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On monotonicity formulæ, fractional operators and strong competition

Doctoral Dissertation of:
Alessandro Zilio

Supervisors:
Prof. Gianmaria Verzini
Prof. Susanna Terracini

The Chair of the Doctoral Program:
Prof. Roberto Lucchetti

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Contents

Abstract and motivations

The present thesis is the result of a three years research project conducted under the supervision of my advisors prof. Susanna Terracini and prof. Gianmaria Verzini, and in collaboration with my colleague Nicola Soave. The main focus of the thesis is on qualitative properties of solutions to system of elliptic semilinear equations which contain competition features. The manuscript is divided in two parts, corresponding to the two main subjects of the thesis. In the first part we will deal with uniform estimates in appropriate Hölder spaces for solutions to fractional elliptic system involving strong competition. In the second and last part, for a system of elliptic equations concerning the study of Bose-Einstein condensates, we shall prove existence of entire solutions which exhibit an exponential growth at infinity. The main theme, common to the two parts, is the use of monotonicity formulæ of Alt-Caffarelli-Friedman and Almgren type in the study of solutions of elliptic systems. In the following, we briefly comment on the two problems.

Uniform Hölder bounds for strongly competing systems

The first part of the thesis is concerned with the common regularity shared by the solutions of fractional elliptic systems which involve a strong competition. The main subject of investigation are systems of the kind

$$\begin{cases} (-\Delta)^s u_i = f_i(x,u_i) - \beta \sum_{j\neq i} c_{ij}(u_i,u_j) & \text{in } \mathbb{R}^N \\ u_i \in H^s(\mathbb{R}^N) & i = 1,\dots,k \end{cases} \tag{$(P)_\beta$}$$

where the term $(-\Delta)^s$ takes into account the underlying anomalous diffusion process, f_i is used to model internal reaction dynamics, while the last term stands for the competition between the different densities (here and in the rest, we will always assume $\beta > 0$). From a modeling point of view, the competition term is represented by a function c_{ij} which in general satisfies the following assumptions: $rc_{ij}(r,t) \geq 0$ for any $r, t \in \mathbb{R}$ (*positivity*, implying no cooperation), $c_{ij}(r,t)$ is monotone increasing in r for t fixed, monotone increasing in t for $t > 0$ while monotone decreasing in t for $t < 0$

for r fixed (*monotonicity*), $c_{ij}(r,t) = 0$ if and only if $rt = 0$ (*segregation condition*). Of the many possible choices of c_{ij} we shall consider two which are most important in applied sciences, namely the variational (quadratic) competition $c_{ij}(r,t) = a_{ij}rt^2$, and symmetric one $c_{ij}(r,t) = a_{ij}rt$ (and its generalization $c_{ij}(r,t) = a_{ij}r|r|^{p-1}|t|^p$ for $p > 0$).

The interest here is in the study of the behaviour of the solutions to $(P)_\beta$ when the parameter β, which empathizes the strength of the competition mechanism with respect to the diffusion and internal reaction features, diverges. The description of the limiting profiles is quite important from an applied point of view, since the limiting systems associated to $(P)_\beta$ appear in many contest, such as pattern formation and optimal partition problems. It should be mentioned that the limiting problem is in general hard to study, due to the sharp transitions imposed by the diverging competition term, thus an approach based on approximations, as proposed here, is useful in this sense. But the importance of the description of the behavior of the solutions as $\beta \to +\infty$ is crucial also by itself, since the limiting configuration can be seen, from a modeling point of view, as an approximation of highly competing systems. Moreover, this analysis helps to identify the correct conditions that the solutions of the limiting system have to satisfy: we shall see in particular that, in a great contrast with the standard diffusion case $s = 1$, the choice of the competition term affects deeply the geometry and the regularity of the solutions.

Part I of this manuscript is devoted to the analysis of this problem. This part is based on the papers [48, 49, 51].

Existence and qualitative properties of solutions to competing elliptic systems

With respect to the fractional diffusion models introduced before, the standard diffusion case has been already object of a deep analysis by the scientific community. In particular, for the system

$$\begin{cases} -\Delta u_i = f_i(x,u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j^p & \text{in } \Omega \subset \mathbb{R}^N \\ u_i \in H_0^1(\Omega) & i = 1,\dots,k \end{cases}$$

it has been shown (both in the symmetric case $p = 1$ and the variational case $p = 2$) that uniform (w.r.t. β) boundedness in L^∞ of a family of solutions is sufficient to guarantee the optimal regularity of the limiting solutions as $\beta \to +\infty$. This and related results have then been used in order to study existence and qualitative properties of the solutions, as well as to describe precisely how the segregation phenomenon occurs.

In particular, it has been shown that, as $\beta \to \infty$, appropriate scaled versions of the segregating functions converge locally to entire solutions of

$$\Delta u_i = u_i \sum_{j \neq i} a_{ij} u_j^p \qquad \text{in } \mathbb{R}^N$$

and thus a complete characterization of such solutions has become an important subject. The case $p = 1$ was already studied in one of the pioneering papers on this subject by Conti, Terracini and Verzini [22] (see Lemmas 4.2-4.3 [22] where the authors show existence and uniqueness of entire solutions). In the case $p = 2$, only recently Berestycki, Lin, Terracini, Wang, Wei, Zhao were able to show in [5, 6] the following.

Theorem (Theorem 1.2 [5] and Theorem 1.3 [6])**.** *Let $\Phi = \Re(z^d)$ for some $d \in \mathbb{N}$. Then there exists at least a solution to*

$$\begin{cases} \Delta u = uv^2 & \text{in } \mathbb{R}^2 \\ \Delta v = u^2 v & \text{in } \mathbb{R}^2 \end{cases} \tag{$(S)_2$}$$

such that $u \geq \Phi^+$, $v \geq \Phi^-$.

Motivated by this result, in Part II we show that the system $(S)_2$ admits also solutions which exhibit an exponential growth. This part is based on the paper [46].

Uniform Hölder bounds for strongly competing systems

Chapter 1

Introduction, main results and some open questions

Regularity issues involving fractional laplacians are very challenging, because of the genuinely non-local nature of such operators, and for this reason they have recently become the object of an intensive research. Above all, the regularity theory of fully nonlinear equations, such as free boundary problems, has found a new interesting field in which the standard (local) analysis can not be applied in its usual form. Thanks to the seminal paper [16], a bridge between standard and fractional elliptic operators was built, creating a new and direct link which made it possible to expand the reach of local analysis also to non-local problems. As a result, in a very short period, a lot of effort by the scientific community culminated in a series of already sharp results in this new field (see for instance [15, 4, 34, 14, 17, 13, 28] for some of the very first results in this new field): it is our purpose to give a contribution in this sense in the context of system of fractional elliptic equations.

Several physical phenomena can be described by a certain number of densities (of mass, population, probability, ...) distributed in a domain and subject to laws of diffusion, reaction, and competitive interaction. Whenever the competition is the prevailing feature, the densities tend to segregate, hence determining a partition of the domain. When anomalous diffusion is involved, one is lead to consider the class of stationary systems of semilinear equations

$$\begin{cases} (-\Delta)^s u_i = f_i(x, u_i) - \beta \sum_{j \neq i} c_{ij}(u_i, u_j) \\ u_i \in H^s(\mathbb{R}^N), \end{cases}$$

thus focusing on the singular limit problem obtained when the (positive) parameter β, accounting for the competitive interactions, diverges to ∞. Among the others, the cases $f_i(r) = g_i r(1 - r/K_i)$, $c_{ij}(r, t) = a_{ij} rt$ (logistic internal dynamics with Lotka-Volterra competition) and $f_i(r) = \omega_i r^3 + \lambda_i r$, $c_{ij}(r, t) = a_{ij} rt^2$ (focusing-defocusing

Gross-Pitaevskii system with competitive interactions, see for instance [19, 18]) are of the highest interest in the applications to population dynamics and theoretical physics, respectively.

For the standard Laplace diffusion operator (namely $s = 1$), the analysis of the qualitative properties of solutions to the corresponding systems has been undertaken, starting from [19, 20, 21], in a series of recent papers [22, 54, 11, 12, 39], also in the parabolic case [53, 24, 25, 26]. In the singular limit one finds a vector $\mathbf{u} = (u_1, \cdots, u_k)$ of limiting profiles with mutually disjoint supports: indeed, *the segregated states* u_i satisfy $u_i \cdot u_j \equiv 0$, for $i \neq j$, and

$$-\Delta u_i = f_i(x, u_i) \qquad \text{whenever} \;\; u_i \neq 0\,, \quad i = 1, \dots, k.$$

Natural questions concern the functional classes of convergence (a priori bounds), optimal regularity of the limiting profiles, equilibrium conditions at the interfaces, and regularity of the nodal set. In [22] (for the Lotka-Volterra competition) and [39] (for the variational Gross-Pitaevskii one) it is proved that L^∞ boundedness implies $\mathcal{C}^{0,\alpha}$ boundedness, uniformly as $\beta \to +\infty$, for every $\alpha \in (0, 1)$. Moreover, it is shown that the limiting profiles are Lipschitz continuous. The proof relies upon elliptic estimates, the blow-up technique, the monotonicity formulæ by Almgren [1] and Alt-Caffarelli-Friedman [2], and it reveals a subtle interaction between diffusion and competition aspects. This interaction mainly occurs at two levels: the validity and exactness of the Alt-Caffarelli-Friedman monotonicity formula and, consequently, the validity of Liouville type theorems for entire solutions to semilinear systems.

As a consequence of the quite precise description of the segregation phenomena in the standard case $s = 1$, we extended the analysis to the case of fractional diffusion $s \in (0, 1)$.

Notation Since the notation used in this part is quite peculiar and contains some different symbols from the standard one, we chose to collect it here. We will agree that any $X \in \mathbb{R}^{N+1}$ can be written as $X = (x, y)$, with $x \in \mathbb{R}^N$ and $y \in \mathbb{R}$, in such a way that $\mathbb{R}^{N+1}_+ := \mathbb{R}^{N+1} \cap \{y > 0\}$. For any $D \subset \mathbb{R}^{N+1}$ we write

$$\begin{aligned} D^+ &:= D \cap \{y > 0\}, \\ \partial^+ D^+ &:= \partial D \cap \{y > 0\}, \\ \partial^0 D^+ &:= D \cap \{y = 0\}. \end{aligned}$$

In most cases, we use this notation with $D = B_r(x_0, 0)$ (the $(N+1)$-dimensional ball centered at a point of $\mathbb{R}^N$). In such case, we denote

$$S^{N-1}_r(x_0, 0) := \{(x, 0) : x \in \mathbb{R}^N,\, |x - x_0| = r\} = \partial B^+_r \setminus \left(\partial^+ B^+_r \cup \partial^0 B^+_r\right).$$

Beyond the usual functional spaces, we will write

$$H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right) := \{v : \forall D \subset \mathbb{R}^{N+1} \text{ open and bounded}, v|_{D^+} \in H^1(D^+)\}.$$

The parameter s will stand mostly to indicate the fractional power of the Laplace operator in analysis, thus $s \in (0,1)$ will be implied when not explicitly remarked. Accordingly, we will let $a := 1-2s \in (-1,1)$. Well known properties of the Muckenhoupt A_2-weights (see for instance [41]) allow to introduce the weighted spaces

$$H^{1;a}(\Omega) := \left\{ v : \int_\Omega y^a \left(|v|^2 + |\nabla v|^2\right) \mathrm{d}x\mathrm{d}y < \infty \right\},$$

with its natural Hilbert structure, and

$$H^{1;a}_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right) := \{v : \forall D \subset \mathbb{R}^{N+1} \text{ open and bounded}, v|_{D^+} \in H^{1;a}(D^+)\}.$$

These functional spaces will be needed to deal with some degenerate elliptic equation encountered in the following, mainly regarding the differential operator (on the $(N+1)$-dimensional space)

$$L_a v := -\mathrm{div}\left(|y|^a \nabla v\right),$$

whose (degenerate) co-normal derivative with respect to the set $\{y = 0\}$ we shall denote as

$$\partial^a_\nu v := \lim_{y\to 0^+} -y^a \partial_y v.$$

1.1 *The variational problem, case $s = 1/2$*

The first kind of system we analyse is given by

$$\begin{cases} (-\Delta)^{1/2} u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} u_j^2 \\ u_i \in H^{1/2}(\mathbb{R}^N). \end{cases} \tag{1.1.1}$$

This class of problems includes the already mentioned Gross-Pitaevskii systems with focusing or defocusing nonlinearities

$$\begin{cases} (-\Delta + m_i^2)^{1/2} u_i = \omega_i u_i^3 + \lambda_{i,\beta} u_i - \beta u_i \sum_{j\neq i} a_{ij} u_j^2 \\ u_i \in H^{1/2}(\mathbb{R}^N), \end{cases}$$

with $a_{ij} = a_{ji} > 0$, which is the relativistic version of the Hartree-Fock approximation theory for mixtures of Bose-Einstein condensates in different hyperfine states. Even though we will perform the proof in the case $m_i = 0$ (and $a_{ij} = 1$), the general case, allowing positive masses $m_i > 0$, follows with minor changes and it is actually a bit simpler.

As it is well known (see e.g. [16]), the N-dimensional half laplacian can be interpreted as a Dirichlet-to-Neumann operator and solutions to problem (1.1.1) as traces of harmonic functions on the $(N+1)$-dimensional half space having the right-hand side of (1.1.1) as normal derivative. For this reason, it is worth stating our main results for harmonic functions with nonlinear Neumann boundary conditions involving strong competition terms.

Theorem 1.1.1 (Local uniform Hölder bounds)**.** *Let the functions $f_{i,\beta}$ be continuous and uniformly bounded (w.r.t. β) on bounded sets, and let $\{\mathbf{v}_\beta = (v_{i,\beta})_{1\le i\le k}\}_\beta$ be a family of $H^1(B_1^+)$ solutions to the problems*

$$\begin{cases} -\Delta v_i = 0 & \text{in } B_1^+ \\ \partial_\nu v_i = f_{i,\beta}(v_i) - \beta v_i \sum_{j\neq i} v_j^2 & \text{on } \partial^0 B_1^+. \end{cases} \tag{$(GP)_\beta$}$$

Let us assume that

$$\|\mathbf{v}_\beta\|_{L^\infty(B_1^+)} \le M,$$

for a constant M independent of β. Then for every $\alpha \in (0,1/2)$ there exists a constant $C = C(M,\alpha)$, not depending on β, such that

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}\left(\overline{B_{1/2}^+}\right)} \le C(M,\alpha).$$

Furthermore, $\{\mathbf{v}_\beta\}_\beta$ is relatively compact in $H^1(B_{1/2}^+) \cap \mathcal{C}^{0,\alpha}\left(\overline{B_{1/2}^+}\right)$ for every $\alpha < 1/2$.

As a byproduct, up to subsequences we have convergence of the above solutions to a limiting profile, which components are segregated on the boundary $\partial^0 B^+$. If furthermore $f_{i,\beta} \to f_i$, uniformly on compact sets, we can prove that this limiting profile satisfies

$$\begin{cases} -\Delta v_i = 0 & \text{in } B_1^+ \\ v_i \partial_\nu v_i = f_i(v_i) v_i & \text{on } \partial^0 B_1^+. \end{cases}$$

One can see that, for solutions of this type of equation, the highest possible regularity correspond to the Hölder exponent $\alpha = 1/2$. As a matter of fact, we can prove that the limiting profiles do enjoy such optimal regularity.

Theorem 1.1.2 (Optimal regularity of limiting profiles)**.** *Under the assumptions above, assume moreover that the locally Lipschitz continuous functions f_i satisfy $f_i(s) = f_i'(0)s + O(|s|^{1+\varepsilon})$ as $s \to 0$, for some $\varepsilon > 0$. Then $\mathbf{v} \in \mathcal{C}^{0,1/2}\left(\overline{B_{1/2}^+}\right)$.*

Once local regularity is established, we can move from $(GP)_\beta$ and deal with global problems, adding suitable boundary conditions. An example of results that we can prove is the following.

Theorem 1.1.3 (Global uniform Hölder bounds)**.** *Let the functions $f_{i,\beta}$ be continuous and uniformly bounded (w.r.t. β) on bounded sets, and let $\{\mathbf{u}_\beta\}_\beta$ be a family of $H^{1/2}(\mathbb{R}^N)$ solutions to the problems*

$$\begin{cases} (-\Delta)^{1/2} u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} u_j^2 & \text{on } \Omega \\ u_i \equiv 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, with sufficiently smooth boundary. Let us assume that

$$\|\mathbf{u}_\beta\|_{L^\infty(\Omega)} \leq M,$$

for a constant M independent of β. Then for every $\alpha \in (0, 1/2)$ there exists a constant $C = C(M, \alpha)$, not depending on β, such that

$$\|\mathbf{u}_\beta\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \leq C(M, \alpha).$$

Analogous results hold, for instance, when the square root of the laplacian is replaced with the spectral fractional laplacian with homogeneous Dirichlet boundary conditions on bounded domains (see [8]). Moreover, note that L^∞ bounds can be derived from $H^{1/2}$ ones, once suitable restrictions are imposed on the growth rate (subcritical) of the nonlinearities and/or on the dimension N, by means of a Brezis-Kato type argument.

In order to pursue the program just illustrated, compared with the case of the standard laplacian, a number of new difficulties has to be overcome. For instance, the polynomial decay of the fundamental solution of $(-\Delta)^{1/2} + 1$ already affects the rate of segregation. Furthermore, since such segregation occurs only in the N-dimensional space, it is natural to expect free boundaries of codimension 2. But, perhaps, the most challenging issue lies in the lack of the validity of an exact Alt-Caffarelli-Friedman monotonicity formula. This reflects, at the spectral level, the lack of convexity of the eigenvalues with respect to domain variations, see Remark 2.1.4 below. To attack these problems new tools are in order, involving different extremality conditions and new monotonicity formulas (associated with trace spectral problems).

1.2 The variational problem, case $s \in (0,1)$

Later, we generalize the theory developed in the case $s = 1/2$ to the system

$$\begin{cases} (-\Delta)^s u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j^2 \\ u_i \in H^s(\mathbb{R}^N), \end{cases} \tag{1.2.1}$$

where $a_{ij} = a_{ji} > 0$, β is positive and large, and the non-local operator

$$(-\Delta)^s u(x) = c_{N,s}\,\mathrm{pv} \int_{\mathbb{R}^N} \frac{u(x) - u(\xi)}{|x-\xi|^{N+2s}}\,\mathrm{d}\xi$$

denotes the s-power of the laplacian. Exploiting again the local realization of $(-\Delta)^s$ as a Dirichlet-to-Neumann map (see [16] for the general case $s \in (0,1)$) and letting $a := 1-2s \in (-1,1)$, we obtain, up to normalization constants, the problem

$$\begin{cases} L_a v_i = 0 & \text{in } B_1^+ \\ \partial_\nu^a v_i = f_{i,\beta}(v_i) - \beta v_i \sum_{j\neq i} a_{ij} v_j^2 & \text{on } \partial^0 B_1^+, \end{cases} \qquad (GP)_\beta^s$$

which is a localized version of (1.2.1), with $u_i(x) = v_i(x,0)$. The main result we prove in this context is the following.

Theorem 1.2.1 (Local uniform Hölder bounds)**.** *Let the functions $f_{i,\beta}$ be continuous and uniformly bounded (w.r.t. β) on bounded sets. There exists $\alpha = \alpha(N,s) > 0$ such that, for every $\{\mathbf{v}_\beta\}_\beta$ family of $H^{1;a}(B_1^+)$ solutions to the problems $(GP)_\beta^s$,*

$$\|\mathbf{v}_\beta\|_{L^\infty(B_1^+)} \leq M \quad \Longrightarrow \quad \|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}\left(\overline{B_{1/2}^+}\right)} \leq C,$$

where $C = C(M,\alpha)$. Furthermore, $\{\mathbf{v}_\beta\}_{\beta>0}$ is relatively compact in $H^{1;a}(B_{1/2}^+) \cap \mathcal{C}^{0,\alpha}\left(\overline{B_{1/2}^+}\right)$.

The above result allows to prove its natural global counterpart, either on the whole of $\mathbb{R}^N$ or on domains with suitable boundary conditions.

Theorem 1.2.2 (Global uniform Hölder bounds)**.** *Let $f_{i,\beta}$ and α be as in the previous theorem, and let $\{\mathbf{u}_\beta\}_\beta$ be a family of $H^s(\mathbb{R}^N)$ solutions to the problems*

$$\begin{cases} (-\Delta)^s u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j^2 & \text{in } \Omega \\ u_i \equiv 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, with smooth boundary. Then

$$\|\mathbf{u}_\beta\|_{L^\infty(\Omega)} \leq M \quad \Longrightarrow \quad \|\mathbf{u}_\beta\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \leq C(M,\alpha).$$

Of course, a natural question regards the optimal regularity of such problems, that is the maximal value of α for the above results to hold. The exponent α is subject to two main restrictions: as before, α is bounded above by the minimal rate of growth for multi-phase segregation profiles; on the other hand, when $s > 1/2$, a new upper threshold must be taken into account, which is related to the phenomenon of self-segregation.

The first restriction is related to the validity of an exact Alt-Caffarelli-Friedmann formula and to the classification of the possible growth of entire solutions to suitable limiting systems. These are analogous restrictions to those found in the case $s = 1/2$, and indeed, using similar arguments to that case, it would seem reasonable to conjecture that the optimal exponent is exactly s. Even though this is a correct conjecture, at least in the case $s \le 1/2$, an interesting phenomenon takes place beyond this threshold.

A second obstruction (to which we shall refer as *self-segregation*), however, is imposed by the simple observation that the fundamental solution of the s-laplacian

$$\Gamma(X) = \frac{C_{N,s}}{|X|^{N-2s}},$$

turns out to be bounded near 0 and $H^{1;a}(B)$, whenever $s > 1/2$, $N = 1$. This implies that, when $s > 1/2$, $N \ge 2$, the function

$$v(x, y) = (x_1^2 + y^2)^{(2s-1)/2}$$

is positive and L_a-harmonic for $y > 0$, $\partial^a_\nu v(x, 0) = 0$ whenever $v(x, 0) \neq 0$, and its trace on $\mathbb{R}^N$ has disconnected positivity regions. Moreover, such self-segregated profile (so called since the same density is on both sides of the free boundary) is globally Hölder continuous, of exponent $\alpha = 2s - 1$ which is arbitrarily small as $s \to (1/2)^+$. The phenomenon of self-segregation can be excluded in some situations, for instance when $s \le 1/2$ (for capacitary reasons), or when suitable minimality conditions are imposed (as in [14]). Nonetheless, in general it is hard to tackle: for the case $s = 1$ it was excluded only recently, in [26]. As a result, for the moment we can show that the exponent $\alpha(N, s)$ satisfies

$$\alpha(N, s) \begin{cases} = s & \text{for } s \in (0, 1/2] \text{ (optimal)} \\ \ge 2s - 1 & \text{for } s \in (1/2, 1) \end{cases}$$

where in the last case, the optimality is still an open question.

1.3 The symmetric problem, case $s \in (0, 1)$

In the last chapter of this part, we will concentrate on the system

$$(-\Delta)^s u_i = f_i(x, u_i) - \beta u_i^p \sum_{j \neq i} a_{ij} u_j^p, \tag{1.3.1}$$

settled in $H^s(\mathbb{R}^N)$, $N \ge 1$, or in a bounded domain with suitable boundary conditions. As we already mentioned, after [19, 20, 21], the case $s = 1$ of standard diffusion has been extensively studied in the last decade. In particular it is known that, both in the case of Lotka-Volterra competition [22, 11] and in the variational one [12, 39], each family of solutions which share a common uniform bounds in the L^∞ norm is precompact in the topology of $H^1 \cap \mathcal{C}^{0,\alpha}$ for every $\alpha < 1$; we highlight that this result is quasi-optimal, in the sense that $\alpha = 1$ is the maximal common regularity allowed for this problem. Furthermore, the limiting profiles (as $\beta \to +\infty$) are solutions of the segregated system

$$u_i \left(-\Delta u_i - f_i(x, u_i)\right) = 0, \qquad u_i u_j = 0 \text{ for } j \neq i, \tag{1.3.2}$$

they are Lipschitz continuous, and they obey to a weak reflection law which roughly says that, on the free boundary separating two components, the corresponding gradients are equal in magnitude (up to suitable scaling factors depending on the matrix (a_{ij})) and opposite in direction [47]. Remarkably, such law is the same for both types of competition [26]. For some related results, in the case of standard diffusion, we also refer to [24, 40] and references therein.

Coming to the anomalous diffusion case $s \in (0, 1)$, until now we have only considered the competition of Gross-Pitaevskii type. In such framework, we have seen that L^∞ uniform bounds imply uniform bounds in $H^s \cap \mathcal{C}^{0,\alpha}$ (for a suitable extension problem), for every $\alpha < \alpha_{\text{opt}}^{\text{GP}}(s)$. Here the optimal exponent

$$\alpha_{\text{opt}}^{\text{GP}}(s) = s,$$

at least when $0 < s \leq 1/2$; for $1/2 < s < 1$ we could only show that $\alpha_{\text{opt}}^{\text{GP}}(s) \geq 2s - 1$, because of the lack of a clean-up lemma appropriate to exclude self-segregation. In any case, this result agrees with the one holding for the standard Laplace operator, since $\alpha_{\text{opt}}^{\text{GP}}(1) = 1$. Moreover the limiting profiles satisfy a natural extension to the fractional setting of the system (1.3.2), that is

$$u_i \left((-\Delta)^s u_i - f_i(x, u_i)\right) = 0, \qquad u_i u_j = 0 \text{ for } j \neq i, \tag{1.3.3}$$

and the validity of an Almgren monotonicity formula across the free boundary ensures a reflection property, as in the case $s = 1$.

Under the perspective just described, we then address the study of system (1.3.1), where a different type of competition takes place. We remark that such range of parameters not only includes the Lotka-Volterra competition (that is, the case $p = 1$), but it is of interest also in the complementary case $p \neq 1$. Indeed, in the case of $k = 2$ components, such competition appears in the modeling of diffusion flames [14], while in the general case the change of variables $U_i = u_i^p$ turns system (1.3.1) into the one for competing densities subject to fast fractional diffusion (when $p > 1$), or to fractional diffusion in a porous medium (when $p < 1$) [27, 7].

As before, we state our results for a localized extension problem [16] related to the nonlocal system (1.3.1), namely the problem

$$\begin{cases} L_a v_i = 0 & \text{in } B_1^+ \\ \partial_\nu^a v_i = f_{i,\beta}(x, v_1, \dots, v_k) - \beta v_i^p \sum_{j \neq i} a_{ij} v_j^p & \text{on } \partial^0 B_1^+. \end{cases} \qquad (LV)_\beta$$

Our first main results concern the full quasi-optimal theory in the case of two densities.

Theorem 1.3.1. *Let $p > 0$, $a_{ij} > 0$ for any $j \neq i$, and the reaction terms $f_{i,\beta}$ be continuous and map bounded sets into bounded sets, uniformly w.r.t. $\beta > 0$.*

If $k = 2$ then, for every

$$\alpha < \alpha_{\text{opt}}(s) = \alpha_{\text{opt}}^{\text{LV}}(s) := \min(2s, 1)$$

and $\bar{m} > 0$, there exists a constant $C = C(\alpha, \bar{m})$ independent of β such that

$$\|\mathbf{v}_\beta\|_{L^\infty(B^+)} \leq \bar{m} \quad \Longrightarrow \quad \|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}\left(\overline{B_{1/2}^+}\right)} \leq C,$$

for every $\mathbf{v}_\beta = (v_{1,\beta}, v_{2,\beta})$ nonnegative solution of problem $(LV)_\beta$.

Furthermore, any sequence of uniformly bounded, nonnegative solutions $\{(v_{1,\beta_n}, v_{2,\beta_n})\}_n$, with $\beta_n \to \infty$, converges (up to subsequences) in $\left(H^{1;a} \cap \mathcal{C}^{0,\alpha}\right)\left(\overline{B_{1/2}^+}\right)$ to a limiting profile (v_1, v_2).

Theorem 1.3.2. *Under the assumption of the previous theorem, let furthermore $f_{i,\beta} \to f_i$ as $\beta \to \infty$, uniformly on compact sets, with f_i Lipschitz continuous. For any limiting profile (v_1, v_2):*

- *$v_1(x, 0)$, $v_2(x, 0)$ are Lipschitz continuous (optimal regularity of the traces);*
- *$v_1(x, 0) \cdot v_2(x, 0) = 0$ (boundary segregation condition);*
- *$L_a v_1 = L_a v_2 = 0$ for $y > 0$;*
- *$\partial_\nu^a (a_{21} v_1 - a_{12} v_2) = a_{21} f_1 - a_{12} f_2$ for $y = 0$.*

Remark 1.3.3. In the previous results $B_{1/2}^+$ can be replaced by any domain $\Omega \cap \{y > 0\}$, where $\overline{\Omega} \subset B_1$.

Remark 1.3.4. Throughout this paper, we restrict our discussion to nonnegative solutions only to avoid technicalities. Reasoning as in Chapter 2 and 3, also changing sign solutions can be considered, once the competition is suitably extended to negative densities.

Remark 1.3.5. The upper bound $\alpha = 2s$ for the regularity of the functions $v_{i,\beta}$ can not be removed: indeed, from any solution of $(LV)_\beta$ we can construct another solution having $(k+1)$ components, by defining $v_{k+1,\beta}(x,y) = y^{2s}$, $f_{k+1,\beta} \equiv -2s$. One may possibly expect to be able to remove such threshold by considering only the regularity of the traces $v_{i,\beta}(x,0)$, as suggested by Theorem 1.3.2.

On the other hand, the Lipschitz regularity is the natural one, at least for the traces, since the last condition in Theorem 1.3.2 implies that $v_i(x,0)$ are (proportional to) the positive/negative parts of a regular function.

Next, we address the case of $k \geq 3$ densities.

Theorem 1.3.6. *Let $k \geq 3$. Then there exists $\alpha^* > 0$ such that Theorem 1.3.1 holds for any $\alpha < \alpha^*$, under the further assumption that*

$$\text{either } p \geq 1 \qquad \text{or} \qquad a_{ij} = 1 \text{ for every } j \neq i.$$

Furthermore, if $a_{ij} = 1$,

$$\alpha^* = \alpha_{\mathrm{opt}}(s) = \min(2s, 1)$$

whenever $s = 1/2$ or $s \in (0, 1/4)$.

Even though we can show quasi-optimality only in some cases, the above regularity result is sufficient to conclude that, as $\beta \to \infty$, solutions of $(LV)_\beta$ accumulate to limiting profiles v_i which properties, apart from optimal regularity, are analogous to those described in Theorem 1.3.2 for the case $k = 2$ (see Section 4.4 for further details). In particular, going back to the segregated traces $u_i(x) = v_i(x,0)$, we can show that

$$u_i \left[(-\Delta)^s \left(u_i - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} u_j \right) - \left(f_i - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} f_j \right) \right] = 0, \quad u_i u_j = 0 \text{ for } j \neq i. \quad (1.3.4)$$

Comparing with equation (1.3.3), we see that, if $s < 1$, the Gross-Pitaevskii competition and the Lotka-Volterra one exhibit deep differences not only from the point of view of the optimal regularity exponent, but also from that of the differential equations satisfied by the segregated limiting profiles. This is in great contrast with the case $s = 1$ where, as we already mentioned, the two competitions can not be distinguished from each other in the limit. Such feature is caused by the non local nature of the diffusion operators: indeed equation (1.3.4) can not be directly reduced to (1.3.3), since in the set $\{u_i = 0\}$ the corresponding fractional laplacian does not necessarily vanish. Nonetheless, letting $s \to 1^-$, we recover the local nature of the equation: as a consequence

$$u_i (-\Delta)^s \left(u_i - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} u_j \right) \to u_i(-\Delta u_i),$$

so that equation (1.3.2) arises also in this case.

To conclude, we mention that the equations just discussed –or, better, the corresponding ones for the extensions v_i– can be used to obtain further regularity for the limiting profiles, also in the case $a_{ij} \neq 1$. In particular, we have the following result.

Theorem 1.3.7. *Let $k \geq 3$, $s = 1/2$, $p > 1$. If furthermore $f_i(x, t_1, \ldots, t_k) = 0$ for $|(t_1, \ldots, t_k)|$ small then every segregated limiting profile v_i is $C^{0,\alpha}$, for every $\alpha < 1$.*

Remark 1.3.8. Collecting together the results of Theorems 1.3.6 and 1.3.7, we have that for $s = 1/2$ the limiting profiles are $C^{0,\alpha}$, for every $\alpha < 1$, when either $a_{ij} = 1$ or $p > 1$. Since for $s = 1/2$ we have that $L_a = (-\Delta)$, one may then try to apply the arguments contained in [11, Section 2] (see also [2, Section 5]). This should eventually imply that the traces of the limiting profiles are indeed Lipschitz continuous.

Chapter 2

Variational competition: the case of the half laplacian

Outline of the chapter

In this chapter we analyse regularity issue for a system of competition densities which are subject to fractional diffusion of power $s = 1/2$ and interact through a variational term. This particular choice of the diffusion exponent has important justification both from the modeling point of view (it is indeed linked to relativistic versions of the Schrödinger equation) and from analytical issues (the extension problem found in [16] being just of harmonic type). We first introduce some generalisation of the already recalled Alt-Caffarelli-Friedman and Almgren monotonicity formulæ in the context of fractional diffusion, together with some implications. Via a blow up analysis, we conclude uniform bounds in some Hölder norm for the components of the system: at this stage the precise value of the exponent is unknown due to the unknown optimal growth of the new monotonicity formulæ. Nevertheless, this first result is then used in combination with a blow down analysis to obtain the sharp regularity exponent, that is, it is shown that the components of the system admit $\mathcal{C}^{0,\alpha}$ uniform bounds for every $\alpha < 1/2$. The chapter is concluded by showing that the limiting profiles are actually $\mathcal{C}^{0,1/2}$ regular.

2.1 Alt-Caffarelli-Friedman type monotonicity formulæ

We begin the investigation with a section devoted to the proof of some monotonicity formulæ of Alt-Caffarelli-Friedman (ACF) type. For references to the case $s = 1$ we recall [2, 9, 22, 39].

2.1.1 Segregated ACF formula

As recalled in the Intoduction, the validity of ACF type formulæ depends on optimal partition problems involving spectral properties of the domain. In the present situation, the spectral problem we consider involves a pair of functions defined on $\mathbb{S}^N_+ := \partial^+ B^+$. As a peculiar fact, here such functions have not disjoint support on the whole $\mathbb{S}^N_+$, but only on its boundary $\mathbb{S}^{N-1}$. In this way we are lead to consider the following optimal partition problem on $\mathbb{S}^{N-1}$.

Definition 2.1.1. For each open subset ω of $\mathbb{S}^{N-1} := \partial\mathbb{S}^N_+$ we define the first eigenvalue associated to ω as

$$\lambda_1(\omega) := \inf\left\{ \frac{\int_{\mathbb{S}^N_+} |\nabla_T u|^2 \, \mathrm{d}\sigma}{\int_{\mathbb{S}^N_+} u^2 \, \mathrm{d}\sigma} : u \in H^1(\mathbb{S}^N_+),\ u \equiv 0 \text{ on } \mathbb{S}^{N-1} \setminus \omega \right\}.$$

Here $\nabla_T u$ stands for the (tangential) gradient of u on $\mathbb{S}^N_+$.

Definition 2.1.2. On $\mathbb{S}^{N-1}$ we define the set of 2-partition $\mathcal{P}^2$ by

$$\mathcal{P}^2 := \left\{(\omega_1, \omega_2) \colon \omega_i \subset \mathbb{S}^{N-1} \text{ open},\ \omega_1 \cap \omega_2 = \emptyset\right\}$$

and the number, only depending on N,

$$\begin{aligned} \nu^{\mathrm{ACF}} : &= \frac{1}{2} \inf_{(\omega_1,\omega_2)\in\mathcal{P}^2} \sum_{i=1}^{2} \left(\sqrt{\left(\frac{N-1}{2}\right)^2 + \lambda_1(\omega_i)} - \frac{N-1}{2} \right) \\ &= \frac{1}{2} \inf_{(\omega_1,\omega_2)\in\mathcal{P}^2} \sum_{i=1}^{2} \gamma(\lambda_1(\omega_i)). \end{aligned}$$

Remark 2.1.3. As it is well known, u achieves $\lambda_1(\omega)$ if and only if it is one signed, and its $\gamma(\lambda_1(\omega))$-homogeneous extension to $\mathbb{R}^{N+1}_+$ is harmonic.

Remark 2.1.4. By symmetrization arguments, one may try to restrict the study of the above optimal partition problem to the case when both ω_i are spherical caps. In such a situation, writing $\Gamma(\vartheta) := \gamma(\lambda_1(\omega_\vartheta))$ for the spherical cap ω_ϑ with opening ϑ, one is lead to minimize the quantity

$$\varphi(\vartheta) := \frac{1}{2}\left[\Gamma(\vartheta) + \Gamma(\pi - \vartheta)\right], \qquad \vartheta \in [0, \pi].$$

It is worthwhile noticing that the function φ is not convex, indeed one can prove that

$$\varphi(0) = \varphi\left(\frac{\pi}{2}\right) = \varphi(\pi) = \frac{1}{2}$$

(for details, see the proofs of Lemma 2.1.5 and Proposition 2.1.12 below). Thus, in particular, it is not clear whether the minimum of φ may be strictly less that $1/2$. As already mentioned, this marks a notable difference with respect to the standard diffusion case.

Lemma 2.1.5. *For every dimension N, it holds $0 < \nu^{\mathrm{ACF}} \leq \frac{1}{2}$.*

Proof. The bound from above easily follows by comparing with the value corresponding to the partition $(\mathbb{S}^{N-1}, \emptyset)$: indeed, it holds $\lambda_1(\mathbb{S}^{N-1}) = 0$, achieved by $u(x,y) \equiv 1$, and $\lambda_1(\emptyset) = 2N$, achieved by $u(x,y) = y$. In order to prove the estimate from below, let us first observe that, for each pair $(\omega_1, \omega_2) \in \mathcal{P}^2$, there exist two functions u_1 and u_2 in $H^1(\mathbb{S}^N_+)$ such that $u_i \equiv 0$ on $\mathbb{S}^{N-1} \setminus \omega_i$,

$$\lambda_1(\omega_i) = \int_{\mathbb{S}^N_+} |\nabla_T u_i|^2 \, \mathrm{d}\sigma \quad \text{and} \quad \int_{\mathbb{S}^N_+} u_i^2 \, \mathrm{d}\sigma = 1.$$

This claim is a consequence of the compactness both of the embedding $H^1(\mathbb{S}^N_+) \hookrightarrow L^2(\mathbb{S}^N_+)$ and of the trace operator from $H^1(\mathbb{S}^N_+)$ to $L^2(\mathbb{S}^{N-1})$ (recall that the constraint is continuous with respect to the $L^2(\mathbb{S}^{N-1})$ topology).

We proceed by contradiction, supposing that there exists a sequence of 2-partition $(\omega_1^n, \omega_2^n) \in \mathcal{P}^2$ such that

$$\gamma\left(\lambda_1(\omega_1^n)\right) + \gamma\left(\lambda_1(\omega_2^n)\right) \to 0.$$

Since the function γ is non negative and increasing, it must be that $\lambda_1(\omega_i^n) \to 0$ for $i = 1, 2$, that is, there exist two sequences of functions u_1^n and u_2^n in $H^1(\mathbb{S}^N_+)$ such that $u_i \equiv 0$ on $\mathbb{S}^{N-1} \setminus \omega_i$,

$$\int_{\mathbb{S}^N_+} |\nabla_T u_i|^2 \, \mathrm{d}\sigma \to 0 \quad \text{while} \quad \int_{\mathbb{S}^N_+} u_i^2 \, \mathrm{d}\sigma = 1.$$

Therefore, up to a subsequence, it holds

$$u_1^n, u_2^n \rightharpoonup |\mathbb{S}^N_+|^{1/2} \text{ in } H^1(\mathbb{S}^N_+) \quad \text{and} \quad \int_{\mathbb{S}^{N-1}} u_1^n u_2^n \, \mathrm{d}\sigma = 0$$

which are incompatible. □

Under the previous notations, we can prove the following monotonicity formula.

Theorem 2.1.6. *Let $v_1, v_2 \in H^1(B_R^+(x_0, 0))$ be continuous functions such that*

- $v_1 v_2|_{\{y=0\}} = 0$, $v_i(x_0, 0) = 0$;
- *for every non negative $\phi \in \mathcal{C}_0^\infty(B_R(x_0, 0))$,*

$$\int_{\mathbb{R}^{N+1}_+} (-\Delta v_i) v_i \phi \, \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} (\partial_\nu v_i) v_i \phi \, \mathrm{d}x = \int_{\mathbb{R}^{N+1}_+} \nabla v_i \cdot \nabla(v_i \phi) \, \mathrm{d}x\mathrm{d}y \leq 0.$$

Then the function

$$\Phi(r) := \prod_{i=1}^{2} \frac{1}{r^{2\nu^{\mathrm{ACF}}}} \int\limits_{B_r^+(x_0,0)} \frac{|\nabla v_i|^2}{|X-(x_0,0)|^{N-1}}\, \mathrm{d}x\mathrm{d}y$$

is monotone non decreasing in r for $r \in (0,R)$.

Remark 2.1.7. Since

$$\int\limits_{\mathbb{R}^{N+1}_+} \nabla v_i \cdot \nabla(v_i\phi)\, \mathrm{d}x\mathrm{d}y = \int\limits_{\mathbb{R}^{N+1}_+} \left[|\nabla v_i|^2\phi + \frac{1}{2}\nabla(v_i)^2 \cdot \nabla\phi\right] \mathrm{d}x\mathrm{d}y, \tag{2.1.1}$$

we have that if v_1, v_2 satisfy the assumptions of Theorem 2.1.6 then also $|v_1|, |v_2|$ do.

By the above remark, we can assume without loss of generality that v_1 and v_2 are non negative. Since the theorem is trivial if either $v_1 \equiv 0$ or $v_2 \equiv 0$, we will prove it when both v_1 and v_2 are non zero. Moreover, by translating and scaling, the theorem can be proved under the assumption that $x_0 = 0$ and $R = 1$. We will need the following technical lemmas.

Definition 2.1.8. We define $\Gamma_1 \in \mathcal{C}^1(\mathbb{R}^{N+1}_+; \mathbb{R}^+)$ as

$$\Gamma_1(X) := \begin{cases} \frac{1}{|X|^{N-1}} & |X| \geq 1 \\ \frac{N+1}{2} - \frac{N-1}{2}|X|^2 & |X| < 1. \end{cases}$$

We let also $\Gamma_\varepsilon(X) = \Gamma_1(X/\varepsilon)\varepsilon^{1-N}$, so that $\Gamma_\varepsilon \nearrow \Gamma = |X|^{1-N}$, a multiple of the fundamental solution of the half-laplacian, as $\varepsilon \to 0$.

Remark 2.1.9. Let us observe that each Γ_ε is radial and, in particular, $\partial_\nu \Gamma_\varepsilon = 0$ on $\mathbb{R}^N$. Moreover, they are superharmonic on $\mathbb{R}^{N+1}_+$.

Lemma 2.1.10. *Let v_1, v_2 be as in Theorem 2.1.6. The function*

$$r \mapsto \int\limits_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}}\, \mathrm{d}x\mathrm{d}y \tag{2.1.2}$$

is well defined and bounded in any compact subset of $(0,1)$.

Proof. We proceed as follows: let $\varepsilon > 0$, $\delta > 0$ and let $\eta_\delta \in \mathcal{C}_0^\infty(B_{r+\delta})$ be a smooth, radial cutoff function such that $0 \leq \eta_\delta \leq 1$ and $\eta_\delta = 1$ on B_r. Choosing $\phi = \eta_\delta\Gamma_\varepsilon$ in the second assumption of the theorem, and recalling equation (2.1.1), we obtain

$$\begin{aligned} \int\limits_{\mathbb{R}^{N+1}_+} \left[|\nabla v_i|^2\Gamma_\varepsilon + \frac{1}{2}\nabla(v_i)^2 \cdot \nabla\Gamma_\varepsilon\right] \eta_\delta \mathrm{d}x\mathrm{d}y &\leq - \int\limits_{\mathbb{R}^{N+1}_+} \frac{1}{2}\Gamma_\varepsilon \nabla(v_i)^2 \cdot \nabla\eta_\delta \mathrm{d}x\mathrm{d}y \\ &= \int\limits_r^{r+\delta} \left[-\eta_\delta'(\rho) \int\limits_{\partial^+ B_\rho^+} \Gamma_\varepsilon v_i \nabla v_i \cdot \frac{X}{|X|} \mathrm{d}\sigma\right] \mathrm{d}\rho. \end{aligned}$$

Passing to the limit as $\delta \to 0$ we obtain, for almost every $r \in (0,1)$,

$$\int_{B_r^+} \left[|\nabla v_i|^2 \Gamma_\varepsilon + \frac{1}{2} \nabla (v_i)^2 \cdot \nabla \Gamma_\varepsilon \right] \mathrm{d}x\mathrm{d}y \leq \int_{\partial^+ B_r^+} \Gamma_\varepsilon v_i \partial_\nu v_i \mathrm{d}\sigma,$$

which, combined with the inequality $-\Delta \Gamma_\varepsilon \geq 0$ tested with $v_i^2/2$ leads to

$$\int_{B_r^+} |\nabla v_i|^2 \Gamma_\varepsilon \,\mathrm{d}x\mathrm{d}y \leq \int_{\partial^+ B_r^+} \left(\Gamma_\varepsilon v_i \partial_\nu v_i - \frac{v_i^2}{2} \partial_\nu \Gamma_\varepsilon \right) \mathrm{d}\sigma.$$

Letting $\varepsilon \to 0^+$, by monotone convergence we infer

$$\int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}x\mathrm{d}y \leq \frac{1}{r^{N-1}} \int_{\partial^+ B_r^+} v_i \frac{\partial v_i}{\partial \nu} \,\mathrm{d}\sigma + \frac{N-1}{2r^N} \int_{\partial^+ B_r^+} v_i^2 \,\mathrm{d}\sigma \tag{2.1.3}$$

and this, in turns, proves the lemma. □

Lemma 2.1.11. *Let v_1, v_2 be two non trivial functions satisfying the assumptions of Theorem 2.1.6. It holds*

$$\sum_{i=1}^{2} \frac{\int_{\partial^+ B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}\sigma}{\int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}x\mathrm{d}y} \geq \frac{4}{r} \nu^{\mathrm{ACF}}. \tag{2.1.4}$$

Proof. First we use the estimate (2.1.3) to bound from below the left hand side of (2.1.4):

$$\frac{\int_{\partial^+ B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}\sigma}{\int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}x\mathrm{d}y} \geq \frac{\int_{\partial^+ B_r^+} |\nabla v_i|^2 \,\mathrm{d}\sigma}{\int_{\partial^+ B_r^+} v_i \partial_\nu v_i \,\mathrm{d}\sigma + (N-1)\frac{r}{2} \int_{\partial^+ B_r^+} v_i^2 \,\mathrm{d}\sigma}$$

$$= \frac{1}{r} \frac{\int_{\mathbb{S}_+^N} |\nabla v_i^{(r)}|^2 \,\mathrm{d}\sigma}{\int_{\mathbb{S}_+^N} v_i^{(r)} \partial_\nu v_i^{(r)} \,\mathrm{d}\sigma + \frac{N-1}{2} \int_{\mathbb{S}_+^N} (v_i^{(r)})^2 \,\mathrm{d}\sigma},$$

where $v_i^{(r)} \colon \mathbb{S}_+^{N-1} \to \mathbb{R}$ is defined as $v_i^{(r)}(\xi) = v_i(r\xi)$. We now estimate the right hand side as follows: the numerator writes

$$\int_{\mathbb{S}^N_+} |\nabla v_i^{(r)}|^2 \,\mathrm{d}\sigma = \int_{\mathbb{S}^N_+} |\partial_\nu v_i^{(r)}|^2 \,\mathrm{d}\sigma + \int_{\mathbb{S}^N_+} |\nabla_T v_i^{(r)}|^2 \,\mathrm{d}\sigma$$
$$= \int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma \left(\underbrace{\frac{\int_{\mathbb{S}^N_+} |\partial_\nu v_i^{(r)}|^2 \,\mathrm{d}\sigma}{\int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma}}_{t^2} + \underbrace{\frac{\int_{\mathbb{S}^N_+} |\nabla_T v_i^{(r)}|^2 \,\mathrm{d}\sigma}{\int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma}}_{\mathcal{R}} \right).$$

where $\mathcal{R}$ stands for the Rayleigh quotient of $v_i^{(r)}$ on $\mathbb{S}^N_+$. On the other hand, by the Cauchy-Schwarz inequality, the denominator may be estimated from above by

$$\int_{\mathbb{S}^N_+} v_i^{(r)} \partial_\nu v_i^{(r)} \,\mathrm{d}\sigma + r\frac{N-1}{2} \int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma$$
$$\leq \left(\int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma \right)^{1/2} \left(\int_{\mathbb{S}^N_+} \partial_\nu v_i^{(r)} \,\mathrm{d}\sigma \right)^{1/2} + r\frac{N-1}{2} \int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma$$
$$\leq \int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma \left[\underbrace{\left(\frac{\int_{\mathbb{S}^N_+} |\partial_\nu v_i^{(r)}|^2 \,\mathrm{d}\sigma}{\int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma} \right)^{1/2}}_{t} + \frac{N-1}{2} \right].$$

As a consequence

$$\frac{\int_{\partial^+ B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}\sigma}{\int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}x\mathrm{d}y} \geq \frac{1}{r} \min_{t \in \mathbb{R}^+} \frac{\mathcal{R} + t^2}{t + \frac{N-1}{2}}.$$

A simple computation shows that the minimum is achieved when

$$t = \gamma(\mathcal{R}) = \sqrt{\left(\frac{N-1}{2}\right)^2 + \mathcal{R}} - \frac{N-1}{2},$$

and it is equal to $2\gamma(\mathcal{R})$. Summing over $i = 1, 2$, we obtain

$$\sum_{i=1}^2 \frac{\int_{\partial^+ B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}\sigma}{\int_{B_r^+} \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}x\mathrm{d}y} \geq \frac{2}{r} \inf_{(\omega_1,\omega_2) \in \mathcal{P}^2} \sum_{i=1}^2 \gamma\left(\lambda_1(\omega_i)\right) = \frac{4}{r} \nu^{\mathrm{ACF}}$$

where the inequality follows by substituting each $\mathcal{R}$ with their optimal value, that is, the eigenvalue $\lambda_1(\omega_i)$. $\square$

Proof of Theorem 2.1.6. As already noticed, we may assume that $x_0 = 0$ and $R = 1$ and that both v_1 and v_2 are non trivial and non negative. We start observing that the function $\Phi(r)$ is positive and absolutely continuous for $r \in (0,1)$, since it is the product of functions which are positive and absolutely continuous in $(0,1)$. Therefore, the theorem follows once we prove that $\Phi'(r) \geq 0$ for almost every $r \in (0,1)$. A direct computation of the logarithmic derivative of Φ shows that

$$\frac{\Phi'(r)}{\Phi(r)} = -\frac{4\nu^{\mathrm{ACF}}}{r} + \sum_{i=1}^{2} \frac{\int\limits_{\partial^+ B_r^+} |\nabla v_i|^2/|X|^{N-1}\,\mathrm{d}\sigma}{\int\limits_{B_r^+} |\nabla v_i|^2/|X|^{N-1}\,\mathrm{d}x\mathrm{d}y} \geq 0$$

where the last inequality follows by Lemma 2.1.11. $\square$

As we mentioned, Theorem 2.1.6 will be crucial in proving interior regularity estimates. We now provide a related result, suitable to treat regularity up to the boundary. Differently from before, in this case we can show that the optimal exponent in the corresponding monotonicity formula is exactly $\gamma = 1/2$.

Proposition 2.1.12. *Let $v \in H^1(B_R^+)$ be a continuous function such that*

- *$v_1(x,0) = 0$ for $x_1 \leq 0$;*
- *for every non negative $\phi \in \mathcal{C}_0^\infty(B_R)$,*

$$\int\limits_{\mathbb{R}^{N+1}_+} (-\Delta v) v\phi\,\mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}^N} (\partial_\nu v) v\phi\,\mathrm{d}x = \int\limits_{\mathbb{R}^{N+1}_+} \nabla v \cdot \nabla(v\phi)\,\mathrm{d}x\mathrm{d}y \leq 0.$$

Then the function

$$\Phi(r) := \frac{1}{r}\int\limits_{B_r^+} \frac{|\nabla v|^2}{|X|^{N-1}}\,\mathrm{d}x\mathrm{d}y$$

is monotone non decreasing in r for $r \in (0,R)$.

Proof. Let $\bar\omega := \mathbb{S}^{N-1} \cap \{x_1 > 0\}$, and let v denote the $1/2$ homogeneous, harmonic extension of $v(x,0) = \sqrt{x_1^+}$ to $\mathbb{R}^{N+1}_+$, that is

$$v(x,y) = \sqrt{\frac{\sqrt{x_1^2 + y^2} + x_1}{2}}.$$

Since v is positive for $y > 0$, Remark 2.1.3 implies that $v|_{\mathbb{S}^N_+}$ is an eigenfunction associated to $\lambda_1(\bar\omega)$, providing

$$\gamma(\lambda_1(\bar\omega)) = \frac{1}{2}.$$

But then, reasoning as in the proofs of Lemma 2.1.11 and Theorem 2.1.6, we readily obtain that

$$\frac{\Phi'(r)}{\Phi(r)} \geq \frac{2}{r}\left[-\frac{1}{2} + \gamma(\lambda_1(\bar{\omega}))\right] = 0. \qquad \square$$

2.1.2 Perturbed ACF formula

We now move from Theorem 2.1.6 and introduce a perturbed version of the monotonicity formula, suitable for functions which coexist on the boundary, rather than having disjoint support.

Theorem 2.1.13. *Let ν^{ACF} be as in Definition 2.1.2, and let $v_1, v_2 \in H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ be continuous functions such that, for every non negative $\phi \in \mathcal{C}^\infty_0\left(\overline{\mathbb{R}^{N+1}_+}\right)$ and $j \neq i$,*

$$\int_{\mathbb{R}^{N+1}_+} (-\Delta v_i) v_i \phi \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} (\partial_\nu v_i + v_i v_j^2) v_i \phi \,\mathrm{d}x$$
$$= \int_{\mathbb{R}^{N+1}_+} \nabla v_i \cdot \nabla(v_i\phi)\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} v_i^2 v_j^2 \phi \,\mathrm{d}x \leq 0.$$

For any $\nu' \in (0, \nu^{\mathrm{ACF}})$ there exists $\bar{r} > 1$ such that the function

$$\Phi(r) := \prod_{i=1}^{2} \Phi_i(r)$$

is monotone non decreasing in r for $r \in (\bar{r}, \infty)$, where

$$\Phi_i(r) := \frac{1}{r^{2\nu'}}\left(\int_{B_r^+} |\nabla v_i|^2 \Gamma_1 \,\mathrm{d}x\mathrm{d}y + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1 \,\mathrm{d}x\right), \qquad \text{for } j \neq i.$$

Remark 2.1.14. We observe that, analogously to Remark 2.1.7, the main assumption of Theorem 2.1.13 can be equivalently rewritten as

$$\int_{\mathbb{R}^{N+1}_+} \left[|\nabla v_i|^2 \phi + \frac{1}{2}\nabla(v_i)^2 \cdot \nabla\phi\right] \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} v_i^2 v_j^2 \phi \,\mathrm{d}x \leq 0,$$

for every compactly supported $\phi \geq 0$. In particular, if v_1, v_2 satisfy such assumption, so $|v_1|, |v_2|$ do. Moreover, reasoning as in Lemma 2.1.10, we obtain that, for every $\phi \geq 0$ and almost every r,

$$\int_{B_r^+} \left[|\nabla v_i|^2 \phi + \frac{1}{2}\nabla(v_i)^2 \cdot \nabla\phi\right] \mathrm{d}x\mathrm{d}y + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \phi \,\mathrm{d}x \leq \int_{\partial^+ B_r^+} (\partial_\nu v_i) v_i \phi \,\mathrm{d}\sigma, \qquad (2.1.5)$$

The proof of Theorem 2.1.13 follows the lines of the one of Theorem 2.1.6.

Lemma 2.1.15. *Let v_1, v_2 be two non trivial functions satisfying the assumptions of Theorem 2.1.13. Then, for any $r > 1$, it holds*

$$\sum_{i=1}^{2} \frac{\int\limits_{\partial B_r^+} |\nabla v_i|^2 \Gamma_1 \, \mathrm{d}\sigma + \int\limits_{r\mathbb{S}^{N-1}} v_i^2 v_j^2 \Gamma_1 \, \mathrm{d}\sigma}{\int\limits_{B_r^+} |\nabla v_i|^2 \Gamma_1 \, \mathrm{d}x\mathrm{d}y + \int\limits_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1 \, \mathrm{d}x} \geq \frac{2}{r} \sum_{i=1}^{2} \gamma\left(\Lambda_i(r)\right), \tag{2.1.6}$$

where

$$\Lambda_i(r) = \frac{\int\limits_{\mathbb{S}^N_+} |\nabla_T v_i^{(r)}|^2 \, \mathrm{d}\sigma + r \int\limits_{\mathbb{S}^{N-1}} (v_i^{(r)} v_j^{(r)})^2 \, \mathrm{d}\sigma}{\int\limits_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \, \mathrm{d}\sigma}$$

(again, $v_i^{(r)} \colon \mathbb{S}^{N-1}_+ \to \mathbb{R}$ is such that $v_i^{(r)}(\xi) = v_i(r\xi)$).

Proof. By choosing $\phi = \Gamma_1$ (Definition 2.1.8) in equation (2.1.5) we obtain, for a.e. $r > 0$,

$$\int\limits_{B_r^+} \left[|\nabla v_i|^2 \Gamma_1 + \frac{1}{2} \nabla (v_i)^2 \cdot \nabla \Gamma_1 \right] \mathrm{d}x\mathrm{d}y + \int\limits_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1 \, \mathrm{d}x \leq \int\limits_{\partial^+ B_r^+} v_i \partial_\nu v_i \Gamma_1 \, \mathrm{d}\sigma.$$

The superharmonicity of Γ_1 yields then

$$\int\limits_{B_r^+} |\nabla v_i|^2 \Gamma_1 \, \mathrm{d}x\mathrm{d}y + \int\limits_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1 \, \mathrm{d}x \leq \int\limits_{\partial^+ B_r^+} \left(v_i \partial_\nu v_i \Gamma_1 - \frac{v_i^2}{2} \partial_\nu \Gamma_1 \right) \mathrm{d}\sigma.$$

Recalling that $r > 1$ we can use the previous estimate to bound from below the left hand side of equation (2.1.6), obtaining

$$\frac{\int\limits_{\partial B_r^+} |\nabla v_i|^2 \Gamma_1 \, \mathrm{d}\sigma + \int\limits_{r\mathbb{S}^{N-1}} v_i v_j^2 \Gamma_1 \, \mathrm{d}\sigma}{\int\limits_{B_r^+} |\nabla v_i|^2 \Gamma_1 \, \mathrm{d}x\mathrm{d}y + \int\limits_{\partial^0 B_r^+} v_i v_j^2 \Gamma_1 \, \mathrm{d}x} \geq \frac{1}{r} \frac{\int\limits_{\mathbb{S}^N_+} |\nabla v_i^{(r)}|^2 \, \mathrm{d}\sigma + r \int\limits_{\mathbb{S}^{N-1}} (v_i^{(r)} v_j^{(r)})^2 \, \mathrm{d}\sigma}{\int\limits_{\mathbb{S}^N_+} v_i^{(r)} \partial_\nu v_i^{(r)} \, \mathrm{d}\sigma + \frac{N-1}{2} \int\limits_{\mathbb{S}^N_+} (v_i^{(r)})^2 \, \mathrm{d}\sigma}.$$

We now estimate the right hand side as follows: the numerator writes

$$\begin{aligned}
&\int_{\mathbb{S}^N_+} |\nabla v_i^{(r)}|^2 \,\mathrm{d}\sigma + r \int_{\mathbb{S}^{N-1}} (v_i^{(r)} v_j^{(r)})^2 \,\mathrm{d}\sigma \\
&\qquad\qquad = \int_{\mathbb{S}^N_+} |\partial_\nu v_i^{(r)}|^2 \,\mathrm{d}\sigma + \int_{\mathbb{S}^N_+} |\nabla_T v_i^{(r)}|^2 \,\mathrm{d}\sigma + r \int_{\mathbb{S}^{N-1}} (v_i^{(r)} v_j^{(r)})^2 \,\mathrm{d}\sigma \\
&\qquad = \int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma \left(\underbrace{\frac{\int_{\mathbb{S}^N_+} |\partial_\nu v_i^{(r)}|^2 \,\mathrm{d}\sigma}{\int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma}}_{t^2} + \underbrace{\frac{\int_{\mathbb{S}^N_+} |\nabla_T v_i^{(r)}|^2 \,\mathrm{d}\sigma + r \int_{\mathbb{S}^{N-1}} (v_i^{(r)} v_j^{(r)})^2 \,\mathrm{d}\sigma}{\int_{\mathbb{S}^N_+} |v_i^{(r)}|^2 \,\mathrm{d}\sigma}}_{\mathcal{R}} \right).
\end{aligned}$$

We may bound the denominator as in Lemma (2.1.11). As a consequence

$$\frac{\int_{\partial B_r^+} |\nabla v_i|^2 \Gamma_1 \,\mathrm{d}\sigma + \int_{r\mathbb{S}^{N-1}} v_i^2 v_j^2 \Gamma_1 \,\mathrm{d}\sigma}{\int_{B_r^+} |\nabla v_i|^2 \Gamma_1 \,\mathrm{d}x\mathrm{d}y + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1 \,\mathrm{d}x} \geq \frac{1}{r} \min_{t \in \mathbb{R}^+} \frac{\mathcal{R} + t^2}{t + \frac{N-1}{2}}.$$

Minimizing with respect to t as in Lemma (2.1.11) and summing over $i = 1, 2$, we obtain equation (2.1.6). ☐

Proof of Theorem 2.1.13. Without loss of generality, we assume that both v_1 and v_2 are non trivial. As in Theorem 2.1.6, we will prove that the logarithmic derivative of Φ is non negative for any $\nu' \in (0, \nu^{\mathrm{ACF}})$ and r sufficiently large. Again, a direct computation shows that

$$\begin{aligned}
\frac{\Phi'(r)}{\Phi(r)} &= -\frac{4\nu'}{r} + \sum_{i=1}^{2} \frac{\int_{\partial B_r^+} |\nabla v_i|^2 \Gamma_1 \,\mathrm{d}\sigma + \int_{r\mathbb{S}^{N-1}} v_i^2 v_j^2 \Gamma_1 \,\mathrm{d}\sigma}{\int_{B_r^+} |\nabla v_i|^2 \Gamma_1 \,\mathrm{d}x\mathrm{d}y + \int_{\partial^0 B_r^+} v_i^2 v_j^2 \Gamma_1 \,\mathrm{d}x} \\
&\geq \frac{4}{r} \left[-\nu' + \frac{1}{2} \sum_{i=1}^{2} \gamma \left(\Lambda(v_i^{(r)}) \right) \right]
\end{aligned}$$

and thus it is sufficient to prove that there exists $\bar{r} > 1$ such that, for every $r > \bar{r}$, the last term is nonnegative. Of course if $\Lambda_i(r) \to +\infty$ for some i then there is nothing to prove; thus we can suppose that each $\Lambda_i(r)$ is bounded uniformly. To begin with, we see that, for r large,

$$H(r) := \|v_i^{(r_n)}\|^2_{L^2(\mathbb{S}^N_+)} = \int_{\mathbb{S}^N_+} (v_i^{(r)})^2 \mathrm{d}\sigma \geq C > 0. \tag{2.1.7}$$

Indeed, the choice of $\phi \equiv 1$ in equation (2.1.5) yields

$$H'(r) = \int_{\mathbb{S}^N_+} r\partial_\nu (v_i^2)(r\xi)\mathrm{d}\sigma \geq 0,$$

and, since the functions are non trivial, H cannot be identically 0.

Let us suppose by contradiction that there exists a sequence $r_n \to \infty$ such that

$$\frac{1}{2}\sum_{i=1}^{2} \gamma\left(\Lambda_i(r_n)\right) \leq \nu' < \nu^{\mathrm{ACF}}. \tag{2.1.8}$$

We introduce the renormalized sequence

$$w_{i,n} = \frac{v_i^{(r_n)}}{\left(\int\limits_{\mathbb{S}^N_+} (v_i^{(r_n)})^2 \mathrm{d}\sigma\right)^{1/2}}, \quad \text{so that} \quad \|w_{i,n}\|_{L^2(\mathbb{S}^N_+)} = 1.$$

Recall that $\Lambda_i(r_n)$ is uniformly bounded, that is

$$K \geq \Lambda_i(r_n) = \int_{\mathbb{S}^N_+} |\nabla_T w_{i,n}|^2 \mathrm{d}\sigma + \int_{\mathbb{S}^{N-1}} r_n w_{i,n}^2 w_{i,n}^2 \|v_i^{(r_n)}\|_{L^2(\mathbb{S}^N_+)} \mathrm{d}\sigma$$

and, together with (2.1.7), this yields

$$\int_{\mathbb{S}^N_+} |\nabla_T w_{i,n}|^2 \,\mathrm{d}\sigma \leq K \quad \text{and} \quad \int_{\mathbb{S}^{N-1}} w_{i,n}^2 w_{i,n}^2 \,\mathrm{d}\sigma \leq \frac{1}{r_n} K'. \tag{2.1.9}$$

Hence there exist functions $\bar{w}_i \in H^1(\mathbb{S}^N_+)$ such that, up to subsequences, $w_{i,n_k} \rightharpoonup \bar{w}_i$, weakly in $H^1(\mathbb{S}^N_+)$, with $\|\bar{w}_i\|_{L^2(\mathbb{S}^N_+)} = 1$. Moreover, from the weak lower semicontinuity of the norm,

$$\liminf_{k\to\infty} \Lambda_i(r_{n_k}) \geq \int_{\mathbb{S}^N_+} |\nabla_T \bar{w}_i|^2 \mathrm{d}\sigma_N \geq \lambda_1(\{\bar{w}_i|_{y=0} > 0\}).$$

From (2.1.9) we have that $w_i^{(r)} w_j^{(r)} \to 0$ a.e. on $\mathbb{S}^{N-1}$ and $\bar{w}_i\bar{w}_j = 0$ on $\mathbb{S}^{N-1}$. This means that the limit configuration (w_1, w_2) induces a partition of $\mathbb{S}^N_+$, for which we have

$$\liminf_{k\to\infty} \frac{1}{2}\sum_{i=1}^{2} \gamma\left(\Lambda_i(r_{n_k})\right) \geq \nu^{\mathrm{ACF}}$$

in contradiction with (2.1.8). $\square$

2.2 Almgren type monotonicity formulæ

In the following, we will be concerned with a number of entire profiles, that is k-tuples of functions defined on the whole $\mathbb{R}^{N+1}_+$, which will be obtained from solutions to problem $(GP)_\beta$, by suitable limiting procedures. This section is devoted to the proof of some monotonicity formulæ of Almgren type, related to such profiles.

2.2.1 Almgren's formula for segregation entire profiles

To start with, we consider k-tuples $\mathbf{v}$ having components with segregated traces on $\mathbb{R}^N$. In such a situation, on one hand each component of $\mathbf{v}$, when different from zero, satisfies a limiting version of $(GP)_\beta$, where the internal dynamics are trivialized; on the other hand, the interaction between different components is now described by the validity of some Pohozaev type identity. We recall that, in order to prove the Almgren formula, it is sufficient to require the Pohozaev identity to hold only in spherical domains. Nonetheless, we prefer to assume its validity in the broader class of cylindrical domains, that is domains which are products of spherical and cubic ones. This choice will be useful in classifying the possible limiting profiles, when we will be involved in a procedure of dimensional reduction.

More precisely, let $C^+_{r,l}(x_0,0) \subset \mathbb{R}^{N+1}_+$ be any set such that there exists $h \in \mathbb{N}$, $h \leq N$, and a decomposition $\mathbb{R}^{N+1}_+ = \mathbb{R}^{h+1}_+ \oplus \mathbb{R}^{N-h}$ such that, writing

$$\mathbb{R}^{N+1}_+ \ni X = (x', x'', y), \text{ with } (x',y) \in \mathbb{R}^{h+1}_+,\ x'' \in \mathbb{R}^{N-h},$$

it holds

$$C^+_{r,l}(x_0,0) = B^+_r(x'_0,0) \times Q_l(x''_0).$$

Here, $B^+_r \subset \mathbb{R}^{h+1}_+$ denotes an half ball of radius r, and $Q_l \subset \mathbb{R}^{N-h}$ a cube of edge length equal to $2l$.

Definition 2.2.1 (Segregation entire profiles)**.** We denote with $\mathcal{G}_s$ the set of functions $\mathbf{v} \in H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}; \mathbb{R}^k\right)$, $\mathbf{v} = (v_1, \dots, v_k)$ continuous, which satisfy the following assumptions:

1. $v_i v_j|_{y=0} = 0$ for every $j \neq i$;

2. for every i,

$$\begin{cases} -\Delta v_i = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v_i \partial_\nu v_i = 0 & \text{on } \mathbb{R}^N \times \{0\}; \end{cases} \tag{2.2.1}$$

3. for any $x_0 \in \mathbb{R}^N$ and a.e. $r > 0$, $l > 0$,

$$\int_{C^+_{r,l}} \sum_i 2|\nabla_{(x',y)} v_i|^2 - (h+1)|\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y + r \int_{\partial^+ B^+_r \times Q_l} \sum_i |\nabla v_i|^2 \,\mathrm{d}\sigma +$$
$$= 2r \int_{\partial^+ B^+_r \times Q_l} \sum_i |\partial_\nu v_i|^2 \,\mathrm{d}\sigma - 2 \int_{B^+_r \times \partial^+ Q_l} \sum_i \partial_\nu v_i \nabla_{(x',y)} v_i \cdot (x' - x'_0, y) \,\mathrm{d}\sigma, \tag{2.2.2}$$

where $\nabla_{(x',y)}$ is the gradient with respect to the directions in $\mathbb{R}^{h+1}_+$.

Remark 2.2.2. Let $\mathbf{v} \in \mathcal{G}_s$. By choosing $h = N$ in the above definition, we obtain that the spherical Pohozaev identity holds, namely

$$(1-N) \int_{B^+_r} \sum_i |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y + r \int_{\partial^+ B^+_r} \sum_i |\nabla v_i|^2 \,\mathrm{d}\sigma = 2r \int_{\partial^+ B^+_r} \sum_i |\partial_\nu v_i|^2 \,\mathrm{d}\sigma \tag{2.2.3}$$

for a.e. $r > 0$.

Let us define, for every $x_0 \in \mathbb{R}^N$ and $r > 0$,

$$E(x_0, r) := \frac{1}{r^{N-1}} \int_{B^+_r(x_0,0)} \sum_i |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y$$
$$H(x_0, r) := \frac{1}{r^N} \int_{\partial^+ B^+_r(x_0,0)} \sum_i v_i^2 \,\mathrm{d}\sigma.$$

Let x_0 be fixed. Since $\mathbf{v} \in H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}, \mathbb{R}^k\right)$, both E and H are locally absolutely continuous functions on $(0, +\infty)$, that is, both E' and H' are $L^1_{\mathrm{loc}}(0,\infty)$ (here, $' = \mathrm{d}/\mathrm{d}r$).

Theorem 2.2.3. *Let $\mathbf{v} \in \mathcal{G}_s$, $\mathbf{v} \not\equiv 0$. For every $x_0 \in \mathbb{R}^N$ the function (Almgren frequency function)*

$$N(x_0, r) := \frac{E(x_0, r)}{H(x_0, r)}$$

is well defined on $(0,\infty)$, absolutely continuous, non decreasing, and it satisfies the identity

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(r) = \frac{2N(r)}{r}. \tag{2.2.4}$$

Moreover, if $N(r) \equiv \gamma$ on an open interval, then $N \equiv \gamma$ for every r, and $\mathbf{v}$ is a homogeneous function of degree γ.

Proof. Up to a translation, we may suppose that $x_0 = 0$. Obviously $H \geq 0$, and $H > 0$ on a nonempty interval (r_1, r_2), otherwise $\mathbf{v} \equiv 0$. As a consequence, either $\mathbf{v}$

is a nontrivial constant, and the theorem easily follows; or, by harmonicity, $\mathbf{v}$ is not constant in the whole $B^+_{r_2}$, and also $E > 0$ for $r < r_2$. Passing to the logarithmic derivatives, the monotonicity of N will be a consequence of the claim

$$\frac{N'(r)}{N(r)} = \frac{E'(r)}{E(r)} - \frac{H'(r)}{H(r)} \geq 0 \qquad \text{for } r \in (r_1, r_2).$$

Deriving E and using the Pohozaev identity (2.2.3), we have that

$$\begin{aligned} E'(r) &= \frac{1-N}{r^N} \int\limits_{B^+_r} \sum_i |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y + \frac{1}{r^{N-1}} \int\limits_{\partial^+ B^+_r} \sum_i |\nabla v_i|^2 \,\mathrm{d}\sigma \\ &= \frac{2}{r^{N-1}} \int\limits_{\partial_+ B^+_r} \sum_i |\partial_\nu v_i|^2 \,\mathrm{d}\sigma, \end{aligned}$$

while testing equation (2.2.1) with v_i in B^+_r and summing over i, we obtain

$$E(r) = \frac{1}{r^{N-1}} \int\limits_{B^+_r} \sum_i |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y = \frac{1}{r^{N-1}} \int\limits_{\partial^+ B^+_r} \sum_i v_i \partial_\nu v_i \,\mathrm{d}\sigma.$$

As far as H is concerned, we find

$$H'(r) = \frac{2}{r^N} \int\limits_{\partial^+ B^+_r} \sum_i v_i \partial_\nu v_i \,\mathrm{d}\sigma.$$

As a consequence, by the Cauchy-Schwarz inequality, we have

$$\frac{1}{2}\frac{N'(r)}{N(r)} = \frac{\int\limits_{\partial^+ B^+_r} \sum_i |\partial_\nu v_i|^2 \,\mathrm{d}\sigma}{\int\limits_{\partial^+ B^+_r} \sum_i v_i \partial_\nu v_i \,\mathrm{d}\sigma} - \frac{\int\limits_{\partial^+ B^+_r} \sum_i v_i \partial_\nu v_i \,\mathrm{d}\sigma}{\int\limits_{\partial^+ B^+_r} \sum_i v_i^2 \,\mathrm{d}\sigma} \geq 0 \qquad \text{for } r \in (r_1, r_2). \quad (2.2.5)$$

Moreover, on the same interval,

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(r) = \frac{H'(r)}{H(r)} = \frac{2E(r)}{rH(r)} = \frac{2N(r)}{r}.$$

Let us show that we can choose $r_1 = 0$, $r_2 = +\infty$. On one hand, the above equation provides that, if $\log H(\bar{r}) > -\infty$, then $\log H(r) > -\infty$ for every $r > \bar{r}$, so that $r_2 = +\infty$. On the other hand, let us assume by contradiction that

$$r_1 := \inf\{r : H(r) > 0 \text{ on } (r, +\infty)\} > 0.$$

By monotonicity, we have that $N(r) < N(2r_1)$ for every $r_1 < r \leq 2r_1$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(r) \leq \frac{2N(2r_1)}{r} \implies \frac{H(2r_1)}{H(r)} \leq \left(\frac{2r_1}{r}\right)^{2N(2r_1)}$$

and, since H is continuous, $H(r_1) > 0$, a contradiction.

Now, let us assume $N(r) \equiv \gamma$ on some interval I. Recalling equation (2.2.5), we see that

$$\left(\int\limits_{\partial^+ B_r^+} \sum_i v_i \partial_\nu v_i \, \mathrm{d}\sigma \right)^2 = \int\limits_{\partial^+ B_r^+} \sum_i v_i^2 \, \mathrm{d}\sigma \int\limits_{\partial^+ B_r^+} \sum_i |\partial_\nu v_i|^2 \, \mathrm{d}\sigma,$$

which is true, by the Cauchy-Schwarz inequality, if and only if $\mathbf{v}$ and $\partial_\nu \mathbf{v}$ are parallel, that is

$$v_i = \lambda(r) \partial_\nu v_i = \frac{\lambda(r)}{r} X \cdot \nabla v_i, \quad \text{for every } r \in I.$$

Using the definition of N, we have $\gamma = r/\lambda(r)$ for every $r \in I$, so that

$$\gamma v_i = X \cdot \nabla v_i \quad \forall i = 1, \dots, k.$$

But this is the Euler equation for homogeneous functions, and it implies that $\mathbf{v}$ is homogeneous of degree γ. Since each v_i is also harmonic in $\mathbb{R}^{N+1}_+$, the homogeneity extends to the whole of $\mathbb{R}^{N+1}_+$, yielding $N(r) \equiv \gamma$ for every $r > 0$. $\square$

In a standard way, from Theorem 2.2.3 we infer that the growth properties of the elements of $\mathcal{G}_s$ are related with their Almgren quotient.

Lemma 2.2.4. *Let $\mathbf{v} \in \mathcal{G}_s$, and let γ, $\bar{r}$ and C denote positive constants.*

1. *If $|\mathbf{v}(X)| \leq C|X - (x_0, 0)|^\gamma$ for every $X \notin B^+_{\bar{r}}(x_0, 0)$, then $N(x_0, r) \leq \gamma$ for every $r > 0$.*

2. *If $|\mathbf{v}(X)| \leq C|X - (x_0, 0)|^\gamma$ for every $X \in B^+_{\bar{r}}(x_0, 0)$, then $N(x_0, r) \geq \gamma$ for every $r > 0$.*

Proof. Let $\mathbf{v} \in \mathcal{G}_s$, and let us assume the growth condition for $r \geq \bar{r}$. We observe that it implies, for r large, $H(r) \leq C r^{2\gamma}$. Arguing by contradiction, let us suppose that there exists $R > \bar{r}$ such that $N(x_0, R) \geq \gamma + \varepsilon$. By monotonicity of N we have

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(r) \geq \frac{2}{r}(\gamma + \varepsilon) \quad \forall r \geq R$$

and, integrating in (R, r), we find

$$C r^{2(\gamma+\varepsilon)} \leq H(r) \leq C r^{2\gamma},$$

a contradiction for r large enough. On the other hand, if the growth condition holds for $r < \bar{r}$, we can argue in an analogous way, assuming that

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(r) \leq \frac{2(\gamma - \varepsilon)}{r}$$

for r small enough and obtaining again a contradiction. $\square$

Corollary 2.2.5. *If* $\mathbf{v} \in \mathcal{G}_s$ *is globally Hölder continuous of exponent* γ *on* $\mathbb{R}^{N+1}_+$, *then it is homogeneous of degree* γ *with respect to any of its (possible) zeroes, and*

$$\mathcal{Z} := \{x \in \mathbb{R}^N : \mathbf{v}(x,0) = 0\} \quad \textit{is an affine subspace of } \mathbb{R}^N.$$

Furthermore, if $\gamma < 1$, *then*

$$\mathcal{Z} = \emptyset \qquad \Longleftrightarrow \qquad \mathbf{v} \textit{ is a (nontrivial) constant.}$$

Proof. On one hand, if $(x_0, 0) \in \mathcal{Z}$, Lemma 2.2.4 implies $N(x_0, r) = \gamma$ for every r, and the first part easily follows. On the other hand, let $\mathcal{Z} = \emptyset$. By continuity, up to a relabeling, we have that $v_1(x,0) = \cdots = v_{k-1}(x,0) = 0$ on $\mathbb{R}^N$, so that their odd extension across $\{y = 0\}$ are harmonic and globally Hölder continuous of exponent $\gamma < 1$ on the whole of $\mathbb{R}^{N+1}$; but then the classical Liouville Theorem implies that they are all trivial. Finally, by continuity, $v_k(x,0)$ is always different from zero, so that $\partial_\nu v_k(x,0) \equiv 0$ on $\mathbb{R}^N$. As a consequence, Liouville Theorem applies also to the even extension of v_k across $\{y = 0\}$, concluding the proof. □

Remark 2.2.6. We observe that $\mathbf{v} = (1, y, 0, \ldots, 0)$ belongs to $\mathcal{G}_s$ and it is globally Lipschitz continuous, but it is not homogeneous. This does not contradict the previous Corollary 2.2.5, indeed its zero set is empty.

To conclude this section, we observe that the monotonicity of $N(x,r)$ implies that both for r small and for r large the corresponding limits are well defined.

Lemma 2.2.7. *Let* $\mathbf{v} \in \mathcal{G}_s$. *Then*

1. $N(x, 0^+)$ *is a non negative upper semicontinuous function on* $\mathbb{R}^N$*;*
2. $N(x, \infty)$ *is constant (possibly* ∞*).*

Proof. The first assertion follows because $N(x, 0^+)$ is the infimum of continuous functions. On the other hand, let

$$\nu := \lim_{r \to \infty} N(0, r) > 0;$$

we prove the second assertion in the case $\nu < \infty$, the other case following with minor changes. Let us suppose by contradiction that there exists $x_0 \in \mathbb{R}^N$ such that $\sup_{r>0} N(x_0, r) = \nu - 2\varepsilon$ for some $\varepsilon > 0$. Let moreover $r_0 > 0$ be such that $N(0, r_0) \geq \nu - \varepsilon$. Reasoning as in the proof of Lemma 2.2.4 we see that, when R_1, R_2 are sufficiently large, both $H(x_0, R_1) \leq C R_1^{2(\nu - 2\varepsilon)}$ and $H(0, R_2) \geq C R_2^{2(\nu - \varepsilon)}$. By definition

$$\int\limits_{B^+_{R_1}(x_0,0) \setminus B^+_{r_0}(x_0,0)} \sum_i v_i^2 \,\mathrm{d}x\mathrm{d}y = \int\limits_{r_0}^{R_1} H(x_0, s) s^N \mathrm{d}s \leq C R_1^{N+2(\nu-2\varepsilon)}$$

and

$$\int\limits_{B^+_{R_2}(0,0)\setminus B^+_{r_0}(0,0)} \sum_i v_i^2 \,\mathrm{d}x\mathrm{d}y = \int\limits_{r_0}^{R_2} H(0,s)s^N \mathrm{d}s \geq CR_2^{N+2(\nu-\varepsilon)}.$$

Now, if we let $R_1 = R_2 + |x_0|$, we obtain

$$\begin{aligned} CR_2^{N+2(\nu-\varepsilon)} &\leq \int\limits_{B^+_{R_2}(0,0)\setminus B^+_{r_0}(0,0)} \sum_i v_i^2 \,\mathrm{d}x\mathrm{d}y \\ &\leq \int\limits_{B^+_{r_0}(x_0,0)} \sum_i v_i^2 \,\mathrm{d}x\mathrm{d}y - \int\limits_{B^+_{r_0}(0,0)} \sum_i v_i^2 \,\mathrm{d}x\mathrm{d}y + \int\limits_{B^+_{R_1}(x_0,0)\setminus B^+_{r_0}(x_0,0)} \sum_i v_i^2 \,\mathrm{d}x\mathrm{d}y \\ &\leq C + C'(R_2+|x_0|)^{N+2(\nu-2\varepsilon)} \end{aligned}$$

and we find a contradiction for R_2 sufficiently large. Exchanging the role of 0 and x_0 we can conclude.

□

2.2.2 Almgren's formula for coexistence entire profiles

We now shift our attention to the case in which $\mathbf{v}$ is a k-tuple of functions which a priori are not segregated, but satisfy a boundary equation on $\mathbb{R}^N$. In this setting, the validity of the Pohozaev identities is a consequence of the boundary equation.

Definition 2.2.8 (Coexistence entire profiles)**.** We denote with $\mathcal{G}_c$ the set of functions $\mathbf{v} \in H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ which are solutions to

$$\begin{cases} -\Delta v_i = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu v_i + v_i \sum_{j\neq i} v_j^2 = 0 & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{2.2.6}$$

for every $i = 1, \dots, k$.

Remark 2.2.9. Of course, if $\mathbf{v} \in H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ solves

$$\begin{cases} -\Delta v_i = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu v_i + \beta v_i \sum_{j\neq i} v_j^2 = 0 & \text{on } \mathbb{R}^N \times \{0\}, \end{cases}$$

for some $\beta > 0$, then a suitable multiple of $\mathbf{v}$ belongs to $\mathcal{G}_s$.

Lemma 2.2.10. *Let $\mathbf{v} \in \mathcal{G}_c$. For any $x_0 \in \mathbb{R}^N$ and $r > 0$, the following identity* *holds*

$$(1-N)\int\limits_{B_r^+}\sum_i|\nabla v_i|^2\,\mathrm{d}x\mathrm{d}y + r\int\limits_{\partial^+B_r^+}\sum_i|\nabla v_i|^2\,\mathrm{d}\sigma - N\int\limits_{\partial^0B_r^+}\sum_{i,j<i}v_i^2v_j^2\,\mathrm{d}x$$
$$+ r\int\limits_{S_r^{N-1}}\sum_{i,j<i}v_i^2v_j^2\,\mathrm{d}\sigma = 2r\int\limits_{\partial^+B_r^+}\sum_i|\partial_\nu v_i|^2\,\mathrm{d}\sigma.$$

Proof. The proof follows by testing equation (2.2.6) with $\nabla v_i\cdot X$ in B_r^+ and exploiting some standard integral identities (see also Lemma 2.4.2 for a similar proof in a more general case). □

As before, we introduce the functions

$$E(x_0,r) := \frac{1}{r^{N-1}}\int\limits_{B_r^+(x_0,0)}\sum_i|\nabla v_i|^2\,\mathrm{d}x\mathrm{d}y + \frac{1}{r^{N-1}}\int\limits_{\partial^0B_r^+(x_0,0)}\sum_{i,j<i}v_i^2v_j^2\,\mathrm{d}x$$
$$H(x_0,r) := \frac{1}{r^N}\int\limits_{\partial^+B_r^+(x_0,0)}\sum_i v_i^2\,\mathrm{d}\sigma.$$

Theorem 2.2.11. *Let* $\mathbf{v}\in\mathcal{G}_c$*. For every* $x_0\in\mathbb{R}^N$ *the function*

$$N(x_0,r) := \frac{E(x_0,r)}{H(x_0,r)}$$

is non decreasing, absolutely continuous and strictly positive for $r>0$*. Moreover it holds*

$$\frac{\mathrm{d}}{\mathrm{d}r}\log H(r) \geq \frac{2N(r)}{r}.$$

Proof. The proof runs exactly as the one of Theorem 2.2.3, by using Lemma 2.2.10 instead of equation (2.2.3). □

As in the case of entire profiles of segregation, we can state some first consequence of Theorem 2.2.11.

Lemma 2.2.12. *Let* $\mathbf{v}\in\mathcal{G}_c$*, and let* γ *and* C *denote positive constants. If* $|\mathbf{v}(X)|\leq C(1+|X|^\gamma)$ *for every* X*, then* $N(x,\infty)$ *is constant and less than* γ*.*

Proof. The proof follows reasoning as in the ones of Lemmas 2.2.4 and 2.2.7. □

2.3 Liouville type theorems

By combining the results obtained in Sections 2.1 and 2.2, we are in a position to prove that nontrivial entire profiles, both of segregation and of coexistence, exhibit a minimal rate of growth connected with the Alt-Caffarelli-Friedman exponent ν^{ACF}. To be precise, the result concerning coexistence entire profiles only relies on the arguments developed in Section 2.1.

Proposition 2.3.1. *Let $\mathbf{v} \in \mathcal{G}_c$ and ν^{ACF} be defined according to Definitions 2.2.8 and 2.1.2. If for some $\gamma \in (0, \nu^{\mathrm{ACF}})$ there exists C such that*

$$|\mathbf{v}(X)| \leq C\left(1 + |X|^{\gamma}\right),$$

for every X, then $k-1$ components of $\mathbf{v}$ annihilate and the last is constant.

Proof. We start by proving that only one component of $\mathbf{v}$ can be different from zero. Let us suppose by contradiction that two components, say v_1 and v_2, are non trivial: indeed, we observe that $|v_1|$, $|v_2|$ fit in the setting of Theorem 2.1.13 (recall Remark 2.1.14). Let r be large accordingly, and let η be a non negative, smooth and radial cut-off function supported in B^+_{2r} with $\eta = 1$ in B^+_r and $|\nabla\eta| \leq Cr^{-1}$, $|\Delta\eta| \leq Cr^{-2}$. Moreover, let Γ_1 be defined as in 2.1.8 (in particular, it is radial and superharmonic). Testing the equation for v_i with $\Gamma_1 v_i \eta$ we obtain

$$\int_{B^+_{2r}} |\nabla v_i|^2 \Gamma_1 \eta \mathrm{d}x\mathrm{d}y + \int_{\partial^0 B^+_{2r}} v_i^2 v_j^2 \Gamma_1 \eta \mathrm{d}x \leq \int_{B^+_{2r}\setminus B^+_r} \frac{1}{2} v_i^2 \left[\Gamma_1 \Delta\eta + 2\nabla\eta \cdot \nabla\Gamma_1\right] \mathrm{d}x\mathrm{d}y,$$

where in the last step we used that η is constant in B^+_r. Since $\Gamma_1(X) = |X|^{1-N}$ outside B_1, and $|v_i(X)| \leq Cr^{\gamma}$ outside a suitable $B_{\bar{r}}$, using the notations of Theorem 2.1.13 we infer

$$\Phi_i(r) = \frac{1}{r^{2\nu'}} \left(\int_{B^+_r} |\nabla v_i|^2 \Gamma_1 \, \mathrm{d}x\mathrm{d}y + \int_{\partial^0 B^+_r} v_i^2 v_j^2 \Gamma_1 \, \mathrm{d}x \right) \leq \frac{1}{r^{2\nu'}} \cdot Cr^{2\gamma},$$

with C independent of $r > \bar{r}$. Fixing $\gamma < \nu' < \nu^{\mathrm{ACF}}$ and possibly taking $\bar{r}$ larger, Theorem 2.1.13 states that

$$0 < \Phi(\bar{r}) \leq \Phi(r) = \prod_{i=1}^{2} \Phi_i(r) \leq Cr^{4(\gamma-\nu')},$$

a contradiction for r large enough. Finally, if v_1 is the unique non trivial component of $\mathbf{v}$, an even extension of v_1 through $\mathbb{R}^N$ is harmonic in $\mathbb{R}^{N+1}$ and bounded everywhere by a function growing less than linearly, implying that v_1 is constant. □

Turning to segregation entire profiles, the results of Section 2.2 become crucial.

Proposition 2.3.2. *Let $\mathbf{v} \in \mathcal{G}_s$ and ν^{ACF} be defined according to Definitions 2.2.1 and 2.1.2.*

1. *If for some $\gamma \in (0, \nu^{\mathrm{ACF}})$ there exists C such that*

$$|\mathbf{v}(X)| \leq C\left(1 + |X|^{\gamma}\right),$$

for every X, then $k-1$ components of $\mathbf{v}$ annihilate;

2. *if furthermore* $\mathbf{v} \in \mathcal{C}^{0,\gamma}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ *then the only possibly nontrivial component is constant.*

Remark 2.3.3. We notice that the uniform Hölder continuity of exponent γ required in 2. readily implies the growth condition in 1., which we may not require explicitly. On the other hand, from the proof it will be clear that, once $k-1$ components annihilate, 2. follows by assuming uniform Hölder continuity of *any* exponent $\gamma' \in (0,1)$, not necessarily related to ν^{ACF}.

Proof of Proposition 2.3.2. To prove 1., we start as above by assuming by contradiction that there exist two components, v_1 and v_2, which are non trivial. We deduce that they must have a common zero on $\mathbb{R}^N$. As a consequence, we can reason as in the proof of Proposition 2.3.1, using Theorem 2.1.6 (and Remark 2.1.7) instead of Theorem 2.1.13, and obtain a contradiction. Turning to 2., let v denote the only non trivial component. By Corollary 2.2.5, we have that the set

$$\mathcal{Z} = \{x \in \mathbb{R}^N : v(x,0) = 0\}$$

is an affine subspace of $\mathbb{R}^N$. Now, if $\mathcal{Z} = \mathbb{R}^N$, then v satisfies

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v = 0 & \text{on } \mathbb{R}^N, \end{cases}$$

so that the odd extension of v through $\{y = 0\}$ is harmonic in $\mathbb{R}^{N+1}$ and bounded everywhere by a function growing less than linearly, implying that v is constant. On the other hand, if $\dim \mathcal{Z} \leq N-1$, then

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu v = 0 & \text{on } \mathbb{R}^N \setminus \mathcal{Z}, \end{cases}$$

and the even reflection of v through $\{y = 0\}$ is harmonic in $\mathbb{R}^{N+1} \setminus \mathcal{Z}$; since $\mathcal{Z}$ has null capacity with respect to $\mathbb{R}^{N+1}$, we infer that v is actually harmonic in $\mathbb{R}^{N+1}$, and the conclusion follows again since, by assumption, v is bounded everywhere by a function growing less than linearly. □

In the same spirit of the previous theorems, we provide now a result concerning single functions, rather than k-tuples.

Proposition 2.3.4. *Let* $v \in H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ *be continuous and satisfy*

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v\partial_\nu v \leq 0 & \text{on } \mathbb{R}^N \\ v(x,0) = 0 & \text{on } \{x_1 \leq 0\}, \end{cases}$$

and let us suppose that for some $\gamma \in [0, 1/2)$, $C > 0$ it holds

$$|v(X)| \leq C(1 + |X|^{\gamma})$$

for every X. Then v is constant.

Proof. It is trivial to check that v as above fulfills the assumptions of Proposition 2.1.12. Now, assuming that v is not constant, we can argue as in the proof of Proposition 2.3.1 obtaining a contradiction. □

To conclude the section, we provide other two theorems of Liouville type concerning single functions. The first one relies on the construction of a supersolution of a suitable problem, as done in the following lemma.

Lemma 2.3.5. *Let $M > 0$ and $\delta > 0$ be fixed and let $h \in L^{\infty}(\partial^0 B_1^+)$ with $\|h\|_{L^{\infty}} \leq \delta$. Any $v \in H^1(B_1^+) \cap \mathcal{C}\left(\overline{B_1^+}\right)$ non negative solution to*

$$\begin{cases} -\Delta v \leq 0, & \text{in } B_1^+ \\ \partial_{\nu} v \leq -Mv + h, & \text{on } \partial^0 B_1^+ \end{cases}$$

verifies

$$\sup_{\partial^0 B_{1/2}^+} v \leq \frac{1+\delta}{M} \sup_{\partial^+ B_1^+} v.$$

Proof. The proof follows from a simple comparison argument, once one notices that, for any $\delta > 0$, the function

$$w_{\delta} := \delta\frac{1}{M} + \frac{1}{N}\sum_{i=1}^{N}\frac{2}{\pi}\left[\frac{\pi}{2} - \arctan\left(\frac{x_i + 1}{y + \frac{2}{M}}\right) + \frac{\pi}{2} - \arctan\left(\frac{1 - x_i}{y + \frac{2}{M}}\right)\right]$$

satisfies the following system

$$\begin{cases} -\Delta w_{\delta} = 0 & \text{in } B_1^+ \\ \partial_{\nu} w_{\delta} \geq -M w_{\delta} + \delta & \text{on } \partial^0 B_1^+ \\ w_{\delta} \geq 1 & \text{on } \partial^+ B_1^+ \\ w_{\delta} \leq \frac{1+\delta}{M} & \text{in } \partial^0 B_{1/2}^+. \end{cases}$$

The claim can be proved by direct checking. For the reader's convenience, we sketch it in the case $N = 1$, $\delta = 0$.

For notation convenience, let us denote w_M by w. It is a straightforward computation to verify that w is positive and harmonic in $\mathbb{R}^2_+$. Using the elementary inequality $\frac{\pi}{2} - \arctan t \geq \frac{1}{1+t}$ for all $t \geq 0$, we can estimate

$$w(x, 0) \geq \frac{2}{\pi}\left[\frac{1}{1 + \frac{M}{2}(x+1)} + \frac{1}{1 + \frac{M}{2}(1-x)}\right].$$

On the other hand, using the inequality $\frac{t}{1+t^2} \le \frac{2}{1+t}$ for $t \ge 0$, we have

$$w_y(x,0) \le \frac{2}{\pi} M \left[\frac{1}{1+\frac{M}{2}(x+1)} + \frac{1}{1+\frac{M}{2}(1-x)} \right].$$

Therefore, $\partial_\nu w(x,0) = -w_y(x,0) \ge -Mw(x,0)$. For $(x,y) \in \overline{B_1^+}$ we have

$$\arctan\left(\frac{x+1}{y+\frac{2}{M}}\right) + \arctan\left(\frac{1-x}{y+\frac{2}{M}}\right) \le \frac{\pi}{2},$$

that is $w(x,y) \ge 1$ in B_1^+. Finally, we observe that $w(x,0)$, as a function of x, is strictly convex and even in $(-1,1)$. Consequently, if $|x| \le \frac{1}{2}$, using the elementary inequality $\frac{\pi}{2} - \arctan t \le \frac{1}{t}$ for $t \ge 0$, we obtain

$$w(x,0) \le \frac{1}{M}. \qquad \square$$

Remark 2.3.6. One of the peculiar difficulties in dealing with fractional operators with respect to the standard local case is due to the slow decay of supersolutions. Indeed, in the pure laplacian case, it is well known that positive solutions of

$$-\Delta u \le -Mu \quad \text{in } B \subset \mathbb{R}^N$$

exhibit an exponential decay, that is $u|_{B_{1/2}} \le e^{-\frac{1}{2}\sqrt{M}} \sup_{\partial B} u$; see, for instance, [22, 39]. In great contrast with this result, in the previous lemma we proved that non negative solutions of

$$(-\Delta)^{1/2} u \le -Mu \quad \text{in } B \subset \mathbb{R}^N$$

exhibit only polynomial decay, that is $u|_{B_{1/2}} \le \frac{1}{M} \sup_{\mathbb{R}^N \setminus B} u$. This estimate is sharp, since

$$\begin{cases} -\Delta v = 0 & \text{in } B^+ \\ v \ge 0 & \text{in } B^+ \\ \partial_\nu v = -Mv & \text{on } \partial^0 B^+ \end{cases}$$

implies

$$\inf_{\partial^0 B^+_{1/2}} v \ge \frac{1}{1+M} \inf_{\partial^+ B^+} v.$$

This last fact follows by a comparison between v and the subsolution $w = \frac{1}{1+M}(1 + My) \inf_{\partial^+ B^+} v$.

The previous estimate allows to prove the following.

Proposition 2.3.7. *Let v satisfy*

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu v = -\lambda v & \text{on } \mathbb{R}^N \end{cases}$$

for some $\lambda \geq 0$ and let us suppose that for some $\gamma \in [0,1)$, $C > 0$ it holds

$$|v(X)| \leq C(1 + |X|^\gamma)$$

for every X. Then v is constant.

Proof. If $\lambda = 0$, using an even reflection through $\{y = 0\}$, we extend v to a harmonic function in all $\mathbb{R}^{N+1}$, and we conclude as usual using the growth assumption. If $\lambda > 0$ let either $z = v^+$ or $z = v^-$. In both cases,

$$\begin{cases} -\Delta z \leq 0, & \text{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu z \leq -Mz, & \text{on } \mathbb{R}^N. \end{cases}$$

By translating and scaling, Lemma 2.3.5 implies that

$$z(x_0, 0) \leq \sup_{\partial^0 B_{r/2}(x_0,0)} z \leq \frac{1}{\lambda r} \sup_{\partial^+ B_r(x_0,0)} z \leq C \frac{1 + r^\gamma}{r}.$$

Letting $r \to \infty$ the proposition follows. □

Finally, we have the following.

Proposition 2.3.8. *Let v satisfy*

$$\begin{cases} -\Delta v = 0 & \textit{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu v = \lambda & \textit{on } \mathbb{R}^N \end{cases}$$

for some $\lambda \in \mathbb{R}$ and let us suppose that for some $\gamma \in [0,1)$, $C > 0$ it holds

$$|v(X)| \leq C(1 + |X|^\gamma)$$

for every X. Then v is constant.

Proof. For $h \in \mathbb{R}^N$, let $w(x, y) := v(x + h, y) - v(x, y)$. Then w solves

$$\begin{cases} -\Delta w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu w = 0 & \text{on } \mathbb{R}^N \end{cases}$$

and, as usual, we can reflect and use the growth condition to infer that w has to be constant, that is $v(x+h, y) = c_h + v(x, y)$. Deriving the previous expression in x_i, we find that

$$v(x, y) = \sum_{i=1}^{k} c_i(y) x_i + c_0(y).$$

Using again the growth condition, we see that $c_i \equiv 0$ for $i = 1, \dots, k$, while c_0 is constant. We observe that, consequently, $\lambda = 0$. □

2.4 Some approximation results

In the following, we want to apply the Liouville type theorems obtained in the previous section to suitable limiting profiles, obtained from solutions to the problem

$$\begin{cases} -\Delta v_i = 0 & \text{in } B^+ \\ \partial_\nu v_i = f_{i,\beta}(v_i) - \beta v_i \sum_{j\neq i} v_j^2 & \text{on } \partial^0 B^+, \end{cases} \tag{$(GP)_\beta$}$$

through some blow up and blow down procedures. From this point of view we have seen that, in the case of entire profiles of segregation, the key property is the validity of some Pohozaev identities, which imply that the Almgren formula holds. In this section we prove that such identities can be obtained by passing to the limit in the corresponding identities for $(GP)_\beta$, under suitable assumptions about the convergence. To be more precise, we will prove the following.

Proposition 2.4.1. *Let* $\mathbf{v}_n \in H^1\left(B^+_{r_n}\right)$ *solve problem* $(GP)_{\beta_n}$ *on* $B^+_{r_n}$, $n \in \mathbb{N}$, *and* $\mathbf{v} \in H^1_{\text{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$, *be such that, as* $n \to \infty$,

1. $\beta_n \to \infty$;
2. $r_n \to \infty$;
3. *for every compact* $K \subset \mathbb{R}^{N+1}_+$, $\mathbf{v}_n \to \mathbf{v}$ *in* $H^1(K) \cap C(K)$;
4. *the continuous functions* f_{i,β_n} *are such that, for every* $\bar{m} > 0$,
$$|f_{i,\beta_n}(s)| \le C_n(\bar{m}) \quad \text{for } |s| < \bar{m},$$
where $C_n(\bar{m}) \to 0$.

Then $\mathbf{v} \in \mathcal{G}_s$.

We start by stating the basic identities for problem $(GP)_\beta$. We recall that S^{N-1}_r denotes the $(N-1)$-dimensional boundary of $\partial^0 B^+_r$ in $\mathbb{R}^N$.

Lemma 2.4.2 (Pohozaev identity)**.** *Let* $\mathbf{v}$ *solve problem* $(GP)_\beta$ *on* B^+. *For every* $B^+_r := B^+_r(x_0, 0) \subset B^+$ *the following Pohozaev identity holds:*

$$\begin{aligned}(1-N)\int_{B^+_r} \sum_i |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y + r\int_{\partial^+ B^+_r} \sum_i |\nabla v_i|^2 \,\mathrm{d}\sigma + \\ + 2N\int_{\partial^0 B^+_r} \sum_i F_{i,\beta}(v_i)\,\mathrm{d}x - N\beta\int_{\partial^0 B^+_r} \sum_{i,j<i} v_i^2 v_j^2 \,\mathrm{d}x - 2r\int_{S^{N-1}_r} \sum_i F_{i,\beta}(v_i)\,\mathrm{d}\sigma + \\ + r\beta\int_{S^{N-1}_r} \sum_{i,j<i} v_i^2 v_j^2\,\mathrm{d}\sigma = 2r\int_{\partial^+ B^+_r} \sum_i |\partial_\nu v_i|^2\,\mathrm{d}\sigma.\end{aligned}$$

Proof. Let the functions v_i solve problem $(GP)_\beta$. Up to translations, we assume that $x_0 = 0$. By multiplying the equation with $X \cdot \nabla v_i$ and integrating by parts over B_r^+, we obtain

$$\int_{B_r^+} \nabla v_i \cdot \nabla (X \cdot \nabla v_i)\,\mathrm{d}x\mathrm{d}y = r \int_{\partial^+ B_r^+} |\partial_\nu v_i|^2\,\mathrm{d}\sigma + \int_{\partial^0 B_r^+} (\partial_\nu v_i)(x \cdot \nabla_x v_i)\,\mathrm{d}x.$$

Using the identity

$$\nabla v_i \cdot \nabla (X \cdot \nabla v_i) = |\nabla v_i|^2 + X \cdot \nabla \left(\frac{1}{2}|\nabla v_i|^2\right)$$

and integrating again by parts, we can write the right hand side as

$$\int_{B_r^+} \nabla v_i \cdot \nabla (X \cdot \nabla v_i)\,\mathrm{d}x\mathrm{d}y = \frac{1-N}{2} \int_{B_r^+} |\nabla v_i|^2\,\mathrm{d}x\mathrm{d}y + \frac{r}{2} \int_{\partial^+ B_r^+} |\nabla v_i|^2\,\mathrm{d}\sigma$$

and this yields

$$\frac{1-N}{2} \int_{B_r^+} |\nabla v_i|^2\,\mathrm{d}x\mathrm{d}y + \frac{r}{2} \int_{\partial^+ B_r^+} |\nabla v_i|^2\,\mathrm{d}\sigma - \int_{\partial^0 B_r^+} f_{i,\beta}(v_i)(x \cdot \nabla_x v_i)\,\mathrm{d}x +$$
$$+ \frac{\beta}{2} \int_{\partial^0 B_r^+} (x \cdot \nabla_x v_i^2) \sum_{j \neq i} v_j^2\,\mathrm{d}x = r \int_{\partial^+ B_r^+} |\partial_\nu v_i|^2\,\mathrm{d}\sigma.$$

Summing the identities for $i = 1, \dots, k$ we obtain

$$\begin{aligned} &\frac{1-N}{2} \int_{B_r^+} \sum_i |\nabla v_i|^2\,\mathrm{d}x\mathrm{d}y + \frac{r}{2} \int_{\partial^+ B_r^+} \sum_i |\nabla v_i|^2\,\mathrm{d}\sigma \\ &- \int_{\partial^0 B_r^+} (x \cdot \nabla_x) \sum_i F_{i,\beta}(v_i)\,\mathrm{d}x + \frac{\beta}{2} \int_{\partial^0 B_r^+} (x \cdot \nabla_x) \sum_{i,j<i} v_i^2 v_j^2\,\mathrm{d}x = r \int_{\partial^+ B_r^+} \sum_i |\partial_\nu v_i|^2\,\mathrm{d}\sigma. \end{aligned} \tag{2.4.1}$$

The terms on $\partial^0 B_r^+$ can be further simplified: by an application of the divergence theorem on $\mathbb{R}^N$ we have

$$\begin{aligned} \int_{\partial^0 B_r^+} (x \cdot \nabla_x) \sum_{i,j<i} v_i^2 v_j^2\,\mathrm{d}x &= \int_{\partial^0 B_r^+} \mathrm{div}\left(x \sum_{i,j<i} v_i^2 v_j^2\right)\mathrm{d}x - \int_{\partial^0 B_r^+} \mathrm{div}x \sum_{i,j<i} v_i^2 v_j^2\,\mathrm{d}x \\ &= r \int_{S_r^{N-1}} \sum_{i,j<i} v_i^2 v_j^2\,\mathrm{d}\sigma - N \int_{\partial^0 B_r^+} \sum_{i,j<i} v_i^2 v_j^2\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} \int_{\partial^0 B_r^+} (x \cdot \nabla_x) \sum_i F_{i,\beta}(v_i)\,\mathrm{d}x &= \int_{\partial^0 B_r^+} \mathrm{div}\left(x \sum_i F_{i,\beta}(v_i)\right)\mathrm{d}x - \int_{\partial^0 B_r^+} \mathrm{div}x \sum_i F_{i,\beta}(v_i)\mathrm{d}x \\ &= r \int_{rS^{N-1}} \sum_i F_{i,\beta}(v_i)\,\mathrm{d}\sigma - N \int_{\partial^0 B_r^+} \sum_i F_{i,\beta}(v_i)\,\mathrm{d}x; \end{aligned}$$

the lemma follows by substituting into equation (2.4.1). □

In a similar way, it is possible to prove the validity of the Pohozaev identities in cylinders (we use the notations introduced in the discussion at the beginning of Section 2.2.1).

Lemma 2.4.3 (Pohozaev identity in cylinders)**.** *Let* $\mathbf{v} \in H^1(B^+)$ *be a solution to problem* $(GP)_\beta$*. For every* $x \in \partial^0 B^+$ *and* $r > 0$, $l > 0$ *such that* $C^+_{r,l} \subset B^+$ *the following Pohozaev identity holds:*

$$\begin{aligned}\int\limits_{C^+_{r,l}} \Big(\sum_i 2|\nabla_{(x',y)} v_i|^2 - (h+1)|\nabla v_i|^2\Big)\,\mathrm{d}x\mathrm{d}y + r \int\limits_{\partial^+ B^+_r \times Q_l} \sum_i |\nabla v_i|^2\,\mathrm{d}\sigma + \\ + 2h \int\limits_{\partial^0 C^+_{r,l}} \sum_i F_{i,\beta}(v_i)\,\mathrm{d}x - h\beta \int\limits_{\partial^0 C^+_{r,l}} \sum_{i,j<i} v_i^2 v_j^2\,\mathrm{d}x \\ - 2r \int\limits_{S^{h-1}_r \times Q_l} \sum_i F_{i,\beta}(v_i)\,\mathrm{d}\sigma + r\beta \int\limits_{S^{h-1}_r \times Q_l} \sum_{i,j<i} v_i^2 v_j^2\,\mathrm{d}\sigma \\ = 2r \int\limits_{\partial^+ B^+_r \times Q_l} \sum_i |\partial_\nu v_i|^2\,\mathrm{d}\sigma - 2 \int\limits_{B^+_r \times \partial^+ Q_l} \sum_i \partial_\nu v_i \nabla_{(x',y)} v_i \cdot (x',y)\,\mathrm{d}\sigma,\end{aligned}$$

where $\nabla_{(x',y)}$ *is the gradient with respect to the directions in* $\mathbb{R}^{h+1}_+$*.*

Remark 2.4.4. Even though the mentioned Pohozaev identities are enough for our purposes, we would like to point out that they are nothing but special cases of a more general class of identities, namely the domain variation formulas, see for instance [26]. They may be obtained by testing the equation of $(GP)_\beta$ by $\nabla \mathbf{v} \cdot Y$ in a smooth domain $\omega \subset \mathbb{R}^{N+1}_+$, where $Y \in \mathcal{C}^1(\mathbb{R}^{N+1}_+; \mathbb{R}^{N+1}_+)$ is a smooth vector field such that $Y|_{y=0} \in \mathcal{C}^1(\mathbb{R}^N; \mathbb{R}^N)$.

To proceed, we need the following standard result.

Lemma 2.4.5. *Let* $f, \lambda \in L^\infty(\partial^0 B^+)$*. If* $w \in H^1(B^+)$ *is a solution to*

$$\begin{cases} -\Delta w = 0 & \text{in } B^+ \\ \partial_\nu w = f - \lambda w & \text{on } \partial^0 B^+, \end{cases}$$

then $|w| \in H^1(B^+)$ *and for any* $\phi \in H^1(B^+)$, $\phi|_{\partial^+ B^+} = 0$, $\phi \geq 0$ *it holds*

$$\int\limits_{B^+} \nabla |w| \cdot \nabla \phi\,\mathrm{d}x\mathrm{d}y - \int\limits_{\partial^0 B^+} (|f| - \lambda |w|)\phi \mathrm{d}x \leq 0$$

Proof. Let $g_\varepsilon(s) = \sqrt{s^2+\varepsilon} \in \mathcal{C}^\infty(\mathbb{R})$ such that $g_\varepsilon(s) \to |s|$ and $g'_\varepsilon(s) \to \mathrm{sgn}(s)$. By Stampacchia Lemma,

$$g_\varepsilon(w) \to |w| \quad \text{in } H^1(B^+)$$

while, by Lebesgue theorem

$$g'_\varepsilon(w)w \to |w| \quad \text{in } L^2(\partial^0 B^+).$$

Thus, for any $\phi \in H^1(B^+)$, $\phi|_{\partial^+ B^+} = 0$, $\phi \geq 0$, we have

$$\begin{aligned}
&\int_{B^+} \nabla g_\varepsilon(w)\cdot\nabla\phi \,\mathrm{d}x\mathrm{d}y - \int_{\partial^0 B^+} g'_\varepsilon(v)(f-\lambda w)\phi \mathrm{d}x \\
&\qquad = \int_{B^+} g'_\varepsilon(w)\nabla w\cdot\nabla\phi \,\mathrm{d}x\mathrm{d}y - \int_{\partial^0 B^+} g'_\varepsilon(w)\partial_\nu v\phi \mathrm{d}x \\
&= \int_{B^+} -\mathrm{div}\left(g'_\varepsilon(w)\nabla w\right)\phi \,\mathrm{d}x\mathrm{d}y = \int_{B^+} \left(-g''_\varepsilon(w)|\nabla w|^2 - g'_\varepsilon(w)\Delta w\right)\phi \,\mathrm{d}x\mathrm{d}y \leq 0.
\end{aligned}$$

Passing to the limit for $\varepsilon \to 0$ we obtain the lemma. □

Going back to the notations of Proposition 2.4.1, we have the following lemma.

Lemma 2.4.6. *For every K compact subset of $\mathbb{R}^N$, it holds*

$$\lim_{n\to\infty} \beta_n \int_K v_{i,n}^2 \sum_{j\neq i} v_{j,n}^2 \,\mathrm{d}x = 0.$$

Moreover, for every $x_0 \in \mathbb{R}^N$, and for almost every $r > 0$,

$$\beta_n \int_{S_r^{N-1}} v_{i,n}^2 \sum_{j\neq i} v_{j,n}^2 \mathrm{d}\sigma \to 0.$$

Proof. Let $\eta \in \mathcal{C}_0^\infty(B_r)$ be a positive smooth cutoff function with the property that $\eta \equiv 1$ on K. Taking into account Lemma 2.4.5, we obtain that

$$0 \leq \beta_n \int_K |v_{i,n}| \sum_{j\neq i} v_{j,n}^2 \,\mathrm{d}x \leq \int_{\partial^0 B_r^+} \left(|f_{i,n}|\eta - |v_{i,n}|\partial_\nu\eta\right)\mathrm{d}x + \int_{B_r^+} |v_{i,n}|\Delta\eta \,\mathrm{d}x\mathrm{d}y \leq C.$$

In particular, on one hand this implies that

$$\beta_n \int_K |v_{i,n}| \sum_{j\neq i} v_{j,n}^2 \,\mathrm{d}x \leq C,$$

while on the other hand, by passing to the limit, we infer that $\{v_i = 0\} \cup \{v_j = 0\}$ contains K, for every $i \neq j$. As a consequence, each term in the sum can be estimated as follows

$$\begin{aligned}\beta_n \int\limits_K v_{i,n}^2 v_{j,n}^2 \,\mathrm{d}x \leq& \beta_n \int\limits_{K\cap\{v_i=0\}} v_{i,n}^2 v_{j,n}^2 \,\mathrm{d}x + \beta_n \int\limits_{K\cap\{v_j=0\}} v_{j,n}^2 v_{i,n}^2 \,\mathrm{d}x \\ \leq& \|v_{i,n}\|_{L^\infty(K\cap\{v_i=0\})} \beta_n \int\limits_{K\cap\{v_i=0\}} |v_{i,n}| v_{j,n}^2 \,\mathrm{d}x \\ &+ \|v_{j,n}\|_{L^\infty(K\cap\{v_j=0\})} \beta_n \int\limits_{K\cap\{v_j=0\}} |v_{j,n}| v_{i,n}^2 \,\mathrm{d}x \to 0,\end{aligned}$$

and the first conclusion follows by summing over all $j \neq i$. As far as the second one is concerned, it follows by applying Fubini's Theorem to the previous conclusion when $K = \partial^0 B_R^+$. $\square$

Proof of Proposition 2.4.1. First we notice that, by Lemma 2.4.6, it holds $v_i v_j \equiv 0$ for every $i \neq j$. Moreover, since the uniform limit of harmonic functions is harmonic itself, $\Delta v_i = 0$ on $\mathbb{R}_+^{N+1}$. In order to obtain (2.2.1), we observe that, for any $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have

$$\int\limits_{\mathbb{R}^N} v_{i,n} \partial_\nu v_{i,n} \phi \,\mathrm{d}x = \int\limits_{\mathbb{R}^N} \left(v_{i,n} f_{i,\beta_n}(v_{i,n}) - \beta_n v_{i,n}^2 \sum_{j\neq i} v_{j,n}^2 \right) \phi \,\mathrm{d}x \to 0$$

by assumption 4. and Lemma 2.4.6. Finally, to prove that (2.2.2) holds, we are going to show that, for every $x_0 \in \mathbb{R}^N$ and almost every $r > 0$, the Pohozaev identity of Lemma 2.4.2 passes to the limit (the general case following by analogous arguments). Let us recollect the terms of the identity as

$$\begin{aligned}&\underbrace{(1-N) \int\limits_{B_r^+} \sum_i |\nabla v_{i,n}|^2 \,\mathrm{d}x\mathrm{d}y}_{A_n} + \underbrace{r \int\limits_{\partial^+ B_r^+} \sum_i |\nabla v_{i,n}|^2 \,\mathrm{d}\sigma}_{B_n^1} + \\ &\underbrace{+2N \int\limits_{\partial^0 B_r^+} \sum_i F_{i,n}(v_{i,n}) \,\mathrm{d}x - 2r \int\limits_{S_r^{N-1}} \sum_i F_{i,n}(v_{i,n}) \,\mathrm{d}\sigma}_{I_n} + \\ &\underbrace{-N\beta_n \int\limits_{\partial^0 B_r^+} \sum_{i,j<i} v_{i,n}^2 v_{i,n}^2 \,\mathrm{d}x + r\beta \int\limits_{S_r^{N-1}} \sum_{i,j<i} v_{i,n}^2 v_{j,n}^2 \,\mathrm{d}\sigma}_{C_n} = \underbrace{2r \int\limits_{\partial^+ B_r^+} \sum_i |\partial_\nu v_{i,n}|^2 \,\mathrm{d}\sigma}_{B_n^2}.\end{aligned}$$

On one hand, by strong H^1_{loc} convergence,

$$A_n \to (1-N) \int\limits_{B_r^+} \sum_i |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y.$$

Moreover, both $I_n \to 0$ (by assumption 4.) and $C_n \to 0$ for a.e. r (by Lemma 2.4.6). We claim that

$$\lim_{n\to\infty} B_n^1 = r \int\limits_{\partial^+ B_r^+} \sum_i |\nabla v_i|^2 \, \mathrm{d}\sigma \quad \text{and} \quad \lim_{n\to\infty} B_n^2 = 2r \int\limits_{\partial^+ B_r^+} \sum_i |\partial_\nu v_i|^2 \, \mathrm{d}\sigma$$

in $L^1_{\mathrm{loc}}[0,\infty)$: in particular, this will imply convergence for a.e. r. Let us prove the former limit, which implies also the latter. The strong convergence $\mathbf{v}_n \to \mathbf{v}$ in $H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ implies that

$$\int_0^R \int\limits_{\partial^+ B_r^+} \sum_i |\nabla v_{i,n} - \nabla v_i|^2 \, \mathrm{d}\sigma \mathrm{d}r \to 0,$$

so that $\int\limits_{\partial^+ B_r^+} |\nabla v_{i,n}|^2 \mathrm{d}\sigma \to \int\limits_{\partial^+ B_r^+} |\nabla v_{i,n}|^2 \mathrm{d}\sigma$ for a.e. r and there exists an integrable function $f \in L^1(0,R)$ such that, up to a subsequence

$$\int\limits_{\partial^+ B_r^+} |\partial_\nu v_{i,n_k}|^2 \mathrm{d}\sigma \le \int\limits_{\partial^+ B_r^+} |\nabla v_{i,n_k}|^2 \mathrm{d}\sigma \le f(r) \quad \text{a.e. } r \in (0,R)$$

for every $i = 1, \dots, k$. We can then use the Dominated Convergence Theorem. Since every subsequence of $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ admits a convergent sub-subsequence, and the limit is the same, we conclude the convergence for the entire approximating sequence. □

2.5 Local $C^{0,\alpha}$ uniform bounds, α small

In this section we begin our regularity analysis with a first partial result. We will obtain a localized version of the uniform Hölder regularity for solutions to problem $(GP)_\beta$, when the Hölder exponent is sufficiently small. We recall that, here and in the following, the functions $f_{i,\beta}$ are assumed to be continuous and uniformly bounded, with respect to β, on bounded sets.

Remark 2.5.1. By standard regularity results (see for instance the book [50]), we already know that for every $r < 1$, $\alpha \in (0,1)$, $\bar{m} > 0$ and $\bar{\beta} > 0$, there exists a constant $C = C(r, \alpha, \bar{m}, \bar{\beta})$ such that

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}\left(\overline{B_r^+}\right)} \le C,$$

for every $\mathbf{v}_\beta$ solution of problem $(GP)_\beta$ on B_1^+, satisfying

$$\beta \le \bar{\beta} \quad \text{and} \quad \|\mathbf{v}_\beta\|_{L^\infty(B_1^+)} \le \bar{m}.$$

The main result of this section is the following.

Theorem 2.5.2. *Let $\{\mathbf{v}_\beta\}_{\beta>0}$ be a family of solutions to problem $(GP)_\beta$ on B_1^+ such that*

$$\|\mathbf{v}_\beta\|_{L^\infty(B_1^+)} \le \bar{m},$$

with $\bar{m}$ independent of β. Then for every $\alpha \in (0, \nu^{\mathrm{ACF}})$ there exists a constant $C = C(\bar{m}, \alpha)$, not depending on β, such that

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}\left(\overline{B_{1/2}^+}\right)} \le C.$$

Furthermore, $\{\mathbf{v}_\beta\}_{\beta>0}$ is relatively compact in $H^1(B_{1/2}^+) \cap \mathcal{C}^{0,\alpha}\left(\overline{B_{1/2}^+}\right)$ for every $\alpha < \nu^{\mathrm{ACF}}$.

Remark 2.5.3. Even though we prove it in $\overline{B_{1/2}^+}$, Theorem 2.5.2 holds also when replacing $\overline{B_{1/2}^+}$ with $K \cap B_1^+$, for every compact set $K \subset B_1$.

For easier notation, we write $B^+ = B_1^+$. Inspired by [39, 52], we proceed by contradiction and develop a blow up analysis. First, let η denote a smooth function such that

$$\begin{cases} \eta(X) = 1 & 0 \le |X| \le \frac{1}{2} \\ 0 < \eta(X) \le 1 & \frac{1}{2} \le |X| \le 1 \\ \eta(X) = 0 & |X| = 1 \end{cases} \tag{2.5.1}$$

(in particular, η vanishes on $\partial^+ B^+$ but is strictly positive $\partial^0 B^+$). We will prove that

$$\|\eta \mathbf{v}\|_{\mathcal{C}^{0,\alpha}(\overline{B^+})} \le C,$$

and the theorem will follow by the regularity of η.

Let us assume by contradiction the existence of sequences $\{\beta_n\}_{n\in\mathbb{N}}$, $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$, solutions to $(GP)_{\beta_n}$, such that

$$L_n := \max_{i=1,\dots,k} \max_{X' \ne X'' \in \overline{B^+}} \frac{|(\eta v_{i,n})(X') - (\eta v_{i,n})(X'')|}{|X' - X''|^\alpha} \to \infty,$$

for some $\alpha \in (0, \nu^{\mathrm{ACF}})$, which from now on we will consider as fixed. By Remark 2.5.1 we readily infer that $\beta_n \to \infty$. Moreover, up to a relabelling, we may assume that L_n is achieved by $i = 1$ and a sequence of points $(X_n', X_n'') \in \overline{B^+} \times \overline{B^+}$. We start showing some first properties of such sequences.

Lemma 2.5.4. *Let $X_n' \ne X_n''$ and $r_n := |X_n' - X_n''|$ satisfy*

$$L_n = \frac{|(\eta v_{1,n})(X_n') - (\eta v_{i,n})(X_n'')|}{r_n^\alpha}.$$

Then, as $n \to \infty$,

1. $r_n \to 0$;
2. $\dfrac{\mathrm{dist}(X'_n, \partial^+ B^+)}{r_n} \to \infty$, $\dfrac{\mathrm{dist}(X''_n, \partial^+ B^+)}{r_n} \to \infty$.

Proof. By the uniform control on $\|\mathbf{v}_n\|_{L^\infty}$ we have

$$L_n \leq \frac{\bar{m}}{r_n^\alpha} \left(\eta(X'_n) + \eta(X''_n) \right),$$

which immediately implies $r_n \to 0$. Since η vanishes on $\partial^+ B^+$, we have that, for every $X \in \overline{B^+}$, it holds

$$\eta(X) \leq \ell \mathrm{dist}(X, \partial^+ B^+),$$

where ℓ denotes the Lipschitz constant of η. As a consequence, the first inequality becomes

$$\frac{\mathrm{dist}(X'_n, \partial^+ B^+)}{r_n} + \frac{\mathrm{dist}(X''_n, \partial^+ B^+)}{r_n} \geq \frac{L_n r_n^{\alpha-1}}{\bar{m}\ell} \to \infty$$

(recall that $\alpha < 1$), and the lemma follows by recalling that $\mathrm{dist}(X'_n, X''_n) = r_n$. $\square$

Our analysis is based on two different blow up sequences, one having uniformly bounded Hölder quotient, the other satisfying a suitable problem. Let $\{P_n\}_{n\in\mathbb{N}} \subset \overline{B^+}$, $|P_n| < 1$, be a sequence of points, to be chosen later. We write

$$\tau_n B^+ := \frac{B^+ - P_n}{r_n},$$

remarking that $\tau_n B^+$ is a hemisphere, not necessarily centered on the hyperplane $\{y = 0\}$. We introduce the sequences

$$w_{i,n}(X) := \eta(P_n) \frac{v_{i,n}(P_n + r_n X)}{L_n r_n^\alpha} \quad \text{and} \quad \bar{w}_{i,n}(X) := \frac{(\eta v_{i,n})(P_n + r_n X)}{L_n r_n^\alpha},$$

where $X \in \tau_n B^+$. With this choice, on one hand it follows immediately that, for every i

$$\max_{X' \neq X'' \in \overline{\tau_n B^+}} \frac{|\bar{w}_{i,n}(X') - \bar{w}_{i,n}(X'')|}{|X' - X''|^\alpha} \leq \left| \bar{w}_{1,n}\left(\frac{X'_n - P_n}{r_n} \right) - \bar{w}_{1,n}\left(\frac{X''_n - P_n}{r_n} \right) \right| = 1,$$

in such a way that the functions $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ share an uniform bound on Hölder seminorm, and at least their first components are not constant. On the other hand, since $\eta(P_n) > 0$, each $\mathbf{w}_n$ solves

$$\begin{cases} -\Delta w_{i,n} = 0 & \text{in } \tau_n B^+ \\ \partial_\nu w_{i,n} = f_{i,n}(w_{i,n}) - M_n w_{i,n} \sum_{j\neq i} w_{j,n}^2 & \text{on } \tau_n \partial^0 B^+, \end{cases} \tag{2.5.2}$$

with $f_{i,n}(s) = \eta(P_n) r_n^{1-\alpha} L_n^{-1} f_{i,\beta_n}(L_n r_n^\alpha s / \eta(P_n))$ and $M_n = \beta_n L_n^2 r_n^{2\alpha+1} / \eta(P_n)^2$.

Remark 2.5.5. The uniform bound of $\|\mathbf{v}_\beta\|_{L^\infty}$ imply that

$$\sup_{\tau_n \partial^0 B^+} |f_{i,n}(w_{i,n})| = \eta(P_n) r_n^{1-\alpha} L_n^{-1} \sup_{\partial^0 B^+} |f_{i,\beta_n}(v_{i,n})| \leq C(\bar{m}) r_n^{1-\alpha} L_n^{-1} \to 0$$

as $n \to \infty$.

A crucial property is that the two blow up sequences defined above have asymptotically equivalent behavior, as enlighten in the following lemma.

Lemma 2.5.6. *Let $K \subset \mathbb{R}^{N+1}$ be compact. Then*

1. $\max_{X \in K \cap \overline{\tau_n B^+}} |\mathbf{w}_n(X) - \bar{\mathbf{w}}_n(X)| \to 0$*;*
2. *there exists C, only depending on K, such that $|\mathbf{w}_n(X) - \mathbf{w}_n(0)| \leq C$, for every $x \in K$.*

Proof. Again, this is a consequence of the Lipschitz continuity of η and of the uniform boundedness of $\{\mathbf{v}_\beta\}_\beta$. Indeed we have, for every $i = 1, \ldots, k$,

$$|w_{i,n}(X) - \bar{w}_{i,n}(X)| \leq \bar{m} r_n^{-\alpha} L_n^{-1} |\eta(X_n + r_n X) - \eta(X_n)| \leq \ell \bar{m} r_n^{1-\alpha} L_n^{-1} |X|$$

and the right hand side vanishes in n, implying the first part. Moreover, by definition, $\mathbf{w}_n(0) = \bar{\mathbf{w}}_n(0)$, and $|\bar{\mathbf{w}}_n(X) - \bar{\mathbf{w}}_n(0)| \leq C|X|^\alpha$ for every $X \in \tau_n B^+$. But then we can conclude noticing that

$$|\mathbf{w}_n(X) - \mathbf{w}_n(0)| \leq |\mathbf{w}_n(X) - \bar{\mathbf{w}}_n(X)| + |\bar{\mathbf{w}}_n(X) - \bar{\mathbf{w}}_n(0)|$$

and applying the first part. $\square$

Lemma 2.5.7. *Let, up to subsequences, $\Omega_\infty := \lim \tau_n B^+$ and let*

$$\mathbf{W}_n(X) := \mathbf{w}_n(X) - \mathbf{w}_n(0) \quad \textit{and} \quad \bar{\mathbf{W}}_n(X) := \bar{\mathbf{w}}_n(X) - \bar{\mathbf{w}}_n(0).$$

Then there exists a function $\mathbf{W} \in \mathcal{C}^{0,\alpha}(\Omega_\infty)$ which is harmonic and such that $\mathbf{W}_n \to \mathbf{W}$ and $\bar{\mathbf{W}}_n \to \mathbf{W}$ uniformly in every compact set $K \subset \Omega_\infty$. Moreover, if we choose $\{P_n\}_{n\in\mathbb{N}}$ such that $|X_n' - P_n| < C r_n$ for some constant C and for every n, then $\mathbf{W}$ is non constant.

Proof. Let $K \subset \Omega_\infty$ be any fixed compact set. Then, by definition, K is contained in the half sphere $\tau_n B^+$, for every n sufficiently large. By definition, $\{\bar{\mathbf{W}}_n\}_{n\in\mathbb{N}}$ is a sequence of functions which share the same $\mathcal{C}^{0,\alpha}$-seminorm and are uniformly bounded in K, since $\bar{\mathbf{W}}_n(0) = 0$. By the Ascoli-Arzelà theorem, there exists a function $\mathbf{W} \in C(K)$ which, up to a subsequence, is the uniform limit of $\{\bar{\mathbf{W}}_n\}_{n\in\mathbb{N}}$: taking a countable compact exhaustion of Ω_∞ we find that $\bar{\mathbf{W}}_n \to \mathbf{W}$ uniformly in every compact set. By Lemma 2.5.6, we also find that $\mathbf{W}_n \to \mathbf{W}$ and, since the uniform limit of harmonic function is harmonic, we conclude that $\mathbf{W}$ is harmonic. Let $X, Y \in \Omega_\infty$ be any pair of points. By definition, there exists $n_0 \in \mathbb{N}$ such that $X, Y \in \tau_n B^+$ for every $n \geq n_0$, and so

$$|\bar{\mathbf{W}}_n(X) - \bar{\mathbf{W}}_n(Y)| \leq \sqrt{k}|X - Y|^\alpha \quad \text{for every } n \geq n_0.$$

Passing to the limit in n the previous expression, we obtain $\mathbf{W} \in \mathcal{C}^{0,\alpha}(\Omega_\infty)$. Let now $C > 0$ be fixed, and let us choose $\{P_n\}_{n\in\mathbb{N}}$ be such that $|X_n' - P_n| < Cr_n$. It follows that, up to a subsequence,

$$\frac{X_n' - P_n}{r_n} \to X' \quad \text{and} \quad \frac{X_n'' - P_n}{r_n} \to X'',$$

where $X', X'' \in \overline{B_{C+1} \cap \Omega_\infty}$. Therefore, by equicontinuity and uniform convergence,

$$\left| \bar{W}_{1,n}\left(\frac{X_n' - P_n}{r_n}\right) - \bar{W}_{1,n}\left(\frac{X_n'' - P_n}{r_n}\right) \right| = 1 \implies |W_1(X') - W_1(X'')| = 1$$

and the lemma follows. □

In Lemma 2.5.4 we have shown that X_n', X_n'' can not accumulate too fast towards $\partial^+ B^+$. Now we can prove that they converge to $\partial^0 B^+$.

Lemma 2.5.8. *There exists $C > 0$ such that, for every n sufficiently large,*

$$\frac{\operatorname{dist}(X_n', \partial^0 B^+) + \operatorname{dist}(X_n'', \partial^0 B^+)}{r_n} \leq C.$$

Proof. We argue by contradiction. Taking into account the second part of Lemma 2.5.4, this forces

$$\frac{\operatorname{dist}(X_n', \partial B^+) + \operatorname{dist}(X_n'', \partial B^+)}{r_n} \to \infty.$$

Choosing $P_n = X_n'$ in the definition of $\mathbf{w}_n$, $\bar{\mathbf{w}}_n$, we can apply Lemma 2.5.7. First of all, we notice that $\tau_n B^+ \to \Omega_\infty = \mathbb{R}^{N+1}$. But then $\mathbf{W}$ as in the aforementioned lemma is harmonic, globally Hölder continuous on $\mathbb{R}^{N+1}$ and non constant, in contradiction with Liouville theorem. □

We are in a position to choose P_n in the definition of $\mathbf{w}_n$, $\bar{\mathbf{w}}_n$: from now one let us define

$$P_n := (x_n', 0),$$

where as usual $X_n' = (x_n', y_n')$. With this choice, it is immediate to see that $\tau_n B^+ \to \Omega_\infty = \mathbb{R}^{N+1}_+$, and that all the above results, and in particular Lemma 2.5.7, apply. This last fact follows from Lemma 2.5.8, since

$$Cr_n \geq \operatorname{dist}(X_n', \partial^0 B^+) = |X_n' - P_n|.$$

Our next aim is to prove that $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$, $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ are uniformly bounded. This will be done by contradiction, in a series of lemmas.

Lemma 2.5.9. *Under the previous blow up configuration, if $\bar{w}_{i,n}(0) \to \infty$ for some i, then*

$$M_n w_{i,n}^2(0) = M_n \bar{w}_{i,n}^2(0) \le C$$

for a constant C independent of n. In particular, $M_n \to 0$.

Proof. Let $r > 0$ be fixed, and let B_{2r}^+ be the half ball of radius $2r$: by Lemma 2.5.8, for n sufficiently large we have that $B_{2r}^+ \subset \tau_n B^+$. Since the sequence $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ is made of continuous functions which share the same $\mathcal{C}^{0,\alpha}$-seminorm, we have that $\inf_{B_{2r}^+} |\bar{w}_{i,n}| \to \infty$. Furthermore, by Lemma 2.5.6 $\inf_{B_{2r}^+} |w_{i,n}| \to \infty$ as well.

Proceeding by contradiction, we assume that

$$I_n := \inf_{\partial^0 B_{2r}^+} M_n w_{i,n}^2 \to \infty.$$

We first show that for $j \neq i$, both the sequence $\{w_{j,n}\}_{n\in\mathbb{N}}$ and $\{\bar{w}_{j,n}\}_{n\in\mathbb{N}}$ are bounded in B_{2r}^+. We recall that $|w_{j,n}|$ is a subsolution of problem (2.5.2). More precisely, by Lemma 2.4.5, we have that

$$\int\limits_{B_{2r}^+} \nabla |w_{j,n}| \cdot \nabla \varphi \, \mathrm{d}x\mathrm{d}y - \int\limits_{\partial^0 B_{2r}^+} \left(\|f_{j,n}\|_{L^\infty(B_{2r})} - I_n |w_{j,n}| \right) \varphi \, \mathrm{d}x \le 0, \tag{2.5.3}$$

for every $\varphi \in H_0^1(B_{2r})$, $\varphi \ge 0$. Letting $\eta \in \mathcal{C}_0^\infty(B_{2r})$, we can choose $\varphi = \eta^2 |w_{j,n}|$ in the above equation, obtaining

$$\int\limits_{B_{2r}^+} \left(|\nabla(\eta |w_{j,n}|)|^2 - |\nabla \eta|^2 |w_{j,n}|^2 \right) \mathrm{d}x\mathrm{d}y + I_n \int\limits_{\partial^0 B_{2r}^+} \eta^2 |w_{j,n}|^2 \, \mathrm{d}x \\ \le \|f_{j,n}\|_{L^\infty} \int\limits_{\partial^0 B_{2r}^+} \eta^2 |w_{j,n}| \, \mathrm{d}x.$$

As a consequence

$$\begin{aligned} I_n \int\limits_{\partial^0 B_{2r}^+} \eta^2 |w_{j,n}|^2 \mathrm{d}x &\le \int\limits_{B_{2r}^+} |\nabla \eta|^2 |w_{j,n}|^2 \mathrm{d}x\mathrm{d}y + \|f_{j,n}\|_{L^\infty} \int\limits_{\partial^0 B_{2r}^+} \eta^2 |w_{j,n}| \, \mathrm{d}x \\ &\le \sup_{B_{2r}^+} |w_{j,n}|^2 \int\limits_{B_{2r}^+} |\nabla \eta|^2 \mathrm{d}x\mathrm{d}y + \|f_{j,n}\|_{L^\infty} \int\limits_{\partial^0 B_{2r}^+} \eta^2 \frac{1}{2} \left(1 + |w_{j,n}|^2\right) \mathrm{d}x \\ &\le \sup_{B_{2r}^+} |w_{j,n}|^2 \left(\int\limits_{B_{2r}^+} |\nabla \eta|^2 \mathrm{d}x\mathrm{d}y + C(r) \|f_{j,n}\|_{L^\infty} \right) + C(r) \|f_{j,n}\|_{L^\infty}, \end{aligned} \tag{2.5.4}$$

where, by Remark 2.5.5, $C(r)\|f_{j,n}\|_{L^\infty} \to 0$. On the other hand, using again the uniform Hölder bounds of the sequence $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ and the uniform control given by Lemma 2.5.6, we infer

$$\begin{aligned} I_n \int_{\partial^0 B_{2r}^+} \eta^2 |w_{j,n}|^2 \mathrm{d}x &\geq I_n \inf_{\partial^0 B_{2r}^+} |w_{j,n}|^2 \int_{\partial^0 B_{2r}^+} \eta^2 \mathrm{d}x \\ &\geq C I_n \left(\sup_{B_{2r}^+} |w_{j,n}|^2 - (2r)^{2\alpha} \right) \int_{\partial^0 B_{2r}^+} \eta^2 \mathrm{d}x. \end{aligned} \tag{2.5.5}$$

Combining (2.5.4) with (2.5.5) we deduce the uniform boundedness of $\sup_{\partial^+ B_{2r}^+} |w_{j,n}|$ for $j \neq i$. Equation (2.5.3) fits into (the variational counterpart of) Lemma 2.3.5, which implies

$$|w_{j,n}| \leq \frac{C}{I_n} \sup_{\partial^+ B_{2r}^+} |w_{j,n}| \qquad \text{on } \partial^0 B_r^+ \tag{2.5.6}$$

for a constant C independent of n. From the uniform bound it follows that $w_{j,n} \to 0$ uniformly in $\partial^0 B_r^+$ for every $r > 0$, and the same is true for $\bar{w}_{j,n}$, $j \neq i$: in particular, since $|\bar{w}_{1,n}(\tau_n X_n') - \bar{w}_{1,n}(\tau_n X_n'')| = 1$, we deduce that, necessarily, $i = 1$.

Now, $w_{1,n}$ satisfies

$$\int_{B_r^+} \nabla w_{1,n} \cdot \nabla \varphi \, \mathrm{d}x\mathrm{d}y = \int_{\partial^0 B_r^+} \left(f_{1,n} - M_n w_{1,n} \sum_{j\neq 1} w_{j,n}^2 \right) \varphi \, \mathrm{d}x$$

for every $\varphi \in H_0^1(B_r)$. From the previous estimates and the definition of I_n we find

$$\begin{aligned} \left| f_{1,n} - M_n w_{1,n} \sum_{j\neq 1} w_{j,n}^2 \right| &\leq \|f_{1,n}\|_{L^\infty} + M_n(|w_{1,n}|^2 + 1) \sum_{j\neq 1} w_{j,n}^2 \\ &\leq \|f_{1,n}\|_{L^\infty} + C \frac{I_n + M_n(r^{2\alpha} + 1)}{I_n^2} \to 0 \end{aligned}$$

on $\partial^0 B_r^+$, and this holds for every r. As a consequence, we can define $\{\mathbf{W}_n\}_{n\in\mathbb{N}}$ as in Lemma 2.5.7, obtaining that $W_{1,n}$ converges to W_1, which is a nonconstant, globally Hölder continuous function on $\mathbb{R}^{N+1}_+$, which satisfies

$$\begin{cases} -\Delta W_1 = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu W_1 = 0 & \text{on } \mathbb{R}^N. \end{cases}$$

But then the even extension of W_1 through $\{y = 0\}$ contradicts the Liouville theorem.

□

Lemma 2.5.10. *Under the previous blow up setting, if there exists i such that $\bar{w}_{i,n}(0) \to \infty$, then*

$$M_n |\bar{w}_{i,n}(0)| \sum_{j\neq i} \bar{w}_{j,n}^2(0) \leq C$$

for a constant C independent of n.

Proof. Let $r > 1$ be any fixed radius. Multiplying equation (2.5.2) by $w_{i,n}$ and integrating on B_r^+ we obtain the identity

$$\int\limits_{B_r^+} |\nabla w_{i,n}|^2 \,\mathrm{d}x\mathrm{d}y + \int\limits_{\partial^0 B_r^+} \left(-f_{i,n}w_{i,n} + M_n w_{i,n}^2 \sum_{j\neq i} w_{j,n}^2\right) \mathrm{d}x = \int\limits_{\partial^+ B_r^+} w_{i,n}\partial_\nu w_{i,n}\,\mathrm{d}\sigma.$$

Defining

$$E_i(r) := \frac{1}{r^{N-1}}\left(\int\limits_{B_r^+} |\nabla w_{i,n}|^2 \,\mathrm{d}x\mathrm{d}y + \int\limits_{\partial^0 B_r^+} \left(-f_{i,n}w_{i,n} + M_n w_{i,n}^2 \sum_{j\neq i} w_{j,n}^2\right) \mathrm{d}x\right)$$

and

$$H_i(r) := \frac{1}{r^N}\int\limits_{\partial^+ B_r^+} w_{i,n}^2 \mathrm{d}\sigma,$$

a straightforward computation shows that $H_i \in AC(r/2, r)$ and

$$H_i'(r) = \frac{2}{r}E_i(r).$$

In particular, integrating from $r/2$ to r, we obtain the following identity

$$H_i(r) - H_i\left(\frac{r}{2}\right) = \int\limits_{r/2}^{r} \frac{2}{s}E_i(s)\,\mathrm{d}s. \tag{2.5.7}$$

If r is suitably chosen, and n is large, after a scaling in the definition of H_i, we have that the left hand side of (2.5.7) writes as

$$\begin{aligned}
H_i(r) - H_i\left(\frac{r}{2}\right) &= \int\limits_{\partial^+ B^+} \left[w_{i,n}^2(rx) - w_{i,n}^2\left(\frac{r}{2}x\right)\right] \mathrm{d}\sigma \\
&= \int\limits_{\partial^+ B^+} \left[w_{i,n}(rx) - w_{i,n}\left(\frac{r}{2}x\right)\right]\left[w_{i,n}(rx) + w_{i,n}\left(\frac{r}{2}x\right)\right] \mathrm{d}\sigma \\
&\leq C(r)(|w_{i,n}(0)| + 1),
\end{aligned}$$

where we used the first part of Lemma 2.5.6 to estimate the difference in the integral above, and the second part of the same lemma for the sum. In a similar way, we obtain a lower bound of the right hand side of equation (2.5.7)

$$\begin{aligned}
\frac{1}{r}\int\limits_{r/2}^{r} \frac{2}{s}E_i(s)\mathrm{d}s &\geq \min_{s\in[r/2,r]} \frac{1}{s}E_i(s) \\
&\geq M_n \min_{s\in[r/2,r]} \frac{1}{s^N}\int\limits_{\partial^0 B_s^+} \sum_{j\neq i} w_{i,n}^2 w_{j,n}^2 \,\mathrm{d}x - \max_{s\in[r/2,r]} \frac{1}{s^N}\int\limits_{\partial^0 B_s^+} |f_{i,n}w_{i,n}|\,\mathrm{d}x \\
&\geq M_n C(r)\left(\sum_{j\neq i} w_{i,n}^2(0)w_{j,n}^2(0) - 1\right) - C(r)\|f_{j,n}\|_{L^\infty}(|w_{i,n}(0)| + 1),
\end{aligned}$$

where $C(r)\|f_{j,n}\|_{L^\infty} \to 0$ as $n \to \infty$. Putting the two estimates together and recalling that M_n is bounded, we find

$$M_n \sum_{j\neq i} w_{i,n}^2(0) w_{j,n}^2(0) \leq C(r)(|w_{i,n}(0)| + 1).$$

The conclusion follows dividing by $|w_{i,n}(0)|$, using the uniform control of the sequence $\{w_{i,n}\}_{n\in\mathbb{N}}$ and $\{\bar{w}_{i,n}\}_{n\in\mathbb{N}}$ and the assumption that $|\bar{w}_{i,n}(0)| \to \infty$. □

Lemma 2.5.11. *If $\{\bar{\mathbf{w}}_n(0)\}_{n\in\mathbb{N}}$ is unbounded, then also $\{\bar{w}_{1,n}(0)\}_{n\in\mathbb{N}}$ is.*

Proof. By Lemma 2.5.9, $M_n \to 0$. Let i be such that $\{w_{i,n}(0)\}_{n\in\mathbb{N}}$ is bounded. Reasoning as in the proof of Lemma 2.5.7, we have that both $w_{i,n}$ and $\bar{w}_{i,n}$ converge to some w_i, uniformly on compact sets. Furthermore, w_i is harmonic, globally Hölder continuous, and non constant in the possible case $i = 1$. We claim that there exists a constant $\lambda \geq 0$ such that

$$\partial_\nu w_{i,n} = f_{i,n} - M_n w_{i,n} \sum_{j\neq i} w_{j,n}^2 \to -\lambda w_i$$

uniformly on compact sets. This, combined with Proposition 2.3.7, proves the lemma.

To prove the claim:

- let $j \neq i$ be an index such that $\bar{w}_{j,n}(0)$ is unbounded: from Lemma 2.5.10 we see that $M_n \bar{w}_{j,n}^2(0)$ is bounded. Moreover, by uniform Hölder bounds,

$$M_n \left|\bar{w}_{j,n}^2(x,0) - \bar{w}_{j,n}^2(0,0)\right| \leq \underbrace{M_n \bar{w}_{j,n}^2(0,0)}_{\leq C \text{ (Lemma 2.5.9)}} \left|\frac{\bar{w}_{j,n}^2(x,0)}{\bar{w}_{j,n}^2(0,0)} - 1\right| \to 0,$$

  since $\bar{w}_{j,n}(x)/\bar{w}_{j,n}(0) \to 1$ uniformly, implying $M_n \bar{w}_{j,n}^2(x,0) \to \lambda_j \geq 0$;

- let now $j \neq i$ be an index such that $\bar{w}_{j,n}(0)$ is bounded. Then, again by uniform convergence,

$$M_n \bar{w}_{i,n} \bar{w}_{j,n}^2 \to 0$$

  uniformly in every compact set.

It follows that

$$f_{i,n} - M_n \bar{w}_{i,n} \sum_{j\neq i} \bar{w}_{j,n}^2 \to -\lambda w_i$$

uniformly in every compact set, and the same limit holds for $\{w_{i,n}\}_{n\in\mathbb{N}}$ by uniform convergence. □

Lemma 2.5.12. *The sequence $\{\bar{\mathbf{w}}_n(0)\}_{n\in\mathbb{N}}$ is bounded.*

Proof. By contradiction, let $\{\mathbf{w}_n(0)\}_{n\in\mathbb{N}}$ be unbounded. Then, by the above lemmas, $M_n \to 0$, $\{w_{1,n}(0)\}_{n\in\mathbb{N}}$ is unbounded, while $M_n w_{1,n}^2(0)$ is bounded. This implies that $M_n|w_{1,n}| \to 0$ uniformly on compact sets.

Now, if $j \neq 1$ is such that $\{w_{j,n}(0)\}_{n\in\mathbb{N}}$ is bounded, then $M_n w_{1,n} w_{j,n}^2 \to 0$ uniformly in every compact set.

On the other hand, if $j \neq 1$ is such that $\{w_{j,n}(0)\}_{n\in\mathbb{N}}$ is unbounded, then Lemma 2.5.10 provides

$$C \geq M_n |w_{1,n}(0)| \int_{\partial^0 B_r^+} w_{j,n}^2 \,\mathrm{d}x = M_n |w_{1,n}(0)| w_{j,n}^2(0) \int_{\partial^0 B_r^+} \frac{w_{j,n}^2}{w_{j,n}^2(0)} \,\mathrm{d}x,$$

so that $M_n|w_{1,n}(0)|w_{j,n}^2(0)$ is uniformly bounded. Of course, since if $\{w_{j,n}(0)\}_{n\in\mathbb{N}}$ is unbounded then also $\{w_{j,n}(x)\}_{n\in\mathbb{N}}$ is, for any fixed x, the same argument shows that $M_n|w_{1,n}(x)|w_{j,n}^2(x)$ is bounded. Now,

$$\begin{aligned} &M_n \left| |w_{1,n}(x)| w_{j,n}^2(x) - |w_{1,n}(0)| w_{j,n}^2(0) \right| \\ &\quad \leq M_n |w_{1,n}(x)| w_{j,n}^2(x) \left| 1 - \frac{w_{j,n}^2(0)}{w_{j,n}^2(x)} \right| + M_n |w_{1,n}(0)| w_{j,n}^2(0) \left| \frac{w_{1,n}(x)}{w_{1,n}(0)} - 1 \right| \to 0. \end{aligned}$$

This shows the existence of a constant $\lambda_j \in \mathbb{R}$ such that $M_n w_{1,n} w_{j,n}^2 \to \lambda_j$ uniformly in every compact set.

Summing up, at least up to a subsequence,

$$f_{1,n} - M_n w_{1,n} \sum_{h\neq 1} w_{h,n}^2 \to \lambda \in \mathbb{R},$$

uniformly on every compact subset of $\mathbb{R}^N$. Thus, as usual, $W_{1,n} = w_{1,n} - w_{1,n}(0)$ converges to W_1, a nonconstant, globally Hölder continuous solution to

$$\begin{cases} -\Delta W_1 = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu W_1 = \lambda & \text{on } \mathbb{R}^N. \end{cases}$$

Appealing to Proposition 2.3.8, we obtain a contradiction. □

The uniform bound on $\{\bar{\mathbf{w}}_n(0)\}_{n\in\mathbb{N}}$ allows to prove the following convergence result.

Lemma 2.5.13. *Under the previous blow up setting, there exists* $\mathbf{w} \in (H^1_{\mathrm{loc}} \cap \mathcal{C}^{0,\alpha})\left(\overline{\mathbb{R}^{N+1}_+}\right)$ *such that, up to a subsequence,*

$$\mathbf{w}_n \to \mathbf{w} \text{ in } (H^1 \cap C)(K)$$

for every compact $K \subset \overline{\mathbb{R}^{N+1}_+}$.

Proof. Reasoning as in the proof of Lemma 2.5.7 we can easily obtain that, up to subsequences, both $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ converge uniformly on compact sets to the same limit $\mathbf{w} \in \mathcal{C}^{0,\alpha}\left(\overline{\mathbb{R}^{N+1}_+}\right)$, hence we are left to show the H^1_{loc} convergence of the latter sequence.

Let K be compact, r be such that $K \subset \overline{B_r^+}$, and let us consider $\eta \in \mathcal{C}_0^\infty(B_r^+)$ any smooth cutoff function, such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on K. Testing the equation for $w_{i,n}$ by $w_{i,n}\eta^2$, we obtain

$$\begin{aligned}
0 \le \int_K |\nabla w_{i,n}|^2 \mathrm{d}x\mathrm{d}y + M_n \int_{\partial^0 K} w_{i,n}^2 \sum_{j\neq i} w_{j,n}^2 \mathrm{d}x \\
\le \int_{B_r^+} |\nabla w_{i,n}|^2 \eta^2 \mathrm{d}x\mathrm{d}y + M_n \int_{\partial^0 B_r^+} w_{i,n}^2 \sum_{j\neq i} w_{j,n}^2 \eta^2 \mathrm{d}x \\
\le \frac{1}{2} \int_{B_r^+} w_{i,n}^2 |\Delta \eta^2| \mathrm{d}x\mathrm{d}y + \frac{1}{2} \int_{\partial^0 B_r^+} \left(w_{i,n}^2 |\partial_\nu \eta^2| + f_{i,n} w_{i,n} \eta^2\right) \mathrm{d}x.
\end{aligned}$$

Since the right hand side is bounded uniformly in n (recall Lemmas 2.5.12 and 2.5.6), we deduce that, up to subsequences, $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ weakly converges in $H^1(K)$. Since this holds for every K, we deduce that $\mathbf{w}_n \rightharpoonup \mathbf{w}$ in $H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$. To prove the strong convergence, let us now test the equation by $\eta^2(w_{i,n} - w_i)$. We obtain

$$\int_{B_r^+} \nabla w_{i,n} \cdot \nabla \left[\eta^2 (w_{i,n} - w_i)\right] \mathrm{d}x\mathrm{d}y = \int_{\partial^0 B_r^+} \eta^2 (w_{i,n} - w_i) \partial_\nu w_{i,n} \, \mathrm{d}x. \tag{2.5.8}$$

We can estimate the right hand side as

$$\begin{aligned}
\int_{\partial^0 B_r^+} \eta^2 (w_{i,n} - w_i) \partial_\nu w_{i,n} \, \mathrm{d}x \\
\le \sup_{x\in B_r^+} |w_{i,n} - w_i| \int_{\partial^0 B_r^+} \eta^2 \left[M_n |w_{i,n}| \sum_{i,j<i} w_{j,n}^2 + |f_{i,n}|\right] \mathrm{d}x \\
\le C(r) \sup_{x\in B_r^+} |w_{i,n} - w_i|,
\end{aligned}$$

where the last step holds since the inequality for $|w_{i,n}|$ (Lemma 2.4.5) tested by η^2 yields

$$\begin{aligned}
\int_{\partial^0 B_r^+} \eta^2 M_n |w_{i,n}| \sum_{i,j<i} w_{j,n}^2 \, \mathrm{d}x \\
\le \int_{\partial^0 B_r^+} \left(|f_{i,n}|\eta^2 + |w_{i,n} \partial_\nu \eta^2|\right) \mathrm{d}x + \int_{B_r^+} |w_{i,n} \Delta \eta^2| \, \mathrm{d}x\mathrm{d}y \le C(r).
\end{aligned}$$

Resuming, equation (2.5.8) implies

$$\int_{B_r^+} |\nabla(\eta w_{i,n})|^2 \, \mathrm{d}x\mathrm{d}y \leq \int_{B_r^+} \left(\eta^2 \nabla w_{i,n} \cdot \nabla w_i + 2\eta w_i \nabla w_{i,n} \cdot \nabla \eta + |\nabla \eta|^2 w_i^2\right) \mathrm{d}x\mathrm{d}y$$
$$+ C(r) \sup_{x \in B_r^+} |w_{i,n} - w_i|.$$

Using both the weak H^1 convergence and the uniform one, we have that

$$\limsup_{n\to\infty} \int_{B_r^+} |\nabla(\eta w_{i,n})|^2 \, \mathrm{d}x\mathrm{d}y \leq \int_{B_r^+} |\nabla(\eta w_i)|^2 \, \mathrm{d}x\mathrm{d}y$$

and we conclude the strong convergence in $H^1(B_r^+)$ of $\{\eta \mathbf{w}_n\}_{n\in\mathbb{N}}$ to $\eta\mathbf{w}$, that is, since η was arbitrary, the strong convergence of $\mathbf{w}_n$ to $\mathbf{w}$ in $H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$. $\square$

End of the proof of Theorem 2.5.2. Summing up, we have that $\mathbf{w}_n \to \mathbf{w}$ in $(H^1 \cap C)_{\mathrm{loc}}$, and that the limiting blow up profile $\mathbf{w}$ is a nonconstant vector of harmonic, globally Hölder continuous functions. To reach the final contradiction, we distinguish, up to subsequences, between the following three cases.

Case 1: $M_n \to 0$. In this case also the equation on the boundary passes to the limit, and the nonconstant component w_1 satisfies $\partial_\nu w_1 \equiv 0$ on $\mathbb{R}^N$, so that its even extension through $\{y = 0\}$ contradicts Liouville theorem.

Case 2: $M_n \to C > 0$. Even in this case the equation on the boundary passes to the limit, and $\mathbf{w}$ solves

$$\begin{cases} -\Delta w_i = 0 & x \in \mathbb{R}^{N+1}_+ \\ \partial_\nu w_i = -C w_i \sum_{j\neq i} w_j^2 & \text{on } \mathbb{R}^N \times \{0\} \end{cases}$$

The contradiction is now reached using Proposition 3.2.9, since $\mathbf{w} \in \mathcal{G}_c \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^{N+1}_+)$ and $\alpha < \nu^{\mathrm{ACF}}$.

Case 3: $M_n \to \infty$. By Proposition 2.4.1, we infer $\mathbf{w} \in \mathcal{G}_s \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^{N+1}_+)$ with $\alpha < \nu^{\mathrm{ACF}}$, in contradiction with Proposition 2.3.2.

As of now, the contradictions we have obtained imply that $\{\mathbf{v}_\beta\}_{\beta>0}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}\left(\overline{B^+_{1/2}}\right)$, for every $\alpha < \nu^{\mathrm{ACF}}$. But then the relative compactness in $\mathcal{C}^{0,\alpha}\left(\overline{B^+_{1/2}}\right)$ follows by Ascoli-Arzelà Theorem, while the one in $H^1(B^+_{1/2})$ can be shown by reasoning as in the proof of Lemma 2.5.13. $\square$

Remark 2.5.14. It is worthwhile noticing that, in proving Theorem 2.5.2, the only part in which we used the assumption $\alpha < \nu^{\mathrm{ACF}}$ is the concluding argument, while in the rest of the proof it is sufficient to suppose $\alpha < 1$.

As we mentioned, even though we are not able to show that $\nu^{\mathrm{ACF}} = 1/2$, nonetheless we will prove that the uniform Hölder bound hold for any $\alpha < 1/2$. In view of the previous remark, this can be done by means of some sharper Liouville type results, which will be obtained in the next section. To conclude the present discussion, we observe that a result analogous to Theorem 2.5.2 holds, when entire profiles of segregation are considered, instead of solutions to $(GP)_\beta$.

Proposition 2.5.15. *Let $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ be a subset of $\mathcal{G}_s \cap \mathcal{C}^{0,\alpha}\left(\overline{B_1^+}\right)$, for some $0 < \alpha \leq \nu^{\mathrm{ACF}}$, such that*

$$\|\mathbf{v}_n\|_{L^\infty(B_1^+)} \leq \bar{m},$$

with $\bar{m}$ independent of n. Then for every $\alpha' \in (0, \alpha)$ there exists a constant $C = C(\bar{m}, \alpha')$, not depending on n, such that

$$\|\mathbf{v}_n\|_{\mathcal{C}^{0,\alpha'}\left(\overline{B_{1/2}^+}\right)} \leq C.$$

Furthermore, $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ is relatively compact in $H^1(B_{1/2}^+) \cap \mathcal{C}^{0,\alpha'}\left(\overline{B_{1/2}^+}\right)$ for every $\alpha' < \alpha$.

Proof. The proof follows the line of the one of Theorem 2.5.2, being in fact easier, since we do not have to handle any competition term. We proceed by contradiction, assuming that, up to subsequences,

$$L_n := \max_{i=1,\dots,k} \sup_{X',X'' \in \overline{B^+}} \frac{|\eta(X')v_{i,n}(X') - \eta(X'')v_{i,n}(X'')|}{|X' - X''|^{\alpha'}} \to \infty,$$

where again η is a smooth cutoff function of the ball $B_{1/2}$ and $\alpha' < \alpha$. If L_n is achieved by (X_n', X_n''), we introduce the sequences

$$w_{i,n}(X) := \eta(X_n)\frac{v_{i,n}(P_n + r_n X)}{L_n r_n^{\alpha'}} \quad \text{and} \quad \bar{w}_{i,n}(X) := \frac{(\eta v_{i,n})(P_n + r_n X)}{L_n r_n^{\alpha'}},$$

for $X \in \tau_n B^+$, where, as usual, on one hand $\bar{\mathbf{w}}_n$ has Hölder seminorm (and oscillation) equal to 1, while on the other hand $\mathbf{w}_n$ belongs to $\mathcal{G}_s$. All the preliminary properties of (X_n', X_n''), up to Lemma 2.5.8, are still valid, since they depend only on the harmonicity of $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$. It follows that the choice $P_n = (x_n', 0)$ for every $n \in \mathbb{N}$ guarantees the convergence of the rescaled domains $\tau_n B^+$ to $\mathbb{R}^{N+1}_+$, while on any compact set the sequences $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ and $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ shadow each other. Up to relabelling and subsequences, we are left with two alternatives:

1. either for any compact set $K \in \mathbb{R}^N$ we have $\mathbf{w}_{1,n}(x, 0) \neq 0$ for every $n \geq n_0(K)$ and $x \in K$;

2. or there exists a bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\mathbf{w}_n(x_n, 0) = 0$ for every n.

In the first case, if we define $\mathbf{W}_n = \mathbf{w}_n - \mathbf{w}_n(0)$ and $\bar{\mathbf{W}}_n = \bar{\mathbf{w}}_n - \bar{\mathbf{w}}_n(0)$, we obtain that the sequence $\{\bar{\mathbf{W}}_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}^{0,\alpha'}$, and hence $\{\mathbf{W}_n\}_{n\in\mathbb{N}}$ converges uniformly on compact sets to a non constant, globally Hölder continuous function $\mathbf{W}$, with $\partial_\nu W_1(x,0) \equiv 0$ and $W_i(x,0) \equiv 0$ for $i > 1$, on $\mathbb{R}^N$. Extending properly the vector $\mathbf{W}$ to the whole $\mathbb{R}^{N+1}$, we find a contradiction with the Liouville theorem.

Coming to the second alternative, this time $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ itself converges, uniformly on compact sets, to a non constant, globally Hölder continuous function $\mathbf{w}$. We want to show that the convergence is also strong in H^1_{loc}: this will imply that also $\mathbf{w} \in \mathcal{G}_s$ (recall also the end of the proof of Proposition 2.4.1), in contradiction with Proposition 2.3.2. To prove the strong convergence let us consider, for any i, the even extension of $|w_{i,n}|$ through $\{y = 0\}$, which we denote again with $|w_{i,n}|$. We have that there exists a non negative Radon measure $\mu_{i,n}$ such that

$$-\Delta |w_{i,n}| = -\mu_{i,n} \qquad \text{in } \mathcal{D}'(\tau_n B):$$

indeed, on one hand, for $X \in \{w_{i,n} \neq 0\}$, there exists a radius $r > 0$ such that the even extension of $w_{i,n}$ through $\{y = 0\}$ is harmonic in $B_r(X)$, providing

$$|w_{i,n}|(X) \leq \frac{1}{|B_r|} \int\limits_{B_r(X)} |w_{i,n}|(Y)\mathrm{d}Y;$$

on the other hand $X \in \{w_{i,n} = 0\}$ immediately implies the same inequality, and the consequent subharmonicity of $|w_{i,n}|$. At this point, we can reason as in [47], showing that the L^∞ uniform bounds on compact sets of $|w_{i,n}|$ implies that the measures $\mu_{i,n}$ are bounded on compact sets [47, Lemma 3.7]; and that this, together with the uniform convergence of $\{|\mathbf{w}_n|\}_{n\in\mathbb{N}}$, implies its strong H^1_{loc}-convergence [47, Lemma 3.11].

As a consequence of the previous contradiction argument, we deduce both the uniform bounds and the pre-compactness of $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}^{0,\alpha'}(B_{1/2})$. Once we have (the uniform L^∞ bounds and) the uniform convergence of $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$, the strong H^1 pre-compactness can be obtained exactly as in the last part of the proof, replacing $|w_{i,n}|$ with $|v_{i,n}|$. □

2.6 Liouville type theorems, reprise: the optimal growth

In Section 2.5 we proved that non existence results of Liouville type imply uniform bounds in corresponding Hölder norms. This section is devoted to the study of the optimal Liouville exponents, which will allow to enhance the regularity estimates. Our aim is to prove the following result.

Theorem 2.6.1. *Let $\nu \in \left(0, \frac{1}{2}\right)$. If*

1. *either $\mathbf{v} \in \mathcal{G}_s \cap \mathcal{C}^{0,\nu}\left(\overline{\mathbb{R}^{N+1}_+}\right)$,*
2. *or $\mathbf{v} \in \mathcal{G}_c$ and $|\mathbf{v}(X)| \leq C(1 + |X|^\nu)$ for every X,*

then $\mathbf{v}$ is constant.

The rest of the section is devoted to the proof of the above theorem. As of now, we already know by Propositions 2.3.1 and 3.2.10 that such theorem holds true whenever ν is smaller than ν^{ACF}. In order to refine such result, we will prove that it holds for ν smaller than ν^{Liou}, according to the following definition.

Definition 2.6.2. For $\nu > 0$ and for every dimension N, we define the class

$$\mathcal{H}(\nu, N) := \left\{ \mathbf{v} \in \mathcal{G}_s : \begin{array}{l} \mathbf{v} \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right), \text{ for some } \alpha > 0 \\ \mathbf{v} \text{ is non trivial and } \nu\text{-homogeneous} \end{array} \right\},$$

and the critical value

$$\nu^{\mathrm{Liou}}(N) = \inf\{\nu > 0 : \mathcal{H}(\nu, N) \text{ is non empty}\}$$

Remark 2.6.3. Since $(y, 0, \dots, 0) \in \mathcal{H}(1, N)$, for every N, we have that $\nu^{\mathrm{Liou}}(N) \leq 1$.

Remark 2.6.4. By Corollary 2.2.5 we have that, if $\mathbf{v}$ is non constant and satisfies assumption (1) in Theorem 2.6.1 for some ν, then $\mathbf{v} \in \mathcal{H}(\nu, N)$.

To prove Theorem 2.6.1, we start by showing that, given any non constant $\mathbf{v}$ satisfying assumption (2) in Theorem 2.6.1 for some ν, we can construct a function $\bar{\mathbf{v}} \in \mathcal{H}(\nu', N)$, for a suitable $\nu' \leq \nu$. This, together with the previous remark, will imply the equivalence between Theorem 2.6.1 and the inequality

$$\nu^{\mathrm{Liou}}(N) \geq \frac{1}{2}.$$

To construct such $\bar{\mathbf{v}}$, we will use the blow down method. For any (non trivial) $\mathbf{v} \in \mathcal{G}_c$ we denote with $N_{\mathbf{v}}(x_0, r)$, $H_{\mathbf{v}}(x_0, r)$ the related quantities involved in the Almgren frequency formula, defined in Section 2.2. For any $r > 0$, let us define

$$\mathbf{v}_r(X) := \frac{1}{\sqrt{H_{\mathbf{v}}(0, r)}} \mathbf{v}(rX).$$

Since H is a strictly positive increasing function in $\mathbb{R}_+$ (recall Theorem 2.2.11), $\mathbf{v}_r$ is well defined. We have the following.

Lemma 2.6.5 (Blow down method)**.** *Let* $\mathbf{v}$ *be a non constant function, satisfying assumption (2) in Theorem 2.6.1 for some* ν*, and let*

$$0 < \nu' = \lim_{r\to\infty} N_{\mathbf{v}}(r) \leq \nu.$$

Then there exists $\bar{\mathbf{v}} \in \mathcal{H}(\nu', N)$ *such that, for a suitable sequence* $r_n \to \infty$*,*

$$\mathbf{v}_{r_n} \to \bar{\mathbf{v}} \text{ in } (H^1 \cap C)(K)$$

for every compact $K \subset \overline{\mathbb{R}^{N+1}_+}$*.*

Proof. First of all, by construction, we have that

$$\|\mathbf{v}_r\|_{L^2(\partial^+ B^+)} = 1 \qquad \text{so that} \qquad \|v_{i,r}\|_{L^2(\partial^+ B^+)} \leq 1 \quad \text{for } i = 1, \dots, k.$$

Each $\mathbf{v}_r$ is solution to the system

$$\begin{cases} -\Delta v_{i,r} = 0 & \text{in } B^+ \\ \partial_\nu v_{i,r} + rH(r) v_{i,r} \sum_{j\neq i} v_{j,r}^2 = 0 & \text{on } \partial^0 B^+, \end{cases}$$

where $rH(r) \to \infty$ monotonically as $r \to \infty$. Reasoning as in Lemma 2.4.5, we have that the even reflection of $|v_{i,r}|$ through $\{y = 0\}$ satisfies

$$\begin{cases} -\Delta |v_{i,r}| \leq 0 & \text{in } B \\ \|v_{i,r}\|_{L^2(\partial B)} \leq \sqrt{2}. \end{cases}$$

By the Poisson representation formula, it follows that there exists a constant C, not depending on r, such that

$$\|\mathbf{v}_r\|_{L^\infty(B^+_{3/4})} \leq C$$

for every r. Thus we are in a position to apply Theorem 2.5.2 in order to see that the family $\{\mathbf{v}_r\}_{r>1}$ is relatively compact in $(H^1 \cap \mathcal{C}^{0,\alpha})(B^+_{1/2})$, for all $\alpha < \nu^{\mathrm{ACF}}$. Furthermore, Proposition 2.4.1 implies that any limiting point of the family is an element of $\mathcal{G}_s$ on $B^+_{1/2}$. In order to find a non trivial limiting point, we claim that there exists a sequence of radii $\{r_n\}_{n\in\mathbb{N}}$, $r_n \to \infty$, and a positive constant C such that

$$H(r_n) \leq CH(r_n/2) \qquad \forall r_n > 0.$$

Indeed, we can argue by contradiction, assuming that there exists $r_0 > 0$ such that

$$H(r) \geq 3^{2\nu} H(r/2) \qquad \forall r \geq r_0.$$

Using the diadic sequence of radii $\{2^j r_0\}_{j\in\mathbb{N}}$ we see that

$$3^{2\nu j} H(r_0) \leq H(2^j r_0) \leq C(2^j)^{2\nu},$$

by assumption. The above inequality provides a contradiction for j sufficiently large, yielding the validity of the claim. If we denote with $\bar{\mathbf{v}}$ a limiting point of the sequence $\{\mathbf{v}_{r_n}\}_{n\in\mathbb{N}}$, it follows that

$$\|\mathbf{v}_{r_n}\|_{L^2(\partial^+ B^+_{1/2})} = \sqrt{\frac{H(r_n/2)}{H(r_n)}} \geq \sqrt{\frac{1}{C}},$$

implying, in particular, that $\bar{\mathbf{v}}$ is a nontrivial element of $\mathcal{G}_s$. Moreover, its Almgren quotient $N_{\bar{\mathbf{v}}}(0,r)$ is constant for all $r \in (0,1/2)$, indeed

$$N_{\bar{\mathbf{v}}}(0,r) = \lim_{r_n\to\infty} N_{\mathbf{v}_{r_n}}(0,r) = \lim_{r_n\to\infty} N_{\mathbf{v}}(0,r_n r) = \lim_{r\to\infty} N_{\mathbf{v}}(0,r) = \nu',$$

where the latter limit exists by the monotonicity of N (Theorem 2.2.11); moreover, since $\mathbf{v}$ is not constant we have that $\nu' > 0$, while $\nu' \leq \nu$ by Lemma 2.2.12. Since $N(0,r)$ is constant, we conclude by Theorem 2.2.3 that $\bar{\mathbf{v}}$ is homogeneous of degree ν', and then it can be extended on the whole $\mathbb{R}^{N+1}_+$ to a $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}$ function, for every $\alpha < \nu^{\mathrm{ACF}}$. □

By the previous lemma, if we show that $\nu^{\mathrm{Liou}}(N) \geq 1/2$ then Theorem 2.6.1 will follow. The next step in this direction consists in reducing such problem to the one of estimating $\nu^{\mathrm{Liou}}(1)$.

Lemma 2.6.6 (Dimensional descent)**.** *For any dimension $N \geq 2$, it holds*

$$\nu^{\mathrm{Liou}}(N) \geq \nu^{\mathrm{Liou}}(N-1).$$

Proof. For every $\nu > 0$ such that there exists $\mathbf{v} \in \mathcal{H}(\nu, N)$, we will prove that $\nu^{\mathrm{Liou}}(N-1) \leq \nu$. Let ν, $\mathbf{v}$ as above. By homogeneity, we have that $\mathbf{v}(0,0) = 0$ and $N(0,r) = \nu$ for all $r > 0$. Since the function $\mathbf{v}$ is homogeneous, its boundary nodal set

$$\mathcal{Z} = \{x \in \mathbb{R}^N : \mathbf{v}(x,0) = 0\}$$

is a cone at $(0,0)$. We can easily rule out two degenerate situations:

1. $\mathcal{Z} = \mathbb{R}^N$, in which case all the components of $\mathbf{v}$ have trivial trace on $\mathbb{R}^N$. As a consequence, the odd extension of $\mathbf{v}$ through $\{y = 0\}$ is a nontrivial vector of harmonic functions on $\mathbb{R}^{N+1}$, forcing $\nu \geq 1 \geq \nu^{\mathrm{Liou}}(N-1)$ by Remark 2.6.3;

2. $\mathcal{Z} = \{(0,0)\}$, in which case all the components of $\mathbf{v}$ but one have trivial trace, and the last one has necessarily a vanishing normal derivative in $\{y = 0\}$. As before, extending the former functions oddly and the latter evenly through $\{y = 0\}$, we obtain again that $\nu \geq 1 \geq \nu^{\mathrm{Liou}}(N-1)$.

We are left to analyze the third and most delicate case, namely the one in which the boundary $\partial\mathcal{Z}$ is non trivial. Let $x_0 \in \partial\mathcal{Z} \setminus \{(0,0)\}$, and let us introduce the following blow up family (here $r \to 0$)

$$\mathbf{v}_r(X) = \frac{1}{\sqrt{H(x_0, r)}} \mathbf{v}\left((x_0, 0) + rX\right).$$

We want to apply Proposition 2.5.15 to (a subsequence of) $\{\mathbf{v}_r\}_r$: the only assumption non trivial to check is the uniform L^∞ bound. To prove it, we observe that the even extension of $|v_{i,r}|$ through $\{y = 0\}$ (denoted with the same writing) is subharmonic, indeed the inequality

$$|v_{i,r}|(X) \leq \frac{1}{|B_\rho|} \int\limits_{B_\rho(X)} |v_{i,r}|(Y) \mathrm{d}Y$$

holds true, if ρ is sufficiently small, both when $v_{i,r}(X) = 0$ and when $v_{i,r}(X) \neq 0$; once we know that each $|v_{i,r}|$ is non negative and subharmonic, arguing as in the first part of the proof of Lemma 2.6.5, one can show that $w_{i,r}$ is uniformly bounded in $L^\infty(B_{3/4})$. Applying Proposition 2.5.15 we obtain that, up to subsequences, $\mathbf{v}_r$ converges uniformly and strongly in H^1 to $\bar{\mathbf{v}}$, an element $\mathcal{G}_s(N)$ on $B^+_{1/2}$. Reasoning as in the end of the proof of Lemma 2.6.5, we infer that $\bar{\mathbf{v}}$ is non trivial, locally $\mathcal{C}^{0,\alpha}$, and that

$$N_{\bar{\mathbf{v}}}(0, \rho) = \lim_{r \to 0} N_{\mathbf{v}_r}(0, \rho) = \lim_{r \to} N_{\mathbf{v}}(x_0, r\rho) = \lim_{r \to 0} N_{\mathbf{v}}(x_0, r) =: \nu',$$

where

$$\alpha \leq N_{\mathbf{v}}(x_0, 0^+) = \nu' \leq N_{\mathbf{v}}(x_0, \infty) = \nu$$

by Lemmas 2.2.4, 2.2.7 and the monotonicity of N. In particular, $\bar{\mathbf{v}}$ is homogeneous of degree ν'.

To conclude the proof, we will show that $\bar{\mathbf{v}}$ is constant along the direction parallel to $(x_0, 0)$, and that its restriction on the orthogonal half plane belongs to $\mathcal{G}_s(N-1)$. Let $(x, y) \in \mathbb{R}^{N+1}_+$ and $h \in \mathbb{R}$ be fixed. By the homogeneity of $\mathbf{v}$ we have

$$\begin{aligned}
&\left|\mathbf{v}_r(x + h(x_0 + rx), (1 + hr)y) - \mathbf{v}_r(x, y)\right| \\
&\qquad\qquad = \frac{\left|\mathbf{v}((1 + hr)(x_0 + rx, ry)) - \mathbf{v}(x_0 + rx, ry)\right|}{\sqrt{H(x_0, r)}} \\
&\qquad\qquad = |(1 + hr)^\nu - 1| \frac{|\mathbf{v}(x_0 + rx, ry)|}{\sqrt{H(x_0, r)}} = |(1 + hr)^\nu - 1| \, |\mathbf{v}_r(x, y)|.
\end{aligned}$$

As $r \to 0$ (up to subsequences) we infer, by uniform convergence,

$$|\bar{\mathbf{v}}(x + hx_0, y) - \bar{\mathbf{v}}(x, y)| = 0, \qquad \text{for every } h \in \mathbb{R}.$$

Let us denote by $\hat{\mathbf{v}}$ a section of $\bar{\mathbf{v}}$ with respect to the direction $\{h(x_0, 0)\}_{h\in\mathbb{R}}$: we claim that $\hat{\mathbf{v}} \in \mathcal{H}(\nu', N-1)$. It is a direct check to verify that $\hat{\mathbf{v}}$ is nontrivial, ν'-homogeneous, and $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}$. In order to show that $\hat{\mathbf{v}} \in \mathcal{G}_s(N-1)$, we observe that the equations and the segregation conditions are trivially satisfied, therefore we only need to prove the Pohozaev identities on cylindrical domains (recall the discussion before Definition 2.2.1). To this aim, let C' denote one of such domains in $\mathbb{R}^N_+$, and C'' the corresponding domain in $\mathbb{R}^{N+1}_+$ having C' as N-dimensional section, and the further axis parallel to $(x_0, 0)$. But then the Pohozaev identity for $\hat{\mathbf{v}}$ on C' immediately follows by the one for $\bar{\mathbf{v}}$ on C'', using Fubini theorem. □

We are ready to obtain the proof of Theorem 2.6.1 as a byproduct of the following classification result, which completely characterize the elements of $\mathcal{H}(\nu, 1)$ and shows that $\nu^{\mathrm{Liou}}(1) = 1/2$.

Proposition 2.6.7. *Let $\nu > 0$. The following holds.*

1. *$\mathcal{H}(\nu, 1) = \emptyset \iff 2\nu \notin \mathbb{N}$;*
2. *if $\nu \in \mathbb{N}$ any element of $\mathcal{H}(\nu, 1)$ consists in homogeneous polynomial, and only one of its components may have non trivial trace on $\{y = 0\}$;*
3. *if $\nu = k + 1/2$, $k \in \mathbb{N}$, any element of $\mathcal{H}(\nu, 1)$ has exactly two non trivial components, say v and w, and there exists $c \neq 0$ such that*

$$v(\rho, \theta) = c\,\rho^{\frac{1}{2}+k} \cos\left(\frac{1}{2} + k\right)\theta, \qquad w(\rho, \theta) = \pm c\,\rho^{\frac{1}{2}+k} \sin\left(\frac{1}{2} + k\right)\theta$$

(here (ρ, θ) denote polar coordinates in $\mathbb{R}^2_+$ around the homogeneity pole).

Proof. Let $\nu > 0$ be such that $\mathcal{H}(\nu, 1)$ is not empty, and $\mathbf{v} \in \mathcal{H}(\nu, 1)$. Since, by assumption, $\mathbf{v}$ is homogeneous, the Almgren quotient $N(0, r)$ is equal to ν for every $r > 0$. Moreover, for topological reasons, no more than two components of $\mathbf{v}$ can have non trivial trace on $\{y = 0\}$. We will classify $\mathbf{v}$, and hence ν, according to the number of such components.

As a first case, let us suppose that two components of $\mathbf{v}$, say v and w, have non trivial trivial trace, in such a way that they solve

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^2_+ \\ v(x, 0) = 0 & \text{on } x < 0 \\ \partial_\nu v(x, 0) = 0 & \text{on } x > 0, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta w = 0 & \text{in } \mathbb{R}^2_+ \\ w(x, 0) = 0 & \text{on } x > 0 \\ \partial_\nu w(x, 0) = 0 & \text{on } x < 0. \end{cases}$$

By homogeneity, we can easily find v and w; indeed, for instance, v must be of the form $v(\rho,\theta)=\rho^\nu g(\theta)$ with ν and g solutions to

$$\begin{cases} \nu^2 g + g'' = 0 & \text{in } (0,\pi) \\ g(\pi)=0,\ g'(0)=0, \end{cases}$$

and an analogous argument holds for w. We conclude that

$$v(\rho,\theta) = c\rho^{\frac{1}{2}+k}\cos\left(\frac{1}{2}+k\right)\theta, \qquad w(\rho,\theta) = d\rho^{\frac{1}{2}+k}\sin\left(\frac{1}{2}+k\right)\theta,$$

with $c,d\neq 0$ and $k\in\mathbb{N}$, forcing $\nu = k+1/2$. All the other components of $\mathbf{v}$ must satisfy

$$\begin{cases} -\Delta v_i = 0 & \text{in } \mathbb{R}^2_+ \\ v_i = 0 & \text{on } \mathbb{R}\times\{0\}, \end{cases}$$

with homogeneity degree equal to $k+1/2$, which is impossible unless they are null. Let $\bar v$ be the function

$$\bar v(\rho,\theta) = \rho^{\frac{1}{2}+k}\cos\left(\frac{1}{2}+k\right)\theta,$$

in such a way that $v(x,y) = c\bar v(x,y)$, while $w(x,y) = d\bar v(-x,y)$. Since $\mathbf{v}$ must satisfy the Pohozaev identities for the elements of $\mathcal{G}_s$, we infer that

$$\int_{\partial^+ B_r^+(x_0,0)} |\nabla v|^2 + |\nabla w|^2 \,\mathrm{d}\sigma = 2\int_{\partial^+ B_r^+(x_0,0)} |\partial_\nu v|^2 + |\partial_\nu w|^2\,\mathrm{d}\sigma,$$

for every $x_0\in\mathbb{R}$ and $r>0$. Considering the choices $x_0=1$ and $x_1=-1$, and using the symmetries, one has

$$\begin{aligned} A_+c^2 + A_-d^2 &= 2B_+c^2 + 2B_-d^2, \\ A_-c^2 + A_+d^2 &= 2B_-c^2 + 2B_+d^2, \end{aligned}$$

where

$$A_\pm = \int_{\partial^+ B_r^+(\pm 1,0)} |\nabla \bar v|^2\,\mathrm{d}\sigma, \qquad B_\pm = \int_{\partial^+ B_r^+(\pm 1,0)} |\partial_\nu \bar v|^2\,\mathrm{d}\sigma.$$

Since $A_\pm - 2B_\pm \neq 0$, at least for some r, the above equalities force $c^4-d^4=0$, that is $d=\pm c$. We want to show that this condition is also sufficient for $(v,w,0,\dots,0)$ to belong to $\mathcal{H}(\nu,1)$. To this aim, we only need to prove the actual validity of the Pohozaev identity for any x_0 and R. We begin by observing that v and w are conjugated harmonic functions, thus in particular it holds

$$\nabla v\cdot\nabla w = 0 \quad\text{and}\quad |\nabla v| = |\nabla w| \quad \text{in } \mathbb{R}^2_+.$$

Hence, for any unitary vector $\mathbf{n} \in \mathbb{R}^2$ we have

$$|\nabla v|^2 = |\nabla w|^2 = |\nabla v \cdot \mathbf{n}|^2 + |\nabla w \cdot \mathbf{n}|^2 = |\partial_{\mathbf{n}} v|^2 + |\partial_{\mathbf{n}} w|^2,$$

and the Pohozaev identity follows by integrating over half circles, and choosing ν as the outer normal. Resuming, the case in which $\mathbf{v}$ has two components with non trivial trace on $\{y = 0\}$ always fall into alternative (3) of the statement.

Secondly, let us assume that only one component, say v, has non trivial trace on $\{y = 0\}$. Then $\{v(x, 0) > 0\}$ is either a half line or the entire real line. The first case never happens, since v would solve

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^2_+ \\ v(x, 0) = 0 & \text{on } x < 0 \\ \partial_\nu v = 0 & \text{on } x > 0. \end{cases}$$

Reasoning as before, we would obtain that v is of the form

$$v(\rho, \theta) = c\rho^{\frac{1}{2}+k} \cos\left(\frac{1}{2} + k\right)\theta,$$

with $c \in \mathbb{R}$ and $k \in \mathbb{N}$, while all the (odd extensions of the) other components should be harmonic on $\mathbb{R}^2$ and homogeneous of degree $k + 1/2$, that is null; the Pohozaev identity would force $c = 0$, and $\mathbf{v}$ would be trivial. In the second case, if $v(x, 0) \neq 0$ for every $x \neq 0$, then v is of the form

$$v(\rho, \theta) = c\rho^k \cos(k\theta)$$

with $c \in \mathbb{R} \setminus \{0\}$ and $k \in \mathbb{N}$, while all the other components of $\mathbf{v}$ are of the form

$$v_i(\rho, \theta) = c_i \rho^k \sin(k\theta)$$

for some $c_i \in \mathbb{R}$. Then the case of one non trivial trace on $\{y = 0\}$ always falls into alternative (2) of the statement.

As the last case, let us suppose that $v_i(x, 0) \equiv 0$ for every i. Then each of them is a ν-homogeneous solution to

$$\begin{cases} -\Delta v_i = 0 & \text{in } \mathbb{R}^2_+ \\ v_i = 0 & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

that is, for some $k \in \mathbb{N}$ and $c_i \in \mathbb{R}$, $\nu = 1 + k$ and

$$v_i(\rho, \theta) = c_i \rho^{1+k} \sin((1 + k)\theta).$$

Also this case always falls into alternative (2) of the statement, and the proposition follows. $\square$

2.7 $\mathcal{C}^{0,\alpha}$ uniform bounds, $\alpha < 1/2$

This section is devoted to the proof of the uniform Hölder bounds, with every exponent less that $1/2$, for the problem with exterior boundary Dirichlet data. In this direction, let us consider the problem

$$\begin{cases} -\Delta v_i = 0 & \text{in } B^+ \\ \partial_\nu v_i = f_{i,\beta}(v_i) - \beta v_i \sum_{j\neq i} v_j^2 & \text{on } \partial^0 B^+ \cap \Omega \\ v_i = 0 & \text{on } \partial^0 B^+ \setminus \Omega, \end{cases} \tag{$(PD)_\beta$}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ and the functions $f_{i,\beta}$ are continuous and uniformly bounded, with respect to β, on bounded sets.

Remark 2.7.1. For $(PD)_\beta$ it is known that, if Ω is of class $\mathcal{C}^3$, then any L^∞ solution is in fact $\mathcal{C}^{0,\alpha}$ for every $\alpha < 1/2$, see [44]. Furthermore, a uniform bound holds when β is bounded, similarly to Remark 2.5.1. Actually, the assumption on the smoothness of Ω can be weakened, at least when considering global problems for $u(\cdot) = v(\cdot, 0)$, as done in the recent paper [43].

We prove the following.

Theorem 2.7.2. *Let $\{\mathbf{v}_\beta\}_{\beta>0}$ be a family of solutions to problem $(PD)_\beta$ on B_1^+ such that*

$$\|\mathbf{v}_\beta\|_{L^\infty(B_1^+)} \leq \bar{m},$$

with $\bar{m}$ independent of β. Then for every $\alpha \in (0, 1/2)$ there exists a constant $C = C(\bar{m}, \alpha)$, not depending on β, such that

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}\left(\overline{B_{1/2}^+}\right)} \leq C.$$

Furthermore, $\{\mathbf{v}_\beta\}_{\beta>0}$ is relatively compact in $H^1(B_{1/2}^+) \cap \mathcal{C}^{0,\alpha}\left(\overline{B_{1/2}^+}\right)$ for every $\alpha < 1/2$.

Actually, two particular cases of the above theorem can be obtained in a rather direct way.

Remark 2.7.3. If $\partial^0 B^+ \cap \Omega = \emptyset$ then Theorem 2.7.2 holds true. Indeed, the family of functions obtained from $\{\mathbf{v}_\beta\}_{\beta>0}$ by odd reflection across $\{y = 0\}$ consists in harmonic, L^∞ uniformly bounded functions on B_1.

Remark 2.7.4 (Proof of Theorem 1.1.1)**.** If $\partial^0 B^+ \subset \Omega$ then Theorem 2.7.2 holds true. This is indeed the content of Theorem 1.1.1, that is the one of Theorem 2.5.2

with ν^{ACF} replaced by $1/2$. In order to prove this result, one can reason as in the proof of such theorem, by using Theorem 2.6.1 instead of Propositions 2.3.1 and 3.2.10 (also recall Remark 2.5.14).

Proof of Theorem 2.7.2. The outline of the proof follows the one of Theorem 2.5.2, to which we refer the reader for further details. To start with, let η be a smooth cutoff function as in equation (2.5.1), and let $\alpha \in (0, 1/2)$ be fixed. We assume by contradiction that

$$
\begin{aligned}
L_n &:= \max_{i=1,\dots,k} \max_{X' \neq X'' \in \overline{B^+}} \frac{|(\eta v_{i,n})(X') - (\eta v_{i,n})(X'')|}{|X' - X''|^\alpha} \\
&= \frac{|(\eta v_{1,n})(X'_n) - (\eta v_{i,n})(X''_n)|}{r_n^\alpha} \to \infty,
\end{aligned}
$$

where, as usual, $\mathbf{v}_n$ solves $(PD)_{\beta_n}$, $\beta_n \to \infty$, and $r_n := |X'_n - X''_n| \to 0$. Furthermore, reasoning as in Lemmas 2.5.4 and 2.5.8, one can prove that the sequences $\{X'_n\}_{n\in\mathbb{N}}$ and $\{X''_n\}_{n\in\mathbb{N}}$ accumulate near $\partial^0 B^+$ and far away from $\partial^+ B^+$, at least in the scale of r_n.

Under the previous notations, we define the blow up sequences

$$
w_{i,n}(X) := \eta(P_n) \frac{v_{i,n}(P_n + r_n X)}{L_n r_n^\alpha} \quad \text{and} \quad \bar{w}_{i,n}(X) := \frac{(\eta v_{i,n})(P_n + r_n X)}{L_n r_n^\alpha},
$$

where

$$
P_n := (x'_n, 0) \quad \text{and} \quad X \in \tau_n B^+ := \frac{B^+ - P_n}{r_n}.
$$

Such sequences satisfy the following properties:

- $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ have uniformly bounded Hölder quotient on $\tau_n \overline{B^+}$, and $\operatorname{osc} w_{1,n} = 1$ for every n on a suitable compact set;

- each $\mathbf{w}_n$ solves

$$
\begin{cases}
-\Delta w_{i,n} = 0 & \text{in } \tau_n B^+ \\
\partial_\nu w_{i,n} = f_{i,n}(w_{i,n}) - M_n w_{i,n} \sum_{j\neq i} w_{j,n}^2 & \text{on } \tau_n(\partial^0 B^+ \cap \Omega) \\
w_{i,n} = 0 & \text{on } \tau_n(\partial^0 B^+ \setminus \Omega),
\end{cases}
$$

  where $\sup |f_{i,n}(w_{i,n})| \to 0$ as $n \to \infty$;

- $|\mathbf{w}_n - \bar{\mathbf{w}}_n| \to 0$ uniformly, as $n \to \infty$, on every compact set.

By the regularity assumption on $\partial\Omega$ we infer that, up to translations, rotations and subsequences, one of the following three cases must hold.

Case 1: $\tau_n(\partial^0 B^+ \setminus \Omega) \to \mathbb{R}^N$. In particular, we have that $\mathbf{w}_n(0) = \bar{\mathbf{w}}_n(0) = 0$ for n large. Reasoning as in Section 2.5 we obtain that both $\mathbf{w}_n$ and $\bar{\mathbf{w}}_n$ converge, uniformly on compact sets, to the same $\mathbf{w}$ which is harmonic and globally Hölder continuous on $\mathbb{R}^{N+1}_+$, vanishing on $\mathbb{R}^N$ and nonconstant. But then the odd extension of $\mathbf{w}$ across $\{y = 0\}$ contradicts Liouville theorem.

Case 2: $\tau_n(\partial^0 B^+ \cap \Omega) \to \mathbb{R}^N$. In this case, for every compact set $K \subset \overline{\mathbb{R}^{N+1}_+}$, we have that $\{\mathbf{w}_n|_K\}_{n\in\mathbb{N}}$ and $\{\bar{\mathbf{w}}_n|_K\}_{n\in\mathbb{N}}$, for n large, fit in the setting of Section 2.5. Consequently we can argue exactly in the same way, recalling that the regularity for every $\alpha < 1/2$ is obtained by means of Theorem 2.6.1 (see also Remark 2.7.4).

Case 3: $\tau_n(\partial^0 B^+ \cap \Omega) \to \{x \in \mathbb{R}^N : x_1 > 0\}$. As in the first case, we have that $\mathbf{w}_n(0) = \bar{\mathbf{w}}_n(0) = 0$ for n large, implying that $w_{1,n} \to w_1$, uniformly on compact sets of $\overline{\mathbb{R}^{N+1}_+}$, with w_1 non constant, harmonic, and such that $w_1(x, 0) = 0$ for $x_1 \leq 0$. Finally, reasoning as in Lemma 2.5.13, we have that $w_{1,n} \to w_1$ also strongly in H^1_{loc}, thus $w_1 \partial_\nu w_1 \leq 0$. We are in a position to apply Proposition 2.3.4 to w_1 and reach a contradiction. $\square$

Using the above result, we can prove the following global theorem.

Theorem 2.7.5. *Let $\{\mathbf{v}_\beta\} \in H^1_{\mathrm{loc}}(\mathbb{R}^N \times (0, 1))$ solve*

$$\begin{cases} -\Delta v_{i,\beta} = 0 & \text{in } \mathbb{R}^N \times (0,1) \\ \partial_\nu v_{i,\beta} = f_{i,\beta}(v_{i,\beta}) - \beta v_{i,\beta} \sum_{j\neq i} v_{j,\beta}^2 & \text{on } \Omega \\ v_{i,\beta} = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

If there exists a constant $\bar{m}$ such that

$$\|v_{i,\beta}\|_{L^\infty(\mathbb{R}^N\times(0,1))} \leq \bar{m}$$

then for any $\alpha \in (0, 1/2)$

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N\times[0,1/3])} \leq C(\bar{m}, \alpha).$$

Furthermore, $\{\mathbf{v}_\beta\}_{\beta>0}$ is relatively compact in $(H^1 \cap \mathcal{C}^{0,\alpha})_{\mathrm{loc}}$ for every $\alpha < 1/2$.

Proof. The proof easily follows by a covering argument. Indeed, we can cover $\mathbb{R}^N \times [0, 1/3]$ with a countable number of half-balls of radius $1/2$, centered on $\mathbb{R}^N$, and apply Theorem 2.7.2 to each of the corresponding half-ball of radius 1. $\square$

Proof of Theorem 1.1.3. This is actually a corollary of Theorem 2.7.5: indeed, if $u \in (H^{1/2} \cap L^\infty)(\mathbb{R}^N)$, and $v \in H^1(\mathbb{R}^{N+1}_+)$ is its unique harmonic extension satisfying

$$(-\Delta)^{1/2} u(\cdot) = -\partial_y v(\cdot, 0),$$

then v is uniformly bounded in L^∞. $\square$

Remark 2.7.6. Analogous results can be proved, with minor changes, when the fractional operator considered is the spectral square root of the laplacian, as studied in [8]. Indeed, in such situation, the corresponding extension problem is given by

$$\begin{cases} -\Delta v_{i,\beta} = 0 & \text{in } \Omega \times (0,\infty) \\ \partial_\nu v_{i,\beta} = f_{i,\beta}(v_{i,\beta}) - \beta v_{i,\beta} \sum_{j\neq i} v_{j,\beta}^2 & \text{on } \Omega \times \{0\} \\ v_{i,\beta} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

and the starting regularity for β bounded is even finer. As a consequence, one can consider the extension of v which is trivial outside $\Omega \times (0,\infty)$, and conclude by using a modified version of Proposition 2.3.4, suitable for subharmonic functions.

2.8 $\mathcal{C}^{0,1/2}$ regularity of the limiting profiles

In this section we consider the regularity of the limiting profiles, that is, the accumulation points of solutions to problem $(GP)_\beta$ as $\beta \to \infty$. In Section 2.5 we proved that, if $\{\mathbf{v}_\beta\}_{\beta>0}$ is a family of solutions to problem $(GP)_\beta$, and $\|\mathbf{v}_\beta\|_{L^\infty(B^+)} \leq \bar{m}$ for a constant $\bar{m}$ independent of β, then there exists a sequence $\mathbf{v}_n := \mathbf{v}_{\beta_n}$ such that $\beta_n \to \infty$ and

$$\mathbf{v}_n \to \mathbf{v} \qquad \text{in } (H^1 \cap \mathcal{C}^{0,\alpha})(K \cap B^+),$$

for every compact set $K \subset B$ and every $\alpha \in (0,1/2)$. Now we turn to the proof of Theorem 1.1.2, that is, we show that $\mathbf{v} \in \mathcal{C}^{0,1/2}_{\mathrm{loc}}(B^+ \cup \partial^0 B^+)$. Actually, we will prove such theorem under a more general assumption: from now on we will assume that the reaction terms in problem $(GP)_\beta$ satisfy

$$\lim_{n\to\infty} f_{i,n} = f_i \quad \text{uniformly in every compact set,}$$

where $(f_1,\dots,f_k)$ are locally Lipschitz, and such that, for some $\varepsilon > 0$,

$$2F_i(s) - sf_i(s) \geq -C|s|^{2+\varepsilon} \text{ for } s \text{ sufficiently small,} \tag{2.8.1}$$

for every i, where $F_i(s) = \int_0^s f_i(t)\mathrm{d}t$ (in particular, $f_i(0) = 0$).

Remark 2.8.1. If $f_i \in \mathcal{C}^{1,\varepsilon}$ in a neighborhood of 0, and $f_i(0) = 0$, then assumption (2.8.1) holds true. Indeed this implies that $2F_i(s) - sf_i(s) = O(s^{2+\varepsilon})$ as $s \to 0$.

We will obtain Theorem 1.1.2 as a byproduct of a stronger result, in the form of the following proposition.

Proposition 2.8.2. *Let $\mathbf{v} \in H^1(B^+)$ be such that*

1. $\mathbf{v} \in (H^1 \cap \mathcal{C}^{0,\alpha})(K \cap B^+)$, *for every compact set* $K \subset B$ *and every* $\alpha \in (0, 1/2)$*;*

2. $v_i v_j|_{\partial^0 B^+} = 0$ *for every* $j \neq i$ *and*

$$\begin{cases} -\Delta v_i = 0 & \text{in } B^+ \\ v_i \partial_\nu v_i = v_i f_i(v_i) & \text{on } \partial^0 B^+, \end{cases}$$

where f_i *is locally Lipschitz continuous and satisfies* (2.8.1)*, for every* $i = 1, \dots, k$*;*

3. *for every* $x_0 \in \partial^0 B^+$ *and a.e.* $r > 0$ *such that* $B_r^+(x_0, 0) \subset B^+$*, the following Pohozaev identity holds*

$$(1 - N) \int_{B_r^+} \sum_i |\nabla v_i|^2 \, \mathrm{d}x\mathrm{d}y + r \int_{\partial^+ B_r^+} \sum_i |\nabla v_i|^2 \, \mathrm{d}\sigma + \\ + 2N \int_{\partial^0 B_r^+} \sum_i F_i(v_i) \, \mathrm{d}x - 2r \int_{S_r^{N-1}} \sum_i F_i(v_i) \, \mathrm{d}\sigma = 2r \int_{\partial^+ B_r^+} \sum_i |\partial_\nu v_i|^2 \, \mathrm{d}\sigma.$$

Then $\mathbf{v} \in \mathcal{C}^{0,1/2}(K \cap B^+)$*, for every compact* $K \subset B$*.*

As we mentioned, Theorem 1.1.2 will follow from the above proposition by virtue of the following result.

Lemma 2.8.3. *Let* $\beta_n \to \infty$ *and* $\mathbf{v}_n$ *solve problem* $(GP)_{\beta_n}$*, for every* n*, be such that*

$$\mathbf{v}_n \to \mathbf{v} \qquad \text{in } (H^1 \cap \mathcal{C}^{0,\alpha})(K \cap B^+),$$

for every compact set $K \subset B$ *and every* $\alpha \in (0, 1/2)$*. Moreover, let the corresponding reaction terms* $f_{i,n}$ *converge, uniformly on compact sets, to the locally Lipschitz functions* f_i *satisfying* (2.8.1)*. Then* $\mathbf{v}$ *fulfills the assumptions of the Proposition 2.8.2.*

Proof. The proof follow the line of the one of Proposition 2.4.1, with minor changes. □

In view of the previous lemma, with a slight abuse of terminology, we will denote as limiting profiles also functions which simply satisfy the assumptions of Proposition 2.8.2. For the rest of this section we will denote with $\mathbf{v}$ a fixed limiting profile.

In the proof of Proposition 2.8.2 we shall use a further monotonicity formula of Almgren type. For every $x_0 \in \partial^0 B^+$ and $r > 0$ such that $B_r^+(x_0, 0) \subset B^+$, we introduce the functions

$$E(x_0,r) := \frac{1}{r^{N-1}}\left(\int\limits_{B_r^+(x_0,0)} \sum_i |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y - \int\limits_{\partial^0 B_r^+(x_0,0)} \sum_i f_i(v_i)v_i \,\mathrm{d}x\right)$$

$$H(x_0,r) := \frac{1}{r^N}\int\limits_{\partial^+ B_r^+(x_0,0)} \sum_i v_i^2 \,\mathrm{d}\sigma.$$

As usual, the function $E(x_0,r)$ admits an equivalent expression: indeed, multiplying the equation in assumption (2) by v_i, integrating over $B_r^+(x_0,0)$ and summing over $i=1,\dots,k$ we obtain

$$E(x_0,r) = \frac{1}{r^{N-1}}\int\limits_{\partial^+ B_r^+(x_0,0)} \sum_i v_i\partial_\nu v_i \mathrm{d}\sigma = \frac{2}{r}H'(x_0,r). \tag{2.8.2}$$

The presence of internal reaction terms in the definition of the function E has to be dealt with. To this end, the next two lemmas will provide a crucial estimate in order to bound the Almgren quotient. Before we state them, let us recall the following Poincaré inequality: for every $p\in[2,p^\#]$, where $p^\# = 2N/(N-1)$ denotes the critical Sobolev exponent for trace embedding (or simply $p\geq 2$ in dimension $N=1$), there exists a constant $C_P = C_P(N,p)$ such that, for every $w\in H^1(B_r^+)$,

$$\left[\frac{1}{r^N}\int\limits_{\partial^0 B_r^+} |w|^p \,\mathrm{d}x\right]^{\frac{2}{p}} \leq C_P\left[\frac{1}{r^{N-1}}\int\limits_{B_r^+} |\nabla w|^2 \,\mathrm{d}x\mathrm{d}y + \frac{1}{r^N}\int\limits_{\partial^+ B_r^+} w^2 \,\mathrm{d}\sigma\right] \tag{2.8.3}$$

(such an inequality follows by the one on B^+ by scaling arguments).

Lemma 2.8.4. *For every $p\in[2,p^\#]$ there exist constants $C>0$, $\bar r>0$ such that*

$$\left[\frac{1}{r^N}\int\limits_{\partial^0 B_r^+} \sum_i |v_i|^p \,\mathrm{d}x\right]^{\frac{2}{p}} \leq C\left[E(r)+H(r)\right] \quad \textit{for every } r\in(0,\bar r).$$

Proof. Since $\mathbf{v}\in L^\infty(B^+)$, and each f_i is locally Lipschitz continuous with $f_i(0)=0$, we have

$$\begin{aligned}\left|\frac{1}{r^{N-1}}\int\limits_{\partial^0 B_r^+} \sum_i f_i(v_i)v_i \,\mathrm{d}x\right| &\leq C\frac{1}{r^{N-1}}\int\limits_{\partial^0 B_r^+} \sum_i v_i^2 \,\mathrm{d}x \\ &\leq C'r\left[\frac{1}{r^{N-1}}\int\limits_{B_r^+} \sum_i |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y + \frac{1}{r^N}\int\limits_{\partial^+ B_r^+} \sum_i v_i^2 \,\mathrm{d}\sigma\right],\end{aligned}$$

where we used inequality (2.8.3) with $p = 2$. As a consequence,

$$E(r) + H(r) \geq (1 - Cr) \left[\frac{1}{r^{N-1}} \int\limits_{B_r^+} \sum_i |\nabla v_i|^2 \, \mathrm{d}x\mathrm{d}y + \frac{1}{r^N} \int\limits_{\partial^+ B_r^+} \sum_i v_i^2 \, \mathrm{d}\sigma \right], \quad (2.8.4)$$

and the lemma follows by taking into account equation (2.8.3) and choosing $\bar{r}$ sufficiently small. □

For the following lemma we introduce, for $p \in (2, p^\#]$, the auxiliary function

$$\psi(x_0, r) := \left(\frac{1}{r^N} \int\limits_{\partial^0 B_r^+(x_0,0)} \sum_i |v_i|^p \, \mathrm{d}x \right)^{1-\frac{2}{p}},$$

which is bounded for r small. We have the following.

Lemma 2.8.5. *For every $p \in (2, p^\#]$ there exist constants $C > 0$, $\bar{r} > 0$ such that*

$$\frac{1}{r^{N-1}} \int\limits_{S_r^{N-1}} \sum_i |v_i|^p \, \mathrm{d}\sigma \leq C \, [E(r) + H(r)] \cdot \frac{\mathrm{d}}{\mathrm{d}r}(r\psi(r)) \quad \textit{for every } r \in (0, \bar{r}).$$

Proof. A direct computation yields the identity

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\psi(r) &= \left(1 - \frac{2}{p}\right) \psi^{-2/(p-2)} \left(\frac{1}{r^N} \int\limits_{\partial^0 B_r^+(x_0,0)} \sum_i |v_i|^p \, \mathrm{d}x \right)' \\ &= \left(1 - \frac{2}{p}\right) \psi(r) \frac{\left(r^{-N} \int_{\partial^0 B_r^+} \sum_i |v_i|^p \, \mathrm{d}x \right)'}{r^{-N} \int_{\partial^0 B_r^+} \sum_i |v_i|^p \, \mathrm{d}x}. \end{aligned}$$

As a consequence we infer

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\psi(r)) = \psi(r) \left[r \left(1 - \frac{2}{p}\right) \frac{\int_{S_r^{N-1}} \sum_i |v_i|^p \, \mathrm{d}\sigma}{\int_{\partial^0 B_r^+} \sum_i |v_i|^p \, \mathrm{d}\sigma} + \left(1 - N\left(1 - \frac{2}{p}\right)\right) \right].$$

Now, $p \leq p^\#$ implies $N\left(1 - \frac{2}{p}\right) \leq 1$, so that

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\psi(r)) \geq r\psi(r) \left(1 - \frac{2}{p}\right) \frac{\int_{S_r^{N-1}} \sum_i |v_i|^p \, \mathrm{d}\sigma}{\int_{\partial^0 B_r^+} \sum_i |v_i|^p \, \mathrm{d}\sigma}.$$

Recalling the definition of ψ and using Lemma 2.8.4, we finally obtain

$$(E(r) + H(r)) \frac{\mathrm{d}}{\mathrm{d}r}(r\psi(r)) \geq C \frac{1}{r^{N-1}} \int\limits_{S_r^{N-1}} \sum_i |v_i|^p \, \mathrm{d}\sigma,$$

where, since $p > 2$, $C > 0$. □

As a matter of fact, we need to estimate the Almgren quotient only on the zero set of $\mathbf{v}$ (which is well defined since $\mathbf{v}$ is continuous).

Definition 2.8.6. We define the boundary zero set of the limiting profile $\mathbf{v}$ as

$$\mathcal{Z} = \{x \in \partial^0 B^+ : \mathbf{v}(x,0) = 0\}.$$

Remark 2.8.7. A natural notion of free boundary, associated to a limiting profile $\mathbf{v}$, is the set in which the boundary condition of assumption (2) does not reduce to

$$\partial_\nu v_i = f_i(v_i), v_j \equiv 0 \quad \text{for some } j \neq i,$$

that is, *a posteriori*, the support of the singular part of the measure $\partial_\nu \mathbf{v}$. It is then clear that the free boundary is a subset of $\mathcal{Z} \subset \mathbb{R}^N$.

We are now in a position to state the Almgren type result which we use in this framework. As we mentioned, we prove it only at points of $\mathcal{Z}$; furthermore, it concerns boundedness of a (modified) Almgren quotient, rather than its monotonicity. More precisely, let us consider the function

$$N(x_0, r) := \frac{E(x_0, r)}{H(x_0, r)} + 1.$$

We have the following.

Lemma 2.8.8. *There exist constants $C > 0$, $\bar{r} > 0$ such that, for every $x_0 \in \mathcal{Z}$, $r \in (0, \bar{r})$ and $B_r^+(x_0, 0) \subset B^+$, we have:*

1. *$H(r) > 0$, $N(r) > 0$ on $(0, \bar{r})$;*

2. *the function $r \mapsto e^{Cr(1+\psi(r))} N(x_0, r)$ is monotone non decreasing;*

3. *$N(x_0, 0^+) \geq 1 + \dfrac{1}{2}$.*

Proof. The proof is similar to the one of Theorem 2.2.3, but in this case the internal reaction terms do not vanish. Let $x_0 \in \mathcal{Z}$ and let $\bar{r}$ be such that both Lemma 2.8.4 and Lemma 2.8.5 hold. First, we ensure that the Almgren quotient, where defined, is non negative. Indeed, by Lemma 2.8.4,

$$E(r) + H(r) \geq 0 \implies N(r) = \frac{E}{H} + 1 \geq 0,$$

whenever $H(r) \neq 0$. By continuity of H we can consider, as in the proof of Theorem 2.2.3, a neighborhood of r where H does not vanish. We compute the derivative of E

and we use the Pohozaev identity (assumption (3) of Proposition 2.8.2), to obtain

$$
\begin{aligned}
E'(r) &= \frac{1-N}{r^N}\left(\int_{B_r^+}\sum_i |\nabla v_i|^2\,\mathrm{d}x\mathrm{d}y - \int_{\partial^0 B_r^+}\sum_i v_i f_i(v_i)\,\mathrm{d}x\right)\\
&\qquad + \frac{1}{r^{N-1}}\left(\int_{\partial^+ B_r^+}\sum_i |\nabla v_i|^2\,\mathrm{d}x\mathrm{d}y - \int_{S_r^{N-1}}\sum_i v_i f_i(v_i)\,\mathrm{d}x\right)\\
&= \underbrace{\frac{2}{r^{N-1}}\int_{\partial^+ B_r^+}\sum_i |\partial_\nu v_i|^2\,\mathrm{d}\sigma}_{T} + \underbrace{\frac{1}{r^N}\int_{\partial^0 B_r^+}\left[(N-1)\sum_i v_i f_i(v_i) - 2N\sum_i F_i(v_i)\right]\mathrm{d}x}_{I}\\
&\qquad + \underbrace{\frac{1}{r^{N-1}}\int_{S_r^{N-1}}\left[-\sum_i v_i f_i(v_i) + 2\sum_i F_i(v_i)\right]\mathrm{d}\sigma}_{Q}\,.
\end{aligned}
$$

Since $\mathbf{v} \in L^\infty$, f_i are locally Lipschitz and $f_i(0) = 0$, there exists a positive constant C, such that

$$|f(v_i)v_i| \leq Cv_i^2 \text{ and } |F(v_i)| \leq Cv_i^2.$$

The direct application of Lemma 2.8.4 (with $p = 2$) provides

$$I \geq -C(E + H).$$

On the other hand, by assumption (2.8.1) and Lemma 2.8.5 (it is sufficient to choose $p = \min\{2 + \varepsilon, p^\#\}$), we obtain

$$Q \geq -C(E + H)(r\psi)'.$$

The two estimates yield

$$E' \geq T - C\,[1 + (r\psi)']\,(E + H).$$

Therefore, differentiating the Almgren quotient and using the Cauchy-Schwarz inequality, we obtain

$$\frac{N'}{N} = \frac{E' + H'}{E + H} - \frac{H'}{H} \geq \frac{TH - EH'}{H(E + H)} - C\,[1 + (r\psi)'] \geq -C\,[1 + (r\psi)']\,,$$

which implies that the function $e^{Cr(1+\psi(r))}N(r)$ is non decreasing as far as $H(r) \neq 0$. Equation (2.8.2) directly implies

$$\frac{\mathrm{d}}{\mathrm{d}r}\log H(r) = \frac{H'(r)}{H(r)} = \frac{2E(r)}{rH(r)} = \frac{2(N(r)-1)}{r};$$

reasoning as in the proof of Theorem 2.2.3, we can use this formula, together with the bound

$$N(r) \leq e^{Cr^*(1+\psi(r^*))} N(r^*) \qquad \text{for every } r \leq r^*,$$

in order to obtain the strict positivity of H for $r \in (0, \bar{r})$ (for a possibly smaller $\bar{r}$). Finally, reasoning as in the proof of Lemma 2.2.4, part (2), let us assume by contradiction that, for some $r^* < \bar{r}$ and $\varepsilon > 0$, $e^{Cr^*(1+\psi(r^*))} N(r^*) \leq \frac{3}{2} - \varepsilon$. By the above bound we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(r) \leq \frac{2(e^{Cr^*(1+\psi(r^*))} N(r^*) - 1)}{r} \leq \frac{1-2\varepsilon}{r}$$

for every $r \in (0, r^*)$. But this is in contradiction with the fact that $\mathbf{v}$ is in $\mathcal{C}^{0,\alpha}$ for $\alpha = (1-\varepsilon)/2$. □

The proof Proposition 2.8.2 is based on a contradiction argument, involving Morrey inequality. Indeed, let $K \subset B$ be compact, and let us define, for every $X \in K \cap \{y \geq 0\}$ and every $r < \operatorname{dist}(K, \partial B)$, the function

$$\Phi(X, r) := \frac{1}{r^N} \int\limits_{B_r(X) \cap \{y>0\}} \sum_i |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y.$$

It is well known that if Φ is bounded then $\mathbf{v} \in \mathcal{C}^{0,1/2}(K \cap B^+)$.

As a consequence of Lemma 2.8.8, we can prove a first estimate on Φ.

Lemma 2.8.9. *For every compact $K \subset B$ there exists constants $C > 0$, $\bar{r} > 0$, such that for every $x_0 \in \mathcal{Z} \cap K$ and $r \in (0, \bar{r})$, it holds*

$$\Phi(x_0, r) \leq C.$$

Proof. If $\bar{r}$ is sufficiently small, from Lemma 2.8.8 we know that

$$\frac{3}{2} e^{-Cr(1+\psi(r))} \leq N(r) \leq C$$

for every $r \in (0, \bar{r})$. Since $E + H = NH$, equation (2.8.4) implies that

$$\frac{1}{r^N} \int\limits_{B_r^+(x_0,0)} \sum_i |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}y \leq C \frac{H(r)}{r}.$$

On the other hand, by the lower estimate on N,

$$\frac{\mathrm{d}}{\mathrm{d}r} \log \frac{H(r)}{r} \geq 3 \frac{e^{-Cr(1+\psi(r))} - 1}{r} \geq -3C(1+\psi(r)) \geq -C.$$

Integrating, we obtain

$$\frac{H(r)}{r} \leq e^{C\bar{r}} \frac{H(\bar{r})}{\bar{r}},$$

and the lemma follows. □

The above result can be complemented by the following lemma.

Lemma 2.8.10. *For every compact $K \subset B$ there exist constants, $C > 0$, $\bar{r} > 0$, such that for every $x_0 \in (K \cap \{y = 0\}) \setminus \mathcal{Z}$ and*

$$0 < r < d := \min\{\mathrm{dist}(x_0, \mathcal{Z}), \bar{r}\},$$

it holds

$$\Phi(x_0, d) \geq C\Phi(x_0, r).$$

Proof. Since $x_0 \notin \mathcal{Z}$ and $r \leq \mathrm{dist}(x_0, \mathcal{Z})$, we can assume that $v_j \equiv 0$ on $\partial^0 B_r^+(x_0, 0)$ for, say, $j \geq 2$. As a consequence, the odd extension of v_j across $\{y = 0\}$ is harmonic on $B_r(x_0, 0)$, and the mean value property applied to the subharmonic function $|\nabla v_j|^2$ provides

$$\frac{1}{r^N} \int\limits_{B_r^+(x_0,0)} |\nabla v_j|^2 \,\mathrm{d}x\mathrm{d}y \leq \frac{r}{d}\frac{1}{d^N} \int\limits_{B_d^+(x_0,0)} |\nabla v_j|^2 \,\mathrm{d}x\mathrm{d}y, \qquad \text{for every } j \geq 2. \tag{2.8.5}$$

We now show that a similar estimate holds true also for v_1. Indeed, let $u := |\nabla v_1|^2$; by a straightforward computation, we have that

$$\begin{cases} -\Delta u \leq 0 & \text{in } B_d^+ \\ \partial_\nu u \leq au & \text{in } \partial^0 B_d^+, \end{cases}$$

where $a := 2||f_1'(v_1)||_{L^\infty(B^+)}$ is bounded by assumption. Now, by scaling, one can show that if $\bar{r} = \bar{r}(a)$ is sufficiently small, then the equation

$$\begin{cases} -\Delta \varphi = 0 & \text{in } B_{\bar{r}}^+ \\ \partial_\nu \varphi = a\varphi & \text{on } \partial^0 B_{\bar{r}}^+ \end{cases}$$

admits a strictly positive (and smooth) solution. By the definition of d we deduce that

$$\begin{cases} -\mathrm{div}\left(\varphi^2 \nabla \frac{u}{\varphi}\right) \leq 0 & B_d^+ \\ \varphi^2 \partial_\nu \frac{u}{\varphi} \leq 0 & \partial^0 B_d^+, \end{cases}$$

so that the even extension of u is a solution to

$$-\mathrm{div}\left(\varphi^2 \nabla \frac{u}{\varphi}\right) \leq 0 \quad \text{in } B_d.$$

Integrating such equation on any ball B_r, we obtain

$$\int\limits_{\partial B_r} \varphi^2 \partial_\nu \frac{u}{\varphi} \mathrm{d}\sigma \geq 0$$

If we introduce the function

$$H(r) = \frac{1}{r^N} \int\limits_{\partial B_r} \varphi u \mathrm{d}\sigma = \int\limits_{\partial B} \varphi^2(rx) \frac{u(rx)}{\varphi(rx)} \mathrm{d}\sigma,$$

a straightforward computation shows that

$$H'(r) = \frac{2}{r^N} \int\limits_{\partial B_r} u\varphi \frac{\partial_\nu \varphi}{\varphi} \mathrm{d}\sigma + \frac{1}{r^N} \int\limits_{\partial B_r} \varphi^2 \partial_\nu \frac{u}{\varphi} \mathrm{d}\sigma \geq -2 \left\| \frac{\partial_\nu \varphi}{\varphi} \right\|_{L^\infty(B)} H(r) \geq -CH(r),$$

that is, the function $r \mapsto e^{Cr} H(r)$ is monotone non decreasing in r. Hence, for every $0 < r_1 \leq r_2 \leq d$, we obtain that $H(r_1) \leq CH(r_2)$. Multiplying by $r_1^N r_2^N$ and integrating in $(0, r) \times (r, d)$, with $r \leq d$, we obtain

$$\left(1 - \frac{r^{N+1}}{d^{N+1}}\right) \frac{1}{r^{N+1}} \int\limits_{B_r} \varphi u \, \mathrm{d}x\mathrm{d}y \leq \frac{C}{d^{N+1}} \int\limits_{B_d \setminus B_r} \varphi u \, \mathrm{d}x\mathrm{d}y.$$

Adding $Cd^{-N-1} \int\limits_{B_r} \varphi u \, \mathrm{d}x\mathrm{d}y$, we infer

$$\frac{1}{r^{N+1}} \int\limits_{B_r} \varphi u \mathrm{d}x\mathrm{d}y \leq C \frac{1}{d^{N+1}} \int\limits_{B_d} \varphi u \mathrm{d}x\mathrm{d}y.$$

Recalling that φ is positive and bounded, and that $u = |\nabla v_1|^2$, we finally obtain that

$$\frac{1}{r^N} \int\limits_{B_r^+(x_0,0)} |\nabla v_1|^2 \, \mathrm{d}x\mathrm{d}y \leq C \frac{r}{d} \frac{1}{d^N} \int\limits_{B_d^+(x_0,0)} |\nabla v_1|^2 \, \mathrm{d}x\mathrm{d}y.$$

The lemma now follows by summing up with inequality (2.8.5), for $j = 2, \ldots, k$, and recalling that $d/r \geq 1$. □

End of the proof of Proposition 2.8.2. Let us assume by contradiction that there exists a sequence $\{(X_n, r_n)\}_{n \in \mathbb{N}}$ such that $X_n = (x_n, y_n) \in K \cap \{y \geq 0\}$, $r_n < \mathrm{dist}(K, \partial B)$, and

$$\Phi(X_n, r_n) \to +\infty, \qquad \text{as } n \to \infty.$$

It is immediate to prove that $r_n \to 0$ and $y_n \to 0$: indeed, $\mathbf{v}$ is H^1 and harmonic for $\{y > 0\}$. In particular, the sequence $\{X_n\}_{n \in \mathbb{N}}$ accumulates at $\partial^0 K$. First we observe that, thanks to the subharmonicity of $\sum_i |\nabla v_i|^2$, if $r_n < y_n$ then

$$\Phi(X_n, y_n) \geq \frac{y_n}{r_n} \Phi(X_n, r_n) \geq \Phi(X_n, r_n);$$

as a consequence we can assume without loss of generality that $r_n \geq y_n$. Analogously, once $r_n \geq y_n$, we have that

$$\Phi((x_n, 0), 2r_n) \geq \frac{1}{2^N} \Phi(X_n, r_n),$$

and again, without loss of generality, we can assume that $y_n = 0$ for every n, and drop it from our notation.

Now, by the result of Lemma 2.8.10, the sequence (x_n, r_n) can be replaced by a sequence of points in $\mathcal{Z}$. Indeed, if $\mathrm{dist}(x_n, \mathcal{Z}) > \bar{r}$ for every $n \in \mathbb{N}$, then

$$\Phi(x_n, r_n) \leq C\Phi(x_n, \bar{r})$$

and the right hand side is bounded since $\mathbf{v} \in H^1(B^+)$. Consequently, it must be $\mathrm{dist}(x_n, \mathcal{Z}) \leq \bar{r}$, and then

$$\Phi(x_n, r_n) \leq C\Phi(x_n, \mathrm{dist}(x_n, \mathcal{Z})).$$

Since the set $\mathcal{Z}$ is locally closed and $\mathrm{dist}(K, \partial^+ B) > 0$, for n sufficiently large, to each x_n we can associate $x_n' \in \mathcal{Z}$ such that $\mathrm{dist}(x_n, \mathcal{Z}) = |x_n - x_n'| \leq \frac{1}{2}\mathrm{dist}(x_n, \partial^+ B)$ and we can substitute the sequence $(x_n, \mathrm{dist}(x_n, \mathcal{Z}))$ with $(x_n', 2\mathrm{dist}(x_n, \mathcal{Z}))$. We are in position to apply Lemma 2.8.9 and find a contradiction to the unboundedness of the Morrey quotient. $\square$

Chapter 3

Variational competition: the case of the s-laplacian

Outline of the chapter

The second chapter is devoted to a partial extension of the results found in the previous one for the case $s = 1/2$, to the general case $s \in (0, 1)$. Though at a first thought this generalization may seem direct and straightforward, new interesting phenomena (mostly in the case $s > 1/2$) appear and have to be dealt with.

3.1 Monotonicity formulæ

This section is devoted to the introduction of some monotonicity formulæ, which will provide suitable estimates in order to prove some Liouville type results. Our first aim is to prove monotonicity formulæ of Alt-Caffarelli-Friedman type for the one phase problem: these will imply non existence results for L_a-harmonic functions under different assumptions on their growth at infinity and on the geometry of their null set.

Secondly, we will concentrate on systems of degenerate elliptic equations, providing monotonicity formulæ of Alt-Caffarelli-Friedman type with two phases, and of Almgren type.

3.1.1 One phase Alt-Caffarelli-Friedman formulæ

We first deal with single L_a-harmonic functions (on $\mathbb{R}^{N+1}_+$) which vanish on the whole $\mathbb{R}^N$.

Proposition 3.1.1. *Let $v \in H^{1;a}(B^+_R)$ be a continuous function such that*

- *$v(x,0)=0$ for $x \in \mathbb{R}^N$;*
- *for every non negative $\phi \in \mathcal{C}_0^\infty(B_R)$,*

$$\int_{\mathbb{R}^{N+1}_+} (L_a v) v\phi \, \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} (\partial^a_\nu v) v\phi \, \mathrm{d}x = \int_{\mathbb{R}^{N+1}_+} y^a \nabla v \cdot \nabla(v\phi) \, \mathrm{d}x\mathrm{d}y \leq 0.$$

Then the function

$$\Phi(r) := \frac{1}{r^{4s}} \int_{B_r^+} y^a \frac{|\nabla v|^2}{|X|^{N-2s}} \, \mathrm{d}x\mathrm{d}y$$

is monotone non decreasing in r for $r \in (0,R)$.

Remark 3.1.2. Since

$$\int_{\mathbb{R}^{N+1}_+} y^a \nabla v \cdot \nabla(v\phi) \, \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{N+1}_+} y^a \left[|\nabla v|^2 \phi + \frac{1}{2} \nabla v^2 \cdot \nabla \phi \right] \mathrm{d}x\mathrm{d}y, \tag{3.1.1}$$

we have that if v satisfies the assumptions of Proposition 3.1.1 then also $|v|$ does.

Definition 3.1.3. We define $\Gamma_1^s \in \mathcal{C}^1(\mathbb{R}^{N+1}_+; \mathbb{R}^+)$ as

$$\Gamma_1^s(X) := \begin{cases} \frac{1}{|X|^{N-2s}} & |X| \geq 1 \\ \frac{N+2(1-s)}{2} - \frac{N-2s}{2}|X|^2 & |X| < 1. \end{cases}$$

We let also $\Gamma_\varepsilon^s(X) = {\Gamma_1}^s(X/\varepsilon)\varepsilon^{2s-N}$, so that $\Gamma_\varepsilon^s \nearrow \Gamma^s = |X|^{2s-N}$, a multiple of the fundamental solution of the s-laplacian, as $\varepsilon \to 0$.

Remark 3.1.4. We observe that each Γ_ε^s is radial and, in particular, $\partial^a_\nu \Gamma^s_\varepsilon = 0$ on $\mathbb{R}^N$. Moreover, if $N-2s>0$, they are L_a-superharmonic on $\mathbb{R}^{N+1}_+$. On the other hand, in the case $N<2s$, that is $N=1$ and $s>1/2$, the fundamental solution is already $H^{1;a}_{\mathrm{loc}}(\mathbb{R}^2_+)$, thus there is no need to regularize it. However, in this case the monotonicity formulæ are hard to prove and they hold true under the additional assumption that the functions involved are a priori known to be homogeneous: indeed, it follows that the lowest possible degree of homogeneity γ for non trivial functions that satisfy the assumptions of monotonicity formulæ is $\gamma \geq 2s-1$ (see the proof of Proposition 3.1.7), and thus the estimates following from equation (3.1.3) ahead are valid. This additional assumption is true in our setting, thanks to the validity of an Almgren monotonicity formula (see Corollary 3.1.12). In addition to this possible way out, we suggest a simpler strategy: given a configuration $v_1, \cdots, v_k \in H^{1;a}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+})$ which falls into the assumptions of any of the monotonicity formulæ of this section, we can construct a new configuration $\tilde{v}_1, \cdots, \tilde{v}_k \in H^{1;a}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+2}_+})$, by extension in a constant way along the new direction. At this point, we can the apply to the new configuration the monotonicity formulæ proved in the case $N-2s>0$.

The proof of Proposition 3.1.1 is based on the following calculation. Incidentally, we observe that also the following monotonicity results rest on a similar argument.

Lemma 3.1.5. *Let v be as in Proposition 3.1.1. The function*

$$r \mapsto \int_{B_r^+} y^a \frac{|\nabla v|^2}{|X|^{N-2s}} \, \mathrm{d}x\mathrm{d}y$$

is well defined and bounded in any compact subset of $(0,1)$.

Proof. We proceed as follows: let $\varepsilon > 0$, $\delta > 0$ and let $\eta_\delta \in \mathcal{C}_0^\infty(B_{r+\delta})$ be a smooth, radial cutoff function such that $0 \le \eta_\delta \le 1$ and $\eta_\delta = 1$ on B_r. Choosing $\phi = \eta_\delta \Gamma_\varepsilon^s$ in the second assumption of Proposition 3.1.1, and recalling equation (3.1.1), we obtain

$$\begin{aligned}\int_{\mathbb{R}_+^{N+1}} y^a \left[|\nabla v|^2 \Gamma_\varepsilon^s + \frac{1}{2} \nabla v^2 \cdot \nabla \Gamma_\varepsilon^s \right] \eta_\delta \mathrm{d}x\mathrm{d}y &\le - \int_{\mathbb{R}_+^{N+1}} \frac{1}{2} y^a \Gamma_\varepsilon^s \nabla v^2 \cdot \nabla \eta_\delta \mathrm{d}x\mathrm{d}y \\ &= \int_r^{r+\delta} \left[-\eta_\delta'(\rho) \int_{\partial^+ B_\rho^+} y^a \Gamma_\varepsilon^s v \nabla v \cdot \frac{X}{|X|} \mathrm{d}\sigma \right] \mathrm{d}\rho.\end{aligned}$$

Passing to the limit as $\delta \to 0$ we obtain, for almost every $r \in (0,1)$,

$$\int_{B_r^+} y^a \left[|\nabla v|^2 \Gamma_\varepsilon^s + \frac{1}{2} \nabla (v)^2 \cdot \nabla \Gamma_\varepsilon^s \right] \mathrm{d}x\mathrm{d}y \le \int_{\partial^+ B_r^+} y^a \Gamma_\varepsilon^s v \partial_\nu v \mathrm{d}\sigma,$$

which, combined with the inequality $L_a \Gamma_\varepsilon^s \ge 0$ tested with $v^2/2$ leads to

$$\int_{B_r^+} y^a |\nabla v|^2 \Gamma_\varepsilon^s \, \mathrm{d}x\mathrm{d}y \le \int_{\partial^+ B_r^+} y^a \left(\Gamma_\varepsilon^s v \partial_\nu v - \frac{v^2}{2} \partial_\nu \Gamma_\varepsilon^s \right) \mathrm{d}\sigma.$$

Letting $\varepsilon \to 0^+$, by monotone convergence we infer

$$\int_{B_r^+} y^a \frac{|\nabla v|^2}{|X|^{N-2s}} \, \mathrm{d}x\mathrm{d}y \le \frac{1}{r^{N-2s}} \int_{\partial^+ B_r^+} y^a v \frac{\partial v}{\partial \nu} \, \mathrm{d}\sigma + \frac{N-2s}{2r^{N+1-2s}} \int_{\partial^+ B_r^+} y^a v^2 \, \mathrm{d}\sigma \tag{3.1.2}$$

and this, in turns, proves the lemma. □

Proof of Proposition 3.1.1. By Remark 3.1.1 we can assume, without loss of generality, that v is (non trivial and) non negative, and that $R = 1$. We start observing that the function $\Phi(r)$ is positive and absolutely continuous for $r \in (0,1)$. Therefore, the proposition follows once we prove that $\Phi'(r) \ge 0$ for almost every $r \in (0,1)$. A direct computation of the logarithmic derivative of Φ shows that

$$\frac{\Phi'(r)}{\Phi(r)} = -\frac{4s}{r} + \frac{\int_{\partial^+ B_r^+} y^a |\nabla v|^2 / |X|^{N-2s} \, \mathrm{d}\sigma}{\int_{B_r^+} y^a |\nabla v|^2 / |X|^{N-2s} \, \mathrm{d}x\mathrm{d}y}.$$

First we use the estimate (3.1.2) to bound from below the left hand side:

$$\frac{\int\limits_{\partial^+ B_r^+} y^a|\nabla v|^2/|X|^{N-2s}\,\mathrm{d}\sigma}{\int\limits_{B_r^+} y^a|\nabla v|^2/|X|^{N-2s}\,\mathrm{d}x\mathrm{d}y} \geq \frac{\int\limits_{\partial^+ B_r^+} y^a|\nabla v|^2\,\mathrm{d}\sigma}{\int\limits_{\partial^+ B_r^+} v y^a \partial_\nu v\,\mathrm{d}\sigma + (N-2s)\frac{r}{2}\int\limits_{\partial^+ B_r^+} y^a v^2\,\mathrm{d}\sigma}$$
$$= \frac{1}{r}\frac{\int\limits_{\mathbb{S}^N_+} \xi_{N+1}^a|\nabla v^{(r)}|^2\,\mathrm{d}\sigma}{\int\limits_{\mathbb{S}^N_+} v^{(r)}\xi^a_{N+1}\partial_\nu v^{(r)}\,\mathrm{d}\sigma + \frac{N-2s}{2}\int\limits_{\mathbb{S}^N_+}\xi^a_{N+1}(v^{(r)})^2\,\mathrm{d}\sigma},$$

where $v^{(r)}\colon \mathbb{S}^{N-1}_+ \to \mathbb{R}$ is defined as $v^{(r)}(\xi) = v(r\xi)$, so that $y = r\xi_{N+1}$. We now estimate the right hand side as follows: the numerator writes

$$\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|\nabla v^{(r)}|^2\,\mathrm{d}\sigma = \int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|\partial_\nu v^{(r)}|^2\,\mathrm{d}\sigma + \int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|\nabla_T v^{(r)}|^2\,\mathrm{d}\sigma$$
$$= \int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|v^{(r)}|^2\,\mathrm{d}\sigma\left(\underbrace{\frac{\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|\partial_\nu v^{(r)}|^2\,\mathrm{d}\sigma}{\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|v^{(r)}|^2\,\mathrm{d}\sigma}}_{t^2} + \underbrace{\frac{\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|\nabla_T v^{(r)}|^2\,\mathrm{d}\sigma}{\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|v^{(r)}|^2\,\mathrm{d}\sigma}}_{\mathcal{R}}\right).$$

where $\mathcal{R}$ stands for the Rayleigh quotient of $v^{(r)}$ on $\mathbb{S}^N_+$. On the other hand, by the Cauchy-Schwarz inequality, the denominator may be estimated from above by

$$\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}v^{(r)}\partial_\nu v^{(r)}\,\mathrm{d}\sigma + \frac{N-2s}{2}\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|v^{(r)}|^2\,\mathrm{d}\sigma$$
$$\leq \left(\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|v^{(r)}|^2\,\mathrm{d}\sigma\right)^{1/2}\left(\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}(\partial_\nu v^{(r)})^2\,\mathrm{d}\sigma\right)^{1/2} + \frac{N-2s}{2}\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|v^{(r)}|^2\,\mathrm{d}\sigma$$
$$\leq \int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|v^{(r)}|^2\,\mathrm{d}\sigma\left[\underbrace{\left(\frac{\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|\partial_\nu v^{(r)}|^2\,\mathrm{d}\sigma}{\int\limits_{\mathbb{S}^N_+} \xi^a_{N+1}|v^{(r)}|^2\,\mathrm{d}\sigma}\right)^{1/2}}_{t} + \frac{N-2s}{2}\right].$$

As a consequence

$$\frac{\int\limits_{\partial^+ B_r^+} y^a|\nabla v|^2/|X|^{N-2s}\,\mathrm{d}\sigma}{\int\limits_{B_r^+} y^a|\nabla v|^2/|X|^{N-2s}\,\mathrm{d}x\mathrm{d}y} \geq \frac{1}{r}\min_{t\in\mathbb{R}^+}\frac{\mathcal{R}+t^2}{t+\frac{N-2s}{2}}. \tag{3.1.3}$$

A simple computation shows that the minimum is achieved when

$$t = \gamma(\mathcal{R}) = \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mathcal{R}} - \frac{N-2s}{2},$$

and it is equal to $2\gamma(\mathcal{R})$. Recalling the definition of $\lambda_1^s(\emptyset)$ (equation (3.1.4)) we obtain

$$\frac{\Phi'(r)}{\Phi(r)} + \frac{4s}{r} \geq \frac{2}{r}\gamma(\lambda_1^s(\emptyset))$$

and the proposition follows observing that $\lambda_1^s(\emptyset)$ is achieved by $v(x,y) = y^{2s}$, in such a way that

$$\gamma\left(\lambda_1^s\left(\emptyset\right)\right) = 2s. \qquad \square$$

Now we turn to functions which vanish only on a half space.

Proposition 3.1.6. *Let $v \in H^{1;a}(B_R^+)$ be a continuous function such that*

- $v(x,0) = 0$ *for* $x_1 \leq 0$;
- *for every non negative* $\phi \in \mathcal{C}_0^\infty(B_R)$,

$$\int_{\mathbb{R}^{N+1}_+} (L_a v)v\phi \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} (\partial_\nu^a v)v\phi \,\mathrm{d}x = \int_{\mathbb{R}^{N+1}_+} y^a \nabla v \cdot \nabla(v\phi)\,\mathrm{d}x\mathrm{d}y \leq 0.$$

Then the function

$$\Phi(r) := \frac{1}{r^{2s}} \int_{B_r^+} y^a \frac{|\nabla v|^2}{|X|^{N-2s}}\,\mathrm{d}x\mathrm{d}y$$

is monotone non decreasing in r for $r \in (0,R)$.

Proof. The proof follows the line of the one of Proposition 3.1.1, recalling that

$$v(x,y) = \left(\frac{\sqrt{x_1^2+y^2}+x_1}{2}\right)^s$$

achieves $\gamma(\lambda_1^s(\mathbb{S}^{N-1} \cap \{x_1 > 0\})) = s$ (see, for instance, [15, page 442]). $\square$

In the previous propositions, we considered functions vanishing on the whole $\mathbb{R}^N$, or on a half-space. Now, in great contrast with the case $s \leq 1/2$, it is known that, if $s > 1/2$, then also $(N-1)$-dimensional subsets may have positive capacity. This motivates the following formula, which is the analogous of the previous ones, for functions which vanish on subspaces of $\mathbb{R}^N$ of codimension 1.

Proposition 3.1.7. *Let $s > 1/2$ and let $v \in H^{1;a}(B_R^+)$ be a continuous function such that*

- *$v(x,0)=0$ for $x_1=0$;*
- *for every non negative $\phi \in \mathcal{C}_0^\infty(B_R)$,*

$$\int_{\mathbb{R}^{N+1}_+} (L_a v) v\phi \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} (\partial^a_\nu v) v\phi \,\mathrm{d}x = \int_{\mathbb{R}^{N+1}_+} y^a \nabla v \cdot \nabla(v\phi)\,\mathrm{d}x\mathrm{d}y \leq 0.$$

Then the function

$$\Phi(r) := \frac{1}{r^{4s-2}} \int_{B_r^+} y^a \frac{|\nabla v|^2}{|X|^{N-2s}}\,\mathrm{d}x\mathrm{d}y$$

is monotone non decreasing in r for $r \in (0,R)$.

Proof. Let $\bar\omega = \mathbb{S}^{N-1} \setminus \{x_1 = 0\}$, and let us consider the function

$$v(x,y) = |(x_1, 0, y)|^{2s-1},$$

that is the fundamental solution in dimension 1, extended in a constant way to the other directions. Then v is $(2s-1)$-homogeneous, positive and L_a-harmonic for $y>0$. We deduce that its restriction to $\partial^+ B_1^+ = \mathbb{S}^N_+$ is an eigenfunction associated to $\lambda_1^s(\bar\omega)$, so that

$$\gamma(\lambda_1^s(\bar\omega)) = 2s-1.$$

As a consequence, also in this case the proposition follows by reasoning as in the proof of Proposition 3.1.1. □

3.1.2 Two phases Alt-Caffarelli-Friedman monotonicity formulæ

Now we turn to the multi-component ACF formulæ. We start with some definitions which extend to the case $s \in (0,1)$ those contained in Section 2.1. More precisely, let $\mathbb{S}^N_+ := \partial^+ B^+$. For each open $\omega \subset \mathbb{S}^{N-1} := \partial \mathbb{S}^N_+$ we define the first s-eigenvalue associated to ω as

$$\lambda_1^s(\omega) := \inf\left\{ \frac{\int_{\mathbb{S}^N_+} y^a |\nabla_T u|^2 \,\mathrm{d}\sigma}{\int_{\mathbb{S}^N_+} y^a u^2 \,\mathrm{d}\sigma} : u \in H^{1;a}(\mathbb{S}^N_+),\ u \equiv 0 \text{ on } \mathbb{S}^{N-1} \setminus \omega \right\}, \tag{3.1.4}$$

where $\nabla_T u$ is the tangential gradient of u on $\mathbb{S}^N_+$. The minimal rate of growth for multi-phase segregation profiles is given by the number

$$\nu^{\mathrm{ACF}} := \inf\left\{ \frac{\gamma(\lambda_1^s(\omega_1)) + \gamma(\lambda_1^s(\omega_2))}{2} : \omega_1 \cap \omega_2 = \emptyset \right\}, \tag{3.1.5}$$

where, as usual,

$$\gamma(t) := \sqrt{\left(\frac{N-2s}{2}\right)^2 + t} - \frac{N-2s}{2}$$

is defined in such a way that u achieves $\lambda_1^s(\omega)$ if and only if it is one signed, and its $\gamma(\lambda_1^s(\omega))$-homogeneous extension to $\mathbb{R}^{N+1}_+$ is L_a-harmonic. As a peculiar difference with respect to the case $s=1$, we recall that the eigenfunctions achieving ν^{ACF} have not disjoint support on the whole $\mathbb{S}^N_+$, but only on its boundary $\mathbb{S}^{N-1}$. In particular, the degenerate partition $(\emptyset, \mathbb{S}^{N-1})$ is admissible, and one can show that it has the same level than the equatorial cut one:

$$\frac{\gamma(\lambda_1^s(\emptyset)) + \gamma(\lambda_1^s(\mathbb{S}^{N-1}))}{2} = \frac{\gamma(\lambda_1^s(\mathbb{S}^{N-1}_+)) + \gamma(\lambda_1^s(\mathbb{S}^{N-1}_-))}{2} = s.$$

We start by proving that the constant ν^{ACF} defined in equation (3.1.5) is not 0.

Lemma 3.1.8. *For any $N \geq 2$, $0 < \nu^{\mathrm{ACF}} \leq s$.*

Proof. The bound from above easily follows by comparing with the value corresponding to the partition $(\mathbb{S}^{N-1}, \emptyset)$: indeed, it holds $\lambda_1^s(\mathbb{S}^{N-1}) = 0$, achieved by $u(x,y) \equiv 1$, and $\lambda_1^s(\emptyset) = 2sN$, achieved by $u(x,y) = y^{1-a}$. In order to prove the estimate from below, one can argue by contradiction, as in the proof of Lemma 2.1.5, exploiting the compactness both of the embedding $H^{1;a}(\mathbb{S}^N_+) \hookrightarrow L^{2;a}(\mathbb{S}^N_+)$ and of the trace operator from $H^{1;a}(\mathbb{S}^N_+)$ to $L^2(\mathbb{S}^{N-1})$. □

We will prove two multi-component formulæ, the first regarding entire profiles which are segregated on $\mathbb{R}^N$, the second regarding profiles which coexist on $\mathbb{R}^N$.

Proposition 3.1.9. *Let $v_1, v_2 \in H^{1;a}(B^+_R(x_0,0))$ be continuous functions such that*

- $v_1 v_2|_{\{y=0\}} = 0$, $v_i(x_0,0) = 0$;
- *for every non negative $\phi \in \mathcal{C}^\infty_0(B_R(x_0,0))$,*

$$\int_{\mathbb{R}^{N+1}_+} (L_a v_i) v_i \phi \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} (\partial^a_\nu v_i) v_i \phi \,\mathrm{d}x = \int_{\mathbb{R}^{N+1}_+} y^a \nabla v_i \cdot \nabla(v_i \phi) \,\mathrm{d}x\mathrm{d}y \leq 0.$$

Then the function

$$\Phi(r) := \prod_{i=1}^{2} \frac{1}{r^{2\nu^{\mathrm{ACF}}}} \int_{B^+_r(x_0,0)} y^a \frac{|\nabla v_i|^2}{|X|^{N-2s}} \,\mathrm{d}x\mathrm{d}y$$

is monotone non decreasing in r for $r \in (0,R)$.

Proof. Applying the same estimates developed for the proof of Proposition 3.1.1, it is easy to see that the proposition is equivalent to (summing equation (3.1.3) for the two functions)

$$\Phi'(r) \geq 0 \Leftrightarrow \sum_{i=1}^{2} \frac{\displaystyle\int\limits_{\partial^+ B_r^+} y^a \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}\sigma}{\displaystyle\int\limits_{B_r^+} y^a \frac{|\nabla v_i|^2}{|X|^{N-1}} \,\mathrm{d}x\mathrm{d}y} \geq \frac{2}{r} \inf_{(\omega_1,\omega_2)\in\mathcal{P}^2} \sum_{i=1}^{2} \gamma\left(\lambda_1^s(\omega_i)\right) = \frac{4}{r}\nu^{\mathrm{ACF}}$$

In particular, the last inequality follows by the definition of ν^{ACF}. ☐

Proposition 3.1.10. *Let $v_1, v_2 \in H^{1;a}_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ be continuous functions such that, for every non negative $\phi \in \mathcal{C}_0^\infty\left(\overline{\mathbb{R}^{N+1}_+}\right)$ and $j \neq i$,*

$$\begin{aligned}\int\limits_{\mathbb{R}^{N+1}_+} (L_a v_i) v_i \phi \,\mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}^N} (\partial^a_\nu v_i + a_{ij} v_i v_j^2) v_i \phi \,\mathrm{d}x \\ = \int\limits_{\mathbb{R}^{N+1}_+} y^a \nabla v_i \cdot \nabla(v_i\phi) \,\mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}^N} a_{ij} v_i^2 v_j^2 \phi \,\mathrm{d}x \leq 0.\end{aligned}$$

For any $\nu' \in (0, \nu^{\mathrm{ACF}})$ there exists $\bar{r} > 1$ such that the function

$$\Phi(r) := \prod_{i=1}^{2} \Phi_i(r)$$

is monotone non decreasing in r for $r \in (\bar{r}, \infty)$, where

$$\Phi_i(r) := \frac{1}{r^{2\nu'}} \left(\int\limits_{B_r^+} y^a |\nabla v_i|^2 \Gamma_1 \,\mathrm{d}x\mathrm{d}y + \int\limits_{\partial^0 B_r^+} a_{ij} v_i^2 v_j^2 \Gamma_1 \,\mathrm{d}x \right), \qquad \textit{for } j \neq i.$$

The proof of Proposition 3.1.10 is based on a contradiction argument, and follows the lines of the one of Proposition 3.1.9. We do not report the details, referring the reader to [39, Lemma 2.5] and Theorem 2.1.13, where similar computations were developed for the case $s = 1$ and $s = 1/2$, respectively.

3.1.3 Almgren type monotonicity formula

To conclude this section on monotonicity formulæ, we focus our attention on an Almgren quotient defined for a suitable class a functions: these will come into play as limits of a blow up sequence. First, for any

$$\mathbf{v} \in H^{1;a}_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right) := \{v : \forall D \subset \mathbb{R}^{N+1} \text{ open and bounded}, v|_{D^+} \in H^{1;a}(D^+)\},$$

$\mathbf{v} = (v_1, \ldots, v_k)$ continuous, let use define

$$E(x_0, r) := \frac{1}{r^{N-2s}} \int\limits_{B_r^+(x_0,0)} y^a \sum_i |\nabla v_i|^2 \, \mathrm{d}x\mathrm{d}y,$$
$$H(x_0, r) := \frac{1}{r^{N+1-2s}} \int\limits_{\partial^+ B_r^+(x_0,0)} y^a \sum_i v_i^2 \, \mathrm{d}\sigma,$$

where $x_0 \in \mathbb{R}^N$ and $r > 0$. By assumption, both E and H are locally absolutely continuous functions on $(0, +\infty)$, that is, both E' and H' are $L^1_{\mathrm{loc}}(0, \infty)$ (here, $' = \mathrm{d}/\mathrm{d}r$). Let us also consider the function (Almgren frequency function)

$$N(x_0, r) := \frac{E(x_0, r)}{H(x_0, r)}.$$

We have the following result, which proof we omit since it follows with minor changes from the one of Theorem 2.2.3.

Proposition 3.1.11. *Let* $\mathbf{v} \in H^{1;a}_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}; \mathbb{R}^k\right)$, $\mathbf{v} = (v_1, \ldots, v_k)$ *continuous, and let us assume that:*

1. $v_i v_j|_{y=0} = 0$ *for every* $j \neq i$;

2. *for every* i,
$$\begin{cases} L_a v_i = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v_i \partial^a_\nu v_i = 0 & \text{on } \mathbb{R}^N \times \{0\}; \end{cases}$$

3. *for any* $x_0 \in \mathbb{R}^N$ *and a.e.* $r > 0$, *the following (Pohozaev type) identity holds*
$$(2s-N) \int\limits_{B_r^+} y^a \sum_i |\nabla v_i|^2 \, \mathrm{d}x\mathrm{d}y + r \int\limits_{\partial^+ B_r^+} y^a \sum_i |\nabla v_i|^2 \, \mathrm{d}\sigma = 2r \int\limits_{\partial^+ B_r^+} y^a \sum_i |\partial_\nu v_i|^2 \, \mathrm{d}\sigma.$$

Then for every $x_0 \in \mathbb{R}^N$ *the Almgren frequency function* $N(x_0, r)$ *is well defined on* $(0, \infty)$, *absolutely continuous, non decreasing, and it satisfies the identity*

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(r) = \frac{2N(r)}{r}.$$

Moreover, if $N(r) \equiv \gamma$ *on an open interval, then* $N \equiv \gamma$ *for every* r, *and* $\mathbf{v}$ *is a homogeneous function of degree* γ.

Of the many consequences that the validity of an Almgren monotonicity formula carries, at this stage we are mostly interested in the following, which states a rigidity property implied by Hölder continuity.

Corollary 3.1.12. *If* $\mathbf{v}$ *satisfies the assumptions of Proposition 3.1.11 and is globally Hölder continuous of exponent* γ *on* $\mathbb{R}^{N+1}_+$*, then it is homogeneous of degree* γ *with respect to any of its (possible) zeroes, and thus*

$$\mathcal{Z} := \{x \in \mathbb{R}^N : \mathbf{v}(x,0) = 0\} \quad \textit{is an affine subspace of } \mathbb{R}^N.$$

Proof. The proof relies on the fact that the Almgren centered at any point of $\mathcal{Z}$ has to be constant and equal to γ. Indeed letting $x_0 \in \mathcal{Z}$, we argue by contradiction and suppose that $N(x_0, R) > \gamma$ for some R. By monotonicity of N we have

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(r) \geq \frac{2}{r} N(x_0, R) \quad \forall r \geq R$$

and, integrating in (R, r), we find

$$Cr^{2N(x_0,R)} \leq H(r) \leq Cr^{2\gamma},$$

a contradiction for r large enough. The same reasoning provides a contradiction in the case $N(x_0, R) < \gamma$ and $r \leq R$. □

3.2 Liouville type results

Relying on the previous monotonicity formulæ, in this section we will prove some Liouville type theorems for solution to either equations or systems involving the operator L_a. As a first result, we have the following.

Proposition 3.2.1. *Let* $v \in H^{1;a}_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ *be continuous and satisfy*

$$\begin{cases} L_a v = 0 & \textit{in } \mathbb{R}^{N+1}_+ \\ v(x,0) = 0 & \textit{on } \mathbb{R}^N, \end{cases}$$

and let us suppose that for some $\gamma \in [0, 2s)$*,* $C > 0$ *it holds*

$$|v(X)| \leq C(1 + |X|^\gamma)$$

for every X*. Then* v *is identically zero.*

Proof. We remark that v satisfies the assumptions of Proposition 3.1.1 for any R. For $r > 0$ large enough, we choose η non negative, smooth and radial cut-off function supported in B^+_{2r} with $\eta = 1$ in B^+_r such that

$$\int_{\mathbb{R}^{N+1}_+} y^a |\nabla \eta| \leq Cr^{N+1-2s}, \quad \int_{\mathbb{R}^{N+1}_+} |L_a \eta| \leq Cr^{N-2s}$$

(for instance, we can take η as a smooth approximation of the function $\frac{1}{r}(2r - |X|)$ in $B_{2r} \setminus B_r$). Moreover, let Γ_1^s be defined as in Definition 3.1.3 (in particular, it is radial and superharmonic). Testing the equation for v with $\Gamma_1^s v\eta$ we obtain

$$\int\limits_{B_{2r}^+} y^a |\nabla v|^2 \Gamma_1^s \eta \mathrm{d}x\mathrm{d}y \leq \int\limits_{B_{2r}^+ \setminus B_r^+} \frac{1}{2} v^2 \left[-L_a \eta \Gamma_1^s + 2y^a \nabla \eta \cdot \nabla \Gamma_1^s \right] \mathrm{d}x\mathrm{d}y,$$

where we used that η is constant in B_r^+. Since $\Gamma_1^s(X) = |X|^{2s-N}$ outside B_1, and $|v(X)| \leq Cr^\gamma$ outside a suitable $B_{\bar{r}}$, using the notations of Proposition 3.1.1 we infer

$$\Phi(r) = \frac{1}{r^{4s}} \left(\int\limits_{B_r^+} y^a |\nabla v|^2 \Gamma_1^s \, \mathrm{d}x\mathrm{d}y \right) \leq \frac{1}{r^{4s}} \cdot Cr^{2\gamma},$$

with C independent of $r > \bar{r}$. Due to the monotonicity of Φ, we then find

$$0 \leq \Phi(\bar{r}) \leq Cr^{2(\gamma - 2s)}.$$

for every $r > \bar{r}$ sufficiently large. This forces v to be constant. $\square$

The previous proposition allows to prove an analogous result of the classical Liouville Theorem, which holds for L_a-harmonic functions.

Proposition 3.2.2. *Let v be an entire L_a-harmonic function defined on $\mathbb{R}^{N+1}$. If there exists $\gamma < 1$ such that*

$$|v(X)| \leq C \left(1 + |X|^\gamma\right),$$

then $v|_{y=0}$ is constant. Moreover, if $\gamma < \min(2s, 1)$, then v is constant.

Proof. It is well known (see [15]) that L_a-harmonic functions enjoy the mean value property ($C > 0$)

$$v(x, 0) = \frac{C}{r^{N+a}} \int\limits_{\partial B_r(x,0)} |y|^a v \, \mathrm{d}\sigma$$

and, equivalently

$$v(x, 0) = \frac{C}{R^{N+1+a}} \int\limits_{B_R(x,0)} |y|^a v \, \mathrm{d}\sigma.$$

It follows, by the growth condition, that

$$\begin{aligned} |v(x', 0) - v(x'', 0)| &\leq \frac{C}{R^{N+1+a}} \int_{B_R(x',0) \triangle B_R(x'',0)} y^a |v(x, y)| \, \mathrm{d}x\mathrm{d}y \\ &\leq \frac{C}{R^{N+1+a}} \int_{B_R(x',0) \triangle B_R(x'',0)} y^a |X|^\gamma \, \mathrm{d}x\mathrm{d}y \leq CR^{\gamma - 1} \end{aligned}$$

and the first conclusion follows since $\gamma < 1$. Let us now assume $\gamma < \min(2s, 1)$: since $v|_{y=0}$ is constant, we can assume $v|_{y=0} \equiv 0$ and apply Proposition 3.2.1. $\square$

We can obtain the analogous of the classical Liouville Theorem for s-harmonic functions by applying the previous result to the even reflection through $\{y = 0\}$ of their L_a-harmonic extensions.

Corollary 3.2.3. *Let $v \in H^{1;a}_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ be continuous and satisfy*

$$\begin{cases} L_a v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu v(x,0) = 0 & \text{on } \mathbb{R}^N, \end{cases}$$

and let us suppose that for some $\gamma < \min(2s, 1)$, $C > 0$ it holds

$$|v(X)| \leq C(1 + |X|^\gamma)$$

for every X. Then v is constant.

By the way, a stronger result in the direction of the above corollary is contained in [15, Lemma 2.7].

In the same spirit of Proposition 3.2.1, we provide a result concerning L_a-harmonic functions which vanish on a half space of $\mathbb{R}^N$.

Proposition 3.2.4. *Let $v \in H^{1;a}_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ satisfy the assumptions of Proposition 3.1.6. Let us suppose that for some $\gamma \in [0, s)$, $C > 0$ it holds*

$$|v(X)| \leq C(1 + |X|^\gamma)$$

for every X. Then v is identically zero.

Proof. Again, v as above fulfills the assumptions of Proposition 3.1.6. Now, assuming that v is not constant, we can argue as in the proof of Proposition 3.2.9 obtaining a contradiction. □

We proceed with a lemma regarding the decay of subsolutions to a linear equation involving L_a.

Lemma 3.2.5. *Let $M > 0$ and $\delta > 0$ be fixed and let $h \in L^\infty(\partial^0 B_1^+)$ with $\|h\|_{L^\infty} \leq \delta$. Any $v \in H^{1;a}(B_1^+)$ non negative solution to*

$$\begin{cases} L_a v \leq 0 & \text{in } B_1^+ \\ \partial^a_\nu v \leq -Mv + h & \text{on } \partial^0 B_1^+ \end{cases}$$

verifies

$$\sup_{\partial^0 B^+_{1/2}} v \leq \frac{1+\delta}{M} \sup_{\partial^+ B_1^+} v.$$

The proof of Lemma 3.2.5 follows by a comparison argument. In order to construct an appropriate supersolution, we need a technical lemma. Let $f \in AC(\mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R})$ be defined as

$$f(x) = C \int_{-\infty}^{x} \frac{1}{(1+t^2)^{1-a/2}} \mathrm{d}t,$$

where C is such that $f(+\infty) = 1$.

Lemma 3.2.6. *There exists $c > 0$ such that*

$$(-\Delta)^s f(x) \geq -cf(x)$$

for any $x < 0$.

Proof. The function f under consideration is increasing, smooth and such that there exist $c, C > 0$ with

$$\lim_{|t|\to\infty} f'(t)|t|^{2-a} = C > 0 \quad \text{and} \quad \lim_{|t|\to\infty} f''(t)|t|^{3-a} = c.$$

The s-laplacian of the function f is well-defined. Thanks to the extension representation of the fractional laplacian, we can consider

$$\begin{aligned} v(x,y) &= \int_{\mathbb{R}} P_a(\xi, y) f(x-\xi) \mathrm{d}\xi = \int_{\mathbb{R}} y^{1-a} \frac{f(x-\xi)}{(\xi^2+y^2)^{1-a/2}} \mathrm{d}\xi \\ &= \{t = \xi/y\} = \int_{\mathbb{R}} \frac{f(x-ty)}{(1+t^2)^{1-a/2}} \mathrm{d}t \end{aligned}$$

so that

$$\begin{aligned} \partial_\nu^a v(x,0) &= \lim_{y\to0^+} -y^a \frac{\partial}{\partial y} \int \frac{f(x-ty)}{(1+t^2)^{1-a/2}} \mathrm{d}t = \lim_{y\to0^+} \int y^a t \frac{f'(x-ty)}{(1+t^2)^{1-a/2}} \mathrm{d}t \\ &= \{r = yt\} = \lim_{y\to0^+} \int \frac{r}{(y^2+r^2)^{1-a/2}} f'(x-r) \mathrm{d}r \\ &= \mathrm{pv} \int \frac{|r|^a}{r} f'(x-r) \mathrm{d}r = \mathrm{pv} \int \frac{|x-r|^a}{x-r} f'(r) \mathrm{d}r. \end{aligned}$$

Let us observe that, due to the decay properties of f' at infinity, the last principal value acts only around the singularity $x = r$, that is

$$(-\Delta)^s f(x) = \lim_{\varepsilon\to0^+} \int_{\mathbb{R}\setminus(r-\varepsilon, r+\varepsilon)} \frac{|x-r|^a}{x-r} f'(r) \mathrm{d}r.$$

We aim at proving that there exists a positive $c > 0$ such that the estimate

$$(-\Delta)^s f(x) \geq -cf(x)$$

holds for every $x \leq 0$. As a first step, we are going to estimate the asymptotic behavior of the right hand side as $x \to -\infty$. To this end, letting $K > 0$ be a fixed number, we write

$$(-\Delta)^s f(x) = \mathrm{pv} \int_{-\infty}^{-K} \frac{|x-r|^a}{x-r} f'(r) \mathrm{d}r + \int_{-K}^{\infty} \frac{|x-r|^a}{x-r} f'(r) \mathrm{d}r \tag{3.2.1}$$

(this decomposition is possible thanks to the prescribed decay of f'). We estimate the two contributions separately. First ($a<1$)

$$\int_{-K}^{\infty} \frac{|x-r|^a}{x-r} f'(r)\mathrm{d}r \geq -(-K-x)^{a-1} \int_{-K}^{\infty} f'(r)\mathrm{d}r \geq -C|x|^{a-1}.$$

We further decompose the second integral in (3.2.1), to find

$$\begin{aligned}\mathrm{pv}\int_{-\infty}^{-K} \frac{|x-r|^a}{x-r} f'(r)\mathrm{d}r = \{t=r/|x|\} &= -|x|^a\,\mathrm{pv}\int_{-\infty}^{-K/|x|} \frac{|1+t|^a}{1+t} f'(t|x|)\mathrm{d}t \\ &= -|x|^a\left[\int_{-\infty}^{-3/2} \dots\,\mathrm{d}t + \mathrm{pv}\int_{-3/2}^{-1/2} \dots\,\mathrm{d}t + \int_{-1/2}^{-K/|x|} \dots\,\mathrm{d}t\right]\end{aligned}$$

In the first part we use the estimate

$$f'(t|x|) \geq c|t|^{a-2}|x|^{a-2}$$

in order to obtain

$$-|x|^a \int_{-\infty}^{-3/2} \frac{|1+t|^a}{1+t} f'(t|x|)\mathrm{d}t \geq -c|x|^{2a-2} \int_{-\infty}^{-3/2} \frac{|1+t|^a}{1+t}|t|^{a-2}\mathrm{d}t \geq -C|x|^{2a-2}.$$

In the principal value we write

$$-|x|^a\,\mathrm{pv}\int_{-3/2}^{-1/2} \frac{|1+t|^a}{1+t} f'(t|x|)\mathrm{d}t = -|x|^{2a-2}\,\mathrm{pv}\int_{-3/2}^{-1/2} \frac{|1+t|^a}{1+t} f'(t|x|)|x|^{2-a}\mathrm{d}t.$$

Since

$$f'(t|x|)|x|^{2-a} \to C|t|^{a-2} \quad \text{in } \mathcal{C}^1\left(-\frac{3}{2}, -\frac{1}{2}\right) \text{ as } |x| \to \infty$$

and

$$\mathrm{pv}\int_{-3/2}^{-1/2} \frac{|1+t|^a}{1+t}|t|^{a-2}\mathrm{d}t = \{r=-1-t\} = \mathrm{pv}\int_{-1/2}^{1/2} -\frac{|r|^a}{r}(r+1)^{a-2}\mathrm{d}r > 0,$$

we obtain the lower bound

$$-|x|^a\,\mathrm{pv}\int_{-3/2}^{-1/2} \frac{|1+t|^a}{1+t} f'(t|x|)\mathrm{d}t \geq -C|x|^{2a-2}.$$

To estimate the last integral we use

$$f'(t|x|) \leq C|t|^{a-2}|x|^{a-2}$$

to obtain

$$\begin{aligned}-|x|^a \int_{-1/2}^{-K/|x|} \frac{|1+t|^a}{1+t} f'(t|x|)\mathrm{d}t &\geq -C|x|^{2a-2} \int_{-1/2}^{-K/|x|} \frac{|1+t|^a}{1+t}|t|^{a-2}\mathrm{d}t \\ &\geq -C|x|^{2a-2}\left(1+\frac{1}{|x|^{a-1}}\right) \geq -C|x|^{a-1}.\end{aligned}$$

As a consequence

$$(-\Delta)^s f(x) \geq -C\left(|x|^{a-1} + |x|^{2a-2}\right) \geq -C|x|^{a-1}.$$

On the other hand, by a direct estimate we have ($x \ll 0$)

$$f(x) \leq C\frac{1}{|x|^{1-a}}$$

which immediately yields that for $x \ll 0$ there exists $c > 0$ such that

$$(-\Delta)^s f(x) \geq -cf(x).$$

Due to the positivity and regularity of f, this estimates extends to every $x \leq 0$. □

We can conclude with the proof of Lemma 3.2.5.

Proof of Lemma 3.2.5. Let us first consider, for $M > 0$, the scaling $x \mapsto M^{1/2s}x$ and let us introduce the function $f_M(x) := f(M^{1/2s}x)$. It follows that

$$(-\Delta)^s f_M(x) = M^{2s/2s}\left[(-\Delta)^s f\right](M^{1/2s}x) \geq -cMf_M(x)$$

It is then clear that if we let

$$g_M(x) := f_M(t-1) + f_M(-t-1)$$

then for any $M > 0$ it holds

$$\begin{cases} (-\Delta)^s g_M(x) \geq -cMg_M(x) & \text{in } (-1,1) \\ g_M(x) \geq \frac{1}{2} & \text{in } \mathbb{R}\setminus(-1,1) \\ g_M(x) \leq CM^{-1} & \text{in } \left(-\frac{1}{2},\frac{1}{2}\right). \end{cases}$$

The proof follows by a comparison argument between v and the supersolution

$$w_\delta := \delta\frac{1}{M} + \int_{\mathbb{R}} P_a(\xi,y)g_M(x-\xi)\mathrm{d}\xi.$$ □

The previous estimate allows to prove the following.

Proposition 3.2.7. *Let v satisfy*

$$\begin{cases} L_a v = 0 & \textit{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu v = -\lambda v & \textit{on } \mathbb{R}^N \end{cases}$$

for some $\lambda > 0$ and let us suppose that for some $\gamma < \min(1,2s)$, $C > 0$ it holds

$$|v(X)| \leq C(1+|X|^\gamma)$$

for every X. Then v is constant.

Proof. Let either $z = v^+$ or $z = v^-$. In both cases,

$$\begin{cases} L_a z \le 0, & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu z \le -\lambda z, & \text{on } \mathbb{R}^N. \end{cases}$$

By translating and scaling, Lemma 3.2.5 implies that

$$z(x_0, 0) \le \sup_{\partial^0 B_{r/2}(x_0,0)} z \le \frac{1}{\lambda r^{2s}} \sup_{\partial^+ B_r(x_0,0)} z \le C \frac{1 + r^\gamma}{r^{2s}}.$$

Letting $r \to \infty$ the proposition follows. □

Proposition 3.2.8. *Let v satisfy*

$$\begin{cases} L_a v = 0 & \textit{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu v = \lambda & \textit{on } \mathbb{R}^N \end{cases}$$

for some $\lambda \in \mathbb{R}$ and let us suppose that for some $\gamma < \min(1, 2s)$, $C > 0$ it holds

$$|v(X)| \le C(1 + |X|^\gamma)$$

for every X. Then v is constant.

Proof. For $h \in \mathbb{R}^N$, let $w(x, y) := v(x + h, y) - v(x, y)$. Then w solves

$$\begin{cases} L_a w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu w = 0 & \text{on } \mathbb{R}^N \end{cases}$$

and, as usual, we can reflect and use the growth condition to infer that w has to be constant, that is $v(x+h, y) = c_h + v(x, y)$. Deriving the previous expression in x_i, we find that

$$v(x, y) = \sum_{i=1}^{k} c_i(y) x_i + c_0(y).$$

Using again the growth condition, we see that $c_i \equiv 0$ for $i = 1, \ldots, k$, while c_0 is constant. We observe that, consequently, $\lambda = 0$. □

Proposition 3.2.9. *Let $\mathbf{v} \in H^{1;a}_{\mathrm{loc}}\left(\overline{\mathbb{R}^{N+1}_+}\right)$ be continuous and satisfy*

$$\begin{cases} L_a v_i = 0 & \textit{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu v_i = -v_i \sum_{j \ne i} a_{ij} v_j^2 & \textit{on } \mathbb{R}^N, \end{cases}$$

and let ν^{ACF} be defined according to (3.1.5). If for some $\gamma \in (0, \nu^{\mathrm{ACF}})$ there exists C such that

$$|\mathbf{v}(X)| \le C\left(1 + |X|^\gamma\right),$$

for every X, then $k - 1$ components of $\mathbf{v}$ annihilate and the last is constant.

Proof. We only sketch the proof, referring to Proposition 2.3.1 for a detailed proof in the case $s = 1/2$. To start with, we observe that any pair of components of $\mathbf{v}$ satisfy the assumptions of Proposition 3.1.10; as a consequence, if $\mathbf{v}$ had two nontrivial components, then one could argue as in the proof of Proposition 3.2.1 in order to obtain a contradiction. Once we know that all but one component are trivial, we can conclude by applying Corollary 3.2.3 to the last one. □

Proposition 3.2.10. *Let* $\mathbf{v}$ *satisfy the assumptions of Proposition 3.1.11 and* $\gamma \in (0, \nu^{\mathrm{ACF}})$.

1. *If there exists* C *such that*

$$|\mathbf{v}(X)| \leq C\left(1 + |X|^{\gamma}\right),$$

for every X*, then* $k-1$ *components of* $\mathbf{v}$ *annihilate;*

2. *if furthermore* $\mathbf{v} \in \mathcal{C}^{0,\gamma}\left(\overline{\mathbb{R}^{N+1}_{+}}\right)$ *and*

$$\gamma < \begin{cases} \nu^{\mathrm{ACF}} & 0 < s \leq \dfrac{1}{2} \\ \min(\nu^{\mathrm{ACF}}, 2s-1) & \dfrac{1}{2} < s < 1, \end{cases}$$

then the only possibly nontrivial component is constant.

Proof. To prove 1., we can reason as in the proof of Proposition 3.2.9, using Proposition 3.1.9 instead of Proposition 3.1.10. Turning to 2., let v denote the only non trivial component. If $v(x,0) \neq 0$ for every x, then we deduce that $\partial^a_\nu v(x,0) \equiv 0$, and we can conclude by using Corollary 3.2.3. On the other hand, let

$$\mathcal{Z} = \{x \in \mathbb{R}^N : v(x,0) = 0\} \neq \emptyset.$$

By Corollary 3.1.12, we have that v is γ-homogeneous about any point of $\mathcal{Z}$, which is then an affine subspace of $\mathbb{R}^N$, and that v solves

$$\begin{cases} L_a v = 0 & \text{in } \mathbb{R}^{N+1}_{+} \\ v = 0 & \text{on } \mathcal{Z} \\ \partial^a_\nu v = 0 & \text{on } \mathbb{R}^N \setminus \mathcal{Z}. \end{cases} \tag{3.2.2}$$

Now, if $\mathcal{Z} = \mathbb{R}^N$, then Proposition 3.2.1 applies. On the other hand, if $\dim \mathcal{Z} \leq N - 2s$, we obtain that $\mathcal{Z}$ has null L_a-capacity (this can be seen directly for the fractional laplacian in $\mathbb{R}^N$, see for instance [37, Theorem 3.14]), and the conclusion follows by Proposition 3.2.2. Finally, we are left to deal with the case

$$\dim \mathcal{Z} = N - 1 \quad \text{and} \quad \frac{1}{2} < s < 1.$$

In this situation, the previous capacitary reasoning fails, see Remark 3.2.11 below. Nonetheless, assuming without loss of generality that $\mathcal{Z} = \{x \in \mathbb{R}^N : x_1 = 0\}$, we have that v satisfies the assumptions of Proposition 3.1.7. As a consequence, one can reason once again as in the proof of Proposition 3.2.1, obtaining a contradiction with the fact that $\gamma < 2s - 1$. $\square$

Remark 3.2.11. As we already mentioned in the introduction, in great contrast with the case $s \le 1/2$, if $s > 1/2$ the fundamental solution of the s-laplacian in $\mathbb{R}$ is bounded in a neighborhood of $x = 0$. As a consequence, the function $\Gamma(x, y) = |(x_1, y)|^{2s-1}$ solves (3.2.2). This implies that, for $s > 1/2$, the sets of codimension 1 in $\mathbb{R}^N$ have positive s-capacity.

3.3 $\mathcal{C}^{0,\alpha}$ uniform bounds

In this section we turn to the proof of some regularity results, analogous to those of Section 2.5. In the next session, this result will be used to obtain a sharper bound on the optimal regularity exponent $\alpha(N, s)$: these same steps can be used then to obtain the proof of Theorem 1.2.1. We recall that, here and in the following, the functions $f_{i,\beta}$ appearing in problem $(GP)^s_\beta$ are assumed to be continuous and uniformly bounded, with respect to β, on bounded sets. We start by recalling the regularity results which hold for β bounded. For easier notation, we write $B^+ = B^+_1$.

Lemma 3.3.1. *There exists $\alpha^* \in (0, 1)$ such that, for every $\alpha \in (0, \alpha^*)$, $\bar{m} > 0$ and $\bar{\beta} > 0$, there exists a constant $C = C(\alpha, \bar{m}, \bar{\beta})$ such that*

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}\left(B^+_{1/2}\right)} \le C,$$

for every $\mathbf{v}_\beta$ solution of problem $(GP)_\beta$ on B^+, satisfying

$$\beta \le \bar{\beta} \quad \text{and} \quad \|\mathbf{v}_\beta\|_{L^\infty(B^+)} \le \bar{m}.$$

Proof. The above regularity issue can be rephrased for a general $h \in H^{1;a}(B^+)$ with

$$\begin{cases} L_a h = 0 & \text{in } B^+ \\ h = f \in L^\infty & \text{on } \partial^+ B^+ \\ \partial^a_\nu h = g \in L^\infty & \text{on } \partial^0 B^+. \end{cases}$$

Denoting

$$\tilde{f}(x, y) := f(x, |y|) \quad \text{and} \quad \tilde{g}(x) = \begin{cases} g(x) & x \in \partial^0 B^+ \\ 0 & x \in \mathbb{R}^N \setminus \partial^0 B^+, \end{cases}$$

we can write $h = h_1 + h_2$, where

$$\begin{cases} L_a h_1 = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu h_1 = \tilde{g} & \text{on } \mathbb{R}^N \end{cases} \quad \text{and} \quad \begin{cases} L_a h_2 = 0 & \text{in } B \\ h_2 = \tilde{f} - h_1 & \text{on } \partial B. \end{cases}$$

But then the regularity of h_1 (depending on $\|\tilde{g}\|_{L^\infty}$) follows by [45, Proposition 2.9] [], while the one of h_2 is proved in [30] (see also [15, Section 2]). □

From now on, without loss of generality, we will fix $\alpha^* > 0$ in such a way that Lemma 3.3.1 holds, and furthermore

$$\alpha^* \leq \begin{cases} \nu^{\mathrm{ACF}} & 0 < s \leq \dfrac{1}{2} \\ \min(\nu^{\mathrm{ACF}}, 2s-1) & \dfrac{1}{2} < s < 1. \end{cases}$$

We will obtain Theorem 1.2.1 for any fixed $\alpha \in (0, \alpha^*)$. Following the outline of Section 2.5, we proceed by contradiction and develop a blow up analysis. Let η denote a smooth function such that

$$\begin{cases} \eta(X) = 1 & 0 \leq |X| \leq \frac{1}{2} \\ 0 < \eta(X) \leq 1 & \frac{1}{2} \leq |X| \leq 1 \\ \eta(X) = 0 & |X| = 1 \end{cases}$$

(in particular, η vanishes on $\partial^+ B^+$ but is strictly positive $\partial^0 B^+$). We will show that

$$\|\eta \mathbf{v}\|_{\mathcal{C}^{0,\alpha}(\overline{B^+})} \leq C,$$

and the theorem will follow by the definition of η. Let us assume by contradiction the existence of sequences $\{\beta_n\}_{n\in\mathbb{N}}$, $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$, solutions to $(GP)^s_{\beta_n}$, such that

$$L_n := \max_{i=1,\dots,k} \max_{X' \neq X'' \in \overline{B^+}} \frac{|(\eta v_{i,n})(X') - (\eta v_{i,n})(X'')|}{|X' - X''|^\alpha} \to \infty.$$

By Lemma 3.3.1 (and the regularity of η) we infer that $\beta_n \to \infty$. Moreover, up to a relabeling, we may assume that L_n is achieved by $i = 1$ and by two sequences of points $(X'_n, X''_n) \in \overline{B^+} \times \overline{B^+}$. The first properties of such sequences have already been obtained in Section 2.5.

Lemma 3.3.2 (Lemma (2.5.4))**.** *Let $X'_n \neq X''_n$ and $r_n := |X'_n - X''_n|$ satisfy*

$$L_n = \frac{|(\eta v_{1,n})(X'_n) - (\eta v_{1,n})(X''_n)|}{r_n^\alpha}.$$

Then, as $n \to \infty$,

1. *$r_n \to 0$;*

2. $\dfrac{\operatorname{dist}(X_n', \partial^+ B^+)}{r_n} \to \infty, \; \dfrac{\operatorname{dist}(X_n'', \partial^+ B^+)}{r_n} \to \infty.$

Our analysis is based on two different blow up sequences, one having uniformly bounded Hölder quotient, the other satisfying a suitable problem. Let $\{\hat{X}_n\}_{n\in\mathbb{N}} \subset \overline{B^+}$, $|\hat{X}_n| < 1$, be a sequence of points, to be chosen later. We write

$$\tau_n B^+ := \frac{B^+ - \hat{X}_n}{r_n},$$

remarking that $\tau_n B^+$ is a hemisphere, not necessarily centered on the hyperplane $\{y = 0\}$. We introduce the sequences

$$w_{i,n}(X) := \eta(\hat{X}_n)\frac{v_{i,n}(\hat{X}_n + r_n X)}{L_n r_n^\alpha} \quad \text{and} \quad \bar{w}_{i,n}(X) := \frac{(\eta v_{i,n})(\hat{X}_n + r_n X)}{L_n r_n^\alpha},$$

where $X \in \tau_n B^+$. With this choice, on one hand it follows immediately that, for every i and $X' \neq X'' \in \overline{\tau_n B^+}$,

$$\frac{|\bar{w}_{i,n}(X') - \bar{w}_{i,n}(X'')|}{|X' - X''|^\alpha} \leq \left| \bar{w}_{1,n}\left(\frac{X_n' - \hat{X}_n}{r_n}\right) - \bar{w}_{1,n}\left(\frac{X_n'' - \hat{X}_n}{r_n}\right)\right| = 1,$$

in such a way that the functions $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ share an uniform bound on Hölder semi-norm, and at least their first components are not constant. On the other hand, since $\eta(\hat{X}_n) > 0$, each $\mathbf{w}_n$ solves

$$\begin{cases} L_a^{\tau_n} w_{i,n} = 0 & \text{in } \tau_n B^+ \\ \partial_\nu^{a,\tau_n} w_{i,n} = f_{i,n}(w_{i,n}) - M_n w_{i,n} \sum_{j\neq i} a_{ij} w_{j,n}^2 & \text{on } \tau_n \partial^0 B^+, \end{cases}$$

where the new operators write ($\hat{X}_n = (\hat{x}_n, \hat{y}_n)$)

$$L_a^{\tau_n} = -\operatorname{div}\left(\left(\hat{y}_n r_n^{-1} + y\right)^a \nabla\right), \qquad \partial_\nu^{a,\tau_n} = \lim_{y\to(-\hat{y}_n r_n^{-1})^+} -\left(\hat{y}_n r_n^{-1} + y\right)^a \partial_y,$$

and $f_{i,n}(t) = \eta(\hat{X}_n) r_n^{2s-\alpha} L_n^{-1} f_{i,\beta_n}(L_n r_n^\alpha t/\eta(\hat{X}_n))$, $M_n = \beta_n L_n^2 r_n^{2\alpha+2s}/\eta(\hat{X}_n)^2$.

Remark 3.3.3. The uniform bound of $\|\mathbf{v}_\beta\|_{L^\infty}$ imply that

$$\sup_{\tau_n \partial^0 B^+} |f_{i,n}(w_{i,n})| = \eta(\hat{X}_n) r_n^{2s-\alpha} L_n^{-1} \sup_{\partial^0 B^+} |f_{i,\beta_n}(v_{i,n})| \leq C(\bar{m}) r_n^{2s-\alpha} L_n^{-1} \to 0$$

as $n \to \infty$.

A crucial property is that the two blow up sequences defined above have asymptotically equivalent behavior, as enlighten in the following lemma.

Lemma 3.3.4 (Lemma 2.5.6)**.** *Let $K \subset \mathbb{R}^{N+1}$ be compact. Then*

1. $\displaystyle\max_{X\in K\cap\overline{\tau_n B^+}} |\mathbf{w}_n(X) - \bar{\mathbf{w}}_n(X)| \to 0;$

2. *there exists C, only depending on K, such that $|\mathbf{w}_n(X) - \mathbf{w}_n(0)| \le C$, for every $x \in K$.*

Now we show that the sequences (X_n', X_n'') accumulates towards $\{y = 0\}$.

Lemma 3.3.5. *There exists $C > 0$ such that, for every n sufficiently large,*

$$\frac{\operatorname{dist}(X_n', \partial^0 B^+) + \operatorname{dist}(X_n'', \partial^0 B^+)}{r_n} \le C.$$

Proof. We argue by contradiction. Taking into account the second part of Lemma 3.3.2, this forces

$$\frac{\operatorname{dist}(X_n', \partial B^+) + \operatorname{dist}(X_n'', \partial B^+)}{r_n} \to \infty.$$

In the definition of $\mathbf{w}_n$, $\bar{\mathbf{w}}_n$ we choose $\hat{X}_n = X_n'$, so that $\tau_n B^+ \to \mathbb{R}^{N+1}$ and $\hat{y}_n^{-1} r_n \to 0$. Let K be any fixed compact set. Then, by definition, K is contained in the half sphere $\tau_n B^+$, for every n sufficiently large. By defining $\mathbf{W}_n = \mathbf{w}_n - \mathbf{w}_n(0)$, $\bar{\mathbf{W}}_n = \bar{\mathbf{w}}_n - \bar{\mathbf{w}}_n(0)$, we obtain that $\{\bar{\mathbf{W}}_n\}_{n\in\mathbb{N}}$ is a sequence of functions which share the same $\mathcal{C}^{0,\alpha}$-seminorm and are uniformly bounded in K, since $\bar{\mathbf{W}}_n(0) = 0$. By the Ascoli-Arzelà Theorem, there exists a function $\mathbf{W} \in C(K)$ which, up to a subsequence, is the uniform limit of $\{\bar{\mathbf{W}}_n\}_{n\in\mathbb{N}}$: taking a countable compact exhaustion of $\mathbb{R}^{N+1}$ we find that $\bar{\mathbf{W}}_n \to \mathbf{W}$ uniformly in every compact set. Moreover, for any pair X, Y, we have that $X, Y \in \tau_n B^+$ for every n sufficiently large, and so

$$|\bar{\mathbf{W}}_n(X) - \bar{\mathbf{W}}_n(Y)| \le \sqrt{k}|X - Y|^\alpha.$$

Passing to the limit in n the previous expression, we obtain $\mathbf{W} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^{N+1})$. By Lemma 3.3.4, we also find that $\mathbf{W}_n \to \mathbf{W}$ uniformly on compact sets. We want to show that $\mathbf{W}$ is harmonic. To this purpose, let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{N+1})$ be a smooth test function, and let $\bar{n}$ be sufficiently large so that $\operatorname{supp}\varphi \subset \tau_n B^+$ for all $n \ge \bar{n}$. For a fixed $i \in \{1, \dots, k\}$, we test the equation $L_a^{\tau_n} w_{i,n} = 0$ by φ to find

$$\int_{\mathbb{R}^{N+1}} -\operatorname{div}\left(\left(1 + y r_n \hat{y}_n^{-1}\right)^a \nabla\varphi\right) w_{i,n} \, \mathrm{d}x\mathrm{d}y = 0.$$

Passing to the uniform limit and observing that $(1 + y r_n \hat{y}_n^{-1})^a \to 1$ in $\mathcal{C}^\infty(\operatorname{supp}\varphi)$, we obtain at once that $\mathbf{W}$ is indeed harmonic. We will obtain a contradiction with the classical Liouville Theorem once we show that $\mathbf{W}$ is not constant. To this aim we observe that $(X_n' - \hat{X}_n)/r_n = 0$ and, up to a subsequence,

$$\frac{X_n'' - \hat{X}_n}{r_n} = \frac{X_n'' - X_n'}{|X_n'' - X_n'|} \to X'' \in \partial B_1.$$

Therefore, by equicontinuity and uniform convergence,

$$\left|\bar{W}_{1,n}\left(\frac{X_n' - \hat{X}_n}{r_n}\right) - \bar{W}_{1,n}\left(\frac{X_n'' - \hat{X}_n}{r_n}\right)\right| = 1 \implies |W_1(0) - W_1(X'')| = 1. \qquad \square$$

After the result above, we are in a position to choose $\hat{X}_n$ in the definition of $\mathbf{w}_n$, $\bar{\mathbf{w}}_n$ as

$$\hat{X}_n := (x'_n, 0),$$

where as usual $X'_n = (x'_n, y'_n)$. With this choice, it is immediate to see that

$$L_a^{\tau_n} = L_a, \quad \partial_\nu^{a,\tau_n} = \partial_\nu^a, \quad \tau_n B^+ \to \Omega_\infty = \mathbb{R}^{N+1}_+.$$

Moreover, by Lemma 3.3.5, we have that $X'_n, X''_n \in B^+_C$, for some C not depending on n. This will imply that any possible blow up limit can not be constant. Now one can reason as in Section 2.5 in order to prove that the blow up sequences converge. In doing this, a first crucial step consists in proving that $\mathbf{w}_n(0)$ is bounded: to this aim, it is useful to notice that the decay rate for subsolutions which we obtained in Lemma 3.2.5 does not depend on s and completely agrees with the one found in Lemma 2.3.5. Consequently, the uniform bound on the Hölder seminorm allows to prove the following result.

Lemma 3.3.6 (Lemma 2.5.13)**.** *Under the previous blow up setting, there exists* $\mathbf{w} \in (H^{1;a}_{\mathrm{loc}} \cap \mathcal{C}^{0,\alpha})\left(\overline{\mathbb{R}^{N+1}_+}\right)$ *such that, up to a subsequence,*

$$\mathbf{w}_n \to \mathbf{w} \text{ in } (H^{1;a} \cap C)(K)$$

for every compact $K \subset \overline{\mathbb{R}^{N+1}_+}$.

End of the proof of Theorem 1.2.1. Up to now, we have that $\mathbf{w}_n \to \mathbf{w}$ in $(H^{1;a} \cap C)_{\mathrm{loc}}$, and that the limiting blow up profile $\mathbf{w}$ is a nonconstant vector of harmonic, globally Hölder continuous functions. To reach the final contradiction, we distinguish, up to subsequences, between the following three cases.

Case 1: $M_n \to 0$. In this case also the equation on the boundary passes to the limit, and the nonconstant component w_1 satisfies $\partial_\nu^a w_1 \equiv 0$ on $\mathbb{R}^N$, in contradiction with Corollary 3.2.3.

Case 2: $M_n \to C > 0$. Even in this case the equation on the boundary passes to the limit, and $\mathbf{w}$ solves

$$\begin{cases} L_a w_i = 0 & x \in \mathbb{R}^{N+1}_+ \\ \partial_\nu^a w_i = -C w_i \sum_{j \neq i} a_{ij} w_j^2 & \text{on } \mathbb{R}^N \times \{0\} \end{cases}$$

The contradiction is now reached using Proposition 3.2.9.

Case 3: $M_n \to \infty$. In this case we can find a contradiction with Proposition 3.2.10. To this aim, one has to prove the validity of a Pohozaev-type identity for the limits of the blow-up sequences. This can be done by taking into account Lemma 3.3.6 and reasoning as in Section 2.4.

As of now, the contradictions we have obtained imply that $\{\mathbf{v}_\beta\}_{\beta>0}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}\left(\overline{B^+_{1/2}}\right)$, for every $\alpha < \alpha^*$. But then the relative compactness in $\mathcal{C}^{0,\alpha}\left(\overline{B^+_{1/2}}\right)$ follows by Ascoli-Arzelà Theorem, while the one in $H^{1;a}(B^+_{1/2})$ can be shown by reasoning as in the proof of Lemma 3.3.6. □

Incidentally, we remark that similar arguments can be exploited in order to prove the following compactness result, concerning segregated profiles (see Proposition 2.5.15). This result, though technical at this stage, provides a compactness criterion for suitable blow down sequences, and may be useful in proving optimal regularity results, along the scheme explained in the introduction.

Proposition 3.3.7. *Let $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ be a subset of $\mathcal{C}^{0,\alpha}\left(\overline{B^+_1}\right)$, for some $0 < \alpha \leq \alpha^*$, and satisfy the assumptions of Proposition 3.1.11. If*

$$\|\mathbf{v}_n\|_{L^\infty(B^+_1)} \leq \bar{m},$$

with $\bar{m}$ independent of n, then for every $\alpha' \in (0,\alpha)$ there exists a constant $C = C(\bar{m},\alpha')$, not depending on n, such that

$$\|\mathbf{v}_n\|_{\mathcal{C}^{0,\alpha'}\left(\overline{B^+_{1/2}}\right)} \leq C.$$

Furthermore, $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ is relatively compact in $H^{1;a}(B^+_{1/2}) \cap \mathcal{C}^{0,\alpha'}\left(\overline{B^+_{1/2}}\right)$ for every $\alpha' < \alpha$.

To conclude, we mention that the above local result can be used, together with a covering argument and Proposition 3.2.4, to prove Theorem 1.2.2.

Sketch of the proof of Theorem 1.2.2. Basically, one can argue as in the proof of Theorem 2.7.5. There are, however, two different situations to be handled.

First, if one considers the problem (1) set on the whole $\mathbb{R}^N$ (Theorem 1.2.2 in the case $\Omega = \mathbb{R}^N$), then the global uniform bounds on $\mathbf{u}_\beta$ imply, by the representation formula of Caffarelli and Silvestre [16], that also $\mathbf{v}_\beta$ enjoy the same uniform L^∞ bounds. As a consequence, the local uniform bounds extend at once to the global case by a simple covering argument.

In the case of $\Omega \neq \mathbb{R}^N$, one has to deal also with the boundary of Ω. In this situation, the regularity for $\mathbf{u}_\beta$ is ensured by [43], while the uniform Hölder bounds - obtained again via the blow up analysis - follows with similar arguments and the use of the appropriate Liouville type results (Proposition 3.2.4). Further details can be found in Section 2.7. □

3.4 Concluding remarks on the optimal regularity

As of now, we have shown preliminary compactness results in complete analogy to the case $s = 1/2$ of the previous chapter. Arguing in the same way as done in Section 2.6, the value of optimal regularity exponent (up to self-segregation) is a consequence of a classification result regarding one dimensional entire profiles.

Definition 3.4.1 (Definition 2.2.1 and 2.6.2)**.** For each $\nu > 0$, we define the class $\mathcal{H}^s(\nu)$ as the set of functions $\mathbf{v} \in H^{1;a}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ which enjoy the assumptions of Proposition 3.1.11, such that $\mathbf{v} \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}\left(\overline{\mathbb{R}^2_+}\right)$, for some $\alpha > 0$, $\mathbf{v}$ is non trivial and ν-homogeneous.

Lemma 3.4.2. *There exist three strictly increasing sequences of real numbers $\{\nu_n^0\}_{n\in\mathbb{N}}$, $\{\nu_n^1\}_{n\in\mathbb{N}}$ and $\{\nu_n^2\}_{n\in\mathbb{N}}$,*

$$\nu_0^0 = 2s, \quad \nu_0^1 = \begin{cases} 1 & \text{if } s \leq 1/2, \\ 2s-1 & \text{if } s > 1/2 \end{cases}, \quad \nu_0^2 = s$$

such that

- *if $\nu > 0$ is not contained in any of the three sequences, then $\mathcal{H}^s(\nu) = \emptyset$;*
- *if $\nu = \nu_n^0$, any element of $\mathcal{H}^s(\nu)$ has trivial trace on $\{y = 0\}$;*
- *if $\nu = \nu_n^1$, any element of $\mathcal{H}^s(\nu)$ has at most one element with non trivial;*
- *if $\nu = \nu_n^2$, any element of $\mathcal{H}^s(\nu)$ either has trivial trace or has exactly two components with non trivial . In particular, for $\nu = s$ the two possibly non trivial components are given by*

$$v(\rho,\theta) = c\rho^s \left(\cos\frac{\theta}{2}\right)^{2(1-s)}, \qquad w(\rho,\theta) = \pm c\rho^s \left(\sin\frac{\theta}{2}\right)^{2(1-s)}$$

  while in the general case for each $n \in \mathbb{N}$ there exists a function $g_n : [0,\pi] \to \mathbb{R}$ such that g_n has n distinct zeroes in $(0,\pi)$ and

$$v(\rho,\theta) = c\rho^{\nu_n^2} g_n(\theta), \qquad w(\rho,\theta) = \pm c\rho^{\nu_n^2} g_n(\pi - \theta).$$

Proof. We classify the homogeneity degrees for which the class $\mathcal{H}^s$ is not empty. By the topology of the real line, it easily follows that, given an element $\mathbf{v}$ belonging to any class $\mathcal{H}^s$, no more than two of its component can have non trivial trace on $\{y = 0\}$, and thus no more than two components can be non trivial. We deal first with the case of two non trivial components. In this setting, let us suppose that two components of $\mathbf{v}$, say v and w, have non trivial trace, in such a way that they solve

$$\begin{cases} L_a v = 0 & \text{in } \mathbb{R}^2_+ \\ v(x,0) = 0 & \text{on } x < 0 \\ \partial^a_\nu v(x,0) = 0 & \text{on } x > 0, \end{cases} \qquad \text{and} \qquad \begin{cases} L_a w = 0 & \text{in } \mathbb{R}^2_+ \\ w(x,0) = 0 & \text{on } x > 0 \\ \partial^a_\nu w(x,0) = 0 & \text{on } x < 0. \end{cases}$$

Exploiting the homogeneity of the function $\mathbf{v}$, we can try to classify the solutions v and w; to this purpose, v may be written in the form $v(\rho,\theta) = \rho^\nu g(\theta)$, with ν and g solutions to

$$\begin{cases} ((\sin\theta)^a g')' + \nu(\nu+a)(\sin\theta)^a g = 0 & \text{in } (0,\pi) \\ g(\pi) = 0, \ \lim_{\theta\to 0^+}(\sin\theta)^a g'(\theta) = 0, \end{cases}$$

and similarly we can reason for w. We then recognize a Sturm-Liouville eigenvalue problem for the eigenfunction v and eigenvalue $\lambda = \nu(\nu+a)$. In a moment we will show that all the eigenvalues of the problem are positive, but first we wish to observe that, assuming the positivity of λ, the equation

$$\nu(\nu+a) = \lambda$$

defining the homogeneity of the corresponding solution, has a unique admissible solution

$$\nu = \frac{-a + \sqrt{a^2+4\lambda}}{2} \geq 0$$

since the classes $\mathcal{H}^s$ are constituted by continuous functions. Thus, we have established a one-to-one correspondence between eigenvalues λ and homogeneity degrees ν. By the well established theory for the Sturm-Liouville eigenvalue problem, we know the existence of a strictly increasing sequence $\{\lambda_n\}_{n=0}^\infty$ of real eigenvalues, such that each λ_n is associated to a (unique up to renormalization) eigenfunction v_n, and the same holds true for function w. Moreover, each eigenfunction has exactly n zeroes in the set $(0,\pi)$, so that, in order to compute the first eigenvalue λ_0, it is sufficient to find an eigenfunction which does not vanish in $(0,\pi)$. In particular, by a direct check, it follows that the first eigenvalue in the sequence is given by $\lambda = s(s+a) > 0$ and it correspond to the couple

$$v(\rho,\theta) = c\rho^s\left(\cos\frac{\theta}{2}\right)^{2(1-s)} = c\bar{v}(x,y) \qquad w(\rho,\theta) = d\rho^s\left(\sin\frac{\theta}{2}\right)^{2(1-s)} = d\bar{v}(-x,y)$$

while the other functions in the vector $\mathbf{v}$ are trivial (this follows again by the geometric simplicity of the eigenvalues). We need to ensure that $\mathbf{v}$ satisfy the Pohozaev identity, that is

$$\int\limits_{\partial^+ B_r^+(x_0,0)} y^a\left(|\nabla v|^2 + |\nabla w|^2\right) \mathrm{d}\sigma = 2 \int\limits_{\partial^+ B_r^+(x_0,0)} y^a\left(|\partial_\nu v|^2 + |\partial_\nu w|^2\right) \mathrm{d}\sigma,$$

for every $x_0 \in \mathbb{R}$ and $r > 0$. First, considering the points $x_0 = 1$ and $x_1 = -1$, and using the symmetries, we obtain

$$
\begin{aligned}
A_+c^2 + A_-d^2 &= 2B_+c^2 + 2B_-d^2,\\
A_-c^2 + A_+d^2 &= 2B_-c^2 + 2B_+d^2,
\end{aligned}
$$

where

$$
A_\pm = \int\limits_{\partial^+ B_r^+(\pm 1,0)} y^a |\nabla \bar{v}|^2 \, \mathrm{d}\sigma, \qquad B_\pm = \int\limits_{\partial^+ B_r^+(\pm 1,0)} y^a |\partial_\nu \bar{v}|^2 \, \mathrm{d}\sigma.
$$

Since, by direct computation, $A_\pm - 2B_\pm \neq 0$, at least for some r, the above equalities force $c^4 - d^4 = 0$, that is $d = \pm c$.

Imposing the validity of the Pohozaev identity, it follows that $d = \pm c$. In the case of the following eigenvalues in the sequence, one can reasoning in the same way, exploiting the uniqueness of the eigenfunctions and the one-to-one correspondence between eigenvalue and homogeneity degrees, to conclude that the solutions are in the form prescribed in the thesis.

We now turn to the case of only one component, say v, with non trivial trace. At first, we need to deal with two different situation, corresponding to the case $v \neq 0$ on a half line or $v \neq 0$ on the punctured real line. The first case actually never happens: indeed, reasoning as before, we find that v should be of the form

$$
v(\rho, \theta) = c\rho^{\nu_n^2} g_n(\theta)
$$

for a constant c that should be chosen in order to verify the Pohozaev identity. But again the other functions in the vector $\mathbf{v}$ are trivial, and this forces $c = 0$. In the second case, we find that v has to solve

$$
\begin{cases}
L_a v = 0 & \text{in } \mathbb{R}^2_+\\
v(0,0) = 0 & \\
\partial^a_\nu v(x,0) = 0 & \text{on } x \neq 0,
\end{cases}
$$

that is, passing again to polar coordinates, we may write $v(\rho, \theta) = \rho^\nu g(\theta)$, with ν and g solutions to

$$
\begin{cases}
((\sin\theta)^a g')' + \nu(\nu + a)(\sin\theta)^a g = 0 & \text{in } (0, \pi)\\
\lim_{\theta \to 0^+, \theta \to \pi^-} (\sin\theta)^a g'(\theta) = 0. &
\end{cases}
$$

We can make use again of the theory of Strum-Liouville operators, but now we need to distinguish between the case $s \leq 1/2$ and $s > 1/2$. In the first case, the constant solution $g(\theta) = c$ corresponds to

$$
\nu(\nu + a) = 0 \implies \nu = 0 \text{ or } \nu = -a = 2s - 1 < 0
$$

and thus it is not admissible (recall that $v(0,0) = 0$). As a consequence, we need to find the second eigenvalue of the problem, which is obtained by considering the function $g(\theta) = A\cos\theta$, and a direct computation shows that

$$\nu(\nu + a) = 1 + a \implies \nu = 1,\ v(x, y) = cx$$

is the only admissible solution. On the contrary, in the case $s > 1/2$, then the constant solution $g(\theta) = c$ is admissible, and it correspond to

$$v(x, y) = c|(x, y)|^{2s-1}$$

that is, the fundamental solution of the s-Laplace operator, already encountered in the study of the phenomenon of self-segregation.

To analyze the case of trivial trace on $\{y = 0\}$ we just observe that any element of the vector $\mathbf{v}$ has to satisfy

$$\begin{cases} L_a v = 0 & \text{in } \mathbb{R}^2_+ \\ v(x, 0) = 0 & \text{on } \mathbb{R}. \end{cases}$$

The conclusion follows that again passing to polar coordinates and observing that $v(x, y) = cy^{1-a}$ is the solution with the least possible homogeneity degree. $\square$

Chapter 4

The Lotka-Volterra type competition

Outline of the chapter

In the last chapter of this part, we move our attention to system which contain competition terms of Lotka-Volterra type, that is

$$(-\Delta)^s u_i = f_{i,\beta}(x, u_1, \ldots, u_k) - \beta u_i^p \sum_{j \neq i} a_{ij} u_j^p,$$

where $i = 1, \ldots, k$, $s \in (0, 1)$, $p > 0$, $a_{ij} > 0$ and $\beta > 0$. When $k = 2$ we develop a quasi-optimal regularity theory in $\mathcal{C}^{0,\alpha}$, uniformly w.r.t. β, for every $\alpha < \alpha_{\mathrm{opt}} = \min(1, 2s)$; moreover we show that the traces of the limiting profiles as $\beta \to +\infty$ are Lipschitz continuous and segregated. Such results are extended to the case of $k \geq 3$ densities, with some restrictions on s, p and a_{ij}.

4.1 Preliminary results

We devote this section to some results concerning the operator L_a and solutions to some associated differential problem. Most of such results already appeared, even if in slightly different form, in the previous chapters and in the literature, but we chose to restate them for the reader convenience. We refer to Chapter 2, 3 and to [15] for further details.

Lemma 4.1.1 ([15, Lemma 2.7]). *If v is a non constant, global solution of $L_a v = 0$ in $\mathbb{R}^{N+1}$, with the property that*

$$|v(X)| \leq C\left(1 + |X|^\gamma\right),$$

then $\gamma \geq \min(2s, 1)$. If furthermore $v(x, -y) = v(x, y)$ then $\gamma \geq 1$ (and v is a polynomial).

Lemma 4.1.2 (Proposition 3.2.8)**.** *Let v satisfy*

$$\begin{cases} L_a v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu v = \lambda & \text{on } \mathbb{R}^N \end{cases}$$

for some $\lambda \in \mathbb{R}$, and

$$|v(X)| \le C\left(1 + |X|^\gamma\right),$$

for some $0 \le \gamma < \min(2s, 1)$. Then v is constant.

The two last results we need are based on the following comparison principle.

Lemma 4.1.3 (Comparison principle)**.** *Let $u, v \in H^{1;a}(B^+)$ satisfy*

$$\begin{cases} L_a u \le 0 & \text{in } B_1^+ \\ \partial^a_\nu u \le -Mu^p + \delta & \text{on } \partial^0 B_1^+, \end{cases} \qquad \begin{cases} L_a v \ge 0 & \text{in } B_1^+ \\ \partial^a_\nu v \ge -Mv^p + \delta & \text{on } \partial^0 B_1^+, \end{cases}$$

respectively. Then $u \le v$ on $\partial^+ B_1^+$ implies $u \le v$ on B_1^+.

Proof. Letting $w = u - v$, we obtain that w is a solution to

$$\begin{cases} L_a w \le 0 & \text{in } B_1^+ \\ \partial^a_\nu w \le -M(u^p - v^p) & \text{on } \partial^0 B_1^+ \\ w \le 0 & \text{on } \partial^+ B_1^+. \end{cases}$$

Testing the equation with w^+ and recalling that $p > 0$ we find

$$\int_{B_1^+} y^a |\nabla w^+|^2 \mathrm{d}x\mathrm{d}y \le -M \int_{\partial^0 B^+} \frac{u^p - v^p}{u - v}(w^+)^2 \mathrm{d}x \le 0. \qquad \square$$

Lemma 4.1.4. *Let $M > 0$ be any large constant and $\delta > 0$ be fixed and let $h \in L^\infty(\partial^0 B_1^+)$ with $\|h\|_{L^\infty} \le \delta$. Any $v \in H^{1;a}(B_1^+)$ non negative solution to*

$$\begin{cases} L_a v \le 0 & \text{in } B_1^+ \\ \partial^a_\nu v \le -Mv^p + h & \text{on } \partial^0 B_1^+ \end{cases}$$

verifies

$$\sup_{\partial^0 B^+_{1/2}} v \le \frac{1+\delta}{M^{1/p}} \sup_{\partial^+ B_1^+} v.$$

Sketch of proof. The proof is similar to the one of Lemma 3.2.5, the only difference being in the choice of the supersolution. For $a \in (-1, 1)$ and $p > 0$ fixed, let $b = 1 + (1-a)/p > 1$ and $f \in AC(\mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R})$ be defined as

$$f(x) = c \int_{-\infty}^x \frac{1}{(1+t^2)^{b/2}} \mathrm{d}t,$$

where c is chosen in such a way that $f(+\infty) = 1$. Then, for some $C > 0$, the estimate

$$(-\Delta)^s f(x) \geq -Cf(x)^p$$

holds for any $x < 0$. For $M > 0$, the function $f_M(x) := f(M^{1/(2s)}x)$ satisfies

$$(-\Delta)^s f_M(x) = M^{2s/(2s)} \left[(-\Delta)^s f\right](M^{1/(2s)}x) \geq -CMf_M^p(x).$$

Therefore, if we let

$$g_M(x) := f_M(x-1) + f_M(-x-1)$$

then

$$\left(f_M^p(x-1) + f_M^p(-x-1)\right)^{1/p} \leq c_p g_M,$$

for some $c_p > 0$. It follows that, for any $M > 0$, it holds

$$\begin{cases} (-\Delta)^s g_M(x) \geq -CMg_M^p(x) & \text{in } (-1,1) \\ g_M(x) \geq \frac{1}{2} & \text{in } \mathbb{R} \setminus (-1,1) \\ g_M(x) \leq CM^{-1/p} & \text{in } \left(-\frac{1}{2}, \frac{1}{2}\right). \end{cases}$$

The lemma follows by comparison between v and the supersolution (see [16])

$$w_\delta := \delta \frac{1}{M^{1/p}} + \int_{\mathbb{R}} y^{1-a} \frac{g_M(x-\xi)}{(\xi^2 + y^2)^{1-a/2}} \mathrm{d}\xi. \qquad \square$$

Lemma 4.1.5. *Let $\lambda > 0$ and $v \in H^{1;a}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+})$ be non negative and satisfy*

$$\begin{cases} L_a v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu v \leq -\lambda v^p & \text{on } \mathbb{R}^N. \end{cases}$$

If the Hölder quotient of exponent γ of v is uniformly bounded, for some $\gamma \in [0, 2s)$, then v is constant.

Proof. When $p \leq 1$ the lemma follows directly from Lemma 4.1.4 (see also Proposition 3.2.7): indeed, by translating and scaling,

$$v(x_0, 0) \leq \sup_{\partial^0 B_{r/2}(x_0,0)} v \leq \frac{1}{\lambda^{1/p} r^{2s/p}} \sup_{\partial^+ B_r(x_0,0)} v \leq C\frac{1 + r^\gamma}{r^{2s/p}} \to 0 \quad \text{ as } r \to \infty.$$

When $p > 1$, we start by showing that v has a bounded trace on $\mathbb{R}^N$. Let us assume, on the contrary, that $v(x,0)$ is not uniformly bounded from above: by the uniform control on the Hölder seminorm, there exists a sequence $\{x_n\} \subset \mathbb{R}^N$ such that

$$M_n := \inf_{\partial^0 B_1^+(x_n,0)} v^{p-1} \to +\infty.$$

But then, restricting on $B_1^+(x_n,0)$, we have that $v \geq 0$ satisfies

$$\begin{cases} L_a v = 0 & \text{in } B_1^+(x_n,0) \\ \partial_\nu^a v \leq -M_n v & \text{on } \partial B_1^+(x_n,0) \end{cases}$$

and, thanks to Lemma 4.1.4 (with exponent 1 instead of p) and the Hölder continuity, we obtain

$$\inf_{\partial^0 B_1^+(x_n,0)} v \leq \sup_{\partial^0 B_{1/2}^+(x_n,0)} v \leq \frac{1}{M_n} \sup_{\partial^+ B_1^+(x_n,0)} v \leq \frac{1}{M_n} \left(\inf_{\partial^0 B_1^+(x_n,0)} v + C \right),$$

a contradiction. Let now $\{x_n\} \subset \mathbb{R}^N$ be a maximizing sequence of $v(x,0)$, that is

$$\sup_{x \in \mathbb{R}^N} v(x,0) = \lim_{n\to\infty} v(x_n,0) < \infty,$$

and let us also introduce the sequence of functions

$$v_n(x,y) := v(x - x_n, y).$$

The functions v_n share the same uniform bound in $\mathcal{C}^{0,\gamma}$, so that we can pass to the uniform limit and find a limiting function $\bar{v} \in \mathcal{C}^{0,\gamma}(\overline{\mathbb{R}^{N+1}_+})$ which satisfies the assumptions of the lemma, its trace on $\mathbb{R}^N$ achieving the global maximum at $(0,0)$. Let us denote with w the unique bounded L_a-harmonic extension of $\bar{v}(x,0)$ (which is defined since $\bar{v}(x,0)$ is bounded). We see that the odd extension across $\{y = 0\}$ of the difference $w - \bar{v}$ satisfies the assumptions of Lemma 4.1.1, yielding $\bar{v} \equiv w$. From the equation we deduce that

$$\partial_\nu^a \bar{v}(0,0) = -\lambda \bar{v}(0,0)^p = -\lambda \sup_{x \in \mathbb{R}^N} v(x,0)^p \leq 0$$

and the Hopf Lemma implies $\bar{v}(0,0) = 0$, that is $v \equiv 0$. □

4.2 The blow-up argument

As for the variational competition case of Chapter 2 and 3, the proof of the a priori uniform $\mathcal{C}^{0,\alpha}$-bounds of solutions to problem $(LV)_\beta$ is based on a blow-up argument. To perform this technique, we will assume that the solutions are not a priori bounded in a uniform way in some Hölder norms and then, through a series of lemmas, we will show that this implies the existence of entire solutions to some limiting problem. The scheme of the proof here presented may resemble the one contained for instance in [22], which has also inspired the proofs in Chapter 2 and 3. However, in the present situation, some of the steps, which were adopted in the aforementioned papers, actually fail. This phenomenon is consequence of deep differences in the interaction between competition and diffusion features of the models. Once the blow-up procedure is completed, we will reach different contradictions in the next section, depending on the particular choice of k, p and a_{ij}: for the moment, in what follows we will always assume that $p > 0$, $a_{ij} > 0$ for any $j \neq i$, and that the reaction terms $f_{i,\beta}$ are continuous and map bounded sets into bounded sets, uniformly w.r.t. $\beta > 0$ (notice that these are the common assumptions for all the statements in the introduction).

Let $\{\mathbf{v}_\beta\}_\beta = \{(v_{1,\beta}, \dots, v_{k,\beta})\}_\beta$ denote a family of positive solutions to problem $(LV)_\beta$, uniformly bounded in B_1^+. We begin the analysis by recalling the regularity result which holds whenever β is finite. As always, for easier notation, we write $B^+ = B_1^+$.

Lemma 4.2.1 (Lemma 3.3.1)**.** *For every $0 < \alpha < \min(2s, 1)$, $\bar{m} > 0$ and $\bar{\beta} > 0$, there exists a constant $C = C(\alpha, \bar{m}, \bar{\beta})$ such that*

$$\|\mathbf{v}_\beta\|_{\mathcal{C}^{0,\alpha}\left(\overline{B^+_{1/2}}\right)} \leq C,$$

for every $\mathbf{v}_\beta$ solution of problem $(LV)_\beta$ on B^+, satisfying

$$\beta \leq \bar{\beta} \quad \text{and} \quad \|\mathbf{v}_\beta\|_{L^\infty(B^+)} \leq \bar{m}.$$

Let the cut-off function η be smooth, with

$$\eta(X) = \begin{cases} 1 & X \in B_{1/2} \\ 0 & X \in \mathbb{R}^{N+1} \setminus B_1 \end{cases} \quad \text{while} \quad \eta(X) \in (0,1) \text{ elsewhere.}$$

The rest of this section is devoted to the proof of the following proposition.

Proposition 4.2.2. *If there exists $0 < \alpha < \min(2s, 1)$ such that*

$$\sup_{\beta > 0} |\eta \mathbf{v}_\beta|_{\mathcal{C}^{0,\alpha}\left(\overline{B^+}\right)} = +\infty$$

then for a suitable choice of $\{r_\beta\}_\beta \subset \mathbb{R}^+$ and $\{x'_\beta\}_\beta \subset \mathbb{R}^N$, the blow-up family

$$w_{i,\beta}(X) := \eta(x'_\beta, 0) \frac{v_{i,\beta}((x'_\beta, 0) + r_\beta X)}{r_\beta^\alpha |\eta \mathbf{v}_\beta|_{\mathcal{C}^{0,\alpha}\left(\overline{B^+}\right)}}$$

admits a convergent subsequence in the local uniform topology. Moreover the limit $\mathbf{w} \in (H^{1;a}_{\mathrm{loc}} \cap \mathcal{C}^{0,\alpha})\left(\overline{\mathbb{R}^{N+1}_+}\right)$ enjoys the following properties:

1. *each w_i is a L_a-harmonic function of $\mathbb{R}^{N+1}_+$;*
2. *at least one component of $\mathbf{w}$ is non constant, and it attains its maximal Hölder quotient of exponent α at a pair of points in the half-ball $\overline{B^+_1}$;*

3. either there exists $M > 0$ such that

$$\partial^a_\nu w_i = -M w_i^p \sum_{j \neq i} a_{ij} w_j^p \quad \text{on } \mathbb{R}^N$$

or $w_i w_j|_{y=0} = 0$ for every $j \neq i$ and

$$\partial^a_\nu w_i \leq 0, \qquad w_i \partial^a_\nu \left(w_i - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} w_j \right) = 0 \quad \text{on } \mathbb{R}^N.$$

The proof is divided in several steps. First, we choose any subsequence $\mathbf{v}_n := \mathbf{v}_{\beta_n}$ such that

$$\sup_{n \in \mathbb{N}} |\eta \mathbf{v}_n|_{\mathcal{C}^{0,\alpha}(\overline{B^+})} =: \sup_{n \in \mathbb{N}} L_n = +\infty,$$

where by Lemma 4.2.1 both $\beta_n \to \infty$ and the Hölder quotients L_n are achieved, say

$$\begin{aligned} L_n &:= \max_{i=1,\dots,k} \max_{X' \neq X'' \in \overline{B^+}} \frac{|(\eta v_{i,n})(X') - (\eta v_{i,n})(X'')|}{|X' - X''|^\alpha} \\ &= \frac{|(\eta v_{1,n})(X'_n) - (\eta v_{1,n})(X''_n)|}{r_n^\alpha}, \end{aligned}$$

where we have written $r_n := |X'_n - X''_n|$. Finally, we are in a position to define the two blow-up sequences we will work with as

$$w_{i,n}(X) := \eta(x'_n, 0) \frac{v_{i,n}((x'_n, 0) + r_n X)}{L_n r_n^\alpha} \quad \text{and} \quad \bar{w}_{i,n}(X) := \frac{(\eta v_{i,n})((x'_n, 0) + r_n X)}{L_n r_n^\alpha},$$

both defined on the domain

$$\tau_n B^+ := \frac{B^+ - (x'_n, 0)}{r_n}.$$

Accordingly, the corresponding reaction terms can be expressed as

$$f_{i,n}(x, t_1, \dots, t_k) = r_n^{2s} \frac{\eta(x'_n, 0)}{L_n r_n^\alpha} f_{i,\beta_n} \left(X'_n + r_n x, t_1 \frac{L_n r_n^\alpha}{\eta(x'_n, 0)}, \dots, t_k \frac{L_n r_n^\alpha}{\eta(x'_n, 0)} \right).$$

In Section 2.5 and 3.3 we have analyzed in detail the behavior of the two blow-up sequences in the different case of variational competition. In the following lemma we collect the initial remarks about such sequences, the proof of which is independent of the type of competition. In particular, we have that the domains exhaust the whole $\mathbb{R}^{N+1}_+$, and that the two sequences $\{\mathbf{w}_n\}_n$ and $\{\bar{\mathbf{w}}_n\}_n$ – of which the former satisfies an equation and the latter has uniformly bounded Hölder quotient – are close on any compact.

Lemma 4.2.3. *As $n \to \infty$ the following assertions hold*

1. *$r_n \to 0$, $\|f_{i,n}\|_\infty \to 0$, $\tau_n B^+ \to \mathbb{R}^{N+1}_+$ and $\tau_n \partial^0 B^+ \to \mathbb{R}^N \times \{0\}$;*

2. *the sequence* $\{\mathbf{w}_n\}_n$ *satisfies*

$$\begin{cases} L_a w_{i,n} = 0 & \text{in } \tau_n B^+ \\ \partial^a_\nu w_{i,n} = f_{i,n}(x, w_{1,n}, \ldots, w_{k,n}) - M_n w^p_{i,n} \sum_{j \neq i} a_{ij} w^p_{j,n} & \text{on } \tau_n \partial^0 B^+, \end{cases} \tag{4.2.1}$$

   where

$$M_n = \beta_n r_n^{2s} \left(\frac{\eta(x'_n, 0)}{L_n r_n^\alpha} \right)^{1-2p};$$

3. *the sequence* $\{\bar{\mathbf{w}}_n\}_n$ *has uniformly bounded* $\mathcal{C}^{0,\alpha}$*-seminorm, the oscillation of the first component in* B_1^+ *being always* 1*;*

4. *for any compact* $K \subset \mathbb{R}^{N+1}$*,*

$$\max_{X \in K \cap \overline{\tau_n B^+}} |\mathbf{w}_n(X) - \bar{\mathbf{w}}_n(X)| \to 0$$

   (and therefore also $\mathbf{w}_n$ *has uniformly bounded oscillation on* K*).*

In the next series of lemmas we are going to show that both sequences converge to the same blow-up limit. To this end, we have to exclude the case in which the sequences are unbounded at the origin: indeed, the uniform boundedness of a sequence at some point is enough, together with points (3) and (4) of the previous lemma, to conclude the convergence (uniform on compact sets) of the two sequences.

Lemma 4.2.4. *For any* $r > 0$ *there exists a constant* C *such that the estimate*

$$M_n \int_{\partial^0 B_r^+} \sum_{j \neq i} a_{ij} w^{p+1}_{i,n} w^p_{j,n} \, \mathrm{d}x \leq C(r)(|w_{i,n}(0)| + 1)$$

holds uniformly in n*.*

Proof. Let us consider the quantities

$$E(r) := \frac{1}{r^{N+a-1}} \left(\int_{B_r^+} y^a |\nabla w_{i,n}|^2 + \int_{\partial^0 B_r^+} \left(-f_{i,n} w_{i,n} + M_n w^{p+1}_{i,n} \sum_{j \neq i} a_{ij} w^p_{j,n} \right) \right)$$

$$H(r) := \frac{1}{r^{N+a}} \int_{\partial^+ B_r^+} y^a w^2_{i,n},$$

where $H \in AC(R, 2R)$, for any $R > 0$ fixed and n sufficiently large. If we test equation (4.2.1) by $w_{i,n}$ itself in the ball B_r^+, we obtain

$$H'(r) = \frac{2}{r^{N+a}} \int_{\partial^+ B_r^+} y^a w_{i,n} \partial_\nu w_{i,n} = \frac{2}{r} E(r),$$

which can be integrated to infer

$$H(2R) - H(R) = \int_R^{2R} \frac{2}{r} E(r)\,\mathrm{d}r.$$

On the one hand, the left hand side of of the previous identity can be estimated by recalling that $w_{i,n}$ has uniformly bounded oscillation on any compact set (Lemma 4.2.3, (4)):

$$\begin{aligned} H(2R) - H(R) &= \int_{\partial^+ B^+} y^a \left[w_{i,n}^2(2RX) - w_{i,n}^2(RX) \right] \mathrm{d}\sigma \\ &= \int_{\partial^+ B^+} y^a \, w_{i,n}|_{RX}^{2RX} \left[w_{i,n}|_0^{2RX} + w_{i,n}|_0^{RX} + 2w_{i,n}(0) \right] \mathrm{d}\sigma \\ &\le C(R)(|w_{i,n}(0)| + 1). \end{aligned}$$

On the other hand, we obtain a lower bound of the right hand side as

$$\begin{aligned} \int_r^{2r} \frac{2}{s} E(s)\mathrm{d}s &\ge \min_{s\in[r,2r]} E(s) \\ &\ge \frac{1}{r^{N+a-1}} \left(\frac{M_n}{2^{N+a}} \int_{\partial^0 B_r^+} \sum_{j\neq i} a_{ij} w_{i,n}^{p+1} w_{j,n}^p \,\mathrm{d}x - \int_{\partial^0 B_{2r}^+} |f_{i,n}| w_{i,n} \,\mathrm{d}x \right) \\ &\ge C \left(M_n \int_{\partial^0 B_r^+} \sum_{j\neq i} a_{ij} w_{i,n}^{p+1} w_{j,n}^p \,\mathrm{d}x - \|f_{j,n}\|_{L^\infty} (|w_{i,n}(0)| + 1) \right). \end{aligned}$$

□

Lemma 4.2.5. *If $\bar{w}_{i,n}(0) \to \infty$ for some i, then there exists C such that*

$$M_n \bar{w}_{i,n}^p(0) \le C$$

for a constant C independent of n. In particular, $M_n \to 0$.

Proof. Reasoning by contradiction we assume that $M_n \bar{w}_{i,n}^p(0) \to \infty$, at least for a subsequence. For any $r > 0$ fixed, Lemma 4.2.3 forces

$$I_{r,n} := \inf_{\partial^0 B_r^+} M_n w_{i,n}^p \to \infty.$$

From Lemma 4.2.4, we directly obtain

$$M_n \inf_{\partial^0 B_r^+} w_{i,n}^{p+1} \int_{\partial^0 B_r^+} \sum_{j\neq i} a_{ij} w_{j,n}^p \mathrm{d}x \le C(r)(|w_{i,n}(0)| + 1) \tag{4.2.2}$$

that is, since $w_{i,n}(0)/w_{i,n}(x) \to 1$ uniformly in compact sets,

$$I_{r,n} \int_{\partial^0 B_r^+} \sum_{j\neq i} a_{ij} w_{j,n}^p \mathrm{d}x \le C.$$

Let $j \neq i$. Since $I_{r,n} \to \infty$, we deduce that $w_{j,n} \to 0$ in $L^p(\partial^0 B_r^+)$, for every r. Therefore Lemma 4.2.3 implies that both $\{\bar{w}_{j,n}\}_n$ and $\{w_{j,n}\}_n$ converge, uniformly on compact sets, to an L_a-harmonic function $w_{j,\infty} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^{n+1}_+)$ such that

$$w_{j,\infty}(x,0) = 0 \qquad \text{on } \mathbb{R}^N.$$

The Liouville result in Lemma 4.1.1 applies to the odd extension of $w_{j,\infty}$ across $\{y = 0\}$, yielding

$$w_{j,\infty} \equiv 0 \qquad \text{for } j \neq i.$$

In particular, by uniform convergence, the unitary Hölder quotient is not achieved by any of the functions $\bar{w}_{j,n}$ for $j \neq i$ and n large enough: it follows that we must have $i = 1$.

Now, let us recall that each $w_{j,n}$ with $j \neq 1$ satisfies the inequality

$$\begin{cases} L_a w_{j,n} = 0 & \text{in } B_{2r}^+ \\ \partial_\nu^a w_{j,n} \leq \|f_{j,n}\|_{L^\infty(B_{2r})} - a_{ji} I_{2r,n} w_{j,n}^p & \text{in } \partial^0 B_{2r}^+ \end{cases} \tag{4.2.3}$$

so that by Lemma 4.1.4 we have the estimate

$$\sup_{\partial^0 B_r^+} w_{j,n}^p \leq \frac{C(r)}{I_{2r,n}} \sup_{\partial^+ B_{2r}^+} w_{j,n}^p.$$

On the other hand, the function $w_{1,n}$ satisfies a boundary condition that can be estimated as

$$\begin{aligned} \sup_{\partial^0 B_r^+} |\partial_\nu^a w_{1,n}| &\leq \|f_{1,n}\|_{L^\infty(B_{2r})} + I_{r,n} \sum_{i \neq 1} a_{ij} \sup_{\partial^0 B_r^+} w_{j,n}^p \\ &\leq \|f_{1,n}\|_{L^\infty(B_{2r})} + C(r) \frac{I_{r,n}}{I_{2r,n}} \sum_{i \neq 1} \sup_{\partial^+ B_{2r}^+} w_{j,n}^p \to 0, \end{aligned}$$

where we used the fact that

$$\inf_{\partial^0 B_{2r}^+} \bar{w}_{i,n} \leq \inf_{\partial^0 B_r^+} \bar{w}_{i,n} \leq \inf_{\partial^0 B_{2r}^+} \bar{w}_{i,n} + C r^\alpha \quad \Longrightarrow \quad \lim_{n \to \infty} \frac{I_{r,n}}{I_{2r,n}} = 1.$$

Let us now introduce the sequences

$$W_{1,n}(x,y) := w_{1,n}(x,y) - w_{1,n}(0,0), \qquad \bar{W}_{1,n}(x,y) := \bar{w}_{1,n}(x,y) - \bar{w}_{1,n}(0,0).$$

As before, we can use Lemma 4.2.3 to prove that both sequences converge to the same L_a-harmonic function, which is globally Hölder continuous, non constant, and which has trivial conormal derivative on $\mathbb{R}^N$, in contradiction with Lemma 4.1.2. $\square$

Lemma 4.2.6. *The sequence $\{\bar{\mathbf{w}}_n(0)\}_{n \in \mathbb{N}}$ is bounded.*

Proof. By contradiction, let $\{\bar{\mathbf{w}}_n(0)\}_{n\in\mathbb{N}}$ be unbounded. Then, by the previous lemma, $M_n \to 0$. To start with, we claim that for every j there exists a constant $\lambda_j \geq 0$ such that, up to subsequences,

$$M_n \bar{w}_{j,n}^p \to \lambda_j \qquad \text{locally uniformly.}$$

Indeed, if $\bar{w}_{j,n}(0)$ is bounded this follows by uniform Hölder bounds, with $\lambda_j = 0$; if it is unbounded, from Lemma 4.2.5 we obtain that $M_n \bar{w}_{j,n}^p(0) \to \lambda_j$, while

$$\sup_{\partial^0 B_r^+} |M_n \bar{w}_{j,n}^p - M_n \bar{w}_{j,n}^p(0)| = M_n \bar{w}_{j,n}^p(0) \sup_{\partial^0 B_r^+} \left| \left(\frac{\bar{w}_{j,n}}{\bar{w}_{j,n}(0)} \right)^p - 1 \right| \to 0.$$

Now, let i be such that $w_{i,n}(0)$ is bounded. As usual, we can use Lemma 4.2.3 to show that $w_{i,n} \to w_{i,\infty} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^{N+1}_+)$ in the local uniform topology, where, using the claim above, $w_{i,\infty}$ is a solution to

$$\begin{cases} L_a w_{i,\infty} = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu w_{i,\infty} = -w_{i,\infty}^p \sum_j a_{ij}\lambda_j & \text{on } \mathbb{R}^N. \end{cases}$$

Lemma 4.1.5 then implies $w_{i,\infty} \equiv 0$: in particular, we have that $\bar{w}_{1,n}(0)$ is unbounded.

Let us then turn our attention to $w_{1,n}$. Again, if j is such that $\bar{w}_{j,n}(0)$ is bounded, then by the previous discussion $\bar{w}_{j,n} \to 0$ locally uniformly and

$$\underbrace{M_n \bar{w}_{1,n}^p}_{\leq C \text{ (Lemma 4.2.5)}} \quad \bar{w}_{j,n}^p \to 0.$$

Otherwise, if j is such that $\bar{w}_{j,n}(0)$ is unbounded, then Lemma 4.2.4 provides

$$C \geq M_n w_{1,n}(0)^p w_{j,n}(0)^p \int\limits_{\partial^0 B_r^+} \sum_{j\neq 1} a_{1j} \frac{w_{1,n}^{p+1}}{w_{1,n}(0)^p(|w_{1,n}(0)|+1)} \frac{w_{j,n}^p}{w_{j,n}(0)^p} \,\mathrm{d}x$$

so that $M_n w_{1,n}(0)^p w_{j,n}(0)^p$ is uniformly bounded. Since if $\{w_{j,n}(0)\}_{n\in\mathbb{N}}$ is unbounded then also $\{w_{j,n}(x)\}_{n\in\mathbb{N}}$ is, for any fixed x, and the same argument shows that $M_n w_{1,n}(x)^p w_{j,n}(x)^p$ is bounded. Now,

$$\begin{aligned} &M_n \left| w_{1,n}(x)^p w_{j,n}(x)^p - w_{1,n}(0)^p w_{j,n}(0)^p \right| \\ &\leq M_n w_{1,n}(x)^p w_{j,n}(x)^p \left| 1 - \frac{w_{j,n}(0)^p}{w_{j,n}(x)^p} \right| + M_n w_{1,n}(0)^p w_{j,n}(0)^p \left| \frac{w_{1,n}(x)^p}{w_{1,n}(0)^p} - 1 \right| \to 0. \end{aligned}$$

This shows the existence of a constant λ such that, at least up to a subsequence,

$$f_{1,n} - M_n \bar{w}_{1,n}^p \sum_{j\neq 1} a_{1j} \bar{w}_{j,n}^p \to \lambda$$

uniformly on every compact subset of $\mathbb{R}^N$, and the same holds true for the sequence $\{w_{1,n}\}_{n\in\mathbb{N}}$. Thus, as usual, $W_{1,n} = w_{1,n} - w_{1,n}(0)$ converges to W_1 which is nonconstant, globally Hölder continuous of exponent $\alpha < \min(1, 2s)$, and which solves

$$\begin{cases} L_a W_1 = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu W_1 = \lambda & \text{on } \mathbb{R}^N. \end{cases}$$

Invoking Lemma 4.1.2, we obtain a contradiction. $\square$

The boundedness of the sequences $\{\bar{\mathbf{w}}_n(0)\}_{n\in\mathbb{N}}$ implies, by Lemma 4.2.3, the convergence of both $\{\bar{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ to the same blow-up limit. Reasoning as in the proof of Lemma 2.5.13, one can show that the convergence is also strong in the natural Sobolev space.

Lemma 4.2.7. *There exists* $\mathbf{w} \in (H^{1;a}_{\mathrm{loc}} \cap \mathcal{C}^{0,\alpha})\left(\overline{\mathbb{R}^{N+1}_+}\right)$ *such that, up to a subsequence,*

$$\mathbf{w}_n \to \mathbf{w} \ \textit{in} \ (H^{1;a} \cap C)(K)$$

for every compact $K \subset \overline{\mathbb{R}^{N+1}_+}$*. Furthermore, each* w_i *is* L_a*-harmonic, and* w_1 *is non constant.*

Depending on the behavior of the sequence M_n, the limiting functions $\mathbf{w}$ satisfy a different limiting problem: first, we can exclude the case $M_n \to 0$.

Lemma 4.2.8. *There exists* $C > 0$ *such that* $M_n \geq C$*.*

Proof. Let us assume that there exists a subsequence M_{n_k} that converges to 0. Passing to the limit in the sequence, we obtain as a limiting problem

$$\begin{cases} L_a w_i = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu w_i = 0 & \text{on } \mathbb{R}^N. \end{cases}$$

By Lemma 4.1.1 each w_i is constant, and this is in contradiction with the fact that w_1 oscillates in the half-ball B^+. $\square$

Lemma 4.2.9. *If* $M_n \to M > 0$*, then the blow-up profiles* $\mathbf{w}$ *solve*

$$\begin{cases} L_a w_i = 0 & \textit{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu w_i = -M w_i^p \sum_{j\neq i} a_{ij} w_j^p & \textit{on } \mathbb{R}^N. \end{cases}$$

Proof. This is a direct consequence of Lemma 4.2.7. $\square$

To conclude the proof of Proposition 4.2.2 we are left to analyze the case $M_n \to \infty$.

Lemma 4.2.10. *If $M_n \to \infty$, then the blow-up profiles $\mathbf{w}$ are such that $w_i w_j|_{y=0} = 0$, for every $j \neq i$, and*

$$\begin{cases} \partial_\nu^a w_i \le 0 \\ \partial_\nu^a \left(w_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} w_j \right) \ge 0 \\ w_i \partial_\nu^a \left(w_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} w_j \right) = 0, \end{cases} \tag{4.2.4}$$

where the inequalities are understood in the sense of $\mathbb{R}^N$-measures.

Proof. For any nonnegative $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^{N+1})$, we test equation (4.2.1) to find

$$0 \le \int\limits_{\partial^0 B_r} M_n w_{i,n}^p \sum_{j\neq i} a_{ij} w_{j,n}^p \psi \le \int\limits_{\partial^0 B_r} (f_{i,n}\psi - w_{i,n}\partial_\nu^a \psi) - \int\limits_{B_r^+} w_{i,n} L_a \psi.$$

Since the right hand side is bounded by local uniform convergence, we infer that

$$M_n \int_K w_{i,n}^p w_{j,n}^p \,\mathrm{d}x \le C(K) \qquad \forall j \neq i, \tag{4.2.5}$$

for any compact set $K \subset \mathbb{R}^N$. In particular it follows that, at the limit, $w_i w_j|_{y=0} = 0$ for every $j \neq i$. Furthermore, the first and the second inequalities in (4.2.4) follow from equation (4.2.1) and from the fact that, for every n,

$$\partial_\nu^a \left(w_{i,n} - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} w_{j,n} \right) = f_{i,n} - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} f_{j,n} + M_n \sum_{\substack{j\neq i \\ h\neq i,j}} \frac{a_{ij}}{a_{ji}} a_{jh} w_{j,n}^p w_{h,n}^p$$

(we recall that the reaction terms $f_{i,n} \to 0$ uniformly in $\mathbb{R}^N$). Finally, the identity in (4.2.4) can be obtained by multiplying the previous equation by $w_{i,n}$, once one can estimate the terms $M_n w_{i,n} w_{j,n}^p w_{h,n}^p$. To this aim, let $\varepsilon > 0$, and let us define the (possibly empty) set

$$\mathrm{supp}_i^\varepsilon = \{x \in \mathbb{R}^N : w_i(x,0) \ge \varepsilon\}.$$

We observe that for any $K \subset \mathbb{R}^N$ compact set, the local uniform convergence of the sequence $\{\mathbf{w}_n\}$ implies

$$\begin{cases} w_{i,n}(x,0) \ge \frac{\varepsilon}{2} & \forall x \in K \cap \mathrm{supp}_i^\varepsilon \\ w_{i,n}(x,0) \le 2\varepsilon & \forall x \in K \setminus \mathrm{supp}_i^\varepsilon, \end{cases}$$

for any n large enough. As a consequence

$$\begin{aligned} M_n \int_K w_{i,n} w_{j,n}^p w_{h,n}^p \mathrm{d}x &\le M_n \int\limits_{K\setminus \mathrm{supp}_i^\varepsilon} w_{i,n} w_{j,n}^p w_{h,n}^p \mathrm{d}x + M_n \int\limits_{K\cap \mathrm{supp}_i^\varepsilon} w_{i,n} w_{j,n}^p w_{h,n}^p \mathrm{d}x \\ &\le M_n 2\varepsilon \int\limits_{K\setminus \mathrm{supp}_i^\varepsilon} w_{j,n}^p w_{h,n}^p \mathrm{d}x + M_n \int\limits_{K\cap \mathrm{supp}_i^\varepsilon} w_{i,n} 2^{2p} \frac{(1+\|f_{j,n}\|)^p}{M_n \varepsilon^p} \frac{(1+\|f_{h,n}\|)^p}{M_n \varepsilon^p} \mathrm{d}x \\ &\le C\left(\varepsilon + \frac{1}{M_n} \varepsilon^{-2p} \right), \end{aligned}$$

where we used estimate (4.2.5) and Lemma 4.1.4 to bound the two terms. Choosing n sufficiently large so that $\varepsilon^{-2p} \le \varepsilon M_n$, we conclude by the arbitrariness of ε that

$$\lim_{n\to\infty} M_n \int_K w_{i,n} w_{j,n}^p w_{h,n}^p \mathrm{d}x = 0 \qquad \text{for every } i \neq j \neq h. \qquad \square$$

Corollary 4.2.11. *Let $a_{ij} = 1$ for every i, j and $\mathbf{w}$ be a blow-up profile. For every $i \neq j$ the functions $z = w_i - w_j$ are such that*

$$\begin{cases} L_a z^\pm \le 0 & \text{in } \mathbb{R}^{N+1}_+ \\ z^\pm \partial^a_\nu z^\pm \le 0 & \text{on } \mathbb{R}^N. \end{cases}$$

Proof. A subtraction of the equation satisfied by $w_{i,n}$ and $w_{j,n}$ yields

$$\begin{aligned}(w_{i,n} - w_{j,n})^\pm \partial^a_\nu (w_{i,n} - w_{j,n}) &= (f_{i,n} - f_{j,n})(w_{i,n} - w_{j,n})^\pm \\ &\quad - M_n \underbrace{(w_{i,n} - w_{j,n})^\pm (w_{i,n}^p - w_{j,n}^p)}_{\ge 0} \sum_{h \neq i,j} w_{h,n}^p. \qquad \square\end{aligned}$$

4.3 Uniform Hölder bounds

We are ready to show the almost optimal uniform Hölder bounds, in the case of two competing species. This will be a consequence of the following lemma.

Lemma 4.3.1. *Under the assumption of Proposition 4.2.2, at least three components of the blow-up profile $\mathbf{w}$ are non constant, and all the constant components are trivial.*

Proof. We start by observing that each constant component has to be trivial: this is a direct consequence of the segregation condition $w_i w_j|_{y=0} = 0$ in the case $M_n \to \infty$, while, if $M_n \to M > 0$, it is implied by the boundary condition and the fact that at least w_1 is non constant.

If w_1 is the only non constant component, then we obtain a contradiction with Lemma 4.1.2 since in both cases $M_n \to M$ (Lemma 4.2.9) and $M_n \to \infty$ (Lemma 4.2.10), we have $\partial^a_\nu w_1 = 0$.

Let us now assume that only w_1 and, say, w_2 are non constant. Invoking again Lemma 4.2.9 and 4.2.10, we obtain in both cases that

$$\begin{cases} L_a \left(a_{21} w_1 - a_{12} w_2 \right) = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu \left(a_{21} w_1 - a_{12} w_2 \right) = 0 & \text{on } \mathbb{R}^N. \end{cases}$$

The application of Lemma 4.1.2 then implies

$$w_1 = C + \frac{a_{12}}{a_{21}} w_2,$$

where, up to a permutation between w_1 and w_2, we may assume that the constant C is non negative. If $M_n \to \infty$, the segregation condition $w_1 w_2|_{y=0} = 0$ yields

$$\left(C + \frac{a_{12}}{a_{21}} w_2\right) w_2|_{y=0} = 0 \implies C = w_1 = w_2 = 0,$$

a contradiction. In the remaining case, the function w_2 solves

$$\begin{cases} L_a w_2 = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial^a_\nu w_2 = -M w_2^p \left(C + \frac{a_{12}}{a_{21}} w_2\right)^p \le -C' w_2^{2p} & \text{on } \mathbb{R}^N, \end{cases}$$

in contradiction with Lemma 4.1.5. □

Proof of Theorem 1.3.1. The lemma above, combined with Proposition 4.2.2, provides all the results in the theorem but the $H^{1;a}$ convergence; this last property follows from the uniform Hölder bounds, reasoning as in the proof of Lemma 4.2.7. □

Now we turn to the case of $k \ge 3$ densities. We first prove uniform Hölder bounds with small exponent, when the power p is greater or equal than 1. In order to quantify such exponent, we need to introduce some notation. For any $\omega \subset \mathbb{S}^N_+ := \partial^+ B_1^+$ we consider the first eigenvalue of (the angular part of) L_a, defined as

$$\lambda_1(\omega) = \inf \left\{ \int_{\mathbb{S}^N_+} |y|^a |\nabla_T u|^2 \,\mathrm{d}\sigma : u \equiv 0 \text{ on } \mathbb{S}^N_+ \setminus \omega, \int_{\mathbb{S}^N_+} |y|^a u^2 \,\mathrm{d}\sigma = 1 \right\}$$

(here ∇_T denotes the tangential part of the gradient), and the associated characteristic function

$$\gamma(t) = \sqrt{\left(\frac{N-2s}{2}\right)^2 + t} - \frac{N-2s}{2}.$$

We are ready to state the following Liouville type result.

Proposition 4.3.2. *Under the assumption*

$$p \ge 1,$$

let $\mathbf{w}$ *denote a blow-up limit as in Proposition 4.2.2, and let*

$$\nu = \nu(s,N) := \inf \left\{ \frac{\gamma(\lambda_1(\omega_1)) + \gamma(\lambda_1(\omega_2))}{2} : \omega_i \subset \mathbb{S}^N_+, \, \omega_1 \cap \omega_2 \cap \{y=0\} = \emptyset \right\}.$$

If

$$|\mathbf{w}(X)| \le C(1 + |X|^\alpha), \qquad \textit{for some } \alpha < \nu,$$

then $k-1$ *components of* $\mathbf{w}$ *are trivial.*

Remark 4.3.3. As shown in Lemma 2.1.5, Lemma 3.1.8, $\nu(s,N) > 0$ (and $\nu(s,N) \leq s$) for every $0 < s < 1$, $N \geq 1$.

Proof. The proof is a byproduct of arguments already exploited in the subsection 3.1.2 of the previous chapter. The first step consists in obtaining a monotonicity formula of Alt-Caffarelli-Friedman type, with exponent between α and ν. In the case in which $\mathbf{w}$ has segregated traces on $\{y = 0\}$, this is Proposition 3.1.9. When $\mathbf{w}$ satisfies a differential system, this can be done as in Proposition 3.1.10, with minor changes: namely, by replacing the term $v_i^2 v_j^2$ with $v_i^{p+1} v_j^p$ (this can be done as far as $p \geq 1$).

Once the validity of the monotonicity formula holds, one can deduce a related minimal growth rate for $\mathbf{w}$, which is consistent with the one in the assumption only if all the components but one vanish. $\square$

The result above can be improved, also removing the restriction on p, in the case of equal competition rates.

Proposition 4.3.4. *Under the assumption*

$$a_{ij} = 1 \qquad \textit{for every } 1 \leq i,j \leq k,$$

let $\mathbf{w}$ *denote a blow-up limit as in Proposition 4.2.2, and let*

$$\mu = \mu(s,N) := \inf\left\{ \frac{\gamma(\lambda_1(\omega_1)) + \gamma(\lambda_1(\omega_2))}{2} : \omega_i \subset \mathbb{S}^N_+,\ \omega_1 \cap \omega_2 = \emptyset \right\}.$$

If

$$|\mathbf{w}(X)| \leq C(1 + |X|^\alpha), \qquad \textit{for some } \alpha < \mu,$$

then $k-1$ *components of* $\mathbf{w}$ *are trivial.*

Remark 4.3.5. It is immediate to check that $\mu(s,N) \geq \nu(s,N)$ for every s, N. In particular, it is always positive. As a matter of fact, at the end of this section we will show that $1/2 \leq \mu(s,N) \leq 1$ for every $0 < s < 1$, $N \geq 1$. Furthermore, it is proved in [2, 9] that

$$\mu\left(\frac{1}{2}, N\right) = 1, \qquad \text{for every } N \geq 1. \tag{4.3.1}$$

Proof. We start showing that, for any choice $i \neq j$, if $w_i(\cdot,0) \leq w_j(\cdot,0)$ then $w_i \equiv 0$. Indeed, if $\mathbf{w}$ solves the differential system, then

$$\partial^a_\nu w_i \leq -M w_i^p w_j^p \leq -M w_i^{2p},$$

and the claim follows by Lemma 4.1.5; in the case of segregated traces, then $w_i(\cdot,0) \equiv 0$ and one can conclude by applying Lemma 4.1.1 (to the odd extension of w_i across $\{y = 0\}$).

On the other hand, let us assume by contradiction that, for some $i \neq j$, the functions $z^{\pm} := (w_i - w_j)^{\pm}$ are both nontrivial. Then they satisfy the inequalities in Corollary 4.2.11, and furthermore

$$|z^{\pm}(X)| \leq C(1 + |X|^{\alpha}),$$

where $\alpha < \mu$. Under these assumptions, we can obtain a contradiction by reasoning as in the proof of Proposition 4.3.2. To this aim, the only missing ingredient is the following monotonicity formula. $\square$

Lemma 4.3.6. *Let $z_1, z_2 \in H^{1;a}(B_R^+(x_0, 0))$ be continuous nonnegative functions such that*

- $z_1 z_2 = 0$, $z_i(x_0, 0) = 0$;
- *for every non negative $\phi \in \mathcal{C}_0^{\infty}(B_R(x_0, 0))$,*

$$\int_{\mathbb{R}^{N+1}_+} (L_a z_i) z_i \phi \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} (\partial^a_{\nu} z_i) z_i \phi \,\mathrm{d}x = \int_{\mathbb{R}^{N+1}_+} y^a \nabla z_i \cdot \nabla(z_i \phi) \,\mathrm{d}x\mathrm{d}y \leq 0.$$

Then

$$\Phi(r) := \prod_{i=1}^{2} \frac{1}{r^{2\mu}} \int_{B_r^+(x_0,0)} \frac{y^a |\nabla z_i|^2}{|X - (x_0, 0)|^{N-2s}} \,\mathrm{d}x\mathrm{d}y$$

is monotone non decreasing in r for $r \in (0, R)$, where μ is defined as in Proposition 4.3.4.

Proof. First of all we observe that, up to an even extension of the functions z_i across $\{y = 0\}$, the formula above is implied by the analogous one stated on the whole B_r. This latter formula, when $s = 1/2$, is nothing but the classical Alt-Caffarelli-Friedman one [2]. On the other hand, when $s \neq 1/2$, its proof resemble the usual one, as done for instance in [9] (see also Section 3.1 for further details). $\square$

To conclude the proof of Theorem 1.3.6, we provide the following rough elementary estimate of $\mu(s, N)$ for $s \neq 1/2$.

Lemma 4.3.7. *For every $0 < s < 1$ and $N \geq 1$ it holds*

$$\mu(s, N) \geq \frac{1}{2}.$$

Proof. By trivial extension to higher dimensions of the eigenfunctions involved, it is easy to prove that μ is decreasing with respect to N, thus we can assume $N \geq 2$.

Let $\omega_1, \omega_2 \subset \mathbb{S}^N_+$, $\omega_1 \cap \omega_2 = \emptyset$, and let $\phi_i \in H^{1;a}(\mathbb{S}^N_+)$ be the first eigenfunction associated to $\lambda_1(\omega_i)$ enjoying the normalization

$$\int_{\mathbb{S}^N} |y|^a \phi_i^2 \mathrm{d}\sigma = 1 \qquad \text{for } i = 1, 2.$$

If $\mathcal{R}$ denotes the Rayleigh quotient associated to λ_1, then we have that

$$\lambda_2(\mathbb{S}^N_+) := \inf_{\substack{V \subset H^{1;a}(\mathbb{S}^N_+) \\ \dim V \geq 2}} \max_V \mathcal{R} \leq \max_\vartheta \mathcal{R}\left(\phi_1 \cos\vartheta + \phi_2 \sin\vartheta\right) \leq \max(\lambda_1(\omega_1), \lambda_1(\omega_2)).$$

By monotonicity of γ we obtain that

$$\mu(s, N) = \inf_{\omega_1 \cap \omega_2 = \emptyset} \frac{\gamma(\lambda_1(\omega_1)) + \gamma(\lambda_1(\omega_2))}{2} \geq \frac{1}{2}\gamma(\lambda_2(\mathbb{S}^N_+)).$$

To conclude the proof, we show that $\gamma(\lambda_2(\mathbb{S}^N_+)) = 1$. Indeed, let ψ_2 be a second eigenfunction. Then its conormal derivative on $\partial\mathbb{S}^N_+$ is identically zero, and it can be extended in an even way across $\{y = 0\}$ to an eigenfunction of $\mathbb{S}^N$. Moreover, by the well known properties of γ, we have that the function

$$v(X) = |X|^{\gamma(\lambda_2(\mathbb{S}^N_+))} \psi_2\left(\frac{X}{|X|}\right)$$

is L_a-harmonic up to 0 (this is true, actually, because we are assuming $N \geq 2$), is y-even, and has bounded growth. By Lemma 4.1.1 we deduce that, up to a rotation in the x plane, $v = x_1$, concluding the proof. $\square$

Proof of Theorem 1.3.6. The uniform Hölder bounds with exponent α^* are obtained by combining Lemma 4.3.1 with either Proposition 4.3.2 (with $\alpha^* = \min(2s, \nu(s, N))$) or Proposition 4.3.4 (with $\alpha^* = \min(2s, \mu(s, N))$), respectively. In the second case, the exact value of α^* is provided by Remark 4.3.5 when $s = 1/2$, and by Lemma 4.3.7 when $s < 1/4$. $\square$

Remark 4.3.8. By comparison with the nodal partition of $\mathbb{S}^N_+$ associated to the homogeneous, L_a-harmonic function $v(x, y) = x_1$, we infer that

$$\mu(s, N) \leq 1.$$

4.4 Further properties of the segregation profiles

In this last section we deal with the proof of Theorems 1.3.2 and 1.3.7. Together with the previous assumptions, in what follows we further suppose that the reaction terms $f_{i,\beta} \to f_i$ as $\beta \to \infty$, uniformly on compact sets, with f_i Lipschitz continuous.

As a result of the previous sections, we have shown that L^∞ uniform bounds on a family of solutions to the problem $(LV)_\beta$ is enough to ensure equicontinuity of the family independently from the competition parameter β. Reasoning as in the proof of Lemmas 4.2.7 and 4.2.10 we deduce the following result.

Proposition 4.4.1. *Any sequence $\{\mathbf{v}_{\beta_n}\}_{n\in\mathbb{N}}$, $\beta_n \to \infty$, of solutions to $(LV)_\beta$ which is uniformly bounded in $L^\infty(B^+)$ admits a subsequence which converges to a limiting profile $\mathbf{v} \in (H^{1;a} \cap \mathcal{C}^{0,\alpha})_{\mathrm{loc}}(B^+)$, for some $\alpha > 0$. Moreover*

$$\begin{cases} L_a v_i = 0 & \text{in } B^+, \\ \partial^a_\nu v_i \le f_i(x, v_1, \dots, v_k) & \\ \partial^a_\nu \left(v_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} v_j\right) \ge f_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} f_j & \text{on } \partial^0 B^+, \\ v_i \cdot \left[\partial^a_\nu \left(v_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} v_j\right) - f_i + \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} f_j\right] = 0 & \end{cases} \tag{4.4.1}$$

and $v_i(x,0) \cdot v_j(x,0) \equiv 0$ for every $j \neq i$.

After Proposition 4.4.1, the optimal regularity for the case of two densities is almost straightforward.

Proof of Theorem 1.3.2. For a limiting profile $\mathbf{v} = (v_1, v_2)$, let $w = a_{21}v_1 - a_{12}v_2$. Then Proposition 4.4.1 implies that

$$v_1(x,0) = \frac{1}{a_{21}} w^+, \qquad v_2(x,0) = \frac{1}{a_{12}} w^-,$$

and

$$\begin{cases} L_a w = 0 & \text{in } B^+, \\ \partial^a_\nu w = g(w) & \text{on } \partial^0 B^+, \end{cases}$$

where

$$g(x,t) := a_{21} f_1\left(x, \frac{1}{a_{21}} t^+, \frac{1}{a_{12}} t^-\right) - a_{12} f_2\left(x, \frac{1}{a_{21}} t^+, \frac{1}{a_{12}} t^-\right)$$

is Lipschitz continuous. As a consequence, standard regularity (e.g. [45, Proposition 2.8], [29, Lemma 2.1]) applies, providing that $w \in C^{1,\alpha}$ and thus u_1, u_2 are Lipschitz continuous. □

Remark 4.4.2. An important consequence of the argument above is that whenever there are only two species that are segregated, under suitable growth conditions about f_1, f_2 the corresponding free boundary

$$\Gamma := \{x \in \partial^0 B : v_1(x,0) = v_2(x,0) = 0\}$$

is a closed set of empty interior (in the N dimensional topology). Indeed $w = a_{21}v_1 - a_{12}v_2$ satisfies a semilinear equation for which unique continuation holds, see [31, Theorems 1.4, 4.1].

We are left to deal with the case $k \geq 3$ for the half-laplacian, i.e.

$$s = \frac{1}{2}.$$

In this case, by Theorem 1.3.6 we already know that, when $a_{ij} = 1$, the traces of the limiting profiles enjoy almost Lipschitz continuity on $K \cap \{y \geq 0\}$, for every compact $K \subset B$. We are going to show that the same holds also for general a_{ij}, when there are no internal reaction terms in a neighborhood of the free boundary. More precisely, we assume that the Lipschitz continuous functions f_i are such that

$$f_i(x, t_1, \dots, t_k) \equiv 0 \qquad \text{whenever } |(t_1, \dots, t_k)| < \theta,$$

for some $\theta > 0$ (such assumption can be weakened, but we prefer to avoid further technicalities at this point). Finally, $K \subset B$ will denote a fixed compact set.

Remark 4.4.3. As before, since the components of a limiting profile $\mathbf{v}$ are harmonic on B^+, we have that its regularity on K is directly connected to the regularity of the same function in $K \cap \{0 \leq y < \varepsilon\}$ for arbitrarily small $\varepsilon > 0$.

Definition 4.4.4. For any function $\mathbf{v} \in H^1 \cap \mathcal{C}(B^+; \mathbb{R}^k)$ which satisfies (4.4.1) (with $s = 1/2$), we let $\hat{\mathbf{v}} := (\hat{v}_1, \dots, \hat{v}_k)$ where

$$\hat{v}_i(x, y) = v_i(x, y) - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} v_j(x, y).$$

To clarify the effect of the segregation condition, we introduce the definition of multiplicity of boundary points.

Definition 4.4.5. We define the multiplicity of a point $x \in \partial^0 B^+$ as

$$m(x) := \sharp \left\{ i : \mathcal{H}^N(\{v_i(x,0) > 0\} \cap \partial^0 B_r(x,0)) > 0. \forall r > 0 \right\}.$$

We start with a result about the regularity of low multiplicity points.

Lemma 4.4.6. *If $K \cap \{y = 0\} \subset \{x : m(x) \leq 1\}$ then $\mathbf{v} \in \mathcal{C}^{1,1/2}(K \cap \{y \geq 0\})$.*

Proof. According to Remark 4.4.3, we will show local regularity of the functions in $B_r^+(x_0, 0)$, where r is small and $m(x_0) \leq 1$. We have three possibilities.

Case 1: $m(x_0) = 0$. in this case, $\mathbf{v}|_{\partial^0 B_r(x_0,0)} \equiv 0$, and the result is standard.

Case 2: $m(x_0) = 1$ and $v_i(x_0, 0) > 0$. By continuity of v_i, we can assume that $v_i|_{\partial^0 B_r(x_0,0)} > 0$, while by the segregation condition $v_j|_{\partial^0 B_r(x_0,0)} \equiv 0$ for every $j \neq i$. Let $\hat{\mathbf{v}}$ be as in Definition 4.4.4. Since in this case $\hat{v}_i = v_i$ on $\partial^0 B_r(x_0, 0)$, it follows from (4.4.1) that

$$\begin{cases} -\Delta \hat{v}_i = 0 & \text{in } B_r^+(x_0, 0) \\ \partial_\nu \hat{v}_i = f(x, 0, \dots, \hat{v}_i, \dots, 0) & \text{in } \partial^0 B_r^+(x_0, 0). \end{cases}$$

The regularity of $\hat{v}_i$ (and thus of v_i) follows by the well established regularity theory of the semilinear Steklov problem.

Case 3: $m(x_0) = 1$ and the non trivial function v_i is such that $v_i(x_0, 0) = 0$. Also in this case we can assume $v_j|_{\partial^0 B_r(x_0,0)} \equiv 0$ for $j \neq i$ and, as before, $v_i = \hat{v}_i \geq 0$ on $\partial^0 B_r(x_0, 0)$. By continuity of v_i, we can also assume that $f_i = 0$ in $\partial^0 B_r(x_0, 0)$. It follows that

$$\begin{cases} -\Delta \hat{v}_i = 0 & \text{in } B_r(x_0, 0) \\ \partial_\nu \hat{v}_i = 0 & \text{in } \partial^0 B_r(x_0, 0) \cap \{\hat{v}_i|_{y=0} > 0\} \\ \partial_\nu \hat{v}_i \geq 0 & \text{in } \partial^0 B_r(x_0, 0). \end{cases}$$

As a consequence, $\hat{v}_i$ is a solution to the zero thin obstacle problem, for which $\mathcal{C}^{1,1/2}$ regularity (up to the obstacle) has been obtained in [3, Theorem 5]. □

Remark 4.4.7. The analogous of the previous lemma holds true also when $s \neq 1/2$, in which case $\mathcal{C}^{1,s}$ regularity can be shown, as a consequence of [15].

Now, for $X \in B$, we introduce the the Morrey quotient associated to $\mathbf{v}$ as

$$\Phi(X, r) := \frac{1}{r^{N+1-2\varepsilon}} \int\limits_{B_r(X) \cap B^+} \sum_{i=1}^{k} |\nabla v_i|^2 \, \mathrm{d}x\mathrm{d}y.$$

It is well known that if Φ is uniformly bounded for any $X \in K \cap \{y \geq 0\}$ and $r < \mathrm{dist}(K, \partial^+ B^+)$, then $\mathbf{v}$ is Hölder continuous of exponent $1 - \varepsilon$ in $K \cap \{y \geq 0\}$. Thus the proof of Theorem 1.3.7 is based on the contradictory assumption that, for some $\varepsilon > 0$, there is a sequence $\{(X_n, r_n)\}_n$ such that $X_n \in K$, $r_n > 0$ and

$$\lim_{n \to \infty} \Phi(X_n, r_n) = \infty. \tag{4.4.2}$$

To reach a contradiction we will use the following technical lemma.

Lemma 4.4.8 ([23, Lemma 8.2])**.** *Let $\Omega \subset \mathbb{R}^{N+1}$ and $v \in H^1(\Omega)$ and let*

$$\Phi(X, r) := \frac{1}{r^{N+1-2\varepsilon}} \int_{B_r(X) \cap \Omega} |\nabla v|^2 \mathrm{d}x\mathrm{d}y.$$

If $(X_n, r_n) \subset \overline{\Omega} \times \mathbb{R}^+$ is a sequence such that $\Phi(X_n, r_n) \to \infty$, then $r_n \to 0$ and

1. *there exists $\{r_n'\} \subset \mathbb{R}^+$ such that $\phi(X_n, r_n') \to \infty$ and*
$$\int_{\partial B_{r_n'}(X_n) \cap \Omega} |\nabla v|^2 \leq \frac{N + 1 - 2\varepsilon}{r_n'} \int_{B_{r_n'}(X_n) \cap \Omega} |\nabla v|^2; \tag{4.4.3}$$

2. *if $A \subset \overline{\Omega}$ and*
$$\mathrm{dist}(X_n, A) \leq C r_n$$
then there exists a sequence $\{(X_n', r_n')\}$ such that $\phi(X_n', r_n') \to \infty$ and $X_n' \in A$ for every n.

Proof of Theorem 1.3.7. Using the second point of Lemma 4.4.8, together with Remark 4.4.3, we can assume without loss of generality that the contradictory assumption (4.4.2) holds for $\partial^0 B^+ \ni X_n =: (x_n, 0)$, for every n. For lighter notation we write $\Phi(x_n, r_n)$ instead of $\Phi((x_n, 0), r_n)$. Lemma 4.4.8 also implies that $r_n \to 0$, and that we can assume estimate (4.4.3) to hold for any n, with $\Omega = \{y > 0\}$. Furthermore, since $\mathbf{v} \in H^1(B^+)$, the function $r \mapsto \Phi(x_n, r)$ is continuous for $r > 0$ and it is uniformly bounded for r faraway from 0: as a consequence we can assume that

$$\Phi(x_n, r) \leq C\Phi(x_n, r_n) \qquad \forall r_n < r < \operatorname{dist}(K, \partial^+ B^+),$$

for some constat C. Finally, by Lemmas 4.4.6 and 4.4.8 we can assume $m(x_n) \geq 2$ for n large, and thus $f_i(\cdot, v_i) \equiv 0$ on $B^+_{r_n}((x_n, 0))$.

Let us introduce a sequence of scaled function $\mathbf{v}_n$ defined as

$$v_{i,n}(X) := \frac{1}{\Phi(X_n, r_n)^{1/2} r_n^{1-\varepsilon}} v_i((x_n, 0) + r_n X) \qquad \text{for } X \in B.$$

By assumptions, $\|\nabla \mathbf{v}_n\|_{L^2(B^+)} = 1$ for every n, and

$$\frac{1}{r^{N+1-2\varepsilon}} \int_{B_r^+} \sum_{i=1}^k |\nabla v_{i,n}|^2 \leq C \qquad \forall 1 < r < r_n^{-1} \operatorname{dist}(K, \partial^+ B^+). \tag{4.4.4}$$

We divide the rest of the proof in a number of steps.

Step 1: also $\|\mathbf{v}_n\|_{L^2(B^+)}$ is uniformly bounded. We argue by contradiction, assuming that $\|\mathbf{v}_n\|_{L^2(B^+)} \to \infty$. Letting

$$\mathbf{u}_n := \|\mathbf{v}_n\|^{-1}_{L^2(B^+)} \mathbf{v}_n$$

we have that $\|\mathbf{u}_n\|_{L^2(B^+)} = 1$, while $\|\nabla \mathbf{u}_n\|_{L^2(B^+)} \to 0$: there exists $\mathbf{d} \in \mathbb{R}^k$ such that

$$\mathbf{u}_n \to \mathbf{d} \qquad \text{in } H^1(B^+).$$

Using the segregation condition $v_{i,n} \cdot v_{j,n}|_{y=0} = 0$, which passes to the strong limit, we infer that only one among the constant d_i may be non trivial, say $d_1 > 0$. But recalling that the even extension of $\hat{v}_{i,n}$ across $\{y = 0\}$ is superharmonic, we find

$$\hat{v}_{1,n}(0) = 0 \implies \int_{B^+} \sum_{j \neq 1} \frac{a_{ij}}{a_{ji}} v_{j,n} \geq \int_{B^+} v_{1,n}$$

a contradiction, passing to the strong limit in $H^1(B^+)$.

Step 2: the sequence $\mathbf{v}_n$ admits a nontrivial weak limit $\bar{\mathbf{v}} \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+})$. From Step 1 and the uniform estimate (4.4.4) we infer the weak convergence; let us show that $\bar{\mathbf{v}}$ is non trivial. To this end, we recall that

$$\begin{cases} -\Delta v_{i,n} = 0 & \text{in } B^+ \\ v_{i,n} \partial_\nu v_{i,n} \leq 0 & \text{on } \partial^0 B^+. \end{cases}$$

Testing the equation against $v_{i,n}$ and summing over i, we have

$$\int_{B^+} |\nabla \mathbf{v}_n|^2 \leq \int_{\partial^+ B^+} \sum_{i=1}^{k} v_{i,n} \partial_\nu v_{i,n} \leq \left(\int_{\partial^+ B^+} |\mathbf{v}_n|^2 \cdot \int_{\partial^+ B^+} |\nabla \mathbf{v}_n|^2 \right)^{1/2}.$$

Were $\bar{\mathbf{v}}$ trivial, the right hand side would go to zero thanks to the compact embedding of the trace operator and the uniform estimate (4.4.3), which is scaling invariant. This would imply strong convergence, in contradiction with the fact that the L^2 norm of $\nabla \mathbf{v}_n$ is equal to 1.

Step 3: $\bar{\mathbf{v}}(x,0) \equiv 0$ on $\mathbb{R}^N$. Let us consider the sequence $\hat{\mathbf{v}}_n$ (recall Definition 4.4.4). From (4.4.1) (in the case $s = 1/2$), we know that the pair $(\hat{v}_i^+, \hat{v}_i^-)$ is made of two continuous, subharmonic, nonnegative functions such that $\hat{v}_i^+ \cdot \hat{v}_i^- = 0$ in $\mathbb{R}^{N+1}$. As a result, they satisfy the assumption of the Alt-Caffarelli-Friedmann monotonicity formula (Lemma 4.3.6 with $a = 0$ and $\mu = 1$), from which we obtain

$$\begin{aligned} &\frac{1}{r^{N+1}} \int_{B_r^+(x_n,0)} |\nabla \hat{v}_i^+|^2 \mathrm{d}x\mathrm{d}y \cdot \frac{1}{r^{N+1}} \int_{B_{r_n}^+(x_n,0)} |\nabla \hat{v}_i^-|^2 \mathrm{d}x\mathrm{d}y \leq \\ &\quad \frac{1}{r^2} \int_{B_r^+(x_n,0)} \frac{|\nabla \hat{v}_i^+|^2}{|X-(x_n,0)|^{N-1}} \mathrm{d}x\mathrm{d}y \cdot \frac{1}{r^2} \int_{B_r^+(x_n,0)} \frac{|\nabla \hat{v}_i^-|^2}{|X-(x_n,0)|^{N-1}} \mathrm{d}x\mathrm{d}y \leq C, \end{aligned}$$

that is

$$\frac{1}{r^{N+1-2\varepsilon}} \int_{B_r^+(x_n,0)} |\nabla \hat{v}_i^+|^2 \mathrm{d}x\mathrm{d}y \cdot \frac{1}{r^{N+1-2\varepsilon}} \int_{B_r^+(x_n,0)} |\nabla \hat{v}_i^-|^2 \leq C r^{4\varepsilon}.$$

Hence at most one of the two Morrey quotients can be unbounded. Moreover, by the triangular inequality, the possibly unbounded one diverges at most at the same rate of $\Phi(x_n, r_n)$. Scaling to $(\hat{v}_{i,n}^+, \hat{v}_{i,n}^-)$ we can distinguish among three different cases:

- both $\|\nabla \hat{v}_{i,n}^+\|_{L^2(B_r)}$ and $\|\nabla \hat{v}_{i,n}^-\|_{L^2(B_r)}$ are infinitesimal. In this situation, we have that there exists $c \geq 0$ such that $\hat{v}_{i,n} \to c$. Since by even extension

$$\begin{cases} -\Delta \hat{v}_{i,n} \geq 0 & \text{in } B_r \\ \hat{v}_{i,n}(0,0) = 0 \end{cases} \implies \int_{B_r} \hat{v}_{i,n} \leq 0$$

  we have that $\hat{v}_{i,n} \to c \leq 0$;

- there exists $c > 0$ such that $\|\nabla \hat{v}_{i,n}^+\|_{L^2(B_r)} \geq c > 0$ while $\|\nabla \hat{v}_{i,n}^-\|_{L^2(B_r)} \to 0$. Testing the equation

$$\begin{cases} -\Delta \hat{v}_{i,n}^+ \leq 0 & \text{in } B_r^+ \\ \hat{v}_{i,n}^+ \partial_\nu \hat{v}_{i,n}^+ \leq 0 & \text{on } \partial^0 B_r^+ \end{cases}$$

  with $\hat{v}_{i,n}^+$ we obtain that in the limit $\hat{v}_{i,n}^+ \rightharpoonup \hat{v}_i \neq 0$, and thus $\hat{v}_{i,n}^- \to 0$ strongly in $H^1(B_r)$. Using again the superharmonicity of $\hat{v}_{i,n}$ as before, we conclude that $\hat{v}_{i,n} \to 0$, in contradiction with $\hat{v}_i \neq 0$;

- there exists $c > 0$ such that $\|\nabla \hat{v}_{i,n}^-\|_{L^2(B_r)} \geq c > 0$ while $\|\nabla \hat{v}_{i,n}^+\|_{L^2(B_r)} \to 0$. Reasoning as in the previous case, we obtain that $\hat{v}_{i,n}^+ \to 0$ strongly in $H^1(B_r)$, thus $\hat{v}_{i,n} \rightharpoonup \hat{v}_i \leq 0$.

In any case, $\hat{v}_{i,n}^+ \to 0$ and $\hat{v}_{i,n} \rightharpoonup \hat{v}_i \leq 0$ in $H^1_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$ for all i, implying in particular that $v_{i,n}|_{y=0} \to \bar{v}_i|_{y=0} = 0$ in $H^{1/2}_{\mathrm{loc}}(\mathbb{R}^N)$.

Conclusion. If we extend $\bar{\mathbf{v}}$ evenly across $\{y = 0\}$, we obtain a k-tuple of harmonic functions defined on $\mathbb{R}^{N+1}$ for which $\Phi(0, r) \leq C$ for all $r \geq 1$. From the Morrey inequality, we have that for any $X \in \mathbb{R}^{N+1}$, $|X| \geq 1$,

$$|\bar{\mathbf{v}}(X) - \bar{\mathbf{v}}(0)| \leq C|X|^{1-\frac{N+1}{2}} \|\nabla \bar{\mathbf{v}}\|_{L^2(B_{2|X|})}.$$

As a result, we have

$$|\bar{\mathbf{v}}(X) - \bar{\mathbf{v}}(0)| \leq C|X|^{1-\varepsilon}$$

for every $|X| \geq 1$, in contradiction with the fact that $\mathbf{v}$ is harmonic in $\mathbb{R}^{N+1}$ and non trivial, thanks to the classical Liouville theorem. □

Remark 4.4.9. More general nonlinearities should be addressable, using similar arguments as before, once a generalization of the Caffarelli-Jerison-Kenig almost monotonicity formula [10] to this setting were available.

Remark 4.4.10. The case $s \neq 1/2$ could follow as a generalization of the previous proof, if not for the fact that, at the moment, no exact Alt-Caffarelli-Friedman monotonicity formula is available, in this setting: one could only show, by Lemma 4.3.6 and 4.3.7, the $\mathcal{C}^{0,\alpha}$ continuity of the limiting profiles, for every $\alpha < 2s$ and $\alpha \leq \mu$.

Entire solution for elliptic systems

Chapter 5

Entire solutions with exponential growth for elliptic systems

5.1 Introduction and main results

The last part of the thesis is concerned with the construction of solution to the system

$$\begin{cases} \Delta u = uv^2 & \text{in } \mathbb{R}^N \\ \Delta v = u^2 v & \text{in } \mathbb{R}^N \end{cases} \tag{$(S)_2$}$$

which have a super-polynomial growth, namely, they exhibit an exponential growth along some direction. The equation under investigation arises in many contests in applied mathematics involving, for instance, phase separations. In particular, there is a link in the study of Bose-Einstein condensates. As it is well known, bosons constitute a family of particles which do not obey the Pauli exclusion principle, so that it is possible for many particles to occupy the same quantum state. From an experimental point of view, it is possible to produce artificially bosons via ultra-cooling a gas of atoms. The first empirical evidence of condensation was obtained in 1995. In these experiments, diluted atomic gases are initially trapped by magnetic fields and cooled down at very low temperatures. Below a critical temperature T_c, all the atoms of the gas collapse to the ground state and the whole gas evolves as a single quantum particle with a macroscopic de Broglie wavelength Ψ, given thus by

$$\Psi(x_1, \ldots, x_n) \sim \prod_{i=1}^{n} \psi(x_i).$$

Under some approximation, in [36, 42] Gross and Pitaevskii proposed[1] that the wave

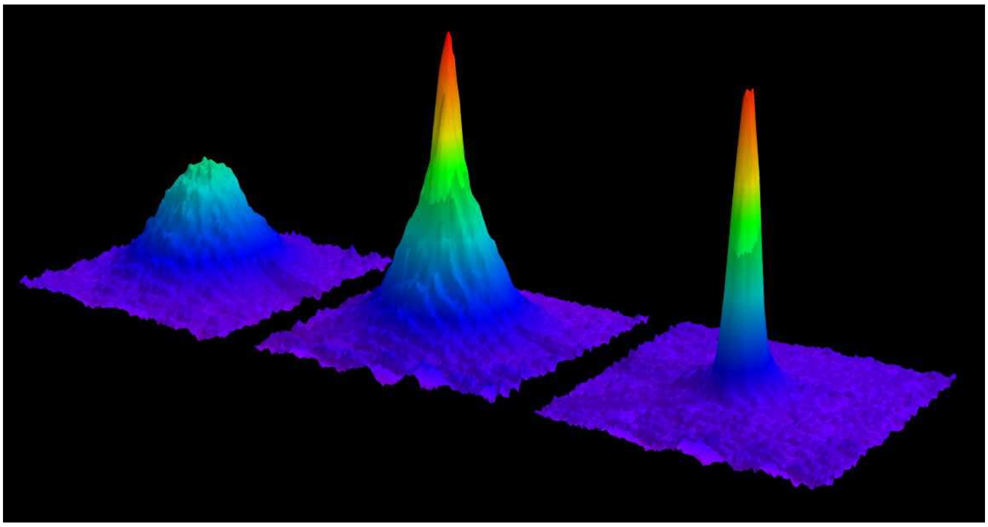

Figure 5.1: Distribution of the velocity during a condensation.

function ψ of a single particle satisfies the equation

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + V(x) - \omega|\psi|^2\right)\psi.$$

When more species of particles are involved, the previous model has to be modified in to the system of Schrödinger equations

$$i\partial_t\psi_i = \left(-\Delta + V(x) - \omega_i|\psi_i|^2 + \sum_{j\neq i}\beta_{ij}|\psi_j|^2\right)\psi_i$$

where the parameters β_{ij} represent the interspecific scattering lengths. When looking for standing wave solutions for the previous system, (that is, solution of the form $\psi_i(x,t) = e^{-i\lambda_i t}u_i(x)$), one finds that they obey to the stationary system

$$\begin{cases} -\Delta u_i + (\lambda_i + V(x))u_i = \omega_i|u_i|^2 u_i - u_i\sum_{j\neq i}\beta_{ij}u_j^2 \\ u_i \in H_0^1(\Omega) \text{ for every } i = 1,\dots,k. \end{cases}$$

As the interspecies scattering lengths $\beta_{ij} = \beta \to +\infty$ the densities tends to segregate, giving birth to a pattern of disjoint condensates. A main issue here is to characterize precisely how this segregation occurs.

More precisely, considering the case of two non negative components, as $\beta \to +\infty$, there is convergence (up to a subsequence) to some limiting profile (u_β, v_β) which are

[1]see also [38] for a rigorous derivation of the Gross-Pitaevskii model

solutions to

$$\begin{cases} -\Delta u + \lambda_1 u = \omega_1 u^3 & \text{in } \Omega_u, \\ -\Delta v + \lambda_2 v = \omega_2 v^3 & \text{in } \Omega_v, \end{cases}$$

where $\Omega_u := \{x \in \Omega : u > 0\}$ and $\Omega_v := \{x \in \Omega : v > 0\}$ are positivity domains composed of finitely disjoint components with positive Lebesgue measure, and letting $w = u - v$, it holds

$$-\Delta w + \lambda_1 w^+ - \lambda_2 w^- = \omega_1 (w^+)^3 - \omega_2 (w^-)^3.$$

To the end of studying the convergence of the solutions, one is lead to exploit a blow up analysis. One considers the points

$$x_\beta \in \Omega \text{ such that } u_\beta(x_\beta) = v_\beta(x_\beta) =: m_\beta$$

and scales the equation accordingly around such points. At least in dimension $N = 1$, in [5] it is shown that $m_\beta^4 \beta \to C \in (0, \infty)$ as $\beta \to \infty$ which gives the correct scaling rate of the equation. It follows that, letting

$$\hat{u}_\beta(x) := \frac{1}{m_\beta} u_\beta(m_\beta x + x_\beta), \quad \hat{v}_\beta(x) := \frac{1}{m_\beta} v_\beta(m_\beta x + x_\beta)$$

there is accumulation of the sequence in $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R})$ to solutions of

$$u'' = uv^2, v'' = u^2 v \quad \text{in } \mathbb{R}.$$

For higher dimensions, an equivalent statement is not available yet, but it is conjectured by the authors of [5] that the same asymptotic for m_β should holds. Under this assumption, it is possible to show that limits of the same scaling converge to a solution of

$$\begin{cases} \Delta u = uv^2 & \text{in } \mathbb{R}^N, \\ \Delta v = u^2 v & \text{in } \mathbb{R}^N. \end{cases} \tag{$(S)_2$}$$

To understand the geometry of the solutions of the previous system will then clarify about the behavior of the segregation. A first result on the solutions of $(S)_2$ can be found in [39] where the following Liouville-type theorem is stated.

Theorem (Proposition 2.6 [39])**.** *Let $(u, v) \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ be entire solutions to $(S)_2$ satisfying the growth condition*

$$|u|(x) + |v|(x) \leq C(1 + |x|^\alpha)$$

for some $\alpha \in (0, 1)$. Then either $u \equiv 0$ and v is constant or $v \equiv 0$ and u is constant.

Remark 5.1.1. Let us mention that Theorem 2.6.1 is the precise equivalent of the previous one in the case $s = 1/2$. There, the critical growth rate seems to be $\alpha = 1/2$, even though at this stage no existence result has been shown in the limiting case. The case $s \in (0, 1)$ is not yet addressed, even if it seems reasonable to conjecture that the critical growth condition should be given by the exponent s itself.

As the growth rate reaches the linear limit, that is $\alpha = 1$, on the other hand it is possible to show existence of solutions, as done in [5] and refined in [6] in the shape of the following two theorems.

Theorem (Theorems 1.2-1.3 [5], Theorem 1.1 [6])**.** *Let $N = 1$, and let (u, v) be a non negative (and non trivial) solution of $(S)_2$. Then there exists a point $x_0 \in \mathbb{R}$ such that $u(x - x_0) = v(x_0 - x)$ while $u(-\infty) = u'(-\infty) = 0$, $u(+\infty) = +\infty$ and $u'(+\infty) = C > 0$. The couple (u, v) is unique up to translation, scaling and exchange and moreover the solution is nondegenerate and stable*[2].

Theorem (Theorem 1.2 [6])**.** *In $\mathbb{R}^2$ any stable solution (U, V) which has at most linear growth at infinity, is one-dimensional, that is, there exists $a \in \mathbb{R}^2$, $|a| = 1$ such that*

$$U(x, y) = u(a \cdot (x, y)) \quad V(x, y) = v(a \cdot (x, y))$$

where (u, v) is a couple of one-dimensional solutions.

At this point, the research has divided into two main streams: a) finding more general conditions which imply one-dimensionality of the solutions (conjectures of De Giorgi and Gibbons type) and b) constructing solutions which are not one-dimensional and do not exhibit linear growth at infinity. The main improvements in the first direction can be briefly synthesised in following three results: a solution (u, v) is one-dimensional if (u, v) grows at most linearly and the couple is a local minimizer to the corresponding energy functional [52]; (u, v) grows at most as a polynomial and the limit

$$\lim_{y\to\pm\infty}(u(x, y) - v(x, y)) = \pm\infty$$

holds uniformly in $x \in \mathbb{R}^{N-1}$ (see [33]); in the case $N = 2$ when (u, v) grows at most as a polynomial and u is strictly monotone in one direction (see [32]).

The non-existence result for non constant solutions which grow less than linearly and the one-dimensional theorems for solution having linear growth suggest a relation

[2] A solution (u, v) is called stable if the second differential of the energy is definite nonnegative, that is

$$\int_{\mathbb{R}^N} |\nabla\phi|^2 + |\nabla\psi|^2 + v^2\phi^2 + u^2\psi^2 + 4uv\phi\psi \geq 0 \quad \forall\phi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^N).$$

between solutions of the system $(S)_2$ and harmonic functions. This point is made clear in [6] where the following classification result is shown.

Theorem (Theorem 1.4 [6]). *Let (u, v) solve $(S)_2$ in $\mathbb{R}^N$, and let*

$$N(r) = \frac{E(r)}{H(r)} := \frac{\dfrac{1}{r^{N-2}} \displaystyle\int_{B_r(0)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2}{\dfrac{1}{r^{N-1}} \displaystyle\int_{\partial B_r(0)} u^2 + v^2}.$$

Then N is a monotone non decreasing function. If one assume that

$$\lim_{r\to\infty} N(r) < +\infty,$$

it must be $N(+\infty) = d \in \mathbb{N}$. Moreover, letting

$$(u_R(x), v_R(x)) := \frac{1}{\sqrt{H(R)}}(u(Rx), v(Rx))$$

then, as $R \to \infty$, the family of scaled functions (u_R, v_R) converges uniformly on compact sets of $\mathbb{R}^N$ to a multiple of (Ψ^+, Ψ^-), where Ψ is a harmonic homogeneous polynomial of degree d.

The quantization of the "growth rate" d and the convergence of the blow up profiles strengthen the link between solution of $(S)_2$ and harmonic function. The authors also present a dual (in dimension $N = 2$) of the previous theorem

Theorem (Theorem 1.3 [6]). *Let $d \in \mathbb{N}_0$ fixed. There exists a solution (u, v) of $(S)_2$ which shadows $\Phi = \Re(z^d)$, in particular $u \geq \Phi^+$ and $v \geq \Phi^-$; u is obtained from v by a rotations and reflections (the symmetries of model function Φ). Moreover, one has*

$$N(+\infty) = d \quad \textit{and} \quad (u_R, v_R) \to (\Phi^+, \Phi^-).$$

These results can be generalized also to systems of more than 2 components.

Motivated by the quoted achievements, we wonder if the system $(S)_2$ has solutions with growth higher than the one of any polynomial. We can give a positive answer to this question proving the existence of solutions with exponential growth. In our construction we adapt the same line of reasoning introduced in the proof of Theorem 1.3 of [6]. There, the authors proved the existence of solutions to $(S)_2$ with the same symmetry of the function $\Re(z^d)$ in any bounded ball $B_R(0) \subset \mathbb{R}^2$, with boundary conditions $u = (\Re(z^d))^+$, $v = (\Re(z^d))^-$ on $\partial B_R(0)$. By means of suitable monotonicity formulæ, they could pass to the limit for $R \to +\infty$ obtaining convergence (up to a subsequence) for the previous family to a nontrivial entire solution. In this sense, their solutions are modeled on the harmonic functions $\Re(z^d)$.

Here, having in mind the construction of solutions with exponential growth, and recalling the relationship between entire solution of our system and harmonic functions, we start by considering

$$\Phi(x, y) := \cosh x \sin y.$$

The first of our main results is the following.

Theorem 5.1.2. *There exists an entire solution $(u, v) \in (\mathcal{C}^\infty(\mathbb{R}^2))^2$ to system $(S)_2$ such that*

1) $u(x, y + 2\pi) = u(x, y)$ *and* $v(x, y + 2\pi) = v(x, y)$*,*

2) $u(-x, y) = u(x, y)$ *and* $v(-x, y) = v(x, y)$*,*

3) *the symmetries*

$$v(x,y) = u(x, y-\pi) \qquad u(x, \pi - y) = v(x, \pi + y)$$
$$u\left(x, \frac{\pi}{2} + y\right) = u\left(x, \frac{\pi}{2} - y\right) \qquad v\left(x, \frac{3}{2}\pi + y\right) = v\left(x, \frac{3}{2}\pi - y\right)$$

   hold,

4) $u - v > 0$ *in* $\{\Phi > 0\}$ *and* $v - u > 0$ *in* $\{\Phi < 0\}$*,*

5) $u > \Phi^+$ *and* $v > \Phi^-$ *in* $\mathbb{R}^2$*,*

6) *the function (*Almgren quotient*)*

$$r \mapsto \frac{\int_{(0,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\{r\}\times[0,2\pi]} u^2 + v^2}$$

   is well-defined for every $r > 0$, is nondecreasing, and

$$\lim_{r\to+\infty} \frac{\int_{(0,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\{r\}\times[0,2\pi]} u^2 + v^2} = 1,$$

7) *there exists the limit*

$$\lim_{r\to+\infty} \frac{\int_{\{r\}\times[0,2\pi]} u^2 + v^2}{e^{2r}} =: \alpha \in (0, +\infty).$$

Remark 5.1.3. This solution is modeled on the harmonic function Φ, in the sense that it inherits the symmetries of (Φ^+, Φ^-) and has the same rate of growth of Φ.

Remark 5.1.4. Point 7) of the Theorem gives a lower and a upper bound to the rate of growth of the quadratic mean of (u, v) on $\{r\} \times [0, 2\pi]$ when r varies:

$$\left(\int_{\{r\}\times[0,2\pi]} u^2 + v^2 \right)^{\frac{1}{2}} = O(e^r) \qquad \text{as } r \to +\infty.$$

The domain of integration takes into account the periodicity of (u, v). The quadratic mean of (u, v) on $\{r\} \times [0, 2\pi]$ has exponential growth, and the rate of growth is the same of the function e^r, which in turns has the same rate of growth of Φ. Note that the coefficient 1 in the exponent of e^r coincides with the limit as $r \to +\infty$ of the Almgren quotient defined in point 6).

Remark 5.1.5. With a scaling argument, it is not difficult to prove the existence of entire solutions with exponential growth of order λ for every $\lambda > 0$ (in the previous sense). To see this, let

$$(u_\lambda(x, y), v_\lambda(x, y)) = (\lambda u(\lambda x, \lambda y), \lambda v(\lambda x, \lambda y)\,.$$

It is straightforward to check that (u_λ, v_λ) is still a solution to $(S)_2$ in the plane, is $\frac{2\pi}{\lambda}$-periodic in y and is such that

$$u_\lambda(x, y) \geq \lambda \left(\cosh(\lambda x)\sin(\lambda y)\right)^+ \quad \text{ and } \quad v_\lambda(x, y) \geq \lambda \left(\cosh(\lambda x)\sin(\lambda y)\right)^- .$$

Moreover,

$$\lim_{r\to+\infty} \frac{\int_{(0,r)\times\left(0,\frac{2\pi}{\lambda}\right)} |\nabla u_\lambda|^2 + |\nabla v_\lambda|^2 + 2u_\lambda^2 v_\lambda^2}{\int_{\{r\}\times\left[0,\frac{2\pi}{\lambda}\right]} u_\lambda^2 + v_\lambda^2} = \lambda, \tag{5.1.1}$$

and

$$\lim_{r\to+\infty} \frac{\int_{\{r\}\times\left[0,\frac{2\pi}{\lambda}\right]} u_\lambda^2 + v_\lambda^2}{e^{2\lambda r}} = \lambda\alpha.$$

One can consider the solution (u_λ, v_λ) as related to the harmonic function $\cosh(\lambda x)\sin(\lambda y)$. This reveals that there exists a correspondence

$$\{(u_\lambda, v_\lambda) : \lambda > 0\} \leftrightarrow \{\sin(\lambda x)\cosh(\lambda y) : \lambda > 0\}.$$

Due to the invariance under translations and rotations of problem $(S)_2$, the family $\{(u_\lambda, v_\lambda) : \lambda > 0\}$ can equivalently be related with the families of harmonic functions $\{\cosh(\lambda x)\,[C_1\cos(\lambda y) + C_2\sin(\lambda y)]\}$ or $\{[C_3\cos(\lambda x) + C_4\sin(\lambda x)]\cosh(\lambda y) : \lambda > 0\}$, where $C_1, C_2, C_3, C_4 \in \mathbb{R}$.

As observed in Remark 5.1.4, the limit of the Almgren quotient in (5.1.1) describes the rate of the growth of the quadratic mean of (u_λ, v_λ) computed on an interval of periodicity in the y variable. The previous computation reveals that for every $\lambda > 0$ we can construct a solution having rate of growth equal to λ. This marks a relevant difference between entire solutions with polynomial growth and entire solutions with exponential growth: while in the former case the admissible rates of growth are quantized (Theorem 1.4 of [6]), in the latter one we can prescribe any positive real value as rate of growth.

Remark 5.1.5 reveals that, starting from the solution found in Theorem 5.1.2, we can build infinitely-many entire solutions with different exponential growth. However, noting that system $(S)_2$ is invariant under rotations, translations and scalings, intuitively speaking they are all the same solution. We wonder if there exists an entire solution of $(S)_2$ having exponential growth which cannot be obtained by the one found in Theorem 5.1.2 through a rotation, a translation or a scaling; the answer is affirmative. We denote

$$\Gamma(x,y) := e^x \sin y.$$

Theorem 5.1.6. *There exists an entire solution $(u,v) \in (\mathcal{C}^\infty(\mathbb{R}^2))^2$ to system $(S)_2$ which enjoys points 1), 3), 4) of Theorem 5.1.2; moreover*

2) for every $r \in \mathbb{R}$

$$\int_{(-\infty,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + u^2v^2 < +\infty, \tag{5.1.2}$$

5) $u > \Gamma^+$ and $v > \Gamma^-$ in $\mathbb{R}^2$ $\;u - v > \Gamma^+$ and $v - u > \Gamma^-$ in $\mathbb{R}^2$,

*6) the function (*Almgren quotient*)*

$$r \mapsto \frac{\int_{(-\infty,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\{r\}\times(0,2\pi)} u^2 + v^2}$$

is well-defined for every $r > 0$, is nondecreasing, and

$$\lim_{r\to+\infty} \frac{\int_{(-\infty,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\{r\}\times(0,2\pi)} u^2 + v^2} = 1,$$

7) there exist the limits

$$\lim_{r\to+\infty} \frac{\int_{\{r\}\times[0,2\pi]} u^2 + v^2}{e^{2r}} =: \beta \in (0,+\infty) \quad \textit{and} \quad \lim_{r\to-\infty} \int_{\{r\}\times[0,2\pi]} u^2 + v^2 = 0.$$

Remark 5.1.7. This solution is modeled on the harmonic function Γ. As explained in Remark 5.1.4, it is possible to obtain a family of entire solutions which is in correspondence with a family of harmonic functions.

Remark 5.1.8. Note that the Almgren quotients that we defined in Theorem 5.1.2 and 5.1.6 are different. They are both different to the Almgren quotient which has been defined in [6].

We can partially generalize our existence result to the case of systems with many components. To be precise, given an integer k, we will construct a solution $(u_1, \dots, u_k)$ of

$$\begin{cases} \Delta u_i = u_i \sum_{j \neq i} u_j^2 & \text{in } \mathbb{R}^2 \\ u_i > 0, & i = 1, \dots, k \end{cases} \tag{$(S)_k$}$$

having the same growth and the same symmetries of Γ. Here and in the following we consider the indexes are meant modulus k.

Theorem 5.1.9. *There exists an entire solution $(u_1, \dots, u_k) \in (\mathcal{C}^\infty(\mathbb{R}^2))^k$ to system $(S)_k$ such that, for every $i = 1, \dots, k$,*

1) $u_i(x, y + k\pi) = u_i(x, y)$,

2) the symmetries

$$u_{i+1}(x,y) = u_i(x, y - \pi) \qquad u_1\left(x, \frac{\pi}{2} + y\right) = u_1\left(x, \frac{\pi}{2} - y\right)$$

hold,

3) for every $r \in \mathbb{R}$

$$\int_{(-\infty,r)\times(0,k\pi)} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1\leq i<j\leq k} u_i^2 u_j^2 < +\infty;$$

*4) the function (*Almgren quotient*)*

$$r \mapsto \frac{\int_{(-\infty,r)\times(0,k\pi)} \sum_{i=1}^k |\nabla u_i|^2 + 2\sum_{1\leq i<j\leq k} u_i^2 u_j^2}{\int_{\{r\}\times[0,k\pi]} \sum_{i=1}^k u_i^2}$$

is well-defined for every $r > 0$, is nondecreasing, and

$$\lim_{r\to+\infty} \frac{\int_{(-\infty,r)\times(0,k\pi)} \sum_{i=1}^k |\nabla u_i|^2 + 2\sum_{1\leq i<j\leq k} u_i^2 u_j^2}{\int_{\{r\}\times[0,k\pi]} \sum_{i=1}^k u_i^2} = 1.$$

5) there exist the limits

$$\lim_{r\to+\infty} \int_{\{r\}\times[0,k\pi]} \sum_{i=1}^k u_i^2 =: \gamma \in (0, +\infty) \quad \textit{and} \quad \lim_{r\to-\infty} \int_{\{r\}\times[0,k\pi]} \sum_{i=1}^k u_i^2 = 0.$$

Our last main result is the counterpart of Theorem 1.4 of [6], regarding the quantization of the growth rates, in our setting. This can be quite surprising because, as we already observed, we cannot expect a quantization of the admissible rates of growth dealing with solutions with exponential growth, see Remark 5.1.5. Nevertheless, if we consider solutions which are periodic in one direction, prescribing a period such a quantization can be recovered.

Theorem 5.1.10. *Let (u, v) be a nontrivial solution of $(S)_2$ in $\mathbb{R}^2$ which is 2π-periodic in y, and such that one of the following situation occurs:*

(i) there holds

$$\lim_{r\to-\infty}\int_{\{r\}\times[0,2\pi]} u^2+v^2=0,$$

and

$$d := \lim_{r\to+\infty}\frac{\int_{(-\infty,r)\times(0,2\pi)}|\nabla u|^2+|\nabla v|^2+u^2v^2}{\int_{\{r\}\times[0,2\pi]}u^2+v^2}<+\infty.$$

(ii) $\partial_x u = 0 = \partial_x v$ on $\{a\}\times[0,2\pi]$ for some $a\in\mathbb{R}$, and

$$d := \lim_{r\to+\infty}\frac{\int_{(a,r)\times(0,2\pi)}|\nabla u|^2+|\nabla v|^2+u^2v^2}{\int_{\{r\}\times[0,2\pi]}u^2+v^2}<+\infty.$$

Then d is a positive integer,

$$\left(\int_{\{r\}\times[0,2\pi]}u^2+v^2\right)^{\frac{1}{2}}=O(e^{dr})\qquad \text{as } r\to+\infty,$$

and the sequence

$$(u_R(x,y),v_R(x,y)) := \frac{1}{\sqrt{\int_{\{r\}\times[0,2\pi]}u^2+v^2}}\,(u(x+R,y),v(x+R,y))$$

converges in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^2)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ to (Ψ^+,Ψ^-), where

$$\Psi(x,y)=e^{dx}\left(C_1\cos(dy)+C_2\sin(dy)\right)\qquad \text{for some } C_1,C_2\in\mathbb{R}.$$

In Section 5.2 we will prove some monotonicity formulæ which will come useful in the rest of the chapter. We can deal with two types of solutions: solutions satisfying a homogeneous Neumann condition defined in a cylinder $C_{(a,b)}$ with $a>-\infty$, or solutions defined in a semi-infinite cylinder of type $C_{(-\infty,b)}$ and decaying at $x\to-\infty$. For the sake of completeness and having in mind to use some monotonicity formulæ in the proof of Theorem 5.1.9, we will always consider the case of systems with k components.

The proof of Theorem 5.1.2 will be the object of Section 5.3. It follows the same sketch of the proof of Theorem 1.3 in [6]: we start by showing that for any $R>0$ there exists a solution (u_R, v_R) to $(S)_2$ in the cylinder C_R, with Dirichlet boundary condition

$$u_R=\Phi^+\quad\text{and}\quad v_R=\Phi^-\quad\text{on } \{-R,R\}\times[0,2\pi],$$

and exhibiting the same symmetries of (Φ^+,Φ^-). In order to obtain a solution defined in the whole C_∞, we wish to prove the $\mathcal{C}^2_{loc}(C_\infty)$ convergence of the family $\{(u_R,v_R):$

$R > 1\}$, as $R \to +\infty$. To show that this convergence occurs, we will exploit the monotonicity formulæ proved in subsection 5.2.1. With respect to Theorem 1.3 of [6], major difficulties arise in the precise characterization of the growth of (u,v), points 6) and 7) of Theorem 5.1.2.

In Section 5.4 we will prove Theorem 5.1.6. One could be tempted to try to adapt the proof of Theorem 5.1.2 replacing Φ with Γ. Unfortunately, in such a situation we could not exploit the results of subsection 5.2.1; this is related to the lack of the even symmetry in the x variable of the function Γ (note that the function Φ enjoys this symmetry). A possible way to overcome this problem is to work in semi-infinite cylinders $C_{(-\infty,R)}$ and use the monotonicity formulæ proved in subsection 5.2.2. But to work in an unbounded set introduces further complications: for instance, the compactness of the Sobolev embedding and of some trace operators, a property that we will use many times in section 5.3, does not hold in $C_{(-\infty,R)}$. Although we believe that this kind of obstacle can be overcome, we propose a different approach for the construction of solutions modeled on Γ, which is based on the elementary limit

$$\lim_{R\to+\infty} \Phi_R(x,y) = \Gamma(x,y) \qquad \forall (x,y) \in \mathbb{R}^2,$$

where $\Phi_R(x,y) = 2e^{-R}\cosh(x+R)\sin y$. We will prove the existence of a solution (u_R, v_R) of $(S)_2$ in $C_{(-3R,R)}$ with Dirichlet boundary condition

$$u_R = \Phi_R^+ \quad \text{and} \quad v_R = \Phi_R^- \quad \text{on } \{-3R, R\} \times [0, 2\pi],$$

and exhibiting the same symmetries of (Φ_R^+, Φ_R^-). Then, using again the results of Section 5.2, we will pass to the limit as $R \to +\infty$ proving the compactness of $\{(u_R, v_R)\}$.

Section 5.5 is devoted to the study of systems with many components. As in [6] the authors could prove in one shot an existence theorem for 2 or k components (there are no substantial changes in the proofs), it is natural to wonder if here we can simply adapt step by step the construction carried on in section 5.3 or 5.4, or not. Unfortunately, the answer is negative: following the sketch of the proof of Theorem 5.1.2, we can adapt most the results of sections 5.3 and 5.4 with minor changes, but in the counterpart of Proposition 5.3.1 we cannot prove the pointwise estimate given by point 4). As a consequence, with respect to subsections 5.3.2 and 5.4.2 we cannot show that the limit of the sequence $(u_{1,R}, \dots, u_{k,R})$ does not vanish. Note that, in the case of two components, this nondegeneracy is ensured precisely by the above pointwise estimate. As far as the case of k component in [6], we observe that they obtained nondegeneracy through their Corollary 5.4, which is the counterpart of point (i) of our Corollary 5.2.5. But, while there the estimate of the growth given by this statement is optimal, in our situation it does not provide any information; this is related to the different expression of the term of rest in the Almgren monotonicity formula, Proposition 5.2.4. This is why we have to use a completely different argument which is not based on the existence of solutions for the system of k components in bounded cylinders (or in semi-infinite cylinders), but rests on Theorem 1.6 of [6]. Roughly speaking, we will obtain the existence of a solution of $(S)_k$ with exponential growth as a limit of solutions of the same system having algebraic growth.

The proof of Theorem 5.1.10 will be the object of Section 5.6.

5.2 Almgren-type monotonicity formulæ

Let $k \geq 2$ be a fixed integer. In this section we are going to prove some monotonicity formulæ for solutions of

$$\begin{cases} \Delta u_i = u_i \sum_{j \neq i} u_j^2 & \text{in } \mathbb{R}^2 \\ u_i > 0, & i = 1, \dots, k \end{cases} \tag{$(S)_k$}$$

considered as defined in a cylinder $C_{(a,b)}$ (this means that we assume from the beginning that $(u_1, \dots, u_k)$ is $k\pi$-periodic in y).

In this section we will use many times the following general result:

Lemma 5.2.1. *Let $(u_1, \dots, u_k)$ be a solution of $(S)_k$ in $C_{(a,b)}$. Then the function*

$$r \mapsto \int_{\Sigma_r} \sum_{i=1}^{k} |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 - 2 \int_{\Sigma_r} \sum_{i=1}^{k} (\partial_x u_i)^2$$

is constant in (a, b).

Proof. Let $a < r_1 < r_2 < b$. We test the equation $(S)_k$ with $(\partial_x u_1, \dots, \partial_x u_k)$ in $C_{(r_1, r_2)}$: for every i it results

$$\int_{C_{(r_1,r_2)}} \frac{1}{2} \partial_x \left(|\nabla u_i|^2 \right) + \left(\sum_{j \neq i} u_j^2 \right) u_i \partial_x u_i = \int_{\Sigma_{r_2}} (\partial_x u_i)^2 - \int_{\Sigma_{r_1}} (\partial_x u_i)^2.$$

Summing for $i = 1, \dots, k$ we obtain

$$\int_{C_{(r_1,r_2)}} \partial_x \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) = 2 \int_{\Sigma_{r_2}} \sum_i (\partial_x u_i)^2 - 2 \int_{\Sigma_{r_1}} \sum_i (\partial_x u_i)^2,$$

which gives the thesis. □

5.2.1 Solutions with Neumann boundary conditions

In this subsection we are interested in solutions to $(S)_k$ defined in $C_{(a,b)}$ (thus $k\pi$-periodic in y), with $a > -\infty$ and $b \in (a, +\infty]$, and satisfying a homogeneous Neumann boundary condition on Σ_a, that is,

$$\partial_x u_i = 0 \qquad \text{on } \Sigma_a, \text{ for every } i = 1, \dots, k. \tag{5.2.1}$$

Firstly, we observed that under this assumption Lemma 5.2.1 implies

Lemma 5.2.2. *Let $(u_1, \dots, u_k)$ be a solution of $(S)_k$ in $C_{(a,b)}$, such that (5.2.1) holds true. For every $r \in (a, b)$ the following identity holds:*

$$\int_{\Sigma_r} \sum_{i=1}^{k} |\nabla u_i|^2 + \sum_{1 \le i < j \le k} u_i^2 u_j^2 = 2 \int_{\Sigma_r} \sum_{i=1}^{k} (\partial_x u_i)^2 + \int_{\Sigma_a} \sum_{i=1}^{k} (\partial_y u_i)^2 + \sum_{1 \le i < j \le k} u_i^2 u_j^2.$$

For a solution $(u_1, \dots, u_k)$ of $(S)_k$ in $C_{(a,b)}$ satisfying (5.2.1), we define

$$\begin{aligned} E^{sym}(r) &:= \int_{C_{(a,r)}} \sum_{i=1}^{k} |\nabla u_i|^2 + 2 \sum_{1 \le i < j \le k} u_i^2 u_j^2, \\ \mathcal{E}^{sym}(r) &:= \int_{C_{(a,r)}} \sum_{i=1}^{k} |\nabla u_i|^2 + \sum_{1 \le i < j \le k} u_i^2 u_j^2, \\ H(r) &:= \int_{\Sigma_r} \sum_{i=1}^{k} u_i^2 \end{aligned}$$

Remark 5.2.3. The index *sym* denotes the fact that, as we will see, the quantities E^{sym} and $\mathcal{E}^{sym}$ are well suited to describe the growth of the solution $(u_1, \dots, u_k)$ only if $(u_1, \dots, u_k)$ satisfies the (5.2.1), which can be considered as a symmetry condition. Indeed, under (5.2.1) one can extend $(u_1, \dots, u_k)$ on $C_{(2a-b,b)}$ by even symmetry in the x variable.

By regularity, E, $\mathcal{E}$ and H are smooth. A direct computation shows that they are nondecreasing functions: in particular

$$H'(r) = 2 \int_{\Sigma_r} \sum_i u_i \partial_\nu u_i = 2E(r), \tag{5.2.2}$$

where the last identity follows from the divergence theorem and the boundary conditions of $(u_1, \dots, u_k)$. Our next result consist in showing that also the ratio between E (or $\mathcal{E}$) and H is nondecreasing.

Proposition 5.2.4. *Let $(u_1, \dots, u_k)$ be a solution of $(S)_k$ in $C_{(a,b)}$ such that (5.2.1) holds true. The* Almgren quotient

$$N^{sym}(r) := \frac{E^{sym}(r)}{H(r)}$$

is well defined and nondecreasing in (a, b). Moreover

$$\int_a^r \frac{\int_{\Sigma_s} \sum_{i<j} u_i^2 u_j^2}{H(s)} \, \mathrm{d}s \le N(r).$$

Analogously, the function (which we will call Almgren quotient, *too)* $\mathfrak{N}^{sym}(r) := \dfrac{E^{sym}(r)}{H(r)}$ *is well defined and nondecreasing in* (a,b), *and*

$$\mathfrak{N}'(r) \geq 2\mathfrak{N}(r)\frac{\int_{C_{(a,r)}}\sum_{i<j}u_i^2u_j^2}{H(r)} + 2\left(\frac{\int_{C_{(a,r)}}\sum_{i<j}u_i^2u_j^2}{H(r)}\right)^2.$$

In the rest of this subsection we will briefly write $E, \mathcal{E}, N$ and $\mathfrak{N}$ instead of $E^{sym}, \mathcal{E}^{sym}, N^{sym}$ and $\mathfrak{N}^{sym}$ to ease the notation.

Proof. Since $(u,v) \in H^1_{\mathrm{loc}}(C_{(a,b)})$ is nontrivial, E and H are positive in (a,b) and bounded for r bounded. We compute, by means of Lemma 5.2.2

$$\begin{aligned} E'(r) &= \int_{\Sigma_r}\sum_i |\nabla u_i|^2 + 2\sum_{i<j}u_i^2u_j^2 \\ &= \int_{\Sigma_r} 2\sum_i(\partial_x u_i)^2 + \sum_{i<j}u_i^2u_j^2 + \int_{\Sigma_a}\sum_i(\partial_y u_i)^2 + \sum_{i<j}u_i^2u_j^2. \end{aligned}$$

Note that $\partial_x u_i = \partial_\nu u_i$ on Σ_r. Using the previous identity and the (5.2.2) we are in position to compute the logarithmic derivative of N:

$$\begin{aligned} \frac{N'(r)}{N(r)} &= \frac{E'(r)}{E(r)} - \frac{H'(r)}{H(r)} \\ &= 2\frac{\int_{\Sigma_r}\sum_i(\partial_\nu u_i)^2}{\int_{\Sigma_r}\sum_i u\partial_\nu u_i} + \frac{2\int_{\Sigma_a}\sum_i(\partial_y u_i)^2 + \sum_{i<j}u_i^2u_j^2 + \int_{\Sigma_r}\sum_{i<j}u_i^2u_j^2}{E(r)} - 2\frac{\int_{\Sigma_r}\sum_i u\partial_\nu u_i}{\int_{\Sigma_r}\sum_i u_i^2} \\ &\geq 2\left(\frac{\int_{\Sigma_r}\sum_i(\partial_\nu u_i)^2}{\int_{\Sigma_r}\sum_i u\partial_\nu u_i} - \frac{\int_{\Sigma_r}\sum_i u\partial_\nu u_i}{\int_{\Sigma_r}\sum_i u_i^2}\right) + \frac{\int_{\Sigma_r}\sum_{i<j}u_i^2u_j^2}{E(r)} \geq \frac{\int_{\Sigma_r}\sum_{i<j}u_i^2u_j^2}{E(r)} \geq 0, \end{aligned}$$

where we used the Cauchy-Schwarz and the Young inequalities. As a consequence, N is nondecreasing in (a,b). Note also that

$$N'(r) \geq \frac{\int_{\Sigma_r}\sum_{i<j}u_i^2u_j^2}{H(r)} \quad \Rightarrow \quad N(r) \geq \int_a^r \frac{\int_{\Sigma_s}\sum_{i<j}u_i^2u_j^2}{H(s)}\,\mathrm{d}s$$

for every $r > a$. The same argument can be adapted with minor changes to prove the monotonicity of $\mathfrak{N}$. $\square$

As a first consequence, we have the following

Corollary 5.2.5. *Let* $(u_1,\dots,u_k)$ *be a solution of* $(S)_k$ *in* $C_{(a,b)}$ *such that* (5.2.1) *holds.*

(i) If $N(r) \geq \underline{d}$ *for* $r \geq s > a$, *then*

$$\frac{H(r_1)}{e^{2\underline{d}r_1}} \leq \frac{H(r_2)}{e^{2\underline{d}r_2}} \qquad \forall\ s \leq r_1 < r_2 < b,$$

ii) If $N(r) \leq \bar{d}$ for $r \leq t < b$, then

$$\frac{H(r_1)}{e^{2\bar{d}r_1}} \geq \frac{H(r_2)}{e^{2\bar{d}r_2}} \qquad \forall\ a < r_1 < r_2 \leq t.$$

Proof. We prove only (ii). Recalling that $H'(r) = 2E(r)$ (see (5.2.2)), we have

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(r) = 2N(r) \leq 2\bar{d} \qquad \forall r \in (a, t].$$

By integrating, the thesis follows. □

The next step is to prove a similar monotonicity property for the function E. Our result rests on Theorem 5.6 of [6] (see also [5]), which we state here for the reader's convenience

Theorem 5.2.6. *Let k be a fixed integer and let $\Lambda > 1$. Let*

$$\mathcal{L}(k,\Lambda) := \min\left\{ \int_0^{2\pi} \sum_{i=1}^{k} (f_i')^2 + \Lambda \sum_{1\leq i<j\leq k} f_i^2 f_j^2 \;\middle|\; \begin{array}{l} f_1, \dots, f_k \in H^1([0,2\pi]),\ \int_0^{2\pi} \sum_{i=1}^k f_i^2 = 1 \\ f_{i+1}(t) = f_i\left(t - \frac{2\pi}{k}\right),\ f_1(\pi + t) = f_1(\pi - t) \end{array} \right\},$$

where the indexes are counted $\mod k$. *There exists $C > 0$ such that*

$$\left(\frac{k}{2}\right)^2 - C\Lambda^{-1/4} \leq \mathcal{L}(k,\Lambda) \leq \left(\frac{k}{2}\right)^2.$$

Remark 5.2.7. Having in mind to apply Theorem 5.2.6 on 2π-periodic functions, note that the condition $f_1(\pi + t) = f_1(\pi - t)$ can be replaced by $f_1(t + \tau) = f_1(\tau - t)$ for any $\tau \in [0, 2\pi)$.

For a fixed $r_0 \in (a, b)$, let us introduce

$$\varphi(r; r_0) := \int_{r_0}^{r} \frac{\mathrm{d}s}{H(s)^{1/4}}.$$

The function φ is positive and increasing in $\mathbb{R}^+$; thanks to point (i) of Corollary 5.2.5 and to the monotonicity of N, whenever (u, v) is nontrivial φ is bounded by a quantity depending only $H(r_0)$ and $N(r_0)$. To be precise:

$$\varphi(r; r_0) \leq 2\frac{e^{\frac{1}{2}N(r_0)r_0}}{H(r_0)^{\frac{1}{4}} N(r_0)} \left[e^{-\frac{1}{2}N(r_0)r_0} - e^{-\frac{1}{2}N(r_0)r} \right]. \tag{5.2.3}$$

This, together with the monotonicity of $\varphi(\cdot; r_0)$, implies that if $b = +\infty$ then there exists the limit

$$\lim_{r \to +\infty} \varphi(r; r_0) < +\infty. \tag{5.2.4}$$

Lemma 5.2.8. *Let $(u_1,\dots,u_k)$ be a solution of $(S)_2$ in $C_{(a,b)}$ such that (5.2.1) holds. Let $r_0 \in (a,b)$, and assume that*

$$u_{i+1}(x,y) = u_i(x,y-\pi) \quad \text{and} \quad u_1(x,\tau+y) = u_1(x,\tau-y) \tag{5.2.5}$$

where $\tau \in [0,k\pi)$. There exists $C>0$ such that the function $r \mapsto \frac{E(r)}{e^{2r}} e^{C\varphi(r;r_0)}$ is nondecreasing in r for $r > r_0$.

Proof. Recalling the (5.2.2), we compute the logarithmic derivative

$$\frac{\mathrm{d}}{\mathrm{d}r}\log\left(\frac{E(r)}{e^{2r}}\right) = -2 + \frac{\int_{\Sigma_r}\sum_i (\partial_\nu u_i)^2 + \int_{\Sigma_r}(\partial_y u_i)^2 + 2\sum_{i<j} u_i^2 u_j^2}{\int_{\Sigma_r}\sum_i u_i \partial_\nu u_i} \tag{5.2.6}$$

To apply Theorem 5.2.6, we observe that $\Sigma_r = \{r\}\times[0,k\pi]$, so that

$$\begin{aligned}\int_{\Sigma_r} (\partial_y u_i)^2 + 2\sum_{i<j} u_i^2 u_j^2 &= \int_0^{k\pi} (\partial_y u_i(r,y))^2 + 2\sum_{i<j} u_i(r,y)^2 u_j(r,y)^2 \,\mathrm{d}y \\ &= \frac{2}{k}\int_0^{2\pi} (\partial_y \tilde u_i(r,y))^2 + 2\left(\frac{k}{2}\right)^2 \sum_{i<j} \tilde u_i(r,y)^2 \tilde u_j(r,y)^2 \,\mathrm{d}y, \end{aligned} \tag{5.2.7}$$

where $\tilde u_i(r,y) = u_i\left(r, \frac{k}{2}y\right)$. By a scaling argument, thanks to assumption (5.2.5) (see also Remark 5.2.7) we can say that for every $\Lambda > \frac{1}{2}$ there holds

$$\begin{aligned}&\int_0^{2\pi} (\partial_y \tilde u_i(r,y))^2 + \left(\frac{k}{2}\right)^2 \frac{2\Lambda}{\int_0^{2\pi}\sum_i \tilde u_i(r,y)^2\,\mathrm{d}y} \sum_{i<j} \tilde u_i(r,y)^2 \tilde u_j(r,y)^2 \,\mathrm{d}y \\ &\qquad \geq \mathcal{L}\left(k, 2\Lambda\left(\frac{k}{2}\right)^2\right)\int_0^{2\pi}\sum_i \tilde u_i(r,y)^2\,\mathrm{d}y = \frac{2}{k}\mathcal{L}\left(k, 2\Lambda\left(\frac{k}{2}\right)^2\right)\int_{\Sigma_r}\sum_i u_i^2\end{aligned}$$

The choice

$$\Lambda = \int_0^{2\pi}\sum_i \tilde u_i(r,y)^2\,\mathrm{d}y = \frac{2}{k}H(r)$$

yields

$$\int_0^{2\pi} (\partial_y \tilde u_i(r,y))^2 + 2\left(\frac{k}{2}\right)^2 \sum_{i<j} \tilde u_i(r,y)^2 \tilde u_j(r,y)^2\,\mathrm{d}y \geq \frac{2}{k}\mathcal{L}(k, kH(r))\int_{\Sigma_r}\sum_i u_i^2,$$

and coming back to (5.2.7) we obtain

$$\int_{\Sigma_r} (\partial_y u_i)^2 + 2\sum_{i<j} u_i^2 u_j^2 \geq \left(\frac{2}{k}\right)^2 \mathcal{L}(k, kH(r))\int_{\Sigma_r}\sum_i u_i^2.$$

Plugging this estimate into the (5.2.6) we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\log\left(\frac{E(r)}{e^{2r}}\right) &\geq -2 + \frac{\int_{\Sigma_r}\sum_i(\partial_\nu u_i)^2 + \left(\frac{2}{k}\right)^2 \mathcal{L}(k,kH(r))\int_{\Sigma_r}\sum_i u_i^2}{\int_{\Sigma_r}\sum_i u_i\partial_\nu u_i} \\ &\geq -2 + 2\frac{2}{k}\sqrt{\mathcal{L}(k,kH(r))} \geq -\frac{C}{H(r)^{1/4}}\end{aligned}$$

where we used Theorem 5.2.6. An integration gives the thesis. □

Lemma 5.2.9. *Let $(u_1, \dots, u_k)$ be a nontrivial solution of $(S)_k$ in $C_{(a,+\infty)}$, and assume that (5.2.1) and (5.2.5) hold. If $d := \lim_{r\to+\infty} N(r) < +\infty$, then $d \geq 1$ and*

$$\lim_{r\to+\infty} \frac{E(r)}{e^{2r}} > 0.$$

Proof. Let us fix $r_0 > a$. Firstly, from the previous Lemma and the (5.2.4), we deduce that there exists the limit

$$l := \lim_{r\to+\infty} \frac{E(r)}{e^{2r}} \geq 0.$$

Recalling that $\varphi(r; r_0)$ is bounded, it results

$$\frac{E(r)}{e^{2r}} \geq e^{-C\varphi(r;r_0)} \frac{E(r_0)}{e^{2r_0}} \geq C > 0 \qquad \forall r > r_0,$$

so that the value l is strictly greater then 0. Now, assume by contradiction that $d = \lim_{r\to+\infty} N(r) < 1$. The monotonicity of N implies $N(r) \leq d$ for every $r > 0$. Hence, from Corollary 5.2.5 we deduce

$$\frac{H(r)}{e^{2dr}} \leq \frac{H(r_0)}{e^{2dr_0}} \quad \forall r > r_0 \quad \Rightarrow \quad \limsup_{r\to+\infty} \frac{H(r)}{e^{2dr}} < +\infty \quad \Rightarrow \quad \lim_{r\to+\infty} \frac{H(r)}{e^{2r}} = 0,$$

which in turns gives

$$0 < l = \lim_{r\to+\infty} \frac{E(r)}{e^{2r}} = \lim_{r\to+\infty} N(r) \lim_{r\to+\infty} \frac{H(r)}{e^{2r}} = 0,$$

a contradiction. □

5.2.2 Solutions with finite energy in unbounded cylinders

In what follows we consider a solution $(u_1, \dots, u_k)$ of $(S)_k$ defined in an unbounded cylinder $C_{(-\infty,b)}$, with $b \in \mathbb{R}$ (the choice $b = +\infty$ is admissible). In this setting we assume that $(u_1, \dots, u_k)$ has a sufficiently fast decay as $x \to -\infty$, in the sense that

$$H(r) := \int_{\Sigma_r} \sum_{i=1}^k u_i^2 \to 0 \quad \text{as } r \to -\infty. \tag{5.2.8}$$

First of all, we can show that under assumption (5.2.8) $(u_1, \dots, u_k)$ has finite energy in $C_{(-\infty,b)}$.

Lemma 5.2.10. *Let $(u_1, \dots, u_k)$ be a solution of $(S)_k$ in $C_{(-\infty,b)}$, such that (5.2.8) holds. Then*

$$\mathcal{E}^{unb}(r) := \int_{C_{(-\infty,r)}} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1\leq i<j\leq k} u_i^2 u_j^2 < +\infty \qquad \forall r < b.$$

The index *unb* stands for the fact that the energy is evaluated in an unbounded cylinder, and will be omitted in the rest of the subsection.

Proof. Firstly, being a solution in $C_{(-\infty,b)}$, it results $(u_1,\dots,u_k)\in H^1_{loc}(C_{(-\infty,b)})$. Thus, under assumption (5.2.8), there exists $C>0$ such that $H(r)\leq C$ for every $r<b$.

Let $r_0<b$. Let us introduce, for $r>0$, the functional

$$e(r):=\int_{C_{(-r+r_0,r_0)}}\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2.$$

For the sake of simplicity, in the rest of the proof we assume $r_0=0$ (thus $b>0$). By direct computation and an application of Lemma 5.2.1, we find

$$e'(r)=\int_{\Sigma_{-r}}\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2=2\int_{\Sigma_{-r}}(\partial_x u_i)^2+\int_{\Sigma_0}\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2-2\int_{\Sigma_0}(\partial_x u_i)^2$$

that is

$$\int_{\Sigma_{-r}}(\partial_x u_i)^2=\frac{1}{2}e'(r)+C_0 \tag{5.2.9}$$

On the other hand, testing the equation $(S)_k$ in $C_{(-r,0)}$ by $(u_1,\dots,u_k)$ and summing for $i=1,\dots,k$, we find

$$\begin{aligned}e(r)&\leq\int_{C_{(-r,0)}}\sum_i|\nabla u_i|^2+2\sum_{i<j}u_i^2u_j^2=\int_{\Sigma_0}\sum_i u_i\partial_x u_i-\int_{\Sigma_{-r}}\sum_i u_i\partial_x u_i\\&\leq\int_{\Sigma_0}\sum_i u_i\partial_x u_i+\left(\int_{\Sigma_{-r}}\sum_i(\partial_x u_i)^2\right)^{\frac12}\left(\int_{\Sigma_{-r}}\sum_i u_i^2\right)^{\frac12}\end{aligned}$$

Let us assume that by contradiction that $e(r)\to+\infty$ as $r\to+\infty$. Taking the square of the previous inequality, using the boundedness of H and the assumption (5.2.8), we have

$$\begin{cases}\frac{1}{C^2}(e(r)+C_1)^2-2C_0\leq e'(r) & \text{for } r>\bar r\\ e(\bar r)>0,\end{cases}$$

for some $C_0,C_1>0$ and $\bar r$ sufficiently large. Any solution to the previous differential inequality blows up in finite time, in contradiction with the fact that $(u_1,\dots,u_k)\in H^1_{\mathrm{loc}}(C_{(-\infty,b)})$. As a consequence e is bounded and, by regularity,

$$\int_{C_{(-\infty,r)}}\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2<+\infty\qquad\forall r<b.$$ □

Remark 5.2.11. As a byproduct of the previous Lemma, if $(u_1,\dots,u_k)$ solves the $(S)_k$ in $C_{(-\infty,b)}$ and (5.2.8) holds, then

$$\lim_{r\to-\infty}\mathcal{E}(r)=0.$$

Having in mind to recover the monotonicity formulæ of the previous subsection in the present situation, we cannot adapt the proof of Lemma 5.2.2, where assumption (5.2.1) played an important role. However, we can obtain a similar result with a different proof.

Lemma 5.2.12. *Let $(u_1,\dots,u_k)$ be a solution to $(S)_2$ in $C_{(-\infty,b)}$, such that (5.2.8) holds. Then*

$$\int_{\Sigma_r}\sum_{i=k}|\nabla u_i|^2+\sum_{1\le i<j\le k}u_i^2u_j^2=2\int_{\Sigma_r}\sum_{i=1}^{k}(\partial_x u_i)^2$$

for every $r<b$.

Proof. We use the method of the variations of the domains: for $\psi\in\mathcal{C}^1_c(-\infty,r)$, we consider

$$u_{i,\varepsilon}(r,y)=u_i(r+\varepsilon\psi(r),y)\qquad i=1,\dots,k.$$

It is possible to see $(u_{1,\varepsilon},\dots,u_{k,\varepsilon})$ as a smooth variations of $(u_1,\dots,u_k)$ with compact support in $C_{(-\infty,r)}$: indeed

$$u_i(x+\varepsilon\psi(x),y)-u_i(x,y)=\varepsilon\partial_x u(\xi_x,y)\psi(x),$$

where $\xi_x\in(x,x+\varepsilon\psi(x))$. To proceed, we explicitly remark that any solution to $(S)_k$ is critical for the energy functional

$$J(v_1,\dots,v_k):=\int_{C_{(-\infty,b)}}\sum_{i=1}^{k}|\nabla v_i|^2+\sum_{1\le i<j\le j}v_i^2v_j^2$$

with respect to variations with compact support in $\mathcal{C}^\infty_c(C_{(-\infty,b)})$. Note that $J(u_1,\dots,u_k)=\mathcal{E}(b)$. As $(u_1,\dots,u_k)$ is a smooth solution of $(S)_k$ with finite energy $\mathcal{E}(r)$, it follows that

$$\begin{aligned}
0&=\lim_{\varepsilon\to0}\frac{\int_{C_{(-\infty,r)}}\sum_i|\nabla u_{i,\varepsilon}|^2+\sum_{i<j}u_{i,\varepsilon}^2u_{j,\varepsilon}^2-\mathcal{E}(r)}{\varepsilon}\\
&=\int_{C_{(-\infty,r)}}\frac{\partial}{\partial\varepsilon}\left(\sum_i|\nabla u_i(x+\varepsilon\psi(x),y)|^2+\sum_{i<j}u_i^2(x+\varepsilon\psi(x),y)u_j^2(x+\varepsilon\psi(x),y)\right)\Bigg|_{\varepsilon=0}\mathrm{d}x\mathrm{d}y\\
&\quad+2\lim_{\varepsilon\to0}\int_{C_{(-\infty,r)}}\psi'(x)\sum_i(\partial_x u_i)^2(x+\varepsilon\psi(x))\,\mathrm{d}x\mathrm{d}y\\
&=\int_{C_{(-\infty,x)}}\left(2\sum_i(\partial_x u_i)^2-\left(\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2\right)\right)\psi'
\end{aligned}\tag{5.2.10}$$

for every $\psi\in\mathcal{C}^1_c(-\infty,x)$. Since $\mathcal{E}(r)<+\infty$, for every $\varepsilon>0$ there exists a compact $K_\varepsilon\subset C_{(-\infty,r)}$ such that

$$\int_{C_{(-\infty,r)}\setminus K_\varepsilon}\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2<\varepsilon.$$

Let $\psi \in \mathcal{C}^1(-\infty, r)$ be such that $\|\psi\|_{\mathcal{C}^1(-\infty,r)} < +\infty$ and $\psi = 0$ in a neighborhood of r. It is possible to write $\psi = \psi_1 + \psi_2$ where $\psi_1 \in \mathcal{C}^1_c(-\infty, r)$ and $\operatorname{supp}\psi_2 \times (\mathbb{R}/k\pi\mathbb{Z}) \subset (C_{(-\infty,r)} \setminus K_\varepsilon)$. Therefore, from (5.2.10) it follows

$$\begin{aligned}\int_{C_{(-\infty,r)}} &\left(2\sum_i (\partial_x u_i)^2 - \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \psi' \\ &= \int_{C_{(-\infty,r)} \setminus K_\varepsilon} \left(2\sum_i (\partial_x u_i)^2 - \left(\sum_i |\nabla u|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \psi_2' \\ &\qquad \leq 3\|\psi\|_{\mathcal{C}^1(-\infty,x)} \int_{C_{(-\infty,r)} \setminus K_\varepsilon} \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) < C\varepsilon.\end{aligned}$$

Since ε has been arbitrarily chosen, we obtain

$$\int_{C_{(-\infty,r)}} \left(2\sum_i (\partial_x u_i)^2 - \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \psi' = 0 \tag{5.2.11}$$

for every $\psi \in \mathcal{C}^1(-\infty, r)$ be such that $\|\psi\|_{\mathcal{C}^1(-\infty,r)} < +\infty$ and $\psi = 0$ in a neighborhood of r.
Now, let $\psi \in \mathcal{C}^1((-\infty, r])$ be such that $\|\psi\|_{\mathcal{C}^1((-\infty,r])} < +\infty$. For a given $\varepsilon > 0$, we introduce a cut-off function $\eta \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\eta(s) = \begin{cases} 1 & \text{if } s \leq r - \varepsilon \\ 0 & \text{if } s \geq r. \end{cases}$$

Since $\eta\psi \in \mathcal{C}^1(-\infty, r)$, $\|\eta\psi\|_{\mathcal{C}^1(-\infty,r)} < +\infty$ and $\eta\psi = 0$ in a neighborhood of r, from (5.2.11) we deduce

$$\begin{aligned}\int_{C_{(-\infty,r)}} &\left(2\sum_i (\partial_x u_i)^2 - \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \eta\psi' \\ &= \int_{C_{(-\infty,r)}} \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 - 2\sum_i (\partial_x u_i)^2 \right) \eta'\psi.\end{aligned} \tag{5.2.12}$$

Denoting by

$$\gamma = \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 - 2\sum_i (\partial_x u_i)^2 \right) \psi,$$

the right hand side is

$$\int_0^{k\pi}\left(\int_{r-\varepsilon}^{r}\eta'(x)\gamma(s,y)\,\mathrm{d}x\right)\,\mathrm{d}y = -\int_0^{k\pi}\gamma(r-\varepsilon,y)\,\mathrm{d}y$$
$$-\int_0^{k\pi}\left(\int_{r-\varepsilon}^{r}\eta(s)\partial_x\gamma(x,y)\,\mathrm{d}x\right)\,\mathrm{d}y$$
$$=\int_{\Sigma_r}\left(2\sum_i(\partial_x u_i)^2-\left(\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2\right)\right)\psi+o(1)$$

as $\varepsilon\to 0$, where the last identity follows from the regularity of $(u_1,\ldots,u_k)$ and from the $\mathcal{C}^1$-boundedness of ψ and η. Passing to the limit as $\varepsilon\to 0$ in the (5.2.12), we deduce that for every $\psi\in\mathcal{C}^1((-\infty,r])$ such that $\|\psi\|_{\mathcal{C}^1((-\infty,r])}<+\infty$ it results

$$\int_{C_{(-\infty,r)}}\left(2\sum_i(\partial_x u_i)^2-\left(\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2\right)\right)\psi'$$
$$=\int_{\Sigma_r}\left(2\sum_i(\partial_x u_i)^2-\left(\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2\right)\right)\psi.$$

Choosing $\psi=1$ we obtain the thesis. □

This result permits to prove an Almgren monotonicity formula for a solution $(u_1,\ldots,u_k)$ of $(S)_k$ in $C_{(-\infty,b)}$ such that (5.2.8) holds. For such a solution, let us set

$$E^{unb}(r):=\int_{C_{(-\infty,r)}}\sum_{i=1}^k|\nabla u_i|^2+2\sum_{1\le i<j\le k}u_i^2u_j^2,$$

We will briefly write E in the rest of the subsection. Clearly, Lemma 5.2.10 and the fact that $\mathcal{E}(r)\to 0$ as $r\to-\infty$ (see Remark 5.2.11) implies that

$$E(r)<+\infty\quad\forall r<b\quad\text{and}\quad\lim_{r\to-\infty}E(r)=0. \tag{5.2.13}$$

By regularity, $E,\mathcal{E}$ and H are smooth. A direct computation shows that E and $\mathcal{E}$ are increasing in r. As far as H is concerned, with respect to the previous subsection we cannot deduce the identity (5.2.2) by means of a simple integration by parts, since we are working in an unbounded domain. However,

Lemma 5.2.13. *Let $(u_1,\ldots,u_k)$ be a solution to $(S)_k$ in $C_{(-\infty,b)}$, such that (5.2.8) holds. Then*

$$H'(r)=2\int_{\Sigma_r}\sum_{i=1}^k u_i\partial_\nu u_i=2E(r)$$

for every $r<b$. In particular, H is nondecreasing.

Proof. For every $s < r < b$, the divergence theorem and the periodicity of $(u_1, \dots, u_k)$ imply that

$$\begin{aligned} E(r) &= E(s) + \int_{C_{(s,r)}} \sum_i |\nabla u_i|^2 + 2\sum_{i<j} u_i^2 u_j^2 \\ &= E(s) - \int_{\Sigma_s} \sum_i u_i \partial_x u_i + \int_{\Sigma_x} \sum_i u_i \partial_\nu u_i. \end{aligned} \tag{5.2.14}$$

We consider the second term on the right hand side. Let $\eta \in C_c^\infty(-1,1)$ be a non negative cut-off function, even with respect to $r = 0$, such that $\eta(0) = 1$ and $\eta \le 1$ in $(-1,1)$. Let $\eta_s(x) = \eta(x-s)$; testing the equation $(S)_k$ with $u_i\eta_s$ in $C_{(s-1,s)}$, we find

$$\int_{C_{(s-1,s)}} \nabla u_i \cdot \nabla(u_i\eta_s) + u_i^2 \sum_{i\neq j} u_j^2 \eta_s = \int_{\Sigma_s} u_i \partial_x u_i$$

Summing up for $i = 1, \dots, k$, we obtain

$$\begin{aligned} \int_{\Sigma_s} \sum_i u_i \partial_x u_i &= \int_{C_{(s-1,s)}} \sum_i \left(u_i \partial_x u_i \eta_s' + |\nabla u_i|^2 \eta_s \right) + 2\sum_{i<j} u_i^2 u_j^2 \eta_s \\ &\le C(\eta') \sum_i \|u_i\|^2_{H^1(C_{(s-1,s)})} + E(s), \end{aligned} \tag{5.2.15}$$

where the last estimate follows from the Hölder inequality. We claim that

$$\sum_i \|u_i\|_{H^1(C_{(s-1,s)})} \to 0 \qquad \text{as } s \to -\infty.$$

This is a consequence of the Poincaré inequality

$$\int_{C_{(s-1,s)}} u^2 \le C \left(\int_{\Sigma_s} u^2 + \int_{C_{(s-1,s)}} |\nabla u|^2 \right) \qquad \forall u \in H^1(C_{(s-1,s)})$$

together with assumption (5.2.8) and the fact that $E(s) \to 0$ as $s \to -\infty$ (see (5.2.13)). Thus, from the (5.2.15) we deduce that

$$\lim_{s\to-\infty} \int_{\Sigma_s} \sum_i u_i \partial_x u_i = 0,$$

which in turns can be used in the (5.2.14) to obtain the thesis:

$$E(r) = \lim_{s\to-\infty} \left(E(s) - \int_{\Sigma_s} \sum_i u_i \partial_x u_i + \int_{\Sigma_x} \sum_i u_i \partial_\nu u_i \right) = \int_{\Sigma_x} \sum_i u_i \partial_\nu u_i. \qquad \square$$

In light of the previous results, the proof of the following statements are straightforward modification of the proofs of Proposition 5.2.4, Corollary 5.2.5 and Lemmas 5.2.8 and 5.2.9.

Proposition 5.2.14. *Let $(u_1, \ldots, u_k)$ be a solution of $(S)_k$ in $C_{(-\infty,b)}$ such that (5.2.8) holds. The*Almgren quotient

$$N^{unb}(r) := \frac{E^{unb}(r)}{H(r)}$$

is well defined in $(-\infty, b)$ and nondecreasing. Moreover,

$$\int_{-\infty}^{r} \frac{\int_{\Sigma_s} \sum_{i<j} u_i^2 u_j^2}{H(s)} \, \mathrm{d}s \leq N(r).$$

Analogously, the function $\mathfrak{N}^{unb}(r) := \dfrac{\mathcal{E}^{unb}(r)}{H(r)}$ is well defined in $(-\infty, b)$ and nondecreasing.

We will briefly write N and $\mathfrak{N}$ instead of N^{unb} and $\mathfrak{N}^{unb}$ in the rest of this subsection.

Corollary 5.2.15. *Let $(u_1, \ldots, u_k)$ be a solution of $(S)_k$ in $C_{(-\infty,b)}$ such that (5.2.8) holds.*

(i) If $N(r) \geq \underline{d}$ for $r \geq s$, then

$$\frac{H(r_1)}{e^{2\underline{d}r_1}} \leq \frac{H(r_2)}{e^{2\underline{d}r_2}} \qquad \forall\ s \leq r_1 < r_2 < b,$$

ii) If $N(r) \leq \bar{d}$ for $r \leq t < b$, then

$$\frac{H(r_1)}{e^{2\bar{d}r_1}} \geq \frac{H(r_2)}{e^{2\bar{d}r_2}} \qquad \forall\ r_1 < r_2 \leq t.$$

For a fixed $r_0 < b$, let us introduce

$$\varphi(r; r_0) := \int_{r_0}^{r} \frac{\mathrm{d}s}{H(s)^{1/4}}.$$

The function φ is positive and increasing in $\mathbb{R}^+$; thanks to point (i) of Corollary 5.2.15 and to the monotonicity of N, whenever (u, v) is nontrivial φ is bounded by a quantity depending only $H(r_0)$ and $N(r_0)$:

$$\varphi(r; r_0) \leq 2 \frac{e^{\frac{1}{2}N(r_0)r_0}}{H(r_0)^{\frac{1}{4}} N(r_0)} \left[e^{-\frac{1}{2}N(r_0)r_0} - e^{-\frac{1}{2}N(r_0)r} \right]. \tag{5.2.16}$$

This, together with the monotonicity of $\varphi(\cdot; r_0)$, implies that if $b = +\infty$ then there exists the limit

$$\lim_{r \to +\infty} \varphi(r; r_0) < +\infty.$$

Lemma 5.2.16. *Let $(u_1, \ldots, u_k)$ be a solution of $(S)_2$ in $C_{(-\infty,b)}$ such that (5.2.8) hold. Let $r_0 \in (-\infty, b)$, and assume that*

$$u_{i+1}(x,y) = u_i(x, y-\pi) \quad \textit{and} \quad u_1(x, \tau+y) = u_1(x, \tau - y) \tag{5.2.17}$$

where $\tau \in [0, k\pi)$. There exists $C > 0$ such that the function $r \mapsto \dfrac{E(r)}{e^{2r}} e^{C\varphi(r;r_0)}$ is nondecreasing in r for $r > r_0$.

Lemma 5.2.17. *Let $(u_1, \ldots, u_k)$ be a nontrivial solution of $(S)_k$ in C_∞, and assume that (5.2.8) and (5.2.17) hold. If $d := \lim_{r\to+\infty} N(r) < +\infty$, then $d \geq 1$ and*

$$\lim_{r\to+\infty} \frac{E(r)}{e^{2r}} > 0.$$

Remark 5.2.18. The achievements of this section hold true for solutions to

$$\begin{cases} -\Delta u_i = -\beta u_i \sum_{j\neq i} u_j^2 \\ u_i > 0 \end{cases}$$

with the energy density

$$\sum_i |\nabla u_i|^2 + 2\sum_{i<j} u_i^2 u_j^2 \quad \text{replaced by F} \sum_i |\nabla u_i|^2 + 2\beta \sum_{i<j} u_i^2 u_j^2.$$

5.2.3 Monotonicity formulæ for harmonic functions

Here we prove some monotonicity formulæ for harmonic functions of the plane which are 2π periodic in one variable. In what follows, in the definition of $C_{(a,b)}$ and Σ_r we mean $k = 2$. The following results will come useful in section 5.6.

Firstly, it is not difficult to obtain the counterpart of Lemma 5.2.1.

Lemma 5.2.19. *Let Ψ be an entire harmonic function in $C_{(a,b)}$. Then*

$$r \mapsto \int_{\Sigma_r} |\nabla\Psi|^2 - 2\Psi_x^2$$

is constant.

Proof. We proceed as in the proof of Lemma 5.2.1: for $a < r_1 < r_2 < b$, we test the equation $-\Delta\Psi = 0$ with Ψ_x in $C_{(r_1,r_2)}$ and integrate by parts. □

In what follows we consider a harmonic function Ψ defined in an unbounded cylinder $C_{(-\infty,b)}$, with $b \in \mathbb{R}$ or $b = +\infty$. We assume that

$$H(r;\Psi) := \int_{\Sigma_r} \Psi^2 \to 0 \quad \text{as } r \to -\infty. \tag{5.2.18}$$

Lemma 5.2.20. *Let Ψ be a harmonic function in $C_{(-\infty,b)}$ such that (5.2.18) holds true. Then*

(i) for every $r \in \mathbb{R}$ it results $E^{unb}(r;\Psi) := \int_{C_{(-\infty,r)}} |\nabla\Psi|^2 < +\infty$

(ii) it results

$$\int_{\Sigma_r} |\nabla\Psi|^2 = 2\int_{\Sigma_r} (\partial_x\Psi)^2 \tag{5.2.19}$$

Proof. In light of Lemma 5.2.19, it is not difficult to adapt the proof of Lemma 5.2.11 and obtain (i). As far as (ii), we can proceed as in the proof of Lemma 5.2.12. □

Proposition 5.2.21. *Let Ψ be a nontrivial harmonic function in $C_{(-\infty,b)}$, such that (5.2.18) holds true. The* Almgren quotient

$$N^{unb}(r;\Psi) := \frac{\int_{C_{(-\infty,r)}} |\nabla\Psi|^2}{\int_{\Sigma_r} \Psi^2}$$

is nondecreasing in r. If $N(\cdot;\Psi)$ is constant for r in some non empty open interval (r_1,r_2), then $N(r;\Psi)$ is constant for all $r \in \mathbb{R}$ and there exists a positive integer $d \in \mathbb{N}$ such that $N(r;\Psi) = d$; furthermore,

$$\Psi(x,y) = [C_1\cos(dy) + C_2\sin(dy)]\, e^{dx}$$

for some $C_1, C_2 \in \mathbb{R}$.

Proof. The Almgren quotient is well defined, thanks to Lemma 5.2.20. To prove its monotonicity, we compute the logarithmic derivative by means of the Pohozaev identity (5.2.19) and the fact that $H'(r;\Psi) = 2E^{unb}(r;\Psi)$ (this follows from (5.2.18)):

$$\frac{(N^{unb})'(r;\Psi)}{N^{unb}(r;\Psi)} = \frac{\int_{\Sigma_r} |\nabla\Psi|^2}{\int_{C_{(-\infty,r)}} |\nabla\Psi|^2} - 2\frac{\int_{\Sigma_r} \Psi\partial_x\Psi}{\int_{\Sigma_r} \Psi^2} = 2\frac{\int_{\Sigma_r} |\partial_x\Psi|^2}{\int_{\Sigma_r} \Psi\partial_x\Psi} - 2\frac{\int_{\Sigma_r} \Psi\partial_x\Psi}{\int_{\Sigma_r} \Psi^2} \geq 0$$

where in the last step we used the Cauchy-Schwarz inequality.

Let us assume now that $N^{unb}(r;\Psi)$ is constant for $r \in (r_1,r_2)$. By the previous computations it follows that necessarily

$$\int_{\Sigma_r} |\partial_x\Psi|^2 \int_{\Sigma_r} \Psi^2 = \left(\int_{\Sigma_r} \Psi\partial_x\Psi\right)^2$$

for every $r \in (r_1,r_2)$. Again from the Cauchy-Schwarz inequality, we evince that it must be

$$\partial_x\Psi = \lambda\Psi \qquad \text{on } \Sigma_r$$

for some constant $\lambda \in \mathbb{R}$ and for every $r \in (r_1,r_2)$. Solving the differential equation, we find that Ψ is of the form

$$\Psi(x,y) = \psi(y)e^{\lambda x}.$$

This, together with the equation $\Delta\Psi = 0$, yields

$$\psi'' + \lambda^2\psi = 0 \quad \Rightarrow \quad \Psi(x,y) = [C_1\cos(\lambda y) + C_2\sin(\lambda y)]\, e^{\lambda x} \qquad \forall (x,y) \in (r_1, r_2)\times\mathbb{R},$$

and Ψ can be uniquely extended to $\mathbb{R}^2$ by the unique continuation principle for harmonic functions. Since Ψ satisfies the condition (5.2.18) and is nontrivial, it follows that $\lambda > 0$. The proof is complete, recalling the periodicity in y of the function Ψ and computing its Almgren quotient. $\square$

5.3 *Proof of Theorem 5.1.2*

In this section we construct a solution to $(S)_2$ modeled on the harmonic function $\Phi(x,y) = \cosh x \sin y$.

5.3.1 *Existence in bounded cylinders*

For every $R > 0$ we construct a solution (u_R, v_R) to

$$\begin{cases} -\Delta u = -uv^2 & \text{in } C_R \\ -\Delta v = -u^2 v & \text{in } C_R \\ u, v > 0 \end{cases} \tag{5.3.1a}$$

(equivalently, we can consider the problem in $(-R,R)\times(0,2\pi)$ with periodic boundary condition on the sides $[-R,R]\times\{0,2\pi\}$) with Dirichlet boundary condition

$$u = \Phi^+, \quad v = \Phi^- \quad \text{on } \Sigma_R \cup \Sigma_{-R}, \tag{5.3.1b}$$

and exhibiting the same symmetries of (Φ^+, Φ^-). To be precise:

Proposition 5.3.1. *There exists a solution (u_R, v_R) to problem (5.3.1a) with the prescribed boundary conditions (5.3.1b), such that*

1) $u_R(-x,y) = u_R(x,y)$ and $v_R(-x,y) = v_R(x,y)$,

2) the symmetries

$$\begin{aligned} v_R(x,y) &= u_R(x, y-\pi) & \qquad u_R(\pi - x, y) &= v_R(\pi + x, y) \\ u_R\left(x, \frac{\pi}{2} + y\right) &= u_R\left(x, \frac{\pi}{2} - y\right) & \qquad v_R\left(x, \frac{3}{2}\pi + y\right) &= v_R\left(x, \frac{3}{2}\pi - y\right) \end{aligned}$$

hold,

3) $u_R - v_R > 0$ in $\{\Phi > 0\}$ and $v_R - u_R > 0$ in $\{\Phi < 0\}$, *4)* $u_R > \Phi^+$ *and* $v_R > \Phi^-$.

Remark 5.3.2. In light of the evenness of (u_R, v_R) in x, it results

$$\partial_x u = 0 = \partial_x v \qquad \text{on } \Sigma_0.$$

As a consequence, the monotonicity formulæ proved in subsection 5.2.1 hold true for (u_R, v_R) in the semi-cylinder $C_{(0,R)}$.

In order to keep the notation as simple as possible, in what follows we will refer to a solution of (5.3.1a)-(5.3.1b) as to a solution of (5.3.1).

Proof. Let

$$\mathcal{U}_R := \left\{ (u,v) \in (H^1(C_R))^2 \;\middle|\; \begin{array}{l} u = \Phi^+,\, v = \Phi^- \text{ on } \Sigma_R \cup \Sigma_{-R},\, u \geq 0, \\ u - v \geq 0 \text{ in } \{\Phi \geq 0\}, \\ v(x,y) = u(x, y-\pi),\, u(-x,y) = u(x,y), \\ u(x, \pi - y) = v(x, \pi + y),\, u\left(x, \frac{\pi}{2} + y\right) = u\left(x, \frac{\pi}{2} - y\right) \end{array} \right\}.$$

Note that if $(u,v) \in \mathcal{U}_R$ then v is nonnegative, even in x and symmetric in y with respect to $\frac{3}{2}\pi$; moreover, $u - v \leq 0$ in $\{\Phi < 0\}$. It is immediate to check that $\mathcal{U}_R$ is weakly closed with respect to the H^1 topology. We seek solutions of (5.3.1) as minimizers of the energy functional

$$J(u,v) := \int_{C_R} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2$$

in $\mathcal{U}_R$. The existence of at least one minimizer is given by the direct method of the calculus of variations; for the coercivity of the functional J, we use the following Poincaré inequality:

$$\int_{C_R} u^2 \leq C \left(\int_{\Sigma_{-R}} u^2 + \int_{C_R} |\nabla u|^2 \right) \qquad \forall u \in H^1(C_R), \tag{5.3.2}$$

where C depends only on R. To show that a minimizer satisfies equation (5.3.1), we consider the parabolic problem

$$\begin{cases} U_t - \Delta U = -UV^2 & \text{in } (0,+\infty) \times C_R \\ V_t - \Delta V = -U^2 V & \text{in } (0,+\infty) \times C_R \\ U = \Phi^+,\ V = \Phi^- & \text{on } (0,+\infty) \times (\Sigma_R \cup \Sigma_{-R}) \end{cases} \tag{5.3.3}$$

with initial condition in $\mathcal{U}_R$. There exists a unique local solution (U,V); by parabolic maximum principle if follows $U, V \geq 0$; hence, the maximum principle gives

$$0 \leq U \leq \sup_{C_R} \Phi^+ \quad \text{and} \quad 0 \leq V \leq \sup_{C_R} \Phi^-.$$

This control reveals that (U,V) can be uniquely extended in the whole $(0,+\infty)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}J(U(t,\cdot),V(t,\cdot)) = -2\int_{C_R}(U_t^2+V_t^2)\le 0, \tag{5.3.4}$$

that is, the energy is a Lyapunov functional, from the parabolic theory it follows that for every sequence $t_i\to+\infty$ there exists a subsequence (t_j) such that $(U(t_j\cdot),V(t_j,\cdot))$ converges to a solution (u,v) of (5.3.1). Therefore, in order to prove that (u_R,v_R) solves (5.3.1), it is sufficient to show that there exists an initial condition in $\mathcal{U}_R$ such that the limiting profile (u,v) coincides with (u_R,v_R). We use the fact that

$$\mathcal{U}_R \text{ is positively invariant under the parabolic flow.} \tag{5.3.5}$$

To prove this claim, we firstly note that by the symmetry of initial and boundary conditions and by the uniqueness of the solution to problem (5.3.3), we have

$$\begin{aligned} V(t,x,y) &= U(t,x,y-\pi), & U(t,-x,y) &= U(t,x,y),\\ V(t,x,\pi+y) &= U(t,x,\pi-y), & U\left(t,x,\frac{\pi}{2}+y\right) &= U\left(t,x,\frac{\pi}{2}-y\right). \end{aligned} \tag{5.3.6}$$

This implies

$$U(t,x,\pi)-V(t,x,\pi)=0 \qquad \forall (t,x)\in(0,+\infty)\times[-R,R].$$

Furthermore, using the (5.3.6) and the periodicity of (U,V)

$$\begin{aligned} U(t,x,0)-V(t,x,0) &= U(t,x,0)-V(t,x,2\pi)=0 \qquad \forall (t,x)\in(0,+\infty)\times[-R,R]\\ U(t,x,2\pi)-V(t,x,2\pi) &= U(t,x,2\pi)-V(t,x,0)=0 \qquad \forall (t,x)\in(0,+\infty)\times[-R,R]. \end{aligned}$$

This means that $U-V=0$ on $\{\Phi=0\}$. Let us introduce $D_R:=\{\Phi>0\}\cap C_R$. For each initial datum in $\mathcal{U}_R$, we have

$$\begin{cases} (U-V)_t-\Delta(U-V)=UV(U-V) & \text{in } (0,+\infty)\times D_R\\ U-V\ge 0 & \text{on } \{0\}\times D_R\\ U-V\ge 0 & \text{on } [0,+\infty)\times\partial D_R. \end{cases} \tag{5.3.7}$$

The parabolic maximum principle implies $U-V\ge 0$ in $(0,+\infty)\times D_R$. This completes the proof of the claim.

Let us consider equation (5.3.3) with the initial conditions $U(0,x,y)=u_R(x,y)$, $V(0,x,y)=v_R(x,y)$; let us denote (U^R,V^R) the corresponding solution. On one side, by minimality,

$$J(u_R,v_R)\le J(U^R(t,\cdot),V^R(t,\cdot)) \qquad \forall t\in(0,+\infty);$$

we point out that this comparison is possible because of (5.3.5). On the other side, by the (5.3.4),

$$J(U^R(t,\cdot), V^R(t,\cdot)) \leq J(u_R, v_R) \qquad \forall t \in (0,+\infty).$$

We deduce that $J(U^R, V^R)$ is constant, which in turns implies (we can use again the (5.3.4)),

$$U_t^R(t,x,y) = V_t^R(t,x,y) \equiv 0 \quad \Rightarrow \quad U^R(t,x,y) = u_R(x,y), \quad V^R(t,x,y) = v_R(x,y).$$

By the above argument, as (u_R, v_R) coincides with the asymptotic profile of a solution of the parabolic problem (5.3.3), it solves (5.3.1). Points 1)-3) of the thesis are satisfied due to the positive invariance of $\mathcal{U}_R$. The strong maximum principle yields $u_R > 0$ and $v_R > 0$. Moreover,

$$\begin{cases} -\Delta(u_R - v_R - \Phi) = u_R v_R (u_R - v_R) \geq 0 & \text{in } D_R \\ u_R - v_R - \Phi = 0 & \text{on } \partial D_R \end{cases} \quad \Rightarrow \quad u_R - v_R - \Phi \geq 0 \quad \text{in } D_R,$$

so that by the strong maximum principle and the fact that $u_R, v_R > 0$ we deduce $u_R > \Phi^+$. Analogously, $v_R > \Phi^-$. □

Remark 5.3.3. The existence of a positive solution of (5.3.1) satisfying the conditions 1)-2) of the Proposition can be proved by means of the celebrated Palais' Principle of Symmetric Criticality. To do this, it is sufficient to minimize the functional J in the weakly closed set

$$\left\{ (u,v) \in (H^1(C_R))^2 \;\middle|\; \begin{array}{l} u = \Phi^+, \; v = \Phi^- \text{ on } \Sigma_R \cup \Sigma_{-R}, \\ v(x,y) = u(x, y-\pi), \; u(-x,y) = u(x,y), \\ u(x, \pi - y) = v(x, \pi + y), \; u\left(x, \frac{\pi}{2} + y\right) = u\left(x, \frac{\pi}{2} - y\right) \end{array} \right\},$$

and apply the maximum principle. We have chose a more complicated proof since we will strongly use the pointwise estimates given by point 4).

5.3.2 Compactness of the family $\{(u_R, v_R)\}$

In this section we aim at proving that, up to a subsequence, the family $\{(u_R, v_R) : R > 1\}$ obtained in Proposition 5.3.1 converges, as $R \to +\infty$, to a solution (u,v) of $(S)_2$ defined in the whole C_∞. Then, by looking at (u,v) as defined in $\mathbb{R}^2$ (this is possible thanks to the periodicity), we obtain a solution of $(S)_2$ satisfying the conditions 1)-5) of Theorem 5.1.2. At a later stage, we will also obtain the estimates of points 6) and 7).

We denote $E_R, \mathcal{E}_R, H_R, N_R$ and $\mathfrak{N}_R$ the functions $E^{sym}, H, \mathcal{E}^{sym}, N^{sym}$ and $\mathfrak{N}^{sym}$ (which have been defined in subsection 5.2.1) when referred to (u_R, v_R). As observed in Remark 5.3.2, for these quantities the results of subsection 5.2.1 apply.

We will obtain compactness of the sequence (u_R, v_R) using some uniform-in-R control on N_R and H_R. We start with a uniform (in both r and R) upper bound for the Almgren quotients $N_R(r)$.

Lemma 5.3.4. *There holds $N_R(r) \leq 2$, for every $R > 0$ and $r \in (0, R)$.*

Proof. It is an easy consequence of the monotonicity of N_R and of the minimality of (u_R, v_R) for the functional J in $\mathcal{U}_R$: noting that $J(u_R, v_R) = \mathcal{E}_R(R)$, we compute

$$N_R(r) \leq N_R(R) \leq \frac{2\mathcal{E}_R(R)}{H_R(R)} \leq \frac{2}{\int_{\Sigma_R} \Phi^2} \int_{C_{(0,R)}} |\nabla \Phi|^2 = 2 \tanh R.$$

We used the fact that the restriction of (Φ^+, Φ^-) in C_R is an element of $\mathcal{U}_R$ for every R, and the boundary condition of (u_R, v_R) on Σ_R. $\square$

In the proof of the following Lemma we will exploit the compactness of the local trace operator $T_{\Sigma_1} : u \in H^1(C_{(0,1)}) \mapsto u|_{\Sigma_1} \in L^2(\Sigma_1)$, see Corollary **??**.

Lemma 5.3.5. *There exists $C > 0$ such that $H_R(1) \leq C$ for every $R > 1$.*

Proof. By contradiction, assume that $H_{R_n}(1) \to +\infty$ for a sequence $R_n \to +\infty$. Let us introduce the sequence of scaled functions

$$(\hat{u}_n(x, y), \hat{v}_n(x, y)) := \frac{1}{\sqrt{H_{R_n}(1)}} (u_{R_n}(x, y), v_{R_n}(x, y)).$$

We wish to prove a convergence result for such a sequence, in order to obtain a uniform lower bound for $N_{R_n}(1)$. In a natural way, the scaling leads us to consider, for $r \in (0, 1)$, the quantities

$$\hat{E}_n(r) := \int_{C_{(0,r)}} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 + 2H_{R_n}(1)\hat{u}_n^2 \hat{v}_n^2,$$
$$\hat{H}_n(r) := \int_{\Sigma_r} \hat{u}_n^2 + \hat{v}_n^2, \quad \hat{N}_n(r) := \frac{\hat{E}_n(r)}{\hat{H}_n(r)}.$$

By construction, it holds $\hat{H}_n(1) = 1$ and $\hat{N}_n(r) = N_{R_n}(r) \leq 2$; therefore, thanks to Lemma 5.3.4

$$\int_{C_{(0,1)}} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 \leq \hat{E}_n(1) = \hat{N}_n(1)\hat{H}_n(1) \leq 2, \tag{5.3.8}$$

which gives a uniform bound in the $H^1(C_{(0,1)})$ norm of the sequence $(\hat{u}_n, \hat{v}_n)$ (we can use a Poincaré inequality of type (5.3.2)). Then, we can extract a subsequence which converges weakly in $H^1(C_{(0,1)})$ to some limiting profile $(\hat{u},\hat{v})$, which is nontrivial in light of the compactness of the local trace operator T_{Σ_1} and of the fact that $\hat{H}_n(1)=1$. Since

$$\mathcal{V}:=\left\{(u,v)\in\left(H^1(C_{(0,1)})\right)^2 \;\middle|\; \begin{array}{l} u-v\geq 0 \text{ in } \Phi\geq 0,\, v(x,y)=u(x,y-\pi), \\ u(x,\pi-y)=v(x,\pi+y),\, u\left(x,\frac{\pi}{2}+y\right)=u\left(x,\frac{\pi}{2}-y\right) \end{array}\right\},$$

is closed in the weak $H^1(C_{(0,1)})$ topology and $(\hat{u}_n|_{C_{(0,1)}},\hat{v}_n|_{C_{(0,1)}})\in\mathcal{V}$ for every n, $\hat{u}$ and $\hat{v}$ are nonnegative functions with the same symmetries of (u_R,v_R); moreover we can show that $(\hat{u},\hat{v})$ satisfies the segregation condition $\hat{u}\hat{v}=0$ a.e. in $C_{(0,1)}$. Indeed, by the compactness of the Sobolev embedding $H^1(C_{(0,1)})\hookrightarrow L^4(C_{(0,1)})$ we deduce that the interaction term

$$I(u,v):=\int_{C_{(0,1)}} u^2v^2$$

is continuous in the weak topology of $(H^1(C_{(0,1)}))^2$. From the estimate (5.3.8), we infer

$$2H_{R_n}(1)I(\hat{u}_n,\hat{v}_n)\leq \hat{E}_n(1)\leq 2;$$

passing to the limit as $n\to+\infty$, we conclude

$$I(\hat{u},\hat{v})=\lim_{n\to\infty} I(\hat{u}_n,\hat{v}_n)=0 \quad\Rightarrow\quad \hat{u}\hat{v}=0 \text{ a.e. in } C_{(0,1)}.$$

Moreover, from the compactness of the local trace operator T_{Σ_1}, we also deduce $\int_{\Sigma_1}\hat{u}^2+\hat{v}^2=1$. Let us consider the functional

$$J^\infty(u,v):=\int_{C_{(0,1)}}|\nabla u|^2+|\nabla v|^2,$$

defined in the set

$$\mathcal{M}:=\left\{(u,v)\in(H^1(C_{(0,1)}))^2 \;\middle|\; \begin{array}{l} \int_{\Sigma_1}u^2+v^2=1, \\ v(x,y)=u(x,y-\pi),\;\; uv=0 \text{ a.e. in } C_1 \end{array}\right\}.$$

Due to the compactness of the trace operator, one can check that $\mathcal{M}$ is closed in the weak $(H^1(C_{(0,1)}))^2$ topology. It is clear that $(\hat{u},\hat{v})\in\mathcal{M}$. We claim that

$$\inf_{(u,v)\in\mathcal{M}} J^\infty(u,v)=:m>0.$$

Indeed, let us assume by contradiction that the infimum is 0: since the set $\mathcal{M}$ is weakly closed and J^∞ is weakly lower semi-continuous and coercive, there exists $(\bar{u},\bar{v})$ such that $J^\infty(\bar{u},\bar{v})=0$. It follows that $(\bar{u},\bar{v})$ is a vector of constant functions; the symmetry and the segregation condition imply that $(\bar{u},\bar{v})\equiv(0,0)$, but this is in contrast with the fact that $(\bar{u},\bar{v})\in\mathcal{M}$. Thus, the weak convergence of the sequence $(\hat{u}_n,\hat{v}_n)$ entails

$$\liminf_{n\to\infty}\hat{N}_n(1)\geq\liminf_{n\to\infty}\int_{C_{(0,1)}}|\nabla\hat{u}_n|^2+|\nabla\hat{v}_n|^2\geq m>0,$$

so that whenever n is sufficiently large

$$N_{R_n}(1) = \hat{N}_n(1) \geq \frac{1}{2}m. \tag{5.3.9}$$

Thanks to Lemma 5.3.4 we know that $\frac{1}{2}m \leq N_{R_n}(1) \leq 2$, and from the assumption $H_{R_n}(1) \to +\infty$ we deduce that (recall the (5.2.3))

$$\begin{aligned}\varphi_{R_n}(r;1) :&= \int_1^r \frac{\mathrm{d}s}{H_{R_n}(s)^{1/4}} \\ &\leq 2\frac{e^{\frac{1}{2}N_{R_n}(1)}}{H_{R_n}(1)^{\frac{1}{4}}N_{R_n}(1)}\left[e^{-\frac{1}{2}N_{R_n}(1)} - e^{-\frac{1}{2}N_{R_n}(1)r}\right] \to 0\end{aligned}$$

as $n \to \infty$, for every $r > 1$. In particular, there exists $C > 0$ such that

$$\varphi_{R_n}(r;1) \leq C \qquad \forall 1 \leq r \leq R_n,\ \forall n. \tag{5.3.10}$$

This implies that the sequence $(E_{R_n}(1))_n$ is bounded. To see this, we firstly note that (u_{R_n}, v_{R_n}) satisfies the symmetry condition (5.2.5) which is necessary to apply Lemma 5.2.8; consequently, the variational characterization of (u_{R_n}, v_{R_n}) (see also the proof of Lemma 5.3.4 and the (5.3.10)) implies that

$$\begin{aligned}\frac{E_{R_n}(1)}{e^2} &\leq e^{C\varphi_{R_n}(R_n;1)}\frac{E_{R_n}(R_n)}{e^{2R_n}} \leq 2C\frac{\mathcal{E}_{R_n}(R_n)}{e^{2R_n}} \\ &\leq C\frac{\int_{C_{(0,R_n)}}|\nabla\Phi|^2}{e^{2R_n}} = C\frac{\sinh R_n \cosh R_n}{e^{2R_n}} \leq C,\end{aligned}$$

where C does not depend on n. Since $(E_{R_n}(1))_n$ is bounded and $(H_{R_n}(1))_n$ tends to infinity, we obtain

$$\lim_{n\to\infty} N_{R_n}(1) = \lim_{n\to\infty}\frac{E_{R_n}(1)}{H_{R_n}(1)} = 0,$$

in contradiction with (5.3.9). □

Proposition 5.3.6. *There exists a subsequence of (u_R, v_R) which converges in $\mathcal{C}^2_{loc}(C_\infty)$, as $R \to +\infty$, to a solution (u,v) of $(S)_2$ in the whole C_∞. This solution satisfies point 2)-5) of Theorem 5.1.2, and its Almgren quotient N is such that*

$$N(r) \leq 2 \quad \forall r > 0 \quad \textit{and} \quad \lim_{r\to+\infty} N(r) \geq 1.$$

Proof. As $H_R(1)$ is bounded in R and $N_R(1) \leq 2$, also $E_R(1)$ is bounded in R. By means of a Poincaré inequality of type (5.3.2), this induces a uniform-in-R bound for the $H^1(C_{(0,1)})$ norm of (u_R, v_R), which in turns, by the compactness of the trace operator, gives a uniform-in-R bound for the $L^2(\partial C_{(0,1)})$ norm. Due to the subharmonicity of (u_R, v_R), the $L^2(\partial C_{(0,1)})$ bound provides a uniform-in-R bound for the L^∞ norm of (u_R, v_R) in every compact subset of $C_{(0,1)}$; the regularity theory for elliptic equations (see [35]) ensures that, up to a subsequence, (u_R, v_R) converges in

$\mathcal{C}^2_{loc}(C_{(0,1)})$, as $R \to +\infty$, to a solution (u^1, v^1) of $(S)_2$ in $C_{(0,1)}$. As each (u_R, v_R) is even in x, this solution can be extended by even symmetry in x to C_1, and here satisfies the conditions 1)-4) of Proposition 5.3.1 (hence both u^1 and v^1 are nontrivial). The previous argument can be iterated: indeed, by Corollary 5.2.5 and Lemma 5.3.4, we deduce

$$H_R(r) \le \frac{H_R(1)}{e^4} e^{4r} \le C e^{4r} \qquad \forall r > 1;$$

that is, a uniform-in-R bound for $H_R(1)$ induces a uniform-in-R bound for $H_R(r)$ for every $r > 1$. As a consequence we obtain, for every $r > 1$, a solution (u^r, v^r) to equation $(S)_2$ in C_r. A diagonal selection gives the existence of a solution (u, v) to $(S)_2$ in the whole C_∞. This solution inherits by (u^r, v^r) the conditions 1)-4) of Proposition 5.3.1, and thanks to the $\mathcal{C}^2_{loc}(C_\infty)$ convergence and Lemma 5.3.4 there holds

$$N(r) = \frac{\int_{C_{(0,r)}} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\Sigma_r} u^2 + v^2} \le 2 \qquad \forall r > 0.$$

From Lemma 5.2.9, which we can apply in light of the symmetries of (u, v), we conclude

$$\lim_{r \to +\infty} N(r) \ge 1. \qquad \square$$

The following Lemma completes the proof of point 6) of Theorem 5.1.2. After that, by means of the pointwise estimates $u > \Phi^+$ and $v > \Phi^-$ and Corollary 5.2.5, it is straightforward to obtain also point 7).

Lemma 5.3.7. *There holds* $d := \lim_{r \to \infty} N(r) = 1$.

Proof. In light of the fact that $d \ge 1$, it is sufficient to show that $d \le 1$. Let (u_{R_n}, v_{R_n}) be the convergent subsequence found in Proposition 5.3.6, which we will simply denote $\{(u_n, v_n)\}$. For $r > 0$ we let

$$f_n(r) := \frac{\int_{C_{(0,r)}} u_n^2 v_n^2}{H_{R_n}(r)}, \quad g_n(r) := \frac{\int_{\Sigma_r} u_n^2 v_n^2}{H_{R_n}(r)}.$$

With f and g we identify the same quantities computed for the limiting profile (u, v). Observe that f_n, g_n, f and g are continuous and nonnegative. By definition,

$$f_n(r) \le \frac{1}{2} N_{R_n}(r) \le 1 \qquad \forall r > 0, \tag{5.3.11}$$

where we used Lemma 5.3.4. The uniform convergence of (u_n, v_n) implies that $f_n \to f$ and $g_n \to g$ uniformly on compact intervals, while by Theorem 5.2.4 we have

$$\int_0^r g_n(s)\,\mathrm{d}s \le N_{R_n}(r) \quad \text{and} \quad \int_0^r g(s)\,\mathrm{d}s \le N(r),$$

so that in particular $g_n \in L^1(0,R)$ and $g \in L^1(\mathbb{R}^+)$. By means of the monotonicity formula for the Almgren quotient $\mathfrak{N}$, Proposition 5.2.4, it is possible to refine the computation in Lemma 5.3.4:

$$N_{R_n}(r) = \mathfrak{N}_{R_n}(r) + f_n(r) \leq \mathfrak{N}_{R_n}(R_n) + f_n(r) \leq 1 + f_n(r).$$

In light of the strong $H^1_{loc}(C_\infty)$ convergence of (u_n, v_n) to (u,v), we deduce

$$N(r) \leq 1 + \lim_{n\to+\infty} f_n(r) = 1 + f(r).$$

We have to show that $f(r) \to 0$ as $r \to +\infty$. To prove this, we begin by computing the logarithmic derivative of f_n:

$$\frac{f_n'(r)}{f_n(r)} = \frac{\int_{\Sigma_r} u_n^2 v_n^2}{\int_{C_{(0,r)}} u_n^2 v_n^2} - 2\frac{E_{R_n}(r)}{H_{R_n}(r)} = \frac{g_n(r)}{f_n(r)} - 2N_{R_n}(r),$$

where we used the fact that $H'_{R_n}(r) = 2E_{R_n}(r)$, see equation (5.2.2). Exploiting the strong H^1 convergence of the sequence $\{(u_n, v_n)\}$ and the fact that $\lim_{r\to+\infty} N(r) \geq 1$, we deduce that there exist $r_0, \delta > 0$ such that $N_{R_n}(r_0) > \delta$ for every n sufficiently large. Consequently, f_n satisfies the inequality

$$f_n'(r) + 2\delta f_n(r) \leq g_n(r) \qquad \text{for } r \in (r_0, R_n).$$

Multiplying for $e^{2\delta r}$ and integrating in (r_1, r_2) for $r_0 < r_1 < r_2 < R_n$, we obtain

$$f_n(r_2) \leq e^{2\delta(r_1 - r_2)} f_n(r_1) + \int_{r_1}^{r_2} g_n(s) e^{2\delta(s-r_2)} \, \mathrm{d}s \leq e^{2\delta(r_1-r_2)} + \int_{r_1}^{r_2} g_n(s) \, \mathrm{d}s,$$

where we used the estimate (5.3.11). This implies

$$f(r_2) \leq e^{2\delta(r_1-r_2)} + \int_{r_1}^{r_2} g(s) \, \mathrm{d}s \qquad \text{for } r_0 < r_1 < r_2.$$

Since $g \in L^1(\mathbb{R}^+)$ and $f \geq 0$, choosing $r_1 = \frac{1}{2} r_2$ we find

$$\limsup_{r\to+\infty} f(r) = 0 = \lim_{r\to+\infty} f(r). \qquad \square$$

5.4 Proof of Theorem 5.1.6

In this section we construct a solution to $(S)_2$ modeled on the harmonic function $\Gamma(x,y) = e^x \sin y$. Our construction is based on the trivial observation that

$$\Phi_R(x,y) := 2\cosh(x+R)e^{-R} \sin y \to \Gamma(x,y) \qquad \text{as } R \to +\infty.$$

5.4.1 Existence in bounded cylinders

As a first step, using the same line of reasoning developed in Proposition 5.3.1, it is possible to show the existence of solution to the system

$$\begin{cases} -\Delta u = -uv^2 & \text{in } C_{(-3R,R)} \\ -\Delta v = -u^2 v & \text{in } C_{(-3R,R)} \\ u, v > 0 \end{cases} \tag{5.4.1a}$$

(equivalently, we can consider the problem in the rectangle $(-3R, R) \times (0, 2\pi)$ with periodic boundary condition on the sides $[-3R, R] \times \{0, 2\pi\}$) and such that

$$u_R = \Phi_R^+, \quad v_R = \Phi_R^- \quad \text{on } \Sigma_R \cup \Sigma_{-3R}. \tag{5.4.1b}$$

More precisely:

Proposition 5.4.1. *There exists a solution (u_R, v_R) to problem (5.4.1a) with the prescribed boundary conditions (5.4.1b), such that*

1) $u_R(-R - x, y) = u_R(-R + x, y)$ and $v_R(-R - x, y) = v_R(-R + x, y)$,

2) the symmetries

$$\begin{aligned} v_R(x,y) &= u_R(x, y-\pi) & \qquad u_R(x, \pi - y) &= v_R(x, \pi + y) \\ u_R\left(x, \frac{\pi}{2} + y\right) &= u_R\left(x, \frac{\pi}{2} - y\right) & \qquad v_R\left(x, \frac{3}{2}\pi + y\right) &= v_R\left(x, \frac{3}{2}\pi - y\right) \end{aligned}$$

hold,

3) $u_R - v_R > 0$ in $\{\Phi_R > 0\}$ and $v_R - u_R > 0$ in $\{\Phi_R < 0\}$,

4) $u_R > (\Phi_R)^+$ and $v_R > (\Phi_R)^-$.

Sketch of proof. One can recast the proof of Proposition 5.3.1 in this setting. □

Remark 5.4.2. In light of point 1) of the Proposition, it results

$$\partial_x u_R = 0 = \partial_x v_R \qquad \text{on } \Sigma_{-R}.$$

Therefore, the monotonicity formulæ proved in subsection 5.2.1 hold true for (u_R, v_R) in the semi-cylinder C_R.

5.4.2 Compactness of the family $\{(u_R, v_R)\}$

As in the previous section, we denote as $E_R, \mathcal{E}_R, N_R$ and $\mathfrak{N}_R$ the functions $E^{sym}, \mathcal{E}^{sym}, N^{sym}$ and $\mathfrak{N}^{sym}$ defined in subsection 5.2.1 when referred to (u_R, v_R). We follow here the same line of reasoning adopted in subsection 5.3.2. Firstly, it is not difficult to modify the proof of Lemmas 5.3.4 and 5.3.5 obtaining the following estimates:

Lemma 5.4.3. *There holds $N_R(r) \leq 2$, for every $R > 0$ and $r \in (-R, R)$.*

Lemma 5.4.4. *There exists $C > 0$ such that $H_R(1) \leq C$ for every $R > 1$.*

We are in position to show that the family $\{(u_R, v_R)\}$ is compact, in the following sense.

Proposition 5.4.5. *There exists a subsequence of $\{(u_R, v_R)\}$ which converges in $\mathcal{C}^2_{loc}(C_\infty)$, as $R \to +\infty$, to a solution (u, v) of $(S)_2$ in the whole C_∞. This solution has the properties 2)-4) of Proposition 5.4.1.*

Proof. As $H_R(1)$ is bounded in R and $N_R(1) \leq 2$, also $E_R(1)$ is bounded in R, and a fortiori

$$\int_{C_1} |\nabla u_R|^2 + |\nabla v_R|^2 \leq C \qquad \forall R > 1.$$

This estimate, the boundedness of $H_R(1)$ and a Poincarè inequality of type (5.3.2) imply that $\{(u_R, v_R)\}$ is bounded in $H^1(C_1)$. Consequently, it is possible to argue as in the proof of Proposition 5.3.6 and obtain the existence of a subsequence of $\{(u_R, v_R)\}$ which converges in $\mathcal{C}^2_{loc}(C_1)$ to a solution (u^1, v^1) of $(S)_2$ in C_1, which inherits by $\{(u_R, v_R)\}$ the properties 2)-4) of Proposition 5.4.1. In light of Corollary 5.2.5 and Lemma 5.4.3, this procedure can be iterated: indeed

$$H_R(r) \leq \frac{H_R(1)}{e^4} e^{4r} \leq C e^{4r} \qquad \forall r > 1,$$

so that applying the previous argument we obtain a subsequence of $\{(u_R, v_R)\}$ which converges in $\mathcal{C}^2_{loc}(C_r)$ to a solution (u^r, v^r) of $(S)_2$ in C_r, and inherits by $\{(u_R, v_R)\}$ the properties 2)-4) of Proposition 5.4.1. A diagonal selection gives the existence of a solution (u, v) of $(S)_2$ in the whole C_∞, and this solution enjoys the properties 2)-4) of Proposition 5.4.1. $\square$

Remark 5.4.6. The monotonicity formulæ proved in subsection 5.2.1 do not apply on (u, v), because passing to the limit we lose the Neumann condition $\partial_x u_R = 0 = \partial_x v_R$ on Σ_{-R}.

In the next Lemma, we show that (u, v) is a solution with finite energy, so that the achievements proved in subsection 5.2.2 applies.

Lemma 5.4.7. *Let (u,v) be the solution found in Proposition 5.4.5. It results*

$$\mathcal{E}^{unb}(r) := \int_{C_{(-\infty,r)}} |\nabla u|^2 + |\nabla v|^2 + u^2v^2 < +\infty \qquad \forall r \in \mathbb{R} \tag{5.4.2}$$

and

$$\lim_{r\to-\infty} H(r) = \lim_{r\to-\infty} \int_{\Sigma_r} u^2 + v^2 = 0.$$

Recall that $\mathcal{E}^{unb}$ has been defined in subsection 5.2.2.

Proof. Let $\{(u_{R_n}, v_{R_n})\}$ be the converging subsequence found in Proposition 5.4.5, which we will simply denote $\{(u_n,v_n)\}$. Since $\{(u_n,v_n)\}$ converges to (u,v) in $\mathcal{C}^2_{loc}(C_\infty)$, it follows that

$$\lim_{n\to\infty} \left(|\nabla u_n|^2 + |\nabla v_n|^2 + u_n^2 v_n^2\right) \chi_{C_{(-R_n,r)}} = \left(|\nabla u|^2 + |\nabla v|^2 + u^2v^2\right) \chi_{C_{(-\infty,r)}} \qquad \text{a.e. in } C_{(-\infty,r)},$$

for every $r > 1$. Therefore, applying Corollary 5.2.5 on (u_n,v_n), Lemma 5.4.4 and the Fatou lemma, we deduce

$$\begin{aligned}\mathcal{E}^{unb}(r) &\le \liminf_{n\to\infty} \int_{C_{(-\infty,r)}} \left(|\nabla u_n|^2 + |\nabla v_n|^2 + u_n^2v_n^2\right)\chi_{C_{(-R_n,r)}} \le \liminf_{n\to\infty} E_{R_n}(r)\\ &= \liminf_{n\to\infty} N_{R_n}(r) H_{R_n}(r) \le \liminf_{n\to\infty} 2\frac{H_{R_n}(1)}{e^4} e^{4r} \le Ce^{4r},\end{aligned}$$

which proves the (5.4.2). To complete the proof, we firstly note that necessarily $\mathcal{E}^{unb}(r) \to 0$ as $r \to -\infty$, and hence the same holds for E^{unb} (which has been defined in subsection 5.2.2). Assume by contradiction that for a sequence $r_n \to -\infty$ it results $H(r_n) \ge C > 0$. We define

$$(\hat{u}_n(x,y), \hat{v}_n(x,y)) := \frac{1}{\sqrt{H(r_n)}} \left(u(x+r_n,y), v(x+r_n,y)\right).$$

A direct computation shows that

$$\int_{C_{(-\infty,0)}} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 \le \int_{C_{(-\infty,0)}} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 + 2H(r_n)\hat{u}_n^2\hat{v}_n^2 = \frac{1}{H(r_n)} E^{unb}(r_n) \to 0$$

as $n \to \infty$. Consequently, $(\hat{u}_n, \hat{v}_n)$ tend to be a pair of constant functions of type $(\hat{u}, \hat{v})$ with $\hat{u} = \hat{v}$ (this follows from the symmetries of (u,v)). As

$$C \int_{C_{(-\infty,0)}} \hat{u}_n^2 \hat{v}_n^2 \le H(r_n) \int_{C_{(-\infty,0)}} \hat{u}_n^2\hat{v}_n^2 \to 0,$$

necessarily $(\hat{u}_n, \hat{v}_n) \to (0,0)$ almost everywhere in $C_{(-\infty,0)}$. This is in contradiction with the fact that $\int_{\Sigma_0} \hat{u}_n^2 + \hat{v}_n^2 = H(r_n) \ge C$. □

So far we proved that the solution (u,v), found in Proposition 5.4.5, enjoys properties 1)-5) of Theorem 5.1.6, and is such that $H(r) \to 0$ as $r \to -\infty$. The previous

Lemma enables us to apply the achievements of subsection 5.2.2 for E^{unb}, H, N^{unb} and $\mathfrak{N}^{unb}$ (which we consider referred to the solution (u,v) found in Proposition 5.4.5), and permits to complete the description of the growth of (u,v), points 6)-7) of Theorem 5.1.6.

Lemma 5.4.8. *Let (u,v) be the solution found in Proposition 5.4.5. It results*

$$\lim_{r\to+\infty} N^{unb}(r) = 1.$$

Proof. Let $\{(u_{R_n}, v_{R_n})\}$be the converging subsequence found in Proposition 5.4.5, , which we will simply denote $\{(u_n, v_n)\}$. Firstly, arguing as in the proof of the previous Lemma, we note that by the $\mathcal{C}^2_{loc}(C_\infty)$ convergence of (u_n, v_n) to (u,v) it follows that

$$N^{unb}(r) \le \liminf_{n\to\infty} N_{R_n}(r) \le 2 \qquad \forall r \in \mathbb{R},$$

thanks to the Fatou lemma. This, together with the symmetries of (u,v), permits to use Lemma 5.2.17, which gives $\lim_{r\to+\infty} N^{unb}(r) \ge 1$. To complete the proof, it is sufficient to show that $\lim_{r\to+\infty} N^{unb}(r) \le 1$. For any $r > 0$, let

$$f_n(r) := \frac{\int_{C_r} u_n^2 v_n^2}{H_{R_n}(r)}, \qquad g_n(r) := \frac{\int_{\Sigma_r \cup \Sigma_{-r}} u_n^2 v_n^2}{H_{R_n}(r)},$$

and let f and g the same quantities referred to the solution (u,v). Observe that f_n, g_n, f and g are continuous and nonnegative. The uniform convergence of (u_n, v_n) to (u,v) implies that $f_n \to f$ and $g_n \to g$, as $n \to \infty$, uniformly on compact intervals. By definition,

$$f_n(r) \le \frac{1}{2} N_{R_n}(r) \le 1 \qquad \forall r > 0. \tag{5.4.3}$$

whenever $R_n \ge r$. We claim that $g \in L^1(\mathbb{R}^+)$. Indeed, by the monotonicity of H and Proposition 5.2.14, it follows that

$$\int_0^r g(s)\,\mathrm{d}s = \int_0^r \frac{\int_{\Sigma_s} u^2v^2}{H(s)}\,\mathrm{d}s + \int_{-r}^0 \frac{\int_{\Sigma_s} u^2v^2}{H(-s)}\,\mathrm{d}s \le \int_{-r}^r \frac{\int_{\Sigma_s} u^2v^2}{H(s)}\,\mathrm{d}s \le \int_{-\infty}^r \frac{\int_{\Sigma_s} u^2v^2}{H(s)}\,\mathrm{d}s \le N^{unb}(r),$$

for every $r > 0$. Let $r > 0$; it is possible to refine the computation on Lemma 5.3.4 to obtain

$$N_{R_n}(r) \le 1 + f_n(r) + \frac{\int_{C_{(-R_n,-r)}} u_n^2 v_n^2}{H_{R_n}(r)} \le 1 + f_n(r) + \frac{E_{R_n}(-r)}{H_{R_n}(r)}$$

Therefore, using again the Fatou lemma we deduce

$$N^{unb}(r) \le \liminf_{n\to\infty} N_{R_n}(r) \le 1 + f(r) + \liminf_{n\to\infty} \frac{E_{R_n}(-r)}{H_{R_n}(r)},$$

and to complete the proof we will show that

$$\lim_{r\to+\infty} \left(f(r) + \liminf_{n\to\infty} \frac{E_{R_n}(-r)}{H_{R_n}(r)} \right) = 0. \tag{5.4.4}$$

Firstly, we note that

$$\liminf_{n\to\infty} \frac{E_{R_n}(-r)}{H_{R_n}(r)} = \liminf_{n\to\infty} \frac{N_{R_n}(-r)H_{R_n}(-r)}{H_{R_n}(r)} \leq 2\liminf_{n\to\infty} \frac{H_{R_n}(-r)}{H_{R_n}(r)}.$$

From the $\mathcal{C}^2_{loc}(C_\infty)$ convergence of (u_n, v_n) to (u, v) it follows

$$2\liminf_{n\to\infty} \frac{H_{R_n}(-r)}{H_{R_n}(r)} = 2\frac{H(-r)}{H(r)} \to 0 \qquad \text{as } r \to +\infty$$

where we used Lemma 5.4.7 and the fact that $H(r) > H(0) > 0$ for every $r > 0$. For the (5.4.4) it remains to prove that $f(r) \to 0$ as $r \to +\infty$. Having observed that $\lim_{r\to+\infty} N(r) \geq 1$ and that $g \in L^1(\mathbb{R}^+)$, it is not difficult to adapt the conclusion of the proof of Lemma 5.3.7. □

5.5 Systems with many components

In this section we are going to prove the existence of entire solutions with exponential growth for the k component system $(S)_k$. Our construction is based on the elementary limit

$$\lim_{d\to+\infty} \Im\left[\left(1+\frac{z}{d}\right)^d\right] = e^x \sin y,$$

which shows that the harmonic function $e^x \sin y$ can be obtained as limit of homogeneous harmonic polynomial. We wish to prove that the same idea applies to solutions of the system $(S)_k$: there exists an entire solution to $(S)_k$ having exponential growth which can be obtained as limit of entire solutions having algebraic growth.

5.5.1 Preliminary results

We recall some results contained in [6]. For $d \in \frac{\mathbb{N}}{2}$, let G_d be the rotation of angle $\frac{\pi}{d}$ in counterclockwise sense.

Theorem 5.5.1 (Theorem 1.6 of [6]). *Let $k \geq 2$ be a positive integer, let $d \in \frac{\mathbb{N}}{2}$ be such that*

$$2d = hk \qquad \textit{for some } h \in \mathbb{N}.$$

There exists a solution $(u_1^d, \dots, u_k^d)$ to the system $(S)_k$ which enjoys the following symmetries

$$\begin{aligned} u_i^d(x,y) &= u_i^d(G_d^k(x,y)) \\ u_i^d(x,y) &= u_{i+1}^d(G_d(x,y)) \\ u_{k+1-i}^d(x,y) &= u_i^d(x,-y) \end{aligned} \tag{5.5.1}$$

where we recall that indexes are meant $\mod k$. *Moreover*

$$\lim_{r\to+\infty} \frac{1}{r^{1+2d}} \int_{\partial B_r} \sum_{i=1}^{k} \left(u_i^d\right)^2 = b \in (0,+\infty),$$

and

$$\lim_{r\to+\infty} \frac{r \int_{B_r} \sum_{i=1}^{k} |\nabla u_i^d|^2 + \sum_{1\le i<j\le k} \left(u_i^d u_j^d\right)^2}{\int_{\partial B_r} \sum_{i=1}^{k} \left(u_i^d\right)^2} = d, \tag{5.5.2}$$

where B_r denotes the ball of center 0 *and radius* r.

The solution $(u_1^d, \dots, u_k^d)$ is modeled on the harmonic function $\Im(z^d)$, as specified by the symmetries (5.5.1). In the quoted statement, the authors modeled their construction on the functions $\Re(z^d)$: it is straightforward to obtain an analogous result replacing the real part with the imaginary one.

Remark 5.5.2. We point out that the symmetries (5.5.1) implies that u_1^d is symmetric with respect to the reflection with the axis $y = \tan\left(\frac{\pi}{2d}\right) x$.

For a solution $(u_1, \dots, u_k)$ of system $(S)_k$ in $\mathbb{R}^2$, we introduce the functionals

$$\begin{aligned} E^{alg}(r;\Lambda) &:= \int_{B_r} \sum_{i=1}^{k} |\nabla u_i|^2 + \Lambda \sum_{1\le i<j\le k} \left(u_i u_j\right)^2 \\ H^{alg}(r) &:= \frac{1}{r} \int_{\partial B_r} \sum_{i=1}^{k} \left(u_i\right)^2 \end{aligned} \tag{5.5.3}$$

The index *alg* denotes the fact that these quantities are well suited to describe the growth of $(u_1, \dots, u_k)$ under the assumption that $(u_1, \dots, u_k)$ has algebraic growth. In particular, as proved in Lemma 2.1 of [32] and Corollary A.8 of [33] for the case $k = 2$, the Almgren quotient

$$N^{alg}(r;1) := \frac{E^{alg}(r;1)}{H^{alg}(r)}$$

is bounded in $r \in \mathbb{R}^+$ if and only if $(u_1, \dots, u_k)$ has algebraic growth.

It is not difficult to adapt the proof of Proposition 5.2 in [6] to obtain the following general result (in the sense that it holds true for an arbitrary solution of $(S)_k$ in $\mathbb{R}^N$, for any dimension $N \ge 2$).

Proposition 5.5.3 (see Proposition 5.2 of [6])**.** *Let* $N \ge 2$,

$$\Lambda \in \begin{cases} \left[1, \frac{N}{N-2}\right] & \text{if } N > 2 \\ [1, +\infty) & \text{if } N = 2, \end{cases}$$

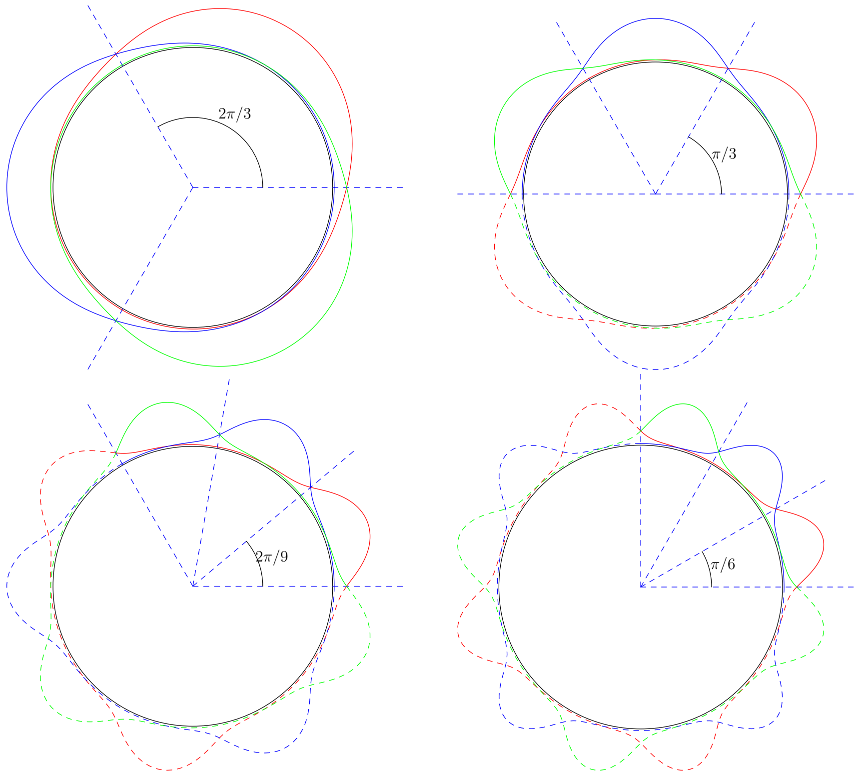

Figure 5.2: In the figure we represent some of the solutions obtained in Theorem 5.5.1. Here the number of components is set as $k = 3$: each component is drawn with a different color. On the other hand the periodicity (that is, how many times the patch of 3-components is replicated in the circle) is given by $h = 1$ (up left), $h = 2$ (up right), $h = 3$ (down left) and $h = 4$ (down right), respectively. As a consequence, the growth rate d varies as $d = \frac{3}{2}, 3, \frac{9}{2}, 6$, following the same order.

and let $(u_1,\dots,u_k)$ *be a solution of* $(S)_k$ *in* $\mathbb{R}^N$*; the Almgren quotient*

$$N^{alg}(r;\Lambda) := \frac{E^{alg}(r;\Lambda)}{H^{alg}(r)} = \frac{r\displaystyle\int_{B_r}\sum_{i=1}^{k}|\nabla u_i|^2 + \Lambda\sum_{1\le i<j\le k}(u_iu_j)^2}{\displaystyle\int_{\partial B_r}\sum_{i=1}^{k}(u_i)^2}$$

is well defined in $(0,+\infty)$ *and nondecreasing in* r*.*

Proof. We observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}E^{alg}(r;\Lambda) &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{N-2}}\int_{B_r}\sum_i|\nabla u_i|^2+\sum_{i<j}(u_iu_j)^2\right)+\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Lambda-1}{r^{N-2}}\int_{B_r}\sum_{i<j}(u_iu_j)^2\right)\\ &= \frac{2}{r^{N-2}}\int_{\partial B_r}\sum_i(\partial_\nu u_i)^2+\frac{2}{r^{N-1}}\int_{B_r}\sum_{i<j}(u_iu_j)^2\\ &\quad+\frac{(2-N)(\Lambda-1)}{r^{N-1}}\int_{B_r}\sum_{i<j}u_i^2u_j^2+\frac{\Lambda-1}{r^{N-2}}\int_{\partial B_r}\sum_{i<j}u_i^2u_j^2,\end{aligned} \tag{5.5.4}$$

where we used equation (5.3) in [6]. Proceeding as in the proof of Proposition 5.2 in [6], one gets

$$\frac{\mathrm{d}}{\mathrm{d}r}N^{alg}(r;\Lambda)\ge(2+(\Lambda-1)(2-N))\frac{\int_{B_r}\sum_{i<j}u_i^2u_j^2}{r^{N-1}H^{alg}(r)}+\frac{(\Lambda-1)\int_{\partial B_r}\sum_{i<j}u_i^2u_j^2}{r^{N-2}H^{alg}(r)},$$

which is ≥ 0 by our assumption on Λ. □

Remark 5.5.4. In [6] the authors consider the case $\Lambda = 1$.

We work in the plane $\mathbb{R}^2$, so that it is possible to choose $\Lambda = 2$ in Proposition 5.5.3. We denote $E_d(\cdot;\Lambda)$ and H_d the quantities defined in (5.5.3) when referred to the functions $(u_1^d,\dots,u_k^d)$ defined in Theorem 5.5.1; also, we denote $N_d(\cdot;\Lambda) := \dfrac{E_d(\cdot;\Lambda)}{H_d}$. In case $\Lambda = 2$, we will simply write E_d and N_d to ease the notation.

Lemma 5.5.5. *Let* $(u_1^d,\dots,u_k^d)$ *be defined in Theorem 5.5.1. There holds* $\lim_{r\to+\infty}N_d(r) = d$*.*

Proof. It is an easy consequence of the (5.5.2) and of Corollary 5.8 in [6], where it is proved that for the solution $(u_1^d,\dots,u_k^d)$ there holds

$$\lim_{r\to+\infty}\frac{E_d(r;2)}{r^{2d}} = \lim_{r\to+\infty}\frac{E_d(r;1)}{r^{2d}}.$$

Therefore,

$$\begin{aligned}\lim_{r\to+\infty}N_d(r) &= \lim_{r\to+\infty}\frac{E_d(r;2)}{H_d(r)} = \lim_{r\to+\infty}\frac{E_d(r;2)}{r^{2d}}\cdot\lim_{r\to+\infty}\frac{r^{2d}}{H_d(r)}\\ &= \lim_{r\to+\infty}\frac{E_d(r;1)}{r^{2d}}\cdot\lim_{r\to+\infty}\frac{r^{2d}}{H_d(r)} = \lim_{r\to+\infty}N_d(r;1) = d.\end{aligned}$$ □

As a consequence, the following doubling property holds true:

Proposition 5.5.6 (See Proposition 5.3 of [6]). *For any $0 < r_1 < r_2$ it holds*

$$\frac{H_d(r_2)}{r_2^{2d}} \le \frac{H_d(r_1)}{r_1^{2d}}.$$

Proof. A direct computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}r} \log \frac{H_d(r)}{r^{2d}} = \frac{2N_d(r)}{r} - \frac{2d}{r} \le 0;$$

an integration gives the thesis. □

Let us consider the scaling

$$(u_{1,R}^d, \dots, u_{k,R}^d) := \left(\frac{2d}{kH_d(R)} \right)^{\frac{1}{2}} \left(u_1^d(Rx, Ry), \dots, u_k^d(Rx, Ry) \right), \tag{5.5.5}$$

where R will be determined later as a function of d. We see that

$$\begin{cases} -\Delta u_{i,R}^d = -\beta_R^d \, u_{i,R}^d \sum_{j \neq i} \left(u_{j,R}^d \right)^2 & \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\partial B_1} \sum_{i=1}^k \left(u_{i,R}^d \right)^2 = \frac{2d}{k} \end{cases} \tag{5.5.6}$$

where $\beta_R^d := \frac{k}{2d} H_d(R) R^2$.

Remark 5.5.7. As a function of R, β_R^d is continuous and such that $\beta_R^d \to 0$ if $R \to 0$ and $\beta_R^d \to \infty$ if $R \to \infty$.

Accordingly with our scaling, we introduce the new Almgren quotient

$$N_{d,R}(r) := \frac{E_{d,R}(r)}{H_R(r)} = \frac{r \displaystyle\int_{B_r} \sum_{i=1}^k |\nabla u_{i,R}^d|^2 + 2\beta_R^d \sum_{1 \le i < j \le k} \left(u_{i,R}^d \, u_{j,R}^d \right)^2}{\displaystyle\int_{\partial B_r} \sum_{i=1}^k \left(u_{i,R}^d \right)^2}.$$

We point out that $N_{d,R}(r) = N_d(Rr)$, so that from Lemma 5.5.5 and the monotonicity of N_d we deduce

$$N_{d,R}(r) \le d \qquad \forall r, R > 0, \tag{5.5.7}$$

for every d. By the symmetries, the solution $(u_{1,R}^d, \dots, u_{k,R}^d)$ is $\frac{k\pi}{d}$-periodic with respect to the angular component, thus it is convenient to restrict our attention to the cones

$$S_r^d := \left\{ (\rho, \theta) : \rho \in (0, r), \theta \in \left(0, \frac{k\pi}{d} \right) \right\} \quad \text{and} \quad S^d := \left\{ (\rho, \theta); \rho > 0, \theta \in \left(0, \frac{k\pi}{d} \right) \right\}.$$

The boundary ∂S^d_r can be decomposed as $\partial S^d_r = \partial_p S^d_r \cup \partial_r S^d_r$, where

$$\partial_p S^d_r := (0,r) \times \left\{0, \frac{k\pi}{d}\right\} \quad \text{and} \quad \partial_r S^d_r := \{r\} \times \left(0, \frac{k\pi}{d}\right).$$

Taking into account the periodicity of $(u^d_{1,R}, \ldots, u^d_{k,R})$, we note that $(u^d_{1,R}, \ldots, u^d_{k,R})$ has periodic boundary conditions on $\partial_p S^d_r$; furthermore

$$\begin{aligned} E_{d,R}(r) &= \frac{2d}{k} \int_{S^d_r} \sum_i |\nabla u^d_{i,R}|^2 + 2\beta^d_R \sum_{i<j} \left(u^d_{i,R}\, u^d_{j,R}\right)^2 \\ H_{d,R}(r) &= \frac{2d}{kr} \int_{\partial_r S^d_r} \sum_i \left(u^d_{i,R}\right)^2 \\ N_{d,R}(r) &= \frac{r \displaystyle\int_{S^d_r} \sum_i |\nabla u^d_{i,R}|^2 + 2\beta^d_R \sum_{i<j} \left(u^d_{i,R}\, u^d_{j,R}\right)^2}{\displaystyle\int_{\partial S^d_r} \sum_i \left(u^d_{i,R}\right)^2}. \end{aligned} \tag{5.5.8}$$

5.5.2 *A blow-up in a neighborhood of* $(1,0)$

In order to pursue our strategy, we consider the further scaling

$$(\hat{u}^d_{1,R}(x,y), \ldots, \hat{u}^d_{k,R}(x,y)) = \frac{\sqrt{\beta^d_R}}{d} \left(u^d_{1,R}\left(1+\frac{x}{d}, \frac{y}{d}\right), \ldots, u^d_{k,R}\left(1+\frac{x}{d}, \frac{y}{d}\right)\right). \tag{5.5.9}$$

Accordingly, we will consider the scaled domains $\hat{S}^d_r = d\left(S^d_r - (1,0)\right)$ and $\hat{S}^d = d\left(S^d - (1,0)\right)$ and the respective boundaries. Having in mind to let $d \to \infty$, we observe that this scaling is a blow-up centered in the point $(1,0)$. It is easy to verify that $(\hat{u}^d_{1,R}, \ldots, \hat{u}^d_{k,R})$ solves (see (5.5.6))

$$\begin{cases} -\Delta \hat{u}^d_{i,R} = -\hat{u}^d_{i,R} \displaystyle\sum_{j \neq i} \left(\hat{u}^d_{j,R}\right)^2 & \text{in } \hat{S}^d \\ \displaystyle\int_{\partial_r \hat{S}^d_1} \sum_{i=1}^k \left(\hat{u}^d_{i,R}\right)^2 = \frac{\beta^d_R}{d}, \end{cases} \tag{5.5.10}$$

with suitable periodic conditions on $\partial \hat{S}^d$. A direct computation shows that from (5.5.8) it follows

$$N_{d,R}(r) = d \frac{r \displaystyle\int_{\hat{S}^d_r} \sum_i |\nabla \hat{u}^d_{i,R}|^2 + 2 \sum_{i<j} \left(\hat{u}^d_{i,R} \hat{u}^d_{j,R}\right)^2}{\displaystyle\int_{\partial_r \hat{S}^d_r} \sum_i \left(\hat{u}^d_{i,R}\right)^2},$$

where in the new coordinates

$$r = \sqrt{\left(1+\frac{x}{d}\right)^2 + \left(\frac{y}{d}\right)^2}. \tag{5.5.11}$$

We are then led to define a new Almgren quotient for the scaled functions $(\hat{u}^d_{1,R}, \dots, \hat{u}^d_{k,R})$:

$$\hat{E}_{d,R}(r) := \int_{\hat{S}^d_r} \sum_{i=1}^k |\nabla \hat{u}^d_{i,R}|^2 + 2 \sum_{1\le i<j\le k} \left(\hat{u}^d_{i,R}\hat{u}^d_{j,R}\right)^2$$

$$\hat{H}_{d,R}(r) := \frac{1}{r}\int_{\partial_r \hat{S}^d_r} \sum_{i=1}^k \left(\hat{u}^d_{i,R}\right)^2$$

$$\hat{N}_{d,R}(r) := \frac{\hat{E}_{d,R}(r)}{\hat{H}_{d,R}(r)} = \frac{1}{d} N_{d,R}(r).$$

From the equation (5.5.7), we deduce

$$\hat{N}_{d,R}(r) \le 1 \qquad \forall r, R > 0,\ \forall d \in \frac{\mathbb{N}}{2}. \tag{5.5.12}$$

In order to understand the behavior of $(\hat{u}^d_{1,R}, \dots, \hat{u}^d_{k,R})$ when $d \to \infty$, we fix $R = R(d)$ to get a non-degeneracy condition.

Lemma 5.5.8. *For every $d \in \frac{\mathbb{N}}{2}$ there exists $R_d > 0$ such that*

$$\hat{H}_{d,R_d}(1) = \int_{\partial_r \hat{S}^d_1} \sum_i \left(\hat{u}^d_{i,R_d}\right)^2 = 1.$$

Proof. By (5.5.10) we know that $\hat{H}_d(1) = \frac{\beta^d_R}{d}$, so that we have to find R_d such that $\beta^d_R = d$. As observed in Remark 5.5.7, this choice is possible. □

We denote $(\hat{u}^d_1, \dots, \hat{u}^d_k) := (\hat{u}^d_{1,R_d}, \dots, \hat{u}^d_{k,R_d})$, $\hat{H}_d := \hat{H}_{d,R_d}$, $\hat{E}_d := \hat{E}_{d,R_d}$, $\hat{N}_d := \hat{N}_{d,R_d}$ and $\beta^d := \beta^d_{R_d}$. We aim at proving that, up to a subsequence, the family $\left\{(\hat{u}^d_1, \dots, \hat{u}^d_k) :\ d \in \frac{\mathbb{N}}{2}\right\}$ converges, as $d \to +\infty$, to a solution of $(S)_k$. To this aim, major difficulties arise from the fact that $\hat{S}^d_r$ and $\hat{S}^d$ depend on d; in the next Lemma we show that this problem can be overcome thanks to a convergence property of these domains.

Lemma 5.5.9. *For any $r > 1$, the sets $\hat{S}^d_r$ converge to $\mathbb{R} \times (0, k\pi)$ as $k \to +\infty$, in the sense that*

$$\mathbb{R} \times (0, k\pi) = \operatorname{int}\left(\bigcap_{n\in\frac{\mathbb{N}}{2}} \bigcup_{d>n} \hat{S}^d_r\right),$$

where for $A \subset \mathbb{R}^2$ we mean that $\operatorname{int}(A)$ *denotes the inner part A. Analogously,*

$$\mathbb{R} \times (0, k\pi) = \operatorname{int}\left(\bigcap_{n\in\frac{\mathbb{N}}{2}} \bigcup_{d>n} \hat{S}^d\right) \quad \text{and} \quad (-\infty, 0) \times (0, k\pi) = \operatorname{int}\left(\bigcap_{n\in\frac{\mathbb{N}}{2}} \bigcup_{d>n} \hat{S}^d_1\right),$$

and for every $\bar{x} \in \mathbb{R}$

$$(-\infty, \bar{x}) \times (0, k\pi) = \operatorname{int}\left(\bigcap_{n\in\frac{\mathbb{N}}{2}} \bigcup_{d>n} \hat{S}^d_{1+\frac{\bar{x}}{d}}\right).$$

Proof. We prove only the first claim. Let $r > 1$.

Step 1) $\mathbb{R}\times(0,k\pi)\subset\bigcap_{n\in\frac{\mathbb{N}}{2}}\bigcup_{d>n}\hat{S}^d_r$.

Let $(x,y)\in\mathbb{R}\times(0,k\pi)$. We show that for every $d\in\frac{\mathbb{N}}{2}$ sufficiently large $(x,y)\in\hat{S}^d_r$, that is, $\left(1+\frac{x}{d},\frac{y}{d}\right)\in S^d_r$, which means

$$\sqrt{\left(1+\frac{x}{d}\right)^2+\left(\frac{y}{d}\right)^2}<r\quad\text{and}\quad\arctan\left(\frac{y}{x+d}\right)\in\left(0,\frac{k\pi}{d}\right).$$

For the first condition it is possible to choose d sufficiently large, as $r>1$. To prove the second condition, we start by considering $d>-x$, so that $\arctan\left(\frac{y}{x+d}\right)>0$. Now, provided d is sufficiently large

$$\arctan\left(\frac{y}{x+d}\right)<\frac{k\pi}{d}\quad\Leftrightarrow\quad y<(x+d)\tan\left(\frac{k\pi}{d}\right).$$

Since $y<k\pi$, there exists $\varepsilon>0$ such that $y\leq k(1-\varepsilon)\pi$. Let $\bar{d}$ be sufficiently large so that

$$x+d>\left(1-\frac{\varepsilon}{2}\right)d\quad\text{and}\quad\frac{d}{k\pi}\tan\left(\frac{k\pi}{d}\right)>1-\frac{\varepsilon}{2}$$

for every $d>\bar{d}$. Then

$$(x+d)\tan\left(\frac{k\pi}{d}\right)>\left(1-\frac{\varepsilon}{2}\right)^2k\pi>(1-\varepsilon)k\pi\geq y$$

whenever $d>\bar{d}$.

Step 2) $\bigcap_{n\in\frac{\mathbb{N}}{2}}\bigcup_{d>n}\hat{S}^d_r\subset\mathbb{R}\times[0,k\pi]$.

We show that $(\mathbb{R}\times[0,k\pi])^c\subset\left(\bigcap_{n\in\frac{\mathbb{N}}{2}}\bigcup_{d>n}\hat{S}^d_r\right)^c$. If $(x,y)\notin\mathbb{R}\times[0,k\pi]$, then $y>k\pi$ or $y<0$. We consider only the case $y>k\pi$; in such a situation

$$y>k\pi=\lim_{d\to\infty}(x+d)\tan\left(\frac{k\pi}{d}\right),$$

so that $(x,y)\notin\hat{S}^d_r$ for every d sufficiently large. $\square$

Remark 5.5.10. As a consequence of the previous result, we see that

$$\partial_r\hat{S}^d_1\to\{0\}\times[0,k\pi]\quad\text{and}\quad\partial_r\hat{S}^d_{1+\frac{\bar{x}}{d}}\to\{\bar{x}\}\times[0,k\pi]$$

for every $\bar{x}\in\mathbb{R}$.

Remark 5.5.11. Recall the expression of r in the new variable, given by (5.5.11). For every $r>0$ and $d\in\frac{\mathbb{N}}{2}$ there exists $\xi(r,d)$ such that

$$r=1+\frac{\xi(r,d)}{d}\quad\Leftrightarrow\quad\xi(r,d)=d(r-1).$$

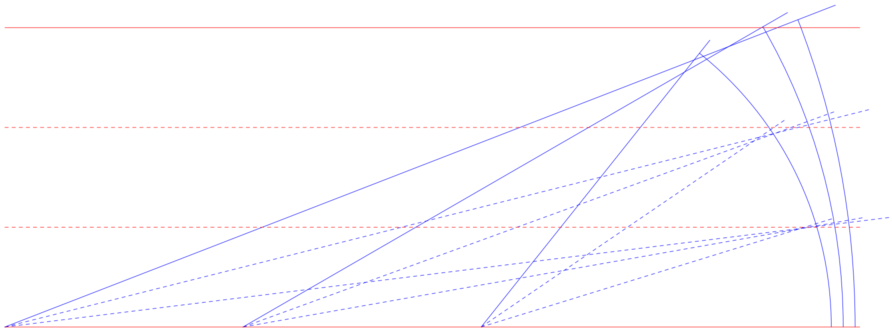

Figure 5.3: Visualization of the construction in Lemma 5.5.9. In red the limiting set $\mathbb{R} \times (0, k\pi)$. In blue some of the scaled domains $\hat{S}_r^d$, for $r > 1$.

Note that for every $(x, y) \in \partial_r \hat{S}_r^d$ it results $x < \xi(r, d)$. On the contrary, fixing $(x, y) \in \partial_r \hat{S}_r^d$ there exists $\zeta(d, x, y)$ such that

$$r = \sqrt{\left(1 + \frac{x}{d}\right)^2 + \left(\frac{y}{d}\right)^2} = 1 + \frac{x}{d} + \zeta(d, x, y).$$

In particular, if $y = 0$ we have $\zeta(d, x, 0) = 0$, while if $y > 0$, $\zeta(d, x, y) \sim d^{-2}$.

We are ready to prove the convergence of $\{(\hat{u}_1^d, \dots, \hat{u}_k^d)\}$ as $d \to \infty$.

Lemma 5.5.12. *Up to a subsequence, $\{(\hat{u}_1^d, \dots, \hat{u}_k^d)\}$ converges in $\mathcal{C}^2_{loc}(C_\infty)$, as $d \to \infty$, to a nontrivial solution $(\hat{u}_1, \dots, \hat{u}_k)$ of $(S)_k$. This solution, which is $k\pi$-periodic in y, enjoys the symmetries*

$$\hat{u}_{i+1}(x, y) = \hat{u}_i(x, y - \pi) \quad \text{and} \quad \hat{u}_1\left(x, y + \frac{\pi}{2}\right) = \hat{u}_1\left(x, y - \frac{\pi}{2}\right)$$

Proof. From Proposition 5.5.6 and Lemma 5.5.8, we deduce that for any $r \geq 1$ and d the inequality

$$\frac{\hat{H}_d(r)}{r^{2d}} = \frac{k\beta^d H_d(r)}{2d^2 r^{2d}} \leq \frac{k\beta^d}{2d^2} H_d(1) = \hat{H}_d(1) = 1$$

holds. For every $x > 0$, let $r = 1 + \frac{x}{d}$; for every d sufficiently large, we have

$$\hat{H}_d\left(1 + \frac{x}{d}\right) \leq \left(1 + \frac{x}{d}\right)^{2d} \leq 2e^{2x} \tag{5.5.13}$$

Recalling the (5.5.12) (which we apply for $R = R_d$), we deduce

$$\hat{E}_d\left(1 + \frac{x}{d}\right) = \hat{N}_d\left(1 + \frac{x}{d}\right) \hat{H}_d\left(1 + \frac{x}{d}\right) \leq 2e^{2x} \tag{5.5.14}$$

for every d sufficiently large. Recall that $(\hat{u}_1^d, \dots, \hat{u}_k^d)$ can be extended by angular periodicity in the whole plane $\mathbb{R}^2$. Let us introduce

$$T_r^d := \left\{(\rho, \theta) : \rho < r,\ \theta \in \left(-\frac{\pi}{d}, (k+1)\frac{\pi}{d}\right)\right\} \supset S_r^d,$$

and let $\hat{T}_r^d := d\left(T_r^d - (1,0)\right) \supset \hat{S}_r^d$. Suitably modifying the argument in Lemma 5.5.9, it is not difficult to see that

$$\operatorname{int}\left(\bigcap_{n\in\frac{\mathbb{N}}{2}}\bigcup_{d>n}\hat{T}^d_{1+\frac{\bar{x}}{d}}\right) = (-\infty,\bar{x})\times(-\pi,(k+1)\pi)$$

for every $\bar{x}\in\mathbb{R}$. Hence, let B an open ball contained in $\mathbb{R}\times(-\pi,(k+1)\pi)$, and let $x_B := \sup\{x : (x,y)\in B\}$, so that $B\subset(-\infty,x_B+1)\times(-\pi,(k+1)\pi)$. Using the same argument in the proof of Lemma 5.5.9, it is possible to show that

$$B\subset\hat{T}^d_{1+\frac{x_B+1}{d}},$$

for every d sufficiently large, and by the (5.5.14) and the periodicity of $(\hat{u}_1,\dots,\hat{u}_k)$ we deduce

$$\int_B\sum_i|\nabla\hat{u}_i^d|^2 \le 3\hat{E}_d\left(1+\frac{x_B+1}{d}\right)\le 6e^{2(x_B+1)}$$

whenever d is sufficiently large. This, together with (5.5.13), implies that $\{(\hat{u}_1^d,\dots,\hat{u}_k^d)\}$ is uniformly bounded in $H^1(B)$, for every $B\subset\mathbb{R}\times(-\pi,(k+1)\pi)$. By the boundedness of the trace operator, this bound provides a uniform-in-d bound on the $L^2(\partial K)$ norm for every compact $K\subset\subset\mathbb{R}\times(-\pi,(k+1)\pi)$, which in turns, due to the subharmonicity of u_i^d, gives a uniform-in-d bound on the $L^\infty(K)$ norm of $\{(\hat{u}_1^d,\dots,\hat{u}_k^d)\}$, for every compact set $K\subset\subset\mathbb{R}\times(-\pi,(k+1)\pi)$. The standard regularity theory for elliptic equations guarantees that when $d\to\infty$ then $\{(\hat{u}_1^d,\dots,\hat{u}_k^d)\}$ converges in $\mathcal{C}^2_{loc}(\mathbb{R}\times(-\pi,(k+1)\pi))$, up to a subsequence, to a function $(\hat{u}_1,\dots,\hat{u}_k)$ which is a solution to $(S)_k$. By the convergence and by the normalization required in Lemma 5.5.8, we deduce that (recall also the convergence of the boundaries $\partial\hat{S}_1^d$, Remark 5.5.10)

$$\int_0^{k\pi}\sum_i\hat{u}_i(0,y)^2\,\mathrm{d}y = 1;$$

in particular, $(\hat{u}_1,\dots,\hat{u}_k)$ is nontrivial. The $k\pi$-periodicity in y follows directly form the convergence of the domains, Lemma 5.5.9. By the pointwise convergence of $(\hat{u}_1^d,\dots,\hat{u}_k^d)$ to $(\hat{u}_1,\dots,\hat{u}_k)$ and by the symmetries of each function $(\hat{u}_1^d,\dots,\hat{u}_k^d)$ (see equation (5.5.1) and Remark 5.5.2) we deduce also that

$$\hat{u}_{i+1}(x,y)=\hat{u}_i\left(x,y-\pi\right)\quad\text{and}\quad\hat{u}_1\left(x,y+\frac{\pi}{2}\right)=\hat{u}_1\left(x,y-\frac{\pi}{2}\right).\qquad\square$$

5.5.3 *Characterization of the growth of* $(\hat{u}_1,\dots,\hat{u}_k)$

So far we proved the existence of a solution $(\hat{u}_1,\dots,\hat{u}_k)$ of $(S)_k$ which enjoys the properties 1) and 2) of Theorem 5.1.9. In this subsection, we are going to complete the proof of the quoted statement, showing that $(\hat{u}_1, \ldots, \hat{u}_k)$ enjoys also the properties 3)-5). We denote as $\hat{\mathcal{E}}, \hat{E}, \hat{H}$ and $\hat{N}$ the quantities $\mathcal{E}^{unb}, E^{unb}, H$ and N^{unb} introduced in subsection 5.2.2 when referred to the function $(\hat{u}_1, \ldots, \hat{u}_k)$. Firstly, we show that $(\hat{u}_1, \ldots, \hat{u}_k)$ has finite energy, point 3) of Theorem 5.1.9, and that $\hat{H}(x) \to -\infty$ as $x \to -\infty$.

Lemma 5.5.13. *For every $x \in \mathbb{R}$ there holds $\hat{\mathcal{E}}(x) < +\infty$. In particular*

$$\hat{\mathcal{E}}(x) \le \liminf_{d\to\infty} \hat{\mathcal{E}}_d\left(1 + \frac{x}{d}\right) \quad \text{and} \quad \hat{E}(x) \le \liminf_{d\to\infty} \hat{E}_d\left(1 + \frac{x}{d}\right).$$

Furthermore, $\lim_{x\to-\infty} \hat{H}(x) = 0$.

Proof. By the $\mathcal{C}^2_{loc}(\mathbb{R}^2)$ convergence of $(\hat{u}_1^d, \ldots, \hat{u}_k^d)$ to $(\hat{u}_1, \ldots, \hat{u}_k)$ and by the convergence properties of the domains $\hat{S}^d_{1+\frac{x}{d}}$, Lemma 5.5.9, we deduce

$$\lim_{d\to\infty} \left(\sum_i |\nabla \hat{u}_i^d|^2 + \sum_{i<j} \left(\hat{u}_i^d \hat{u}_j^d\right)^2\right) \chi_{\hat{S}^d_{1+\frac{x}{d}}} = \left(\sum_i |\nabla \hat{u}_i|^2 + \sum_{i<j} \left(\hat{u}_i \hat{u}_j\right)^2\right) \chi_{C_{(-\infty,x)}} \qquad \text{a. e. in } C_\infty,$$

for every $x \in \mathbb{R}$. As a consequence, we can apply the Fatou lemma obtaining

$$\hat{\mathcal{E}}(x) \le \liminf_{d\to\infty} \hat{\mathcal{E}}_d\left(1 + \frac{x}{d}\right) \le 2e^{2x},$$

where the uniform boundedness of $\hat{\mathcal{E}}_d\left(1 + \frac{x}{d}\right)$ comes from (5.5.14). To prove that $\hat{H}(x) \to 0$ as $x \to -\infty$, we can proceed with the same argument developed in Lemma 5.4.7. $\square$

In light of the previous Lemma, the monotonicity formulæ proved in subsection 5.2.2 applies for $\hat{\mathcal{E}}, \hat{E}, \hat{H}$ and $\hat{N}$.

Lemma 5.5.14. *There holds*

$$\lim_{x\to+\infty} \hat{N}(x) = 1.$$

Proof. By Proposition 5.2.14, we know that $\hat{N}$ is nondecreasing in x, and thanks to the symmetries of $(\hat{u}_1, \ldots, \hat{u}_k)$, see Lemma 5.5.12, Lemma 5.2.17 implies that $\lim_{x\to+\infty} \hat{N}(x) \ge 1$. It remains to show that this limit is smaller then 1. This follows from the estimates of Lemma 5.5.13 and from the strong convergence of $(\hat{u}_1^d, \ldots, \hat{u}_k^d) \to (\hat{u}_1, \ldots, \hat{u}_k)$, which implies that $\hat{H}_d\left(1 + \frac{x}{d}\right) \to \hat{H}(x)$ as $d \to \infty$: therefore, for every $x \in \mathbb{R}$

$$\hat{N}(x) = \frac{\hat{E}(x)}{\hat{H}(x)} \le \frac{\liminf_{d\to\infty} \hat{E}_d(x)}{\lim_{d\to\infty} \hat{H}_d(x)} = \liminf_{d\to\infty} \hat{N}_d(x) \le 1,$$

where we used the (5.5.12). $\square$

In light of this achievement, we can apply Corollary 5.2.15 to complete the proof of point 5) of Theorem 5.1.9. The fact that $\gamma > 0$ follows by Lemmas 5.5.14 and 5.2.17:

$$\lim_{r\to+\infty} \frac{\hat{H}(r)}{e^{2r}} = \lim_{r\to+\infty} \frac{\hat{E}(r)}{e^{2r}} \cdot \lim_{r\to+\infty} \frac{1}{\hat{N}(r)} > 0.$$

Remark 5.5.15. With a similar construction, it is possible to obtain the existence of solutions to $(S)_k$ in $\mathbb{R}^2$ modeled on $\cosh x \sin y$. To do this, we can first construct solutions of $(S)_k$ having algebraic growth defined outside the ball of radius 1, with homogeneous Neumann boundary conditions on ∂B_1. This can be done suitably modifying the proof of Theorem 1.6 in [6]. Then, performing a new blow-up in a neighborhood of $(1, 0)$, we can obtain a solution of $(S)_k$ defined in $\mathbb{R}^2_+$, with homogeneous Neumann condition on $\{x = 0\}$; this solution can be extended by even-symmetry in x in the whole $\mathbb{R}^2$.

5.6 *Asymptotics of solutions which are periodic in one variable*

In this section we prove Theorem 5.1.10.

Proof of Theorem 5.1.10. Let us start with case (i). Since the solution (u, v) is nontrivial $N(0) > 0$: in particular, from point (i) of Corollary 5.2.15 it follows that $H(r) \to +\infty$ as $r \to +\infty$. Let us consider the shifted functions

$$(u_R(x, y), v_R(x, y)) := \frac{1}{\sqrt{H(R)}}(u(x + R, y), v(x + R, y))$$

which solve the system

$$\begin{cases} -\Delta u_R = -H(R) u_R v_R^2 & \text{in } C_\infty \\ -\Delta v_R = -H(R) u_R^2 v_R & \text{in } C_\infty \\ \displaystyle\int_{\Sigma_0} u_R^2 + v_R^2 = 1 \end{cases}$$

and share the same periodicity of (u, v). We introduce

$$E_R(r) := \int_{C_{(-\infty,r)}} |\nabla u_R|^2 + |\nabla_R|^2 + 2H(R) u_R^2 v_R^2,$$

$$H_R(r) := \int_{\Sigma_r} u_R^2 + v_R^2 \quad \text{and} \quad N_R(r) := \frac{E_R(r)}{H_R(r)}.$$

It is easy to see that

$$\left.\begin{aligned} E_R(r) &= \frac{1}{H(R)} E^{unb}(r + R) \\ H_R(r) &= \frac{1}{H(R)} H(r + R) \end{aligned}\right. \quad \Rightarrow \quad N_R(r) = N^{unb}(r + R)$$

for any r (recall that E^{unb} and N^{unb} have been defined in subsection 5.2.2). We point out that, by definition and the monotonicity of N^{unb}, Proposition 5.2.14, $N_{R_1}(r) \leq N_{R_2}(r)$ for every $R_1 < R_2$. Furthermore, $N_R(r) \leq d = \lim_{r\to\infty} N(r)$ for every r, R and $N_R(r) \to d$ as $R \to \infty$ for every $r \in \mathbb{R}$. Therefore, N_R tends to the constant function d in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Thanks to the normalization condition $H_R(0) = 1$ and the uniform bound $N_R(r) \leq d$, applying Corollary 5.2.15 (see also Remark 5.2.18) we deduce that $H_R(r)$ is uniformly bounded in R for every $r > 0$. Consequently, also $E_R(r)$ is uniformly bounded in R for every $r > 0$. By means of a Poincaré inequality of type (5.3.2), we deduce that the sequence (u_R, v_R) is uniformly bounded in $H^1_{\mathrm{loc}}(C_\infty)$ and, by standard elliptic estimates, in $L^\infty_{\mathrm{loc}}(C_\infty)$. From Theorem 2.6 of [52] (it is a local version of Theorem 1.1 of [39]), we evince that the sequence (u_R, v_R) is uniformly bounded also in $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(C_\infty)$ for any $\alpha \in (0,1)$. Consequently, up to a subsequence, (u_R, v_R) converges in $\mathcal{C}^0_{\mathrm{loc}}(C_\infty)$ and in $H^1_{\mathrm{loc}}(C_\infty)$ to a pair (Ψ^+, Ψ^-), where Ψ is a nontrivial harmonic function (this is a combination of the main results in [39] and [24]). By the convergence, Ψ has to be 2π-periodic in y.

Firstly, we prove that $H(r;\Psi) \to 0$ ar $r \to -\infty$, so that the results of subsection 5.2.3 hold true for Ψ. As already observed, $N_R(r) \geq N_{\bar{R}}(r)$ for every $r \in \mathbb{R}$, for every $R > \bar{R}$. By the expression of the logarithmic derivative of H_R, see Corollary 5.2.15 (see also Remark 5.2.18) we have

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H_R(r) = 2N_R(r) \geq 2N_{\bar{R}}(r) = \frac{\mathrm{d}}{\mathrm{d}r} \log H_{\bar{R}}(r) \qquad \forall r.$$

As a consequence, taking into account that $H_R(0) = 1$ for every R, for every $r < 0$ it results

$$\frac{H_R(0)}{H_R(r)} \geq \frac{H_{\bar{R}}(0)}{H_{\bar{R}}(r)} \quad \Leftrightarrow \quad H_{\bar{R}}(r) \geq H_R(r) \qquad \forall R > \bar{R}.$$

Passing to the limit as $R \to +\infty$, by the $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^2)$ convergence of (u_R, v_R) to (Ψ^+, Ψ^-) it follows that $H_{\bar{R}}(r) \geq H(r;\Psi)$, which gives $H(r;\Psi) \to 0$ as $r \to -\infty$ in light of our assumption on (u, v).

Using again the expression of the logarithmic derivative of H_R and $H(\cdot;\Psi)$, we deduce

$$\log \frac{H_R(r_2)}{H_R(r_1)} = 2\int_{r_1}^{r_2} N_R(s)\,\mathrm{d}s \quad \text{and} \quad \log \frac{H(r_2;\Psi)}{H(r_1;\Psi)} = 2\int_{r_1}^{r_2} N(s;\Psi)\,\mathrm{d}s,$$

where $r_1 < r_2$. The left hand side of the first identity converges to the left hand side of the second identity; recalling that $N_R \to d$ in $L^1_{\mathrm{loc}}(\mathbb{R})$, we deduce

$$\int_{r_1}^{r_2} N(s;\Psi)\,\mathrm{d}s = \lim_{R\to+\infty} \int_{r_1}^{r_2} N_R(s)\,\mathrm{d}s = d(r_2 - r_1) \quad \Rightarrow \quad \frac{1}{r_2 - r_1}\int_{r_1}^{r_2} N(s;\Psi)\,\mathrm{d}s = d.$$

for every $r_1 < r_2$. It is well known that, being $N(\cdot;\Psi) \in L^1_{\mathrm{loc}}(\mathbb{R})$, the limit as $r_2 \to r_1$ of the left hand side converges to $N(r_1;\Psi)$ for almost every $r_1 \in \mathbb{R}$. Hence, $N(r;\Psi) = d$ for every $r \in \mathbb{R}$. We are then in position to apply Proposition 5.2.21:

$$\lim_{R\to+\infty} N(R) = \lim_{R\to+\infty} N_R(0) = N(0;\Psi) = d \in \mathbb{N} \setminus \{0\},$$

and $\Psi(x,y) = [C_1 \cos(dy) + C_2 \sin(dy)]\, e^{dx}$ for some constant $C_1, C_2 \in \mathbb{R}$.

As far as case (ii) is concerned, for the sake of simplicity we assume $a = 0$. One can repeat the proof with minor changes replacing E^{unb} and N^{unb} with E^{sym} and N^{sym} (which have been defined in subsection 5.2.1). The unique nontrivial step consists in proving that in this setting $H(r;\Psi) \to 0$ as $r \to -\infty$. To this aim, we note that, as before,

$$H_R(r) \le H_{\bar{R}}(r) \qquad \forall R > \bar{R},$$

for every $r > -\bar{R}$. In particular, if $r \in (1-\bar{R}, 0)$, by Proposition 5.2.4 and Corollary 5.2.5 we deduce

$$H_R(r) \le H_{\bar{R}}(r) = \frac{H(r+\bar{R})}{H(\bar{R})} \le \frac{e^{2N(1)(r+\bar{R})}}{e^{2N(1)\bar{R}}} = e^{2N(1)r} \qquad \forall R > \bar{R}.$$

Passing to the limit as $R \to +\infty$, by $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^2)$ convergence we obtain

$$H(r;\Psi) \le e^{2N(1)r} \qquad \forall r \in (-\infty, 0),$$

which yields $H(r;\Psi) \to 0$ as $r \to -\infty$. $\square$

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