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Mathematical modeling of Controlled Radical Polymerization

Master Thesis

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La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.

Galileo Galilei

Abstract

Controlled radical polymerization (CRP) is a process to form polymers by successive monomers additions. This growing process is mainly made by three events: propagation, if the next monomer is linearly added to the chain, backbiting, when the free radical changes its position and a new branch will start growing perpendicular to the previous one, and termination, if the chain stops to grow.
We have proposed a model describing the CRP process and offered two different approaches for solving it: Partial Differential Equations solutions (PDE) and stochastic simulation algorithm based on Monte Carlo estimations (MC). In this work, the model and the two approaches are summarized and their benefits as well as drawbacks are discussed.
Then, we realize that both the approaches can not explain some particular experimental results. MC method's flexibility allows us to modify the model, varying the hypothesis on which it is built, in order to give an explanation to those experimental results.

This work has been performed at Basque Center for Applied Mathematics (BCAM, Bilbao, Spain), under the supervision of prof. Elena Akhmatskaya from BCAM, prof. Dmitri Sokolovski from University of the Basque Country (UPV/EHU, Bilbao, Spain) and in close collaboration with prof. J.M. Asua from Basque Center for Macromolecular Design and Engineering (POLYMAT, Donostia-San Sebastián, Spain).

Sommario

Polimerizzazione Radicalica Controllata (CRP) è un processo molto diffuso per la sintesi di materiali polimerici. Trova applicazione in molti settori industriali, principalmente nella produzione di materie plastiche.
Questo processo di crescita dei polimeri è costituito da addizioni successive di monomeri. Tre eventi principali ne caratterizzano l'evoluzione: *propagation*, il monomero successivo viene aggiunto linearmente alla catena, *backbiting*, il radicale libero cambia la sua posizione ed un nuovo ramo inizierà a crescere perpendicolarmente al precedente, e *termination*, la catena arresta la sua crescita.
In questo lavoro, si propone un modello per descrivere tale processo. Solo in alcuni casi particolari, è possibile risolvere questo modello tramite soluzioni analitiche di Equazioni a Derivate Parziali (PDE).
Una tecnica alternativa consiste nell'algoritmo di simulazione stocastica (SSA) proposto da D. Gillespie [1]. Questo metodo è stato ampiamente utilizzato, rivisitato e modificato da quando è stato proposto, fino ai giorni nostri.
Il modello ed i due differenti approcci sono descritti in questa tesi, discutendone vantaggi e svantaggi.
All'inizio di questo lavoro, il metodo SSA viene utilizzato nella sua versione originale. Poi, vengono proposte alcune modifiche basate su una dimostrazione alternativa a quella data da Gillespie. Questa prova conduce allo stesso risultato ottenuto da Gillespie [1], ma rende il metodo SSA più flessibile, permettendo di arricchire l'algoritmo con densità di probabilità generiche.
Questo arricchimento permette di spiegare alcuni comportamenti sperimentali che i precedenti approcci non sono in grado di giustificare.

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Chapter 1

Introduction

Controlled Radical Polymerization is a widespread process for the synthesis of polymeric materials, and it finds application in many industrial fields, primarily in producing plastics [2], [3] and [4].
In this work we propose a model to describe this process. The analytical solutions of the corresponding Partial Differential Equations (PDE) can be obtained only for some special cases in this model.
Another useful technique is the basic stochastic simulation algorithm (SSA) proposed by D. Gillespie [1]. This method has been widely used, revised and modified since it was introduced [10], [11], [12], [13], [14], [15], [16], [17], [18], [19] and [20].
At the beginning, we use SSA in its original formulation. Then, we propose some modifications in the algorithm which lead to the equivalent algorithm to the one proposed by Gillespie [1], but it also allows us to use the generic probability density functions.
This enrichment helps us to explain the particular experimental behavior that both the PDE analytical solutions and the original SSA are not able to justify.
The thesis is organized as follows. In Chapter 2, we explain the chemical process that characterizes the reaction of interest.
In Chapter 3, we propose a Markovian model that leads to a pure Poisson process. The original SSA is used to simulate this Markovian-Poisson process.
Chapter 3 is organized as follows.
In Section 3.1, we introduce a model to describe the analyzed phenomenon.
In Section 3.2, we explain in which cases it is possible to get an analytical solution for the proposed model and how to obtain it.
In Section 3.3, first we briefly summarize the original SSA proposed by Gillespie [1]. Then, we give our alternative formulation and explain how to apply the SSA in the case under study. Finally, we set and run a simulation to reproduce the polymerization of Polyvinyl Chloride (PVC).
In Section 3.4, we compare the results provided in Section 3.2 and 3.3, outlining benefits and drawbacks.
In Chapter 4, we introduce a new player in the considered polymerization: the freezing agent. Chapter 4 is organized as follows.
In Section 4.1, we explain what the freezing agent is, its chemical effect on the reaction and also describe the experimental behavior that cannot be re-

produced by the previous approaches.
In Section 4.2, we explain how to introduce the freezing agent in the original SSA algorithm. We can remark that this approach is also not able to give an accurate description of the desired behavior.
In Chapter 5, we introduce Non-Markovian models and Non-Poisson processes. Chapter 5 is organized as follows.
In Section 5.1, we propose the first modification to the original SSA. This first attempt gives the wrong result, but it also gives the idea for the model proposed in Section 5.2. In this section, we can get the desired behavior, but some drawbacks of this approach are outlined. Then, in Section 5.3, we can overcome the drawbacks emerged in Section 5.2 proposing a different model.
In Section 5.4, we explicitly identify non-Markovian and non-Poisson effects that can characterize the polymers growth. In Section 5.5, we apply these ideas to the case of interest and demonstrate that such an approach accurately reproduces the experimental behavior.

Chapter 2

Control Radical Polymerization: the phenomenon

Controlled Radical Polymerization (CRP) is a way to create long molecular chains that are called polymers. The building unit for this molecules is a monomer and the linking tool is the so called "free radical", that is simply an electron attached to the monomers by electro-magnetic fields.
When the first monomer meets another monomer, the free radical creates a link between the two monomers and then it moves forward to the head of the two monomers chain just built.
If this chain meets other monomers, the reaction keeps on and the growth of the chain produces a linear segment, because the free radical always moves forward.
But, this is not the end of the story: after the creation of the link between the chain and a new monomer, the free radical can also move backward inside the chain. The next monomer will be added where the radical is placed, thus this is the start of a new branch.
The reaction creates this branched structure, following the free radical movements and it continues until the occurrence of a termination event. An example for this last event could be the free radical neutralization due to another free radical.

Thus, the considered polymerization is mainly made by the successive occurrence of the following events:

- PROPAGATION: next monomer is linearly added,
- BACKBITING: the radical changes its position and a new branch will grow perpendicular to the previous one,
- TERMINATION: the chain stops to grow.

From chemical description of the process we know that backbiting can only occur after at least three previous propagations in any branch.

In Figure 2.1 the device where such a chemical reaction can take place is shown.

Figure 2.1: An emulsifier where Controlled Radical Polymerization can take place.

The idea of Figure 2.2 is to show the different branched structures that can be created by CRP: the molecule on the top is only made by propagations, because it is a linear segment, whereas the one on the bottom is a result of a propagation followed by a backbiting, since it is made by two perpendicular branches.

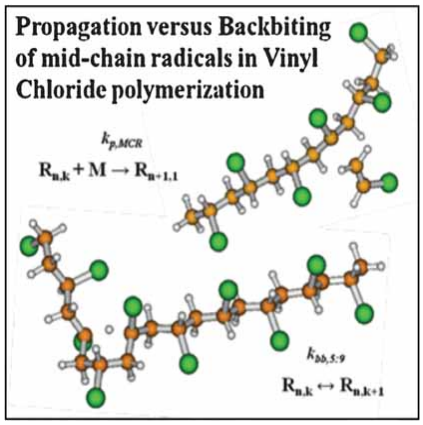

Figure 2.2: Two different branched structures that can be created by the considered polymerization [5].

Chapter 3

Markovian Model

3.1 The model

3.1.1 The hypothesis

Let us define the rates for each event that characterizes the CRP process:

- p is the propagation occurrence rate,
- r is the backbiting occurrence rate,
- q is the termination occurrence rate.

We make these assumptions:

- the rates are constant over time,
- the rates are chain configuration independent.

The idea is that there is such an abundance of monomers to be added that the rates can be regarded as constant.
These are a very strong hypothesis, in particular for the backbiting rate r, but they are needed to solve the model through the PDE approach. The MC approach will allows us to remove these strict assumptions.

3.1.2 The full model

Let us define the model variables:

- $\{N, n_0, .., n_N\}$ is the chain configuration, it means that the chain is made by N branching points and the $N+1$ branches are made by $n_0, .., n_N$ monomers,
- $P(\{N, n_0, .., n_N\}; t)$ is probability to have at time t the configuration $\{N, n_0, .., n_N\}$ and the last branch is still growing,
- $Q(\{N, n_0, .., n_N\}; t)$ is probability to have at time t the configuration $\{N, n_0, .., n_N\}$ and the chain stops to grow before or at time t.

We can write down balance equations:

$$\begin{aligned} P(\{N, n_0, .., n_N\}; t+dt) = & (1-(p+r+q)dt)\ P(\{N, n_0, .., n_N\}; t) \\ & +\ p\ dt\ P(\{N, n_0, .., n_N - 1\}; t) \\ & +\ r\ dt\ P(\{N-1, n_0, .., n_{N-1}\}; t)\ \delta(n_N), \end{aligned} \quad (3.1)$$

$$Q(\{N, n_0, .., n_N\}; t+dt) = Q(\{N, n_0, .., n_N\}; t) + qdtP(\{N, n_0, .., n_N\}; t). \quad (3.2)$$

The equation (3.1) means that the probability to reach at time $t + dt$ the growing configuration $\{N, n_0, .., n_N\}$ is equal to the sum of the probabilities of three disjoint events. Those events are the ones that lead to the previous event:

- the event to be at time t in the same growing configuration $\{N, n_0, .., n_N\}$ and nothing happens between t and $t + dt$,
- the event to be at time t in the growing configuration $\{N, n_0, .., n_N - 1\}$, that is one propagation removed from the considered one $\{N, n_0, .., n_N\}$, and a propagation occurs between t and $t + dt$,
- the event to be in time t in the growing configuration $\{N-1, n_0, .., n_{N-1}\}$, that is one backbiting removed from the considered one $\{N, n_0, .., n_N\}$, and a backbiting occurs between t and $t + dt$.

The $\delta(n_N)$ in equation (3.1) is place to guarantee that the $N + 1^{th}$ branch is made by zero monomers, because the last backbiting has not occurred yet.

The equation (3.2) means that the probability to have at time $t + dt$ the arrested configuration $\{N, n_0, .., n_N\}$ is equal to the sum of the probabilities of two disjoint events. Those events are the ones that lead to the previous event:

- the event that the configuration $\{N, n_0, .., n_N\}$ is terminated before or at time t,
- the event to be at time t in the growing configuration $\{N, n_0, .., n_N\}$ and a termination occurs between t and $t + dt$.

Then, there are trivial conditions that our density functions have to satisfy:

$$\sum_{N\in\mathbb{N}} \sum_{n_0\in\mathbb{N}} .. \sum_{n_N\in\mathbb{N}} [P(\{N, n_0, .., n_N\}; t) + Q(\{N, n_0, .., n_N\}; t)] = 1\ \forall t \geq 0, \quad (3.3)$$

$$P(\{0, 0\}; 0)\ =\ 1. \quad (3.4)$$

The equation (3.3) simply is a normalization condition and the equation (3.4) means that at time $t = 0$ the reaction has to start with an empty polymer structure.

3.2 The Partial Differential Equations approach

3.2.1 The special cases

The PDE approach allows us to solve the model only in these cases:

- FIRST BRANCH the linear segment created before the occurrence of the first event different from propagation (backbiting or termination),
- LAST BRANCH the linear segment that grows after the last backbiting (if backbiting is not happened yet, the last segment is the first one),
- BACKBITING TERMINATED BRANCHES all the linear segments terminated by a backbiting: we consider all these branches as the same kind of branch, regardless the position they have through the chain.

3.2.2 The first branch

Let us define the model variables that hold the first branch:

- $P^f(n_0, t) = P(\{0, n_0\}; t)$ is the probability to have at time t the first branch made by n_0 monomers and it is still growing,
- $Q^f(n_0, t) = Q(\{0, n_0\}; t)$ is the probability to have at time t the first branch made by n_0 monomers and it is terminated by termination before or at time t,
- $R^f(n_0, t)$ is the probability to have at time t the first branch made by n_0 monomers and it is terminated by backbiting before or at time t.

In the case of the first segment, we can rewrite the balance equations (3.1), (3.2) and the conditions (3.3), (3.4):

$$P^f(n_0, t + dt) = \ (1 - (p + r + q)dt)\ P^f(n_0, t)\ + p\ dt\ P^f(n_0 - 1, t), \quad (3.5)$$

$$Q^f(n_0, t + dt) = \ Q^f(n_0, t)\ +\ q\ dt\ P^f(n_0, t), \quad (3.6)$$

$$R^f(n_0, t + dt) = \ R^f(n_0, t)\ +\ r\ dt\ P^f(n_0, t), \quad (3.7)$$

$$\sum_{n_0 \in \mathbb{N}} [P^f(n_0, t) + Q^f(n_0, t) + R^f(n_0, t)] = 1\ \forall t \geq 0, \quad (3.8)$$

$$P^f(0, 0)\ =\ 1. \quad (3.9)$$

The equations (3.5), (3.6) and (3.7) can be rewritten as follows:

$$\frac{P^f(n_0,t+dt)-P^f(n_0,t)}{dt} = \ -p\ [P^f(n_0,t)-P^f(n_0-1,t)]\ -(q+r)\ P^f(n_0,t), \tag{3.10}$$

$$\frac{Q^f(n_0,t+dt)-Q^f(n_0,t)}{dt} = \ q\ P^f(n_0,t), \tag{3.11}$$

$$\frac{R^f(n_0,t+dt)-R^f(n_0,t)}{dt} = \ r\ P^f(n_0,t). \tag{3.12}$$

Now, the next steps are the following:

- take the limit $dt \to 0$,
- replace n_0 by a continuous variable x, assuming $n_0 \gg 1$,

where the first passage to the limit is taken in order to get partial derivatives with respect to the time t on the l.h.s. of (3.10), (3.11), (3.12).
The second passage ($n_0 \to x$) means that the first segment length n_0 is replaced by a continue variable x.
This step is the critical one of the PDE approach, but it is needed to solve the PDE system. We make it now, but later we will see all the consequences.
Finally, if we take both the limits, we get from equations (3.10), (3.11), (3.12), (3.8), (3.9):

$$\partial_t P^f(x,t) = \ -p\ \partial_x P^f(x,t)\ -(q+r)\ P^f(x,t), \tag{3.13}$$

$$\partial_t Q^f(x,t) = \ q\ P^f(x,t), \tag{3.14}$$

$$\partial_t R^f(x,t) = \ r\ P^f(x,t), \tag{3.15}$$

$$\int_0^{+\infty} P^f(x,t)+Q^f(x,t)+R^f(x,t)\ dx = 1\ \forall t \geq 0, \tag{3.16}$$

$$P^f(x,0)\ = \ \delta(x). \tag{3.17}$$

Thus, we can get these solutions:

$$P^f(x,t) = \ \delta(x-pt)\ \exp[-(q+r)t], \tag{3.18}$$

$$Q^f(x,t) = \int_0^t qP^f(x,t')dt' = \ \frac{q}{p}\ \exp[-(q+r)\frac{x}{p}]\ \theta(pt-x), \tag{3.19}$$

$$R^f(x,t) = \int_0^t rP^f(x,t')dt' = \ \frac{r}{p}\ \exp[-(q+r)\frac{x}{p}]\ \theta(pt-x), \tag{3.20}$$

where $\theta(x)$ is 1 for $x > 0$ and 0 otherwise.

Then, we can define:

- $N_P^f(t) = \int_0^{+\infty} P^f(x,t)dx$ is the proportion of still growing first segments at time t, regardless their length x,
- $N_Q^f(t) = \int_0^{+\infty} Q^f(x,t)dx$ is the proportion of terminated segments before or at time t (by termination), regardless their length x,
- $N_R^f(t) = \int_0^{+\infty} R^f(x,t)dx$ is the proportion of terminated segments before or at time t (by backbiting), regardless their length x.

From condition (3.16), these quantities must satisfy this equation:

$$N_P^f(t) + N_Q^f(t) + N_R^f(t) = 1 \ \forall t \geq 0. \tag{3.21}$$

We can solve these quantities and we get:

$$N_P^f(t) = \ \exp[-(q+r)t], \tag{3.22}$$

$$N_Q^f(t) = \ \frac{q}{q+r}\ (1 - \exp[-(q+r)t]), \tag{3.23}$$

$$N_R^f(t) = \ \frac{r}{q+r}\ (1 - \exp[-(q+r)t]). \tag{3.24}$$

It is easy to check that the condition (3.21) is satisfied.

Finally, we can compute the mean lengths of the first segments of each type at a time t:

$$\overline{x}_P^f(t) = \int_0^{+\infty} x\frac{P^f(x,t)}{N_P^f(t)}dx \ = pt, \tag{3.25}$$

$$\overline{x}_Q^f(t) = \int_0^{+\infty} x\frac{Q^f(x,t)}{N_Q^f(t)}dx \ = \frac{p}{q+r}\ \frac{1 - [1+(q+r)t]\exp[-(q+r)t]}{1 - \exp[-(q+r)t]}, \tag{3.26}$$

$$\overline{x}_R^f(t) = \int_0^{+\infty} x\frac{R^f(x,t)}{N_R^f(t)}dx \ = \frac{p}{q+r}\ \frac{1 - [1+(q+r)t]\exp[-(q+r)t]}{1 - \exp[-(q+r)t]}. \tag{3.27}$$

3.2.3 The last branch

Let us define the model variables that describe the last segment:

- $P^\ell(n_N, t) = \sum_{N\in\mathbb{N}} \sum_{n_0\in\mathbb{N}} .. \sum_{n_{N-1}\in\mathbb{N}} P(\{N, n_0, .., n_N\}; t)$
  is the probability to have at time t the last branch made by n_N monomers and it is still growing,

- $Q^\ell(n_N,t) = \sum_{N\in\mathbb{N}} \sum_{n_0\in\mathbb{N}} .. \sum_{n_{N-1}\in\mathbb{N}} Q(\{N,n_0,..,n_N\};t)$
  is the probability to have at time t the last branch made by n_N monomers and the chain is terminated (by termination) before or at time t.

Applying $\sum_{N\in\mathbb{N}} \sum_{n_0\in\mathbb{N}} .. \sum_{n_{N-1}\in\mathbb{N}}$ to the balance equations (3.1) and (3.2), we get:

$$\begin{aligned} P^\ell(n_N,t+dt) = &(1-(p+r+q)dt)\ P^\ell(n_N,t) \\ &+ p\ dt\ P^\ell(n_N-1,t) \\ &+ r\ dt\ \delta(n_N)\ \sum_m P^\ell(m,t), \end{aligned} \tag{3.28}$$

$$Q^\ell(n_N,t+dt) = \ Q^\ell(n_N,t)\ + q\ dt\ P^\ell(n_N,t). \tag{3.29}$$

The conditions (3.3) and (3.4) are still valid:

$$\sum_{n_N\in\mathbb{N}} [P^\ell(n_N,t) + Q^\ell(n_N,t)] = 1\ \forall t \geq 0, \tag{3.30}$$

$$P^\ell(0,0)\ =\ 1. \tag{3.31}$$

The equations (3.28) and (3.29) can be rewritten as follows:

$$\begin{aligned} \frac{P^\ell(n_N,t+dt)-P^\ell(n_N,t)}{dt} = &- p\ [P^\ell(n_N,t) - P^\ell(n_N-1,t)] \\ &- (q+r)\ P^\ell(n_N,t) \\ &+ r\ \delta(n_N)\ \sum_m P^\ell(m,t), \end{aligned} \tag{3.32}$$

$$\frac{Q^\ell(n_N,t+dt)-Q^\ell(n_N,t)}{dt} = \ q\ P^\ell(n_N,t). \tag{3.33}$$

As before, the next steps are:

- take the limit $dt \to 0$,
- take the limit $n_N \to y$,

where the passage to the limit $n_N \to y$ means that the last segment discrete length n_N is replaced by a continuous variable y: this is the critical point of the approach.
Thus, we can get these equations from (3.32), (3.33), (3.30) and (3.31):

$$\partial_t P^\ell(y,t) = -p\partial_y P^\ell(y,t) - (q+r)P^\ell(y,t) + r\delta(y)\int_0^{+\infty} P^\ell(x,t)dx, \tag{3.34}$$

$$\partial_t Q^\ell(y,t) = \ q \ P^\ell(y,t), \tag{3.35}$$

$$\int_0^{+\infty} P^\ell(y,t) + Q^\ell(y,t) \ dy = 1 \ \forall t \geq 0, \tag{3.36}$$

$$P^\ell(y,0) \ = \ \delta(y). \tag{3.37}$$

Finally, we get these solutions:

$$P^\ell(y,t) = \delta(y-pt)\exp[-(q+r)t] + \frac{r}{p}\exp\left(-qt - \frac{r}{p}y\right)\theta(pt-y), \tag{3.38}$$

$$\begin{aligned} Q^\ell(y,t) &= \int_0^t qP^\ell(y,t')dt' = \\ &= \frac{1}{p}\theta(pt-y)\left\{(q+r)\exp\left[-\left(\frac{q+r}{p}\right)y\right] - r\exp\left[-qt-\frac{r}{p}y\right]\right\}. \end{aligned} \tag{3.39}$$

The proportions at a given time t for each kind of last branches, regardless their length, are the following:

$$N_P^\ell(t) = \int_0^{+\infty} P^\ell(y,t)dy = \exp[-qt], \tag{3.40}$$

$$N_Q^\ell(t) = \int_0^{+\infty} Q^\ell(y,t)dy = 1 - \exp[-qt]. \tag{3.41}$$

It is trivial that these proportions satisfy the condition belonging from (3.36):

$$N_P^\ell(t) + N_Q^\ell(t) = 1 \ \forall t \geq 0. \tag{3.42}$$

The mean lengths of the each kind of last branches at time t are:

$$\overline{y}_P^\ell(t) = \int_0^{+\infty} y\frac{P^\ell(y,t)}{N_P^\ell(t)}dy = \frac{p}{r}[1-\exp(-rt)], \tag{3.43}$$

$$\begin{aligned} \overline{y}_Q^\ell(t) &= \int_0^{+\infty} y\frac{Q^\ell(y,t)}{N_Q^\ell(t)}dy = \\ &= \frac{\frac{p}{r}\{\exp[-(q+r)t] - \exp[-qt]\} - \frac{p}{q+r}\{\exp[-(q+r)t]-1\}}{1-\exp[-qt]}. \end{aligned} \tag{3.44}$$

3.2.4 The backbiting terminated branches

Let $R^{bb}(n,t)$ be a measure of the backbiting terminated branches before or at time t made by n monomers.
Thus, this measure at time $t+dt$ is made by the measure there was at time t plus the quantity given by this event: the last segment is still growing at time t, it is made by n monomers at time t and a backbiting occurs between t and $t+dt$.
So, we can write down this balance equation:

$$R^{bb}(n,t+dt) = \; R^{bb}(n,t) \; + r \; dt \; P^{\ell}(n,t). \tag{3.45}$$

Equation (3.45) can easily be rewritten as follows:

$$\frac{R^{bb}(n,t+dt) - R^{bb}(n,t)}{dt} \; = \; r \; P^{\ell}(n,t). \tag{3.46}$$

Taking the two limits $dt \to 0$ and $n \to y$, we get:

$$\partial_t R^{bb}(y,t) \; = \; r \; P^{\ell}(y,t). \tag{3.47}$$

As before, the passage to the limit $n \to y$ means that the discrete length n goes to a continue value y: this is the critical point of the approach, as explained in Section 3.2.2.
Finally, we obtain this solution:

$$\begin{aligned} R^{bb}(y,t) &= \int_0^t r P^{\ell}(y,t')dt' = \\ &= \frac{r}{pq} \; \theta(pt-y) \; \left\{ (q+r) \exp\left[-\left(\frac{q+r}{p} \right) y \right] - r \exp\left[-qt - \frac{r}{p} y \right] \right\}. \end{aligned} \tag{3.48}$$

This measure $R^{bb}(y,t)$ does not integrate to 1 and, in order to deal with probability density functions, we can normalize it:

$$\begin{aligned} R_d^{bb}(y,t) &= \frac{R^{bb}(y,t)}{\int_0^{+\infty} R^{bb}(y',t)dy'} = \\ &= \frac{\frac{1}{p} \; \theta(pt-y) \; \left\{ (q+r) \exp\left[-\left(\frac{q+r}{p} \right) y \right] - r \exp\left[-qt - \frac{r}{p} y \right] \right\}}{1 - \exp\left(-qt\right)}. \end{aligned} \tag{3.49}$$

Thus, we can compute the mean length of the backbiting terminated segments before or at time t:

$$\overline{y}_R^{bb}(t) = \int_0^{+\infty} y R_d^{bb}(y,t)dy = \\ = \frac{p\{\exp[-(q+r)t] - \exp[-qt]\} - \frac{rp}{q+r}\{\exp[-(q+r)t] - 1\}}{r\{1 - \exp[-qt]\}}. \quad (3.50)$$

3.3 The Monte Carlo approach

The idea behind this approach is to simulate the evolution of several growing chains and compute statistics on all the realizations got.
Thus, we have to find a way to sample different realizations of these random variables:

- $T_j :=$ time of the j^{th} event,
- $X_j :=$ j^{th} event.

3.3.1 The Gillespie algorithm

Suppose to arrive at the j^{th} event and to know the configuration created by the first j events and their occurrence times: we want to define the density distributions that model the next event and the time needed for it, in order to sample a realization from those probability density function, and to iteratively keep on the growth of the chain.
A possibility is to use the basic stochastic simulation algorithm (SSA) proposed by D. Gillespie [1]:

- suppose that the $j+1^{th}$ event could be one among $m = m(j+1)$ different events,
- let $k = 0, .., m-1$ be the labels of these m possible next events $[k = k(j+1)]$,
- let λ_k be the k-event rate $[\lambda_k = \lambda_k(j+1)]$.

The algorithm proposed by Gillespie [1] defines these density distributions:

$$T_{j+1} - T_j \sim \text{Exp}\left(\sum_{k=0}^{m-1} \lambda_k\right), \tag{3.51}$$

$$\mathbb{P}\left(X_{j+1} = k\right) = \frac{\lambda_k}{\sum_{i=0}^{m-1} \lambda_i} \quad \forall k = 0, .., m-1. \tag{3.52}$$

The first remark that has to be made is that now the rates λ_k can depend on time and chain configuration, because they can be updated after each event realization. So, the MC approach allow us to remove the hypothesis that the rates are constant over time and chain configuration independent.

3.3.2 An alternative proof

D. Gillespie gives a proof of this algorithm based on the solution of a ODE system [1].
We propose an alternative proof, that gives the same results, but it allows us to make the MC approach very flexible and to have the chance to modify it in order to make the model more and more realistic, as we will see in Section 4.1.

Let $\lambda_0 = \lambda_0(j+1), \lambda_1 = \lambda_1(j+1), .., \lambda_{m-1} = \lambda_{m-1}(j+1)$ be the rates for the m different events that can be the $j+1^{th}$ event.
Once the j^{th} is realized and we know the configuration created by the first j events and their occurrence times, we can update the previous rates and then we can assume that they are constant for the $j+1^{th}$ event.
Let $\hat{T}_0, \hat{T}_1, .., \hat{T}_{m-1}$ be the random variables that model the time needed for each of the m different events that can be the $j+1^{th}$ event.
The $\hat{T}_k$ occurrence rate is equal to λ_k ($\forall k = 0, .., m-1$). Fixing the event $j+1$, we know that the rate λ_k is constant.
Remembering that the only density distribution with respect to the Lebesgue measure that has a constant rate of occurrence is the Exponential one, we can say that:

$$\hat{T}_k \sim \text{Exp}\left(\lambda_k\right) \quad \forall k = 0, .., m-1. \tag{3.53}$$

Then, in order to choose the $j+1^{th}$ event from the m different possible events, we can pick the event whose time of occurrence is the shortest.
So, the time needed for the $j+1^{th}$ event will be the minimum of the random variables $\hat{T}_0, \hat{T}_1, .., \hat{T}_{m-1}$:

$$T_{j+1} - T_j = \min\left\{\hat{T}_0, \hat{T}_1, .., \hat{T}_{m-1}\right\}. \tag{3.54}$$

Assuming independence among the random variables $\hat{T}_0, \hat{T}_1, .., \hat{T}_{m-1}$, it is easy to say that:

$$T_{j+1} - T_j \sim \text{Exp}\left(\sum_{k=0}^{m-1} \lambda_k\right). \tag{3.55}$$

This can be seen by considering the complementary cumulative distribution function:

$$\begin{aligned}
\mathbb{P}\left(\min\left\{\hat{T}_0, \hat{T}_1, .., \hat{T}_{m-1}\right\} > t\right) &= \mathbb{P}\left(\hat{T}_0 > t \wedge \cdots \wedge \hat{T}_{m-1} > t\right) \\
&= \prod_{k=0}^{m-1} \mathbb{P}\left(\hat{T}_k > t\right) \\
&= \prod_{k=0}^{m-1} \exp(-t\lambda_k) = \exp\left(-t\sum_{k=0}^{m-1} \lambda_k\right).
\end{aligned} \tag{3.56}$$

Then, we want to define the random variable X_{j+1} that models the kind of the $j+1^{th}$ event. This X_{j+1} can only take values into the set $\{0,..,m-1\}$, because these are the labels of m different events that can be the $j+1^{th}$ one. So, we have to assign the probability $\mathbb{P}(X_{j+1}=k)$, $\forall k=0,..,m-1$. From the previous choice, we can say that:

$$\{X_{j+1}=k\} \quad \Leftrightarrow \quad \left\{\hat{T}_k < \hat{T}_i\colon\ \forall i=0,..,m-1 \wedge i\neq k\right\}. \tag{3.57}$$

Thus:

$$\mathbb{P}\left(X_{j+1}=k\right) = \mathbb{P}\left(\hat{T}_k < \hat{T}_i\colon\ \forall i=0,..,m-1 \wedge i\neq k\right). \tag{3.58}$$

We follow these simple calculations in order to get the desired result:

$$\begin{aligned}
&\mathbb{P}\left(X_{j+1}=k\right) = \\
&= \mathbb{P}\left(\hat{T}_k < \hat{T}_i\colon\ \forall i=0,..,m-1 \wedge i\neq k\right) \\
&= \int_0^{+\infty} \mathbb{P}\left(\left\{\hat{T}_k < \hat{T}_i\colon\ \forall i=0,..,m-1 \wedge i\neq k\right\} \cap \left\{\hat{T}_k=\tau\right\}\right) d\tau \\
&= \int_0^{+\infty} \mathbb{P}\left(\left\{\hat{T}_k < \hat{T}_i\colon\ \forall i=0,..,m-1 \wedge i\neq k\right\} \Big| \left\{\hat{T}_k=\tau\right\}\right) \mathbb{P}\left(\hat{T}_k=\tau\right) d\tau \\
&= \int_0^{+\infty} \mathbb{P}\left(\left\{\hat{T}_i > \tau\colon\ \forall i=0,..,m-1 \wedge i\neq k\right\} \Big| \left\{\hat{T}_k=\tau\right\}\right) \mathbb{P}\left(\hat{T}_k=\tau\right) d\tau \\
&= \int_0^{+\infty} \mathbb{P}\left(\left\{\hat{T}_i > \tau\colon\ \forall i=0,..,m-1 \wedge i\neq k\right\}\right) \mathbb{P}\left(\hat{T}_k=\tau\right) d\tau \\
&= \int_0^{+\infty} \prod_{\substack{i=0\\ i\neq k}}^{m-1} \mathbb{P}\left(\hat{T}_i > \tau\right) \mathbb{P}\left(\hat{T}_k=\tau\right) d\tau \\
&= \int_0^{+\infty} \prod_{\substack{i=0\\ i\neq k}}^{m-1} \exp(-\lambda_i\tau)\lambda_k \exp(-\lambda_k\tau) d\tau \\
&= \int_0^{+\infty} \lambda_k \exp\left[-\left(\sum_{i=0}^{m-1}\lambda_i\right)\tau\right] d\tau \\
&= \frac{\lambda_k}{\sum_{i=0}^{m-1}\lambda_i} \int_0^{+\infty} \left(\sum_{i=0}^{m-1}\lambda_i\right) \exp\left[-\left(\sum_{i=0}^{m-1}\lambda_i\right)\tau\right] d\tau \\
&= \frac{\lambda_k}{\sum_{i=0}^{m-1}\lambda_i} \quad \forall k=0,..,m-1.
\end{aligned} \tag{3.59}$$

3.3.3 Our implementation

The idea of this section is to apply the proposed algorithm to the case we wish to study, in order to computed desired statistics as solution of the proposed model.
The constant rates and the labels of the events of interest are the following:

- the termination rate is q and its label is 0,
- the propagation rate is p and its label is 1,
- the backbiting rate is r and its label is 2.

We can define the random variables that describe the event and the time needed for it, when termination, propagation and backbiting can occur:

$$X = \begin{cases} 0 & \text{w.p. } \frac{q}{p+r+q} \\ 1 & \text{w.p. } \frac{p}{p+r+q} \\ 2 & \text{w.p. } \frac{r}{p+r+q} \end{cases}, \tag{3.60}$$

$$T \sim \text{Exp}\left(p+r+q\right). \tag{3.61}$$

Then, these are random variables that describe the event and the time needed for it, when backbiting can not occur:

$$X_b = \begin{cases} 0 & \text{w.p. } \frac{q}{p+q} \\ 1 & \text{w.p. } \frac{p}{p+q} \end{cases}, \tag{3.62}$$

$$T_b \sim \text{Exp}\left(p+q\right). \tag{3.63}$$

We define $(\underline{x}, \underline{t})$ the realization of one chain: it means that the j^{th} element of vector $\underline{x}$ is the label of the j^{th} event and the j^{th} element of vector $\underline{t}$ is the occurrence time of the j^{th} event. So, Algorithm 1 explains how to get one chain realization.

In this implementation, we forbid backbiting to be the first event and the event immediately after a previous backbiting, because we know that these events are impossible. Then, from chemistry we know that backbiting can only occur after at least three previous propagations in any branch. Flexibility of the MC approach allows us to put also this information into the simulations, just in case we want to do it. As we explain in Algorithm 2, it is enough to draw the next event and its requested time from X_b and T_b, instead of X and T, if the three propagations of the branch are not occurred yet.

Draw a realization x_b from X_b and put $\underline{x}[1] = x_b$;
Draw a realization t_b from T_b and put $\underline{t}[1] = t_b$;
Initialize $j = 1$;
while $\underline{x}[j] \neq 0$ **do**
    Update $j \leftarrow j + 1$;
    if $\underline{x}[j-1] == 1$ **then**
        Draw a realization x from X and put $\underline{x}[j] = x$;
        Draw a realization t from T and put $\underline{t}[j] = \underline{t}[j-1] + t$;
    end
    if $\underline{x}[j-1] == 2$ **then**
        Draw a realization x_b from X_b and put $\underline{x}[j] = x_b$;
        Draw a realization t_b from T_b and put $\underline{t}[j] = \underline{t}[j-1] + t_b$;
    end
end

Algorithm 1: Our implementation of the Gillespie algorithm (SSA): how to get one chain realization. The random variables X, T, X_b and T_b are (3.60), (3.61), (3.62) and (3.63). The termination label is 0, the propagation label is 1 and the backbiting label is 2.

3.3.4 PVC simulation

We run the Algorithm 1, in order to mimic the polymerization of Polyvinyl Chloride (PVC).
PVC is produced by polymerization of the monomer vinyl chloride (VCM), as shown in Figure 3.1 and Figure 3.2.

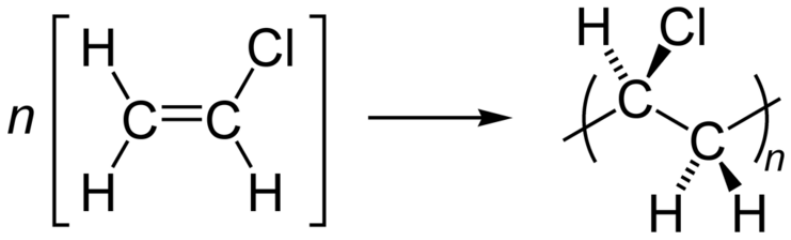

Figure 3.1: PVC polymerization [6].

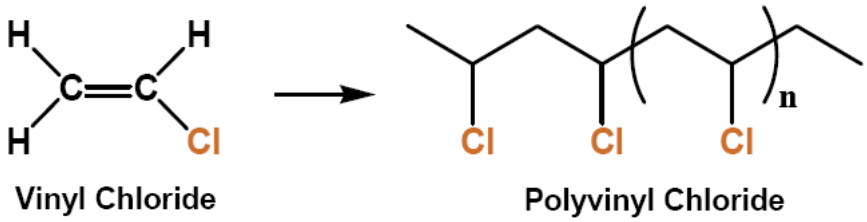

Figure 3.2: PVC polymerization [7].

Thus, a possible resulting shape of PVC is shown in Figure 3.3.

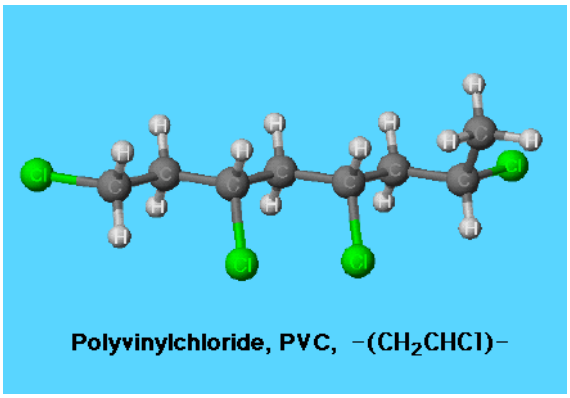

Figure 3.3: A PVC molecule [8].

The simulations can give the results shown in Figure 3.4. The simulated PVC molecule is made by the following elements:

- the blue spheres are carbon atoms [C],
- the white spheres are hydrogen atoms [H],
- the green spheres are chlorine atoms [Cl],
- the red sphere is the free radical.

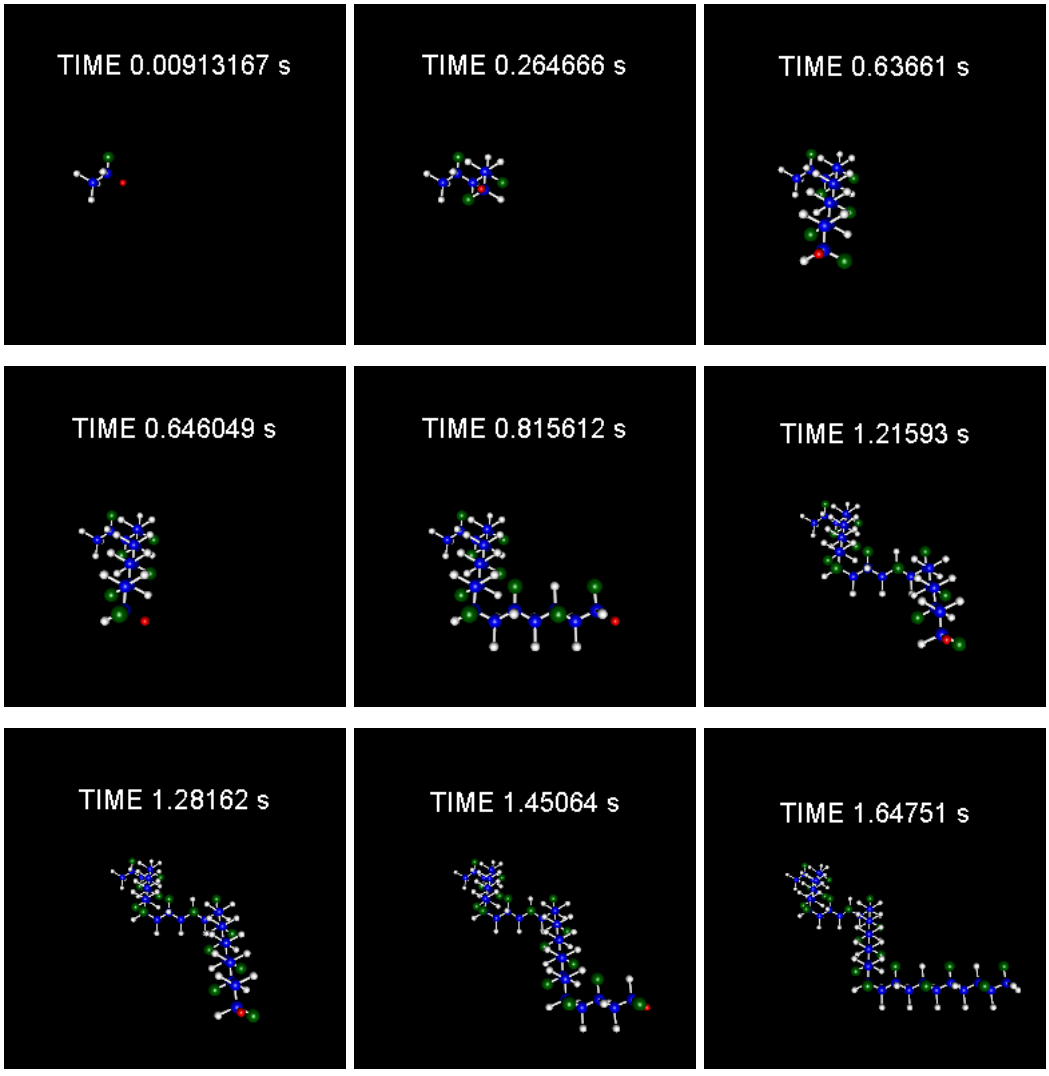

Figure 3.4: A realization of the PVC growing chain.

3.4 Comparison between PDE and MC approaches

In this section we compare the two approaches, asking if there is match between them or if one can give better results.

3.4.1 A case of good match

We can get a very good match between PDE and MC approaches when the backbiting rate and termination rate are much more smaller then the propagation rate.
In Figure 3.5, we show the terminated first segment probability density functions got from both the approach, when the parameters set is (3.64):

$$p = 10s^{-1},\ r = 0.05s^{-1},\ q = 0.1s^{-1}. \tag{3.64}$$

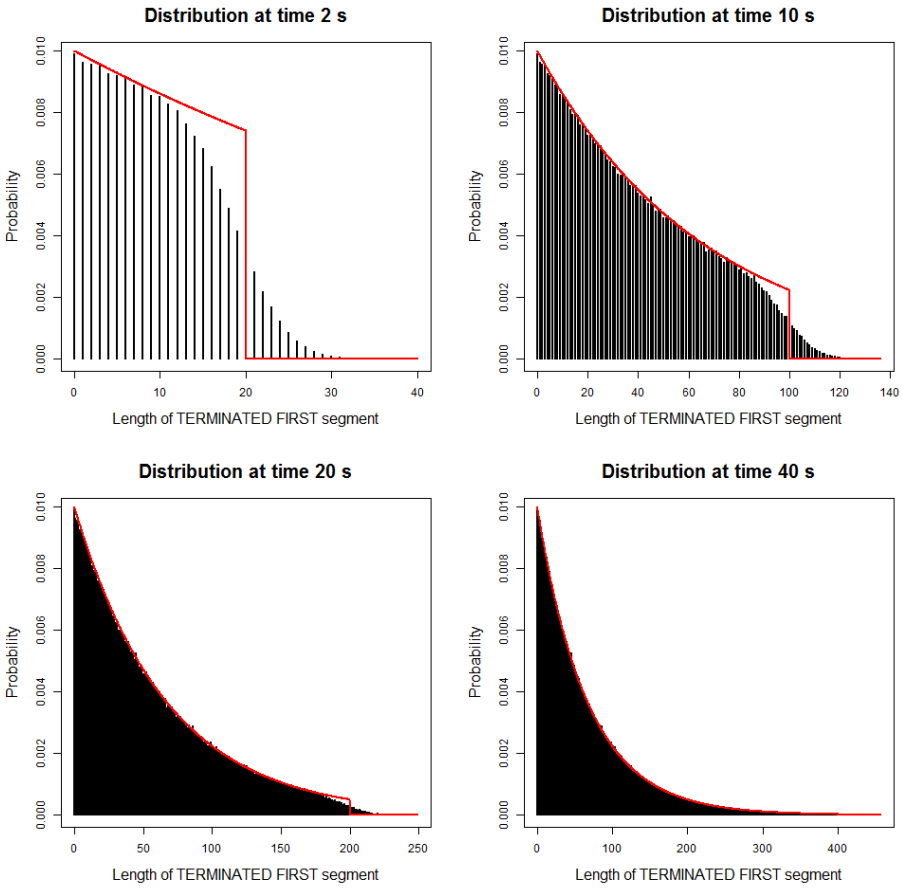

Figure 3.5: Comparison between the PDE solution $Q^f(x,t)$ [■] and the estimations given by MC approach [■], when the rates are equal to (3.64).

As the pdf obtained from the two approaches are very close, the other quantities of interest are very similar, too.
In this case of good match, we prefer the PDE approach. The reason is that

the MC approach cost is its computational time, that mostly depends on the termination rate q. The smaller is q, the bigger is the computational time, but the better could be the match between the two approaches, if the backbiting rate r is small too. Thus, it is useless to wait for the estimations given by MC approach, when it is possible to have an immediate PDE solution.
The drawbacks related to the PDE approach start to be significant when one of the two rates r or q starts to be bigger. In this last case, it is better use the MC approach, as explained in Section 3.4.3.

3.4.2 Deficiencies of the MC approach

The first MC deficiency is its computational time: the smaller is the termination rate q, the bigger is the needed time to complete the simulation, as explained in Section 3.4.1.

The second drawback of this approach is that the longer is the simulated time, the worst is the MC estimate for quantities of interest related to propagating branches.
The reason is that the number or the proportion of propagating branches decreases to zero with time increasing, because a backbiting or a termination occurs sooner or later, giving an end to the propagating growth. Thus, MC estimation fails due to the lack of statistics.
In Figure 3.6, we show the mean length and the proportion of propagating last branches when the parameters set is (3.64).

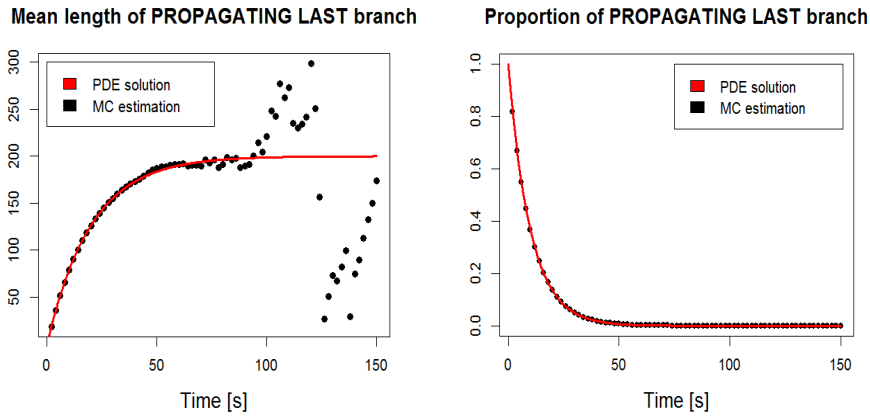

Figure 3.6: Mean length and the proportion of propagating last branches with rates equal to (3.64).

3.4.3 Deficiencies of the PDE approach

The problems related to PDE approach are the more significant the bigger are the rates for backbiting or termination. For example, in this section we use the parameters set (3.65), in order to show the PDE drawbacks and how the MC approach can overcomes these deficiencies.

$$p = 10s^{-1},\ r = 2s^{-1},\ q = 0.1s^{-1}. \tag{3.65}$$

The first remark on PDE approach is that to make the branches discrete length n goes to a continuous value x is an approximation, as explained in Section 3.2.2.
The consequence of this continuous limit is that the most of the times the PDE solutions are probability density functions with respect to the Lebesque measure.
Thus, they allow non zero probability to intervals made by non integer values, because there is a non zero area under PDE solution between two consecutive integer values. This is clearly impossible because the quantity we are studying is discrete.
When the backbiting and termination rates are small, as (3.64), this impossible probability is trivial and we can not see its effects because the approximation properly works.
Whereas, if r or q start to be bigger, like (3.65), this probability is not negligible and the approximation does not work, producing a significant difference between MC and PDE solutions.
The benefit from the MC approach is that it only gives non zero mass on the natural numbers set $\mathbb{N}$, giving the correct description of the branches discrete length whatever are the rates.
In order to visualize this phenomenon, we show in Figure 3.7 the terminated first branch pdf when the parameters set is (3.65).

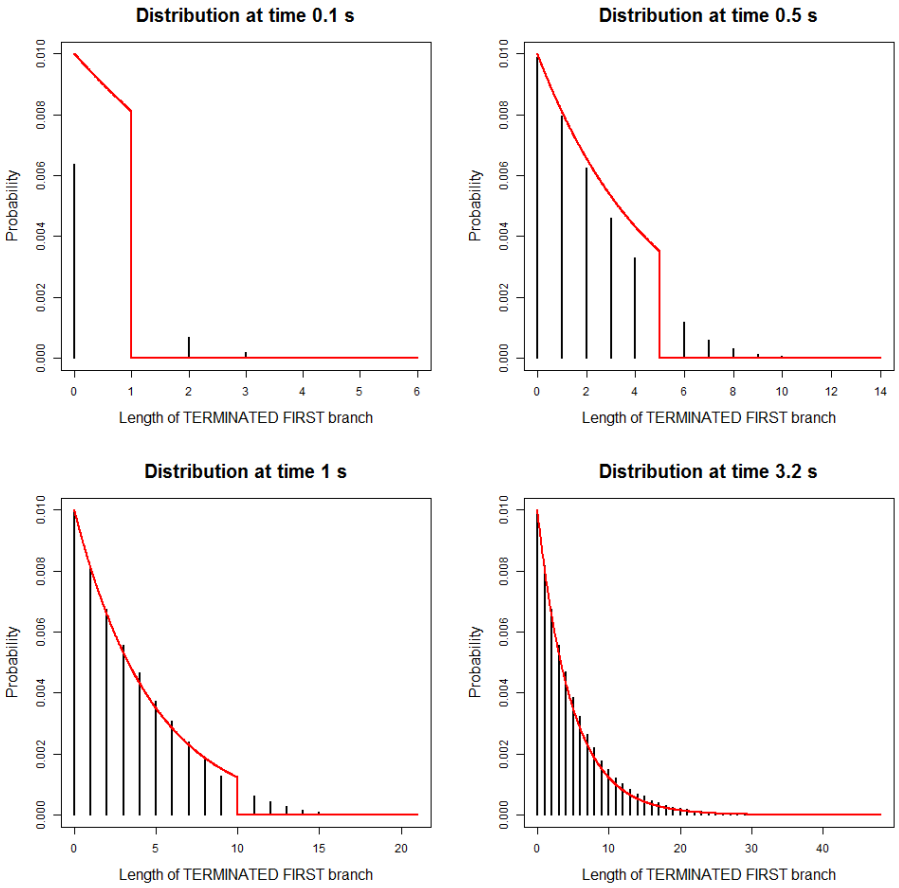

Figure 3.7: If the rates are equal to (3.65), the PDE solution $Q^f(x,t)$ [■] gives non trivial probability to intervals between two consecutive integer values, whereas the MC estimations [■] only gives non zero probability to discrete values.

The second drawback of PDE solution is that it forces unrealistic deterministic values for propagating branches length. For example, the solution (3.18) means that at any given time t the propagating first branches length x must be equal to pt.
The MC estimations give a better description of this length, because they allows fluctuations and variability, as it must be expected from the real world. In Figure 3.8, we show the propagating first branches pdf when the rates are (3.64).

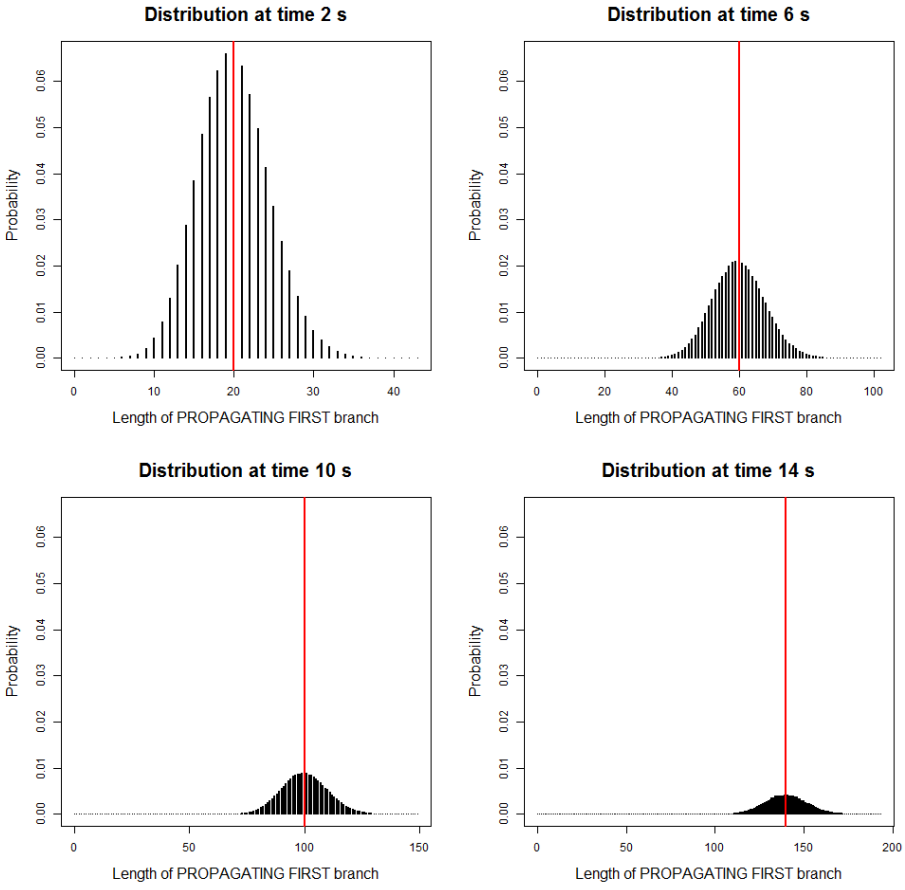

Figure 3.8: The PDE solution $P^f(x,t)$ forces the deterministic values $x = pt$ [■], whereas the MC estimations allows more realistic fluctuations [■]. Here, both approaches are shown when the rates are (3.64).

Then, another PDE problem is that those solutions allow the chain to start from a backbiting, that is an impossible event.
In fact, the PDE solution $R^f(x,t)$ (3.20) gives non zero probability to backbiting to be the first event. If the rates set is (3.65), this probability is not negligible.
Using the MC approach, we can forbid the backbiting to be the first event, as explained in Algorithm 1. Thus, the MC estimation of this event probability always is equal to 0, whatever are the rates.

In order to visualize this difference, we show in Figure 3.9 the backbiting terminated first segments pdf when the rates are (3.65).
An important consequence is that the PDE solution underestimates the mean length of the backbiting terminated first segment, because it gives non zero probability to these branches made by zero monomers.
MC mean is bigger than the PDE one, because the MC estimated pdf gives zero probability to the backbiting terminated first branches made by zero monomers.
The MC mean is a better estimation of the real value, because the backbiting terminated first branches made by zero monomers are impossible to be produced. Thus, their probability must be equal to zero.
It is possible to see this difference in Figure 3.10.

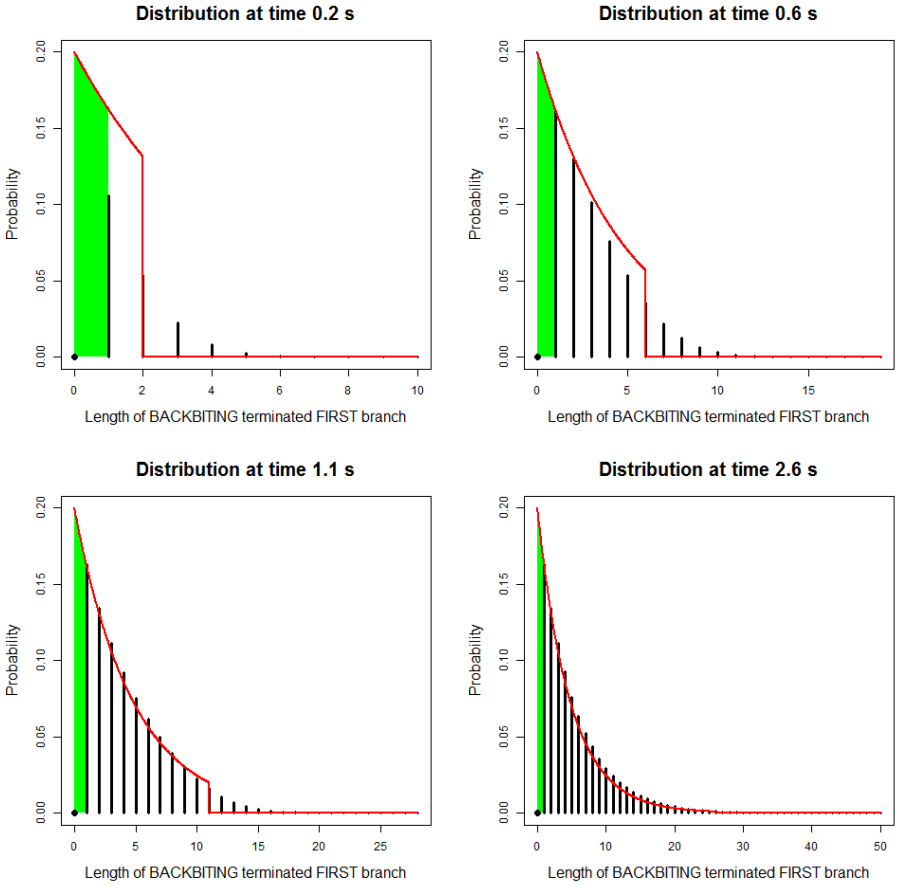

Figure 3.9: If the rates are (3.65), the PDE solution $R^f(x,t)$ [■] gives non trivial probability to backbiting to be the first event [■], whereas the MC estimations [■] always give zero probability to this impossible event [●].

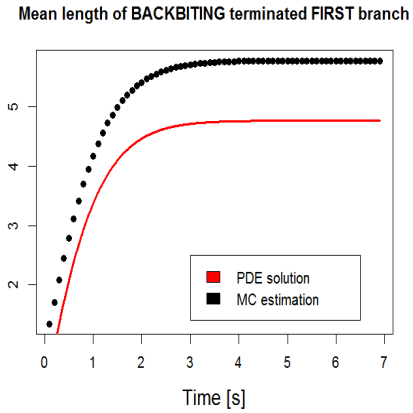

Figure 3.10: The mean length of backbiting terminated first segment, for rates (3.65).

The last PDE drawback is that this approach allows the impossible occurrence of two consecutive backbiting.
In fact, the PDE solution $R_d^{bb}(y,t)$ (3.49) gives non zero probability to this impossible event. If the rates set is (3.65), this probability is not negligible.
Using the MC approach, we can forbid the occurrence of two consecutive backbitings, as explained in Algorithm 1. Thus, the MC estimation of this event probability always is equal to 0, whatever are the rates.
In order to visualize this difference, we show in Figure 3.11 the backbiting terminated branches pdf when the rates are (3.65).
An important consequence is that the PDE solution underestimates the mean length of all backbiting terminated segments, because it gives non zero probability to these branches made by zero monomers.
MC mean is bigger than the PDE one, because the MC estimated pdf gives zero probability to all backbiting terminated segments made by zero monomers.
The MC mean is a better estimation of the real value, because the backbiting terminated branches made by zero monomers are impossible to be produced. So, their probability must be equal to zero.
It is possible to see this difference in Figure 3.12.

In conclusion, it is interesting to remark that, in the case we put into the simulation the information that backbiting can occur only after 3 previous propagation, the difference between PDE and MC approach increases, but now it is clear why.

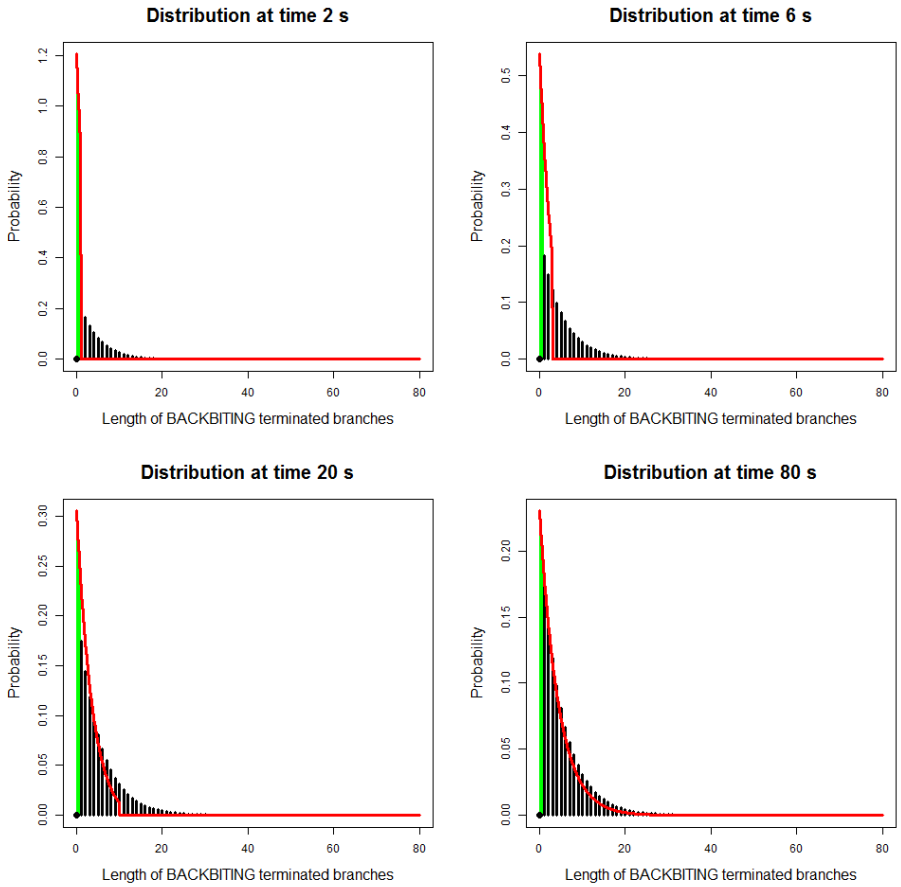

Figure 3.11: If the rates are (3.65), the PDE solution $R_d^{bb}(y,t)$ [■] gives non trivial probability to the occurrence of two consecutive backbiting [■], whereas the MC estimations [■] always give zero probability to this impossible event [•].

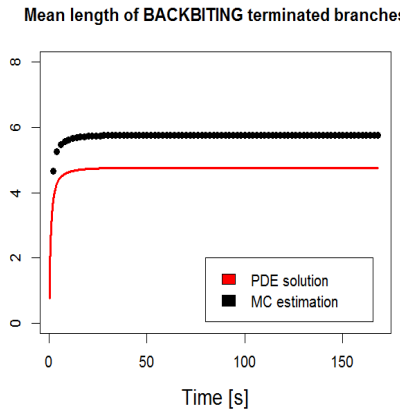

Figure 3.12: The mean length of all backbiting terminated segments, for rates (3.65).

Chapter 4

Control Radical Polymerization with the freezing agent

4.1 The freezing agent

In this section we introduce a new player for the considered polymerization process: the freezing agent. This freezing agent is another kind of monomer, that can be added to the growing chain, as the one added by propagation.
The difference is the following: after the addition of the freezing agent the only possible event is its removal from the chain. Then, the reaction restarts as before. Thus, the occurrence of the freezing event means that the freezing agent is added to the chain and then removed.
There is an experimental evidence for the reduction of the polymers branching fraction with the occurrence of the freezing event. This means that the mean ratio of the number of backbitings to the number of propagations is decreasing with the rate f of freezing event, or it is increasing with its mean occurrence: the Mean Time To Freezing ($MTTF = 1/f$). A schematic behavior for this mean ratio is shown in Figure 4.1.

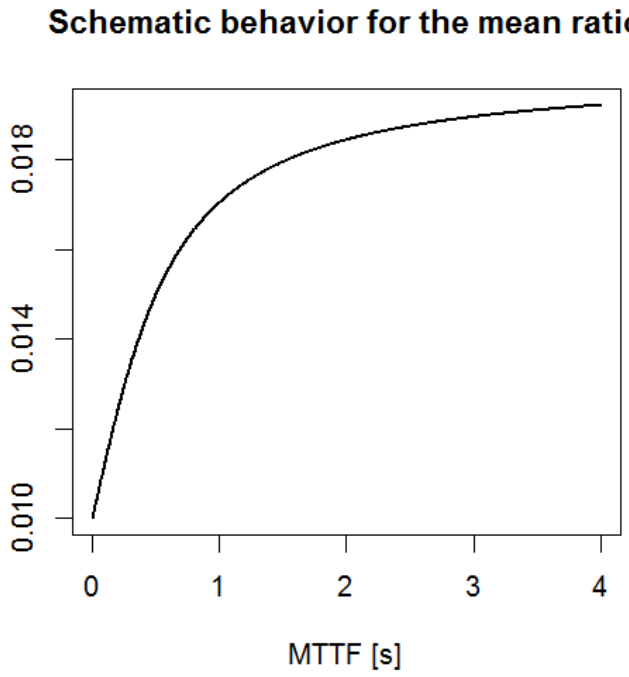

Figure 4.1: Schematic behavior of the ratio of backbitings to propagations with respect to the $MTTF$.

The objective of the following Sections is to implement a proper MC algorithm, which potentially can reproduce the effect of $MTTF$ on the mean ratio of the number of backbitings to the number of propagations.

4.2 The exponential kernel

The first attempt that can be done is to try to implement the MC approach explained in Section 3.3. Following Section 3.3.1, we implement the Gillespie algorithm based on exponential kernel.
We introduce these events: propagation, backbiting, freezing and termination. In order to make the results more realistic, we allow the occurrence of backbiting only if at least three propagations occur in the current branch, as explained in Section 3.3.3.

4.2.1 The algorithm

The rates and the labels of each event are the following:

- the termination rate is q and its label is 0,
- the propagation rate is p and its label is 1,
- the freezing rate is f and its label is 2,
- the backbiting rate is r and its label is 3.

We can define the random variables that describe the event and the time needed for it, when all the events can occur:

$$X = \begin{cases} 0 & \text{w.p. } \frac{q}{p+r+f+q} \\ 1 & \text{w.p. } \frac{p}{p+r+f+q} \\ 2 & \text{w.p. } \frac{f}{p+r+f+q} \\ 3 & \text{w.p. } \frac{r}{p+r+f+q} \end{cases}, \tag{4.1}$$

$$T \sim \text{Exp}\left(p+r+f+q\right). \tag{4.2}$$

Then, these are random variables that describe the event and the time needed for it, when backbiting can not occur:

$$X_b = \begin{cases} 0 & \text{w.p. } \frac{q}{p+f+q} \\ 1 & \text{w.p. } \frac{p}{p+f+q} \\ 2 & \text{w.p. } \frac{f}{p+f+q} \end{cases}, \tag{4.3}$$

$$T_b \sim \text{Exp}\left(p+f+q\right). \tag{4.4}$$

We define $(\underline{x}, \underline{t})$ the realization of one chain: it means that the j^{th} element of a vector $\underline{x}$ is the label of the j^{th} event and the j^{th} element of a vector $\underline{t}$ is the occurrence time of the j^{th} event. So, Algorithm 2 explains how to get one chain realization.

```
Draw a realization x_b from X_b and put x[1] = x_b;
Draw a realization t_b from T_b and put t[1] = t_b;
Initialize the flag for possibility of backbiting B = 0;
Initialize the number of propagations in the current branch n_P = 0;
Initialize j = 1;
if x[1] == 1 then
    Put n_P = 1;
end
while x[j] ≠ 0 do
    Update j ← j + 1;
    if n_P < 3 then
        Put B = 0;
    else
        Put B = 1;
    end
    if B == 1 then
        Draw a realization x from X and put x[j] = x;
        Draw a realization t from T and put t[j] = t[j − 1] + t;
        if x[j] == 3 then
            Put n_P = 0;
        end
    else
        Draw a realization x_b from X_b and put x[j] = x_b;
        Draw a realization t_b from T_b and put t[j] = t[j − 1] + t_b;
        if x[j] == 1 then
            Update n_P ← n_P + 1;
        end
    end
end
```

Algorithm 2: The algorithm to get one chain realization in the exponential kernel case. The random variables X, T, X_b and T_b are (4.1), (4.2), (4.3) and (4.4).

4.2.2 The simulations

The simulations can be run following the Algorithm 2 with the fixed parameters p, r, f and q.
In particular, we run the MC simulation with this set of parameters:

$$p = 10s^{-1},\ r = 0.2s^{-1},\ q = 0.01s^{-1} \tag{4.5}$$

and the freezing rate f is varying according to the set of values:

$$f \in \left\{20s^{-1}, 15s^{-1}, 10s^{-1}, 5s^{-1}, 2s^{-1}, 1.25s^{-1}, 1s^{-1}, 0.75s^{-1}, 0.5s^{-1}\right\}. \tag{4.6}$$

So, it is possible to compute the ratio distribution and its mean, for these different values of $MTTF$:

$$MTTF \in \{0.05s, 0.07s, 0.1s, 0.2s, 0.5s, 0.8s, 1s, 1.33s, 2s\} \tag{4.7}$$

and to find out if the mean ratio increases when $MTTF$ is big. The results are shown in Figure 4.2.

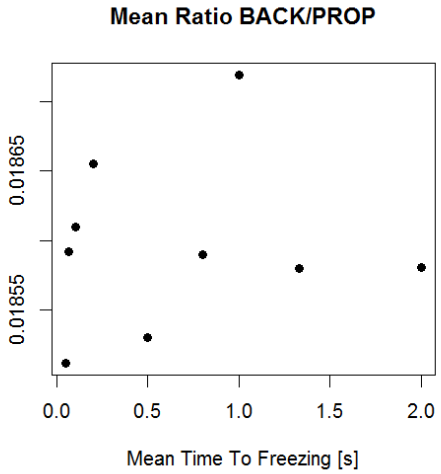

Figure 4.2: Constant mean ratio backbiting over propagation in the case of exponential kernel with rates (4.5) and (4.6).

4.2.3 Important remarks

We have to remark that the mean ratio is constant, because its values corresponding to different freezing rates are identical within the error of $\propto 10^{-4}$. If we neglect the backbiting prohibition, this conclusion can be made directly from the construction of Algorithm 2. Equation (4.1) means that the mean proportions of backbiting and propagation respectively are $\approx \frac{r}{p+r+f+q}$ and $\approx \frac{p}{p+r+f+q}$. Thus, the mean ratio is $\approx \frac{r}{p} = 0.02$, independent from $MTTF$.

The proposed model is not able to correctly describe the experimental behavior of the mean ratio. The reason can be found in Section 3.3.2. The implemented model is based on the hypothesis that the event rates are constant, fixing the time and the chain configuration realized after each event. This hypothesis leads to exponential probability density functions and exponential kernel, as shown in equations (3.53) and (3.55). Thus, the resulting probability of occurrence for each kind of event is proportional to its rate, as shown by (3.59). So, the behavior of the mean ratio has to be constant with respect to the $MTTF$, as explained in the previous paragraph.
In order to reproduce the experimental observations, we can ease the previous hypothesis and try to nominate probability density functions (pdf) with non-constant rate of occurrence. This is the objective of the following Sections.

Moreover, there is another reason to think that the exponential distribution

and its constant rate are not the proper tools to model this phenomenon. In a process characterized by exponential distribution, the probability of occurring an event is maximum when the time goes to zero. This property goes against the intuitive idea that the occurrence probability of a real event approaches to zero when the time goes to zero. Therefore, the modification to the requested time pdf could be the following. We can make the pdf equal to zero for $t = 0$, increasing for small time and then exponentially decreasing for big time.

Thus, the choice of pdf different from the exponential ones leads to non Poisson processes, without the memoryless property. Thus, it is reasonable to introduce non Markovian models, that are not characterized by memoryless properties. In Chapter 5, we analyze non Markovian models and non Poisson processes. We can show that the desired behavior of the mean ratio is strongly related to these non Markovian effects.

We have to stress that we can implement the Gillespie algorithm with a pdf different from the exponential one for the following reason: as we have found in Section 3.3.2, the next event is the one that realize the minimum among the requested times for all the possible events. Thus, we can choose any pdf from which draw the requested times and simply choose the event that realizes the minimum time of occurrence. This is the building idea of the algorithms implemented in the following Sections.

Chapter 5

Non-Markovian Model

5.1 The gamma kernel

In order to find the right shape for the requested time pdf, we rely on the following information. On the one hand, an empirically trial could be the following: to make the pdf be equal to zero for $t = 0$ and increasing for small times. On the other hand, the behavior of the pdf is well known for large times and it is an exponential decay.
One pdf that approximates this behavior is the gamma distribution, Gamma(α,β). Let T be a time needed for a next event with the rate equal to λ and assume that:

$$T \sim \text{Gamma}(\alpha, \beta). \tag{5.1}$$

Equation (5.1) means that the T pdf is the following:

$$f_T(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} \quad \text{for } t \in [0; +\infty) \text{ and } \alpha, \beta > 0. \tag{5.2}$$

The choice of the gamma distribution is an approximation, because the tail of the distribution is not an exponential decay. It is an exponential decay multiplied by a polynomial term, as equation (5.2) shows.

An important remark is that the gamma distribution has not a constant rate of occurrence.
We are looking for the parameters α and β that lead to the desired shape and behavior of the pdf function.

5.1.1 The parameters

These are simple rules that must be always satisfied:

- the expected value of T must be equal to $\frac{1}{\lambda}$,
- the parameter α must be bigger or equal to 1, in order to satisfy the empirical requests.

Since the expected value of a Gamma pdf with parameters α and β is the following:

$$\mathbb{E}[T] = \int_0^{+\infty} t f_T(t) dt = \frac{\alpha}{\beta}, \tag{5.3}$$

we can fix a first rule for the parameters α and β:

$$\frac{\alpha}{\beta} = \frac{1}{\lambda} \quad \Rightarrow \quad \beta = \alpha\lambda. \tag{5.4}$$

Then, $\alpha = 1$, means that the Gamma$(1, \beta)$ pdf is equal to a Exp(β) pdf, so our choice is a generalization of the previous exponential implementation.

5.1.2 The resulting pdf

We introduce these events:

- propagation, with rate p,
- backbiting, with rate r,
- freezing, with rate f,
- termination, with rate q.

Thus, following the criteria from Section 5.1.1, we assign the pdf to the times required for each event in Table 5.1.

event	**needed time pdf**
propagation	$T_p \sim \text{Gamma}(\alpha, \alpha p)$
backbiting	$T_r \sim \text{Gamma}(\alpha, \alpha r)$
freezing	$T_f \sim \text{Gamma}(\alpha, \alpha f)$
termination	$T_q \sim \text{Gamma}(\alpha, \alpha q)$

Table 5.1: The gamma pdf.

5.1.3 The algorithm

Let the next event be the one that realizes the minimum among the times needed for all possible events.
Thus, in order to choose the next event, we follow these steps:

1. draw a realization from each pdf that models the time needed for each possible next event,
2. pick the event that realizes the minimum occurrence time.

Then, we remember that backbiting can only occur after at least three previous propagations in any branch and the events labels are the following:

- the termination label is 0,
- the propagation label is 1,
- the freezing label is 2,
- the backbiting label is 3.

Thus, we consider the random variables T_p, T_r, T_f and T_q from Table 5.1 and we implement Algorithm 3, in order to draw a chain realization $(\underline{x}, \underline{t})$. It means that the j^{th} element of vector $\underline{x}$ is the label of the j^{th} event and the j^{th} element of vector $\underline{t}$ is the occurrence time of the j^{th} event.

Draw a realization t_p from T_p, t_f from T_f and t_q from T_q;
Put $\underline{t}[1] = \min\{t_p, t_f, t_q\}$;
Put $\underline{x}[1] =$ label of argmin$\{t_p, t_f, t_q\}$;
Initialize the flag for possibility of backbiting $B = 0$;
Initialize the number of propagations in the current branch $n_P = 0$;
Initialize $j = 1$;
if $\underline{x}[1] == 1$ **then**
  Put $n_P = 1$;
end
while $\underline{x}[j] \neq 0$ **do**
  Update $j \leftarrow j + 1$;
  if $n_P < 3$ **then**
    Put $B = 0$;
  else
    Put $B = 1$;
  end
  if $B == 1$ **then**
    Draw a realization t_p from T_p, t_r from T_r, t_f from T_f and t_q from T_q;
    Put $\underline{t}[j] = \underline{t}[j-1] + \min\{t_p, t_r, t_f, t_q\}$;
    Put $\underline{x}[j] =$ label of argmin$\{t_p, t_r, t_f, t_q\}$;
    if $\underline{x}[j] == 3$ **then**
      Put $n_P = 0$;
    end
  else
    Draw a realization t_p from T_p, t_f from T_f and t_q from T_q;
    Put $\underline{t}[j] = \underline{t}[j-1] + \min\{t_p, t_f, t_q\}$;
    Put $\underline{x}[j] =$ label of argmin$\{t_p, t_f, t_q\}$;
    if $\underline{x}[j] == 1$ **then**
      Update $n_P \leftarrow n_P + 1$;
    end
  end
end

Algorithm 3: The algorithm to get one chain realization in the gamma kernel case. The random variables T_p, T_r, T_f and T_q are shown in Table 5.1.

5.1.4 The simulations

The simulations can be run following the Algorithm 3 with the fixed parameters α, p, r, f and q.
In particular, we choose the set of parameters (4.5) and (4.6). We repeat simulations for different values of parameter $\alpha \geq 1$. The results are shown in Figure 5.1.

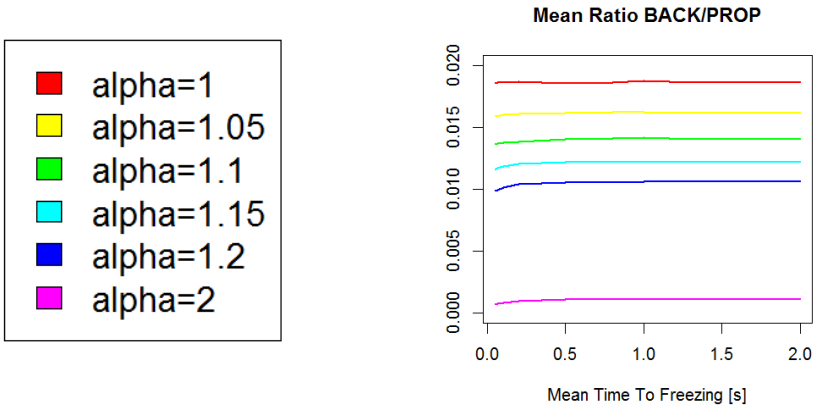

Figure 5.1: Behavior of the ratio of backbitings to propagations with respect to the $MTTF$, in the case of gamma kernel with different fixed parameters $\alpha \geq 1$ and with rates (4.5) and (4.6).

The remark that must be made is that the mean ratio is constant with respect to the $MTTF$ for all fixed $\alpha \geq 1$. It means that the choice of gamma distribution does not explain the desired behavior for the mean ratio between backbiting and propagation events. However, one can observe that the bigger is α, the smaller is the mean ratio. We give an explanation of this phenomenon in Section 5.2.1.

The new idea that comes from these last results is that the parameter α linked to $MTTF$ could explain the not constant behavior of the mean ratio backbiting over propagation. This is the aim of Section 5.2.

The drawback of the gamma distribution choice is that the tail of the distribution is not an exponential decay. It is an exponential decay multiplied by a polynomial term, as equation (5.2) shows.
In order to have a pure exponential decay, we can try with the pdf explained in Section 5.3.

5.2 The delayed gamma kernel

In this section, we keep in the model the propagation, backbiting and termination events, but do not consider the freezing agent occurrence. The idea is try to make the information related to the freezing agent in a way suggested by the results got in Section 5.1.4, but not directly by its occurrence as one of the possible events.
In order to reach this aim, we proposed an exploratory simulation in Section 5.2.1 and then the desired simulation in Section 5.2.2.

5.2.1 The exploratory simulation

We keep in the model the propagation, backbiting and termination events. Following the criteria from Section 5.1.1, we assign in Table 5.2 the pdf to their times required.

event	**needed time pdf**
propagation	$T_p \sim \text{Gamma}(\alpha, \alpha p)$
backbiting	$T_r \sim \text{Gamma}(\alpha, \alpha r)$
termination	$T_q \sim \text{Gamma}(\alpha, \alpha q)$

Table 5.2: The pdf for the exploratory simulation.

We proposed Algorithm 4 in order to draw one chain realization. The important remark is that the freezing agent occurrence is not placed into the algorithm.
We run this simulation with p, r and q values (4.5) and different fixed values of $\alpha \geq 1$. The results are given in Figure 5.2.

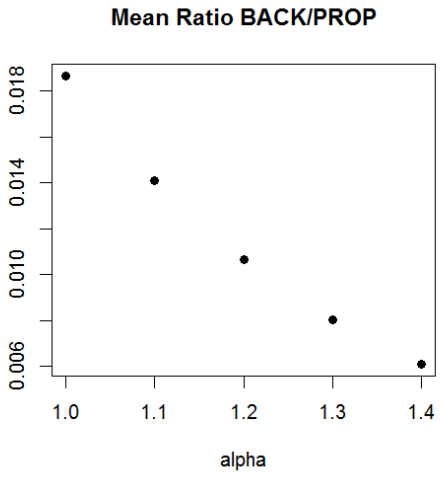

Figure 5.2: Behavior of the asymptotic mean ratio of backbiting to propagation with respect to the parameter α, for the exploratory simulation.

Draw a realization t_p from T_p and t_q from T_q;
Put $\underline{t}[1] = \min\{t_p, t_q\}$;
Put $\underline{x}[1] =$ label of $\text{argmin}\{t_p, t_q\}$;
Initialize the flag for possibility of backbiting $B = 0$;
Initialize the number of propagations in the current branch $n_P = 0$;
Initialize $j = 1$;
if $\underline{x}[1] == 1$ **then**
  Put $n_P = 1$;
end
while $\underline{x}[j] \neq 0$ **do**
  Update $j \leftarrow j + 1$;
  if $n_P < 3$ **then**
    Put $B = 0$;
  else
    Put $B = 1$;
  end
  if $B == 1$ **then**
    Draw a realization t_p from T_p, t_r from T_r and t_q from T_q;
    Put $\underline{t}[j] = \underline{t}[j-1] + \min\{t_p, t_r, t_q\}$;
    Put $\underline{x}[j] =$ label of $\text{argmin}\{t_p, t_r, t_q\}$;
    if $\underline{x}[j] == 3$ **then**
      Put $n_P = 0$;
    end
  else
    Draw a realization t_p from T_p and t_q from T_q;
    Put $\underline{t}[j] = \underline{t}[j-1] + \min\{t_p, t_q\}$;
    Put $\underline{x}[j] =$ label of $\text{argmin}\{t_p, t_q\}$;
    if $\underline{x}[j] == 1$ **then**
      Update $n_P \leftarrow n_P + 1$;
    end
  end
end

Algorithm 4: The algorithm to get one chain realization in the delayed gamma kernel case. The random variables T_p, T_r and T_q are shown in Table 5.2.

The decreasing behavior shown in Figure 5.2 is due to the delay introduced into the model by the parameter α.
In fact, if we fix the pdf average value following (5.4), the bigger is α, the bigger is the arg-maximum point of the pdf and the delay of the model. The introduced delay means that the needed times realizations are likely to be bigger and the reaction is slower. Figure 5.3 shows this behavior.

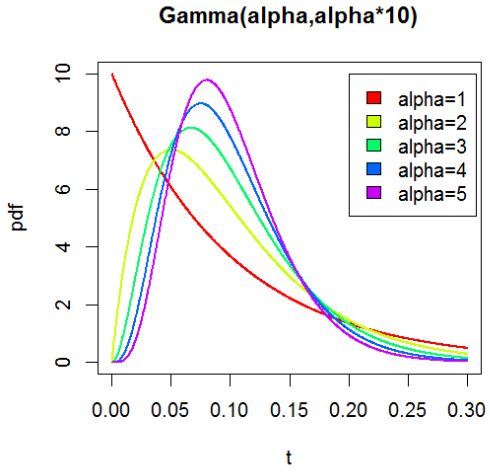

Figure 5.3: The arg-maximum moving behavior due to parameter α.

Since the effect of the parameter α is the same for all the pdf (see Table 5.2), we can say that the same delay affects more the rarest event, because it decreases their competitive probability more than it does for the likely events.
Thus, the bigger delay and parameter α produce more damage for the rarest event and lead to the smaller ratio between a rare event and a likely one.

In order to visualize this concept, we propose the example shown in Figure 5.4 and in Figure 5.5.
Following the criterion (5.4), we define the rare event pdf as Gamma$(\alpha, 2\alpha)$ and the likely event pdf as Gamma$(\alpha, 5\alpha)$.
In an empirical way, we define the rare event competitive probability as the area under the rare event pdf before the not trivial intersection between the two pdf.
In the same way we define the likely event competitive probability: the area under the likely event pdf before the not trivial intersection between the two pdf.
Giving different values to the parameter α, we can see how the two competitive probabilities depend on this parameter: more precisely, the rare event one decreases with α increased, whereas the likely one increases.
This is the reason why the bigger is α the smaller is the ratio between a rare event and a likely one.

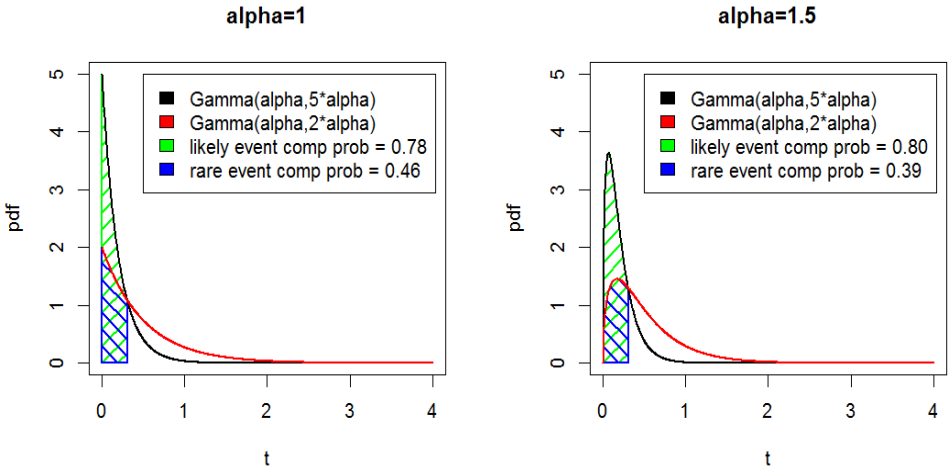

Figure 5.4: The competitive probabilities computed for two different values of parameter α.

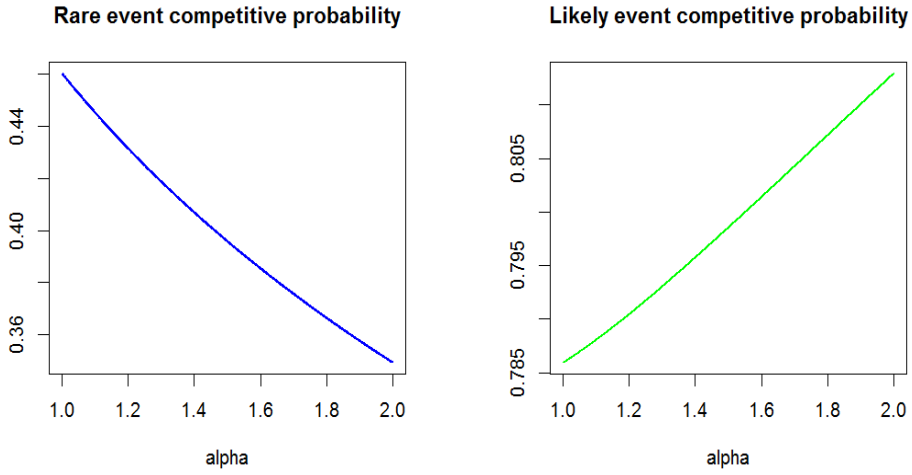

Figure 5.5: The rare event competitive probability as function of α (left) and the likely event competitive probability as function of α (right).

5.2.2 The delay effect

An interesting observation is that the mean ratio depends on α without introducing the freezing agent occurrence.
This could mean that the role of the freezing agent in this phenomenon is in making the reaction's evolution slower. In other words, it introduces a certain delay and thus makes a change in the ratio between backbiting and propagation.
In order to quantify the delay introduced by the freezing event, one can link the freezing rate f to the parameter α, i.e. $\alpha = \alpha(f)$.

The function $\alpha = \alpha(f)$ must satisfy these properties:

- $\alpha(0) = 1$, because if freezing agent occurrence is not introduced into the model, there is experimental evidence for mean ratio values close to the ones got from exponential kernel,
- the function has to increase with increasing freezing rate f, because the ratio behavior is increasing with respect to the $MTTF$.

Another useful, but not mandatory, property can be that the limit for $f \to +\infty$ of function $\alpha = \alpha(f)$ should coincide with the α-value that reproduces the mean ratio obtained at $MTTF$ equal to 0.
A way to fix this function is to watch the estimations shown in Figure 5.2.
As the set of parameters (4.5) is specified, some possible choices for $\alpha = \alpha(f)$ are the following (see Figure 5.6):

$$\alpha_1(f) = 0.2\exp\left(-\frac{1}{f}\right) + 1 \tag{5.5}$$

$$\alpha_2(f) = \begin{cases} -\frac{0.1}{200}f^2 + \frac{0.1}{5}f + 1 & \text{if } f \leq 20s^{-1} \\ 1.2 & \text{if } f > 20s^{-1} \end{cases} \tag{5.6}$$

$$\alpha_3(f) = \begin{cases} \frac{0.2}{20}f + 1 & \text{if } f \leq 20s^{-1} \\ 1.2 & \text{if } f > 20s^{-1} \end{cases} \tag{5.7}$$

$$\alpha_4(f) = \frac{0.2}{\pi}\arctan(f - 10) + 1.1 \tag{5.8}$$

Then, we run the simulations following Algorithm 4 using the pdf in Table 5.2 with p, r and q values (4.5).
Again, the freezing agent occurrence is not introduced in the model. Instead, the values of the parameter α are taken from (5.5), (5.6), (5.7) and (5.8). We repeat a simulation for each α from (5.5), (5.6), (5.7) and (5.8) computed at each value of f in (4.6). Thus, we can check the behavior of the mean ratio with respect to the $MTTF$ corresponding to the rates f (4.6). The results are shown in Figure 5.7.

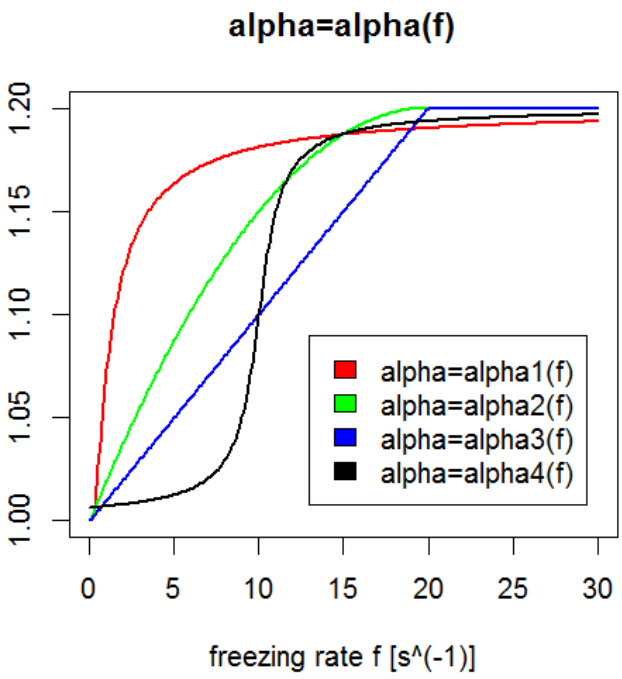

Figure 5.6: The α parameter as a function of f.

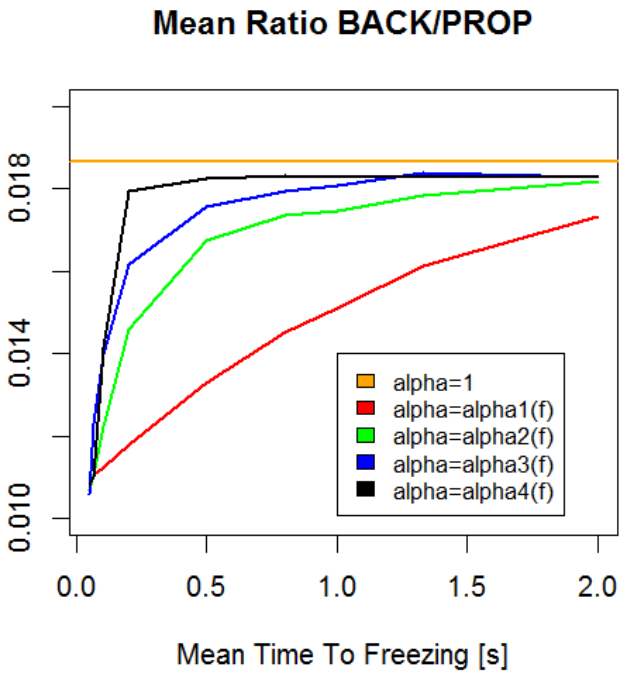

Figure 5.7: The behavior of the mean ratio backbiting over propagation with respect to different functions $\alpha = \alpha(f)$, in the case of delayed gamma kernel.

5.2.3 The drawbacks

We can get the desired behavior of the mean ratio backbiting over propagation, but it implies several drawbacks.
The first drawback of the proposed model is that the function $\alpha = \alpha(f)$ changes if we change the events rates. Also, this function is not unique.
Then, making α a function of f gives more freedom to the model and the sought behavior is obtained, but it has no physical meaning.
The pdf of the different processes should be independent and with this formulation all pdf depend on f.
Moreover, the last drawback of the gamma distribution choice is its not exponential decaying tail.
In Section 5.3, we try to give a solution to all these problems.

5.3 The linear exponential kernel

Let T be the requested time for a possible future event with rate λ. Its pdf has to be equal to zero for $t = 0$, increasing for small times and exponentially decreasing for large times. Thus, a resulting simple choice is the following:

$$f_T(t) = \begin{cases} kt & \text{if } 0 \leq t < b \\ kbe^{-\tau(t-b)} & \text{if } t \geq b \end{cases} \text{ for } t, b \geq 0 \text{ and } k, \tau > 0. \tag{5.9}$$

We are looking for the parameters k, b and τ that lead to the desired behavior of mean ratio backbiting over propagation.

5.3.1 The parameters

First of all, the pdf must be normalized:

$$\int_0^{+\infty} f_T(t)dt = 1. \tag{5.10}$$

Then, the expected value of T must be equal to the event mean occurrence time $\frac{1}{\lambda}$:

$$\int_0^{+\infty} t f_T(t)dt = \frac{1}{\lambda}. \tag{5.11}$$

The equations (5.10) and (5.11) can be rewritten as follow:

$$k = \frac{2\tau}{b\,(\tau b + 2)}, \tag{5.12}$$

$$b\,(2\lambda b - 3)\,\tau^2 + 6\,(\lambda b - 1)\,\tau + 6\lambda = 0. \tag{5.13}$$

Equation (5.13) gives the two solutions:

$$\tau_{1,2} = \frac{-3\,(\lambda b - 1) \pm \sqrt{9 - 3\lambda^2 b^2}}{b\,(2\lambda b - 3)}. \tag{5.14}$$

In order to avoid complex values, b must range within the interval specified by:

$$0 \leq b \leq \frac{\sqrt{3}}{\lambda}. \tag{5.15}$$

In order to close the equations (5.12) and (5.14), we can choose the parameter b as follow:

$$b = \frac{d}{\lambda} \text{ for } 0 \leq d \leq 1, \tag{5.16}$$

where the maximum value for the parameter d is equal to 1 because we have to satisfy (5.15) and it is reasonable that the introduced linear delay doesn't assume bigger values than the mean time $\frac{1}{\lambda}$.

We pick from (5.14) the "minus" solution, because it makes τ be bigger than zero for all $0 \leq d \leq 1$.
Finally, the pdf parameters are the following:

$$b(d, \lambda) = \frac{d}{\lambda}, \tag{5.17}$$

$$\tau(d, \lambda) = \frac{\lambda \left(3d - 3 + \sqrt{9 - 3d^2}\right)}{d\,(3 - 2d)}, \tag{5.18}$$

$$k(d, \lambda) = \frac{2\lambda^2 \left(3d - 3 + \sqrt{9 - 3d^2}\right)}{d^2 \left(\sqrt{9 - 3d^2} - d + 3\right)}. \tag{5.19}$$

Thus, we say that:

$$T \sim \text{Linexp}(d, \lambda), \tag{5.20}$$

if the T pdf is the following:

$$f_T(t) = \begin{cases} k(d, \lambda)t & \text{if } 0 \leq t < b(d, \lambda) \\ k(d, \lambda)b(d, \lambda)e^{-\tau(d,\lambda)[t - b(d,\lambda)]} & \text{if } t \geq b(d, \lambda) \end{cases}, \tag{5.21}$$

where (5.17), (5.18) and (5.19) give the parameters values.

5.3.2 The resulting pdf

We introduce these events: propagation, with rate p, backbiting, with rate r, freezing, with rate f, and termination, with rate q.
So, following the criteria from Section 5.3.1, we assign the pdf to the times required for each event in Table 5.3.

event	**required time pdf**
propagation	$T_p \sim \text{Linexp}(d, p)$
backbiting	$T_r \sim \text{Linexp}(d, r)$
freezing	$T_f \sim \text{Linexp}(d, f)$
termination	$T_q \sim \text{Linexp}(d, q)$

Table 5.3: The linear exponential pdf.

The delay affects all the possible events. This property is due to the intuitive idea that the occurrence probability of all the real events approaches to zero when the time goes to zero. Thus, a delay or a set up time time is needed for all the possible events. Here, the required delay is modeled by the parameter d and by the linear part of all events pdf.

5.3.3 The algorithm

In order to draw different realizations of the growing chains, we implement Algorithm 5, where the random variables T_p, T_r, T_f and T_q are the ones defined into Table 5.3.
In order to draw realizations from these linear-exponential random variables, we have to implement the well known "Inverse Transform Sampling Method" [21].

```
Draw a realization t_p from T_p, t_f from T_f and t_q from T_q;
Put t[1] = min{t_p, t_f, t_q};
Put x[1] = label of argmin{t_p, t_f, t_q};
Initialize the flag for possibility of backbiting B = 0;
Initialize the number of propagations in the current branch n_P = 0;
Initialize j = 1;
if x[1] == 1 then
    Put n_P = 1;
end
while x[j] ≠ 0 do
    Update j ← j + 1;
    if n_P < 3 then
        Put B = 0;
    else
        Put B = 1;
    end
    if B == 1 then
        Draw a realization t_p from T_p, t_r from T_r, t_f from T_f and t_q from T_q;
        Put t[j] = t[j − 1] + min{t_p, t_r, t_f, t_q};
        Put x[j] = label of argmin{t_p, t_r, t_f, t_q};
        if x[j] == 3 then
            Put n_P = 0;
        end
    else
        Draw a realization t_p from T_p, t_f from T_f and t_q from T_q;
        Put t[j] = t[j − 1] + min{t_p, t_f, t_q};
        Put x[j] = label of argmin{t_p, t_f, t_q};
        if x[j] == 1 then
            Update n_P ← n_P + 1;
        end
    end
end
```

Algorithm 5: The algorithm to get one chain realization in the linear exponential kernel case. The random variables T_p, T_r, T_f and T_q are shown in Table 5.3. The termination label is 0, the propagation label is 1, the freezing label is 2, the backbiting label is 3.

5.3.4 The simulations

The simulations can be run following the algorithm explained into Section 5.3.3 with the fixed parameters d, p, r, f and q.
In particular, we choose this set of parameters:

$$p = 10\ s^{-1},\ r = 0.2\ s^{-1},\ q = 0.1\ s^{-1}. \tag{5.22}$$

The usual termination rate $q = 0.01\ s^{-1}$ (4.5) makes the simulations very slow, so we change it into $q = 0.1\ s^{-1}$ (5.22), in order to speed up the computation.

To monitor the behavior of the mean ratio between backbiting and propagation events with respect to the $MTTF$, we repeat simulations for different values of parameter $d \in [0; 1]$ and for the values of the freezing rate f from the usual set (4.6).
The results are shown in Figure 5.8, Figure 5.9 and Figure 5.10.

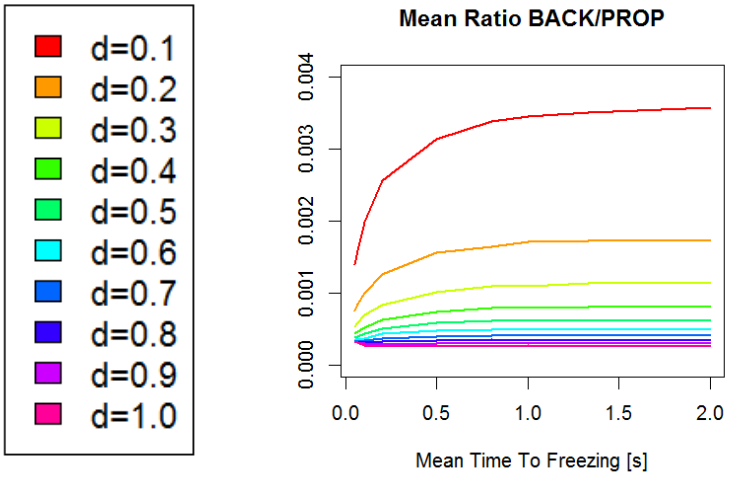

Figure 5.8: Behavior of the mean ratio with respect to the $MTTF$, in the case of linear exponential kernel with rates (5.22), (4.6) and $d \in [0.1, 1]$.

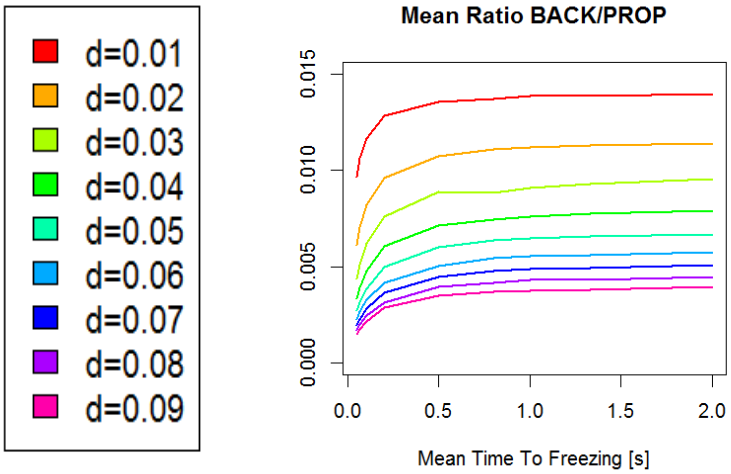

Figure 5.9: Behavior of the mean ratio with respect to the $MTTF$, in the case of linear exponential kernel with rates (5.22), (4.6) and $d \in [0.01, 0.1)$.

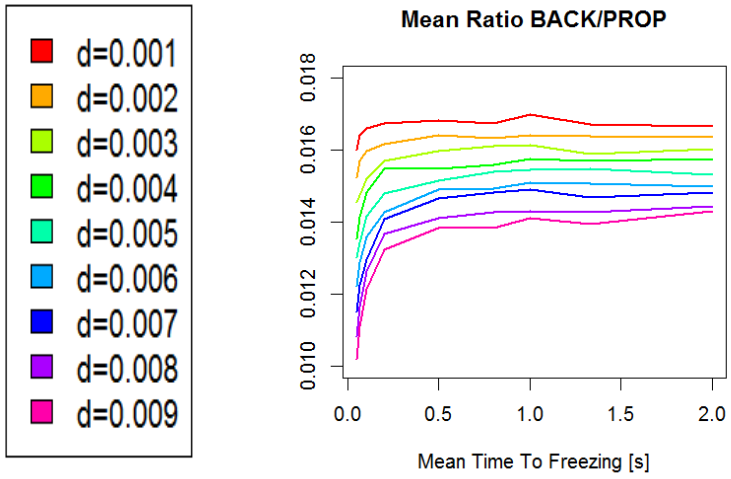

Figure 5.10: Behavior of the mean ratio with respect to the $MTTF$, in the case of linear exponential kernel with rates (5.22), (4.6) and $d \in [0.001, 0.01)$.

If $d \approx 0.01$ we can get the expected behavior, but the mean ratio values are slightly different from the experimental one obtained with the parameters set (4.5).
In fact, there is experimental evidence that the mean ratio reaches the value obtained in the exponential kernel case [Section 4.2] when the $MTTF$ is big. This asymptotic value is ≈ 0.0186 for the parameter set (4.5). The difference could be due to the different choice of the parameter set (5.22).
Then, for smaller d values the mean ratio starts to have a behavior close to the constant one got from the exponential pdf [Section 4.2]. This is because

the pdf (5.55) satisfies the following limit:

$$\lim_{d \to 0} f_T(t) = \lambda \exp(-\lambda t) \quad \forall t \geq 0. \tag{5.23}$$

For $d \to 0$, we can say that $\text{Linexp}(0, \lambda) \approx \text{Exp}(\lambda)$, as shown in Figure 5.11.

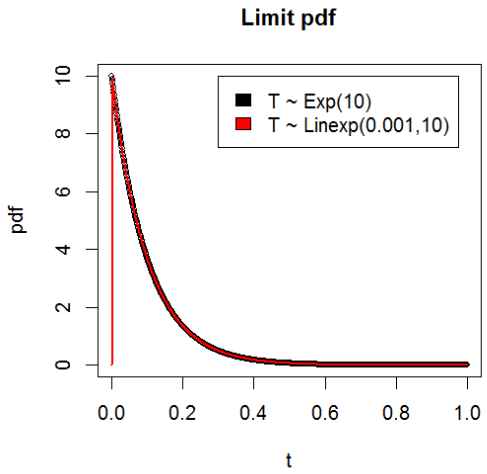

Figure 5.11: The pdf (5.55) behavior when $d \to 0$.

In the end, in order to test the experimental values of the mean ratio, we run few longer simulations with the usual parameters set (4.5) and (4.6). We repeat simulations for different fixed values of parameter d close to 0.01, because it seems to be a good choice from the previous simulations.
The results are shown in Figure 5.12.

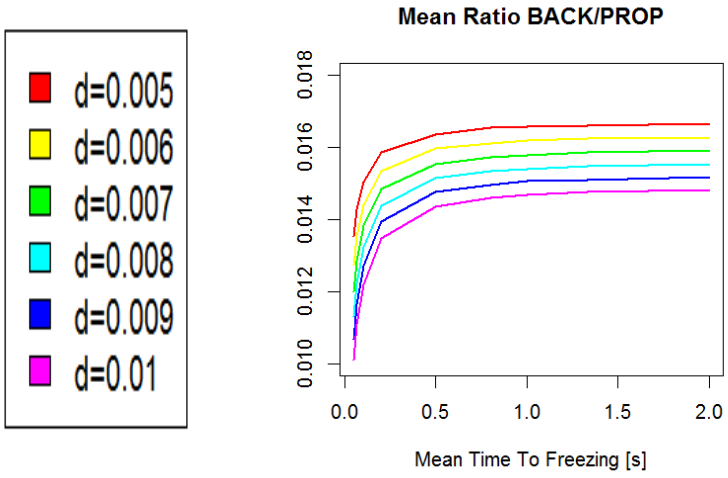

Figure 5.12: Behavior of the mean ratio with respect to the $MTTF$, in the case of linear exponential kernel with the usual rates (4.5), (4.6) and $d \in [0.005, 0.01]$.

5.3.5 Experimental data

In this subsection we want to fit experimental data. A way to enrich the model is to assign a different d value to each event required time pdf. So, we assign these pdf in Table 5.4.

event	**required time pdf**
propagation	$T_p \sim \text{Linexp}(d_p, p)$
backbiting	$T_r \sim \text{Linexp}(d_r, r)$
freezing	$T_f \sim \text{Linexp}(d_f, f)$
termination	$T_q \sim \text{Linexp}(d_q, q)$

Table 5.4: The linear exponential pdf for experimental data fitting.

We want to calculate the values for all the parameters in Table 5.4, taking them from the experimental values.

We can compute the rates values as follow:

$$p = k_p[\text{M}] = 34300 \text{ L mol}^{-1}\text{s}^{-1} \text{ 1 mol L}^{-1} = 34300 \text{ s}^{-1}, \tag{5.24}$$

$$r = k_{bb} = 2500 \text{ s}^{-1}, \tag{5.25}$$

$$q = k_{tr}[\text{CTA}] = 1000 \text{ s}^{-1}. \tag{5.26}$$

Then, we know that:

- reasonable concentrations of the freezing agent, or CRP agent (CRPA), range from 5×10^{-6} mol L^{-1} to 10^{-3} mol L^{-1},
- the constant k for CRPA is $k_f = 10^8$ L mol^{-1} s^{-1},

thus, the freezing rate $f = k_f[\text{CRPA}]$ ranges from 5×10^2 s^{-1} to 10^5 s^{-1}. In our simulations, we choose this set of values for f:

$$f \in \{100000, 75000, 50000, 25000, 10000, 7500, 5000, 2500, 1000, 750, 500\} \text{ s}^{-1}. \tag{5.27}$$

We need to compute the parameters d for each kind of event. We remember that:

$$d = b\lambda, \tag{5.28}$$

where the parameter $b = \frac{d}{\lambda}$ is the border between linear and exponential behavior of the pdf (5.9).
We need the values of d_p, d_r, d_f and d_q, that are the parameters d respectively for propagation, backbiting, freezing and termination.
In order to compute the different values of d, we can use the parameters values valid for all reagents concentrations equal to 1 mol L^{-1}: b'_p, b'_r, b'_f and b'_q

respectively are the borders between linear and exponential behavior of the propagation, backbiting, freezing and termination pdf for $[\,] = 1$ mol L^{-1}. Thus, the d values are:

$$d_p = b'_p p = b'_p k_p[\mathrm{M}] = 4 \times 10^{-6}\mathrm{s}\ 34300\mathrm{L\ mol^{-1}s^{-1}1mol\ L^{-1}} = 0.1372, \quad (5.29)$$

$$d_r = b'_r r = 3 \times 10^{-4}\mathrm{s}\ 2500\mathrm{s^{-1}} = 0.75, \quad (5.30)$$

$$d_f = b'_f f = b'_f k_f[\mathrm{CRPA}] = 10^{-9}\mathrm{s}\ 10^{8}\mathrm{L\ mol^{-1}s^{-1}1mol\ L^{-1}} = 0.1, \quad (5.31)$$

$$d_q = b'_q q = b'_q k_{tr}[\mathrm{CTA}] = 10^{-8}\ \mathrm{s}\ 12900\ \mathrm{L\ mol^{-1}\ s^{-1}1\ mol\ L^{-1}} = 0.000129. \quad (5.32)$$

Table 5.5 shows the resulting parameters set.

event	d	λ
propagation	$d_p = 0.1372$	$p = 34300s^{-1}$
backbiting	$d_r = 0.75$	$r = 2500s^{-1}$
freezing	$d_f = 0.1$	$f \in (5.27)$
termination	$d_q = 0.000129$	$q = 1000s^{-1}$

Table 5.5: The resulting set of parameters for experimental data fitting.

Figure 5.13 shows the behavior of the mean ratio between backbiting and propagation with respect to the Mean Time To Freezing ($= 1/f$), when we run the simulations following the algorithm explained into Section 5.3.3, with pdf from Table 5.4 and parameters set from Table 5.5.

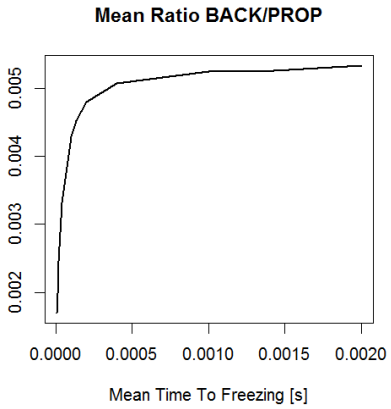

Figure 5.13: The mean ratio behavior with respect to the $MTTF$, in the case of linear exponential kernel for experimental data fitting.

5.3.6 The non-constant rate

The aim of this Section is to remark that, in order to reproduce the experimental observations, we have to ease the hypothesis that the event rates are constant, fixing the time and the chain configuration realized after each event. As explained into Section 4.2.3, we have to choose pdf with non-constant rate of occurrence, as the linear exponential one shown in Section 5.3:

$$f_T(t) = \begin{cases} \frac{2\lambda^2\left(3d-3+\sqrt{9-3d^2}\right)}{d^2\left(\sqrt{9-3d^2}-d+3\right)}t & \text{if } 0 \leq t < \frac{d}{\lambda} \\ \frac{2\lambda\left(3d-3+\sqrt{9-3d^2}\right)}{d\left(\sqrt{9-3d^2}-d+3\right)} \exp\left[-\frac{\lambda\left(3d-3+\sqrt{9-3d^2}\right)}{d(3-2d)}\left(t-\frac{d}{\lambda}\right)\right] & \text{if } t \geq \frac{d}{\lambda} \end{cases}. \tag{5.33}$$

In equation (5.33), λ is the reciprocal of the T mean value, but it is not its constant rate of occurrence! In fact, the T pdf (5.33) does not have a constant rate of occurrence.
We define the T occurrence rate at time t as the so called "hazard function", usually indicated by the symbol $h_T(t)$ [22]:

$$h_T(t) := \lim_{dt \to 0} \frac{\mathbb{P}(t < T \leq t + dt | T > t)}{dt} = \frac{f_T(t)}{1 - F_T(t)}, \tag{5.34}$$

where:

$$F_T(t) := \mathbb{P}(T \leq t). \tag{5.35}$$

In the case of exponential pdf, T has a constant rate of occurrence equal to λ:

$$T \sim \text{Exp}(\lambda) \quad \Leftrightarrow \quad h_T(t) = \lambda \quad \forall t \in \mathbb{R}^+. \tag{5.36}$$

In our case, the T pdf (5.33) means that:

$$h_T(t) = \begin{cases} \frac{2\lambda^2\left(3d-3+\sqrt{9-3d^2}\right)t}{d^2\left(\sqrt{9-3d^2}-d+3\right)-\left(3d-3+\sqrt{9-3d^2}\right)\lambda^2t^2} & \text{if } 0 \leq t < \frac{d}{\lambda} \\ \frac{2\lambda\left(3d-3+\sqrt{9-3d^2}\right)}{6d-4d^2} & \text{if } t \geq \frac{d}{\lambda} \end{cases}, \tag{5.37}$$

that it is a non constant rate of occurrence with respect to time t.

5.4 Non-Markovian effects in the growth of a polymer chain

In Section 4.2, we understand that the model based on memoryless exponential pdf is not able to produce the desired behavior of the mean ratio backbiting over propagation. The generated Poisson process is not able to correctly reproduce the evolution of the reaction. This suggests looking at other models which are not Markovian, as will be done below.
In this section, we want to analyze some non Markovian effects that can affect the growth of a polymer chain [9].

5.4.1 One event growth process

First, we focus on a process with only one possible event: the linear growth event in which one monomers is added to the chain. We introduce the random variable T in order to model the requested time for the next growth event.
We assume that the T probability density function (pdf) is the delayed exponential pdf (5.38): it means that the next event requires a set up time τ, then it can occur with exponentially decaying probability.

$$f_T(t) = \begin{cases} 0 & \text{if } 0 \leq t < \tau \\ c \exp\left[-c\left(t-\tau\right)\right] & \text{if } t \geq \tau \end{cases} \quad \text{for } \tau, c \geq 0. \tag{5.38}$$

If T follows the pdf (5.38), we say that:

$$T \sim \text{Dexp}(c, \tau) \tag{5.39}$$

Let $\underline{t}$ be a vector that holds each event time occurrence: it means that $\underline{t}[j]$ is the time when the j^{th} event occurs.
Let t_{max} be the maximum time for a simulation: we decide to stop the simulation when an event happens after t_{max}.
Algorithm 6 explains how to get one realization from the growth process. In order to draw realizations from this delayed exponential random variable, we implement the well known "Inverse Transform Sampling Method" [21].

```
Draw a realization t from T ~ Dexp(c, 0);
Put t[1] = t;
Initialize j = 1;
while t[j] < t_max do
    Update j ← j + 1;
    Draw a realization t from T ~ Dexp(c, τ);
    Put t[j] = t[j − 1] + t;
end
```

Algorithm 6: The algorithm to draw a realization from the one event growth process.

In order to monitor the mean number of event with respect to time, we draw many realizations from the growing process and we compute statistics on the sample got.
We run the simulations using the following parameters set:

$$c = 1 \text{ s}^{-1} \tag{5.40}$$

$$\tau \in \{0 \text{ s}, 0.5 \text{ s}, 1 \text{ s}, 2 \text{ s}, 3 \text{ s}, 5 \text{ s}, 8 \text{ s}, 10 \text{ s}\} \tag{5.41}$$

The results are shown in Figure 5.14.

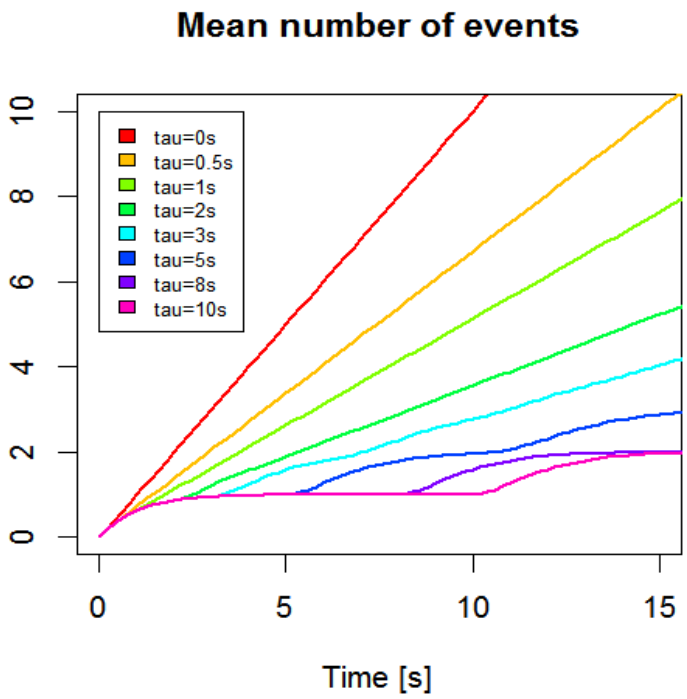

Figure 5.14: Time behavior for the mean number of occurred events, in the one event growth process with parameters $c = 1 \text{ s}^{-1}$ and $\tau \in$ (5.41).

It is worthy to remark that the process is a Poisson one in the case $\tau = 0$ s and the mean number of event is linear with respect to time. Whereas, the bigger is τ, the more the process is far from the Poisson one and the mean number of event starts to assume a strong non linear behavior.

5.4.2 Two competitive events growth process

In this section, we focus on a more complex growth process characterized by two competitive events. In this process two different event can occur: growth and bending. The growth event means that one monomer is added to the chain, whereas the bending event means that the last monomers of the chain moves to a position perpendicular to the last branch. This growth process must satisfy the following constraint: the bending event need at least three previous growth events to occur.

The first model: smooth pdf with implicit constraint

The first model for this process describes the requested time for the next bending with the random variable T.
Let a be the growth rate and let c be the bending rate, we assume that the requested time for a growth event is $T_g \sim \text{Exp}(a)$:

$$f_{T_g}(t) = a \exp(-at) \quad \forall t \geq 0, a > 0. \tag{5.42}$$

Thus, we can write down the pdf for the third growth event requested time equal to τ:

$$w_3(\tau) = \int_0^\tau dt_2 \int_0^{t_2} dt_1 f_{T_g}(\tau - t_2) f_{T_g}(t_2 - t_1) f_{T_g}(t_1) = a^3 \frac{\tau^2}{2} \exp(-a\tau). \tag{5.43}$$

Let τ be the time in which the chain adds three monomers, we assume the following pdf for a bending requested time equal to t:

$$f_{T_b}(t, \tau) = \theta(t - \tau) c \exp[-c(t - \tau)] \quad \forall t, \tau \geq 0, c > 0. \tag{5.44}$$

Then, averaging the delayed pdf for bending (5.44) over all possible delays τ, it's possible to get the pdf for the next bending requested time:

$$\begin{aligned} f_T(t) &= \int_0^{+\infty} w_3(\tau) f_{T_b}(t, \tau) d\tau = \\ &= -\frac{a^3 c}{2(a-c)} \left[t^2 + \frac{2}{a-c} t + \frac{2}{(a-c)^2} \right] \exp(-at) + \frac{a^3 c}{(a-c)^3} \exp(-ct). \end{aligned} \tag{5.45}$$

The cumulative density function (cdf) $F_T(t) := \mathbb{P}(T \leq t)$ is the following:

$$\begin{aligned} F_T(t) = &\left[\frac{a^2 c}{2(a-c)} t^2 + \left(\frac{ac}{a-c} + \frac{a^2 c}{(a-c)^2} \right) t + \left(\frac{c}{a-c} + \frac{ac}{(a-c)^2} + \frac{a^2 c}{(a-c)^3} \right) \right] \times \\ &\times \exp(-at) - \frac{a^3}{(a-c)^3} \exp(-ct) + 1. \end{aligned} \tag{5.46}$$

Let $\underline{t}$ be a vector that holds each bending time occurrence: it means that $\underline{t}[j]$ is the time when the j^{th} bending occurs.
Let t_{max} be the maximum time for a simulation: we decide to stop the simulation when a bending happens after t_{max}.
Algorithm 7 explains how to get one realization from the growth process.

Draw a realization t from $T \sim f_T(t)$ (5.45);
Put $\underline{t}[1] = t$;
Initialize $j = 1$;
while $\underline{t}[j] < t_{max}$ **do**
    Update $j \leftarrow j + 1$;
    Draw a realization t from $T \sim f_T(t)$ (5.45);
    Put $\underline{t}[j] = \underline{t}[j-1] + t$;
end

Algorithm 7: The algorithm to draw a realization from the two competitive events growth process, modeled by a smooth pdf with implicit constraint (5.45).

In order to monitor the mean number of bending with respect to time, we draw many realizations from the growing process and we compute statistics on the sample got.

In order to implement Algorithm 7 and to simulate the growth process, we must be able to draw realizations from the pdf (5.45). The idea is to follow the Inverse Transform Method [21] explained in Algorithm 8.

Draw a realization u from $U \sim \text{Unif}(0, 1)$;
Put $t = F_T^{-1}(u)$;

Algorithm 8: The Inverse Transform Method [21] to get a realization t from the random variable T: the function $F_T^{-1}(u)$ is the inverse of the T cdf.

The cdf (5.46) is not analytically invertible, thus we must use a numeric method to solve the equation $t = F_T^{-1}(u)$. The idea is to see the cdf (5.46) as a increasing monotonic function and, then, to use the Newton method in order to have a good approximation of the desired t. Algorithm 9 explains how to invert the cdf (5.46).

Choose a value $u \in (0, 1)$;
Initialize $t = 0$;
Initialize the value of the time step $\Delta > 0$;
Initialize the value of the tolerance $\varepsilon > 0$;
Initialize $j = 1$;
while $u > F_T(t)$ **do**
  Update $t = j\Delta$;
  Update $j \leftarrow j + 1$;
end
while $|F_T(t) - u| > \varepsilon$ **do**
  Update $t \leftarrow t - \frac{F_T(t)-u}{f_T(t)}$;
end

Algorithm 9: The algorithm to compute the value of $t = F_T^{-1}(u)$, given the value of $u \in (0, 1)$.

In Figure 5.15 we show the sample drawn with the proposed algorithm. We compare it with the pdf (5.45) in order to verify the goodness of the sampling method.

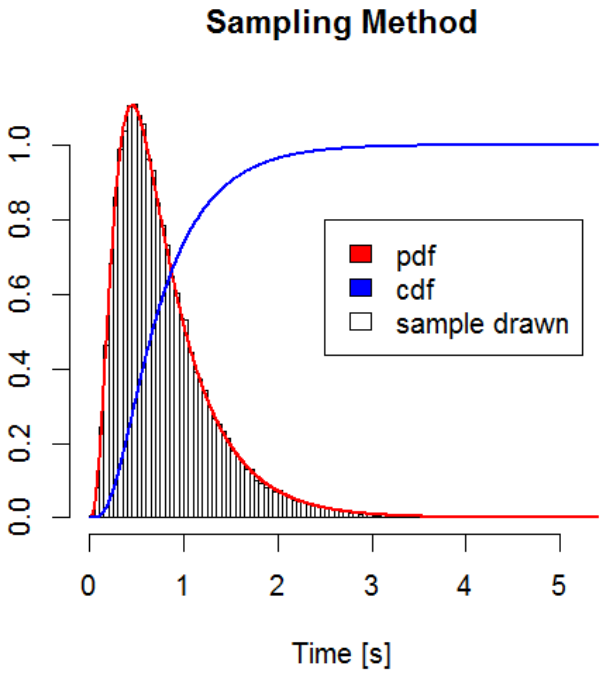

Figure 5.15: Comparison between the sample drawn, the pdf (5.45) and the cdf (5.46).

We run the simulations using the following parameters set:

$$c = 2 \text{ s}^{-1}, \tag{5.47}$$

$$a \in \{10 \text{ s}, 7.5 \text{ s}, 5 \text{ s}, 2.5 \text{ s}, 1 \text{ s}, 0.5 \text{ s}\}. \tag{5.48}$$

The results are shown in Figure 5.16.

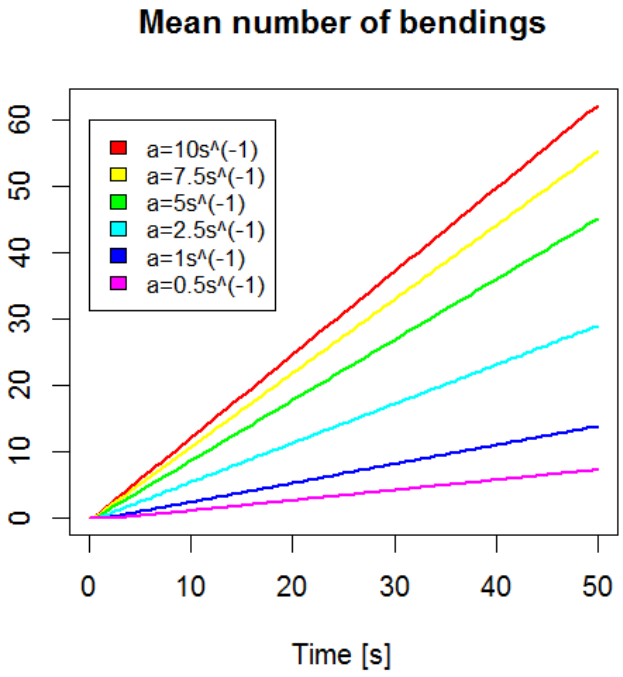

Figure 5.16: Time behavior for the mean number of bendings, in the two competitive events growth process modeled by a smooth pdf with implicit constraint. The used set of parameters is $c = 2 \text{ s}^{-1}$ and $a \in$ (5.48).

The second model: explicit constraint

The second model for this process describes the requested time for a growth with the random variable $T_g \sim \text{Exp}(a)$ and the requested time for a bending with the random variable $T_b \sim \text{Exp}(c)$.
The next event is the one that realizes the minimum among the requested times for all the possible next events. The process must explicitly fulfill the constraint that a bending need at least three previous propagation to occur.
Algorithm 10 explains how to get a chain realization $(\underline{x}, \underline{t})$: $\underline{x}[j]$ is the label of the j^{th} event (1 for growth, 3 for bending) and $\underline{t}[j]$ is the time occurrence of the j^{th} event.

Draw a realization t_g from $T_g \sim \text{Exp}(a)$;
Put $\underline{t}[1] = t_g$;
Put $\underline{x}[1] = 1$;
Initialize the flag for possibility of bending $B = 0$;
Initialize the number of growths in the current branch $n_G = 1$;
Initialize t_{max} as the maximum time for a simulation;
Initialize $j = 1$;
while $\underline{t}[j] < t_{max}$ **do**
  Update $j \leftarrow j + 1$;
  if $n_G < 3$ **then**
    Put $B = 0$;
  else
    Put $B = 1$;
  end
  if $B == 1$ **then**
    Draw a realization t_g from $T_g \sim \text{Exp}(a)$ and t_b from $T_b \sim \text{Exp}(c)$;
    Put $\underline{t}[j] = \underline{t}[j-1] + \min\{t_g, t_b\}$;
    Put $\underline{x}[j] =$ label of $\text{argmin}\{t_g, t_b\}$;
    if $\underline{x}[j] == 3$ **then**
      Put $n_G = 0$;
    end
  else
    Draw a realization t_g from $T_g \sim \text{Exp}(a)$;
    Put $\underline{t}[j] = \underline{t}[j-1] + t_g$;
    Put $\underline{x}[j] = 1$;
    Update $n_G \leftarrow n_G + 1$;
  end
end

Algorithm 10: The algorithm to get one chain realization from the two competitive events growth process, modeled by pdf with explicit constraint.

We run this simulation using the same parameter set for the previous simulation. The results are shown in Figure 5.17.

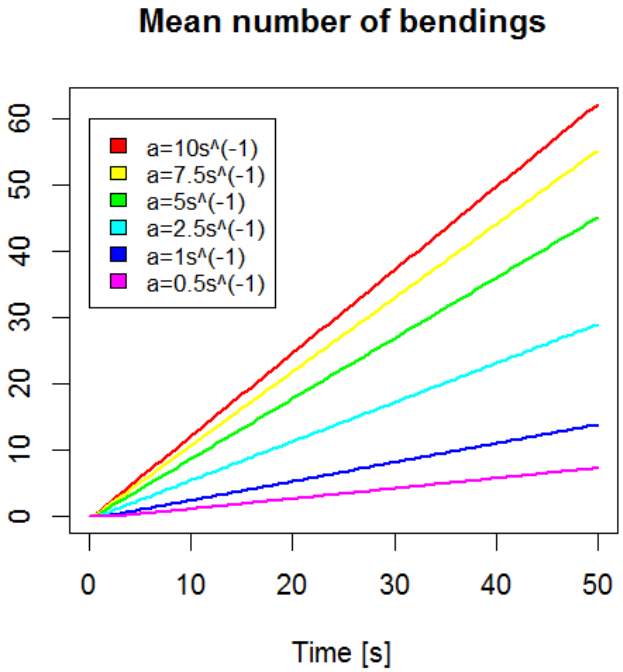

Figure 5.17: Time behavior for the mean number of bendings, in the two competitive events growth process modeled by pdf with explicit constraint. The used set of parameters is $c = 2\ \mathrm{s}^{-1}$ and $a \in$ (5.48).

Comparison

In Figure 5.18, we simply compare the results shown in Figure 5.16 and Figure 5.17.

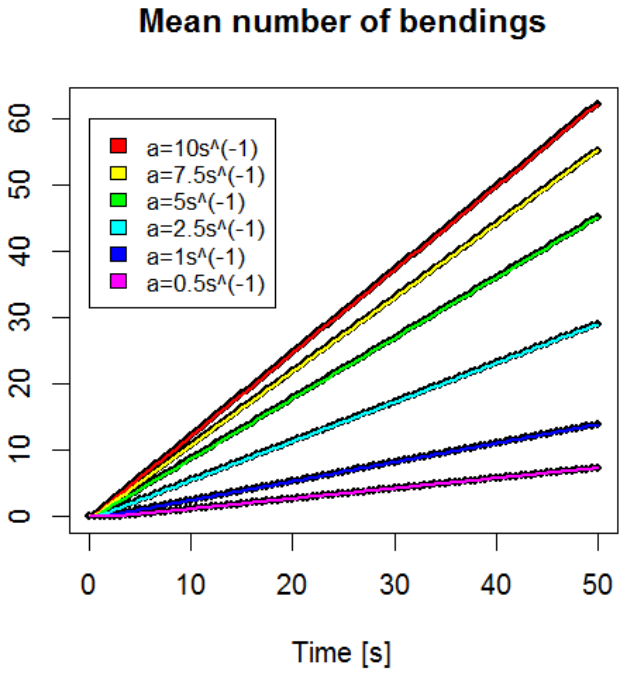

Figure 5.18: Comparison between the two stochastic simulations: the colored lines are the solutions for the first model (smooth pdf with implicit constraint), whereas the corresponding black dots are the solutions for the second model (explicit constraint). The value of c always is equal to $2s^{-1}$ and a takes values into (5.48).

5.5 Releasing delay effect

Controlled Radical Polymerization with a freezing agent is mainly made by the successive occurrence of the following events:

- propagation, with rate p,
- backbiting, with rate r,
- freezing, with rate f,
- termination, with rate q.

The occurrence of the freezing event means that the freezing agent is added to the chain. The chain can add the next monomer if and only if the freezing agent is released. Thus, it means that, after a freezing event, some delay or a set up time are needed to restart the reaction. The aim of this section is to model this delaying phenomenon taking into account the simulations given by Section 5.4. In this section we understand that the non Poisson properties of the growing process are the reasons for the increasing behavior shown in Figure 4.1.

5.5.1 The algorithm

Let the next event be the one that realizes the minimum among the times needed for all possible events.
Thus, in order to choose the next event, we follow these steps:

1. Draw a realization from each pdf that models the time needed for each possible next event, following the well known "Inverse Transform Sampling Method" [21].
2. Pick the event that realizes the minimum occurrence time.

The following general rules should be taken into account:

- Backbiting always requires at least 3 previous propagations to occur.
- The termination event ends each chain sampling.

Thus, we consider the random variables T_p, T_r, T_f and T_q: they respectively are the requested time for a propagation, backbiting, freezing and termination. We implement Algorithm 11 in order to draw a chain realization $(\underline{x}, \underline{t})$. It means that the j^{th} element of vector $\underline{x}$ is the label of the j^{th} event (0 for termination, 1 for propagation, 2 for freezing and 3 for backbiting). The j^{th} element of vector $\underline{t}$ is the occurrence time of the j^{th} event.
In order to monitor the behavior of the mean ratio backbiting over propagation with respect to the $MTTF$, we draw many chain realizations and we compute statistics on the sample got.

Draw a realization t_p from T_p, t_f from T_f and t_q from T_q;
Put $\underline{t}[1] = \min\{t_p, t_f, t_q\}$;
Put $\underline{x}[1]$ = label of argmin$\{t_p, t_f, t_q\}$;
Initialize the flag for possibility of backbiting $B = 0$;
Set the constraint $C = 3$;
Initialize the number of propagations in the current branch $n_P = 0$;
Initialize $j = 1$;
if $\underline{x}[1] == 1$ **then**
  Put $n_P = 1$;
end
while $\underline{x}[j] \neq 0$ **do**
  Update $j \leftarrow j + 1$;
  if $n_P < C$ **then**
    Put $B = 0$;
  else
    Put $B = 1$;
  end
  if $B == 1$ **then**
    Draw a realization t_p from T_p, t_r from T_r, t_f from T_f and t_q from T_q;
    Put $\underline{t}[j] = \underline{t}[j-1] + \min\{t_p, t_r, t_f, t_q\}$;
    Put $\underline{x}[j]$ = label of argmin$\{t_p, t_r, t_f, t_q\}$;
    if $\underline{x}[j] == 3$ **then**
      Put $n_P = 0$;
    end
  else
    Draw a realization t_p from T_p, t_f from T_f and t_q from T_q;
    Put $\underline{t}[j] = \underline{t}[j-1] + \min\{t_p, t_f, t_q\}$;
    Put $\underline{x}[j]$ = label of argmin$\{t_p, t_f, t_q\}$;
    if $\underline{x}[j] == 1$ **then**
      Update $n_P \leftarrow n_P + 1$;
    end
  end
end

Algorithm 11: The algorithm to get one chain realization, in the case of the kernels with releasing delay effects.

5.5.2 Set up time after freezing

Let T be the requested time for a possible future event with rate λ. Its pdf could be the delayed exponential pdf (5.49): it means that the next event requires a set up time τ, then it can occur with exponentially decaying probability.

$$f_T(t) = \begin{cases} 0 & \text{if } 0 \leq t < \tau \\ \lambda e^{-\lambda(t-\tau)} & \text{if } t \geq \tau \end{cases} \quad \text{for } \tau, \lambda \geq 0. \tag{5.49}$$

If T follows the pdf (5.49), we say that:

$$T \sim \text{Dexp}(\lambda, \tau) \tag{5.50}$$

An important remark is that the T pdf comes back to the pure exponential $\text{Exp}(\lambda)$ in the case $\tau = 0$.

The occurrence of the freezing event means that the freezing agent is added to the chain. The reaction can restart if and only if the freezing agent is released. Thus, it means that, after a freezing event, the set up times and the parameters τ are strictly bigger than zero.
After the other events occurrence, set up times are not needed, so the parameters τ are equal to zero.
So, following this idea, we assign the pdf to the times required for each event in Table 5.6.

event	**pdf after freezing**	**pdf after events different from freezing**
propagation	$T_p \sim \text{Dexp}(p, \tau_p)$	$T_p \sim \text{Dexp}(p, 0) = \text{Exp}(p)$
backbiting	$T_r \sim \text{Dexp}(r, \tau_r)$	$T_r \sim \text{Dexp}(r, 0) = \text{Exp}(r)$
freezing	$T_f \sim \text{Dexp}(f, \tau_f)$	$T_f \sim \text{Dexp}(f, 0) = \text{Exp}(f)$
termination	$T_q \sim \text{Dexp}(q, \tau_q)$	$T_q \sim \text{Dexp}(q, 0) = \text{Exp}(q)$

Table 5.6: The exponential pdf delayed by freezing event.

Following the rules given in Section 5.5.1, we can implement Algorithm 11 with the pdf from Table 5.6 and the fixed parameters p, r, f, q and τ_i.
We run the simulations choosing this set of parameters:

$$p = 10\ s^{-1},\ r = 0.2\ s^{-1},\ q = 0.01\ s^{-1}, \tag{5.51}$$

$$\tau_p = 1\ s,\ \tau_r = 5\ s,\ \tau_f = 1\ s,\ \tau_q = 1\ s. \tag{5.52}$$

We repeat simulations for parameters f from the following set:

$$f \in \left\{20s^{-1}, 15s^{-1}, 10s^{-1}, 5s^{-1}, 2s^{-1}, 1.25s^{-1}, 1s^{-1}, 0.75s^{-1}, 0.5s^{-1}\right\}. \tag{5.53}$$

In Figure 5.19 we show the behavior of the mean ratio backbiting over propagation with respect to the $MTTF$ $(= 1/f)$.

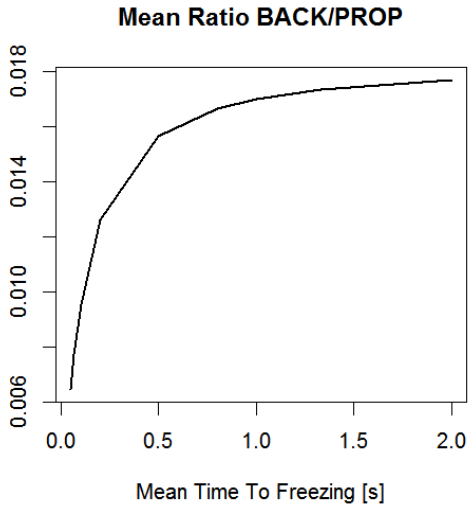

Figure 5.19: Behavior of the mean ratio with respect to the $MTTF$, in the case of exponential kernel delayed by freezing event.

An important remark is that the backbiting set up time τ_r is bigger than the others. This is essential in order to reproduce the desired behavior of the mean ratio.
In order to verify this, we repeat the simulation described above, but we assign to all the set up times τ_p, τ_r, τ_f and τ_q the same value equal to 1s. In this case, we get a constant behavior of the mean ratio, within the error of $\propto 10^{-5}$. The results are shown in Figure 5.20.

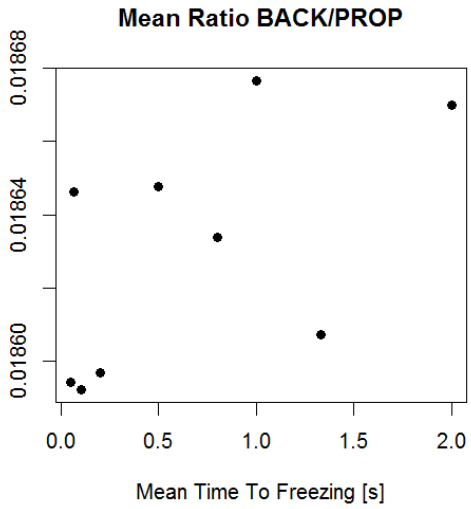

Figure 5.20: Constant behavior of the mean ratio with respect to the $MTTF$, in the case of delayed exponential kernel with equal set up times τ_i.

5.5.3 Linexp pdf after freezing

Let T be the required time for a possible future event with rate λ. We assign its pdf as follow:

$$f_T(t) = \begin{cases} \frac{2\lambda^2\left(3d-3+\sqrt{9-3d^2}\right)}{d^2\left(\sqrt{9-3d^2}-d+3\right)}t & \text{if } 0 \leq t < \frac{d}{\lambda} \\ \frac{2\lambda\left(3d-3+\sqrt{9-3d^2}\right)}{d\left(\sqrt{9-3d^2}-d+3\right)} \exp\left[-\frac{\lambda\left(3d-3+\sqrt{9-3d^2}\right)}{d(3-2d)}\left(t-\frac{d}{\lambda}\right)\right] & \text{if } t \geq \frac{d}{\lambda} \end{cases}, \tag{5.54}$$

Thus, it means that:

$$T \sim \text{Linexp}(d, \lambda). \tag{5.55}$$

An important remark is that the T pdf comes back to the pure exponential $\text{Exp}(\lambda)$ in the case $d = 0$. In the case $d > 0$, the linear behavior for $t < \frac{d}{\lambda}$ models the needed delay after a freezing event.

We assume that backbiting is the only event affected by delay if the previous event is a freezing event. An intuitive justification for this assumption is given in Section 5.5.2, where we can get the desired behavior if and only if backbiting is affected by freezing more than the other events.
So, following this idea, we assign the pdf to the times required for each event in Table 5.7.

event	**pdf after freezing**	**pdf after events different from freezing**
propagation	$T_p \sim \text{Linexp}(0, p) = \text{Exp}(p)$	$T_p \sim \text{Linexp}(0, p) = \text{Exp}(p)$
backbiting	$T_r \sim \text{Linexp}(d_r, r)$	$T_r \sim \text{Linexp}(0, r) = \text{Exp}(r)$
freezing	$T_f \sim \text{Linexp}(0, f) = \text{Exp}(f)$	$T_f \sim \text{Linexp}(0, f) = \text{Exp}(f)$
termination	$T_q \sim \text{Linexp}(0, q) = \text{Exp}(q)$	$T_q \sim \text{Linexp}(0, q) = \text{Exp}(q)$

Table 5.7: The linear exponential pdf as delay after freezing event.

We run the simulations following the rules given in Algorithm 11 with the pdf from Table 5.7 and the fixed parameters p, r, f, q and d_r.
In particular, we choose this set of parameters:

$$p = 10\ s^{-1},\ r = 0.2\ s^{-1},\ q = 0.01\ s^{-1}, \tag{5.56}$$

$$d_r = 0.1. \tag{5.57}$$

We repeat simulations for parameters f from the following set:

$$f \in \left\{20s^{-1}, 15s^{-1}, 10s^{-1}, 5s^{-1}, 2s^{-1}, 1.25s^{-1}, 1s^{-1}, 0.75s^{-1}, 0.5s^{-1}\right\}. \tag{5.58}$$

In Figure 5.21 we show the behavior of the mean ratio backbiting over propagation with respect to the $MTTF$ $(= 1/f)$.

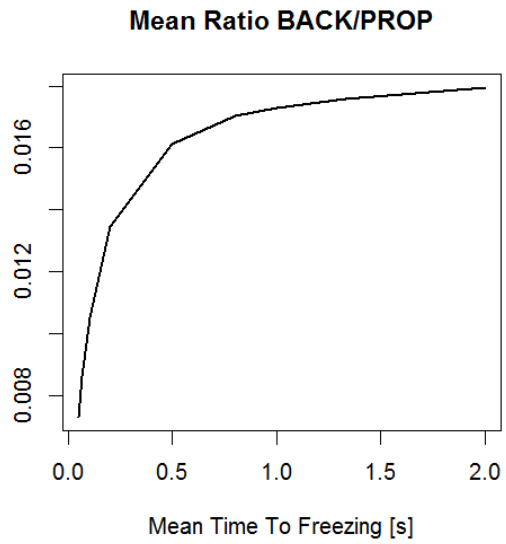

Figure 5.21: Behavior of the mean ratio with respect to the $MTTF$, in the case of linear exponential kernel as delay after freezing event.

5.5.4 Constraint effect

In this section we model the delay after freezing event with the constraint that backbiting has to satisfy. If the previous event is a freezing event, backbiting needs a least $C = 50$ propagations to occurs, otherwise it needs the usual $C = 3$ propagations.
In order to avoid compensation effects we assign the pure exponential pdf to the times required for each event. Thus, the needed time for an event with rate λ is always modeled by the random variable $T \sim \text{Exp}(\lambda)$.
We run the simulation following the Algorithm 11 with the pure exponential pdf, but we change the backbiting constraint as explained before.
We choose this parameters set:

$$p = 10\ s^{-1},\ r = 0.2\ s^{-1},\ q = 0.01\ s^{-1}, \tag{5.59}$$

$$f \in \left\{20s^{-1}, 15s^{-1}, 10s^{-1}, 5s^{-1}, 2s^{-1}, 1.25s^{-1}, 1s^{-1}, 0.75s^{-1}, 0.5s^{-1}\right\}. \tag{5.60}$$

In Figure 5.22 we show the behavior of the mean ratio backbiting over propagation with respect to the $MTTF$ $(= 1/f)$.

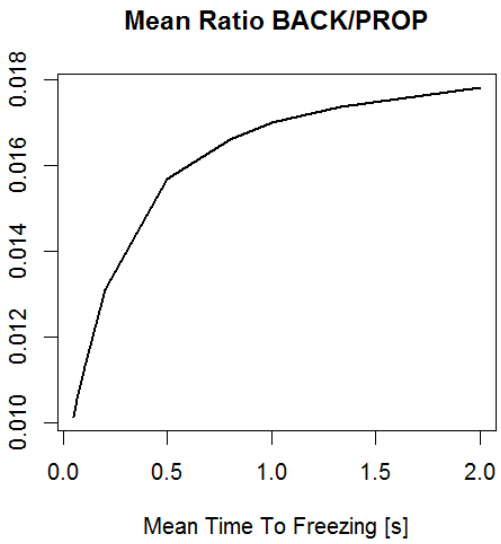

Figure 5.22: Behavior of the mean ratio with respect to the $MTTF$, under the constraint effect.

5.5.5 Important remarks

In this section, we have shown that the desired behavior for the mean ratio is got if the growth process is characterized by non Markovian and non Poisson effects. The freezing event introduces in the reaction some delay or a set up time. This delay is the non Markovian and non Poisson effect that affects the polymer growth process.
It is worthy to remark that the desired behavior for the mean ratio is got if the set up time or the delay affects backbiting more than the other events. In fact, in Section 5.5.2, the backbiting set up time τ_r is bigger than the set up times for the other events. In Section 5.5.3, backbiting is the only event affected by the freezing occurrence. In Section 5.5.4, the freezing occurrence changes the constraint that backbiting has to satisfy. Again, it means that freezing changes the prohibition of occurrence for backbiting, but not for the other events.

Chapter 6

Conclusions

We have studied several mathematical models for Controlled Radical Polymerization. We can divide them in two groups: the models which are Markovian and non-Markovian.

The Markovian models correctly describe the case of polymerization reaction without freezing agent.
The Markovian models studied include the Partial Differential Equations (PDE) approach, explained in Section 3.2, and the original version of the stochastic simulation algorithm (SSA), summarized in Section 3.3.
The PDE approach produces analytical solutions, but it makes strong assumptions, as one needs to approximate the branches discrete length with a continuous variable. These assumptions lead to certain problems. The solutions for discrete quantities are probability density functions (pdf) with respect to the Lebesque measure. The solution for still propagating branches is a delta function of the length, which forces unrealistic deterministic values. The PDE solutions also give non zero probability to impossible events, for example, for the occurrence of backbiting as the first event and the occurrence of two consecutive backbitings.
The original SSA proposed by Gillespie, summarized in Section 3.3, can overcome the problems of PDE approach described above. However, its cost is the computational time required. A PDE analytical solution gives an immediate answer to the problem, whereas the estimations given by SSA need time to be computed.

The Markovian models are not able to correctly reproduce the evolution of the polymerization reaction if a freezing agent is present. For this reason we introduced non-Markovian models.
With a small modification of the original SSA method, we can simulate non-Markovian models and compute their statistics.
As was shown in Section 3.3.2, the original SSA can be reformulated in the way that the next event is the one that realizes the shortest occurrence time among all the possible processes.
Thus, we can choose a pdf of any desired shape in order to model the occurrence time of a process. Then, it is enough to choose the next event to be the

process that realizes the shortest occurrence time. This is the basic idea of the Monte Carlo (MC) approach developed in this thesis.
The just described MC approach is also generic since does not rely on a pdf of any particular type.
In case of an exponential pdf, the method is equivalent to the original SSA.
The choice of a pdf different from the exponential one leads to non Poisson processes and non Markovian models, due to the loss of the memoryless property of a Markovian evolution.
The tested non Markovian models suggest an explanation of why the evolution of the polymerization with freezing agent is not properly described by a Markovian model.
The freezing agent introduces a time delay. As a result, subsequent evolution of the system depends not only on its current state, but also on its recent history.
Thus, the delay is a non Markovian and non Poisson effect which affects the polymerization process in the presence of a freezing agent.

The results are encouraging, but more work is needed. The developed MC approach, tested on different models, will be used to describe existing experimental data for Controlled Radical Polymerization. This will be done in collaboration with J.M. Asua from Basque Center for Macromolecular Design and Engineering (POLYMAT, Donostia-San Sebastián, Spain).

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