \|u\|_{r+1}^2, \quad \forall u \in H^{r+1}.$$

These inequalities, as well as the Hölder and the Young inequalities, will be tacitly used several times in what follows. Next, we introduce the so-called memory spaces

$$\mathcal{M}^r = L^2_\mu(\mathbb{R}^+; H^{r+1})$$

endowed with the weighted L^2-inner products (again, we will write $\mathcal{M}$ in place of $\mathcal{M}^0$)

$$\langle \eta, \xi\rangle_{\mathcal{M}^r} = \int_0^\infty \mu(s)\langle \eta(s), \xi(s)\rangle_{r+1}\,\mathrm{d}s.$$

The infinitesimal generator of the right-translation semigroup on $\mathcal{M}$ is the linear operator

$$T\eta = -D\eta, \qquad \text{with domain} \qquad \mathfrak{D}(T) = \big\{\eta \in \mathcal{M} : D\eta \in \mathcal{M},\ \eta(0) = 0\big\},$$

where D stands for weak derivative, whereas $\eta(0) = 0$ means

$$\lim_{s\to 0} \eta(s) = 0 \quad \text{in } H^1.$$

Denoting by

$$\mu_n = \mu(s_n^-) - \mu(s_n^+) \geq 0$$

the jumps of μ at the points $s = s_n$, for every $\eta \in \mathfrak{D}(T)$ we define the nonnegative functional

$$\Gamma[\eta] = -\int_0^\infty \mu'(s)\|\eta(s)\|_1^2 \, \mathrm{d}s + \sum_n \mu_n \|\eta(s_n)\|_1^2.$$

An integration by parts together with a limiting argument yield the equality (see [11, 41, 69])

$$2\langle T\eta, \eta\rangle_{\mathcal{M}} = -\Gamma[\eta]. \tag{6.2.1}$$

In order to simplify the calculations, we consider the strictly positive operator

$$B = I + \omega A, \qquad \text{with domain} \qquad \mathfrak{D}(B) = \mathfrak{D}(A).$$

The operator B commutes with A and the bilinear form

$$(u, v)_r = \langle Bu, v\rangle_{r-1} = \langle A^{\frac{r-1}{2}} B^{\frac{1}{2}} u, A^{\frac{r-1}{2}} B^{\frac{1}{2}} v\rangle$$

defines an equivalent inner product on the space H^r with induced norm

$$|u|_r^2 = \|u\|_{r-1}^2 + \omega\|u\|_r^2.$$

In light of the Poincaré inequality, $|\cdot|_r$ turns out to be an equivalent norm on the space H^r. Accordingly,

$$(\eta, \xi)_{\mathcal{M}^r} = \int_0^\infty \mu(s)(\eta(s), \xi(s))_{r+1} \, \mathrm{d}s$$

gives rise to an equivalent inner product on the memory space $\mathcal{M}^r$, with corresponding norm $|\cdot|_{\mathcal{M}^r}$. Finally, we introduce the phase spaces

$$\mathcal{H}^r = H^{r+2} \times H^{r+1} \times H^{r+1} \times H^r \times \mathcal{M}^r$$

endowed with the norms

$$\|(u_0, u_1, \alpha_0, \alpha_1, \eta_0)\|_{\mathcal{H}^r}^2 = \|u_0\|_{r+2}^2 + |u_1|_{r+1}^2 + \|\alpha_0\|_{r+1}^2 + \|\alpha_1\|_r^2 + \|\eta_0\|_{\mathcal{M}^r}^2.$$

We conclude the section by recalling a generalized version of the Gronwall lemma (cf. Lemma 4.2.1).

Lemma 6.2.1. *Let $\Lambda : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an absolutely continuous function satisfying for some $\varkappa > 0$ and almost every t the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + 2\varkappa\Lambda(t) \leq \psi(t)\Lambda(t),$$

where $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ fulfills

$$\int_\tau^t \psi(y) \, \mathrm{d}y \leq \varkappa(t - \tau) + m$$

for every $t > \tau \geq 0$ and some $m \geq 0$. Then

$$\Lambda(t) \leq \Lambda(0)\mathrm{e}^m \mathrm{e}^{-\varkappa t}.$$

6.3 The Solution Semigroup

In this section, we study the well-posedness of system (6.1.1) in the phase space $\mathcal{H}$.

6.3.1 The past history formulation

An effective way to deal with the nonlocal character of equations with memory is translating the problem in the history space framework of Dafermos [25]. To this end, setting formally for $t \geq 0$ and $s > 0$

$$\eta^t(s) = \alpha(t) - \alpha(t-s), \tag{6.3.1}$$

we rewrite system (6.1.1) as

$$\begin{cases} B\ddot{u} + A^2 u + f(\|u\|_1^2)Au - A\dot{\alpha} = g, \\ \ddot{\alpha} + A\alpha + \displaystyle\int_0^\infty \mu(s)A\eta(s)\,\mathrm{d}s + A\dot{u} = 0, \\ \dot{\eta} = T\eta + \dot{\alpha}. \end{cases} \tag{6.3.2}$$

Thanks to the introduction of the auxiliary variable η^t, containing the information on the past history of α, we reduce to an abstract ODE on a Hilbert space, for which the powerful machinery of the theory of dynamical systems applies.

6.3.2 Well-posedness

System (6.3.2) possesses a unique weak solution which continuously depends on the initial data.

Proposition 6.3.1. *For all initial data $z \in \mathcal{H}$, problem* (6.3.2) *admits a unique solution*

$$Z \in \mathcal{C}([0,\infty),\mathcal{H}).$$

Moreover, if $z_1, z_2 \in \mathcal{H}$ are such that $\|z_i\|_{\mathcal{H}} \leq R$, then the corresponding solutions fulfill

$$\|Z_1(t) - Z_2(t)\|_{\mathcal{H}} \leq \mathrm{e}^{\mathcal{Q}(R)t}\|z_1 - z_2\|_{\mathcal{H}}, \quad \forall t \geq 0, \tag{6.3.3}$$

for some positive increasing function $\mathcal{Q}$.

Proof. Along the proof, it is crucial to have uniform energy estimates on any finite-time interval. Indeed, these estimates follow directly from the existence of a Lyapunov functional, as shown later in the work. We omit the proof of existence, based on a standard Galerkin approximation procedure. Concerning the continuous dependence estimate (6.3.3), we consider initial data $z_1, z_2 \in \mathcal{H}$ such that $\|z_i\|_{\mathcal{H}} \leq R$ and we call

$$Z_i(t) = (u_i(t), \dot{u}_i(t), \alpha_i(t), \dot{\alpha}_i(t), \eta_i^t)$$

the corresponding solutions. Considering the difference

$$Z_1(t) - Z_2(t) = (v(t), \dot{v}(t), \beta(t), \dot{\beta}(t), \xi^t)$$

and denoting

$$\hat{\mathcal{E}}(t) = \|Z_1(t) - Z_2(t)\|^2_{\mathcal{H}},$$

we multiply in H the first equation of (6.3.2) by $2\dot{v}$ and the second one by $2\dot{\beta}$. Moreover, we multiply in $\mathcal{M}$ the third equation of (6.3.2) by 2ξ. Exploiting (6.2.1), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{E}} &= 2\langle T\xi, \xi\rangle + 2\langle f(\|u_2\|_1^2)Au_2 - f(\|u_1\|_1^2)Au_1, \dot{v}\rangle \\ &\leq 2\|\dot{v}\| \big[\|f(\|u_1\|_1^2)Av\| + \|(f(\|u_1\|_1^2) - f(\|u_2\|_1^2))Au_2\|\big] \\ &\leq \mathcal{Q}(R)\hat{\mathcal{E}},\end{aligned}$$

for some positive increasing function $\mathcal{Q}$. Applying the (standard) Gronwall lemma we are done. □

Hence system (6.3.2) generates a dynamical system

$$S(t) : \mathcal{H} \to \mathcal{H}$$

acting as

$$z = (u_0, u_1, \alpha_0, \alpha_1, \eta_0) \mapsto S(t)z = Z(t),$$

where $Z(t)$ is the unique weak solution to (6.3.2) with initial datum $Z(0) = z$. In particular, the last component η^t of the solution $Z(t)$ has the explicit representation formula (see [76])

$$\eta^t(s) = \begin{cases} \alpha(t) - \alpha(t-s) & 0 < s \leq t, \\ \eta_0(s-t) + \alpha(t) - \alpha_0 & s > t. \end{cases} \tag{6.3.4}$$

Finally, we define (twice) the *energy* at time $t \geq 0$ corresponding to the initial datum $z \in \mathcal{H}$ as

$$\mathcal{E}(t) = \|S(t)z\|^2_{\mathcal{H}}.$$

Proposition 6.3.2 (Energy equality)**.** *For all sufficiently regular initial data z (in particular, with η in the domain of T) we have the energy identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{E} + F(\|u\|_1^2)\big] + \Gamma[\eta] = 2\langle g, \dot{u}\rangle. \tag{6.3.5}$$

Proof. We multiply in H the first equation of (6.3.2) by $2\dot{u}$ and the second one by $2\dot{\alpha}$. Next, we take the product in $\mathcal{M}$ of the third equation of (6.3.2) and 2η. Exploiting (6.2.1), the claim follows. □

6.4 The Main Result

From now on, the memory kernel is supposed to satisfy the additional assumption

$$\mu'(s) + \delta\mu(s) \leq 0 \tag{6.4.1}$$

for some $\delta > 0$ and almost every $s \in \mathbb{R}^+$. Note that μ can be unbounded in a neighborhood of zero. As a direct consequence, we deduce the inequality

$$\delta\|\eta\|^2_{\mathcal{M}} \leq \Gamma[\eta], \quad \forall \eta \in \mathfrak{D}(T). \tag{6.4.2}$$

Remark 6.4.1. *Actually, hypothesis* (6.4.1) *can be relaxed: the results contained in the present chapter hold even if* μ *satisfies for some* $C \geq 1$ *and* $\delta > 0$ *the weaker condition*

$$\mu(t+s) \leq C\mathrm{e}^{-\delta t}\mu(s),$$

for every $t \geq 0$ *and almost every* $s \in \mathbb{R}^+$, *provided that* μ *is not too flat (cf. [34, 68]). In fact, the latter inequality boils down to* (6.4.1) *in the particular case where* $C = 1$.

The set $\mathbb{S}$ of stationary points of $S(t)$ consists of all vectors of the form $(u, 0, 0, 0, 0)$ with u solution to

$$A^2u + f(\|u\|_1^2)Au = g. \tag{6.4.3}$$

In particular, the set $\mathbb{S}$ is bounded in $\mathcal{H}$. To see that, just multiply the equation by u in H and use (6.1.3).

Theorem 6.4.1. *Within assumption* (6.4.1), *the semigroup* $S(t)$ *acting on* $\mathcal{H}$ *possesses the (connected) global attractor* $\mathbb{A}$ *bounded in* $\mathcal{H}^2$. *In addition,* $\mathbb{A}$ *coincides with the unstable set* $W(\mathbb{S})$ *of* $\mathbb{S}$.

As typically occurs in equations with memory, the next proposition holds.

Proposition 6.4.1. *The semigroup* $S(t)$ *fulfills the backward uniqueness property on the whole space* $\mathcal{H}$*; namely, if*

$$S(\tau)z_1 = S(\tau)z_2$$

for some $\tau > 0$ *and* $z_1, z_2 \in \mathcal{H}$, *then* $z_1 = z_2$.

As a straightforward consequence, we have

Corollary 6.4.1. *The restriction of* $S(t)$ *on* $\mathbb{A}$ *extends to a* $\mathcal{C}_0$*-group of operators* $\{S(t)\}_{t\in\mathbb{R}}$, *that is, the map* $S(t)|_{\mathbb{A}}$ *can be extended to negative times by the formula*

$$S(-t)|_{\mathbb{A}} = \big[S(t)|_{\mathbb{A}}\big]^{-1}.$$

Finally, the formal equality (6.3.1) is actually verified for all times by the trajectories lying on the attractor.

Corollary 6.4.2. *For every initial datum* $z \in \mathbb{A}$, *let*

$$Z(t) = (u(t), \dot{u}(t), \alpha(t), \dot{\alpha}(t), \eta^t)$$

be the (unique) CBT *such that* $Z(0) = z$. *Then the equality* (6.3.1) *holds for every* $t \in \mathbb{R}$ *and every* $s \in \mathbb{R}^+$.

6.5 Further Remarks

I. Up to minor modifications in the proofs, it is possible to allow the presence of an external heat supply $h \in H$ in the model; that is, to consider the system

$$\begin{cases} \ddot{u} + \omega A\ddot{u} + A^2u + f(\|A^{\frac{1}{2}}u\|^2)Au - A\dot{\alpha} = g, \\ \ddot{\alpha} + A\alpha + \displaystyle\int_0^\infty \mu(s)A[\alpha(t) - \alpha(t-s)]\,\mathrm{d}s + A\dot{u} = h. \end{cases}$$

In this case, setting

$$\alpha_h = A^{-1}h \qquad \text{and} \qquad z_h = (0, 0, \alpha_h, 0, 0),$$

the corresponding solution semigroup possesses the global attractor $\mathbb{A}_h$, which satisfies

$$\mathbb{A}_h = \mathbb{A} + z_h,$$

where $\mathbb{A}$ is the attractor of system (6.3.2). Accordingly, $\mathbb{A}_h$ is bounded (at least) in the less regular space

$$\mathcal{V} = H^4 \times H^3 \times H^2 \times H^2 \times \mathcal{M}^2.$$

II. As shown in [20], the ball $\mathbb{B}_0$ of $\mathcal{H}$ centered at zero of radius[1]

$$R_0 = 1 + \sup\Big\{\|z\|_{\mathcal{H}} : \Lambda(z) \leq 1 + \sup_{z_0 \in \mathbb{S}} \Lambda(z_0)\Big\}$$

turns out to be an absorbing set for $S(t)$. Note that R_0 can be explicitly calculated in terms of the structural quantities of the system. Nonetheless, given a bounded set $\mathcal{B} \subset \mathcal{H}$, we do not have a procedure to compute the entering time of $\mathcal{B}$ into the absorbing ball $\mathbb{B}_0$. However, within further assumptions on the nonlinear map f (e.g. in the physically relevant case $f(x) = x + b$), it is possible to prove the existence of absorbing sets for $S(t)$ via explicit energy estimates, exploiting a Gronwall-type lemma from the recent paper [70] (cf. Lemma 2.2.1). This provide a precise (uniform) control on the entering time of the trajectories. Incidentally, the argument applies also when time-dependent external forces are present.

III. By means of the regularity proved in Theorem 6.4.1, exploiting [17, Corollary 2.20] one can show that the fractal dimension of the global attractor $\mathbb{A}$ in $\mathcal{H}$ is finite. However, since the embedding $\mathcal{H}^2 \subset \mathcal{H}$ is not compact due to the memory component (see [76] for a counterexample), this requires the introduction of a suitable space compactly embedded into $\mathcal{H}$ (see e.g. [35]). As a matter of fact, it is even possible to prove the existence of a regular exponential attractor having finite fractal dimension.

IV. As detailed in forthcoming Section 6.9, the proof of Theorem 6.4.1 relies on a suitable decomposition of the semigroup into the sum of two *nonlinear* one-parameter operators: the first exponentially decaying to zero, and the other one uniformly bounded in the more regular space $\mathcal{H}^2$. Nevertheless, as shown in the next Section 6.6, the system is of gradient type and, due to the rotational inertia coefficient $\omega > 0$, the nonlinear term is subcritical and compact on the phase space. Hence, the existence of the global attractor depends on the exponential stability of the corresponding *linear* semigroup. According to this approach, once existence is attained, the regularity has to be proved in a second moment. On the contrary, our calculations allow to obtain both existence and (optimal) regularity by the same token. Incidentally, the proof of the exponential stability of the associated linear semigroup is contained in the forthcoming Lemma 6.9.1. The exponential stability of a similar linear model, with the Gurtin-Pipkin heat conduction law, has been proved in [42].

[1] Here Λ is the Lyapunov functional for $S(t)$ of the next Section 6.6.

6.6 The Lyapunov Functional

In this section, we prove the existence of a Lyapunov functional for the semigroup $S(t)$.

Theorem 6.6.1. *Let $z = (u_0, u_1, \alpha_0, \alpha_1, \eta_0) \in \mathcal{H}$. The function*

$$\Lambda(z) = \|z\|_{\mathcal{H}}^2 + F(\|u_0\|_1^2) - 2\langle g, u_0\rangle$$

is a Lyapunov functional for $S(t)$.

Proof. We will prove properties (i)-(iii) of Definition 1.1.10. Exploiting (6.1.4), it is immediate to verify that

$$\varpi\|z\|_{\mathcal{H}}^2 - c \leq \Lambda(z) \leq c\|z\|_{\mathcal{H}}^2 + F(\|u_0\|_1^2) + c, \tag{6.6.1}$$

for some $\varpi > 0$ and $c > 0$, both independent of z. This proves (i). Property (ii) is a direct consequence of the energy identity (6.3.5), which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda = -\Gamma[\eta] \leq 0. \tag{6.6.2}$$

Concerning property (iii), if Λ is constant along a trajectory, from (6.6.2) and (6.4.2) we obtain

$$\delta\|\eta\|_{\mathcal{M}}^2 \leq \Gamma[\eta] = 0,$$

which implies $\eta^t = 0$ for all $t \geq 0$. Therefore $\alpha(t)$ is constant in time and, using the second equation of (6.3.2), we learn that $\dot{u}(t)$ is constant. Accordingly,

$$u(t) = u_0 + w_0 t$$

for some vector $w_0 \in H^1$. At this point, we infer from (6.6.1) that $u(t)$ is bounded. Thus $w_0 = 0$ and then $u(t)$ is constant in time. The second equation now reads $\alpha(t) = 0$ for all $t \geq 0$. Since $\ddot{u}(t) = 0$, from the first equation we also obtain that $u(t)$ solves (6.4.3) and the theorem is proven. □

6.7 An Auxiliary Functional

The next step is introducing a suitable energy functional reflecting the dissipation mechanism of the problem. In order to deal with the possible singularity of μ at zero, we choose $s_\star \in (0, s_1)$ (where $s_1 = \infty$ if μ is jump-free) such that

$$\int_0^{s_\star} \mu(s)\,\mathrm{d}s \leq \frac{1}{4}. \tag{6.7.1}$$

Defining the truncated kernel

$$\rho(s) = \mu(s_\star)\chi_{(0,s_\star]}(s) + \mu(s)\chi_{(s_\star,\infty)}(s),$$

we consider the functional

$$\Theta(t) = -\int_0^\infty \rho(s)\langle\dot{\alpha}(t), \eta^t(s)\rangle\,\mathrm{d}s.$$

It is easily seen that

$$|\Theta(t)| \leq c\mathcal{E}(t), \quad \forall t \geq 0, \tag{6.7.2}$$

for some $c > 0$.

Lemma 6.7.1. *For every fixed $\nu \in (0,1]$, the functional Θ satisfies the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta + \frac{1}{2}\|\dot\alpha\|^2 \leq \frac{\nu}{2}\big[\|\alpha\|_1^2 + |\dot u|_1^2\big] + \frac{c}{\nu}\big[\|\eta\|_{\mathcal{M}}^2 + \Gamma[\eta]\big],$$

for some constant $c \geq 0$ independent of ν.

Proof. Along the proof, $c \geq 0$ indicates a *generic* constant depending only on the structural quantities of the problem but independent of ν. The time derivative of Θ is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Theta &= -\int_0^\infty \rho(s)\langle\ddot\alpha, \eta(s)\rangle\,\mathrm{d}s - \int_0^\infty \rho(s)\langle\dot\alpha, \dot\eta(s)\rangle\,\mathrm{d}s\\ &= \int_0^\infty \rho(s)\langle\alpha, \eta(s)\rangle_1\,\mathrm{d}s + \int_0^\infty \rho(s)\Big(\int_0^\infty \mu(\sigma)\langle\eta(\sigma), \eta(s)\rangle_1\,\mathrm{d}\sigma\Big)\,\mathrm{d}s\\ &\quad + \int_0^\infty \rho(s)\langle\dot u, \eta(s)\rangle_1\,\mathrm{d}s - \int_0^\infty \rho(s)\langle\dot\alpha, T\eta(s)\rangle\,\mathrm{d}s - \|\dot\alpha\|^2\int_0^\infty \rho(s)\,\mathrm{d}s.\end{aligned}$$

Since $\rho(s) \leq \mu(s)$, we estimate

$$\begin{aligned}\int_0^\infty \rho(s)\langle\alpha, \eta(s)\rangle_1\,\mathrm{d}s &\leq \|\alpha\|_1\int_0^\infty \mu(s)\|\eta(s)\|_1\,\mathrm{d}s\\ &\leq \|\alpha\|_1\|\eta\|_{\mathcal{M}} \leq \frac{\nu}{2}\|\alpha\|_1^2 + \frac{1}{2\nu}\|\eta\|_{\mathcal{M}}^2,\end{aligned} \tag{6.7.3}$$

and

$$\begin{aligned}\int_0^\infty \rho(s)\Big(\int_0^\infty \mu(\sigma)\langle\eta(\sigma), \eta(s)\rangle_1\,\mathrm{d}\sigma\Big)\,\mathrm{d}s &\leq \Big(\int_0^\infty \mu(s)\|\eta(s)\|_1\,\mathrm{d}s\Big)^2\\ &\leq \|\eta\|_{\mathcal{M}}^2.\end{aligned} \tag{6.7.4}$$

Moreover

$$\begin{aligned}\int_0^\infty \rho(s)\langle\dot u, \eta(s)\rangle_1\,\mathrm{d}s &\leq \|\dot u\|_1\int_0^\infty \mu(s)\|\eta(s)\|_1\,\mathrm{d}s\\ &\leq \|\dot u\|_1\|\eta\|_{\mathcal{M}} \leq \frac{\nu}{2}|\dot u|_1^2 + \frac{c}{\nu}\|\eta\|_{\mathcal{M}}^2.\end{aligned} \tag{6.7.5}$$

Integrating by parts in s, we infer that

$$\begin{aligned}-\int_0^\infty \rho(s)\langle\dot\alpha, T\eta(s)\rangle\,\mathrm{d}s &= \sum_n \mu_n\langle\dot\alpha, \eta(s_n)\rangle - \int_{s_\star}^\infty \mu'(s)\langle\dot\alpha, \eta(s)\rangle\,\mathrm{d}s\\ &\leq \|\dot\alpha\|\Big(\sum_n \mu_n\|\eta(s_n)\| - \int_{s_\star}^\infty \mu'(s)\|\eta(s)\|\,\mathrm{d}s\Big)\\ &\leq \frac{1}{4}\|\dot\alpha\|^2 + \Big(\sum_n \mu_n\|\eta(s_n)\| - \int_{s_\star}^\infty \mu'(s)\|\eta(s)\|\,\mathrm{d}s\Big)^2.\end{aligned}$$

Since $\sum_n \mu_n \le \mu(s_\star)$ as $s_\star < s_1$, we have

$$\begin{aligned}
&\bigg(\sum_n \mu_n \|\eta(s_n)\| - \int_{s_\star}^\infty \mu'(s)\|\eta(s)\|\,\mathrm{d}s\bigg)^2 \\
&\quad \le 2\bigg(\sum_n \mu_n \|\eta(s_n)\|\bigg)^2 + 2\bigg(\int_{s_\star}^\infty \mu'(s)\|\eta(s)\|\,\mathrm{d}s\bigg)^2 \\
&\quad \le 2\sum_n \mu_n \sum_n \mu_n \|\eta(s_n)\|^2 + 2\int_{s_\star}^\infty \mu'(s)\,\mathrm{d}s \int_{s_\star}^\infty \mu'(s)\|\eta(s)\|^2\,\mathrm{d}s \\
&\quad \le c\mu(s_\star)\Gamma[\eta].
\end{aligned}$$

Thus

$$-\int_0^\infty \rho(s)\langle \dot\alpha, T\eta(s)\rangle\,\mathrm{d}s \le \frac{1}{4}\|\dot\alpha\|^2 + c\Gamma[\eta]. \tag{6.7.6}$$

Finally, using (6.7.1) and the equality $\rho(s) = \mu(s)$ for $s \ge s_\star$ we obtain

$$-\|\dot\alpha\|^2 \int_0^\infty \rho(s)\,\mathrm{d}s \le -\|\dot\alpha\|^2 \int_{s_\star}^\infty \mu(s)\,\mathrm{d}s \le -\frac{3}{4}\|\dot\alpha\|^2. \tag{6.7.7}$$

Collecting (6.7.3)-(6.7.7), the proof is finished. □

6.8 Dissipation Integrals

Let now $R \ge 0$ be fixed. Till the end of the section, $C \ge 0$ will denote a *generic* constant depending only on R, besides the structural quantities of the problem. We consider initial data $z \in \mathcal{H}$ such that $\|z\|_{\mathcal{H}} \le R$. Thanks to Theorem 6.6.1, we draw the control

$$\mathcal{E}(t) \le C. \tag{6.8.1}$$

We have the following dissipation integral for the norm of $\dot\alpha$.

Lemma 6.8.1. *For every $\nu > 0$ small, the integral estimate*

$$\int_\tau^t \|\dot\alpha(y)\|^2\,\mathrm{d}y \le \nu(t-\tau) + \frac{C}{\nu^2}$$

holds for all $t > \tau \ge 0$.

Proof. Define

$$\Lambda(t) = \Lambda(t) + \nu^2\Theta(t),$$

where Λ is the Lyapunov functional and Θ is the functional introduced in Lemma 6.7.1. Exploiting (6.6.2), Lemma 6.7.1 and (6.8.1) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Lambda &= -\Gamma[\eta] + \nu^2 \frac{\mathrm{d}}{\mathrm{d}t}\Theta \\
&\le -\Gamma[\eta] + c\nu\big[\|\eta\|^2_{\mathcal{M}} + \Gamma[\eta]\big] + C\nu^3 - \frac{\nu^2}{2}\|\dot\alpha\|^2.
\end{aligned}$$

For every $\nu > 0$ small enough, invoking (6.4.2) we estimate

$$\begin{aligned}-\Gamma[\eta] + c\nu\big[\|\eta\|_{\mathcal{M}}^2 + \Gamma[\eta]\big] &= (c\nu - 1)\Gamma[\eta] + c\nu\|\eta\|_{\mathcal{M}}^2 \\ &\leq -(\delta - c\nu\delta - c\nu)\|\eta\|_{\mathcal{M}}^2 \leq 0.\end{aligned}$$

Therefore, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \frac{\nu^2}{2}\|\dot{\alpha}\|^2 \leq C\nu^3.$$

Since from (6.7.2) and (6.8.1)

$$|\Lambda(t)| \leq C,$$

integrating on the interval (τ, t) the proof is finished. □

Next, we prove the existence of a dissipation integral for the norm of $\dot{u}$.

Lemma 6.8.2. *For every $\nu > 0$ small, the integral estimate*

$$\int_\tau^t |\dot{u}(y)|_1^2 \,\mathrm{d}y \leq \nu(t - \tau) + \frac{C}{\nu^5} \tag{6.8.2}$$

holds for all $t > \tau \geq 0$.

Proof. By direct calculations, the functional

$$\Psi(t) = \langle B\dot{u}(t), \dot{\alpha}(t)\rangle_{-1} + \big(u(t), \alpha(t)\big)_1$$

satisfies the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi + |\dot{u}|_1^2 = \|\dot{\alpha}\|^2 + \big(u, \dot{\alpha}\big)_1 - \langle u, \dot{\alpha}\rangle_1 - f(\|u\|_1^2)\langle u, \dot{\alpha}\rangle + \langle g, \dot{\alpha}\rangle_{-1} - \big(\dot{u}, \eta\big)_{\mathcal{M}}.$$

In light of (6.8.1), estimating

$$\|\dot{\alpha}\|^2 + \big(u, \dot{\alpha}\big)_1 - \langle u, \dot{\alpha}\rangle_1 - f(\|u\|_1^2)\langle u, \dot{\alpha}\rangle + \langle g, \dot{\alpha}\rangle_{-1} \leq C\|\dot{\alpha}\| + \|\dot{\alpha}\|^2$$

we obtain for every $\nu > 0$ small

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi + |\dot{u}|_1^2 \leq C\|\dot{\alpha}\| + \|\dot{\alpha}\|^2 + |\dot{u}|_1|\eta|_{\mathcal{M}} \leq \nu + \frac{C}{\nu}\|\dot{\alpha}\|^2 + |\dot{u}|_1|\eta|_{\mathcal{M}}.$$

At this point, we define the functional

$$\Phi(t) = \Lambda(t) + \nu\Psi(t).$$

It is apparent from (6.8.1) that

$$|\Phi(t)| \leq C. \tag{6.8.3}$$

Exploiting the inequality above, we infer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \Gamma[\eta] + \nu|\dot{u}|_1^2 &\leq \nu^2 + C\|\dot{\alpha}\|^2 + \nu|\dot{u}|_1|\eta|_{\mathcal{M}} \\ &\leq \nu^2 + C\|\dot{\alpha}\|^2 + \frac{\nu}{2}|\dot{u}|_1^2 + \frac{\nu}{2}|\eta|_{\mathcal{M}}^2.\end{aligned}$$

In conclusion, using (6.4.2), for every $\nu > 0$ small enough we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{\nu}{2}|\dot{u}|_1^2 \leq \nu^2 + C\|\dot{\alpha}\|^2.$$

Recalling (6.8.3), an integration over (τ, t) together with Lemma 6.8.1 give the claim. □

6.9 Proof of Theorem 6.4.1

Aiming to apply Lemma 1.1.1, for a fixed $R \geq 0$ we consider initial data $z \in \mathcal{H}$ with $\|z\|_{\mathcal{H}} \leq R$. As before, along the section $C \geq 0$ will denote a *generic* constant depending only on R. Similarly to [39], we split the solution $S(t)z$ into the sum

$$S(t)z = S_0(t)z + S_1(t)z,$$

where

$$\begin{aligned} S_0(t)z &= (v(t), \dot{v}(t), \beta(t), \dot{\beta}(t), \xi(t)), \\ S_1(t)z &= (w(t), \dot{w}(t), \gamma(t), \dot{\gamma}(t), \zeta(t)) \end{aligned}$$

solve the Cauchy problems

$$\begin{cases} B\ddot{v} + A^2 v + f(\|u\|_1^2)Av + \ell v - A\dot{\beta} = 0, \\ \ddot{\beta} + A\beta + \displaystyle\int_0^\infty \mu(s)A\xi(s)\,\mathrm{d}s + A\dot{v} = 0, \\ \dot{\xi} = T\xi + \dot{\beta}, \\ (v(0), \dot{v}(0), \beta(0), \dot{\beta}(0), \xi^0) = z, \end{cases} \tag{6.9.1}$$

and

$$\begin{cases} B\ddot{w} + A^2 w + f(\|u\|_1^2)Aw - \ell v - A\dot{\gamma} = g, \\ \ddot{\gamma} + A\gamma + \displaystyle\int_0^\infty \mu(s)A\zeta(s)\,\mathrm{d}s + A\dot{w} = 0, \\ \dot{\zeta} = T\zeta + \dot{\gamma}, \\ (w(0), \dot{w}(0), \gamma(0), \dot{\gamma}(0), \zeta^0) = 0. \end{cases} \tag{6.9.2}$$

Here, $\ell = \ell(R) > 0$ is a positive constant chosen large enough that

$$\frac{1}{2}\|v\|_2^2 + f(\|u\|_1^2)\|v\|_1^2 + \ell\|v\|^2 \geq \frac{1}{4}\|v\|_2^2. \tag{6.9.3}$$

This choice is possible due to the interpolation inequality

$$\|v\|_1^2 \leq \|v\|\|v\|_2$$

and the fact that $f(\|u\|_1^2)$ is uniformly bounded thanks to (6.8.1). Finally, we set

$$\mathcal{E}_0(t) = \|S_0(t)z\|_{\mathcal{H}}^2 \qquad \text{and} \qquad \mathcal{E}_1(t) = \|S_1(t)z\|_{\mathcal{H}^2}^2.$$

We will show that $\mathcal{E}_0$ decays exponentially and $\mathcal{E}_1$ is uniformly bounded.

Lemma 6.9.1. *There exists $\varkappa = \varkappa(R) > 0$ such that*

$$\mathcal{E}_0(t) \leq C\mathrm{e}^{-\varkappa t}.$$

Proof. The functional

$$\Upsilon_0(t) = \mathcal{E}_0(t) + f(\|u(t)\|_1^2)\|v(t)\|_1^2 + \ell\|v(t)\|^2$$

fulfills the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_0 + \Gamma[\xi] = 2f'(\|u\|_1^2)\langle Au, \dot{u}\rangle\|v\|_1^2.$$

Exploiting (6.8.1), we estimate

$$2f'(\|u\|_1^2)\langle Au, \dot{u}\rangle\|v\|_1^2 \leq C\|\dot{u}\|\mathcal{E}_0,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_0 + \Gamma[\xi] \leq C\|\dot{u}\|\mathcal{E}_0. \tag{6.9.4}$$

We also consider the functionals

$$\begin{aligned}
\Psi_0(t) &= \langle B\dot{v}(t), \dot{\beta}(t)\rangle_{-1} + \big(v(t), \beta(t)\big)_1,\\
\Phi_0(t) &= \langle B\dot{v}(t), v(t)\rangle,\\
\Sigma_0(t) &= \langle \dot{\beta}(t), \beta(t)\rangle + \langle v(t), \beta(t)\rangle_1.
\end{aligned}$$

In light of (6.8.1), the functional Ψ_0 satisfies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Psi_0 + |\dot{v}|_1^2 &= \|\dot{\beta}\|^2 + \big(v, \dot{\beta}\big)_1 - \langle v, \dot{\beta}\rangle_1 - f(\|u\|_1^2)\langle v, \dot{\beta}\rangle - \ell\langle v, \dot{\beta}\rangle_{-1} - \big(\dot{v}, \xi\big)_{\mathcal{M}}\\
&\leq \frac{1}{16}\|v\|_2^2 + \frac{1}{4}|\dot{v}|_1^2 + C\big[\|\dot{\beta}\|^2 + \|\xi\|_{\mathcal{M}}^2\big].
\end{aligned} \tag{6.9.5}$$

Concerning the functional Φ_0, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0 + \|v\|_2^2 + f(\|u\|_1^2)\|v\|_1^2 + \ell\|v\|^2 = \langle \dot{\beta}, v\rangle_1 + |\dot{v}|_1^2 \leq \frac{1}{4}\|v\|_2^2 + |\dot{v}|_1^2 + C\|\dot{\beta}\|^2.$$

Therefore, applying (6.9.3) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0 + \frac{1}{2}\|v\|_2^2 \leq |\dot{v}|_1^2 + C\|\dot{\beta}\|^2. \tag{6.9.6}$$

Lastly, the functional Σ_0 fulfills

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Sigma_0 + \|\beta\|_1^2 &= \|\dot{\beta}\|^2 + \langle v, \dot{\beta}\rangle_1 - \langle \beta, \xi\rangle_{\mathcal{M}}\\
&\leq \frac{1}{8}\|v\|_2^2 + \frac{1}{2}\|\beta\|_1^2 + C\big[\|\dot{\beta}\|^2 + \|\xi\|_{\mathcal{M}}^2\big].
\end{aligned} \tag{6.9.7}$$

Collecting (6.9.5)-(6.9.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\{2\Psi_0 + \Phi_0 + \Sigma_0\} + \frac{1}{2}|\dot{v}|_1^2 + \frac{1}{4}\|v\|_2^2 + \frac{1}{2}\|\beta\|_1^2 \leq C\big[\|\dot{\beta}\|^2 + \|\xi\|_{\mathcal{M}}^2\big]. \tag{6.9.8}$$

At this point, arguing exactly as in Lemma 6.7.1, we consider the functional

$$\Theta_0(t) = -\int_0^\infty \rho(s)\langle \dot{\beta}(t), \xi^t(s)\rangle\,\mathrm{d}s,$$

and we draw the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_0 + \frac{1}{2}\|\dot\beta\|^2 \leq \frac{\varepsilon}{2}\big[\|\beta\|_1^2 + |\dot v|_1^2\big] + \frac{c}{\varepsilon}\big[\|\xi\|_{\mathcal{M}}^2 + \Gamma[\xi]\big] \tag{6.9.9}$$

for every $\varepsilon \in (0,1]$ and some constant $c \geq 0$ (independent of ε and R). Finally, we define

$$\Lambda_0(t) = \Upsilon_0(t) + 2\varepsilon^3\{2\Psi_0(t) + \Phi_0(t) + \Sigma_0(t)\} + \varepsilon^2\Theta_0(t).$$

It is apparent from (6.9.3) that, for $\varepsilon > 0$ sufficiently small,

$$\frac{1}{2}\mathcal{E}_0(t) \leq \Lambda_0(t) \leq C\mathcal{E}_0(t). \tag{6.9.10}$$

Appealing to (6.9.4), (6.9.8) and (6.9.9), for $\varepsilon > 0$ small enough the functional Λ_0 satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0 + \frac{1}{2}\Gamma[\xi] + \frac{\varepsilon^3}{2}\big[\|v\|_2^2 + |\dot v|_1^2 + \|\beta\|_1^2 + \|\dot\beta\|^2\big] \leq C\|\dot u\|\mathcal{E}_0 + C\varepsilon\|\xi\|_{\mathcal{M}}^2.$$

Estimating

$$C\|\dot u\|\mathcal{E}_0 \leq \frac{\varepsilon^3}{4}\mathcal{E}_0 + C\|\dot u\|^2\mathcal{E}_0,$$

by applying (6.4.2) and (6.9.10), possibly reducing the parameter $\varepsilon > 0$, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0 + 2\varkappa\Lambda_0 \leq C\|\dot u\|^2\Lambda_0$$

for some $\varkappa = \varkappa(R) > 0$. Up to fixing $\nu > 0$ in (6.8.2) sufficiently small in order to have

$$C\int_\tau^t \|\dot u(y)\|^2\,\mathrm{d}y \leq \varkappa(t-\tau) + C,$$

the claim follows from Lemma 6.2.1 together with (6.9.10). □

Lemma 6.9.2. *We have the uniform estimate*

$$\sup_{t\geq 0}\mathcal{E}_1(t) \leq C.$$

Proof. The functional

$$\Upsilon_1(t) = \mathcal{E}_1(t) + f(\|u(t)\|_1^2)\|w(t)\|_3^2 - 2\langle g, w(t)\rangle_2$$

satisfies the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_1 + \Gamma[A\zeta] = 2f'(\|u\|_1^2)\langle \dot u, u\rangle_1\|w\|_3^2 + 2\ell\langle v, \dot w\rangle_2.$$

Due to (6.8.1) and the bound $\|v\|_2 \leq C$ ensured by Lemma 6.9.1, we infer that

$$\|w\|_3^2 \leq \|w\|_2\|w\|_4 \leq (\|u\|_2 + \|v\|_2)\|w\|_4 \leq C\|w\|_4, \tag{6.9.11}$$

and

$$2\ell\langle v, \dot w\rangle_2 \leq C\|\dot w\|_2 \leq C|\dot w|_3.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_1 + \Gamma[A\zeta] \le C\big[\|w\|_4 + |\dot w|_3\big]. \tag{6.9.12}$$

Then, analogously to the proof of the previous lemma, we define the functionals

$$\begin{aligned}
\Psi_1(t) &= \langle B\dot w(t), \dot\gamma(t)\rangle_1 + \big(w(t), \gamma(t)\big)_3,\\
\Phi_1(t) &= \langle B\dot w(t), w(t)\rangle_2,\\
\Sigma_1(t) &= \langle \dot\gamma(t), \gamma(t)\rangle_2 + \langle w(t), \gamma(t)\rangle_3.
\end{aligned}$$

Again, we estimate the time derivative of every single functional, making use of the equations of system (6.9.2). Regarding the functional Ψ_1, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Psi_1 + |\dot w|_3^2\\
&= \|\dot\gamma\|_2^2 + \big(w, \dot\gamma\big)_3 - \langle w, \dot\gamma\rangle_3 - f(\|u\|_1^2)\langle w, \dot\gamma\rangle_2 + \ell\langle v, \dot\gamma\rangle_1 - \big(\dot w, \zeta\big)_{\mathcal{M}^2} + \langle g, \dot\gamma\rangle_1\\
&\le \frac{1}{16}\|w\|_4^2 + \frac{1}{4}|\dot w|_3^2 + C\big[\|\dot\gamma\|_2^2 + \|\zeta\|_{\mathcal{M}^2}^2 + 1\big].
\end{aligned} \tag{6.9.13}$$

Concerning Φ_1, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1 + \|w\|_4^2 = |\dot w|_3^2 + \langle \dot\gamma, w\rangle_3 - f(\|u\|_1^2)\|w\|_3^2 + \ell\langle v, w\rangle_2 + \langle g, w\rangle_2.$$

Applying (6.8.1) and (6.9.11), the right-hand side is controlled by

$$|\dot w|_3^2 + \langle \dot\gamma, w\rangle_3 - f(\|u\|_1^2)\|w\|_3^2 + \ell\langle v, w\rangle_2 + \langle g, w\rangle_2 \le |\dot w|_3^2 + \frac{1}{4}\|w\|_4^2 + C\|\dot\gamma\|_2^2 + C,$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1 + \frac{3}{4}\|w\|_4^2 \le |\dot w|_3^2 + C\big[\|\dot\gamma\|_2^2 + 1\big]. \tag{6.9.14}$$

Moreover, the functional Σ_1 fulfills

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Sigma_1 + \|\gamma\|_3^2 &= \|\dot\gamma\|_2^2 + \langle w, \dot\gamma\rangle_3 - \langle \gamma, \zeta\rangle_{\mathcal{M}^2}\\
&\le \frac{1}{8}\|w\|_4^2 + \frac{1}{2}\|\gamma\|_3^2 + C\big[\|\dot\gamma\|_2^2 + \|\zeta\|_{\mathcal{M}^2}^2\big].
\end{aligned} \tag{6.9.15}$$

Collecting (6.9.13)-(6.9.15) we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\{2\Psi_1 + \Phi_1 + \Sigma_1\} + \frac{1}{2}|\dot w|_3^2 + \frac{1}{2}\|w\|_4^2 + \frac{1}{2}\|\gamma\|_3^2 \le C\big[\|\dot\gamma\|_2^2 + \|\zeta\|_{\mathcal{M}^2}^2 + 1\big]. \tag{6.9.16}$$

Next, with reference to Lemma 6.7.1, we consider the further functional

$$\Theta_1(t) = -\int_0^\infty \rho(s)\langle \dot\gamma(t), \zeta^t(s)\rangle_2 \,\mathrm{d}s,$$

which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_1 + \frac{1}{2}\|\dot\gamma\|_2^2 \le \frac{\varepsilon}{2}\big[\|\gamma\|_3^2 + |\dot w|_3^2\big] + \frac{c}{\varepsilon}\big[\|\zeta\|_{\mathcal{M}^2}^2 + \Gamma[A\zeta]\big] \tag{6.9.17}$$

for every $\varepsilon \in (0,1]$ and some constant $c \geq 0$ (independent of ε and R). Finally, we define

$$\Lambda_1(t) = \Upsilon_1(t) + 2\varepsilon^3\{2\Psi_1(t) + \Phi_1(t) + \Sigma_1(t)\} + \varepsilon^2\Theta_1(t).$$

It is then an easy matter to see that, for $\varepsilon > 0$ small enough,

$$\frac{1}{2}\mathcal{E}_1(t) - C \leq \Lambda_1(t) \leq C\mathcal{E}_1(t) + C. \tag{6.9.18}$$

Exploiting (6.9.12), (6.9.16) and (6.9.17), for $\varepsilon > 0$ sufficiently small we draw the inequalities

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \frac{1}{2}\Gamma[A\zeta] + \frac{\varepsilon^3}{2}\big[\|\gamma\|_3^2 + |\dot{w}|_3^2 + \|\dot{\gamma}\|_2^2 + \|w\|_4^2\big]& \\ \leq C\|w\|_4 + C|\dot{w}|_3 + C\varepsilon\|\zeta\|^2_{\mathcal{M}^2} + C& \\ \leq \frac{\varepsilon^3}{4}\|w\|_4^2 + \frac{\varepsilon^3}{4}|\dot{w}|_3^2 + C\varepsilon\|\zeta\|^2_{\mathcal{M}^2} + C.&\end{aligned}$$

Thus, in light of (6.4.2) and possibly reducing again $\varepsilon > 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \frac{\varepsilon^3}{4}\mathcal{E}_1 \leq C.$$

Since $\Lambda_1(0) = 0$, the claim follows from the controls (6.9.18), together with the standard Gronwall lemma. □

6.9.1 Conclusion of the proof of Theorem 6.4.1

We are now in a position to complete the proof of Theorem 6.4.1. Since Theorem 6.6.1 ensures the existence of a Lyapunov functional for $S(t)$ and the set $\mathbb{S}$ of stationary points is bounded in $\mathcal{H}$, we need to verify only assumption (iii) of Lemma 1.1.1. To this end, recalling Lemma 6.9.1, we are left to show that the solution $S_1(t)z$ to problem (6.9.2) belongs to a compact set $\mathcal{K}_R$ whenever $\|z\|_{\mathcal{H}} \leq R$. Indeed, the third component ζ^t of $S_1(t)z$ admits the explicit representation

$$\zeta^t(s) = \begin{cases}\gamma(t) - \gamma(t-s) & 0 < s \leq t,\\ \gamma(t) & s > t.\end{cases}$$

Therefore,

$$D\zeta^t(s) = \begin{cases}\dot{\gamma}(t-s) & 0 < s \leq t,\\ 0 & s > t.\end{cases}$$

Hence, Lemma 6.9.2 and the summability of the kernel μ give

$$\|D\zeta^t\|_{\mathcal{M}^1} + \sup_{s\in\mathbb{R}^+}\|\zeta^t(s)\|_3 \leq C,$$

and we conclude that

$$S_1(t)z \in \mathcal{K}_R,$$

where

$$\mathcal{K}_R = \Big\{ z = (u_0, u_1, \alpha_0, \alpha_1, \eta_0) : \|z\|_{\mathcal{H}^2} + \|D\eta_0\|_{\mathcal{M}^1} + \sup_{s\in\mathbb{R}^+} \|\eta_0(s)\|_3 \le C, \;\; \eta_0(0) = 0 \Big\}$$

is compact in $\mathcal{H}$ due to a general compactness result for memory spaces (see Lemma 5.5 in [76]). □

Remark 6.9.1. *It is worth noting that, except for the proof of the boundedness of* $\mathbb{S}$*, hypothesis* (6.1.3) *on the nonlinear map* f *is not needed. In fact, the set* $\mathbb{S}$ *turns out to be bounded assuming* (6.1.4) *only (see e.g. [15]).*

6.10 Proofs of Proposition 6.4.1 and Corollary 6.4.2

6.10.1 Proof of Proposition 6.4.1

We consider two initial data $z_1, z_2 \in \mathcal{H}$, and we denote by $Z_1(t)$ and $Z_2(t)$ the corresponding solutions. Calling the difference

$$Z_1(t) - Z_2(t) = (v(t), \dot{v}(t), \beta(t), \dot{\beta}(t), \xi^t),$$

we suppose that $Z_1(\tau) = Z_2(\tau)$ for some $\tau > 0$. The explicit representation formula for ξ^t reads

$$\xi^t(s) = \begin{cases} \beta(t) - \beta(t-s) & 0 < s \le t, \\ \xi^0(s-t) + \beta(t) - \beta(0) & s > t. \end{cases}$$

Since $\xi^\tau = 0$, we learn that

$$0 = \beta(\tau) = \beta(\tau - s), \quad \forall s \in (0, \tau],$$

i.e. $\beta(t) \equiv 0$ on the whole interval $[0, \tau]$. In turn,

$$0 = \xi^\tau(s) = \xi^0(s - \tau), \quad \forall s > \tau,$$

implying $\xi^0 = 0$, and so $\xi^t \equiv 0$ on $[0, \tau]$. At this point, from the second equation of (6.3.2) we get $\dot{v}(t) \equiv 0$ on $[0, \tau]$. Since $v(\tau) = 0$, this gives $v(t) \equiv 0$ on $[0, \tau]$. □

6.10.2 Proof of Corollary 6.4.2

For every $z \in \mathbb{A}$, let

$$Z(t) = (u(t), \dot{u}(t), \alpha(t), \dot{\alpha}(t), \eta^t)$$

be the (unique from Proposition 6.4.1) CBT such that $Z(0) = z$. Assume first $t > 0$, and let $\tau > 0$ be arbitrarily fixed. Naming

$$z_\tau = S(-\tau)z$$

and setting

$$(u_\tau(t), \dot{u}_\tau(t), \alpha_\tau(t), \dot{\alpha}_\tau(t), \eta_\tau^t) = S(t)z_\tau,$$

we have

$$(u_\tau(t+\tau), \dot{u}_\tau(t+\tau), \alpha_\tau(t+\tau), \dot{\alpha}_\tau(t+\tau), \eta_\tau^{t+\tau}) = (u(t), \dot{u}(t), \alpha(t), \dot{\alpha}(t), \eta^t).$$

Hence, the representation formula (6.3.4) gives

$$\eta^t(s) = \eta_\tau^{t+\tau}(s) = \alpha_\tau(t+\tau) - \alpha_\tau(t+\tau-s) = \alpha(t) - \alpha(t-s),$$

whenever $0 < s \leq t+\tau$. From the arbitrariness of $\tau > 0$, we conclude that (6.3.1) holds for all $t > 0$. The proof of the case $t \leq 0$ is analogous and therefore omitted. □

6.11 The Linear Case

In the remaining part of the chapter, we analyze the linear version of system (6.1.1) and the corresponding limit situation without memory in the case $\omega = 0$. For more generality, we also consider a coupling term depending on a fixed parameter

$$\sigma \leq \tfrac{3}{2}.$$

Precisely, we study the two linear systems

$$\begin{cases} \ddot{u} + A^2 u - A^\sigma \dot{\alpha} = 0, \\ \ddot{\alpha} + A\alpha + A\dot{\alpha} + A^\sigma \dot{u} = 0, \end{cases} \tag{6.11.1}$$

and

$$\begin{cases} \ddot{u} + A^2 u - A^\sigma \dot{\alpha} = 0, \\ \ddot{\alpha} + A\alpha + \displaystyle\int_0^\infty \mu(s) A[\alpha(t) - \alpha(t-s)]\,\mathrm{d}s + A^\sigma \dot{u} = 0, \end{cases} \tag{6.11.2}$$

investigating the decay properties of the associated contraction semigroups (that will be denoted by $U_1(t)$ and $U_2(t)$, respectively). When $\sigma \geq \frac{1}{2}$ we prove the exponential decay of $U_1(t)$ via energy methods. We also show that $U_1(t)$ fails to be exponentially stable if $\sigma < \frac{1}{2}$. From the physical viewpoint, this is not surprising. Indeed, since the dissipation mechanism is only thermal, when the coupling is not strong enough the system is not able to convert thermal into mechanical dissipation, which is needed to stabilize the plate. Concerning the semigroup $U_2(t)$, lack of exponential stability is proved for all values of σ.

6.12 The Contraction Semigroups

In this section, we show that systems (6.11.1) and (6.11.2) generate two contraction semigroups on the phase spaces

$$\mathcal{H} = H^2 \times H \times H^1 \times H$$

and

$$\mathcal{V} = H^2 \times H \times H^1 \times H \times \mathcal{M},$$

respectively. It will be clear from the proofs that the limitation $\sigma \leq \frac{3}{2}$ plays an essential role.

6.12.1 The first system

Introducing the state vector $z(t) = (u(t), v(t), \alpha(t), \beta(t))$, we view system (6.11.1) as the ODE in $\mathcal{H}$

$$\dot{z}(t) = \mathbb{A}z(t),$$

where the linear operator $\mathbb{A}$ is defined as

$$\mathbb{A}\begin{pmatrix} u \\ v \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} v \\ -A^2u + A^\sigma\beta \\ \beta \\ -A\alpha - A\beta - A^\sigma v \end{pmatrix}$$

with domain

$$\mathfrak{D}(\mathbb{A}) = \left\{ z \in \mathcal{H} \;\middle|\; \begin{array}{c} v \in H^2 \\ -u + A^{\sigma-2}\beta \in H^4 \\ \beta \in H^1 \\ \alpha + \beta + A^{\sigma-1}v \in H^2 \end{array} \right\}.$$

Theorem 6.12.1. *The operator $\mathbb{A}$ is the infinitesimal generator of a contraction semigroup*

$$U_1(t) = \mathrm{e}^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}.$$

Proof. The proof is based on an application of Lemma 1.2.1. It is easy to see that $\mathbb{A}$ is dissipative, for

$$\langle \mathbb{A}z, z\rangle = -\|\beta\|_1^2 \leq 0, \quad \forall z \in \mathfrak{D}(\mathbb{A}).$$

It remains to show that

$$\mathrm{Range}(\mathbb{I} - \mathbb{A}) = \mathcal{H}.$$

To this aim, let $\hat{z} = (\hat{u}, \hat{v}, \hat{\alpha}, \hat{\beta}) \in \mathcal{H}$. We look for a solution $z = (u, v, \alpha, \beta) \in \mathfrak{D}(\mathbb{A})$ to the equation

$$z - \mathbb{A}z = \hat{z} \tag{6.12.1}$$

which, written in components, reads

$$u - v = \hat{u}, \tag{6.12.2}$$

$$v + A^2u - A^\sigma\beta = \hat{v}, \tag{6.12.3}$$

$$\alpha - \beta = \hat{\alpha}, \tag{6.12.4}$$

$$\beta + A\alpha + A\beta + A^\sigma v = \hat{\beta}. \tag{6.12.5}$$

Plugging (6.12.2) into (6.12.3) and (6.12.4) into (6.12.5) we obtain

$$\begin{cases} v + A^2v - A^\sigma\beta = \psi_1, \\ \beta + 2A\beta + A^\sigma v = \psi_2, \end{cases} \tag{6.12.6}$$

where

$$\psi_1 = \hat{v} - A^2\hat{u} \in H^{-2} \qquad \text{and} \qquad \psi_2 = \hat{\beta} - A\hat{\alpha} \in H^{-1}.$$

Then we associate to (6.12.6) the bilinear form on $H^2 \times H^1$

$$B((v,\beta),(\tilde v,\tilde\beta)) = \langle v,\tilde v\rangle + \langle v,\tilde v\rangle_2 - \langle \beta,\tilde v\rangle_\sigma + \langle \beta,\tilde\beta\rangle + 2\langle \beta,\tilde\beta\rangle_1 + \langle v,\tilde\beta\rangle_\sigma.$$

Clearly, B is coercive on $H^2 \times H^1$. Moreover, since $\sigma \le \frac{3}{2}$,

$$\begin{aligned} |B((v,\beta),(\tilde v,\tilde\beta))| \le& c\big[\|v\|_2\|\tilde v\|_2 + \|\beta\|_1\|\tilde\beta\|_1\big] + \|\beta\|_1\|\tilde v\|_{2\sigma-1} + \|\tilde\beta\|_1\|v\|_{2\sigma-1} \\ \le& c\|(v,\beta)\|_{H^2\times H^1}\|(\tilde v,\tilde\beta)\|_{H^2\times H^1}, \end{aligned}$$

for some constant $c \ge 0$. Hence, by means of the Lax-Milgram lemma, problem (6.12.6) admits a unique (weak) solution $(v,\beta) \in H^2 \times H^1$. Thus, in light of (6.12.2)-(6.12.5), the vector

$$z = (v + \hat u, v, \beta + \hat\alpha, \beta) \in \mathfrak{D}(\mathbb{A})$$

solves equation (6.12.1). □

6.12.2 The second system

In order to carry out the analysis of (6.11.2), we translate the problem in the history space framework. To this end, defining the auxiliary variable

$$\eta^t(s) = \alpha(t) - \alpha(t-s),$$

system (6.11.2) can be given the form

$$\begin{cases} \ddot u + A^2 u - A^\sigma \dot\alpha = 0, \\ \ddot\alpha + A\alpha + \displaystyle\int_0^\infty \mu(s)A\eta(s)\,\mathrm{d}s + A^\sigma \dot u = 0, \\ \dot\eta = T\eta + \dot\alpha. \end{cases}$$

Introducing the state vector $z(t) = (u(t), v(t), \alpha(t), \beta(t), \eta^t)$, we view the latter system as the ODE in $\mathcal{V}$

$$\dot z(t) = \mathbb{B}z(t),$$

where the linear operator $\mathbb{B}$ is defined as

$$\mathbb{B}\begin{pmatrix} u \\ v \\ \alpha \\ \beta \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ -A^2u + A^\sigma\beta \\ \beta \\ -A\alpha - \int_0^\infty \mu(s)A\eta(s)\,\mathrm{d}s - A^\sigma v \\ T\eta + \beta \end{pmatrix}$$

with domain

$$\mathfrak{D}(\mathbb{B}) = \left\{ z \in \mathcal{V} \;\middle|\; \begin{array}{c} v \in H^2 \\ -u + A^{\sigma-2}\beta \in H^4 \\ \beta \in H^1 \\ \alpha + \int_0^\infty \mu(s)\eta(s)\,\mathrm{d}s + A^{\sigma-1}v \in H^2 \\ \eta \in \mathfrak{D}(T) \end{array} \right\}.$$

Theorem 6.12.2. *The operator $\mathbb{B}$ is the infinitesimal generator of a contraction semigroup*

$$U_2(t) = \mathrm{e}^{t\mathbb{B}} : \mathcal{V} \to \mathcal{V}.$$

Proof. Analogously to Theorem 6.12.1, the proof relies on Lemma 1.2.1. It is immediate to see that $\mathbb{B}$ is dissipative, for (see e.g. [69])

$$\langle \mathbb{B}z, z\rangle = \langle T\eta, \eta\rangle_{\mathcal{M}} \leq 0.$$

Next, we prove that

$$\mathrm{Range}(\mathbb{I} - \mathbb{B}) = \mathcal{V}.$$

As before, given $\hat{z} = (\hat{u}, \hat{v}, \hat{\alpha}, \hat{\beta}, \hat{\eta}) \in \mathcal{V}$, we look for a solution $z = (u, v, \alpha, \beta, \eta) \in \mathfrak{D}(\mathbb{B})$ to the equation

$$z - \mathbb{B}z = \hat{z} \tag{6.12.7}$$

which, written in components, reads

$$u - v = \hat{u}, \tag{6.12.8}$$

$$v + A^2 u - A^\sigma \beta = \hat{v}, \tag{6.12.9}$$

$$\alpha - \beta = \hat{\alpha}, \tag{6.12.10}$$

$$\beta + A\alpha + \int_0^\infty \mu(s) A\eta(s)\,\mathrm{d}s + A^\sigma v = \hat{\beta}, \tag{6.12.11}$$

$$\eta - T\eta - \beta = \hat{\eta}. \tag{6.12.12}$$

Integrating (6.12.12) with $\eta(0) = 0$, we find

$$\eta(s) = (1 - \mathrm{e}^{-s})\beta + E(s), \tag{6.12.13}$$

where

$$E(s) = \int_0^s \mathrm{e}^{y-s}\hat{\eta}(y)\,\mathrm{d}y.$$

Substituting (6.12.10) and (6.12.13) into (6.12.11) and (6.12.8) into (6.12.9), we obtain

$$\begin{cases} v + A^2 v - A^\sigma \beta = \psi_1, \\ \beta + A\beta + \gamma A\beta + A^\sigma v = \psi_2, \end{cases} \tag{6.12.14}$$

where we set

$$\gamma = \int_0^\infty \mu(s)(1 - \mathrm{e}^{-s})\,\mathrm{d}s > 0$$

and

$$\psi_1 = \hat{v} - A^2 \hat{u},$$
$$\psi_2 = \hat{\beta} - A\hat{\alpha} - A\int_0^\infty \mu(s) E(s)\,\mathrm{d}s.$$

It is apparent that $\psi_1 \in H^{-2}$. Concerning ψ_2, we note that

$$\begin{aligned}\Big\| \int_0^\infty \mu(s) E(s)\, \mathrm{d}s \Big\|_1 &\leq \int_0^\infty \mu(s) \int_0^s \mathrm{e}^{y-s} \|\hat{\eta}(y)\|_1 \,\mathrm{d}y\, \mathrm{d}s \\ &\leq \int_0^\infty \sqrt{\mu(s)} \int_0^s \mathrm{e}^{y-s} \sqrt{\mu(y)} \|\hat{\eta}(y)\|_1 \,\mathrm{d}y\, \mathrm{d}s \\ &\leq \sqrt{\varkappa}\, \|\hat{\eta}\|_{\mathcal{M}},\end{aligned}$$

which readily yields $\psi_2 \in H^{-1}$. At this point, as in the proof of Theorem 6.12.1, we infer the existence of a unique (weak) solution $(v, \beta) \in H^2 \times H^1$ to problem (6.12.14). Moreover, in light of (6.12.13), we have

$$\begin{aligned}\|\eta\|_{\mathcal{M}}^2 &\leq 2\varkappa \|\beta\|_1^2 + 2 \int_0^\infty \mu(s) \|E(s)\|_1^2 \,\mathrm{d}s \\ &\leq 2\varkappa \|\beta\|_1^2 + 2 \int_0^\infty \Big(\int_0^s \mathrm{e}^{y-s} \sqrt{\mu(y)} \|\hat{\eta}(y)\|_1 \,\mathrm{d}y \Big)^2 \mathrm{d}s \\ &\leq 2\varkappa \|\beta\|_1^2 + 2 \|\hat{\eta}\|_{\mathcal{M}}^2,\end{aligned}$$

implying $\eta \in \mathcal{M}$. Accordingly,

$$T\eta = \eta - \beta - \hat{\eta} \in \mathcal{M}.$$

Finally, it is easy to ascertain that $\eta(s) \to 0$ in H^1 as $s \to 0$. This finishes the proof. □

6.13 Decay Properties of the Semigroup $U_1(t)$

We now analyze the decay properties of the semigroup $U_1(t)$. As we will see, the picture is quite different depending whether or not $\sigma \geq \frac{1}{2}$.

Theorem 6.13.1. *If $\sigma \geq \frac{1}{2}$ the semigroup $U_1(t)$ is exponentially stable.*

Proof. We will perform several multiplications which make sense if the initial data belong to the domain of $\mathbb{A}$. For the general case, a standard approximation argument will do. Along the proof, the *generic* constant $C \geq 0$ will depend only on the structural quantities of the problem. Defining the energy functional

$$\mathcal{E}(t) = \frac{1}{2}\big[\|\dot{u}(t)\|^2 + \|u(t)\|_2^2 + \|\dot{\alpha}(t)\|^2 + \|\alpha(t)\|_1^2\big],$$

multiplying in H the first equation of (6.11.1) by $\dot{u}$ and the second one by $\dot{\alpha}$, we obtain the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \|\dot{\alpha}\|_1^2 = 0. \tag{6.13.1}$$

Then, we introduce the further functionals

$$\begin{aligned}\Phi(t) &= \langle \dot{u}(t), u(t) \rangle, \\ \Theta(t) &= \langle \dot{\alpha}(t), \alpha(t) \rangle + \langle u(t), \alpha(t) \rangle_\sigma, \\ \Psi(t) &= \langle \dot{u}(t), \dot{\alpha}(t) \rangle_{-\sigma} + \langle u(t), \alpha(t) \rangle_{1-\sigma}.\end{aligned}$$

Concerning Φ, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \|u\|_2^2 &= \|\dot{u}\|^2 + \langle u, \dot{\alpha}\rangle_\sigma \\ &\leq \|\dot{u}\|^2 + \|u\|_{2\sigma-1}\|\dot{\alpha}\|_1 \\ &\leq \frac{1}{4}\|u\|_2^2 + \|\dot{u}\|^2 + C\|\dot{\alpha}\|_1^2,\end{aligned} \tag{6.13.2}$$

while, regarding Θ,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Theta + \|\alpha\|_1^2 &= \|\dot{\alpha}\|^2 + \langle u, \dot{\alpha}\rangle_\sigma - \langle \alpha, \dot{\alpha}\rangle_1 \\ &\leq \|\dot{\alpha}\|^2 + \|u\|_{2\sigma-1}\|\dot{\alpha}\|_1 + \|\alpha\|_1\|\dot{\alpha}\|_1 \\ &\leq \frac{1}{8}\|u\|_2^2 + \frac{1}{2}\|\alpha\|_1^2 + C\|\dot{\alpha}\|_1^2.\end{aligned} \tag{6.13.3}$$

Besides, the functional Ψ fulfills

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi + \|\dot{u}\|^2 &= \|\dot{\alpha}\|^2 - \langle u, \dot{\alpha}\rangle_{2-\sigma} + \langle u, \dot{\alpha}\rangle_{1-\sigma} - \langle \dot{u}, \dot{\alpha}\rangle_{1-\sigma} \\ &\leq \|\dot{\alpha}\|^2 + \|u\|_{3-2\sigma}\|\dot{\alpha}\|_1 + \|\dot{u}\|_{1-2\sigma}\|\dot{\alpha}\|_1 + \|u\|_{1-2\sigma}\|\dot{\alpha}\|_1 \\ &\leq \frac{1}{16}\|u\|_2^2 + \frac{1}{4}\|\dot{u}\|^2 + C\|\dot{\alpha}\|_1^2.\end{aligned} \tag{6.13.4}$$

Finally, we set

$$\Lambda(t) = \mathcal{E}(t) + \varepsilon\big[\Phi(t) + \Theta(t) + 2\Psi(t)\big],$$

for ε small enough such that

$$\frac{1}{2}\mathcal{E}(t) \leq \Lambda(t) \leq C\mathcal{E}(t). \tag{6.13.5}$$

On account of (6.13.1)-(6.13.4), we have the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \frac{\varepsilon}{2}\big[\|u\|_2^2 + \|\dot{u}\|^2 + \|\alpha\|_1^2\big] + (1 - C\varepsilon)\|\dot{\alpha}\|_1^2 \leq 0.$$

Up to taking a smaller ε, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \omega\Lambda \leq 0$$

for some $\omega > 0$, and the Gronwall lemma entails

$$\Lambda(t) \leq \Lambda(0)\mathrm{e}^{-\omega t}.$$

Therefore, exploiting (6.13.5),

$$\mathcal{E}(t) \leq C\mathcal{E}(0)\mathrm{e}^{-\omega t},$$

as claimed. □

Remark 6.13.1. *The exponential stability of $U_1(t)$ for $\sigma = 1$ has been actually already obtained in [58] by means of semigroups techniques. The advantage of our approach (via explicit energy estimates) lies in the fact that the argument can be exported to the nonlinear case.*

Theorem 6.13.2. *If $\sigma < \frac{1}{2}$ the semigroup $U_1(t)$ is not exponentially stable.*

Proof. With reference to the abstract Lemma 1.2.2, the strategy consists in showing that condition (1.2.1) fails to hold. It is understood that in this proof $\mathbb{A}$ and $U_1(t)$ stand for the complexifications of the operator $\mathbb{A}$ and the semigroup $U_1(t)$, respectively. Denoting by

$$\lambda_n \to \infty$$

the increasing sequence of the (strictly positive) eigenvalues of A, and by $w_n \in H$ the corresponding normalized eigenvectors, we choose

$$\hat{z}_n = (0, w_n, 0, 0),$$

which satisfies by construction

$$\|\hat{z}_n\|_{\mathcal{H}} = 1.$$

For every $n \in \mathbb{N}$, we claim that the equation

$$\mathrm{i}\lambda_n z_n - \mathbb{A} z_n = \hat{z}_n$$

has a unique solution $z_n = (u_n, v_n, \alpha_n, \beta_n) \in \mathfrak{D}(\mathbb{A})$ such that

$$\lim_{n\to\infty} \|z_n\|_{\mathcal{H}} = \infty,$$

hence violating (1.2.1). Indeed, looking for a solution of the form

$$u_n = p_n w_n, \qquad v_n = q_n w_n, \qquad \alpha_n = a_n w_n, \qquad \beta_n = b_n w_n,$$

for some $p_n, q_n, a_n, b_n \in \mathbb{C}$, we obtain the system

$$\mathrm{i}\lambda_n p_n - q_n = 0, \tag{6.13.6}$$

$$\mathrm{i}\lambda_n q_n + \lambda_n^2 p_n - \lambda_n^\sigma b_n = 1, \tag{6.13.7}$$

$$\mathrm{i}\lambda_n a_n - b_n = 0, \tag{6.13.8}$$

$$\mathrm{i}\lambda_n b_n + \lambda_n a_n + \lambda_n b_n + \lambda_n^\sigma q_n = 0. \tag{6.13.9}$$

Substituting (6.13.6) into (6.13.7) we get

$$b_n = -\lambda_n^{-\sigma}.$$

Plugging then (6.13.8) into (6.13.9) we are led to

$$q_n = \lambda_n^{1-2\sigma} + \mathrm{i}\big(\lambda_n^{1-2\sigma} - \lambda_n^{-2\sigma}\big).$$

Recalling the assumption $\sigma < \frac{1}{2}$, we conclude that

$$|q_n| \geq \mathfrak{Re}(q_n) \to \infty.$$

Since

$$\|z_n\|_{\mathcal{H}} \geq \|v_n\|_H = |q_n|,$$

the claim follows. □

Remark 6.13.2. *Nonetheless, it is not hard to show that stability occurs for all values of σ, namely,*

$$\lim_{t\to\infty} \|U_1(t)x\|_{\mathcal{H}} = 0, \quad \forall x \in \mathcal{H}.$$

6.14 Decay Properties of the Semigroup $U_2(t)$

Contrary to the previous case, the presence of the memory prevents the exponential decay of the semigroup $U_2(t)$ for all σ, at least within a very mild decay assumption on the kernel.

Theorem 6.14.1. *The semigroup $U_2(t)$ is not exponentially stable provided that*

$$\lim_{s\to 0} \sqrt{s}\,\mu(s) = 0 \qquad \textit{if } \sigma \leq 1,$$
$$\lim_{s\to 0} s^{\frac{2\sigma-1}{2\sigma}}\,\mu(s) = 0 \qquad \textit{if } \sigma > 1.$$

Let us denote the Fourier transform of μ by

$$F(\omega) = \int_0^\infty \mu(s)\mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}s.$$

In order to prove the theorem, we will need a technical lemma from [71].

Lemma 6.14.1. *Given $\vartheta \in [0,1)$, assume that*

$$\lim_{s\to 0} s^{1-\vartheta}\mu(s) = 0.$$

Then

$$\lim_{\omega\to\infty} \omega^{\vartheta} F(\omega) = 0.$$

Proof of Theorem 6.14.1. In the same spirit of Theorem 6.13.2, we will show that condition (1.2.1) fails to hold. Again, $\mathbb{B}$ and $U_2(t)$ will denote the complexifications of $\mathbb{B}$ and $U_2(t)$, respectively, and

$$\lambda_n \to \infty$$

the increasing sequence of the (strictly positive) eigenvalues of A, with corresponding normalized eigenvectors $w_n \in H$. Denoting

$$\xi_n = \frac{1}{\sqrt{\lambda_n}}\,w_n,$$

we consider the vector

$$\hat{z}_n = (0,0,0,0,\xi_n)$$

with norm

$$\|\hat{z}_n\|_{\mathcal{V}} = \|\xi_n\|_{\mathcal{M}} = \sqrt{\varkappa}\,. \tag{6.14.1}$$

For every $n \in \mathbb{N}$, we study the equation

$$\mathrm{i}\omega_n z_n - \mathbb{B}z_n = \hat{z}_n,$$

for some $\omega_n \in \mathbb{R}$ to be suitably chosen, in the unknown $z_n = (u_n, v_n, \alpha_n, \beta_n, \eta_n)$. We look for a solution of the form

$$u_n = p_n w_n, \qquad v_n = q_n w_n, \qquad \alpha_n = a_n w_n, \qquad \beta_n = b_n w_n, \qquad \eta_n(s) = \phi_n(s) w_n,$$

for some $p_n, q_n, a_n, b_n \in \mathbb{C}$ and $\phi_n \in L^2_\mu(\mathbb{R}^+)$ with $\phi_n(0) = 0$. Componentwise, we draw the set of equations

$$\mathrm{i}\omega_n p_n - q_n = 0, \tag{6.14.2}$$

$$\mathrm{i}\omega_n q_n + p_n \lambda_n^2 - \lambda_n^\sigma b_n = 0, \tag{6.14.3}$$

$$\mathrm{i}\omega_n a_n - b_n = 0, \tag{6.14.4}$$

$$\mathrm{i}\omega_n b_n + a_n \lambda_n + \lambda_n \int_0^\infty \mu(s)\phi_n(s)\,\mathrm{d}s + q_n \lambda_n^\sigma = 0, \tag{6.14.5}$$

$$\mathrm{i}\omega_n \phi_n(s) + \phi_n'(s) - b_n = \frac{1}{\sqrt{\lambda_n}}. \tag{6.14.6}$$

Substituting (6.14.2) into (6.14.3) and (6.14.4) into (6.14.5) we get, respectively,

$$q_n = \frac{\mathrm{i}\omega_n \lambda_n^\sigma}{\lambda_n^2 - \omega_n^2} b_n \tag{6.14.7}$$

and

$$-\omega_n^2 b_n + \lambda_n b_n + \mathrm{i}\omega_n \lambda_n \int_0^\infty \mu(s)\phi_n(s)\,\mathrm{d}s + \mathrm{i}\omega_n q_n \lambda_n^\sigma = 0. \tag{6.14.8}$$

Besides, an integration of (6.14.6) with $\phi_n(0) = 0$ entails

$$\phi_n(s) = \frac{1}{\mathrm{i}\omega_n}\Big(b_n + \frac{1}{\sqrt{\lambda_n}}\Big)\big(1 - \mathrm{e}^{-\mathrm{i}\omega_n s}\big). \tag{6.14.9}$$

Plugging (6.14.7) and (6.14.9) into (6.14.8), we finally arrive at

$$\bigg(-\omega_n^2 + \lambda_n(1+\varkappa) - \frac{\omega_n^2 \lambda_n^{2\sigma}}{\lambda_n^2 - \omega_n^2}\bigg) b_n + \sqrt{\lambda_n}\big(\varkappa - F(\omega_n)\big) - \lambda_n F(\omega_n) b_n = 0. \tag{6.14.10}$$

At this point, we fix ω_n in such a way that

$$-\omega_n^2 + \lambda_n(1+\varkappa) - \frac{\omega_n^2 \lambda_n^{2\sigma}}{\lambda_n^2 - \omega_n^2} = 0.$$

This is possible if and only if the fourth order equation

$$\omega_n^4 - \omega_n^2\big(\lambda_n^2 + \lambda_n(1+\varkappa) + \lambda_n^{2\sigma}\big) + \lambda_n^3(1+\varkappa) = 0$$

admits a real solution. Indeed, for λ_n large enough, it is immediate to verify that there exists a positive solution

$$\omega_n \sim \begin{cases} \lambda_n & \text{if } \sigma < 1, \\ \sqrt{2}\,\lambda_n & \text{if } \sigma = 1, \\ \lambda_n^\sigma & \text{if } \sigma > 1. \end{cases} \tag{6.14.11}$$

Since $F(\omega_n) \to 0$, we obtain from (6.14.10) and (6.14.11)

$$b_n \sim \frac{\varkappa}{F(\omega_n)} \cdot \begin{cases} \omega_n^{-\frac{1}{2}} & \text{if } \sigma < 1, \\ \sqrt[4]{2}\,\omega_n^{-\frac{1}{2}} & \text{if } \sigma = 1, \\ \omega_n^{-\frac{1}{2\sigma}} & \text{if } \sigma > 1. \end{cases}$$

Using Lemma 6.14.1, we conclude that

$$\|z_n\|_{\mathcal{V}} \geq \|\beta_n\| = |b_n| \to \infty,$$

which, together with (6.14.1), yield the thesis. □

Remark 6.14.1. *As a matter of fact, the semigroup $U_2(t)$ turns out to be stable for all σ, except in the case of a very particular class of memory kernels μ, called resonant, for which the system exhibits trajectories with conserved energy (see e.g. [69]).*

CHAPTER 7

Caginalp Phase-Field Systems

7.1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following nonlinear system of Caginalp type in the unknowns $u = u(\boldsymbol{x}, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ and $\alpha = \alpha(\boldsymbol{x}, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$

$$\begin{cases} u_t - \Delta u + \phi(u) = \alpha_t, \\ \alpha_{tt} - \Delta \alpha_t - \Delta \alpha + g(\alpha) = -u_t, \end{cases} \tag{7.1.1}$$

supplemented with the initial data

$$\begin{cases} u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \\ \alpha(\boldsymbol{x}, 0) = \alpha_0(\boldsymbol{x}), \\ \alpha_t(\boldsymbol{x}, 0) = \alpha_1(\boldsymbol{x}), \end{cases}$$

and Dirichlet boundary conditions,

$$\begin{cases} u(\boldsymbol{x}, t)_{|\boldsymbol{x} \in \partial\Omega} = 0, \\ \alpha(\boldsymbol{x}, t)_{|\boldsymbol{x} \in \partial\Omega} = 0. \end{cases}$$

As detailed in the introduction of the thesis, system (7.1.1) serves as a model in the description of type III phase-field evolutions, with order parameter u and thermal displacement α.

Assumptions on the nonlinearities. *We assume that $\phi, g \in \mathcal{F}$, where $\mathcal{F}$ is the class of functions*

$$\mathcal{F} = \{ f \in \mathcal{C}^1(\mathbb{R}) : f(0) = 0 \ \textit{ and } \ |f'(s)| \le c + c|s|^4, \ \textit{for all } s \in \mathbb{R} \textit{ and some } c \ge 0 \}.$$

Furthermore, we require the following dissipation conditions (in the sense that they allow to prove the dissipativity of the associated semigroup, i.e., the existence of bounded absorbing sets, see below):

$$\liminf_{|s| \to +\infty} \frac{\phi(s)}{s} > -\lambda_1, \tag{7.1.2}$$

$$\liminf_{|s| \to +\infty} g'(s) > -\lambda_1, \tag{7.1.3}$$

where $\lambda_1 > 0$ is the first eigenvalue of the linear operator $-\Delta$ on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H^1_0(\Omega)$. Finally, we assume that

$$\phi'(s) \ge -c, \quad \textit{for every } s \in \mathbb{R}. \tag{7.1.4}$$

Our aim in this chapter is to study the asymptotic behavior of the solution semigroup associated to (7.1.1). The first results concerning well-posedness and existence of global attractors for this model have been obtained in [63] in the case $g = 0$, under the growth restriction

$$|\phi'(s)| \le c + c|s|^r, \quad r = 2,$$

and with (7.1.2) replaced by the less general condition

$$\phi(s)s \ge \int_0^s \phi(y)\mathrm{d}y \ge -c.$$

Here, we develop the analysis under assumption (7.1.2), in the case of two nonlinearities ϕ and g, both of critical order $r = 4$, proving the existence of a global attractor of optimal regularity.

After introducing some preliminaries in Section 7.2, we prove the well-posedness in Section 7.3, yielding a solution semigroup (dynamical system) acting on the natural weak energy space. In Sections 7.4-7.5, we turn to the study of the dissipativity of the semigroup, characterized by the existence of regular absorbing sets. The final Section 7.6 is devoted to the main result concerning existence and regularity of the global attractor.

7.2 Preliminaries

7.2.1 Functional setting

Let A be the strictly positive Dirichlet operator

$$A = -\Delta \qquad \text{with domain} \qquad \mathfrak{D}(A) = H^2(\Omega) \cap H^1_0(\Omega) \Subset L^2(\Omega).$$

Analogously to Chapter 2, for $\sigma \in \mathbb{R}$ we introduce the scale of (compactly) nested Hilbert spaces

$$\mathrm{H}_\sigma = \mathfrak{D}(A^{\frac{\sigma}{2}}),$$

with inner product

$$\langle w, v\rangle_\sigma = \langle A^{\frac{\sigma}{2}} w, A^{\frac{\sigma}{2}} v\rangle$$

and norm

$$\|w\|_\sigma = \|A^{\frac{\sigma}{2}} w\|.$$

Again, for $\sigma > 0$, it is understood that $\mathrm{H}_{-\sigma}$ denotes the completion of the domain, so that $\mathrm{H}_{-\sigma}$ is the dual space of H_σ. We omit the subscript σ whenever it equals zero and the symbol $\langle\cdot,\cdot\rangle$ also stands for the duality product between H_σ and $\mathrm{H}_{-\sigma}$. In particular,

$$\mathrm{H}_2 = H^2(\Omega) \cap H_0^1(\Omega) \Subset \mathrm{H}_1 = H_0^1(\Omega) \Subset \mathrm{H} = L^2(\Omega) \Subset \mathrm{H}_{-1} = H^{-1}(\Omega).$$

We also define the energy spaces

$$\mathcal{H}_\sigma = \mathrm{H}_{\sigma+1} \times \mathrm{H}_{\sigma+1} \times \mathrm{H}_\sigma,$$

endowed with the standard product norms

$$\|\{w_1, w_2, w_3\}\|^2_{\mathcal{H}_\sigma} = \|w_1\|^2_{\sigma+1} + \|w_2\|^2_{\sigma+1} + \|w_3\|^2_\sigma.$$

7.2.2 Technical lemmas

We need the following Gronwall-type lemma.

Lemma 7.2.1. *Let X be a Banach space and let $\mathcal{Z} \subset \mathcal{C}(\mathbb{R}^+, X)$. Let $\mathcal{E} : X \to \mathbb{R}$ be a function such that*

$$\inf_{t\in\mathbb{R}^+} \mathcal{E}(z(t)) \geq -m, \qquad \mathcal{E}(z(0)) \leq M,$$

for some $m, M \geq 0$ and every $z \in \mathcal{Z}$. In addition, assume that, for every $z \in \mathcal{Z}$, the function $t \mapsto \mathcal{E}(z(t))$ is continuously differentiable and satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(z(t)) + \mu\|z(t)\|^2_X \leq k,$$

for some $\mu > 0$ and $k > 0$, both independent of $z \in \mathcal{Z}$. Then, there exists $t_0 = \frac{m+M}{k} \geq 0$ such that

$$\mathcal{E}(z(t)) \leq \sup_{\zeta\in X} \Big\{\mathcal{E}(\zeta) : \mu\|\zeta\|^2_X \leq 2k\Big\}, \qquad \forall t \geq t_0.$$

The proof can be found in [3]. We will also exploit the Uniform Gronwall Lemma (see, e.g., [91], Section 1.1.3) which reads as follows.

Lemma 7.2.2. *Let Λ_0 be an absolutely continuous nonnegative function and Λ_1, Λ_2 be two nonnegative functions satisfying, almost everywhere in $\mathbb{R}^+$, the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0 \leq \Lambda_0\Lambda_1 + \Lambda_2.$$

Assume also that

$$\sup_{t\geq 0} \int_t^{t+1} \Lambda_\imath(\tau)d\tau \leq m_\imath, \qquad \imath = 0, 1, 2,$$

for some positive constants $m_\imath$. Then, there exists $c > 0$ depending only on m_i such that

$$\Lambda_0(t+1) \leq c, \quad \forall t \geq 0.$$

We finally recall a useful decomposition lemma stated in Chapter 2 (Lemma 2.3.1).

Lemma 7.2.3. *Let $g \in \mathcal{F}$ satisfy* (7.1.3). *Then, for every $\lambda < \lambda_1$ sufficiently close to λ_1, the decomposition*

$$g(s) = g_0(s) - \lambda s + g_1(s)$$

holds for some $g_0, g_1 \in \mathcal{C}^1(\mathbb{R})$ with the following properties:

- *g_1 is compactly supported with $g_1(0) = 0$;*
- *g_0 vanishes inside $[-1, 1]$ and fulfills, for some $c \geq 0$ and every $s \in \mathbb{R}$, the bounds*

$$0 \leq g_0'(s) \leq c|s|^4.$$

Defining the following functionals on H_1:

$$\Phi(w) = 2\int_\Omega \int_0^{w(\boldsymbol{x})} \phi(y)\,\mathrm{d}y\mathrm{d}\boldsymbol{x}$$

and

$$G_0(w) = 2\int_\Omega \int_0^{w(\boldsymbol{x})} g_0(y)\,\mathrm{d}y\mathrm{d}\boldsymbol{x},$$

by (7.1.2) and Lemma 7.2.3, the inequalities

$$\Phi(w) \geq -(1-\nu)\|w\|_1^2 - c_1, \tag{7.2.1}$$

$$\langle \phi(w), w\rangle \geq -(1-\nu)\|w\|_1^2 - c_1 \tag{7.2.2}$$

and

$$0 \leq G_0(w) \leq 2\langle g_0(w), w\rangle \tag{7.2.3}$$

hold for some $0 < \nu < 1$ and $c_1 \geq 0$.

Remark 7.2.1. *Along the chapter, we will perform several formal estimates which can be justified within a proper Galerkin scheme. Moreover, the Hölder, Young and Poincaré inequalities will be used without explicit mention, as well as the Sobolev embedding*

$$\mathrm{H}_1 \subset L^6(\Omega).$$

Remark 7.2.2. *We stress that we study the model in the meaningful physical dimension $N = 3$. Nonetheless, all the results are true in lower dimensions and can also be proved for $N > 3$: accordingly, one has to assume the correct critical growth on the nonlinearities ϕ' and g' (namely, $\frac{4}{N-2}$ instead of 4) and to exploit the corresponding Sobolev embeddings.*

7.3 The Solution Semigroup

We start by giving a suitable notion of weak solution.

Definition 7.3.1. *Given $T > 0$, we call weak solution to* (7.1.1) *on $[0, T]$ a pair (u, α),*

$$u \in \mathcal{C}([0,T], \mathrm{H}_1) \cap \mathcal{C}^1([0,T], \mathrm{H}_{-1}),$$
$$\alpha \in \mathcal{C}([0,T], \mathrm{H}_1) \cap \mathcal{C}^1([0,T], \mathrm{H}) \cap \mathcal{C}^2([0,T], \mathrm{H}_{-1}) \cap W^{1,2}(0,T;\mathrm{H}_1),$$

satisfying, for almost every $t \in [0,T]$ and every test function $(v, \theta) \in \mathrm{H}_1 \times \mathrm{H}_1$, the equalities

$$\langle u_t, v\rangle + \langle u, v\rangle_1 + \langle \phi(u), v\rangle = \langle \alpha_t, v\rangle,$$
$$\langle \alpha_{tt}, \theta\rangle + \langle \alpha_t, \theta\rangle_1 + \langle \alpha, \theta\rangle_1 + \langle g(\alpha), \theta\rangle = -\langle u_t, \theta\rangle.$$

Theorem 7.3.1. *For every $T > 0$ and every $z_0 = \{a_1, a_2, a_3\} \in \mathcal{H}$, system* (7.1.1) *admits a unique weak solution (u, α) on $[0, T]$ such that*

$$\{u(0), \alpha(0), \alpha_t(0)\} = z_0.$$

In addition, given $R > 0$, for any pair of initial data $z_1, z_2 \in \mathcal{H}$ such that $\|z_1\|_{\mathcal{H}} \le R$ and $\|z_2\|_{\mathcal{H}} \le R$, the difference $(\bar{u}, \bar{\alpha}) = (u^1 - u^2, \alpha^1 - \alpha^2)$ of the corresponding solutions satisfies

$$\|\{\bar{u}, \bar{\alpha}, \bar{\alpha}_t\}\|_{L^\infty(0,T;\mathcal{H})} \le K\mathrm{e}^{KT}\|z_1 - z_2\|_{\mathcal{H}}, \tag{7.3.1}$$

for some constant $K = K(R) \ge 0$.

Proof. Concerning existence, this follows from a usual Galerkin procedure, considering the solutions (u_n, α_n) to n-dimensional approximating problems. Arguing as in the forthcoming Theorem 7.4.1 and Corollary 7.4.1, with (u_n, α_n) in place of (u, α), we deduce the boundedness of

$$\begin{aligned}
&u_n \quad \text{in } L^\infty(0,T;\mathrm{H}_1) \cap L^2(0,T;\mathrm{H}_2),\\
&\partial_t u_n \quad \text{in } L^2(0,T;\mathrm{H}),\\
&\alpha_n \quad \text{in } L^\infty(0,T;\mathrm{H}_1),\\
&\partial_t \alpha_n \quad \text{in } L^\infty(0,T;\mathrm{H}) \cap L^2(0,T;\mathrm{H}_1),
\end{aligned}$$

and, calling $\Omega_T = \Omega \times (0,T)$, those of $\phi(u_n)$ and $g(\alpha_n)$ in $L^{\frac{6}{5}}(\Omega_T)$. Hence, we can extract (weakly or weakly-$*$) convergent subsequences to some limit (u, α). Proving that this limit solves the original problem is a standard matter, as well as the proof of the regularity

$$u \in \mathcal{C}([0,T], \mathrm{H}_1).$$

Moreover, arguing as in Chapter 2 (proof of Theorem 2.4.1), we obtain

$$\alpha \in \mathcal{C}([0,T], \mathrm{H}_1) \cap \mathcal{C}^1([0,T], \mathrm{H}).$$

The required regularity in H_{-1} of both u and α is now immediately seen by comparison in the equations. It remains to prove (7.3.1). Setting $\bar{u} = u^1 - u^2$ and $\bar{\alpha} = \alpha^1 - \alpha^2$, the difference system reads

$$\begin{cases} \bar{u}_t - \Delta \bar{u} + \phi(u^1) - \phi(u^2) = \bar{\alpha}_t, \\ \bar{\alpha}_{tt} - \Delta \bar{\alpha}_t - \Delta \bar{\alpha} + g(\alpha^1) - g(\alpha^2) = -\bar{u}_t. \end{cases}$$

Along the proof, the *generic* constant $K \geq 0$ may depend on R. Multiplying the first equation by $2\bar{u}_t$, the second one by $2\bar{\alpha}_t$ and summing up, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\|\bar{u}_t\|^2 + 2\|\bar{\alpha}_t\|_1^2 = 2\langle \phi(u^2) - \phi(u^1), \bar{u}_t \rangle + 2\langle g(\alpha^2) - g(\alpha^1), \bar{\alpha}_t \rangle, \tag{7.3.2}$$

where

$$\Lambda = \|\bar{u}\|_1^2 + \|\bar{\alpha}\|_1^2 + \|\bar{\alpha}_t\|^2.$$

The growth of g and Corollary 7.4.1 entail

$$2\langle g(\alpha^2) - g(\alpha^1), \bar{\alpha}_t \rangle \leq K\|\bar{\alpha}\|_1 \|\bar{\alpha}_t\|_1 \leq \|\bar{\alpha}_t\|_1^2 + K\|\bar{\alpha}\|_1^2.$$

Using the embedding $\mathrm{H}_{5/4} \subset L^{12}(\Omega)$, we have

$$2\langle \phi(u^2) - \phi(u^1), \bar{u}_t \rangle \leq K\|\bar{u}\|_1 \|\bar{u}_t\| \big[1 + \|u^2\|_{5/4}^4 + \|u^1\|_{5/4}^4\big].$$

Furthermore, a classical interpolation inequality yields

$$\|u^\imath\|_{5/4}^4 \leq K\|u^\imath\|_1^3 \|u^\imath\|_2. \tag{7.3.3}$$

Thus, accounting for Corollary 7.4.1, the right-hand side of (7.3.2) is controlled by

$$\|\bar{u}_t\|^2 + \|\bar{\alpha}_t\|_1^2 + K\big[1 + \|u^2\|_2^2 + \|u^1\|_2^2\big]\Lambda$$

and we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda \leq K\big[1 + \|u^2\|_2^2 + \|u^1\|_2^2\big]\Lambda.$$

An application of the Gronwall lemma, along with Corollary 7.4.1, completes the proof. □

In light of Theorem 7.3.1, system (7.1.1) generates a dynamical system $S(t)$ on the phase space $\mathcal{H}$, defined by

$$S(t)z_0 = z(t) = \{u(t), \alpha(t), \alpha_t(t)\},$$

where $(u(t), \alpha(t))$ is the solution at time t with initial datum $z_0 = \{u(0), \alpha(0), \alpha_t(0)\} \in \mathcal{H}$.

7.4 A Priori Estimates and Dissipativity

As already said, the presence of some dissipation mechanism in the dynamical system $S(t)$ reflects into the existence of an absorbing set.

Theorem 7.4.1. *There exists a constant* $R_0 > 0$ *with the following property: given any* $R \geq 0$*, there exists* $t_{\rm e} = t_{\rm e}(R) \geq 0$ *such that, whenever*

$$\|z_0\|_{\mathcal{H}} \leq R,$$

the inequality

$$\|S(t)z_0\|_{\mathcal{H}} \leq R_0$$

holds for every $t \geq t_{\rm e}$.

In order to simplify the calculations, according to Lemma 7.2.3, we fix λ sufficiently close to λ_1 and rewrite the nonlinear Caginalp system (7.1.1) as

$$\begin{cases} u_t - \Delta u + \phi(u) = \alpha_t, \\ \alpha_{tt} - \Delta\alpha_t - \Delta\alpha + g_0(\alpha) - \lambda\alpha = -u_t - q, \end{cases} \tag{7.4.1}$$

with $q = g_1(\alpha)$. In addition, given any solution (u, α) to (7.4.1), we introduce the equivalent inner product $(\cdot,\cdot)_1$ on H_1 defined by

$$(w, v)_1 = \langle w, v\rangle_1 - \lambda\langle w, v\rangle,$$

with induced norm $|\cdot|_1$. The corresponding energy reads

$$E(t) = \|u(t)\|_1^2 + |\alpha(t)|_1^2 + \|\alpha_t(t)\|^2.$$

Proof of Theorem 7.4.1. Let $C \geq 0$ be a *generic* constant independent of R and set $S(t)z_0 = z(t)$. Due to (7.2.1)-(7.2.3) and the growth of ϕ and g_0, the functional

$$\begin{aligned} \mathcal{E}(z(t)) =& \|u(t)\|_1^2 + \varepsilon\|u(t)\|^2 + \|\alpha_t(t)\|^2 + |\alpha(t)|_1^2 \\ &+\varepsilon\|\alpha(t)\|_1^2 + 2\varepsilon\langle\alpha_t(t), \alpha(t)\rangle + \Phi(u(t)) + G_0(\alpha(t)) \end{aligned}$$

fulfills, for $\varepsilon > 0$ small,

$$\mathcal{E}(z(t)) \geq \nu E(t) - C \tag{7.4.2}$$

and

$$\mathcal{E}(z(t)) \leq CE(t)\big[1 + E(t)^2\big]. \tag{7.4.3}$$

Multiplying the first equation of (7.4.1) by $2u_t + 2\varepsilon u$, the second one by $2\alpha_t + 2\varepsilon\alpha$ and summing up, we infer

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} &+ 2\|u_t\|^2 + 2\varepsilon\|u\|_1^2 + 2\|\alpha_t\|_1^2 - 2\varepsilon\|\alpha_t\|^2 \\ &+ 2\varepsilon|\alpha|_1^2 + 2\varepsilon\langle\phi(u), u\rangle + 2\varepsilon\langle g_0(\alpha), \alpha\rangle \\ &= 2\varepsilon\langle\alpha_t, u\rangle - 2\varepsilon\langle u_t, \alpha\rangle - \langle q, 2\alpha_t + 2\varepsilon\alpha\rangle. \end{aligned}$$

Thus, by (7.2.2) and (7.2.3),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}+&\|u_t\|^2+\frac{3}{2}\|\alpha_t\|_1^2+2\varepsilon\nu\|u\|_1^2+2\varepsilon|\alpha|_1^2\\ &\leq 2\varepsilon\langle\alpha_t,u\rangle-2\varepsilon\langle u_t,\alpha\rangle-\langle q,2\alpha_t+2\varepsilon\alpha\rangle+2\varepsilon c_1,\end{aligned}$$

for $\varepsilon>0$ small enough. Possibly reducing ε, we have

$$-\langle q,2\alpha_t+2\varepsilon\alpha\rangle\leq\frac{1}{4}\|\alpha_t\|_1^2+\frac{\varepsilon}{4}|\alpha|_1^2+C$$

and

$$2\varepsilon\langle\alpha_t,u\rangle-2\varepsilon\langle u_t,\alpha\rangle\leq\frac{1}{2}\|u_t\|^2+\frac{1}{4}\|\alpha_t\|_1^2+\varepsilon\nu\|u\|_1^2+\frac{\varepsilon}{4}|\alpha|_1^2.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}+\mu\big[\|u\|_1^2+\|\alpha\|_1^2+\|\alpha_t\|^2\big]+\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\alpha_t\|_1^2\leq C,$$

for some $\mu>0$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(z(t))+\mu\|z(t)\|_{\mathcal{H}}^2\leq C.$$

Applying Lemma 7.2.1, together with (7.4.2) and (7.4.3), we finish the proof of the theorem. □

Corollary 7.4.1. *Given any $R\geq 0$, there exists a constant $K=K(R)\geq 0$ such that, whenever $\|z_0\|\leq R$, the corresponding solution $S(t)z_0$ fulfills*

$$\|S(t)z_0\|_{\mathcal{H}}\leq K,\quad\forall t\geq 0,\tag{7.4.4}$$

and

$$\int_t^T\big[\|u_t(\tau)\|^2+\|\alpha_t(\tau)\|_1^2+\|u(\tau)\|_2^2\big]\,\mathrm{d}\tau\leq K+K(T-t),\ \ \forall\ T>t\geq 0.$$

Proof. Along the proof, $K\geq 0$ denotes a *generic* constant depending on R. Recasting word by word the proof of Theorem 7.4.1 with $\varepsilon=0$, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\|u\|_1^2+\|\alpha_t\|^2+|\alpha|_1^2+\Phi(u)+G_0(\alpha)\big]+\|u_t\|^2+\|\alpha_t\|_1^2\leq C,\tag{7.4.5}$$

for some $C\geq 0$ (independent of R). Fix any $t\leq t_{\mathrm{e}}(R)$ and integrate the last inequality over $[0,t]$ to find, in light of (7.2.1)-(7.2.3),

$$E(t)\leq K.$$

Applying Theorem 7.4.1, we conclude that $\|S(t)z_0\|_{\mathcal{H}}\leq K$ for every $t\geq 0$. To prove the remaining bounds, we integrate inequality (7.4.5) over $[t,T]$ and obtain the required estimates for u_t in $L^2(t,T;\mathrm{H})$ and α_t in $L^2(t,T;\mathrm{H}_1)$. The product in H of the first equation of (7.4.1) by $-2\Delta u$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_1^2+2\|u\|_2^2+2\langle\phi'(u)\nabla u,\nabla u\rangle=-2\langle\alpha_t,\Delta u\rangle.$$

Using (7.1.4) and (7.4.4), we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_1^2 + \|u\|_2^2 \le K$$

and, integrating over $[t,T]$, the claim is proved. □

According to (7.4.4), the set

$$\mathbb{B}_0 = \{z_0 \in \mathcal{H} : \|S(t)z_0\|_{\mathcal{H}} \le K(R_0), \ \ \forall t \ge 0\}$$

is an invariant absorbing set for the semigroup $S(t)$.

Remark 7.4.1. *It is worth noticing that the bound from below on ϕ', assumed in* (7.1.4)*, is not needed for the existence of the absorbing set $\mathbb{B}_0$.*

Remark 7.4.2. *All the results obtained in this section are a priori estimates and do not make use of Theorem 7.3.1.*

7.5 Further Dissipativity

A deeper analysis of the dissipation mechanism involved in the system allows to discover a partial regularization effect on the solutions.

Theorem 7.5.1. *There exists an invariant absorbing set $\mathbb{B}$ such that*

$$\sup_{t\ge 0}\sup_{z_0\in\mathbb{B}} \Big[\|u(t)\|_2 + \|u_t(t)\|_1 + \|\alpha_t(t)\|_1 + \|\alpha_{tt}(t)\|\Big] < +\infty.$$

The proof of Theorem 7.5.1 can be divided into two steps. First, we obtain the regularization of u and α_t.

Lemma 7.5.1. *There exists an invariant absorbing set $\mathbb{B}_1$ such that*

$$\sup_{t\ge 0}\sup_{z_0\in\mathbb{B}_1} \Big[\|u(t)\|_2 + \|\alpha_t(t)\|_1\Big] < +\infty.$$

Proof. In what follows, $C \ge 0$ denotes a *generic* constant depending only on the invariant absorbing set $\mathbb{B}_0$ constructed in the previous section. Consider an initial datum $z_0 \in \mathbb{B}_0$. Owing to (7.1.4) and the growth of g_0, the functional

$$\Lambda = \Lambda(S(t)z_0) = \|u\|_2^2 + \|u\|_1^2 + \|\alpha_t\|_1^2 + 2(\alpha,\alpha_t)_1 + 2\langle g_0(\alpha),\alpha_t\rangle + K_0$$

satisfies, for $K_0 = K_0(\mathbb{B}_0) > 0$ sufficiently large,

$$\|u\|_2^2 + \|\alpha_t\|_1^2 \le 2\Lambda \le C\big[1 + \|u\|_2^2 + \|\alpha_t\|_1^2\big]. \tag{7.5.1}$$

Multiplying the first equation of (7.4.1) by $-2\Delta(u_t+u)$, the second one by $2\alpha_{tt}$ and summing up, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Lambda &+ 2\|u\|_2^2 + 2\|u_t\|_1^2 + 2\|\alpha_{tt}\|^2 + 2\langle\phi'(u)\nabla u,\nabla u\rangle\\ &= 2|\alpha_t|_1^2 - 2\langle\phi'(u)\nabla u,\nabla u_t\rangle + 2\langle g_0'(\alpha)\alpha_t,\alpha_t\rangle\\ &\quad - 2\langle\alpha_t,\Delta u_t\rangle - 2\langle\alpha_t,\Delta u\rangle - 2\langle u_t,\alpha_{tt}\rangle - 2\langle q,\alpha_{tt}\rangle.\end{aligned}$$

By (7.1.4), we have

$$2\langle \phi'(u)\nabla u, \nabla u\rangle \geq -C,$$

while, by the growth assumption on ϕ and g_0, exploiting the embedding $\mathrm{H}_{5/4} \subset L^{12}(\Omega)$ and interpolation (see inequality (7.3.3)),

$$\begin{aligned} -2\langle \phi'(u)\nabla u, \nabla u_t\rangle &\leq C\|u_t\|_1 \|u\|_2 \big[1 + \|u\|_{5/4}^4\big] \\ &\leq \|u_t\|_1^2 + C\big[1 + \|u\|_2^2\big]\Lambda \end{aligned}$$

and

$$2\langle g_0'(\alpha)\alpha_t, \alpha_t\rangle \leq C\|\alpha_t\|_1^2 \leq C\Lambda.$$

Next, noting that

$$-2\langle u_t, \alpha_{tt}\rangle - 2\langle q, \alpha_{tt}\rangle \leq C\big[1 + \|u_t\|^2\big] + \|\alpha_{tt}\|^2$$

and

$$2|\alpha_t|_1^2 - 2\langle \alpha_t, \Delta u_t\rangle - 2\langle \alpha_t, \Delta u\rangle \leq C\Lambda + \frac{1}{2}\|u\|_2^2 + \frac{1}{2}\|u_t\|_1^2 + C,$$

we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \frac{1}{2}\|u_t\|_1^2 + \frac{1}{2}\|\alpha_{tt}\|^2 \leq C\big[1 + \|u\|_2^2\big]\Lambda + C\big[1 + \|u_t\|^2\big]. \tag{7.5.2}$$

Hence, setting

$$\Lambda_0(t) = \frac{t}{t+1}\Lambda(S(t)z_0),$$

elementary computations provide

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0(t) \leq h(t)\Lambda_0(t) + k(t),$$

with

$$h(t) = C\big[1 + \|u(t)\|_2^2\big]$$

and

$$k(t) = \frac{1}{[t+1]^2}\Lambda(S(t)z_0) + C\left[\frac{t}{t+1}\big[1 + \|u_t(t)\|^2\big]\right].$$

Since, by Corollary 7.4.1 and (7.5.1), there holds

$$\sup_{t\geq 0}\int_t^{t+1}\big[\Lambda_0(\tau) + h(\tau) + k(\tau)\big]\,\mathrm{d}\tau \leq C,$$

applying Lemma 7.2.2, we have

$$\Lambda_0(t+1) \leq C, \quad \forall t \geq 0,$$

and, in particular,

$$\Lambda(S(1)z_0) \leq C.$$

The set

$$\mathbb{B}_1 = S(1)\mathbb{B}_0 \subset \mathbb{B}_0,$$

is clearly absorbing and invariant. Moreover,

$$\sup_{t\geq 0}\sup_{z_0\in\mathbb{B}_1}\Lambda(S(t)z_0)=\sup_{t\geq 0}\sup_{z_0\in\mathbb{B}_0}\Lambda(S(t+1)z_0)\leq\sup_{z_0\in\mathbb{B}_0}\Lambda(S(1)z_0)\leq C,$$

which, along with (7.5.1), establishes the desired bound

$$\sup_{t\geq 0}\sup_{z_0\in\mathbb{B}_1}\big[\|u(t)\|_2+\|\alpha_t(t)\|_1\big]\leq C.$$

$\square$

Corollary 7.5.1. *We have the uniform estimate*

$$\sup_{t\geq 0}\sup_{z_0\in\mathbb{B}_1}\Big[\|u_t(t)\|+\int_t^{t+1}\big[\|u_t(\tau)\|_1^2+\|\alpha_{tt}(\tau)\|^2\big]\,\mathrm{d}\tau\Big]<+\infty.$$

Proof. The control of u_t in $L^\infty(\mathbb{R}^+;\mathrm{H})$ immediately follows by comparison in the first equation of (7.4.1). Integrating (7.5.2) over $[t,t+1]$, we finish the proof. $\square$

Proof of Theorem 7.5.1. Take an initial datum $z_0\in\mathbb{B}_1$ and consider a *generic* constant $C\geq 0$ depending only on $\mathbb{B}_1$. Differentiating system (7.4.1) with respect to time, we obtain

$$\begin{cases}u_{tt}-\Delta u_t+\phi'(u)u_t=\alpha_{tt},\\ \alpha_{ttt}-\Delta\alpha_{tt}-\Delta\alpha_t+g_0'(\alpha)\alpha_t-\lambda\alpha_t=-u_{tt}-q_t.\end{cases}$$

Multiplying the first equation by $2u_{tt}$, the second one by $2\alpha_{tt}$ and summing up, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda+2\|u_{tt}\|^2+2\|\alpha_{tt}\|_1^2=-2\langle\phi'(u)u_t,u_{tt}\rangle-2\langle g_0'(\alpha)\alpha_t,\alpha_{tt}\rangle-2\langle q_t,\alpha_{tt}\rangle,$$

where

$$\Lambda=\Lambda(S(t)z_0)=\|u_t\|_1^2+\|\alpha_{tt}\|^2+|\alpha_t|_1^2.$$

Due to the growth of ϕ and the Agmon inequality,

$$-2\langle\phi'(u)u_t,u_{tt}\rangle\leq C\|u_{tt}\|\leq\|u_{tt}\|^2+C,$$

while, using the growth of g_0,

$$-2\langle g_0'(\alpha)\alpha_t,\alpha_{tt}\rangle-2\langle q_t,\alpha_{tt}\rangle\leq C\|\alpha_{tt}\|_1\leq\|\alpha_{tt}\|_1^2+C.$$

Summarizing, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda+\|u_{tt}\|^2+\|\alpha_{tt}\|_1^2\leq C.$$

At this point, multiplying both terms of the above inequality by $t\in[0,1]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[t\Lambda(S(t)z_0)\big]\leq\Lambda(S(t)z_0)+C.$$

By Lemma 7.5.1 and Corollary 7.5.1,

$$\int_0^1\Lambda(S(t)z_0)\,\mathrm{d}t\leq C,$$

hence, integrating over $[0,1]$,

$$\Lambda(S(1)z_0) \le C.$$

Setting

$$\mathbb{B} = S(1)\mathbb{B}_1 \subset \mathbb{B}_1,$$

which is absorbing and invariant, we conclude that

$$\sup_{t\ge 0}\sup_{z_0\in\mathbb{B}} \Lambda(S(t)z_0) = \sup_{t\ge 0}\sup_{z_0\in\mathbb{B}_1} \Lambda(S(t+1)z_0) \le \sup_{z_0\in\mathbb{B}_1} \Lambda(S(1)z_0) \le C.$$

□

7.6 The Global Attractor

It follows from Theorem 7.5.1 that $u(t)$ regularizes in finite time. Unfortunately, this cannot occur for $\alpha(t)$, preventing the existence of a compact absorbing set for $S(t)$. Nonetheless, it is still true that there exists a compact attracting set for the semigroup. More precisely, our main result is the existence of a compact set which is exponentially attracting in $\mathcal{H}$ for the semigroup.

Theorem 7.6.1. *There exist three strictly positive constants ϱ, C_1, ω_1 and a closed ball $\mathcal{B}_{\mathcal{H}_1}(\varrho)$ of $\mathcal{H}_1$ such that*

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathbb{B}, \mathcal{B}_{\mathcal{H}_1}(\varrho)) \le C_1 \mathrm{e}^{-\omega_1 t}, \quad t \ge 0.$$

As a corollary we have existence of the global attractor $\mathbb{A}$ for $S(t)$. Besides, since the global attractor is contained in any compact attracting set, it satisfies $\|\mathbb{A}\|_{\mathcal{H}_1} \le \varrho$.

Corollary 7.6.1. *The semigroup $S(t) : \mathcal{H} \to \mathcal{H}$ possesses a (connected) global attractor $\mathbb{A}$ which is bounded in $\mathcal{H}_1$.*

In light of Theorem 7.5.1, which provides a regularization of $u(t)$ in finite time, the proof of Theorem 7.6.1 is an immediate consequence of the following lemma.

Lemma 7.6.1. *There exists a closed ball B in the space $\mathrm{H}_2 \times \mathrm{H}_1$ such that*

$$\sup_{z_0\in\mathbb{B}} \boldsymbol{\delta}_{\mathrm{H}_1\times\mathrm{H}}\big(\{\alpha(t), \alpha_t(t)\}, B\big) \le C_0 \mathrm{e}^{-t}, \quad t \ge 0, \tag{7.6.1}$$

for some positive constant C_0 depending only on $\mathbb{B}$.

Proof. We exploit the "parabolic" approach developed in [73]. Accordingly, we take an initial datum $z_0 = \{a_1, a_2, a_3\} \in \mathbb{B}$ and we write $\alpha = \eta + \zeta$, with

$$\begin{cases} -\Delta\eta_t - \Delta\eta + g_0(\alpha) - g_0(\zeta) = 0, \\ \eta(0) = a_2, \\ \eta_t(0) = a_3, \end{cases} \tag{7.6.2}$$

and

$$\begin{cases} -\Delta\zeta_t - \Delta\zeta + g_0(\zeta) = h, \\ \zeta(0) = 0, \\ \zeta_t(0) = 0, \end{cases} \tag{7.6.3}$$

where, by force of Theorem 7.5.1,

$$h = \lambda\alpha - u_t - q - \alpha_{tt} \in L^\infty(\mathbb{R}^+; \mathrm{H}).$$

In what follows, the *generic* constant $C \geq 0$ only depends on $\mathbb{B}$. Multiplying (7.6.2) by 2η, from the monotonicity of g_0, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_1^2 + 2\|\eta\|_1^2 \leq 0$$

and the Gronwall lemma entails

$$\|\eta(t)\|_1^2 \leq \|\eta(0)\|_1^2\,\mathrm{e}^{-2t}, \quad \forall t \geq 0. \tag{7.6.4}$$

Then, we multiply (7.6.3) by $-2\Delta\zeta$. Using again the monotonicity of g_0, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta\|_2^2 + 2\|\zeta\|_2^2 \leq -2\langle h, \Delta\zeta\rangle \leq C\|\zeta\|_2 \leq C + \|\zeta\|_2^2.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta\|_2^2 + \|\zeta\|_2^2 \leq C$$

and the Gronwall lemma gives the uniform bound

$$\|\zeta(t)\|_2^2 \leq C. \tag{7.6.5}$$

Collecting (7.6.4), (7.6.5) and Theorem 7.5.1, we see that, for every initial datum $z_0 \in \mathbb{B}$,

$$\|\eta(t)\|_1 \leq C\mathrm{e}^{-t}, \qquad \|\zeta(t)\|_2 \leq C, \qquad \|\alpha_t(t)\|_1 \leq C.$$

Hence, calling B the ball of $\mathrm{H}_2 \times \mathrm{H}_1$ of radius $C\sqrt{2}$, we complete the proof. □

Remark 7.6.1. *The existence of a global attractor bounded in $\mathcal{H}_1$ for the semigroup $S(t)$ can also be proved within the weaker dissipativity assumption*

$$\liminf_{|s|\to\infty} \frac{g(s)}{s} > -\lambda_1.$$

In that case, the proof of Theorem 7.6.1 is indeed more complicated and requires an abstract result from [21] (see Theorem A.4 in the appendix). We stress out that, contrary to Lemma 7.6.1 in which the attraction property for α is obtained separately, the alternative technique needs a suitable splitting of the whole semigroup. The interested reader can find the proper decomposition for α in [21, Section 4] (see also [72]) which has to be coupled with the "subcritical" splitting for u as in [21, Example 3.5].

Remark 7.6.2. *In light of Theorem 7.5.1, the global attractor $\mathbb{A}$ is easily seen to be bounded in the more regular space $\mathrm{H}_2 \times \mathrm{H}_2 \times \mathrm{H}_2$.*

CHAPTER 8

Timoshenko Systems with Gurtin-Pipkin Thermal Law

8.1 Introduction

Given a real interval $\mathfrak{I} = [0, \ell]$, we consider the thermoelastic beam model of Timoshenko type [92]

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \end{cases} \tag{8.1.1}$$

where the unknown variables

$$\varphi, \psi, \theta, q : \ (x, t) \in \mathfrak{I} \times [0, \infty) \mapsto \mathbb{R}$$

represent the transverse displacement of a beam with reference configuration $\mathfrak{I}$, the rotation angle of a filament, the relative temperature (i.e. the temperature variation field from an equilibrium reference value) and the heat flux vector, respectively. Here, ρ_1, ρ_2, ρ_3 as well as κ, b, δ are strictly positive fixed constants. The system is complemented with the Dirichlet boundary conditions for φ and θ

$$\varphi(0, t) = \varphi(\ell, t) = \theta(0, t) = \theta(\ell, t) = 0,$$

and the Neumann one for ψ

$$\psi_x(0, t) = \psi_x(\ell, t) = 0.$$

As already said in the introduction of the thesis, such boundary conditions seem to be the most feasible from a physical viewpoint. To complete the picture, a further relation is needed: the so-called constitutive law for the heat flux, establishing a link between q and θ. This is what really characterizes the dynamics, since no mechanical dissipation is present in the system, and any possible loss of energy can be due only to thermal effects.

8.1.1 The Fourier thermal law

A first choice is to assume the classical Fourier law of heat conduction

$$\beta q + \theta_x = 0, \tag{8.1.2}$$

where $\beta > 0$ is a fixed constant. In which case, the third equation of (8.1.1) becomes

$$\rho_3 \theta_t - \frac{1}{\beta}\theta_{xx} + \delta\psi_{tx} = 0.$$

The exponential stability of the resulting Timoshenko-Fourier system has been analyzed in [67]. There, the authors introduce the so-called stability number[1]

$$\chi = \frac{\rho_1}{\kappa} - \frac{\rho_2}{b},$$

representing the difference of the inverses of the propagation speeds. The main result of [67] reads as follows: the contraction semigroup generated by (8.1.1)-(8.1.2) acting on the triplet (φ, ψ, θ) is exponentially stable (in the natural weak energy space) if and only if $\chi = 0$.

8.1.2 The Cattaneo thermal law

As already said, the drawback of the Fourier law lies in the physical paradox of infinite propagation speed of (thermal) signals, a typical side-effect of parabolicity. A different model, removing this paradox, is the Cattaneo law [8], namely, the differential perturbation of (8.1.2)

$$\tau q_t + \beta q + \theta_x = 0, \tag{8.1.3}$$

for $\tau > 0$ small. A natural question is whether the semigroup generated by (8.1.1) coupled with (8.1.3), now acting on the state variable $(\varphi, \psi, \theta, q)$, remains exponentially stable within the condition $\chi = 0$ above. As shown in [31], the answer is negative: exponential stability can never occur when $\chi = 0$. More recently, in [89] a new stability number is introduced in order to deal with the Timoshenko-Cattaneo system, that is,[2]

$$\chi_\tau = \left[\frac{\rho_1}{\rho_3\kappa} - \tau\right]\left[\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right] - \tau\frac{\rho_1\delta^2}{\rho_3\kappa b}.$$

[1]The notion of stability number is actually introduced in the subsequent paper [89], defined there as $\chi = \kappa/\rho_1 - b/\rho_2$. The difference is clearly irrelevant with respect to the relation $\chi = 0$. The motivation of our choice of χ is to render more direct the comparison with the Cattaneo law.

[2]The value of χ_τ in [89] differs from ours for a multiplicative constant.

The system is shown to be exponentially stable if and only if $\chi_\tau = 0$. Quite interestingly, the Fourier case is fully recovered in the limit $\tau \to 0$, when (8.1.3) collapses into (8.1.2). Indeed, for $\tau = 0$ the equality

$$\chi_0 = \frac{\rho_1}{\rho_3 \kappa} \chi$$

holds, which tells at once that

$$\chi_0 = 0 \quad \Leftrightarrow \quad \chi = 0.$$

It is worth mentioning that the proof of exponential stability in [89] is carried out via linear semigroup techniques, whereas the analogous result of [67] for the Fourier case is obtained by constructing explicit energy functionals.

8.1.3 The Gurtin-Pipkin thermal law

The aim of the present chapter is studying the Timoshenko system (8.1.1) assuming the Gurtin-Pipkin heat conduction law for the heat flux [48]. More precisely, we consider the constitutive equation

$$\beta q(t) + \int_0^\infty g(s)\theta_x(t-s)\,\mathrm{d}s = 0, \tag{8.1.4}$$

where g, called the memory kernel, is a (bounded) convex summable function on $[0, \infty)$ of total mass

$$\int_0^\infty g(s)\,\mathrm{d}s = 1,$$

whose properties will be specified in more detail later on. Equation (8.1.4) can be viewed as a memory relaxation of the Fourier law (8.1.2), inducing (similarly to the Cattaneo law) a fully hyperbolic mechanism of heat transfer. In this perspective, it may be considered a more realistic description of physical reality. Accordingly, system (8.1.1) turns into

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \\ \rho_3 \theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s + \delta\psi_{tx} = 0. \end{cases} \tag{8.1.5}$$

Rephrasing system (8.1.5) within the history framework of Dafermos [25], we construct a contraction semigroup $S(t)$ of solutions acting on a suitable Hilbert space $\mathcal{H}$, accounting for the presence of the memory. Then, introducing the stability number

$$\chi_g = \left[\frac{\rho_1}{\rho_3\kappa} - \frac{\beta}{g(0)}\right]\left[\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right] - \frac{\beta}{g(0)}\frac{\rho_1\delta^2}{\rho_3\kappa b},$$

our main theorem can be stated as follows.

Theorem 8.1.1. *The semigroup $S(t)$ is exponentially stable if and only if $\chi_g = 0$.*

As we will see in the next section, Theorem 8.1.1 actually subsumes and generalizes all the previously known results on the exponential decay properties of the thermoelastic Timoshenko system (8.1.1).

Remark 8.1.1. *Actually, exploiting the techniques of [10, 11], it is possible to prove that the contraction semigroup $S(t)$ remains stable*[3] *on $\mathcal{H}$ (although not exponentially stable) also when $\chi_g \neq 0$.*

In Section 8.2 we compare Timoshenko systems of the form (8.1.1) subject to different laws of heat conduction, viewed as particular instances of (8.1.5) for suitable choices of the memory kernel. The comparison with the Timoshenko-Cattaneo system of [31, 89], only formal at this stage, is rendered rigorous in the final Section 8.8. After introducing some notation (Section 8.3), in Section 8.4 we define the semigroup $S(t)$ describing the solutions to (8.1.5). The subsequent three sections are devoted to the proof of Theorem 8.1.1. Firstly, we introduce some auxiliary functionals (Section 8.5), needed in the proof of the sufficiency part of the theorem carried out in Section 8.6. The necessity of the condition $\chi_g = 0$ in order for exponential stability to occur is proved in Section 8.7.

8.2 Comparison with Earlier Results

8.2.1 The Fourier case

The Fourier law (8.1.2) can be seen as a (singular) limit of the Gurtin-Pipkin law (8.1.4). Indeed, defining the ε-scaling of the memory kernel g by

$$g_\varepsilon(s) = \frac{1}{\varepsilon} g\Big(\frac{s}{\varepsilon}\Big), \quad \varepsilon > 0,$$

we consider in place of the original (8.1.4) the constitutive equation

$$\beta q(t) + \int_0^\infty g_\varepsilon(s)\theta_x(t-s)\,\mathrm{d}s = 0. \tag{8.2.1}$$

Since $g_\varepsilon \to \delta_0$ in the distributional sense, where δ_0 denotes the Dirac mass at 0^+, it is clear that, in the limit $\varepsilon \to 0$, equation (8.2.1) reduces to the classical constitutive law (8.1.2). According to Theorem 8.1.1, exponential stability for the Timoshenko-Gurtin-Pipkin model with memory kernel g_ε occurs if and only if

$$\chi_{g_\varepsilon} = \left[\frac{\rho_1}{\rho_3\kappa} - \frac{\beta\varepsilon}{g(0)}\right]\left[\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right] - \frac{\beta\varepsilon}{g(0)}\frac{\rho_1\delta^2}{\rho_3\kappa b} = 0.$$

Letting $\varepsilon \to 0$, we recover the condition

$$\chi_{\delta_0} = \frac{\rho_1}{\rho_3\kappa}\chi = 0$$

of the Fourier case. The convergence of the Timoshenko-Gurtin-Pipkin model to the Timoshenko-Fourier one as $\varepsilon \to 0$ can be made rigorous within the proper functional setting, along the same lines of [22].

[3] We recall that $S(t)$ is said to be stable on $\mathcal{H}$ if

$$\lim_{t\to\infty} \|S(t)z\|_{\mathcal{H}} = 0, \quad \forall z \in \mathcal{H}.$$

8.2.2 The Cattaneo case

The Cattaneo law (8.1.3) can be deduced as a particular instance of (8.1.4), corresponding to the memory kernel

$$g_\tau(s) = \frac{\beta}{\tau}\,\mathrm{e}^{-\frac{s\beta}{\tau}}.$$

Indeed, changing the integration variable, we can write the flux vector q in the form

$$q(t) = -\frac{1}{\beta}\int_{-\infty}^{t} g_\tau(t-s)\theta_x(s)\,\mathrm{d}s.$$

Since

$$g_\tau'(s) = -\frac{\beta}{\tau}\,g_\tau(s),$$

we draw the relation

$$q_t(t) = -\frac{1}{\beta}\int_{-\infty}^{t} g_\tau'(t-s)\theta_x(s)\,\mathrm{d}s - \frac{g_\tau(0)}{\beta}\theta_x(t) = \frac{1}{\tau}\int_{-\infty}^{t} g_\tau(t-s)\theta_x(s)\,\mathrm{d}s - \frac{1}{\tau}\theta_x(t),$$

which is nothing but (8.1.3). Besides, we have the equality of the stability numbers

$$\chi_{g_\tau} = \chi_\tau.$$

8.2.3 The Coleman-Gurtin case

A further interesting model, midway between the Fourier and the Gurtin-Pipkin one, is obtained by assuming the (parabolic-hyperbolic) Coleman-Gurtin law for the heat flux, namely,

$$\beta q(t) + (1-\alpha)\theta_x(t) + \alpha\int_0^\infty g(s)\theta_x(t-s)\,\mathrm{d}s = 0, \quad \alpha\in(0,1). \tag{8.2.2}$$

The limit cases $\alpha = 0$ and $\alpha = 1$ correspond to the fully parabolic Fourier case and the fully hyperbolic Gurtin-Pipkin case. The corresponding Timoshenko-Coleman-Gurtin system, whose third equation now reads

$$\rho_3\theta_t - \frac{1}{\beta}\left[(1-\alpha)\theta_{xx} + \alpha\int_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s\right] + \delta\psi_{tx} = 0,$$

generates (similarly to the Timoshenko-Gurtin-Pipkin system) a contraction semigroup $\Sigma(t)$ on $\mathcal{H}$. For this system, the following theorem holds.

Theorem 8.2.1. *The semigroup $\Sigma(t)$ is exponentially stable if and only if $\chi = 0$.*

Hence, the picture is exactly the same as in the Fourier case. This, as observed in [89], is due to the predominant character of parabolicity. Theorem 8.2.1 can be given a direct proof, following the lines of the next sections. In fact, the situation here is much simpler, do to the presence of instantaneous dissipation given by the term $-\theta_{xx}$ in the equation. However, it is also possible to obtain Theorem 8.2.1 as a byproduct of

Theorem 8.1.1. To this end, it is enough to consider the Timoshenko-Gurtin-Pipkin system with kernel

$$g_\varepsilon(s) = \frac{1-\alpha}{\varepsilon} g\Big(\frac{s}{\varepsilon}\Big) + \alpha g(s),$$

whose exponential stability takes place if and only if

$$\chi_{g_\varepsilon} = \Big[\frac{\rho_1}{\rho_3\kappa} - \frac{\beta\varepsilon}{(1-\alpha+\alpha\varepsilon)g(0)}\Big]\Big[\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\Big] - \frac{\beta\varepsilon}{(1-\alpha+\alpha\varepsilon)g(0)}\frac{\rho_1\delta^2}{\rho_3\kappa b} = 0.$$

Performing the limit $\varepsilon \to 0$, we obtain the distributional convergence

$$g_\varepsilon \to (1-\alpha)\delta_0 + \alpha g,$$

yielding in turn

$$\int_0^\infty g_\varepsilon(s)\theta_{xx}(t-s)\,\mathrm{d}s \to (1-\alpha)\theta_{xx} + \alpha\int_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s.$$

Accordingly, we see that

$$\chi_{(1-\alpha)\delta_0+\alpha g} = \frac{\rho_1}{\rho_3\kappa}\,\chi,$$

and we recover the same stability condition of the Timoshenko-Fourier system.

8.2.4 Heat conduction of type III

We finally mention the model resulting from the constitutive law of type III of Green-Naghdi for the heat flux

$$\beta q + \theta_x + dp_x = 0, \quad d > 0, \tag{8.2.3}$$

where

$$p(t) = p(0) + \int_0^t \theta(r)\,\mathrm{d}r$$

is the thermal displacement. Plugging (8.2.3) into (8.1.1), one obtains

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x+\psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x+\psi) + \delta p_{tx} = 0, \\ \rho_3 p_{tt} - \frac{1}{\beta}p_{txx} - \frac{d}{\beta}p_{xx} + \delta\psi_{tx} = 0. \end{cases}$$

In [61], the system is shown to be exponentially stable if $\chi = 0$. Again, the partial parabolicity of the model prevails, so that the exponential stability condition is the same as in the Fourier case. Though, the constitutive law (8.2.3) cannot be deduced from (8.1.4), not even as a limiting case. Possibly, this feature may reflect the fact that the theory of heat conduction of type III seems to be at the limit of thermodynamic admissibility (see the analysis of [36]).

8.3 Functional Setting and Notation

8.3.1 Assumptions on the memory kernel

Calling

$$\mu(s) = -g'(s),$$

the *prime* denoting the derivative with respect to s, let the following conditions hold.

(i) μ is a nonnegative nonincreasing absolutely continuous function on $\mathbb{R}^+$ such that

$$\mu(0) = \lim_{s\to 0} \mu(s) \in (0, \infty).$$

(ii) There exists $\nu > 0$ such that the differential inequality

$$\mu'(s) + \nu\mu(s) \leq 0$$

holds for almost every $s > 0$.

Remark 8.3.1. *For every $k > 0$, the exponential kernel $g(s) = k\mathrm{e}^{-ks}$ meets the hypotheses (i)-(ii).*

In particular, μ is summable on $\mathbb{R}^+$ with

$$\int_0^\infty \mu(s)\,\mathrm{d}s = g(0).$$

Besides, the requirement that g has total mass 1 translates into

$$\int_0^\infty s\mu(s)\,\mathrm{d}s = 1.$$

8.3.2 Functional spaces

In what follows, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are the standard inner product and norm on the Hilbert space $L^2(\mathfrak{I})$. We introduce the Hilbert subspace

$$L^2_*(\mathfrak{I}) = \left\{ f \in L^2(\mathfrak{I}) : \int_0^\ell f(x)\,\mathrm{d}x = 0 \right\}$$

of zero-mean functions, along with the Hilbert spaces

$$H^1_0(\mathfrak{I}) \qquad \text{and} \qquad H^1_*(\mathfrak{I}) = H^1(\mathfrak{I}) \cap L^2_*(\mathfrak{I}),$$

both endowed with the gradient norm, due to the Poincaré inequality. We also consider the space $H^2(\mathfrak{I})$ and so-called memory space

$$\mathcal{M} = L^2(\mathbb{R}^+; H^1_0(\mathfrak{I}))$$

of square summable H^1_0-valued functions on $\mathbb{R}^+$ with respect to the measure $\mu(s)\mathrm{d}s$, endowed with the inner product

$$\langle \eta, \xi \rangle_{\mathcal{M}} = \int_0^\infty \mu(s) \langle \eta_x(s), \xi_x(s) \rangle\,\mathrm{d}s.$$

The infinitesimal generator of the right-translation semigroup on $\mathcal{M}$ is the linear operator

$$T\eta = -D\eta$$

with domain

$$\mathfrak{D}(T) = \Big\{\eta \in \mathcal{M} : D\eta \in \mathcal{M},\ \lim_{s\to 0} \|\eta_x(s)\| = 0\Big\},$$

where D stands for weak derivative with respect to the internal variable $s \in \mathbb{R}^+$. The phase space of our problem will be

$$\mathcal{H} = H_0^1(\mathfrak{I}) \times L^2(\mathfrak{I}) \times H_*^1(\mathfrak{I}) \times L_*^2(\mathfrak{I}) \times L^2(\mathfrak{I}) \times \mathcal{M}$$

normed by

$$\|(\varphi,\tilde{\varphi},\psi,\tilde{\psi},\theta,\eta)\|_{\mathcal{H}}^2 = \kappa\|\varphi_x+\psi\|^2+\rho_1\|\tilde{\varphi}\|^2+b\|\psi_x\|^2+\rho_2\|\tilde{\psi}\|^2+\rho_3\|\theta\|^2+\frac{1}{\beta}\|\eta\|_{\mathcal{M}}^2.$$

8.3.3 Basic facts on the memory space

For every $\eta \in \mathfrak{D}(T)$, the nonnegative functional

$$\Gamma[\eta] = -\int_0^\infty \mu'(s)\|\eta_x(s)\|^2\,\mathrm{d}s$$

is well defined, and the following identity holds (see [11, 41, 69])

$$2\langle T\eta,\eta\rangle_{\mathcal{M}} = -\Gamma[\eta]. \tag{8.3.1}$$

Moreover, in light of assumption (ii) on μ, we deduce the inequality

$$\nu\|\eta\|_{\mathcal{M}}^2 \le \Gamma[\eta], \tag{8.3.2}$$

which will be crucial for our purposes.

8.4 The Contraction Semigroup

Firstly, we introduce the auxiliary variable

$$\eta = \eta^t(x,s) : (x,t,s) \in \mathfrak{I}\times[0,\infty)\times\mathbb{R}^+ \mapsto \mathbb{R},$$

accounting for the integrated past history of θ and formally defined as (see [25, 41])

$$\eta^t(x,s) = \int_0^s \theta(x,t-\sigma)\,\mathrm{d}\sigma,$$

thus satisfying the Dirichlet boundary condition

$$\eta^t(0,s) = \eta^t(\ell,s) = 0$$

and the further "boundary condition"

$$\lim_{s\to 0}\eta^t(x,s) = 0.$$

Hence, η satisfies the equation

$$\eta_t^t = -\eta_s^t + \theta(t).$$

The way to render the argument rigorous is recasting (8.1.5) in the history space framework devised by C.M. Dafermos [25]. This amounts to considering the partial differential system in the unknowns $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\theta = \theta(t)$ and $\eta = \eta^t$

$$\rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \tag{8.4.1}$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \tag{8.4.2}$$

$$\rho_3 \theta_t - \frac{1}{\beta}\int_0^\infty \mu(s)\eta_{xx}(s)\,\mathrm{d}s + \delta\psi_{tx} = 0, \tag{8.4.3}$$

$$\eta_t = T\eta + \theta. \tag{8.4.4}$$

Remark 8.4.1. *The analogy with the original system* (8.1.5) *is not merely formal, and can be made rigorous within the proper functional setting (see [41] for more details).*

Introducing the state vector

$$z(t) = (\varphi(t), \tilde{\varphi}(t), \psi(t), \tilde{\psi}(t), \theta(t), \eta^t),$$

we view (8.4.1)-(8.4.4) as the ODE in $\mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \mathbb{A} z(t), \tag{8.4.5}$$

where the linear operator $\mathbb{A}$ is defined as

$$\mathbb{A}\begin{pmatrix}\varphi\\ \tilde{\varphi}\\ \psi\\ \tilde{\psi}\\ \theta\\ \eta\end{pmatrix} = \begin{pmatrix}\tilde{\varphi}\\ \frac{\kappa}{\rho_1}(\varphi_x+\psi)_x\\ \tilde{\psi}\\ \frac{b}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}(\varphi_x+\psi) - \frac{\delta}{\rho_2}\theta_x\\ \frac{1}{\beta\rho_3}\int_0^\infty \mu(s)\eta_{xx}(s)\,\mathrm{d}s - \frac{\delta}{\rho_3}\tilde{\psi}_x\\ T\eta + \theta\end{pmatrix}$$

with domain

$$\mathfrak{D}(\mathbb{A}) = \left\{ (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, \theta, \eta) \in \mathcal{H} \;\middle|\; \begin{array}{c} \varphi \in H^2(\mathfrak{I}) \\ \tilde{\varphi} \in H_0^1(\mathfrak{I}) \\ \psi_x \in H_0^1(\mathfrak{I}) \\ \tilde{\psi} \in H_*^1(\mathfrak{I}) \\ \theta \in H_0^1(\mathfrak{I}) \\ \eta \in \mathfrak{D}(T) \\ \int_0^\infty \mu(s)\eta(s)\,\mathrm{d}s \in H^2(\mathfrak{I}) \end{array} \right\}.$$

Theorem 8.4.1. *The operator $\mathbb{A}$ is the infinitesimal generator of a contraction semigroup*

$$S(t) = \mathrm{e}^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}.$$

The proof of this fact is based on Lemma 1.2.1, and is omitted (however, the argument is similar to the one of Theorem 6.12.2). Thus, for every initial datum

$$z_0 = (\varphi_0, \tilde{\varphi}_0, \psi_0, \tilde{\psi}_0, \theta_0, \eta_0) \in \mathcal{H}$$

given at time $t = 0$, the unique solution at time $t > 0$ to (8.4.5) reads

$$z(t) = (\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), \theta(t), \eta^t) = S(t)z_0.$$

Besides, η^t fulfills the explicit representation formula (see [41])

$$\eta^t(s) = \begin{cases} \int_0^s \theta(t-\sigma)\,\mathrm{d}\sigma & s \le t, \\ \eta_0(s-t) + \int_0^t \theta(t-\sigma)\,\mathrm{d}\sigma & s > t. \end{cases}$$

Remark 8.4.2. *As observed in [67], the choice of the spaces of zero-mean functions for the variable ψ and its derivative is consistent. Indeed, calling*

$$\Theta(t) = \int_0^\ell \psi(x,t)\,\mathrm{d}x$$

and integrating (8.4.2) *on $\mathfrak{I}$ we obtain the differential equation*

$$\rho_2 \ddot{\Theta}(t) + \kappa \Theta(t) = 0.$$

Hence, if $\Theta(0) = \dot{\Theta}(0) = 0$ it follows that $\Theta(t) \equiv 0$.

Remark 8.4.3. *For the existence of the contraction semigroup $S(t)$ the hypotheses (i)-(ii) on the kernel are overabundant. It is actually enough to require that μ be a (nonnull and nonnegative) nonincreasing absolutely continuous summable function on $\mathbb{R}^+$, possibly unbounded in a neighborhood of zero.*

For any fixed initial datum $z_0 \in \mathcal{H}$, we define (twice) the energy as

$$E(t) = \|S(t)z_0\|^2_{\mathcal{H}}.$$

The natural multiplication of equation (8.4.5) by $z(t)$ in the weak energy space, along with an exploitation of (8.3.1), provide the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 2\langle \mathbb{A}z(t), z(t)\rangle_{\mathcal{H}} = \frac{2}{\beta}\langle T\eta^t, \eta^t\rangle_{\mathcal{M}} = -\frac{1}{\beta}\Gamma[\eta^t], \tag{8.4.6}$$

valid for all $z_0 \in \mathfrak{D}(\mathbb{A})$.

As anticipated in the introduction, the main Theorem 8.1.1 of this chapter tells that

$$S(t) \text{ exp. stable} \quad \Leftrightarrow \quad \chi_g = 0.$$

The proof of the result is carried out in the next Sections 8.5-8.7.

Remark 8.4.4. *We mention that an alternative approach is also possible. Namely, to set the problem in the so-called minimal state framework [29], rather than in the past history one. In which case, the necessary and sufficient condition $\chi_g = 0$ of exponential decay remains the same. In fact, one can show in general that the exponential decay in the history space (i.e. what proved here) implies the analogous decay in the minimal state space (cf. [19, 29]).*

8.5 Some Auxiliary Functionals

In this section, we define some auxiliary functionals needed in the proof of the sufficiency part of Theorem 8.1.1. As customary, it is understood that we work with (regular) solutions arising from initial data belonging to the domain of the operator $\mathbb{A}$. Along the section, $C \geq 0$ will denote a *generic* constant depending only on the structural quantities of the problem. Besides, we will tacitly use several times the Hölder, Young and Poincaré inequalities. In particular we will exploit the inequality

$$\int_0^\infty \mu(s)\|\eta_x(s)\|\,\mathrm{d}s \leq \left(\int_0^\infty \mu(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^\infty \mu(s)\|\eta_x(s)\|^2\,\mathrm{d}s\right)^{\frac{1}{2}} = \sqrt{g(0)}\,\|\eta\|_{\mathcal{M}}.$$

8.5.1 The functional *I*

Let

$$I(t) = -\frac{2\rho_3}{g(0)}\int_0^\infty \mu(s)\langle\theta(t),\eta^t(s)\rangle\,\mathrm{d}s.$$

Lemma 8.5.1. *For every $\varepsilon_I > 0$ small, I satisfies the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}I + \rho_3\|\theta\|^2 + \|\eta\|^2_{\mathcal{M}} \leq \varepsilon_I\|\psi_t\|^2 + \frac{c_I}{\varepsilon_I}\Gamma[\eta]$$

for some $c_I > 0$ independent of ε_I.

Proof. In light of (8.4.3) and (8.4.4), we have the identity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I + 2\rho_3\|\theta\|^2 + \|\eta\|^2_{\mathcal{M}} = &-\frac{2\rho_3}{g(0)}\int_0^\infty \mu(s)\langle T\eta(s),\theta\rangle\,\mathrm{d}s\\ &+\frac{2}{g(0)\beta}\left\|\int_0^\infty \mu(s)\eta_x(s)\,\mathrm{d}s\right\|^2\\ &-\frac{2\delta}{g(0)}\int_0^\infty \mu(s)\langle\eta_x(s),\psi_t\rangle\,\mathrm{d}s + \|\eta\|^2_{\mathcal{M}}.\end{aligned}$$

Integrating by parts in s, we infer that (as shown in [41], the boundary terms vanish)

$$\begin{aligned}-\frac{2\rho_3}{g(0)}\int_0^\infty \mu(s)\langle T\eta(s),\theta\rangle\,\mathrm{d}s &= -\frac{2\rho_3}{g(0)}\int_0^\infty \mu'(s)\langle\eta(s),\theta\rangle\,\mathrm{d}s\\ &\leq C\|\theta\|\sqrt{\Gamma[\eta]}\\ &\leq \rho_3\|\theta\|^2 + C\Gamma[\eta].\end{aligned}$$

Thus, exploiting the inequalities

$$\frac{2}{g(0)\beta}\left\|\int_0^\infty \mu(s)\eta_x(s)\,\mathrm{d}s\right\|^2 \leq C\|\eta\|^2_{\mathcal{M}}$$

and

$$-\frac{2\delta}{g(0)}\int_0^\infty \mu(s)\langle\eta_x(s),\psi_t\rangle\,\mathrm{d}s \leq C\|\psi_t\|\|\eta\|_{\mathcal{M}},$$

appealing to (8.3.2) we obtain for every $\varepsilon_I > 0$ small the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} I + \rho_3 \|\theta\|^2 + \|\eta\|^2_{\mathcal{M}} \leq C\|\psi_t\|\Gamma[\eta] + C\Gamma[\eta] \leq \varepsilon_I \|\psi_t\|^2 + \frac{C}{\varepsilon_I}\Gamma[\eta],$$

where C is independent of ε_I. □

8.5.2 The functional *J*

Defining the primitive[4]

$$\Psi(x,t) = \int_0^x \psi(y,t)\,\mathrm{d}y,$$

let

$$J(t) = -\frac{2\rho_2\rho_3}{\delta}\langle \theta(t), \Psi_t(t)\rangle.$$

Lemma 8.5.2. *For every $\varepsilon_J > 0$ small, J satisfies the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} J + \rho_2\|\psi_t\|^2 \leq \varepsilon_J\big[\|\psi_x\|^2 + \|\varphi_x + \psi\|^2\big] + \frac{c_J}{\varepsilon_J}\big[\|\theta\|^2 + \Gamma[\eta]\big]$$

for some $c_J > 0$ independent of ε_J.

Proof. By means of (8.4.2) and (8.4.3), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} J + 2\rho_2\|\psi_t\|^2 \\ &= \frac{2\rho_2}{\beta\delta}\int_0^\infty \mu(s)\langle \eta_x(s), \psi_t\rangle\,\mathrm{d}s - \frac{2\rho_3 b}{\delta}\langle\theta,\psi_x\rangle + \frac{2\rho_3\kappa}{\delta}\langle\theta,\varphi+\Psi\rangle + 2\rho_3\|\theta\|^2.\end{aligned}$$

Estimating the terms in the right-hand side as (here we use again (8.3.2))

$$\frac{2\rho_2}{\beta\delta}\int_0^\infty \mu(s)\langle \eta_x(s), \psi_t\rangle\,\mathrm{d}s \leq C\|\psi_t\|\|\eta\|_{\mathcal{M}} \leq \rho_2\|\psi_t\|^2 + C\Gamma[\eta]$$

and, for every $\varepsilon_J > 0$ small,

$$\begin{aligned}&-\frac{2\rho_3 b}{\delta}\langle\theta,\psi_x\rangle + \frac{2\rho_3\kappa}{\delta}\langle\theta,\varphi+\Psi\rangle + 2\rho_3\|\theta\|^2 \\ &\leq C\big[\|\psi_x\| + \|\varphi_x+\psi\|\big]\|\theta\| + C\|\theta\|^2 \\ &\leq \varepsilon_J\big[\|\psi_x\|^2 + \|\varphi_x+\psi\|^2\big] + \frac{C}{\varepsilon_J}\|\theta\|^2,\end{aligned}$$

with C independent of ε_J, the claim follows. □

[4] In particular, $\Psi \in H^1_0(\mathfrak{I})$.

8.5.3 The functional K

We introduce the number

$$\gamma_g = \kappa - \frac{g(0)\rho_1}{\beta\rho_3}$$

depending on the memory kernel g. It is readily seen that

$$\chi_g = 0 \quad \Rightarrow \quad \gamma_g \neq 0.$$

Then, assuming $\chi_g = 0$ and calling

$$\begin{aligned}
K_1(t) &= \frac{\rho_1 b}{\kappa}\langle \psi_x(t), \varphi_t(t)\rangle + \rho_2\langle \psi_t(t), \varphi_x(t) + \psi(t)\rangle, \\
K_2(t) &= \int_0^\infty \mu(s)\langle \eta_x^t(s), \varphi_x(t) + \psi(t)\rangle \,\mathrm{d}s, \\
K_3(t) &= -\frac{\delta\rho_1}{\kappa}\langle \theta(t), \varphi_t(t)\rangle,
\end{aligned}$$

we set

$$K(t) = \frac{2\kappa}{\gamma_g}\bigg[\frac{\gamma_g}{\kappa}K_1(t) - \frac{\rho_1\delta}{\beta\rho_3\kappa}K_2(t) + K_3(t)\bigg].$$

Lemma 8.5.3. *Suppose that $\chi_g = 0$. Then K satisfies the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}K + \kappa\|\varphi_x + \psi\|^2 \leq c_K\big[\|\psi_t\|^2 + \Gamma[\eta]\big]$$

for some $c_K > 0$.

Proof. In light of (8.4.1) and (8.4.2), we obtain the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}K_1 + \kappa\|\varphi_x + \psi\|^2 = \Big(\rho_2 - \frac{\rho_1 b}{\kappa}\Big)\langle \psi_t, \varphi_{tx}\rangle + \rho_2\|\psi_t\|^2 - \delta\langle \theta_x, \varphi_x + \psi\rangle. \tag{8.5.1}$$

By (8.4.4),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}K_2 = &-\int_0^\infty \mu(s)\langle T\eta(s), (\varphi_x + \psi)_x\rangle \,\mathrm{d}s - g(0)\langle \theta, (\varphi_x + \psi)_x\rangle \\
&-\int_0^\infty \mu(s)\langle \eta_{xx}(s), \varphi_t\rangle \,\mathrm{d}s + \int_0^\infty \mu(s)\langle \eta_x(s), \psi_t\rangle \,\mathrm{d}s.
\end{aligned}$$

From (8.4.3) we learn that

$$-\int_0^\infty \mu(s)\langle \eta_{xx}(s), \varphi_t\rangle \,\mathrm{d}s = -\beta\rho_3\langle \theta_t, \varphi_t\rangle + \beta\delta\langle \psi_t, \varphi_{tx}\rangle,$$

while an integration by parts in s yields (again, the boundary terms vanish)

$$-\int_0^\infty \mu(s)\langle T\eta(s), (\varphi_x + \psi)_x\rangle \,\mathrm{d}s = \int_0^\infty \mu'(s)\langle \eta_x(s), \varphi_x + \psi\rangle \,\mathrm{d}s.$$

We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}K_2 = \int_0^\infty \mu'(s)\langle \eta_x(s), \varphi_x + \psi\rangle\,\mathrm{d}s + \int_0^\infty \mu(s)\langle \eta_x(s), \psi_t\rangle\,\mathrm{d}s \qquad (8.5.2)$$
$$- \beta\rho_3\langle \theta_t, \varphi_t\rangle + \beta\delta\langle \psi_t, \varphi_{tx}\rangle + g(0)\langle \theta_x, \varphi_x + \psi\rangle.$$

Finally, exploiting once more (8.4.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}K_3 = -\frac{\delta\rho_1}{\kappa}\langle \theta_t, \varphi_t\rangle + \delta\langle \theta_x, \varphi_x + \psi\rangle. \qquad (8.5.3)$$

At this point, reconstructing K from (8.5.1)-(8.5.3), we are led to the differential identity

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}K + 2\kappa\|\varphi_x + \psi\|^2 \\
&= \chi_g \frac{2g(0)\kappa b}{\beta\gamma_g}\langle \psi_t, \varphi_{tx}\rangle + 2\rho_2\|\psi_t\|^2 \\
&\quad - \frac{2\rho_1\delta}{\beta\gamma_g\rho_3}\bigg[\int_0^\infty \mu'(s)\langle \eta_x(s), \varphi_x + \psi\rangle\,\mathrm{d}s + \int_0^\infty \mu(s)\langle \eta_x(s), \psi_t\rangle\,\mathrm{d}s\bigg].
\end{aligned}$$

Since $\chi_g = 0$ by assumption, we are left to control the integral terms in the right-hand side. We have

$$\begin{aligned}
-\frac{2\rho_1\delta}{\beta\gamma_g\rho_3}\int_0^\infty \mu'(s)\langle \eta_x(s), \varphi_x + \psi\rangle\,\mathrm{d}s &\le C\|\varphi_x + \psi\|\int_0^\infty -\mu'(s)\|\eta_x(s)\|\,\mathrm{d}s \\
&\le C\|\varphi_x + \psi\|\sqrt{\Gamma[\eta]} \\
&\le \kappa\|\varphi_x + \psi\|^2 + C\Gamma[\eta],
\end{aligned}$$

and, recalling (8.3.2),

$$-\frac{2\rho_1\delta}{\beta\gamma_g\rho_3}\int_0^\infty \mu(s)\langle \eta_x(s), \psi_t\rangle\,\mathrm{d}s \le C\|\psi_t\|\|\eta\|_{\mathcal{M}} \le C\|\psi_t\|^2 + C\Gamma[\eta].$$

The proof is completed. □

8.5.4 The functional *L*

Let

$$L(t) = 2\rho_2\langle \psi_t(t), \psi(t)\rangle - 2\rho_1\langle \varphi_t(t), \varphi(t)\rangle.$$

Lemma 8.5.4. *The functional L satisfies the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}L + \rho_1\|\varphi_t\|^2 + b\|\psi_x\|^2 \le c_L\big[\|\varphi_x + \psi\|^2 + \|\psi_t\|^2 + \|\theta\|^2\big]$$

for some $c_L > 0$.

Proof. By means of (8.4.1) and (8.4.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}L + 2\rho_1\|\varphi_t\|^2 + 2b\|\psi_x\|^2 = 2\rho_2\|\psi_t\|^2 + 2\delta\langle \theta, \psi_x\rangle + 2\kappa\|\varphi_x + \psi\|^2 - 4\kappa\langle \varphi_x + \psi, \psi\rangle.$$

Since the right-hand side is easily controlled by

$$b\|\psi_x\|^2 + C\big[\|\varphi_x + \psi\|^2 + \|\psi_t\|^2 + \|\theta\|^2\big],$$

we are done. □

8.6 Proof of Theorem 8.1.1 (Sufficiency)

Within the condition $\chi_g = 0$, we are now in the position to prove the exponential stability of $S(t)$. In what follows, E is (twice) the energy, whereas I, J, K, L denote the functionals of the previous Section 8.5.

8.6.1 A further energy functional

For $\varepsilon > 0$, we define

$$M_\varepsilon(t) = I(t) + \varepsilon J(t) + \frac{\varepsilon\rho_2}{2c_K} K(t) + \varepsilon\sqrt{\varepsilon}\, L(t),$$

where $c_K > 0$ is the constant of Lemma 8.5.3.

Lemma 8.6.1. *For every $\varepsilon > 0$ sufficiently small, the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} M_\varepsilon + \varepsilon^2 E \leq \frac{c_M}{\varepsilon}\Gamma[\eta]$$

holds for some $c_M > 0$ independent of ε.

Proof. Collecting the inequalities of Lemmas 8.5.1, 8.5.2, 8.5.3 and 8.5.4, we end up with

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} M_\varepsilon + \varepsilon\Big(\frac{\kappa\rho_2}{2c_K} - \varepsilon_J - \sqrt{\varepsilon}\, c_L\Big)\|\varphi_x + \psi\|^2 + \varepsilon\sqrt{\varepsilon}\,\rho_1\|\varphi_t\|^2 + \varepsilon(\sqrt{\varepsilon}\, b - \varepsilon_J)\|\psi_x\|^2\\ &\quad + \Big(\frac{\varepsilon\rho_2}{2} - \varepsilon_I - \varepsilon\sqrt{\varepsilon}\, c_L\Big)\|\psi_t\|^2 + \Big(\rho_3 - \varepsilon\sqrt{\varepsilon}\, c_L - \frac{\varepsilon c_J}{\varepsilon_J}\Big)\|\theta\|^2 + \|\eta\|^2_{\mathcal{M}}\\ &\leq \Big(\frac{c_I}{\varepsilon_I} + \frac{\varepsilon c_J}{\varepsilon_J} + \frac{\varepsilon\rho_2}{2}\Big)\Gamma[\eta].\end{aligned}$$

At this point, we choose

$$\varepsilon_I = \frac{\varepsilon\rho_2}{4} \qquad \text{and} \qquad \varepsilon_J = \frac{2\varepsilon c_J}{\rho_3}.$$

Taking $\varepsilon > 0$ sufficiently small, the claim follows. ☐

8.6.2 Conclusion of the proof of Theorem 8.1.1

By virtue of (8.4.6) and Lemma 8.6.1, for $\varepsilon > 0$ sufficiently small the functional

$$G_\varepsilon(t) = E(t) + \varepsilon^2 M_\varepsilon(t)$$

fulfills the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} G_\varepsilon + \varepsilon^4 E \leq -\Big(\frac{1}{\beta} - c_M\varepsilon\Big)\Gamma[\eta] \leq 0.$$

It is also clear from the definition of the functionals involved that, for all $\varepsilon > 0$ small,

$$\frac{1}{2}E(t) \leq G_\varepsilon(t) \leq 2E(t).$$

Therefore, an application of the Gronwall lemma entails the required exponential decay of the energy. ☐

Remark 8.6.1. *It is worth observing that, contrary to what done in [89], here the proof of exponential stability is based on the construction of explicit energy-like functionals. The advantage (with respect to linear semigroups techniques) is that the same calculations apply to the analysis of nonlinear version of the problem, allowing, for instance, to prove the existence of absorbing sets.*

8.7 Proof of Theorem 8.1.1 (Necessity)

In this section, we prove that the semigroup $S(t)$ is not exponentially stable when the stability number χ_g is different from zero. According to the abstract Lemma 1.2.2 of Chapter 1, we show that condition (1.2.1) fails to hold assuming, without loss of generality, $\ell = \pi$. Accordingly, for every $n \in \mathbb{N}$, the vector

$$\zeta_n = \big(0, \tfrac{\sin nx}{\rho_1}, 0, 0, 0, 0\big)$$

satisfies

$$\|\zeta_n\|_{\mathcal{H}} = \sqrt{\tfrac{\pi}{2\rho_1}}\,.$$

For all $n \in \mathbb{N}$, we denote for short

$$\lambda_n = \sqrt{\tfrac{\kappa}{\rho_1}}\,n$$

and we study the equation

$$\mathrm{i}\lambda_n z_n - \mathbb{A} z_n = \zeta_n$$

in the unknown variable

$$z_n = (\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, \theta_n, \eta_n).$$

Our conclusion is reached if we show that z_n is not bounded in $\mathcal{H}$, since this would violate (1.2.1). Straightforward calculations entail the system

$$\begin{cases} \rho_1\lambda_n^2\varphi_n + \kappa(\varphi_{nx} + \psi_n)_x = -\sin nx, \\ \rho_2\lambda_n^2\psi_n + b\psi_{nxx} - \kappa(\varphi_{nx} + \psi_n) - \delta\theta_{nx} = 0, \\ \mathrm{i}\rho_3\lambda_n\theta_n - \dfrac{1}{\beta}\displaystyle\int_0^\infty \mu(s)\eta_{nxx}(s)\,\mathrm{d}s + \mathrm{i}\delta\lambda_n\psi_{nx} = 0, \\ \mathrm{i}\lambda_n\eta_n - T\eta_n - \theta_n = 0. \end{cases}$$

We now look for solutions (compatible with the boundary conditions) of the form

$$\begin{aligned} \varphi_n &= A_n \sin nx, \\ \psi_n &= B_n \cos nx, \\ \theta_n &= C_n \sin nx, \\ \eta_n &= \phi_n(s) \sin nx, \end{aligned}$$

for some $A_n, B_n, C_n \in \mathbb{C}$ and some complex square summable function ϕ_n on $\mathbb{R}^+$ with respect to the measure $\mu(s)\mathrm{d}s$, satisfying $\phi_n(0) = 0$. This yields

$$\begin{cases} \kappa n B_n = 1, \\ knA_n + \big(-\rho_2\lambda_n^2 + bn^2 + \kappa\big)B_n + \delta n C_n = 0, \\ \mathrm{i}\rho_3\lambda_n C_n + \dfrac{n^2}{\beta}\displaystyle\int_0^\infty \mu(s)\phi_n(s)\,\mathrm{d}s - \mathrm{i}\delta\lambda_n n B_n = 0, \\ \mathrm{i}\lambda_n\phi_n + \phi_n' - C_n = 0. \end{cases}$$

An integration of the last equation gives

$$\phi_n(s) = \frac{C_n}{\mathrm{i}\lambda_n}\big(1 - \mathrm{e}^{-\mathrm{i}\lambda_n s}\big).$$

Substituting the result into the third equation above, and denoting by

$$\hat{\mu}(\lambda_n) = \int_0^\infty \mu(s)\mathrm{e}^{-\mathrm{i}\lambda_n s}\,\mathrm{d}s$$

the Fourier transform[5] of μ, we find the explicit solution

$$A_n = \frac{\rho_2\kappa n^2 - \rho_1 bn^2 - \rho_1\kappa}{\rho_1\kappa^2 n^2} - \frac{\delta^2\beta}{\rho_3\kappa\beta\gamma_g + \rho_1\kappa\hat{\mu}(\lambda_n)},$$

where, according to the notation of Section 8.5,

$$\gamma_g = \kappa - \frac{g(0)\rho_1}{\beta\rho_3}.$$

At this point, we consider separately two cases.

Case $\gamma_g = 0$

We have

$$A_n = \frac{\rho_2\kappa n^2 - \rho_1 bn^2 - \rho_1\kappa}{\rho_1\kappa^2 n^2} - \frac{\delta^2\beta}{\rho_1\kappa\hat{\mu}(\lambda_n)}.$$

Due to the convergence $\hat{\mu}(\lambda_n) \to 0$, ensured by the Riemann-Lebesgue lemma, we find the asymptotic expression as $n \to \infty$

$$A_n \sim -\frac{\delta^2\beta}{\rho_1\kappa\hat{\mu}(\lambda_n)}.$$

Since

$$\|z_n\|_{\mathcal{H}}^2 \geq \kappa\|\varphi_{nx} + \psi_n\|^2 + b\|\psi_{nx}\|^2,$$

there exists $\varpi > 0$ such that

$$\|z_n\|_{\mathcal{H}} \geq \varpi\|\varphi_{nx}\| = \varpi n|A_n|\bigg(\int_0^\pi \cos^2 nx\,\mathrm{d}x\bigg)^{\frac{1}{2}} = \frac{\varpi\sqrt{\pi}}{\sqrt{2}}n|A_n| \to \infty.$$

[5]Since μ is continuous nonincreasing and summable, it is easy to see that $\hat{\mu}(\lambda_n) \neq 0$ for every n.

Case $\gamma_g \neq 0$

Exploiting again the Riemann-Lebesgue lemma, we now get

$$A_n \to \frac{1}{\kappa}\left(\frac{\rho_2}{\rho_1} - \frac{b}{\kappa}\right) - \frac{\delta^2}{\rho_3\kappa\gamma_g} = \frac{\rho_3 b g(0)}{\rho_1\rho_3\beta\gamma_g}\,\chi_g \neq 0,$$

as $\chi_g \neq 0$ by assumption. As before, we end up with

$$\|z_n\|_{\mathcal{H}} \geq \frac{\varpi\sqrt{\pi}}{\sqrt{2}} n|A_n| \to \infty.$$

This finishes the proof. □

Remark 8.7.1. *The proof above actually holds within the same minimal assumptions on the memory kernel ensuring the existence of $S(t)$, i.e. μ nonnull, nonnegative, nonincreasing, absolutely continuous and summable on $\mathbb{R}^+$.*

8.8 More on the Comparison with the Cattaneo Model

As previously observed in Section 8.2, at a formal level it is possible to recover the exponential stability (as well as the lack of exponential stability) of the Timoshenko-Cattaneo system from our main Theorem 8.1.1. Here, we give a rigorous proof of this fact. To this aim, let us write explicitly the system studied in [31, 89]

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases}$$

which generates a contraction semigroup $\hat{S}(t)$ acting on the phase space

$$\hat{\mathcal{H}} = H_0^1(\mathfrak{I}) \times L^2(\mathfrak{I}) \times H_*^1(\mathfrak{I}) \times L_*^2(\mathfrak{I}) \times L^2(\mathfrak{I}) \times L^2(\mathfrak{I})$$

normed by

$$\|(\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, \theta, q)\|_{\hat{\mathcal{H}}}^2 = \kappa\|\varphi_x + \psi\|^2 + \rho_1\|\tilde{\varphi}\|^2 + b\|\psi_x\|^2 + \rho_2\|\tilde{\psi}\|^2 + \rho_3\|\theta\|^2 + \tau\|q\|^2.$$

According to the article [89], the semigroup $\hat{S}(t)$ is exponentially stable if and only if the stability number χ_τ equals zero.

Along with $\hat{S}(t)$, we consider the semigroup $S(t)$ on $\mathcal{H}$ generated by (8.4.5) for the particular choice of the kernel

$$\mu(s) = -g_\tau'(s) = \frac{\beta^2}{\tau^2}\,\mathrm{e}^{-\frac{s\beta}{\tau}}.$$

Then, we define the map $\Lambda : \mathcal{M} \to L^2(\mathfrak{I})$ as

$$\Lambda\eta = -\frac{1}{\beta}\int_0^\infty \mu(s)\eta_x(s)\,\mathrm{d}s.$$

On account of the Hölder inequality,

$$\tau\|\Lambda\eta\|^2 \le \frac{\tau}{\beta^2}\left[\int_0^\infty \mu(s)\|\eta_x(s)\|\,\mathrm{d}s\right]^2 \le \frac{1}{\beta}\|\eta\|^2_{\mathcal{M}}. \tag{8.8.1}$$

Due to the peculiar form of the kernel, the following result is a direct consequence of the equations. The easy proof is left to the reader.

Lemma 8.8.1. *Let $z_0 = (u_0, \eta_0) \in \mathcal{H}$ be any initial datum, where u_0 subsumes the first 5 components of z_0, and call $\hat{z}_0 = (u_0, \Lambda\eta_0) \in \hat{\mathcal{H}}$. Then the first 5 components of $S(t)z_0$ and $\hat{S}(t)\hat{z}_0$ coincide. Besides, the last component $q(t)$ of $\hat{S}(t)\hat{z}_0$ fulfills the equality*

$$q(t) = \Lambda\eta^t,$$

where η^t is the last component of $S(t)z_0$.

The full equivalence between the two models is established in the next two propositions.

Proposition 8.8.1. *If $S(t)$ is exponentially stable on $\mathcal{H}$, then so is $\hat{S}(t)$ on $\hat{\mathcal{H}}$.*

Proof. Let $\hat{z}_0 = (u_0, q_0) \in \hat{\mathcal{H}}$ be fixed. Choosing $\eta_0 \in \mathcal{M}$ of the form

$$\eta_0(x,s) = -\tau\int_0^x q_0(y)\,\mathrm{d}y,$$

it is readily seen that $\Lambda\eta_0 = q_0$ and

$$\frac{1}{\beta}\|\eta_0\|^2_{\mathcal{M}} = \frac{\tau^2}{\beta}\|q_0\|^2\int_0^\infty \mu(s)\,\mathrm{d}s = \tau\|q_0\|^2.$$

Lemma 8.8.1 yields the identity

$$\|\hat{S}(t)\hat{z}_0\|_{\hat{\mathcal{H}}} = \|\hat{S}(t)(u_0, \Lambda\eta_0)\|_{\hat{\mathcal{H}}} = \|(u(t), \Lambda\eta^t)\|_{\hat{\mathcal{H}}},$$

where $u(t)$ denotes the first 5 components of either solution. On the other hand, we infer from (8.8.1) and the exponential stability of $S(t)$ that

$$\|(u(t), \Lambda\eta^t)\|_{\hat{\mathcal{H}}} \le \|(u(t), \eta^t)\|_{\mathcal{H}} \le C\|(u_0, \eta_0)\|_{\mathcal{H}}\mathrm{e}^{-\omega t},$$

for some $\omega > 0$ and $C \ge 1$. Since

$$\|(u_0, \eta_0)\|_{\mathcal{H}} = \|\hat{z}_0\|_{\hat{\mathcal{H}}},$$

we are finished. □

Proposition 8.8.2. *If $\hat{S}(t)$ is exponentially stable on $\hat{\mathcal{H}}$, then so is $S(t)$ on $\mathcal{H}$.*

Proof. To simplify the notation, we introduce the 5-component space

$$\mathcal{V} = H_0^1(\mathfrak{I}) \times L^2(\mathfrak{I}) \times H_*^1(\mathfrak{I}) \times L_*^2(\mathfrak{I}) \times L^2(\mathfrak{I})$$

normed by

$$\|(\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, \theta)\|_{\mathcal{V}}^2 = \kappa\|\varphi_x + \psi\|^2 + \rho_1\|\tilde{\varphi}\|^2 + b\|\psi_x\|^2 + \rho_2\|\tilde{\psi}\|^2 + \rho_3\|\theta\|^2.$$

For a fixed $z_0 = (u_0, \eta_0) \in \mathcal{H}$, we set

$$\hat{z}_0 = (u_0, \Lambda\eta_0) \in \hat{\mathcal{H}}.$$

The exponential stability of $\hat{S}(t)$ and (8.8.1) imply that

$$\|u(t)\|_{\mathcal{V}}^2 \leq \|\hat{S}(t)\hat{z}_0\|_{\hat{\mathcal{H}}}^2 \leq C\mathrm{e}^{-\omega t}\|\hat{z}_0\|_{\hat{\mathcal{H}}}^2 \leq C\mathrm{e}^{-\omega t}\|z_0\|_{\mathcal{H}}^2,$$

for some $\omega > 0$ and $C \geq 1$. Again, on account of Lemma 8.8.1, $u(t)$ denotes the first 5 components of either solution. Thus, exploiting the energy identity (8.4.6) together with (8.3.2), we arrive at the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|S(t)z_0\|_{\mathcal{H}}^2 + \nu\|S(t)z_0\|_{\mathcal{H}}^2 \leq \nu\|u(t)\|_{\mathcal{V}}^2 \leq C\nu\mathrm{e}^{-\omega t}\|z_0\|_{\mathcal{H}}^2.$$

A standard application of the Gronwall entails the sought exponential decay estimate.
□

Appendix

In this appendix, we report some abstract results on existence and regularity of global and exponential attractors needed in the analysis carried out in Chapters 2-5. The presentation will be given in the more general context of a semigroup, that is, a family of maps $S(t)$ satisfying assumptions (S.1) and (S.2) of Definition 1.1.1 (no continuity properties are required).

On the Existence of Global Attractors

We present a slightly modified version of a classical theorem on the existence of global attractors for semigroups in metric spaces. Compared to analogous results in the current literature (cf. [17,49]), the main advantage of our approach is that we require very mild continuity-like assumptions on the semigroup.

I. Statement of the result

Given a complete metric space (X, d), let

$$S(t) : X \to X$$

be a semigroup possessing an absorbing set $\mathbb{B}$. In order to state the theorem, we need a definition: a map $S : X \to X$ is *closed* on a set $\mathcal{B} \subset X$ if the implication

$$x_n \to x, \quad Sx_n \to y \quad \Rightarrow \quad Sx = y$$

holds for every sequence $x_n \in \mathcal{B}$. The set $\mathcal{B}$ itself is neither assumed to be closed, nor invariant under the action of S.

Theorem A.1. *Assume there exist $T > 0$, $\nu < 1$ and a precompact pseudometric $\boldsymbol{\rho}$ on $\mathbb{B}$ such that the map $S = S(T)$ is closed on $\mathbb{B}$ and satisfies the inequality*

$$\mathrm{d}(Sx, Sy) \leq \nu\, \mathrm{d}(x, y) + \boldsymbol{\rho}(x, y), \quad \forall\, x, y \in \mathbb{B}.$$

Then $S(t)$ has a (unique) global attractor $\mathbb{A}$.

Since $\mathbb{B}$ is bounded, in comply with our definition of absorbing set, $\boldsymbol{\rho}$ precompact on $\mathbb{B}$ means that any sequence in $\mathbb{B}$ admits a $\boldsymbol{\rho}$-Cauchy subsequence. As a byproduct, for every $\varepsilon > 0$, there exists a finite cover $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ of $\mathbb{B}$ such that

$$\boldsymbol{\rho}(x, y) < \varepsilon, \quad \forall\, x, y \in \mathcal{G}_k.$$

II. Proof of Theorem A.1

In view of our scopes, the main tool is the following abstract result from [75] (cf. also Theorem 1.1.1).

Theorem A.2. *If $\mathbb{B}$ is an absorbing set and*

$$\lim_{n\to\infty} \alpha(S(t_n)\mathbb{B}) = 0$$

for some $t_n \to \infty$, then the ω-limit set $\mathbb{A}$ of $\mathbb{B}$ is nonempty, compact and attracting for the semigroup $S(t)$ on X.

We point out that, other than being a semigroup, no assumptions on $S(t)$ are made. The proof is divided in four steps.

Step 1. We preliminarily show that

$$\alpha(S\mathcal{B}) \leq \nu\alpha(\mathcal{B}), \quad \forall \mathcal{B} \subset \mathbb{B}.$$

Actually, this is well known. For the reader's convenience, we recast the classical argument of [49] (see Lemma 2.3 therein). Given $\mathcal{B} \subset \mathbb{B}$, select an arbitrary $\varepsilon > 0$. Then

$$\mathcal{B} \subset \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n \qquad \text{and} \qquad \mathcal{B} \subset \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m,$$

for some $\mathcal{F}_i$ and $\mathcal{G}_k$ satisfying

$$\mathrm{diam}(\mathcal{F}_i) \leq \alpha(\mathcal{B}) + \varepsilon \qquad \text{and} \qquad \boldsymbol{\rho}(x, y) < \varepsilon, \quad \forall\, x, y \in \mathcal{G}_k.$$

Setting $\mathcal{M}_{ik} = \mathcal{F}_i \cap \mathcal{G}_k$, from the properties of α (see Chapter 1) we deduce the inequality

$$\alpha(S\mathcal{B}) \leq \alpha\big(S \textstyle\bigcup_{ik} \mathcal{M}_{ik}\big) = \alpha\big(\textstyle\bigcup_{ik} S\mathcal{M}_{ik}\big) = \max_{ik} \alpha(S\mathcal{M}_{ik}) \leq \max_{ik} \mathrm{diam}(S\mathcal{M}_{ik}).$$

On the other hand, for every $x, y \in \mathcal{M}_{ik}$, we have

$$\mathrm{d}(Sx, Sy) \leq \nu\, \mathrm{d}(x, y) + \boldsymbol{\rho}(x, y) \leq \nu\alpha(\mathcal{B}) + \varepsilon(\nu + 1).$$

Consequently,

$$\alpha(S\mathcal{B}) \leq \max_{ik} \mathrm{diam}(S\mathcal{M}_{ik}) \leq \nu\alpha(\mathcal{B}) + \varepsilon(\nu + 1),$$

and a final limit $\varepsilon \to 0$ will do.

Step 2. Since $\mathbb{B}$ is absorbing, for some $m \in \mathbb{N}$ large enough we know that

$$S^n\mathbb{B} \subset \mathbb{B}, \quad \forall n \geq m.$$

In the usual notation, $S^n = S \circ \cdots \circ S$ (n-times). Therefore, exploiting Step 1,

$$\alpha(S^n\mathbb{B}) = \alpha(SS^{n-1}\mathbb{B}) \leq \nu\alpha(S^{n-1}\mathbb{B}) \leq \cdots \leq \nu^{n-m}\alpha(S^m\mathbb{B}) \leq \nu^{n-m}\alpha(\mathbb{B}).$$

Choosing then $t_n = nT$, we draw the convergence

$$\lim_{n\to\infty} \alpha(S(t_n)\mathbb{B}) = \lim_{n\to\infty} \alpha(S^n\mathbb{B}) = 0,$$

and from Theorem A.2 we learn that the ω-limit set $\mathbb{A}$ of $\mathbb{B}$ is compact and attracting.

Step 3. We claim that the equality $S\mathbb{A} = \mathbb{A}$ implies the full invariance of $\mathbb{A}$. Indeed, let $t \geq 0$ be arbitrarily fixed. Since $\mathbb{A}$ is closed, attracting and (in the above assumption)

$$\mathbb{A} = S^n\mathbb{A} = S(nT)\mathbb{A},$$

we get

$$\boldsymbol{\delta}_X(S(t)\mathbb{A}, \mathbb{A}) = \lim_{n\to\infty} \boldsymbol{\delta}_X(S(t+nT)\mathbb{A}, \mathbb{A}) = 0 \quad \Rightarrow \quad S(t)\mathbb{A} \subset \mathbb{A}.$$

Once $\mathbb{A}$ is known to be invariant, for n large we can write

$$\mathbb{A} = S(nT)\mathbb{A} = S(t)S(nT-t)\mathbb{A} \subset S(t)\mathbb{A} \subset \mathbb{A} \quad \Rightarrow \quad S(t)\mathbb{A} = \mathbb{A}.$$

Step 4. Finally, we show that $S\mathbb{A} = \mathbb{A}$. Let then $x \in \mathbb{A}$. By the definition of ω-limit set, there exist sequences $t_n \to \infty$ and $\xi_n \in \mathbb{B}$ satisfying

$$x_n \doteq S(t_n)\xi_n \to x.$$

At the same time, as $\mathbb{A}$ is compact and attracting, there exist $y, z \in \mathbb{A}$ such that (up to subsequences)

$$y_n \doteq S(t_n - T)\xi_n \to y \qquad \text{and} \qquad z_n \doteq S(t_n + T)\xi_n \to z.$$

Summarizing,

$$y_n \to y, \qquad Sy_n = x_n \to x, \qquad x_n \to x, \qquad Sx_n = z_n \to z.$$

Since the sequences x_n, y_n, z_n eventually fall in the absorbing set $\mathbb{B}$ and the map S is closed on $\mathbb{B}$, we deduce the equalities $Sy = x$ and $Sx = z$.

In conclusion, the set $\mathbb{A}$ is compact, attracting and fully invariant for $S(t)$. The proof of Theorem A.1 is finished. □

On the Existence of Exponential Attractors

We recall a well-known abstract result on existence of exponential attractors, first devised in [27]. Let $\mathcal{H}_1 \Subset \mathcal{H}$ be compactly embedded Banach spaces, and let $S(t) : \mathcal{H} \to \mathcal{H}$ be a semigroup. Finally, let $\mathbb{B} \subset \mathcal{H}$ be a bounded closed invariant set.

I. The discrete case

Firstly, we focus on the discrete semigroup

$$S^n = S \circ \cdots \circ S \quad \text{(n-times)}$$

generated by a map $S : \mathbb{B} \to \mathbb{B}$. In this case, we talk of a discrete exponential attractor, namely, a compact invariant set $\mathbb{E}_{\rm d} \subset \mathbb{B}$ of finite fractal dimension such that

$$\boldsymbol{\delta}_{\mathcal{H}}(S^n\mathbb{B}, \mathbb{E}_{\rm d}) \leq C\mathrm{e}^{-\omega n}$$

for some $\omega > 0$ and $C \geq 0$.

We have the following lemma [17, Corollary 2.23].

Lemma A.1. *For some $\vartheta < 1$ and $\Theta \geq 0$, let the map S admit the decomposition*

$$S = L + K,$$

where

$$\|Lz_1 - Lz_2\|_{\mathcal{H}} \leq \vartheta \|z_1 - z_2\|_{\mathcal{H}} \tag{A.1}$$

and

$$\|Kz_1 - Kz_2\|_{\mathcal{H}_1} \leq \Theta \|z_1 - z_2\|_{\mathcal{H}} \tag{A.2}$$

for every $z_1, z_2 \in \mathbb{B}$. Then there exists a discrete exponential attractor $\mathbb{E}_{\rm d}$.

II. The continuous case

By definition, an exponential attractor for the semigroup $S(t)$ on $\mathbb{B}$ is a compact invariant set $\mathbb{E} \subset \mathcal{H}$ of finite fractal dimension attracting the set $\mathbb{B}$ at an exponential rate, namely

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathbb{B}, \mathbb{E}) \leq C\mathrm{e}^{-\omega t}$$

for some $\omega > 0$ and some $C \geq 0$.

The next result appears in several slightly different forms in the literature (see e.g. [66]).

Theorem A.3. *Assume there exists $t_\star > 0$ such that the map $S = S(t_\star)$ fulfills the hypotheses of Lemma A.1. Moreover, let the map*

$$(t, z) \mapsto S(t)z$$

be Lipschitz continuous on $[0, t_\star] \times \mathbb{B}$ into $\mathcal{H}$. Then there exists an exponential attractor $\mathbb{E}$ on $\mathbb{B}$.

The exponential attractor $\mathbb{E}$ on $\mathbb{B}$ is obtained by setting

$$\mathbb{E} = \bigcup_{\tau \in [0, t_\star]} S(\tau)\mathbb{E}_{\rm d},$$

where $\mathbb{E}_{\rm d}$ is the discrete exponential attractor of the map $S = S(t_\star)$, whose existence is ensured by Lemma A.1. Since $\mathbb{B}$ is invariant and $\mathbb{E}_{\rm d} \subset \mathbb{B}$, we obtain in particular $\mathbb{E} \subset \mathbb{B}$.

On the Regularity of Global Attractors

Let $\mathcal{V} \Subset \mathcal{H}$ be reflexive Banach spaces with compact embedding, and let $S(t) : \mathcal{H} \to \mathcal{H}$ be a semigroup possessing a (bounded) absorbing set $\mathbb{B}_0$. Within some continuity assumptions (e.g. the strong continuity of the semigroup), a standard strategy to prove the existence of the global attractor $\mathbb{A}$ for $S(t)$ is showing that

$$\lim_{t\to\infty} \big[\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathbb{B}_0, \mathcal{C}(t))\big] = 0, \tag{A.3}$$

where $\mathcal{C}(t)$ is a bounded subset of $\mathcal{V}$ for every fixed t. If, in addition, one has the uniform-in-time estimate

$$\sup_{t\geq 0} \|\mathcal{C}(t)\|_{\mathcal{V}} < \infty, \tag{A.4}$$

then $\mathbb{A}$ is bounded in $\mathcal{V}$ as well. Unfortunately, the latter conclusion cannot be merely inferred from (A.3). To this end, we report here a version of an abstract result from [21], tailored for our scopes, allowing to deduce regularity properties for the global attractor without making use of estimate (A.4), generally hard to obtain in several concrete situations. We proceed with a definition.

Definition A.1. *A solution operator on $\mathcal{H}$ is a family of maps $U(t) : \mathcal{H} \to \mathcal{H}$, depending on the parameter $t \geq 0$, such that $U(0)z = z$ for all $z \in \mathcal{H}$.*

Assuming the strong continuity of the semigroup $S(t)$ and calling $r = \|\mathbb{B}_0\|_{\mathcal{H}} < \infty$, the theorem reads as follows.

Theorem A.4. *For every $z \in \mathbb{B}_0$ let there exist two solution operators $V_z(t)$ on $\mathcal{H}$ and $U_z(t)$ on $\mathcal{V}$ such that the following hold.* [6]

(i) For any $x \in \mathsf{B}_{\mathcal{H}}(r)$ and $y \in \mathcal{V}$ satisfying $x + y = z$, we have

$$S(t)z = V_z(t)x + U_z(t)y.$$

(ii) There is $\alpha \in \mathfrak{D}$ such that, for every $x \in \mathsf{B}_{\mathcal{H}}(r)$,

$$\sup_{z\in\mathbb{B}_0} \|V_z(t)x\|_{\mathcal{H}} \leq \alpha(t)\|x\|_{\mathcal{H}}.$$

(iii) There are $\beta \in \mathfrak{D}$ and $J \in \mathfrak{I}$ such that, for every $y \in \mathcal{V}$,

$$\sup_{z\in\mathbb{B}_0} \|U_z(t)y\|_{\mathcal{V}} \leq \beta(t)\|y\|_{\mathcal{V}} + J(t).$$

Then there exist constants $K, \omega, \varrho > 0$ such that

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)\mathbb{B}_0, \mathsf{B}_{\mathcal{V}}(\varrho)) \leq K\mathrm{e}^{-\omega t}.$$

As a byproduct, $S(t)$ possesses the global attractor $\mathbb{A}$ bounded in $\mathcal{V}$.

Remark A.1. *Theorem A.4 still works if $S(t)$ is only a semigroup of closed operators (see [75]).*

[6] See Chapter 5, Section 5.2 for the definition of $\mathfrak{D}$, $\mathfrak{I}$ and $\mathsf{B}_{\mathcal{H}}(r)$.

Conclusions

In this thesis, we have studied the asymptotic behavior of some evolution equations with nonclassical heat conduction. After a brief introduction to the theory of infinite-dimensional dynamical systems and linear semigroups, in Chapters 2-5 we have analyzed the strongly damped nonlinear wave equation

$$u_{tt} - \Delta u_t - \Delta u + f(u_t) + g(u) = h,$$

proving existence and regularity of global and exponential attractors and improving all the results available in the previous literature. In particular, the nonlinearity f acting on u_t was allowed to exhibit a critical growth of polynomial order 5. In this situation, the techniques based on the classical Gronwall lemma do not work and it is necessary to exploit novel Gronwall-type lemmas with parameters from [70]. In the fully critical case (i.e. when also the nonlinear function g has a critical growth) existence and optimal regularity of the attractor were obtained under the essential-monotonicity assumption

$$\inf_{s\in\mathbb{R}} f'(s) > -\lambda_1.$$

Still, the existence of the global attractor in the fully critical case within the weaker dissipativity condition

$$\liminf_{|s|\to\infty} f'(s) > -\lambda_1$$

remains an open (and possibly quite challenging) question. Another interesting problem is related to singular limits in type III heat conduction. More precisely, assuming $g \equiv 0$ and restoring the physical constants according to the model equation discussed in the introduction of the thesis, we end up with

$$u_{tt} - \kappa\Delta u_t - \omega\Delta u + f(u_t) = h.$$

It is then of great interest to understand what happens in the "Fourier limit" $\omega \to 0$, formally leading to the classical nonlinear heat equation

$$\vartheta_t - \kappa\Delta\vartheta + f(\vartheta) = h,$$

with the position $\vartheta = u_t$. Quite naturally, one might expect to deduce some kind of convergence of the respective solutions. On the contrary, as shown in [36], for quite general source terms the solutions to the type III equation diverge from the solutions to the Fourier heat equation. Thus, the type III theory of heat conduction cannot be considered as comprehensive of the Fourier one in a proper sense.

In Chapters 6 we have studied the system

$$\begin{cases} u_{tt} - \omega\Delta u_{tt} + \Delta^2 u - \big[b + \|\nabla u\|^2_{L^2(\Omega)}\big]\Delta u + \Delta\alpha_t = g, \\ \alpha_{tt} - \Delta\alpha - \displaystyle\int_0^\infty \mu(s)\Delta[\alpha(t) - \alpha(t-s)]\,\mathrm{d}s - \Delta u_t = 0, \end{cases}$$

modeling type III thermoelastic extensible beams or Berger plates with memory. The main results are existence and regularity of the global attractor, together with some characterizations of exponential and lack of exponential stability for the associated linear semigroup in the limit situation $\omega = 0$. In this context, it would be interesting to develop the analysis by considering more general nonlinearities or boundary conditions on the variable u different from the hinged ones.

In Chapter 7 we have proved the existence of the regular global attractor for the solution semigroup generated by the nonlinear Caginalp phase-field system

$$\begin{cases} u_t - \Delta u + \phi(u) = \alpha_t, \\ \alpha_{tt} - \Delta\alpha_t - \Delta\alpha + g(\alpha) = -u_t. \end{cases}$$

In light of the results obtained in Chapters 2-5, possible future development may concern with the study of more general situations, adding for instance a further nonlinearity depending on the variable α_t and analyzing the asymptotic properties of the corresponding dynamical system.

Finally, in Chapters 8, we have studied the Timoshenko system with Gurtin-Pipkin thermal law

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\vartheta_x = 0, \\ \rho_3\vartheta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g(s)\vartheta_{xx}(t-s)\,\mathrm{d}s + \delta\psi_{tx} = 0, \end{cases}$$

establishing a necessary and sufficient condition for the exponential stability in terms of the structural parameters of the equations, and generalizing previously known results on the Fourier-Timoshenko and the Cattaneo-Timoshenko beam models. However, in spite of this complete characterization of the uniform decay of the solutions, several questions are still open. For instance, the polynomial stability (particularly challenging due to the presence of the memory component) or the asymptotic analysis of nonlinear versions of the system. In particular, in the latter situation, it would be interesting to prove existence and regularity of global and exponential attractors, and to show some kind of convergence in the singular limit when the Gurtin-Pipkin thermal law reduces to the Fourier one.

Bibliography

[1] A.V. Babin and B. Nicolaenko, *Exponential attractors of reaction-diffusion systems in an unbounded domain*, J. Dynam. Differential Equations **7** (1995), 567–590.

[2] A.V. Babin and M.I. Vishik, *Attractors of evolution equations*, North-Holland, Amsterdam, 1992.

[3] V. Belleri and V. Pata, *Attractors for semilinear strongly damped wave equations on* $\mathbb{R}^3$, Discrete Contin. Dynam. Systems **7** (2001), 719–735.

[4] H.M. Berger, *A new approach to the analysis of large deflections of plates*, J. Appl. Mech. **22** (1955), 465–472.

[5] F. Bucci and I. Chueshov, *Long-time dynamics of a coupled system of nonlinear wave and thermoelastic plate equations*, Discrete Contin. Dyn. Syst. **22** (2008), 557–586.

[6] G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Rational Mech. Anal. **92** (1989), 205–245.

[7] A.N. Carvalho and J.W. Cholewa, *Attractors for strongly damped wave equations with critical nonlinearities*, Pacific J. Math. **207** (2002), 287–310.

[8] C. Cattaneo, *Sulla conduzione del calore*, Atti Semin. Matemat. Univ. Modena **3** (1948), 83–101.

[9] V.V. Chepyzhov, M. Conti and V. Pata, *A minimal approach to the theory of global attractors*, Discrete Contin. Dyn. Syst. **32** (2012), 2079–2088.

[10] V.V. Chepyzhov, E. Mainini and V. Pata, *Stability of abstract linear semigroups arising from heat conduction with memory*, Asymptot. Anal. **50** (2006), 269–291.

[11] V.V. Chepyzhov and V. Pata, *Some remarks on stability of semigroups arising from linear viscoelasticity*, Asymptot. Anal. **46** (2006), 251–273.

[12] V.V. Chepyzhov, V. Pata and M.I. Vishik, *Averaging of 2D Navier-Stokes equations with singularly oscillating forces*, Nonlinearity **22** (2009), 351–370.

[13] V.V. Chepyzhov and M.I. Vishik, *Attractors for equations of mathematical physics*, Amer. Math. Soc., Providence, 2002.

[14] C.I. Christov and P.M. Jordan, *Heat conduction paradox involving second-sound propagation in moving media*, Phys. Rev. Lett. **94** (2005), p.154301.

[15] I. Chueshov, *Introduction to the theory of infinite-dimensional dissipative systems*, Acta, Kharkov, 2002.

[16] I. Chueshov and I. Lasiecka, *Attractors and long-time behavior of von Karman thermoelastic plates*, Appl. Math. Optim. **58** (2008), 195–241.

[17] I. Chueshov and I. Lasiecka, *Long-time behaviour of second order evolution equations with nonlinear damping*, Mem. Amer. Math. Soc. **195** (2008), viii+183 pp.

[18] B.D. Coleman and M.E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys. **18** (1967), 199–208.

[19] M. Conti, E.M. Marchini and V. Pata, *Semilinear wave equations of viscoelasticity in the minimal state framework*, Discrete Contin. Dyn. Syst. **27** (2010), 1535–1552.

[20] M. Conti and V. Pata, *Weakly dissipative semilinear equations of viscoelasticity*, Commun. Pure Appl. Anal. **4** (2005), 705–720.

[21] M. Conti and V. Pata, *On the regularity of global attractors,* Discrete Contin. Dyn. Syst. **25** (2009), 1209–1217.

[22] M. Conti, V. Pata and M. Squassina, *Singular limit of differential systems with memory*, Indiana Univ. Math. J. **55** (2006), 170–213.

[23] M. Coti Zelati, V. Pata and R. Quintanilla, *Regular global attractors of type III thermoelastic extensible beams*, Chin. Ann. Math. Series B **31** (2010), 619–630.

[24] R.F. Curtain and H.J. Zwart, *An introduction to infinite-dimensional linear system theory*, Springer-Verlang, New York, 1995.

[25] C.M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37** (1970), 297–308.

[26] A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential attractors for dissipative evolution equations*, Paris, Masson, 1994.

[27] M. Efendiev, A. Miranville and S. Zelik, *Exponential attractors for a nonlinear reaction-diffusion system in* $\mathbb{R}^3$, C.R. Acad. Sci. Paris Sér. I Math. **330** (2000), 713–718.

[28] P. Fabrie, C. Galusinski, A. Miranville and S. Zelik, *Uniform exponential attractors for a singularly perturbed damped wave equation*, Discrete Contin. Dyn. Syst. **10** (2004), 211–238.

[29] M. Fabrizio, C. Giorgi and V. Pata, *A new approach to equations with memory*, Arch. Rational Mech. Anal. **198** (2010), 189–232.

[30] M. Fabrizio, B. Lazzari and J.E. Muñoz Rivera, *Asymptotic behaviour of thermoelastic plates of weakly hyperbolic type*, Differential Integral Equation **13** (2000), 1347–1370.

[31] H.D. Fernández Sare and R. Racke, *On the stability of damped Timoshenko systems: Cattaneo versus Fourier law*, Arch. Rational Mech. Anal. **194** (2009), 221–251.

[32] G. Fichera, *Is the Fourier theory of heat propagation paradoxical?*, Rend. Circ. Mat. Palermo **41** (1992), 5–28.

[33] J. Fort, and V. Méndez, *Wavefront in time-delayed reaction-diffusion systems. Theory and comparison to experiments*, Rep. Prog. Phys. **65** (2002), 895–954.

[34] S. Gatti, A. Miranville, V. Pata and S. Zelik, *Attractors for semilinear equations of viscoelasticity with very low dissipation*, Rocky Mountain J. Math. **38** (2008), 1117–1138.

[35] S. Gatti, A. Miranville, V. Pata and S. Zelik, *Continuous families of exponential attractors for singularly perturbed equations with memory*, Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), 329–366.

[36] C. Giorgi, D. Grandi and V. Pata, *On the Green-Naghdi type III heat conduction model*, Discrete Contin. Dyn. Syst. Ser. B (in press).

[37] C. Giorgi, M.G. Naso and V. Pata, *Exponential stability in linear heat conduction with memory: a semigroup approach*, Comm. Appl. Anal. **5** (2001), 121–134.

[38] C. Giorgi, M.G. Naso, V. Pata and M. Potomkin, *Global attractors for the extensible thermoelastic beam system*, J. Differential Equations **246** (2009), 3496–3517.

[39] C. Giorgi, V. Pata and E. Vuk, *On the extensible viscoelastic beam*, Nonlinearity **21** (2008), 713–733.

[40] M. Grasselli, J.E. Muñoz Rivera and V. Pata, *On the energy decay of the linear thermoelastic plate with memory*, J. Math. Anal. Appl. **309** (2005), 1–14.

[41] M. Grasselli and V. Pata, *Uniform attractors of nonautonomous systems with memory*, in "Evolution Equations, Semigroups and Functional Analysis" (A. Lorenzi and B. Ruf, Eds.), pp.155–178, Progr. Nonlinear Differential Equations Appl. no.50, Birkhäuser, Boston, 2002.

[42] M. Grasselli and M. Squassina, *Exponential stability and singular limit for a linear thermoelastic plate with memory effects*, Adv. Math. Sci. Appl. **16** (2006), 15–31.

[43] A.E. Green and P.M. Naghdi, *On undamped heat waves in an elastic solid*, J. Thermal Stresses **15** (1992), 253–264.

[44] A.E. Green and P.M. Naghdi, *Thermoelasticity without energy dissipation*, J. Elasticity **31** (1993), 189–208.

[45] A.E. Green and P.M. Naghdi, *A unified procedure for construction of theories of deformable media. I. Classical continuum physics*, Proc. Roy. Soc. London A **448** (1995), 335–356.

[46] A.E. Green and P.M. Naghdi, *A unified procedure for construction of theories of deformable media. II. Generalized continua*, Proc. Roy. Soc. London A **448** (1995), 357–377.

[47] A.E. Green and P.M. Naghdi, *A unified procedure for construction of theories of deformable media. III. Mixtures of interacting continua*, Proc. Roy. Soc. London A **448** (1995), 379–388.

[48] M.E. Gurtin and A.C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal. **31** (1968), 113–126.

[49] J.K. Hale, *Asymptotic behavior of dissipative systems*, Amer. Math. Soc., Providence, 1988.

[50] A. Haraux, *Systèmes dynamiques dissipatifs et applications*, Masson, Paris, 1991.

[51] S. Jiang and R. Racke, *Evolution equations in thermoelasticity*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics n.112, Boca Raton, 2000.

[52] V. Kalantarov and S. Zelik, *Finite-dimensional attractors for the quasi-linear strongly-damped wave equation*, J. Differential Equations **247** (2009), 1120–1155.

[53] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1980.

[54] A.Kh. Khanmamedov, *On the existence of a global attractor for the wave equation with nonlinear strong damping perturbed by nonmonotone term*, Nonlinear Anal. **69** (2008), 3372–3385.

[55] J.U. Kim, *On the energy decay of a linear thermoelastic bar and plate*, SIAM J. Math. Anal. **23** (1992), 889–899.

[56] O.A. Ladyzhenskaya, *Finding minimal global attractors for the Navier-Stokes equations and other partial differential equations*, Russian Math. Surveys **42** (1987), 27–73.

[57] J. Lagnese, *Boundary stabilization of thin plates*, SIAM Stud. Appl. Math. n.10, SIAM, Philadelphia, 1989.

[58] Z. Liu and R. Quintanilla, *Analyticity of solutions in type III thermoelastic plates*, IMA J. Appl. Math. **75** (2010), 637–646.

[59] V. Méndez, J. Fort and J. Farjas, *Speed of wavefront solutions to hyperbolic reaction-diffusion systems*, Phys. Rev. E **60** (1999), 5231–5243.

[60] V. Méndez and J.E. Llebot, *Hyperbolic reaction-diffusion equations for a forest fire model*, Phys. Rev. E **56** (1997), 6557–6563.

[61] S.A. Messaoudi and B. Said-Houari, *Energy decay in a Timoshenko-type system of thermoelasticity of type III*, J. Math. Anal. Appl. **348** (2008), 298–307.

[62] A. Miranville and R. Quintanilla, *A generalization of the Caginalp phase-field system based on the Cattaneo law*, Nonlinear Anal. **71** (2009), 2278–2290.

[63] A. Miranville and R. Quintanilla, *Some generalizations of the Caginalp phase-field system*, Appl. Anal. **88** (2009), 877–894.

[64] A. Miranville and R. Quintanilla, *A phase-field model based on a three-phase-lag heat conduction*, Appl. Math. Optim. **63** (2011), 133–150.

[65] A. Miranville and R. Quintanilla, *A type III phase-field system with a logarithmic potential*, Appl. Math. Letters **24** (2011), 1003–1008.

[66] A. Miranville and S. Zelik, *Attractors for dissipative partial differential equations in bounded and unbounded domains*, in "Handbook of Differential Equations: Evolutionary Equations", Vol. IV, Elsevier/North-Holland, Amsterdam, 2008.

[67] J.E. Muñoz Rivera and R. Racke, *Mildly dissipative nonlinear Timoshenko systems–global existence and exponential stability*, J. Math. Anal. Appl. **276** (2002), 248–278.

[68] V. Pata, *Exponential stability in linear viscoelasticity*, Quart. Appl. Math. **64** (2006), 499–513.

[69] V. Pata, *Stability and exponential stability in linear viscoelasticity*, Milan J. Math. **77** (2009), 333–360.

[70] V. Pata, *Uniform estimates of Gronwall type*, J. Math. Anal. Appl. **373** (2011), 264–270.

[71] V. Pata and R. Quintanilla, *On the decay of solutions in nonsimple elastic solids with memory*, J. Math. Anal. Appl. **363** (2010), 19–28.

[72] V. Pata and M. Squassina, *On the strongly damped wave equation*, Comm. Math. Phys. **253** (2005), 511–533.

[73] V. Pata and S. Zelik, *Smooth attractors for strongly damped wave equations*, Nonlinearity **19** (2006), 1495–1506.

[74] V. Pata and S. Zelik, *A remark on the damped wave equation*, Commun. Pure Appl. Anal. **5** (2006), 609–614.

[75] V. Pata and S. Zelik, *A result on the existence of global attractors for semigroups of closed operators*, Commun. Pure Appl. Anal. **6** (2007), 481–486.

[76] V. Pata and A. Zucchi, *Attractors for a damped hyperbolic equation with linear memory*, Adv. Math. Sci. Appl. **11** (2001), 505–529.

[77] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[78] M. Potomkin, *Asymptotic behavior of thermoviscoelastic Berger plate*, Commun. Pure Appl. Anal. **9** (2010), 161–192.

[79] J. Prüss, *On the spectrum of C_0-semigroups*, Trans. Amer. Math. Soc. **284** (1984), 847–857.

[80] R. Quintanilla, *Damping of end effects in a thermoelastic theory*, Appl. Math. Letters **14** (2001), 137–141.

[81] R. Quintanilla, *On existence in thermoelasticity without energy dissipation*, J. Thermal Stresses **25** (2002), 195–202.

[82] R. Quintanilla, *Impossibility of localization in linear thermoelasticity*, Proc. Roy. Soc. London A **463** (2007), 3311–3322.

[83] R. Quintanilla and R. Racke, *Stability in thermoelasticity of type III*, Discrete Contin. Dyn. Syst. Ser. B **3** (2003), 383–400.

[84] R. Quintanilla and B. Straughan, *Growth and uniqueness in thermoelasticity*, Proc. Roy. Soc. London A **456** (2000), 1419–1429.

[85] R. Quintanilla and B. Straughan, *A note on discontinuity waves in type III thermoelasticity*, Proc. Roy. Soc. London A **460** (2004), 1169–1175.

[86] R. Quintanilla and B. Straughan, *Energy bounds for some non-standard problems in thermoelasticity*, Proc. Roy. Soc. London A **461** (2005), 1147–1162.

[87] R. Quintanilla and B. Straughan, *Nonlinear waves in a Green-Naghdi dissipationless fluid*, J. Non-Newtonian Fluid Mech. **154** (2008), 207–210.

[88] J.C. Robinson, *Infinite-dimensional dynamical systems*, Cambridge University Press, Cambridge, 2001.

[89] M.L. Santos, D.S. Almeida Júnior and J.E. Muñoz Rivera, *The stability number of the Timoshenko system with second sound*, J. Differential Equations **253** (2012), 2715–2733.

[90] B. Straughan, *Heat waves*, Springer, New York, 2011.

[91] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer, New York, 1997.

[92] S.P. Timoshenko, *On the correction for shear of a differential equation for transverse vibrations of prismatic bars*, Philos. Magazine **41** (1921), 744–746.

[93] S. Woinowsky-Krieger, *The effect of an axial force on the vibration of hinged bars*, J. Appl. Mech. **17** (1950), 35–36.