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On the Swift-Hohenberg equation with slow and fast dynamics: well-posedness and long-time behavior

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Abstract

In this thesis we discuss the mathematical analysis of the Swift-Hohenberg equation that describes the phenomenon of Rayleigh-Bénard convection in which a fluid is confined between a hot and a cold plate. Thanks to its rich spatio-temporal dynamics, it represents a paradigm in the study of pattern formation. The Swift-Hohenberg equation is also employed in the phase field theory to model the transition from an unstable to a (meta)stable state. We consider a recent generalisation of the original equation, obtained by introducing an inertial term to predict fast degrees of freedom in the system.

We first establish the existence, the uniqueness and the regularity of the solutions with respect to the data to both the equations. Then the solutions are interpreted as dynamical systems in suitable phase space. By making use of the theory of attractors for infinite dimensional dissipative dynamical systems, we analyse the long-time behavior of the solutions. In particular, the main results concern the existence of the global and exponential attractors. Finally, reading the equation with the inertial term as a singular perturbation of the original equation, we prove the upper semicontinuity of the global attractor and we construct a family of exponential attractors which is Hölder continuous with respect to the perturbative parameter of the system.

Sommario

In questa tesi trattiamo l'analisi matematica dell'equazione di Swift- Hohenberg che descrive il fenomeno della convezione di Rayleigh-Bénard in cui un fluido è confinato tra una piastra calda e una fredda. Grazie alla sue ricche dinamiche spazio-temporali, essa rappresenta un paradigma nello studio di formazione di pattern. L'equazione di Swift-Hohenberg viene anche impiegata nella teoria del campo di fase per modellizzare la transizione da uno stato instabile a uno (meta)stabile. Noi consideriamo una recente generalizzazione dell'equazione originale, ottenuta introducendo un termine inerziale per prevedere i gradi di libertà veloci nel sistema.

Anzitutto determiniamo l'esistenza, l'unicità e la regolarità delle soluzioni rispetto ai dati iniziali per entrambe le equazioni. In seguito le soluzioni vengono interpretate come sistemi dinamici in appropriati spazi delle fasi. Facendo uso della teoria degli attratori per sistemi dinamici dissipativi infinito dimensionali, analizziamo il comportamento per tempi grandi delle soluzioni. In particolare, i principali risultati riguardano l'esistenza dell'attratore globale e dell'attratore esponenziale. Infine, leggendo l'equazione con il termine inerziale come una perturbazione singolare dell'equazione originale, proviamola continuità superiore dell'attratore globale e costruiamo una famiglia di attratori esponenziali la quale è Hölder continua rispetto al parametro perturbativo del sistema.

Contents

1 Introduction

The beauty of nature lies in infinite mechanisms which generate fascinating and complex phenomena in the world around us. A scientist is conscious that a model may only describe the main aspects of the phenomenon concerned. At the same time, being an abstraction of the reality, a model can be involved in problems related to different areas. This versatility occurred for the Swift-Hohenberg equation, which has become a famous model in fluid dynamic, in pattern formation and in phase field theory.

Our aim is to explain the importance of the Swift-Hohenberg model in the three different areas of application, following an historical approach. We will also provide a detailed exposition of the general method to deduce this equation and its meaningful variant with the phase field methodology. Lastly, we will change our point of view from the physical modelling to the mathematical analysis with the purpose to motivate the ideas which we will discuss in the following chapters.

1.1 Rayleigh-Bénard convection

The Swift-Hohenberg equation was originally proposed in [17] to describe the effects of thermal fluctuations on the convective instability. In Rayleigh-Bénard experiment, a fluid is confined between two horizontal plates and is heated from the bottom. The heating upsets the thermodynamic equilibrium, the particles of fluid near the bottom plate expand and so their density decrease, due to the absorption of heat. Conversely, the particles near the upper plate lose heat, contract their volume and show a density increase. Then the colder and heavier fluid naturally tends to go toward the heated plate under the gravity force. This motion is contrasted by two dissipative mechanisms: the fluid viscosity and the heat conduction. If the

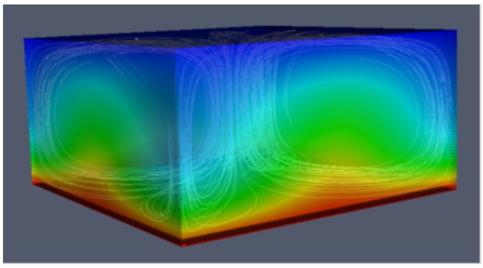

Figure 1.1: Convecting rolls formation in Rayleigh-Bénard experiment.

temperature difference between the two plates is not large enough, these mechanisms have a stabilizing effect and only the diffusion of particles is observable. However, when a critical value of the temperature difference is reached, the phenomenon is unstable and, since all the cold fluid cannot go downward simultaneously, the microscopic random movement becomes a macroscopic pattern consisting in convective parallel rolls. The experimental criterion for the onset of convection is established by the threshold

$$R > R_c \approx 1708 \tag{1.1}$$

where the dimensionless parameter R is the so-called Rayleigh number, defined by

$$R = \frac{\alpha g d^3 \Delta T}{\nu \kappa}. \tag{1.2}$$

In this expression $\alpha = -(1/\rho)(\partial\rho/\partial T)|_p$ is the fluid's coefficient of the thermal expansion at constant pressure, ρ is the density, g represents the gravitational acceleration, d is the distance between the two plates, ΔT is the difference of temperature, ν represents the kinematic viscosity and κ the thermal diffusivity.

In ([17]) the Swift-Hohenberg equation was obtained from the Boussinesq equations for convection, which consider a fluid trapped between two infinite horizontal plates, separated by a distance d, at a temperature T_1 and $T_1 + \Delta T$ respectively. Swift and Hohenberg have considered the linearized equation for the velocity field perpendicular to the plates v_z and $\theta = T - T_1 + (\Delta T/d)_z$, which describes the deviation of the temperature from the uniform gradient $\Delta T/d$. The solution of the linearized equation contains stable and unstable eigenvalues. The Swift-Hohenberg equation was developed considering only the wavelengths near the unstable wavelengths obtained in the linear analysis. In the limit $R \to R_c$, the equation takes the following form

$$\phi_t = [\epsilon - (1+\Delta)^2]\phi - \phi^3 + \eta \tag{1.3}$$

where ϕ represents a dimensionless form of a linear combination of the fluid's velocity and the temperature. The parameter $\epsilon > 0$ is proportional to the distance of the Rayleigh number R with the critical value R_c and acts as a control parameter. η is a Gaussian random field, with zero mean and correlations

$$< \eta(x,t)\eta(x',t') >= 2\lambda\delta(x-x')\delta(t-t'), \tag{1.4}$$

where λ is the intensity of noise. In this present work, we will only consider the deterministic Swift-Hohenberg equation (we refer the reader to [9], [13], [18] for a stochastic analysis).

1.2 Pattern formation

From biology to chemistry, from physic to computer graphic, our eyes have always been attracted by formation of ordered structure in Nature. Self-organized patterns represent an long-standing challenge from mathematical point of view which tries to describe these phenomena, arising from different areas, on the basis of the same mathematical structure.

Numerical simulations and laboratory experiments highlight the surprising aspects of certain dynamical systems where nonlinear field evolves from an irregular initial condition to a regular and symmetric pattern. In many situations, the final state is characterized by repeated steady structure, such as stripes, squares or hexagons, or by time-dependent spirals. In other examples the dynamic can degenerate in a spatio-temporal chaos, namely the nontransient dynamic is bounded but not stationary, periodic or quasi-periodic. The behavior of these systems strongly depends on the value of the parameters, which are involved into the model. Understanding how a certain value may determine the dynamic is not only an interesting issue from a mathematical point of view. Indeed, it represents an important aspect in engineering because the prediction of the dynamic allows to control actively the entire system by applying an external perturbation.

In this framework, numerical calculations have revealed how the Swift-Hohenberg equation possesses a rich spatio-temporal dynamics and soon has become a paradigm model in the study of pattern formation(see [4], [19], [24][25] for more details). Indeed, on the one hand it contains specific symmetries, structures at a preferred length scale, nonlinearity that saturates exponentially growing modes and reproduces qualitative features of the experiments. On the other hand, it is a relatively simplified model to test theoretical formalisms.

A main property of the Swift-Hohenberg equation derives from its form. Equation 1.3 can be written as follow

$$\phi_t = -\frac{\partial F}{\partial \phi} \tag{1.5}$$

where F is a potential which depends on ϕ and has the property that its value decrease monotonically in time. This is an important feature, especially in numerical simulations, because the patterns are determined by the minima of these functional. However, this functional may have a great variety of different minima and the study of this systems become highly interesting.

The simple case of stripes pattern-formation involves the Swift-Hohenberg equation. This pattern has been viewed in magnetic films, liquid crystals and eutectic growth. In particular, numerical simulations are used to understand the influence of the boundary in the orientation of the stripes into the whole domain. As it is shown in Figure 1.2, the wide variety of defects introduced by boundaries and curvature is evident.

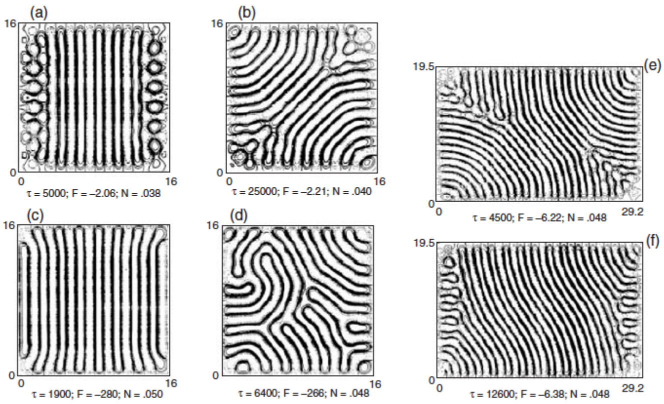

Figure 1.2: Numerical simulations of the Swift-Hohenberg equation with different value of ϵ and time τ. (a), (b), (e) and (f) correspond to $\epsilon = 0.1$, (c) and (d) to $\epsilon = 0.9$. The initial conditions are stripes for (a) and (c) and random for others. (e) is evolving in time, eventually giving the pattern in (f), while the other states have reached a steady state.

The equation is also applied to the coarsening dynamic that occurs in gradient flow. During the evolution, the pattern is characterized by domains of relatively well ordered stripes with an increasing of the average domain size (see Figure 1.3). Furthermore, equation 1.3 has been widely used to understand other problems of pattern formation, namely the dynamics of localized patterns, such as dislocation, grain boundaries and other defects.

To incorporate additional physics aspects observed in nature, the model has been enriched with additional terms. A more generic form of the Swift-Hohenberg equation has been considered to avoid the symmetry $u \to -u$ in (1.3). The simplest way introduces a quadratic nonlinear term

$$\phi_t = [\epsilon - (1 + \Delta)^2]\phi + \gamma\phi^2 - \phi^3 \tag{1.6}$$

A first application of the equation (1.6) in pattern formation occurs in studying supercritical Rayleigh-Bénard convection in two-dimension geometry. With different initial condition and $g_2 > g_{2c}$, the system may reach a pattern with hexagonal structure as highlighted in the here below Figure (1.4).

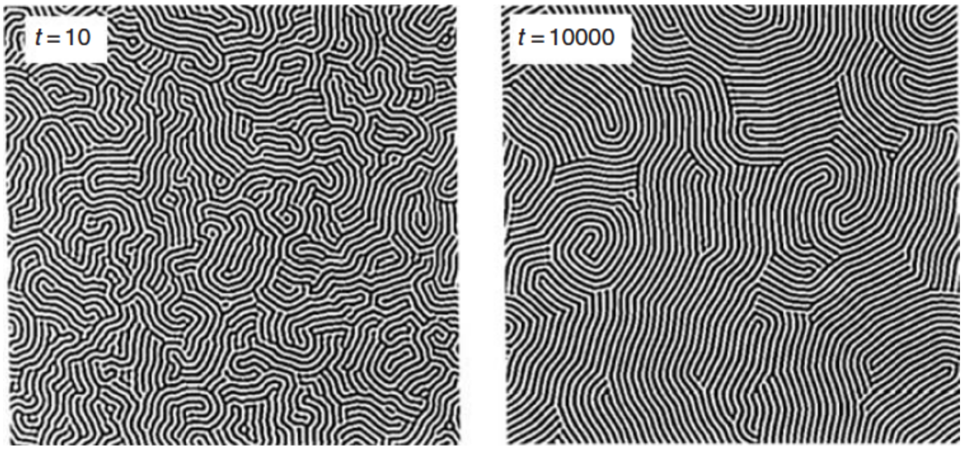

Figure 1.3: Numerical simulation of the Swift-Hohenberg equation in a large periodic geometry with $\epsilon = 0.25$, starting from random initial conditions and stopped at time $t = 10$ and $t = 10000$.

It is also displayed the formation of a crystalline lattice in the whole domain if the initial condition possesses a single hexagonal structure. These observations have led to study equation 1.6 to questions related to the competition between stripe and hexagonal states, wave number selection as well as front propagation.

More complicated generalizations of the Swift-Hohenberg model (1.6), including nonlinear terms of the gradient of the spatial distribution, such as $\phi(\nabla\phi)^2$, $\phi(\Delta\phi)^2$ or $\phi(\Delta\phi^2)$, are involved in crystal formation. The result is the formation of square cells in a circular cavity at a sufficiently large time.

An interesting connection between the Swift-Hohenberg equation and Rayleigh-Bénard convection is the formation of stable rotating spiral into the stable convective rolls. This phenomenon has been studied with the introduction of a new field, the so-called mean flow, which allows to put in touch different regions of the pattern. The importance of mean flow to explain spirals was demonstrated in many experiments. The equation 1.3 is generalized with the inclusion of an advection term

$$\phi_t + U \cdot \nabla\phi = [\epsilon - (1 + \Delta)^2]\phi - \phi^3 \tag{1.7}$$

with mean flow velocity

$$U = \frac{\partial \Psi}{\partial y}\mathbf{e}_x - \frac{\partial \Psi}{\partial x}\mathbf{e}_y \tag{1.8}$$

where Ψ satisfy

$$\left[\frac{\partial}{\partial t} - Pr\left(\Delta - c^2\right)\right]\Delta\Psi = [\nabla(\Delta\phi) \times \nabla\phi] \cdot \mathbf{e}_z. \tag{1.9}$$

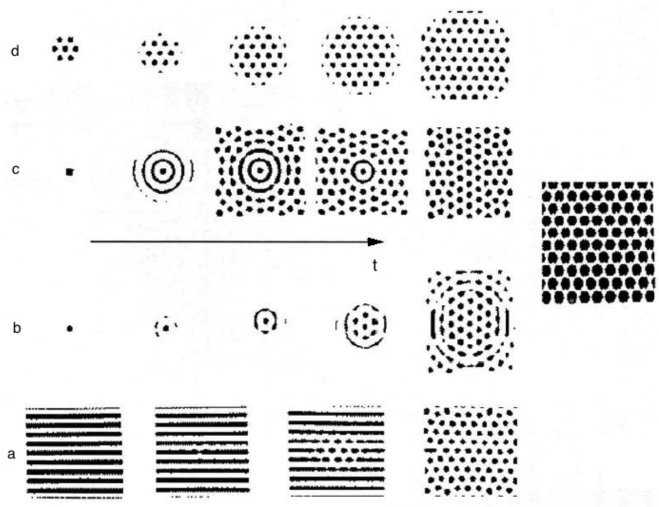

Figure 1.4: Numerical simulations of the Swift-Hohenberg equation showing the evolution of hexagonal structure. The initial conditions are (a) rolls, (b) a disk, (c) a square, (d) seven symmetrically distributed circles.

In this last equation, Pr is the Prandtl number defined by

$$Pr = \frac{\nu}{\kappa} \tag{1.10}$$

and may be viewed as the ratio of the vertical thermal diffusion time τ_κ and the vertical viscous relaxation time τ_ν. Its value depends on the examined fluid. The introduction of the mean flow in system 1.7 and 1.9 eliminates the gradient flow nature of 1.3 and allows also to consider spatio-temporal chaos in accord to experiments (see Figure 1.5).

1.3 Phase-field theory

A relevant aspect of the modern engineering is the study of the materials properties, which establishes the material suitability in applications and provide the quality of performance. The material science investigates the connection between the macroscopic properties, such as strength, ductility, corrosion resistance, hardness, and the microstructures developed at small scale. The particular features of the microstructures arise during the processes of solidification, re-crystallization and thermo-mechanical processing.

Historically, to explain the physics governing such microstructure formation, this phenomenon was modelled by evolution mathematical relations describing the heat diffusion, the transport of impurities and by means complex boundary conditions which take into account the thermodynamic at the interface. In solidification, commonly guaranteed boundary conditions are

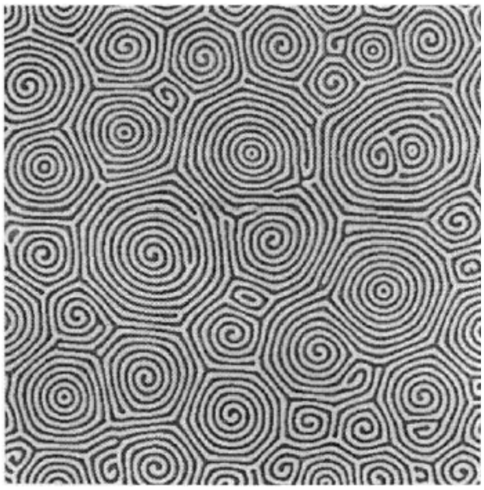

Figure 1.5: Spiral chaos in numerical simulations of the generalized Swift-Hohenberg system (1.7), (1.9).

a balance of flux of heat and the Gibb's Thomson condition for the temperature. This problem was called sharp interface model because the free boundary has zero thickness.

In literature, the most known problems are Stefan or Hele-Shaw problem, respectively for solidification and fluid mechanics. However, the laws at the interface are not easily determined for different phenomena and the analysis, especially the numerical simulation, is difficult when the topology of the interface becomes complicated or multiply connected.

In order to overcome these obstacles, it was introduced a new class of models to describe these phenomena, the so-called diffuse-interface models. The main idea was to consider the interface with a finite thickness where a fast but smooth change of the relevant physical quantities occurs. In this direction, following the basic principles by Poisson, Maxwell and Gibbs, the theory of the gradient in diffuse-interface by Lord Rayleigh, van der Waals and Korteweg, and the phase transition formalism of the Landau theory, a model describing anomalous sound absorption of liquid helium was presented by Landau and Khalatnikov. Subsequently, diffuse-interface models have been developed with the Ginzburg-Landau form of free-energy functional and the theory of critical phenomena by Halperin, Hohenberg and Ma.

The main aspect which characterized these models is the introduction of an additional function, called order parameter or phase-field variable. In solidification, the order parameter ϕ assumes different constant value in each phases, namely $\phi = 1$ in the initial unstable phase which is changing in the final phase with $\phi = -1$. In the intermediate region between the different phases, the order parameter is continuously interpolated. From the physical point of view, the meaning of the new field was to identify the moving interfacial boundary. After, this concept was generalized and actually can be used to describe different phenomena, such as the degree of crystallinity and the atomic order or disorder in a phase (see [8] and reference given there) .

The other two common features which play a relevant role are the free energy functional and the type of dynamic of the phase field. The former acts on the phase field and the other scalar fields, such as temperature, concentration and strain, and explains the thermodynamic connections between the fields. The second aspect describes how the phase-field evolves during the time. A conserved dynamic of phase transition, which takes the form of a flux-conserving equation, implies that an integral of the phase field does not change in time. As a specific example, this evolution occurs in the Cahn-Hilliard equation for spinodal decomposition. Conversely, in nonconserved dynamic, the evolution of the phase field does not fulfil a conservation law, then global quantities can change their value during the time evolution. In this case, the dynamic of ϕ can be commonly described by a Langevin type equation, where the time variation of the phase field is proportional to the variational derivative of the free energy with respect to ϕ. Many interesting examples are isothermal solidification of pure material or magnetic domain growth.

It is worth to point out that it does exists a connection between the phase-filed models and the major models of the sharp interface. Indeed, by means an asymptotic analysis, Caginalp has proved how Stefan and Hele-Shaw models with any set of physical parameters in any dimension can be approximated with arbitrary accuracy by a set of phase-field equations.

In the last decades, the phase-field methodology was successful in explaining various phenomena on the mesoscopic scale, such as solidification, fluid convection, multiphase and multiscale trasformation. Subsequently, it has been introduced a new type of model in the phase-field theory, the so-called phase-field crystal (PFC), which describes the system at atomic length and diffusive time scale in periodic systems. The aim of substituting mesoscopic with microscopic scale in the model is to capture the mechanisms of creation, destruction, and interaction of dislocations in polycrystalline materials. Several PFC models have been proposed to simulation liquid-solid transition, diffusion defects and glass formation.

1.4 Slow and fast dynamics in phase transition

From thermodynamic formalism, intensive properties are features which do not depend on the size of the system. They establish the exchanges within the system and with the outside. Thanks to these properties, the thermodynamic state of equilibrium can be defined as global or local. In the first case, the intensive parameters are homogeneous throughout the whole system. The second case allows to assume the global equilibrium in a neighborhood of any point of the system.

A common hypothesis of the main phase field models is the local equilibrium within a volume of the system, consistently with the ideas of the classic irreversible thermodynamic (CIT). Therefore, only system near an equilibrium can be predicted. The local equilibrium assumption is also valid when the characteristic time scale of the phenomenon is higher than the transient period or the period of intense forced oscillations. It is commonly said that these models describe the slow phase transition .

Recently, a new relevant front of investigation are systems far beyond the equilibrium, characterized by rapid changes in time. In rapid transformation, the local equilibrium is not respected within the bulk phase and at the interface. These systems describe the fast dynamic with the formalism of extended irreversible thermodynamic (EIT). It replaces the classic theory and gives a causal description of transport processes. Moreover, EIT avoids the paradox of propagation of disturbances with an infinite speed. In this framework, the variable selection becomes necessary for an accurate description of a nonequilibrium state. The classical work of a rapid phase transformation within a diffuse interface in a binary system is given by Galenko and Jou in [12].

1.5 The physical model

We explain a general approach to describe non-conserved dynamics with slow and fast phase transition introduced in ([11]).

Let $\phi(\vec{r}, t)$ be the order parameter which defines the state of the system, where $\vec{r}$ is the vector position of a point in the bulk system and t is the time. To describe the evolution from an unstable to a metastable or stable phase state, including short time period and macroscopic time, it is used the prehistory of response between the driving force $\delta F/\delta\phi$ and the first derivative of the order parameter with respect to the time

$$\phi_t = -\int_{-\infty}^{t} M(t-t^*)\frac{\delta F(t^*, \vec{r})}{\delta\phi}dt^* \tag{1.11}$$

where $M(t-t^*)$ is the memory function which connects the dynamic at the time t with the past moment t^* by setting different weights in the past.

1. We consider
$$M(t-t^*) = M^{(1)}(0)\delta(t-t^*), \tag{1.12}$$
where $M^{(1)}(0)$ is a constant and $\delta(t-t^*)$ is the Dirac delta function. This first kernel provides a concrete evolution with zero memory, establishing an instantaneous correlation with the driving force at the

present time. Substituting 1.12 into the equation (1.11), we get

$$\phi_t = -M^{(1)}(0)\frac{\delta F}{\delta \phi}. \tag{1.13}$$

From the mathematical point of view, the obtained equation has a parabolic nature and describes a dissipative phase transition. Indeed, the form of the memory function neglects the past driving force and the instantaneous interaction is given by the proportionality $\phi_t \sim \delta F/\delta\phi$. The equation (1.13) is a typical example model of slow dynamic, which, roughly speaking, means to ignore fast degrees of freedom.

2. The memory function is defined by

$$M(t - t^*) = const. \tag{1.14}$$

Contrary to the first case, the kernel (1.14) gives the same importance to all driving forces in the past in order to predict the change of the phase field at the present time. The order parameter evolution is modelled by the following equation

$$\phi_{tt} = -M(0)\frac{\delta F}{\delta \phi}, \tag{1.15}$$

where $M(0)$ is the mobility at $t = t^*$. This constant memory function describes a transition with infinite memory. The obtained model is an undamped wave equation and, due to the constant memory, the system oscillates around the equilibrium.

3. In the last case, we assume a memory function with an exponential relaxation of the Maxwell type

$$M(t - t^*) = \frac{M^2(0)}{\tau_R} exp\left(-\frac{t - t^*}{\tau_R}\right), \tag{1.16}$$

where τ_R is the characteristic relaxation time and represents the time of the system to reach pure dissipative dynamics described by (1.13) for a motion without inertia. In applications related to pure or binary systems, there exist different models to determine a numeric value of τ_R depending by considered alloy. The kernel of the Maxwell type connects in a intermediate way the present with the past. Indeed, going to the past, the contribution of the driving force becomes less effective. After substitution into equation (1.11), we obtain

$$\tau_R\phi_{tt} + \phi_t = -M^2(0)\frac{\delta F}{\delta \phi}. \tag{1.17}$$

From thermodynamic conditions linked to entropy and stability, $M^2(0)$ is positive as well as τ_R. The acceleration term ϕ_{tt} represents the inertia effects into the diffuse interface and is a consequence of considering

ϕ and ϕ_t as independent variables. This choice directly implies that an equation for ϕ_{tt} should be found. Moreover, the relaxation term $\tau_R\phi_{tt}$ leads to a maximum possible value for the speed of the interface and allows oscillatory phenomena in the width of the interface.

Many physical systems present a periodic pattern in their microstructure. Their particular features, such as the specific length scale that characterizes stationary states or the symmetry, which promotes some direction of the gradient, are taken into account in the free energy functional. We assume the following form of the free energy

$$\mathcal{F}(\phi) = \int \left\{ -a\Delta_0 \frac{\phi^2}{2} + u\frac{\phi^4}{4} + \frac{\lambda}{2}\phi(q_0^2 + \Delta)^2\phi \right\} \, dv, \tag{1.18}$$

where v is the subvolume of the system, $a > 0$ is the parameter of the system periodicity, q_0 is the wave number, $u > 0$ and λ are parameters and $\Delta_0 = T_c - T$ is the quench depth representing the control parameter with the critical temperature T_c and the actual temperature T. This form of free energy functional is minimized by spatial periodic structures of the order parameter and is used especially to describe periodic systems arising from the elasticity. The main examples involve the elasticity in growing crystals, the striple-bubble transition and for crystalline or copolymeric chains.

To obtain the final form of the Swift-Hohenberg equation with slow and fast dynamics, we introduce the free energy (1.18) into the equation 1.17 and we proceed with the definition of the suitable variables to get the dimensionless form of the model

$$\tilde{t} = tM^2(0)\lambda q_0^4, \quad \tilde{\nabla} = q_0\nabla, \quad \tilde{\phi} = \phi\sqrt{\frac{u}{\lambda q_0^4}} \tag{1.19}$$

and parameters

$$\sigma = \tau_R M^2(0)\lambda q_0^4, \qquad \epsilon = \frac{a\Delta_0}{\lambda q_0^4}. \tag{1.20}$$

Substituting these variables into equation (1.17) and omitting the "tilde", we have the modified Swift-Hohenberg equation

$$\sigma\phi_{tt} + \phi_t = [\epsilon - (1 + \Delta)^2]\phi - \phi^3 \tag{1.21}$$

In an analogous way, replacing 1.18 in 1.13 and considering the above variables, it is possible to compute the dimensionless form of the Swift-Hohenberg with slow dynamic, obtaining 1.3.

1.6 Thermodynamic consistency

We verify the consistency of the equation (1.21) with the thermodynamic (macroscopic) theory. Three principle conditions need to be respect:

1. the free energy must be at a minimum in the equilibrium state,
2. the free energy must not be an increase function of the time, i.e. $\partial\mathcal{F}/\partial t \leq 0$,
3. the second differential of $\mathcal{F}$ with respect to ϕ must be positive, i.e. $\delta^2\mathcal{F} > 0$, to guarantee stability in dynamical solutions.

To control the above conditions, we consider a free energy in dimensionless form, which is more general than (1.18) because takes into account both the contributions of the slow variable ϕ and the fast variable ϕ_t.

$$\mathcal{F}(\phi, \partial\phi/\partial t) = \int \left\{ g(\phi) - |\nabla\phi|^2 + \frac{1}{2}|\Delta\phi|^2 + \frac{\sigma}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 \right\} dv \tag{1.22}$$

where

$$g(\phi) = -\frac{\epsilon - 1}{2}\phi^2 + \frac{1}{4}\phi^4.$$

The first condition is obviously verified by the form of the equation (1.21) and the free energy (1.22). In order to guarantee the second condition, we split the $\partial\mathcal{F}/\partial t$ into the sum of the external exchange of free energy

$$\left(\frac{\partial\mathcal{F}}{\partial t}\right)_{ex} = \oint_s \left\{ \left(\nabla_n \frac{\partial\phi}{\partial t}\right)\Delta\phi - \frac{\partial\phi}{\partial t}\nabla_n(\Delta\phi) - 2\frac{\partial\phi}{\partial t}\nabla_n\phi \right\} ds$$

and the internal change of free energy

$$\left(\frac{\partial\mathcal{F}}{\partial t}\right)_{in} = \int_v \left\{ \frac{dg(\phi)}{d\phi} + 2\Delta\phi + \Delta^2\phi + \sigma\frac{\partial^2\phi}{\partial t^2} \right\} \frac{\partial\phi}{\partial t}\, dv, \tag{1.23}$$

where ∇_n represents gradient normal vector to the surface s. We observe that natural boundary conditions to ensure zero exchange of free energy are

$$\phi \equiv const, \quad \nabla_n\phi = 0$$

or

$$\phi \equiv const, \quad \Delta\phi = 0. \tag{1.24}$$

Following the ideas of the extended irreversible thermodynamic and by the sign of (1.23), it follows that thermodynamic fluxes J_i and their conjugated forces X_i are proportional in first approximation. Thus we can write

$$J_i \equiv \frac{\partial\phi}{\partial t} = -MX_i \equiv -M\left(\frac{dg(\phi)}{d\phi} + 2\Delta\phi + \Delta^2\phi + \sigma\frac{\partial^2\phi}{\partial t^2}\right).$$

Observing that $(\partial\mathcal{F}/\partial t)_{in} = \int_v (J_i X_i)\, dv$, this implies that

$$\frac{\partial\mathcal{F}}{\partial t} = \left(\frac{\partial\mathcal{F}}{\partial t}\right)_{in} = -\int_v M\left(\frac{dg(\phi)}{d\phi} + 2\Delta\phi + \Delta^2\phi + \sigma\frac{\partial^2\phi}{\partial t^2}\right)^2 dv \leq 0.$$

Furthermore, from mathematical point of view, we can infer that the free energy (1.22) is a Lyapunov function for the modified Swift-Hohenberg equation.

To check the third condition, we calculate $\delta^2\mathcal{F}$, assuming that $\delta^2\phi$ and $\delta^2(\partial\phi/\partial t) = 0$ as a consequence of the independency of ϕ and $\partial\phi/\partial t$. The result is

$$\delta^2\mathcal{F} = \int \left\{ \mathcal{L}(\phi, \Delta)(\delta\phi)^2 + \sigma \left(\delta \frac{\partial \phi}{\partial t} \right)^2 \right\} \, dv,$$

where

$$\mathcal{L}(\phi, \Delta) = \frac{d^2 g}{d\phi^2} + \Delta + \frac{1}{2}\Delta^2.$$

Due to the non convexity of g, the operator $\mathcal{L}$ has not fixed sign and the solutions, corresponding to some wave vectors, may present instability.

1.7 Plan

Complex physical and mechanical phenomena are usually modelled by a set of time-dependent fields which satisfy nonlinear evolution equations. The idea of discovering an exact solution immediately fails due to the present of the nonlinear term, so a qualitative theory is formulated to provide a consistent description of the physical events. In addition, in study of partial differential equations, it is not possible to obtain a general result, valid for each equation as well as in ordinary differential equations. Therefore, the first level is the analysis of the well-posedness of the physical models concerned. We are interested in finding a formulation and prove the results establishing existence, uniqueness and regularity of the solutions with respect to the data of the problem. Moreover, we investigate the well-posedness of the associated stationary problem. Generally for these models, arising from phase field theory with non convex free energy, the cardinality of the stationary solutions may be very high, also countable or continuum.

As we have seen in the previous section, the introduction of the inertial term ϕ_{tt} allows a description of the transition from an unstable to a (meta)stable phase state, including microscopic and mesoscopic scale, concerning system far from the equilibrium and taking into account the transient phenomena. It is worth pointing out that this additional term is not only a modelling generalization, but it has also a relevant impact on the mathematical analysis. Indeed, as it comes out at the well-posedness level, the Swift-Hohenberg equation has the intrinsic property to regularize the solution in finite-time, which does not appear in the modified Swift-Hohenberg equation.

When a solution is defined for all times, an interesting investigation is to characterize its permanent regime, namely the behavior of the solution when the influence of the initial data has vanished after a long time. The simplest form may be to prove that the time-dependent solution converges, in a suitable sense, to its stationary solution. This strategy fully describes the long-time behavior of a system with uniqueness of the equilibrium, but it is a rare situation. Furthermore, many physical systems, for example in fluidodynamic, reveal important time-dependent permanent regime, such as periodic, quasi-periodic or chaotic solutions. For these reasons, we are motivated to leave the idea of single solution and to consider the approach of the dynamical systems in infinite dimension.

In dynamical systems governed by PDEs, as well as in finite dimension, the solution is considered as a trajectory (curve) contained in a suitable phase space of infinite dimension. Thanks to the well-posedness of the models, it is possible to define a family of maps $S(t)$, the so-called semigroup, which takes the initial data in input and returns the solution at time t. We observe that the trajectory,corresponding to initial data u, is the image of $S(t)u$ as a function of t. The couple composed by phase space and semigroup is commonly called a dynamical system.

In this framework, the description of the long-time behavior is performed by means the existence of phase space subsets with particular properties with respect to the semigroup. For PDEs governed systems, the dynamic is described with inequalities in terms of the function norm, which establish the uniform behavior of the solutions with initial data in a bounded set of the phase space (see [26], [27] for more details).

An important class of dynamical systems involves the so-called dissipative systems. This means that exists a set $\mathcal{B}$ in the phase space and the trajectories, which start from a generic bounded set, enter in finite time in $\mathcal{B}$. The set $\mathcal{B}$ is called absorbing set and its existence expresses the dissipation (coming from friction, viscosity and diffusion) in mathematical words. However, experiments and numerical simulations have pointed out how the permanent regime of the physical relevant quantities of this class of systems is localized inside *thin* subsets compared to the infinite dimension of the phase space. These sets may have very complicated geometries described with a notion of fractal dimension. This remark contributed significantly to define the relevant objects in theory of dynamical systems and it was conjectured that they are invariant in time and finite-dimensional.

The main object is the so-called global attractor, namely a compact set which *attracts* the bounded sets and is completely invariant under the action of the semigroup (the trajectories which start into the attractor remain into the attractor). From its properties directly follows that this attractor is unique and contains the equilibrium points as well as the time-dependent permanent regimes (periodic solution). Historically, many different definitions of attractors were proposed but this type became universally accepted after 1982, when Ladyzhenskaya showed its existence for the two-dimensional Navier-Stokes system. In the subsequent years, the theory was developed by many authors (see [1] or [22] and the references given there) and applied to a large class of models, such as reaction-diffusion, damped wave, phase-field equations and Navier-Stokes system. Now there are many techniques to prove the existence of this set and their application depends to the PDEs concerned. After having established the existence of the global attractor, it is possible to investigate further regularity properties of this set, as an example determine the fractal dimension in term of the physical parameters of the system.

All considered models arise from real physical phenomena, therefore we are interested in studying the rate of attraction of the trajectories towards the global attractor or evaluate as it may be *robust* under perturbation of significant parameters. Many results demonstrated how the global attractor may not satisfy these requirements positively. For example, it may not depend with continuity from the parameters or attracts the trajectories arbitrary slowly. Moreover, fast attraction is a useful information to approximate the long-time behavior with numerical simulations.

To avoid these defects, it is introduced the new concept of exponential attractor, namely a compact set with finite fractal dimension, positively invariant which attracts the trajectories with an exponential rate. This set, if it exists, contains the global attractor but it is not necessarily unique. The first construction was provided by Eden, Foias, Nicolaenko and Temam (see [5]) in Hilbert spaces but it was generalized in the Banach space setting by Efendiev, Miranville and Zelik in [7].

Frequently, existing models are modified and enriched of new terms with the aim to describe further aspects of the phenomenon. From mathematical point of view, we expect that the long-time behavior of the latter models is closed to the one of the former under small perturbation. The exponential attractor gives a positive answer to this problem: for both cases of regular or singular perturbation (the phase space of the perturbed system is or not the same of the unpertubed system), it is possible to construct a family of exponential attractors which satisfy a propriety of *continuity* with respect to the perturbative parameter.

Let Ω be a bounded domain in $\mathbb{R}^3$ and $T > 0$ the final time. We consider the nonlinear term f as a real function for which some restrictions will be given in the next chapter. This work is devoted to a mathematical analysis, in line with the plan above presented, of the following models

1. the Swift-Hohenberg equation (SHE)

$$\begin{cases} \phi_t + \Delta^2\phi + 2\Delta\phi + f(\phi) = 0 & \text{in } \Omega \times (0,T) \\ \phi = \Delta\phi = 0 & \text{on } \partial\Omega \times (0,T) \\ \phi(0) = \phi_0 & \text{in } \Omega \end{cases} \tag{1.25}$$

2. the modified Swift-Hohenberg equation (MSHE)

$$\begin{cases} \sigma\phi_{tt} + \phi_t + \Delta^2\phi + 2\Delta\phi + f(\phi) = 0 & \text{in } \Omega \times (0,T) \\ \phi = \Delta\phi = 0 & \text{on } \partial\Omega \times (0,T) \\ \phi(0) = \phi_0 & \text{in } \Omega \\ \phi_t(0) = \phi_1 & \text{in } \Omega \end{cases} \tag{1.26}$$

The thesis is organized as follows. In Chapter 2 we formulate and prove results of existence, uniqueness and regularity from initial data to the Swift-Hohenberg and the modified Swift-Hohenberg equations with a more general nonlinear term than the physical relevant function. In particular, to show a continuity dependence from initial data for the equation with the inertial term $\sigma\phi_{tt}$, we need to obtain an energy identity valid for the associated linear models. We also prove the existence of at least a solution for the stationary problem associated to these evolution models. In Chapter 3 we briefly give an introduction to the theory of dissipative dynamical system in infinite dimension with particular focus on the role of the global attractor and on the main abstract results to prove its existence. Afterwords, thanks to the well posedness of the two models, we are able to define the strong semigroups and to apply the strategies to the existence of the global attractor to both the equations. Chapter 4 is devoted to the study of the exponential attractor. We give a short introduction of this object and explain the general methods to prove its existence. After, we apply this strategies to the Swift-Hohenberg and the modified Swift-Hohenberg equation. Finally, in Chapter 5, we investigate the robustness property of the global and exponential attractor with respect to the parameter σ. In particular, we read the modified Swift-Hohenberg equation as a singular perturbation of the original Swift-Hohenberg model. In this framework we are able to prove the upper semicontinuity of the global attractor and the existence of a robust family of exponential attractors with respect to the perturbation.

2 Well-posedness

In this chapter our aim is to investigate the well-posedness of the Swift-Hohenberg and the modified Swift-Hohenberg equations presented in the previous chapter. In particular, we will formulate results of existence, uniqueness and regularity with a more general nonlinear term than the physical relevant function. We will be able to prove necessary regularity properties, especially a continuous dependence from initial data of the solutions, in order to define a solution map in the suitable phase spaces and to study the long-time behavior. Lastly, we will also prove the existence of a stationary solution associated to the Swift-Hohenberg equation.

2.1 Functional setting and useful results

Let Ω be a regular domain in $\mathbb{R}^3$. Let V be a Banach space endowed with the norm $||\cdot||_V$ and let V^* be its dual space. By $<,>_{V^*,V}$ we indicate the duality pairing between V and V^*. We introduce the Hilbert space $L^2(\Omega)$ with its standard inner product $(u,v) = \int_\Omega uv\ dx$ and the induced norm $||u||_{L^2(\Omega)} = \sqrt{(u,u)}$. For $m \in \mathbb{N}$, we denote by $H^m(\Omega)$ the Sobolev space with the following scalar product and its associated norm

$$(u,v)_m = \sum_{|k|\leq m} \int_\Omega D^k u D^k v\ dx, \qquad ||u||_{H^m(\Omega)} = ((u,u)_m)^{\frac{1}{2}},$$

where D^k is the distributional derivative of order k.
We indicate by $A : D(A) \to L^2(\Omega)$ the Laplace operator with homogeneous Dirichlet boundary condition. A is a self-adjoint positive operator in $L^2(\Omega)$. The space $D(A)$ can be fully characterized by using the regularity theory of linear elliptic systems,

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

where $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$, defined as the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. The spectral theory of this operator allows us to define the powers of A, namely A^s: $D(A^s) \to L^2(\Omega)$, for $s \in \mathbb{R}$. $D(A^s)$ is a Hilbert space with the following inner product and norm

$$(u, v)_{D(A^s)} = (A^s u, A^s v), \quad ||u||_{D(A^s)} = ||A^s u||_{L^2(\Omega)}.$$

In particular, we consider the following Hilbert spaces

$$\mathcal{H}_0 = L^2(\Omega), \quad \mathcal{H}_1 = H_0^1(\Omega), \quad \mathcal{H}_2 = H^2(\Omega) \cap H_0^1(\Omega),$$

$$\mathcal{H}_3 = D(A^{\frac{3}{2}}), \quad \mathcal{H}_4 = D(A^2),$$

endowed with the norms, induced by the inner products,

$$||v||_{\mathcal{H}_0} = ||v||_{L^2(\Omega)}, \quad ||v||_{\mathcal{H}_1} = ||\nabla v||_{L^2(\Omega)}, \quad ||v||_{\mathcal{H}_2} = ||\Delta v||_{L^2(\Omega)},$$

$$||v||_{\mathcal{H}_3} = ||\nabla \Delta v||_{L^2(\Omega)}, \quad ||v||_{\mathcal{H}_4} = ||\Delta^2 v||_{L^2(\Omega)}.$$

We observe that the norm of the space $\mathcal{H}_k$ is equivalent to $|| \cdot ||_{H^k(\Omega)}$. Moreover, we introduce the product spaces

$$\mathcal{E}_0 = \mathcal{H}_2 \times \mathcal{H}_0, \quad \mathcal{E}_1 = \mathcal{H}_3 \times \mathcal{H}_1, \quad \mathcal{E}_2 = \mathcal{H}_4 \times \mathcal{H}_2,$$

endowed with the graph norms

$$||(u,v)||_{\mathcal{E}_0} = \left(||u||^2_{\mathcal{H}_2} + ||v||^2_{\mathcal{H}_0}\right)^{\frac{1}{2}}, \quad ||(u,v)||_{\mathcal{E}_1} = \left(||u||^2_{\mathcal{H}_3} + ||v||^2_{\mathcal{H}_1}\right)^{\frac{1}{2}},$$

$$||(u,v)||_{\mathcal{E}_2} = \left(||u||^2_{\mathcal{H}_4} + ||v||^2_{\mathcal{H}_2}\right)^{\frac{1}{2}}.$$

Identifying $\mathcal{H}_0$ with its dual space $\mathcal{H}_0^*$, we have the continuous and dense embeddings $\mathcal{H}_2 \hookrightarrow \mathcal{H}_0 \equiv \mathcal{H}_0^* \hookrightarrow \mathcal{H}_2^*$. We also note that the Rellich-Kondrachov Theorem implies $\mathcal{H}_k \overset{c}{\hookrightarrow} \mathcal{H}_{k-1}$.
Let a be the bilinear, symmetric and bounded form on $\mathcal{H}_2 \times \mathcal{H}_2$ defined by

$$a\,(u, v) = \int_\Omega \Delta u \Delta v - 2\nabla u \cdot \nabla v \; dx.$$

Thanks to the spectral Theorem for bilinear forms, there exist a sequence $\{u_m\}_{m \geqslant 1} \subset \mathcal{H}_2$ and a sequence $\{\lambda_m\}_{m \geqslant 1} \in \mathbb{R}$, nondecreasing and tending to $+\infty$ as $m \nearrow +\infty$, such that:

1. $\Delta^2 u_m + 2\Delta u_m = \lambda_m u_m$ in Ω in the weak sense, $\forall m$,

2. u_m is an orthonormal base in $\mathcal{H}_0$ and is an orthogonal base in $\mathcal{H}_2$ with respect to the scalar product $(u, v)_{\mathcal{H}_2} = a(u, v) + \lambda(u, v)$, where λ is a suitable positive constant.

We assume that the nonlinear function f satisfies the following properties:

$$\begin{array}{ll} \textbf{(H1)} & f \in C^3(\mathbb{R}),\ f(0) = 0, \\ \textbf{(H2)} & \exists \delta > 0 :\ f(s)s \geq (1+\delta)s^2 - K_1, \end{array}$$

where the positive constant K_1 is independent of s.
These conditions are not particularly restrictive because they are satisfied for cubic nonlinearities $f(\phi) = \phi^3 - (1-\varepsilon)\phi$. Furthermore, we want to emphasize how our request on the regularity of f is not always necessary. Indeed, more precisely, we need a continuously differentiable function to obtain the existence and the uniqueness. In order to get asymptotic results, it is necessary to put some restrictions on f: we use a C^2 regularity in the analysis of the global and exponential attractors, C^3 in the study of a robust family of exponential attractors with respect to the parameter σ.
Let F be the primitive for f

$$F(s) = \int_0^s f(x)\ dx, \qquad \forall s \in \mathbb{R}.$$

As a consequence of the above hypothesis, the following inequality holds

$$\textbf{(C)} \qquad F(s) \geq (\frac{2+\delta}{4})s^2 - K_2,$$

where K_2 is a positive constant that may depends on K_1, δ and f.

In this work we consider equations endowed with Navier boundary conditions. We observe that all formulated results can be generalized with small changes for different boundary conditions, for example periodic boundary conditions.

We resume two useful results for infinite-dimensional vector functions establishing a criterion to obtain a compactness property. In particular, these Theorems allow us to pass to the limit in approximating problems and recover some regularity properties of the solutions.

Theorem 2.1.1. *Let Y, X, Z be reflexive and separable Banach spaces*

$$Y \overset{c}{\hookrightarrow} X \hookrightarrow Z.$$

Let $\{u_m\}_{m\geq 1}$ be a sequence such that

- *u_m is uniformly bounded in $L^{p_1}(0,T;Y)$, $1 < p_1 < \infty$,*
- *$\dot{u}_m$ is uniformly bounded in $L^{p_2}(0,T;Z)$, $1 < p_2 < \infty$.*

Then there exists a subsequence that strongly converge in $L^{p_1}(0,T;X)$.

Theorem 2.1.2. *Let $\{u_m\}_{m\geq 1}$ be a sequence bounded in $L^p(Q)$ such that $u_m \to u$ a.e. in Q. Then $u_m \to u$ weakly in $L^p(Q)$.*

Lastly, we introduce a result which provides a condition to extend the uniform estimates from approximating solutions ϕ_m to the limit solution ϕ.

Lemma 2.1.3. *Let (V, H, V^*) be a Hilbert triple with V compactly embedded into H. Let $\{u_m\}_{m\geq 1}$ be a sequence such that u_m is uniformly bounded in $L^\infty(0,T;V)$ by a constant C and $u_m \to u$ weakly in $L^2(0,T;V)$. Then there holds*

$$\underset{t\in[0,T]}{esssup}\ ||u(t)||_V < C.$$

Throughout this thesis, Q denotes a generic positive monotone function while C, C_i, R_i, ρ_i and κ_i stand for positive constants that may be estimate according to the parameters of the system. We will subsequently indicate their dependencies. In particular, C_Ω denotes some constant depending only on Ω. To simplify notation in the proofs of this chapter, we use $\dot{\phi}$ to indicate the derivative of ϕ with respect to the time. Moreover in the next chapters, $B_X(0,R)$ stands for the closed ball in X centered in 0 with radius R.

2.2 Swift-Hohenberg equation

We consider the Swift-Hohenberg equation

$$\begin{cases} \phi_t + \Delta^2\phi + 2\Delta\phi + f(\phi) = 0 & \text{in } \Omega\times(0,T), \\ \phi = \Delta\phi = 0 & \text{on } \partial\Omega\times(0,T), \\ \phi(0) = \phi_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

with the above assumptions on the nonlinearity term.
We give the following definition of weak formulation of the problem (2.1).

Definition 2.2.1. Let $T > 0$ be given. ϕ is a weak solution if $\phi \in L^2(0,T;\mathcal{H}_2)$, $\phi_t \in L^2(0,T;\mathcal{H}_2^*)$ such that

(1) $< \phi_t(t), v >_{\mathcal{H}_2^*,\mathcal{H}_2} + a(\phi(t), v) + (f(\phi(t)), v) = 0,$
$\qquad\qquad \forall v \in \mathcal{H}_2, \text{ a.e. } t \in (0,T),$

(2) $\phi(0) = \phi_0$ in $\mathcal{H}_0$.

In this section we proceed with the study of the next result.

Theorem 2.2.2. *Let $\phi_0 \in \mathcal{H}_0$. Then the problem* (2.1) *admits a unique weak solution $\phi \in C([0,T];\mathcal{H}_0)$. Moreover the following estimates holds*

$$||\phi_1 - \phi_2||_{C([0,T];\mathcal{H}_0)} + ||\phi_1 - \phi_2||_{L^2(0,T;\mathcal{H}_2)} \leqslant C_1 e^{C_2 T} ||\phi_{01} - \phi_{02}||_{\mathcal{H}_0}, \tag{2.2}$$

where ϕ_1 and ϕ_2 are weak solutions to (2.1)*, respectively with initial conditions ϕ_{01}, ϕ_{02}, and the positive constant C_i depend on Ω, T, f, ϕ_{01}, ϕ_{02}.*

The main idea of the proof is to apply the Galerkin method (see [20] for more details). Initially, we will define a family of solutions to suitable finite-dimensional problems by making use of the eigenfunctions u_m. Then, we will provide the uniform energy estimates, which allow us to pass to the limit into the equation. The main difficulty comes out from the presence of a nonlinear term, more precisely we will need to use the above compactness results to guarantee that $f(\phi_m)$ converge to $f(\phi)$, and not to a generic function.

Proof. **Galerkin approximation scheme**
We built a sequence of solutions with the Galerkin method: let us fix $m \in \mathbb{N}$, let $\mathcal{H}_{2,m}$ be the finite dimensional subspace of $\mathcal{H}_2$, $\mathcal{H}_{2,m}$ =span$\{u_1, ..., u_m\}$. We find a function $\phi_m \in C^1([0,T]; \mathcal{H}_{2,m})$ of the form

$$\phi_m(t) = \sum_{k=1}^{m} c_k(t)\, u_k,$$

solution to problem

$$\begin{aligned} &(1)\ (\dot{\phi}_m(t), v) + a(\phi_m(t), v) + (f(\phi_m(t)), v) = 0, \\ &\qquad\qquad\qquad\qquad \forall v \in \mathcal{H}_{2,m},\ \text{a.e. } t \in (0,T), \\ &(2)\ \ \phi_m(0) = \phi_{0m} \text{ in } \mathcal{H}_0, \end{aligned} \tag{2.3}$$

where $\phi_{0m} = \sum_{k=1}^{m} \alpha_k u_k \to \phi_0$ in $\mathcal{H}_0$.
Writing the problem (2.3) for each basis function of $\mathcal{H}_{2,m}$, we obtain a nonlinear system of ODEs :

$$\begin{cases} \dot{c}_s(t) = -a(u_s, u_s) c_s(t) - (f(\phi_m(t)), u_s), & \text{a.e. } t \in (0,T), \\ c_s(0) = \alpha_s, \end{cases}$$

for all s= $1, ..., m$.
The right hand side of the system is a continuous and locally Lipschitz function of $c_s, \forall s$, so it follows that there exists a unique vector solution

$$c_m(t) = (c_1(t), ..., c_m(t)) \in C^1([0,T^*); \mathbb{R}^m).$$

Consequently, the approximating problem (2.3) admits a unique solution $\phi_m \in C^1([0,T^*); \mathcal{H}_{2,m})$.

Energy estimates
We want to ensure that the approximating solutions are defined in $[0,T]$ and satisfy some uniform boundedness properties. Using $\phi_m(t)$ as a test function in (2.3), we get

$$(\dot{\phi}_m(t), \phi_m(t)) + a(\phi_m(t), \phi_m(t)) + (f(\phi_m(t)), \phi_m(t)) = 0$$

From integration by parts, the Cauchy-Schwarz and Young inequalities, we have

$$2||\nabla v||^2_{\mathcal{H}_0} \leq 2||\Delta v||_{\mathcal{H}_0}||v||_{\mathcal{H}_0} \leq (1-\gamma)||v||^2_{\mathcal{H}_2} + \frac{1}{1-\gamma}||v||^2_{\mathcal{H}_0}. \qquad (2.4)$$

where γ is a fixed real value such that $0 < \gamma < 1$. According to the last inequality with $\gamma = \frac{\delta}{1+\delta}$ and the property (**H2**) of the nonlinear term, yields

$$\frac{1}{2}\frac{d}{dt}||\phi_m(t)||^2_{\mathcal{H}_0} + \gamma||\phi_m(t)||^2_{\mathcal{H}_2} \leq K_1|\Omega|. \qquad (2.5)$$

In particular, applying the Gronwall Lemma and the embedding property $\mathcal{H}_2 \hookrightarrow \mathcal{H}_0$, we can rewrite 2.5 as

$$||\phi_m(t)||^2_{\mathcal{H}_0} \leq ||\phi_0||^2_{\mathcal{H}_0}e^{-C_1 t} + C_2[1 - e^{-C_1 t}] \leq C,$$

where C depends on Ω, T, ϕ_0, f but is independent of m. Integrating from 0 to T in (2.5), we can assert that

$$||\phi_m(T)||^2_{\mathcal{H}_0} + \int_0^T ||\phi_m(t)||^2_{\mathcal{H}_2}\, dt \leq ||\phi_0||^2_{\mathcal{H}_0} + K_1|\Omega|T. \qquad (2.6)$$

These estimates allow us to conclude that

i. ϕ_m is uniformly bounded in $L^\infty(0,T;\mathcal{H}_0)$,

ii. ϕ_m is uniformly bounded in $L^2(0,T;\mathcal{H}_2)$,

For simplicity of notation, we will use the same constant C to bound ϕ_m in already cited spaces. As a consequence of (i), we can extend the approximating solutions to problem (2.3) on the interval $[0,T]$. It follows from the Gagliardo-Niremberg inequality that the following estimate holds

$$\begin{aligned} ||\phi_m(t)||^N_{L^N(\Omega)} &\leq C_\Omega||\phi_m(t)||^{N-1}_{\mathcal{H}_0}||\phi_m(t)||_{\mathcal{H}_2} \\ &\leq \frac{1}{2}C_\Omega||\phi_m(t)||^{2(N-1)}_{\mathcal{H}_0} + \frac{1}{2}||\phi_m(t)||^2_{\mathcal{H}_2}, \qquad \forall N \geq 1. \end{aligned}$$

Combining (i), (ii) and last inequality, we obtain

$$||\phi_m||_{L^N(Q)} \leq C, \qquad N \geq 1,$$

where $Q = \Omega \times (0,T)$ and C may depend on Ω, T, ϕ_0, f but it is independent of N. Thus we deduce that

iii. ϕ_m is uniformly bounded in $L^\infty(Q)$,

and, as a consequence of (**H1**), we conclude that

iv. $f(\phi_m)$ is uniformly bounded in $L^\infty(Q)$.

Now we prove an energy estimate for $\dot{\phi}_m$: let $v \in \mathcal{H}_2$ be, by the projection Theorem $v = z + w$, where $z \in \mathcal{H}_{2,m}, w \in \mathcal{H}_{2,m}^{\perp}$. In a standard way, we get

$$\begin{aligned} | < \dot{\phi}_m(t), v >_{\mathcal{H}_2^*,\mathcal{H}_2} | &= | < \dot{\phi}_m(t), z >_{\mathcal{H}_2^*,\mathcal{H}_2} | \\ &= | - a(\phi_m(t), z) - (f(\phi_m(t)), z)| \\ &\leq [C_3||\phi_m(t)||_{\mathcal{H}_2} + C_4||f(\phi_m(t))||_{\mathcal{H}_0}] \, ||z||_{\mathcal{H}_2} \\ &\leq [C_3||\phi_m(t)||_{\mathcal{H}_2} + C_4||f(\phi_m(t))||_{\mathcal{H}_0}] \, ||v||_{\mathcal{H}_2}. \end{aligned}$$

This implies that

$$||\dot{\phi}_m(t)||^2_{L^2(0,T;\mathcal{H}_2^*)} \leq \int_0^T C_5||\phi_m(t)||^2_{\mathcal{H}_2} \, dt + \int_0^T C_6||f(\phi_m(t))||^2_{\mathcal{H}_0} \, dt \leq C,$$

where C depends on Ω, T, ϕ_0, f but is independent of m.
Therefore we have

v. $\dot{\phi}_m$ is uniformly bounded in $L^2(0,T;\mathcal{H}_2^*)$.

Passage to the limit
From energy estimates and by compactness arguments, we recover sufficient convergence properties of the approximating solutions and we can pass to the limit in the discrete problem (2.3). For abbreviation, we continue to write ϕ_m also for subsequence.
By the energy estimates (ii) and (v), we conclude that

$$\phi_m \to \phi \text{ weakly in } L^2(0,T;\mathcal{H}_2),$$

$$\dot{\phi}_m \to \dot{\phi} \text{ weakly in } L^2(0,T;\mathcal{H}_2^*).$$

Using Theorem 2.1.1, we have

$$\phi_m \to \phi \text{ in } L^2(0,T;\mathcal{H}_0) \quad \Rightarrow \quad \phi_m \to \phi \text{ a.e. in } Q$$

$$\Rightarrow f(\phi_m) \to f(\phi) \text{ a.e. in } Q,$$

by continuity of the nonlinear term f. Thanks to (iv), we apply Theorem 2.1.2 with $p = 2$ and we obtain

$$f(\phi_m) \to f(\phi) \text{ weakly in } L^2(Q).$$

Now we are ready to prove that the limit function ϕ is a solution to problem (2.1). Let us consider $v \in L^2(0,T;\mathcal{H}_2)$ and let us define

$$v_N(t) = \sum_{k=1}^{N} b_k(t) u_k \to v(t) \text{ in } \mathcal{H}_2.$$

In a standard way, we fix N such that $m > N$ and we take $v_N(t)$ as a test function in the discrete problem (2.3)

$$< \dot{\phi_m}(t), v_N(t) >_{\mathcal{H}_2^*,\mathcal{H}_2} + a(\phi_m(t), v_N(t)) + (f(\phi_m(t)), v_N(t)) = 0.$$

Integrating from 0 to T

$$\int_0^T < \dot{\phi_m}(t), v_N(t) >_{\mathcal{H}_2^*,\mathcal{H}_2} \, dt + \int_0^T a(\phi_m(t), v_N(t)) \, dt + \int_0^T (f(\phi_m(t)), v_N(t)) \, dt = 0. \tag{2.7}$$

We can now pass to the limit as $m \nearrow \infty$ in (2.7) and we get

$$\int_0^T < \dot{\phi}(t), v_N(t) >_{\mathcal{H}_2^*,\mathcal{H}_2} \, dt + \int_0^T a(\phi(t), v_N(t)) \, dt + \int_0^T (f(\phi(t)), v_N(t)) \, dt = 0.$$

Passing also to the limit as $N \nearrow \infty$ and using that $v_N \to v$ in $L^2(0, T; \mathcal{H}_2)$, we have

$$\int_0^T < \dot{\phi}(t), v(t) >_{\mathcal{H}_2^*,\mathcal{H}_2} \, dt + \int_0^T a(\phi(t), v(t))dt + \int_0^T (f(\phi(t)), v(t))dt = 0. \tag{2.8}$$

Since v is arbitrary, the same equality holds true if $v = w\chi_{[s,s+h]}(t)$, where $w \in \mathcal{H}_2$,

$$\int_s^{s+h} < \dot{\phi}(t), w >_{\mathcal{H}_2^*,\mathcal{H}_2} \, dt + \int_s^{s+h} a(\phi(t), w) \, dt + \int_s^{s+h} (f(\phi(t)), w) \, dt = 0.$$

Multiplying for h^{-1}, passing to the limit to $h \to 0$ and using Lebesgue differentiation Theorem, we obtain

$$< \phi_t(t), w >_{\mathcal{H}_2^*,\mathcal{H}_2} + a(\phi(t), w) + (f(\phi(t)), w) = 0, \quad \forall w \in \mathcal{H}_2, \text{ a.e. } t \in (0, T).$$

To conclude that the limit function ϕ is a weak solution in the sense of Definition 2.2.1, we need to check the initial condition. In this way, using the well known Theorem for vector function in $H^1(0, T; \mathcal{H}_2, \mathcal{H}_2^*)$, we have that $\phi \in C([0, T], \mathcal{H}_0)$. An easy computation shows that $\phi(0) = \phi_0$ in $\mathcal{H}_0$: indeed taking $v \in C^1([0, T]; \mathcal{H}_2)$ with $v(T) = 0$ and subtracting (2.7) to

(2.8), we can use integration by parts in time. Hence, passing to the limit to $m \nearrow \infty$ and then to $N \nearrow \infty$, we get

$$(\phi(0), v(0)) = (\phi_0, v(0)) \ \Rightarrow \ \phi(0) = \phi_0.$$

Uniqueness and continuous dependence from initial data

Finally, we want to prove the estimate (2.2), which allows us to conclude that the weak solution to problem (2.1) is unique. As a consequence, we observe that the whole sequence ϕ_m converge to ϕ and not only a subsequence ϕ_{m_j}. Furthermore an estimate of this type proves the continuity of the solution from initial data.

Let us consider two solutions ϕ_1 and ϕ_2, respectively with data ϕ_{01} and ϕ_{02}. We define $\phi = \phi_1 - \phi_2$ and we consider the following equation for ϕ

$$< \phi_t(t), w >_{\mathcal{H}_2^*, \mathcal{H}_2} + a(\phi(t), w) + (f(\phi_1(t)) - f(\phi_2(t)), w) = 0, \qquad \forall w \in \mathcal{H}_2, \ \text{a.e. } t \in (0,T).$$

Testing by $w = \phi(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}||\phi(t)||^2_{\mathcal{H}_0} + ||\phi\,(t)\,||^2_{\mathcal{H}_2} - 2||\nabla\phi\,(t)\,||^2_{\mathcal{H}_0} \leq || \int_0^1 f'(\tau\phi_1(t) + (1-\tau)\phi_2(t))(\phi_1(t) - \phi_2(t)) \ d\tau||_{\mathcal{H}_0}||\phi(t)||_{\mathcal{H}_0}.$$

Since $\phi_1, \phi_2 \in L^\infty(Q)$ and using inequality (2.4), we have

$$\frac{1}{2}\frac{d}{dt}||\phi(t)||^2_{\mathcal{H}_0} + \frac{1}{2}||\phi\,(t)\,||^2_{\mathcal{H}_2} \leq C_7||\phi(t)||^2_{\mathcal{H}_0}. \tag{2.9}$$

Applying the Gronwall Lemma, we get

$$||\phi||_{\mathbb{C}(0,T;\mathcal{H}_0)} \leq ||\phi_{01} - \phi_{02}||_{\mathcal{H}_0} e^{C_8 T}.$$

Using last inequality and integrating from 0 to T in (2.9), we have

$$\int_0^T ||\phi(t)||^2_{\mathcal{H}_2} \ dt \leq C_9||\phi_{01} - \phi_{02}||^2_{\mathcal{H}_0} e^{C_{10}T}.$$

Combining these inequalities, the proof is complete. □

Let us proceed with a regularity result.

Theorem 2.2.3. *Let $\phi_0 \in \mathcal{H}_2$. Then the problem* (2.1) *admits a unique strong solution, namely*

$$\begin{cases} \phi_t + \Delta^2\phi + 2\Delta\phi + f\,(\phi) = 0 & \textit{in } \mathcal{H}_0, \textit{ a.e. } t \in (0,T) \\ \phi = \Delta\phi = 0 & \textit{on } \partial\Omega \times (0,T) \\ \phi(0) = \phi_0 & \textit{in } \Omega, \end{cases} \tag{2.10}$$

such that

$$\phi \in C([0,T];\mathcal{H}_2) \cap L^2(0,T;\mathcal{H}^4), \quad \phi_t \in L^2([0,T];\mathcal{H}_0).$$

Moreover, the following estimate holds

$$||\phi_1 - \phi_2||_{C([0,T];\mathcal{H}_2)} + ||\phi_{1,t} - \phi_{2,t}||_{L^2(0,T;\mathcal{H}_0)} \leqslant C||\phi_{01} - \phi_{02}||_{\mathcal{H}_2}. \quad (2.11)$$

where ϕ_1, ϕ_2 are strong solutions to (2.10)*, respectively with initial data ϕ_{01}, ϕ_{02} and C may depend on the $\mathcal{H}_2$-norms of the initial data as well as on Ω, T, f.*

Proof. Considering the approximating problem (2.3), we have that $\phi_m \in C^1([0,T);\mathcal{H}_{2,m})$ so this allows us to take $v = \dot{\phi}_m(t)$ as test function, getting

$$||\dot{\phi_m}(t)||^2_{\mathcal{H}_0} + \frac{d}{dt}\left\{\frac{1}{2}||\Delta\phi_m(t)||^2_{\mathcal{H}_0} - ||\nabla\phi_m(t)||^2_{\mathcal{H}_0} + \int_\Omega F(\phi_m(t))\, dx\right\} = 0.$$

We recall the following inequality

$$||\nabla v||^2_{\mathcal{H}_0} \leq ||\Delta v||_{\mathcal{H}_0}||v||_{\mathcal{H}_0} \leq \frac{(1-\gamma)}{2}||v||^2_{\mathcal{H}_2} + \frac{1}{2(1-\gamma)}||v||^2_{\mathcal{H}_0}. \quad (2.12)$$

Integrating from 0 to T, using (2.12) with $\gamma = \frac{\delta}{2+\delta}$ and the hypothesis on the nonlinear term, we obtain

$$\int_0^T ||\dot{\phi_m}(t)||^2_{\mathcal{H}_0}\, ds + \frac{\gamma}{2}||\phi_m(T)||^2_{\mathcal{H}_2} \leq C_1. \quad (2.13)$$

where C_1 depends on f, ϕ_0 and Ω. This allows us to conclude that

vi. $\dot{\phi}_m$ is uniform bounded in $L^2(0,T;\mathcal{H}_0)$,

vii. ϕ_m is uniform bounded in $L^\infty(0,T;\mathcal{H}_2)$.

Thanks to (vi), we have that $\dot{\phi}_m \to \dot{\phi}$ weakly in $L^2(0,T;\mathcal{H}_0)$ for the uniqueness of the limit in weaker spaces. Moreover, we can infer form (vii) and Lemma 2.1.3 that ϕ is bounded in $L^\infty(0,T;\mathcal{H}_2)$. In particular we can write the problem in the following form

$$a(\phi(t), v) = -(\dot{\phi}(t) + f(\phi(t), v), \qquad \forall v \in \mathcal{H}_2, \text{ a.e. } t \in (0,T).$$

Applying the regularity results for elliptic equations, we obtain that $\phi(t) \in H^4(\Omega) \cap H^1_0(\Omega)$. Then we are able to apply the Green formula with $v \in C^\infty_0(\Omega)$ and we deduce, by means the du Bois-Raymond Lemma, that

$$\phi_t + \Delta^2\phi + 2\Delta\phi + f(\phi) = 0, \qquad \text{in } \mathcal{H}_0, \text{ a.e. } t \in (0,T).$$

Considering a function test $v \in \mathcal{H}_2$, we can conclude that $\Delta\phi = 0$ on $\partial\Omega$. Consequently, using 2.13, an easy computation shows that

$$\phi \in C([0,T];\mathcal{H}_2) \cap L^2(0,T;\mathcal{H}^4).$$

Let us consider two strong solutions ϕ_1, ϕ_2 with initial data ϕ_{01},ϕ_{02} in $\mathcal{H}_2$. Rewriting the problem for the difference $\phi = \phi_1 - \phi_2$, we obtain

$$\phi_t(t) + \Delta^2\phi(t) + 2\Delta\phi(t) + f(\phi_1(t)) - f(\phi_2(t)) = 0, \quad \text{in } \mathcal{H}_0, \text{ a.e } t \in (0,T).$$

We multiply by ϕ_t and integrate on Ω, getting

$$||\phi_t(t)||^2_{\mathcal{H}_0} + (\Delta^2\phi(t), \phi_t(t)) + (2\Delta\phi(t), \phi_t(t)) + (f(\phi_1(t)) - f(\phi_2(t)), \phi_t(t)) = 0.$$

In a standard way, we deduce that the following inequality holds

$$\begin{aligned}||\phi_t(t)||^2_{\mathcal{H}_0} + &\frac{d}{dt}\left\{\frac{1}{2}||\phi(t)||^2_{\mathcal{H}_2} - ||\nabla\phi(t)||^2_{\mathcal{H}_0}\right\} \\ &\leq || \int_0^1 f'(\tau\phi_1(t) + (1-\tau)\phi_2(t))(\phi(t))\ d\tau||_{\mathcal{H}_0}||\phi_t(t)||_{\mathcal{H}_0} \\ &\leq C_2||\phi(t)||_{\mathcal{H}_0}||\phi_t(t)||_{\mathcal{H}_0}.\end{aligned}$$

Integrating from 0 to t and using the inequality (2.12), we have

$$\frac{1}{2}\int_0^t ||\phi_t(t)||^2_{\mathcal{H}_0}\ ds + \frac{\gamma}{2}||\phi(t)||^2_{\mathcal{H}_2} \leq ||\phi_0||^2_{\mathcal{H}} + ||\phi(t)||^2_{\mathcal{H}_0} + C_3\int_0^t ||\phi(s)||^2_{\mathcal{H}_0}\ ds.$$

Combining last inequality with (2.2), we conclude that

$$||\phi_t||_{L^2(0,T;\mathcal{H}_0)} + ||\phi||_{C^0(0,T;\mathcal{H}_2)} \leq C_4 e^{C_5 T}||\phi_0||_{\mathcal{H}_2}.$$

□

2.3 Modified Swift-Hohenberg equation

In this section we are interested in finding results of existence, uniqueness and regularity of the solution to the modified Swift-Hohenberg equation

$$\begin{cases} \sigma\phi_{tt} + \phi_t + \Delta^2\phi + 2\Delta\phi + f(\phi) = 0 & \text{in } \Omega \times (0,T) \\ \phi = \Delta\phi = 0 & \text{on } \partial\Omega \times (0,T) \\ \phi(0) = \phi_0 & \text{in } \Omega \\ \phi_t(0) = \phi_1 & \text{in } \Omega, \end{cases} \tag{2.14}$$

where σ is a positive constant.

Linear problems and their features are essential in order to achieve one of the main tasks of this section, namely the result of continuity of the weak solutions to problem (2.14) from initial data. Accordingly, we review some of the standard facts on the following linear hyperbolic equation of the fourth-order

$$\begin{cases} \sigma u_{tt} + \Delta^2 u = g(t) & \text{in } \Omega \times (0,T) \\ u = \Delta u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = u_0 & \text{in } \Omega \\ u_t(0) = u_1 & \text{in } \Omega. \end{cases} \tag{2.15}$$

We introduce the definition of weak formulation of the problem (2.15).

Definition 2.3.1. Let $T > 0$ be given. A pair (u, u_t), is a weak solution if $(u, u_t) \in L^\infty(0,T;\mathcal{E}_0)$, $u_{tt} \in L^\infty(0,T;\mathcal{H}_2^*)$ such that

(1) $< \sigma u_{tt}(t), v >_{\mathcal{H}_2^*,\mathcal{H}_2} + (\Delta u(t), \Delta v) = (g(t), v),$
$\forall v \in \mathcal{H}_2,$ a.e. $t \in (0,T),$

(2) $u(0) = u_0$ in $\mathcal{H}_2, u_t(0) = u_1$ in $\mathcal{H}_0$.

Before stating the result to be proved, we want to motivate the study of the linear problem (2.15). The regularity of the weak solution to the modified Swift-Hohenberg equation will not be sufficient to use ϕ_t as test function, so it will not be possible to apply the approach adopted for the Swift-Hohenberg equation. To overcome this difficulty, we will prove an energy identity for (2.15) and we will read (2.14) as a linear problem. In this way, we introduce the following functional associated to the solution (u, u_t)

$$\mathcal{J}(t) = \frac{1}{2}\int_\Omega \left(\sigma|u_t(t)|^2 + |\Delta u(t)|^2\right)\, dx.$$

Theorem 2.3.2. *Let us consider $u_0 \in \mathcal{H}_2$, $u_1 \in \mathcal{H}_0$ and $g \in L^2(0,T;\mathcal{H}_0)$. Then the problem* (2.15) *admits a unique weak solution*

$$u \in C([0,T];\mathcal{H}_2) \cap C^1([0,T];\mathcal{H}_0).$$

Moreover, the following energy identity holds $\forall t \in [0,T]$

$$\mathcal{J}(t) = \mathcal{J}(0) + \int_0^t \int_\Omega g(s) u_t(s)\, dx\, ds. \tag{2.16}$$

Proof. As in the proof of Theorem 2.2.2, we use the Galerkin method to define a family of approximating solutions. The discretization is performed by means of the eigenfunctions $\{w_k\}_{k\geq 1}$, corresponding to the bilinear form associated to Δ^2 in $\mathcal{H}_2$. In particular, we can write

$$u_{0m} = \sum_{k=1}^m u_{0k} w_k \to u_0 \text{ in } \mathcal{H}_2, \quad u_{1m} = \sum_{k=1}^m u_{1k} w_k \to u_1 \text{ in } \mathcal{H}_0.$$

Since $C([0,T];\mathcal{H}_0)$ is dense in $L^2(0,T;\mathcal{H}_0)$, there exists a sequence $g_n \subset C([0,T];\mathcal{H}_0)$ such that $g_n \to g$ in $L^2(0,T;\mathcal{H}_0)$. We fix $m \in \mathbb{N}$ and we define $\mathcal{H}_{2,m}$ =span$\{w_1,...,w_m\}$. We find a function of the form

$$u_m(t) = \sum_{k=1}^{m} c_k(t)\, w_k,$$

solution to the linear problem

$$\begin{aligned} &(1)\ (\sigma\ddot{u}_m(t),v) + (\Delta u_m(t),\Delta v) = (g_m(t),v),\\ &\qquad\qquad\qquad\qquad \forall v \in \mathcal{H}_{2,m},\ \text{a.e. } t \in (0,T), \\ &(2)\ \ u_m(0) = u_{0m} \text{ in } \mathcal{H}_2, \dot{u}_m(0) = u_{1m} \text{ in } \mathcal{H}_0. \end{aligned} \tag{2.17}$$

The linear problem (2.17) is equivalent to the following linear system of ODEs

$$\begin{cases} \sigma\ddot{\mathbf{C}}_m(t) + \Lambda\mathbf{C}_m(t) = \mathbf{G}_m(t) \\ \mathbf{C}_m(0) = \mathbf{C}_{0m} \\ \dot{\mathbf{C}}_m(0) = \mathbf{C}_{1m}, \end{cases} \tag{2.18}$$

where

$$\begin{aligned} \mathbf{C}_m(t) &= (c_1(t),...,c_m(t)), \qquad\qquad \Lambda = diag((\Delta w_k,\Delta w_k)),\\ \mathbf{G}_m(t) &= ((g_1(t),w_1),...,(g_m(t),w_m)),\\ \mathbf{C}_{0m} &= (u_{01},...,u_{0m}), \qquad\qquad\quad \mathbf{C}_{1m} = (u_{11},...,u_{1m}). \end{aligned}$$

We observe that $\mathbf{G}_m(t) \in C([0,T];\mathbb{R}^m)$, so there exists a unique global solution $\mathbf{C}_m(t) \in C^2([0,T];\mathbb{R}^m)$. This implies that there exists a unique solution to problem (2.17), $u_m \in C^2([0,T];\mathcal{H}_2)$. Testing by $v = \dot{u}_m$ in (2.17) we have

$$\frac{\sigma}{2}\frac{d}{dt}||\dot{u}_m(t)||^2_{\mathcal{H}_0} + \frac{1}{2}\frac{d}{dt}||u_m(t)||^2_{\mathcal{H}_2} = (g_m(t),\dot{u_m}(t)).$$

Integrating from 0 to t, we obtain the discrete energy identity

$$\int_\Omega \frac{\sigma\dot{u}_m(t)^2}{2} + \frac{\Delta u_m(t)^2}{2}dx = \int_\Omega \frac{\sigma u_{1m}^2}{2} + \frac{\Delta u_{0m}^2}{2}dx + \int_0^t\int_\Omega g_m(s)\dot{u}_m(s)\ dxds.$$

Let us consider $m \geqslant n$, we compute the difference between the problems related to u_m and u_n and we take $v = \dot{u}_m(t) - \dot{u}_n(t)$ as test function

$$\begin{aligned} \frac{\sigma}{2}\frac{d}{dt}||\dot{u}_m(t) - \dot{u}_n(t)||^2_{\mathcal{H}_0} &+ \frac{1}{2}\frac{d}{dt}||u_m(t) - u_n(t)||^2_{\mathcal{H}_2}\\ &= (g_m(t) - g_n(t), \dot{u}_m(t) - \dot{u}_n(t)). \end{aligned}$$

After applying the Cauchy-Schwarz inequality and the Gronwall Lemma, we obtain

$$\sigma||\dot{u}_m(t) - \dot{u}_n(t)||^2_{\mathcal{H}_0} + ||u_m(t) - u_n(t)||^2_{\mathcal{H}_2}$$
$$\leq e^T\{||u_{1m} - u_{1n}||^2_{\mathcal{H}_0} + ||u_{0m} - u_{0n}||^2_{\mathcal{H}_2} + \int_0^t \frac{1}{\sigma}||g_m(s) - g_n(s)||^2_{\mathcal{H}_0}\ ds\}.$$

Since $u_{1m} \to u_1$ in $\mathcal{H}_0$, $u_{0m} \to u_0$ in $\mathcal{H}_2$, $g_m \to g$ in $L^2(0, T, \mathcal{H}_0)$, we conclude that u_m is a Cauchy sequence in $C([0, T]; \mathcal{H}_2) \cap C^1([0, T]; \mathcal{H}_0)$ and we call u its limit. Let us now prove that u is the solution of the problem (2.15) and it fulfils the energy identity (2.16): first of all it is easy to verify that $\ddot{u}_m$ is uniformly bounded in $L^2(0, T; \mathcal{H}_2^*)$ so, at least for a subsequence for $n \nearrow \infty$, $\ddot{u}_m \to \ddot{u}$ weakly in $L^2(0, T; \mathcal{H}_2^*)$. Repeating the same argument of the Theorem 2.2.2 for the passage to the limit, we can conclude that u is a weak solution of (2.15). Thanks to convergence properties of the discrete solution as well as of g_m, u_{0m} and u_{1m}, we can also pass to the limit in the discrete energy identity and we conclude that 2.16 holds for the solution u. To complete the proof we get the uniqueness of the solution u: let u_1 and u_2 be two solution with the same initial data and we consider $u(t) = u_1(t) - u_2(t)$.
Let define the function $v(t)$ as follows

$$\begin{cases} \int_s^t u(\tau)d\tau, & \text{if } 0 \leq t \leq s, \\ 0, & \text{if } s \leq t \leq T. \end{cases} \tag{2.19}$$

We observe that $v(t) \in \mathcal{H}_2$, $\forall t \in [0, T]$, thus we can use $v(t)$ as a test function and we integrate from 0 to s

$$\int_0^s < \ddot{u}(t), v(t) >_{\mathcal{H}_2^*, \mathcal{H}_2}\ dt + \int_0^s a(u(t), v(t))\ dt = 0.$$

Integrating by parts in time in both terms, we have

$$\int_0^s \frac{d}{dt}[\sigma||u(t)||^2_{\mathcal{H}_0} - ||\Delta v(t)||^2_{\mathcal{H}_0}]\ dt = 0.$$

This implies that

$$\sigma||u(s)||^2_{\mathcal{H}_0} + ||\Delta v(0)||^2_{\mathcal{H}_0} = 0\ \forall s \in [0, T] \Rightarrow u(s) \equiv 0.$$

□

Now we study the well-posedness of the modified Swift-Hohenberg equation in terms of the next definition.

Definition 2.3.3. Let $T > 0$ be given. A pair (ϕ, ϕ_t) is a weak solution if $(\phi, \phi_t) \in L^\infty(0, T; \mathcal{E}_0)$, $\phi_{tt} \in L^\infty(0, T; \mathcal{H}_2^*)$ such that

(1) $< \sigma\phi_{tt}(t), v >_{\mathcal{H}_2^*, \mathcal{H}_2} + (\phi_t(t), v) + a(\phi(t), v) + (f(\phi), v) = 0,$
$\forall v \in \mathcal{H}_2$, a.e. $t \in (0, T)$, (2.20)

(2) $\phi(0) = \phi_0$ in $\mathcal{H}_2, \phi_t(0) = \phi_1$ in $\mathcal{H}_0$.

Next, we state our main result of this section.

Theorem 2.3.4. *Let* $(\phi_0, \phi_1) \in \mathcal{E}_0$*. Then the problem* (2.14) *has a unique weak solution* $\phi \in C([0,T];\mathcal{H}_2) \cap C^1([0,T];\mathcal{H}_0)$*. Moreover, any weak solution satisfies the following inequality,* $\forall t \in [0,T]$,

$$||(\phi_1 - \phi_2, \phi_{1,t} - \phi_{2,t})(t)||^2_{\mathcal{E}_0} \leq ||(\phi_{10} - \phi_{20}, \phi_{11} - \phi_{21}||^2_{\mathcal{E}_0} C_1 e^{C_2 t}, \tag{2.21}$$

where ϕ_1*,* ϕ_2 *are weak solutions to* (2.14)*, respectively with initial data* ϕ_{10}, ϕ_{11} *and* ϕ_{20}, ϕ_{21}*.* C_1, C_2 *are positive constants depending on the norm of the initial data as well as on* σ, Ω *and* f*.*

This result will be proved in much the same way as Theorem 2.2.2.

Proof. **Galerkin approximation scheme**
Applying the same discretization technique used for the Swift-Hohenberg equation, we obtain the approximating problem

$$(1)\ (\sigma\ddot{\phi}_m(t), v) + (\dot{\phi}_m(t), v) + a(\phi_m(t), v\ ) + (f(\phi_m(t)), v) = 0, \quad \forall v \in \mathcal{H}_{2,m} \text{ a.e. } t \in [0,T] \tag{2.22}$$
$$(2)\ \phi_m(0) = \phi_{0m} \text{in } \mathcal{H}_2, \dot{\phi}_m(0) = \phi_{1m} \text{ in } \mathcal{H}_0,$$

where $\phi_{0m} = \sum_{k=1}^{m} \alpha_k u_k \to \phi_0$ in $\mathcal{H}_2$, $\phi_{1m} = \sum_{k=1}^{m} \beta_k u_k \to \phi_1$ in $\mathcal{H}_0$.
This is equivalent to a system of ODEs which admits a local unique solution in $C^2([0,T^*], \mathbb{R}^m)$, so we deduce the existence of a unique solution $\phi_m \in C^2([0,T^*], \mathcal{H}_{2,m})$ to problem (2.22).

Energy estimates of the solutions
Taking $v = \dot{\phi}_m(t)$ as a test function, we have

$$\frac{d}{dt}\left\{\frac{\sigma}{2}||\dot{\phi}_m(t)||^2_{\mathcal{H}_0} + \frac{1}{2}a(\phi_m(t), \phi_m(t)) + \int_\Omega F(\phi_m(t))dx\right\} + ||\dot{\phi}_m(t)||^2_{\mathcal{H}_0} = 0.$$

Integrating in time from 0 to t, using the inequality (2.12) with $\gamma = \frac{\delta}{2+\delta}$ and the property (**C**), yields that

$$\begin{aligned}\sigma||\dot{\phi}_m(t)||^2_{\mathcal{H}_0} + 2\int_0^t ||\dot{\phi}_m(s)||^2_{\mathcal{H}_0}\, ds + \frac{\gamma}{2}||\phi_m(t)||^2_{\mathcal{H}_2} \\ \leq K_2|\Omega| + \sigma||\phi_{1m}||^2_{\mathcal{H}_0} + ||\phi_{0m}||^2_{\mathcal{H}_2} + 2\int_\Omega F(\phi_{0m})\, dx \\ \leq K_2|\Omega| + \mathcal{Q}(||\phi_0||_{\mathcal{H}_2}, ||\phi_1||_{\mathcal{H}_0}).\end{aligned} \tag{2.23}$$

This implies that

i. ϕ_m is uniformly bounded in $L^\infty(0,T;\mathcal{H}_2)$,

ii. $\dot{\phi}_m$ is uniformly bounded in $L^\infty(0,T;\mathcal{H}_0)$.

Thanks to (i) and to the global existence Theorem for ODEs, we can conclude that $\phi_m, \dot{\phi}_m$ are defined in $[0,T]$. Moreover, by the Sobolev embedding $\mathcal{H}_2 \hookrightarrow L^\infty(\Omega)$ and the continuity of the nonlinear term, we get

iii. $f(\phi_m)$ is uniformly bounded in $L^\infty(Q)$.

To obtain an energy estimate for $\ddot{\phi}_m$, we take $v \in \mathcal{H}_2$ and let $v = z + w$, where $z \in \mathcal{H}_{2,m}$ and $w \in \mathcal{H}_{2,m}^\perp$ by the projection Theorem. In a standard way, we have

$$\begin{aligned}
| < \ddot{\phi}_m(t), v >_{\mathcal{H}_2^*,\mathcal{H}_2}| = \ & |(\ddot{\phi}_m(t), v)| \\
&= | - (\dot{\phi}_m(t), z) - a(\phi_m(t), z) - (f(\phi_m(t)), z)| \\
&\leq C\left(||\dot{\phi}_m(t)||_{\mathcal{H}_0} + ||\phi_m(t)||_{\mathcal{H}_2} + ||f(\phi_m(t))||_{\mathcal{H}_0}\right)||z||_{\mathcal{H}_2} \\
&\leq C\left(||\dot{\phi}_m(t)||_{\mathcal{H}_0} + ||\phi_m(t)||_{\mathcal{H}_2} + ||f(\phi_m(t))||_{\mathcal{H}_0}\right)||v||_{\mathcal{H}_2}.
\end{aligned}$$

Using (i), (ii), (iii) in the last expression, we deduce that

iv. $\ddot{\phi}_m$ is uniformly bounded in $L^\infty(0,T;\mathcal{H}_2^*)$.

Passage to the limit

As a consequence of the uniform estimates (i)-(iv) and by means compactness argument, the following convergences hold true

$$\phi_m \to \phi \text{ weakly star in } L^\infty(0,T;\mathcal{H}_2),$$

$$\dot{\phi}_m \to \dot{\phi} \text{ weakly star in } L^\infty(0,T;\mathcal{H}_0),$$

$$\ddot{\phi}_m \to \ddot{\phi} \text{ weakly star in } L^\infty(0,T;\mathcal{H}_2^*).$$

Combining (i) and (ii) with Theorem 2.1.1, we can assert that

$$\phi_m \to \phi \text{ in } L^2(0,T;\mathcal{H}_0) \quad \Rightarrow \quad \phi_m \to \phi \text{ a.e. in } Q$$

$$\Rightarrow f(\phi_m) \to f(\phi) \text{ a.e. in } Q,$$

by continuity of the nonlinear term f.
From Theorem 2.1.2 and (iii), we obtain

$$f(\phi_m) \rightharpoonup f(\phi) \text{ in } L^2(Q).$$

Now we are able to prove that the limit function ϕ is a weak solution to problem (2.14). We take $v \in L^2(0,T;\mathcal{H}_2)$ and we define $v_N(t)$ as in the parabolic case. Then, we fix N such that $m > N$ and we take $v_N(t)$ as a test function in (2.22). Integrating in time from 0 to T and passing to the limit as $m \nearrow \infty$ and also as $N \nearrow \infty$, we finally have

$$\int_0^T < \sigma\ddot{\phi}(t), v(t) >_{\mathcal{H}_2^*,\mathcal{H}_2} \; dt + \int_0^T (\dot{\phi}(t), v(t)) \; dt + \int_0^T a(\phi(t), v(t)) \; dt + \int_0^T (f(\phi(t)), v(t)) \; dt = 0. \quad (2.24)$$

Since v is arbitrary, we consider $v(t) = w\chi_{[s,s+h]}(t)$, where $w \in \mathcal{H}_2$, and by means of Lebesgue differentiation Theorem, we get

$$< \sigma\ddot{\phi}(t), w >_{\mathcal{H}_2^*,\mathcal{H}_2} + (\dot{\phi}(t), w) + a(\phi(t), w) + (f(\phi(t)), w) = 0 \qquad \forall w \in \mathcal{H}_2, \text{ a.e. } t \in [0,T].$$

At this level, we also know that $\phi \in C([0,T]; \mathcal{H}_0)$ and $\phi_t \in C([0,T; \mathcal{H}_2^*)$. Thus, performing the standard method, namely integrating by parts in time two times in both discrete problem and (2.24), computing the difference and passing to the limit, we can conclude that

$$\phi_t(0) = \phi_1, \qquad \phi(0) = \phi_0.$$

Uniqueness

Let us consider two weak solutions ϕ_1, ϕ_2, with the same initial data, and their difference u which satisfies the following equation

$$< \sigma\ddot{u}(t), v >_{\mathcal{H}_2^*,\mathcal{H}_2} + (\dot{u}(t), v) + a(u(t), v) + (f(\phi_1(t)) - f(\phi_2(t)), v) = 0.$$

with $u(0) = \dot{u}(0) = 0$. We define the function $w(t)$ as follows

$$\begin{cases} \int_s^t u(\tau)d\tau, & \text{if } 0 \le t \le s, \\ 0, & \text{if } s \le t \le T. \end{cases} \quad (2.25)$$

and we observe that $w(t) \in \mathcal{H}_2$ and $\dot{w}(t) = -u(t)$.
Integrating from 0 to T the equation for u and taking $v = w(t)$, we have

$$\int_0^s < \sigma\ddot{u}(t), w(t) >_{\mathcal{H}_2^*,\mathcal{H}_2} \; dt + \int_0^s (\dot{u}(t), w(t)) \; dt + \int_0^s a(u(t), w(t)) \; dt + \int_0^s (f(\phi_1(t)) - f(\phi_2(t)), w(t)) \; dt = 0.$$

Using integrations by parts in the first three terms, we obtain

$$\begin{aligned} \frac{\sigma}{2}||u(s)||^2_{\mathcal{H}_0} + \int_0^s ||u(t)||^2_{\mathcal{H}_0} \; dt + \frac{1}{2}||\Delta w(0)||^2_{\mathcal{H}_0} - ||\nabla w(0)||^2_{\mathcal{H}_0} \\ = -\int_0^s (f(\phi_1(t)) - f(\phi_2(t)), w(t)) \; dt \\ \le \int_0^s || \int_0^1 f'(\tau\phi_1 + (1-\tau)\phi_2)u(t) \; d\tau||_{\mathcal{H}_0} ||w(t)||_{\mathcal{H}_0} \\ \le \int_0^s M||u(t)||_{\mathcal{H}_0}||w(t)||_{\mathcal{H}_0} \; dt. \end{aligned}$$

where M is a positive constant depending on f, ϕ_1, ϕ_2.
We observe that

$$||w(t)||^2_{\mathcal{H}_0} \le \int_\Omega (\int_0^s |u(r)|\ dr)^2\ dx \le s \int_0^s ||u(r)||^2_{\mathcal{H}_0}\ dr, \tag{2.26}$$

and it follows that

$$\int_0^s ||w(t)||^2_{\mathcal{H}_0}\ dt \le \int_0^s s \int_0^s ||u(r)||^2_{\mathcal{H}_0}\ dr\ dt \le C_1 \int_0^s ||u(r)||^2_{\mathcal{H}_0}\ dr.$$

Thus we deduce that

$$\begin{aligned} \frac{\sigma}{2}||u(s)||^2_{\mathcal{H}_0} + \int_0^s ||u(t)||^2_{\mathcal{H}_0}\ dt + \frac{1}{2}||\Delta w(0)||^2_{\mathcal{H}_0} - ||\nabla w(0)||^2_{\mathcal{H}_0} \\ \le C_2 \int_0^s ||u(t)||^2_{\mathcal{H}_0}\ dt. \end{aligned}$$

Using the inequality (2.4), we get

$$\begin{aligned} \frac{\sigma}{2}||u(s)||^2_{\mathcal{H}_0} + \int_0^s ||u(t)||^2_{\mathcal{H}_0}\ dt + \frac{1}{4}||\Delta w(0)||^2_{\mathcal{H}_0} \\ \le C_2 \int_0^s ||u(t)||^2_{\mathcal{H}_0}\ dt + ||w(0)||^2_{\mathcal{H}_0}. \end{aligned}$$

Applying once more the inequality (2.26) and the Gronwall Lemma, we conclude that $u(s) \equiv 0\ \forall s \in [0, T]$.

Continuous dependence from initial data

Let us consider two weak solutions ϕ_1, ϕ_2, respectively with initial data ϕ_{10}, ϕ_{11} and ϕ_{20}, ϕ_{21}. We define their difference u and we rewrite the problem for u as follows

$$\begin{cases} \sigma u_{tt} + \Delta^2 u = -2\Delta u - u_t - f(\phi_1) + f(\phi_2) & \text{in } \Omega \times (0, T) \\ u = \Delta u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(0) = \phi_{10} - \phi_{20} & \text{in} \Omega \\ u_t(0) = \phi_{11} - \phi_{21} & \text{in } \Omega. \end{cases}$$

Since $u_t \in L^2(Q)$, $\Delta u \in L^2(Q)$ and $f(\phi_i) \in L^2(Q)$ for $i = 1, 2$, we read u as the solution to the linear problem (2.15) with

$$g := -2\Delta u - u_t - f(\phi_1) + f(\phi_2) \in L^2(Q).$$

Using the energy identity (2.16) and the Sobolev embedding $\mathcal{H}_2 \hookrightarrow L^\infty(\Omega)$, we have

$$\begin{aligned} \sigma||u_t(t)||^2_{\mathcal{H}_0} + ||u(t)||^2_{\mathcal{H}_2} &= -2\int_0^t ||u_t(s)||^2_{\mathcal{H}_0}\ ds - 4\int_0^t (\Delta u(s), u_t(s))\ ds \\ &\quad - 2\int_0^t (f(\phi_1) - f(\phi_2), u_t(s))\ ds \\ &\quad + ||u_t(0)||^2_{\mathcal{H}_0} + ||u(0)||^2_{\mathcal{H}_2} \\ &\le ||u_t(0)||^2_{\mathcal{H}_0} + ||u(0)||^2_{\mathcal{H}_2} + C_1 \int_0^t ||u(s)||^2_{\mathcal{H}_2}\ ds. \end{aligned}$$

By the Gronwall Lemma, we finally get

$$||\phi_{1,t} - \phi_{2,t}||^2_{\mathcal{H}_0} + ||\phi_1 - \phi_2||^2_{\mathcal{H}_2} \leq \left\{ ||\phi_{11} - \phi_{21}||^2_{\mathcal{H}_0} + ||\phi_{10} - \phi_{20}||^2_{\mathcal{H}_2} \right\} e^{C_1 t}.$$

□

2.4 Stationary Swift-Hohenberg equation

We want to establish the existence of a weak solution to the stationary equation associated to the evolution problems previously studied.

$$\begin{cases} \Delta^2\phi + 2\Delta\phi + f(\phi) = 0 & \text{in } \Omega \subseteq \mathbb{R}^3 \\ \phi = \Delta\phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.27}$$

We start by introducing the weak formulation of the problem (2.27).

Definition 2.4.1. $\phi \in \mathcal{H}_2$ is a weak solution if

$$\int_\Omega \left[\Delta\phi\Delta v - 2\nabla\phi \cdot \nabla v + f(\phi)\, v\right] dx = 0, \qquad \forall v \in \mathcal{H}_2. \tag{2.28}$$

Theorem 2.4.2. *The stationary problem admits a weak solution.*

Proof. The strategy is based on the direct method of the calculus of variation. Let us consider the following functional associated to the problem

$$G(v) = \int_\Omega \left[\frac{1}{2}(\Delta v)^2 - |\nabla u|^2 + F(v)\right] dx, \qquad \forall v \in \mathcal{H}_2. \tag{2.29}$$

Using the inequality (2.12) with $\gamma = \frac{\delta}{2+\delta}$ and the hypothesis on the nonlinear term, we see that

$$\begin{aligned} G(\phi) &= \frac{1}{2}||\Delta\phi||^2_{\mathcal{H}_0} - ||\nabla\phi||^2_{\mathcal{H}_0} + \int_\Omega F(\phi)\, dx \\ &\geq \frac{\gamma}{2}||\phi||^2_{\mathcal{H}_2} - \frac{2+\delta}{4}||\phi||^2_{\mathcal{H}_0} + \int_\Omega F(\phi)\, dx \\ &\geq \frac{\gamma}{2}||\phi||^2_{\mathcal{H}_2} - K_2|\Omega|. \end{aligned} \tag{2.30}$$

Hence $G(\phi)$ is bounded from below and $G(\phi) \to +\infty$ as $\phi \to +\infty$ in $\mathcal{H}_2$.
If we set

$$\lambda = \inf_{v \in \mathcal{H}_2} G(v),$$

by definition there exists a sequence $v_n \in \mathcal{H}_2$ such that $G(v_n) \to \lambda$. It follows that there exists C such that $|G(v_n)| \leq C$ and we conclude from (2.30) that v_n is bounded in $\mathcal{H}_2$. Thanks to the Banach-Alaoglu Theorem, we get, at least for subsequence of $n \nearrow \infty$, $v_n \to \phi$ weakly in $\mathcal{H}_2$ and, by the Sobolev embedding, $v_n \to \phi$ in $\mathcal{H}_1$ as well as v_n is bounded in $L^\infty(\Omega)$.

Moreover, since F is continuous, $F(v_n) \to F(\phi)$ a.e. in Ω and we conclude from dominated convergence Theorem that

$$\int_\Omega F(v_n)\; dx \to \int_\Omega F(\phi)\; dx.$$

Consequently, we obtain

$$G(\phi) \le \liminf_n\; G(v_n) = \lambda.$$

Therefore, ϕ is a minimum of $G(v)$ in $\mathcal{H}_2$ and we have from the Fermat Theorem that

$$G'(\phi)[v] = 0 \qquad \forall v \in \mathcal{H}_2,$$

where with G' we indicate the Gâteaux derivative. An easy computation shows that last equality is equivalent to (2.28). □

3 The global attractor

In this chapter we start to investigate the long-time behavior of the solutions corresponding to the Swift-Hohenberg equation and the modified Swift-Hohenberg equation. In particular, we study the permanent regime of the trajectories independently of their initial data. In the first section we will introduce the definitions and results of the theory of dissipative evolution equations in infinite-dimensional space. Firstly, we focus on the role of the global attractor. In the second and third sections we will prove the existence of the global attractor for the models concerned.

3.1 Basic definitions and existence results

Let X be a real Banach space and let $u(t) \in X$ the solution to the autonomous Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t) = \mathcal{F}(u(t)) \quad \forall t > 0, \\ u(0) = u_0 \in X. \end{cases} \tag{3.1}$$

The basic issue to analyse a physical or mechanical phenomenon governed by evolution equations is to establish their well-posedness in terms of existence, uniqueness and regularity properties. These information allow us to define an abstract family of maps

$$S(t) : X \to X, \qquad S(t)u_0 = u(t), \ \ \forall t \geq 0.$$

In this approach, X consists of all possible states of the system (phase space) and $S(t)$ determines the state at time t associated to an initial state in $t = 0$. Existence and uniqueness of global solutions to (3.1) imply that $S(t)$ is well-defined for each $t \geq 0$. Since $u(t)$ satisfies the initial condition, it is clear that $S(0) = \mathcal{I}$, where $\mathcal{I}$ denotes the identity in X. In addition, the system is autonomous, namely the time does not appear explicitly in $\mathcal{F}$, thus we can write that $S(t+\tau) = S(t) \circ S(\tau)$. Furthermore, solutions of partial differential equations are often continuous functions of t or with respect to the initial data, so $S(t)u_0$ turns out to be continuous in t or u_0.

Definition 3.1.1. A one parameter family of maps $S(t) : X \to X$, $\forall t \geq 0$, is a strongly continuous semigroup of operators if

S.1 $S(0) = \mathbb{I}$,

S.2 $S(t+\tau) = S(t)S(\tau), \quad \forall t, \tau \geq 0$,

S.3 $S(\cdot)x \in C([0,\infty); X), \quad \forall x \in X$,

S.4 $S(t) \in C(X; X), \quad \forall t \geq 0$,

A pair $(X, S(t))$ is called dynamical system.

In what follows, we only consider semigroups which satisfy the above Definition, because the semigroups associated to the equations of the previous chapter fulfil these properties. More generally, X can be a compete metric space and (**S.4**) can be replaced by the closed semigroup property, i.e.

$$\text{if } x_k \to x \text{ and } S(t)x_k \to y, \text{ then } y = S(t)x.$$

The analysis to describe the long-time behavior of solutions of physical models is usually carried out by means the existence of subsets of the phase space which satisfy specific properties of attraction with respect to the semigroup (see [1], [2] or [22] for more details). In this framework, by considering the evolutions of the whole system, it is essential to think to a single trajectory as one of a family which starts from a generic subset of X. In particular, we will prove the so-called dissipative estimates, which represent the distance between the trajectories and a suitable set, independently of the features of the single solution. In application of the theory, we will often confine the dynamics on a subset of the phace space, that is the restriction of $S(t)$ on $B \subset X$, so we define the following properties.

Definition 3.1.2. A nonempty set $B \subset$ X is positively invariant for $S(t)$ if

$$S(t)B \subset B, \quad \forall t \geq 0. \tag{3.2}$$

A nonempty set $B \subset$ X is fully invariant for $S(t)$ if

$$S(t)B = B, \quad \forall t \geq 0, \tag{3.3}$$

where $S(t)B = \bigcup_{x \in B} S(t)x$.

A first useful example in description of the long-time behavior is the ω-limit set.

Definition 3.1.3. The ω-limit set of a nonempty set $B \subset X$ is given by

$$\omega(B) = \{x \in X : \exists t_n \to \infty, x_n \in B \text{ with } S(t_n)x_n \to x\}.$$

This set is essential to construct the global attractor and it satisfies the properties summarized in the next statement.

Proposition 3.1.4. *Let $B \subset X$ and $\omega(B)$ be a nonempty set. Then we have*

1. $\omega(B) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} S(\tau)B}^X$,

2. $\omega(S(t)B) = \omega(B), \quad \forall t \geq 0,$

3. $\omega(B)$ *is positively invariant.*

We now introduce two important sets which define the dissipative property of dynamical systems from mathematical point of view.

Definition 3.1.5. A nonempty set $B_0 \subset X$ is called absorbing set for $(X, S(t))$ if for every bounded set $B \subset X$ there exists $t_B \geq 0$ such that

$$S(t)B \subset B_0, \quad \forall t \geq t_B. \tag{3.4}$$

A dynamical system is said to be a dissipative system if it has a bounded absorbing set. Given an absorbing set B_0, it is possible to construct a bounded positively invariant absorbing set for the semigroup $S(t)$ setting

$$B_1 = \bigcup_{t \geq t_0} S(t)B_0,$$

where $t \geq t_0$ implies $S(t)B_0 \subset B_0$.

Definition 3.1.6. If A and B are nonempty subsets of X, the Haursdoff semidistance between A and B is defined as follows

$$dist_H(A, B) = \sup_{x \in A} \inf_{y \in B} ||x - y||_X. \tag{3.5}$$

We remark that the Haursdoff semidistance is not symmetric and it holds

$$dist_H(A, B) = 0 \Rightarrow A \subset \overline{B}.$$

Definition 3.1.7. A nonempty set K is called an attracting set for $(X, S(t))$ if for every bounded set $B \subset X$ there holds

$$\lim_{t \to \infty} dist_H(S(t)B, K) = 0. \tag{3.6}$$

A dynamical system is said to be asymptotically compact if it has a compact attracting set. It is not difficult to prove that an asymptotically compact dynamical system is also a dissipative system.

Definition 3.1.8. A nonempty set $\mathcal{A}$ is the global attractor for the system $(X, S(t))$ if

G.1 $\mathcal{A}$ is fully invariant,

G.2 $\mathcal{A}$ is a compact set,

G.3 $\mathcal{A}$ is an attracting set.

The study of a set, which satisfies Definition 3.1.8, is justified in relation to the consequence that can be inferred. Indeed, the global attractor, when it exists, is unique, is the largest fully invariant bounded set and is the smallest closed attracting set. Furthermore, it contains equilibrium points and periodic orbits of the system. The global attractor can be characterized as the section at time $t = 0$ of all the complete bounded trajectories. It is also possible to prove the following topological property of the global attractor.

Theorem 3.1.9. *Let $(X, S(t))$ be a dynamical system with X connected. Then the global attractor $\mathcal{A}$, if it exists, is connected.*

The ω-limit set of a bounded set has an important connection with the global attractor which is stated in the next result.

Theorem 3.1.10. *Let $B \subset X$ be a nonempty bounded set. Suppose that $\omega(B)$ is nonempty, compact and attracting for $(X, S(t))$. Then $\omega(B)$ is the global attractor of $(X, S(t))$.*

In order to give some results establishing the existence of the global attractor under hypothesis effectively satisfied in applications, we need to introduce additional tools.

Definition 3.1.11. Let $B \subset X$ be a bounded set. The Kuratowski measure of noncompacteness $\alpha(B)$ is defined by

$$\alpha(B) = \inf\{d : B \text{ has a finite cover of balls of } X \text{ of diameter less than } d\}.$$

The measure of noncompacteness satisfies the following properties:

K.1 $\alpha(B) = 0$ if and only if $\overline{B}^X$ is compact,

K.2 $B_1 \subset B_2$ implies $\alpha(B_1) \leq \alpha(B_2)$,

K.3 $\alpha(B_1 \cup B_2) \leq max\{\alpha(B_1), \alpha(B_2)\}$,

K.4 $\alpha(B) = \alpha(\overline{B}^X)$,

K.5 Let $\{B_t\}_{t\geq 0}$ be a family of nonempty bounded closed sets such that $B_{t_2} \subset B_{t_1}$ for $t_1 < t_2$ and $\lim\limits_{t\to\infty} \alpha(B_t) = 0$. Set $B = \bigcap\limits_{t\geq 0} B_t$. Then

1. B is nonempty,
2. B is compact,
3. if the sets B_t are connected for all t, then B is connected.

Now we state the main existence result of the global attractor for the semigroup $S(t)$.

Theorem 3.1.12. *Let us assume that the dynamical system $(X, S(t))$ has a bounded absorbing set $B_0 \subset X$ and there exists a sequence $t_n \geq 0$ such that*

$$\lim_{t_n\to\infty} \alpha(S(t_n)B_0) = 0.$$

Then $\omega(B_0)$ is the global attractor of $(X, S(t))$.

Let us mention some important consequences of the Theorem 3.1.12 involved in application to PDEs. It is worth noting how they are only two of the results which may be formulated.

Theorem 3.1.13. *If $(X, S(t))$ has a compact absorbing set, then there exists the global attractor $\mathcal{A}$.*

Theorem 3.1.14. *Let $(X, S(t))$ be dissipative and let B_0 be an absorbing set. Suppose that $S(t)$ is such that*

$$S(t) = S_1(t) + S_2(t),$$

where

$$\lim_{t\to\infty} ||S_1(t)B_0||_X \to 0,$$

and

$$S_2(t)B_0 \subset K \text{ compact}, \quad \forall t \geq 0.$$

Then there exists the global attractor $\mathcal{A} \subset K$.

We will use two different strategies to the Swift-Hohenberg and to the modified Swift-Hohenberg equations. Indeed, in the former case, the parabolic nature of the equation allows us to prove the existence of a compact absorbing set for the semigroup and we conclude the existence of the global attractor by means of Theorem 3.1.13. Conversely, in the latter case, due to the presence of term ϕ_{tt}, this approach seems not work, so we will split the semigroup as in Theorem 3.1.14 .

The global attractor seems to be the most important object in terms of information to describe the long-time behavior of a solution to dissipative models; nevertheless it presents some defects. A first problem is the rate of attraction of the global attractor, which may be small and not estimable with the physical parameters of the problem. A second drawback is a lack of robustness with respect to perturbations. Actually it is possible to prove, as we will see in the fifth chapter, the upper semicontinuity of the global attractor, namely

$$dist_X(\mathcal{A}_\varepsilon, \mathcal{A}) \to 0, \quad \text{as } \varepsilon \to 0^+,$$

where $\mathcal{A}$ is the global attractor of the system and $\mathcal{A}_\varepsilon$ is the global attractor of the perturbed system. Roughly speaking, this property means that the global attractor cannot explode under perturbations. It is more difficult to prove and it may not hold that the global attractor be lower semicontinuous, i.e.

$$dist_X(\mathcal{A}, \mathcal{A}_\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0^+.$$

Conversely, this means that the global attractor cannot implode. To overcome the above defects, it was introduced a bigger set which contains the global attractor, attracts the trajectories with a very fast rate and is more robust under perturbations. This object is called exponential attractor and it will be the main theme of the next chapters.

3.2 Swift-Hohenberg equation

On account of the Theorem 2.2.2 of the previous chapter, we define the strongly continuous semigroup $S(t)$ on the phace space $\mathcal{H}_0$ such that

$$S(t) : \mathcal{H}_0 \to \mathcal{H}_0, \qquad S(t)\phi_0 = \phi(t) \quad \forall t \geq 0,$$

where $\phi(t)$ is the weak solution in the sense of Definition 2.2.1 to problem (2.1) corresponding to the initial data ϕ_0. In this section, we consider the dynamical system $(\mathcal{H}_0, S(t))$ and we prove our main result.

Theorem 3.2.1. *The dynamical system $(\mathcal{H}_0, S(t))$ has the connected global attractor $\mathcal{A}$ which is bounded in $\mathcal{H}_2$.*

The idea of the proof is to apply the general method for parabolic problems as we stated above. Firstly, we built an absorbing set $\mathcal{B}$ in the phase space $\mathcal{H}_0$, therefore $(\mathcal{H}_0, S(t))$ is a dissipative dynamical system. After that we prove how the trajectories, which get into $\mathcal{B}$ in finite time, are uniformly contained in a bounded set of $\mathcal{H}_2$. In order to prove the last property, we will need the following technical result, called Uniform Gronwall Lemma (see [27] for the proof).

Lemma 3.2.2. *Let g, h, y be three positive locally integrable functions on (t_0, ∞) which satisfy*

$$\frac{dy}{dt} \leq gy + h, \quad \forall t \geq t_0,$$

$$\int_t^{t+r} g(s)\ ds \leq a_1, \quad \int_t^{t+r} h(s)\ ds \leq a_2, \quad \int_t^{t+r} y(s)\ ds \leq a_3, \quad t \geq t_0,$$

where r, a_1, a_2, a_3 are positive constants. Then there holds

$$y(t+r) \leq (\frac{a_3}{r} + a_2)e^{a_1}, \quad \forall t \geq t_0.$$

In the study of the long-time behavior of the solutions to the Swift-Hohenberg and the modified Swift-Hohenberg equations, we prove some different estimates under the assumption that ϕ, ϕ_t are sufficiently regular to guarantee the correctness of each passage. This method is only formal, however it can be exactly applied to the approximating solutions introduced in the previous chapter and then pass to the limit as $m \nearrow \infty$.

Lemma 3.2.3. *Let $\phi(t)$ be the weak solution to problem* (2.14)*. Then, for every $t \geq 0$, the following estimate holds*

$$||\phi(t)||^2_{\mathcal{H}_0} \leq ||\phi_0||^2_{\mathcal{H}_0} e^{-\frac{2\gamma}{C_p}t} + \rho_0^2[1 - e^{-\frac{2\gamma}{C_p}t}], \tag{3.7}$$

where ρ_0 is a positive constant depending on K_1, γ and C_p. Moreover, if $||\phi_0||_{\mathcal{H}_0} \leq R$ and $t \geq t_0(R, \rho_1)$, then

$$\int_t^{t+r} ||\phi(s)||^2_{\mathcal{H}_2}\ ds \leq \frac{C_p}{2\gamma}[2K_1 r + \rho_1^2], \tag{3.8}$$

where $\rho_1 > \rho_0$ and $r > 0$.

Proof. Taking $v = \phi$ as a test function, we obtain

$$\frac{1}{2}\frac{d}{dt}||\phi||^2_{\mathcal{H}_0} + ||\phi||^2_{\mathcal{H}_2} - 2||\phi||^2_{\mathcal{H}_1} + (f(\phi), \phi) = 0.$$

We recall the following inequality for $v \in \mathcal{H}_2$

$$2||\nabla v||^2_{L^2(\Omega)} \leq 2||\Delta v||_{L^2(\Omega)}||v||_{L^2(\Omega)} \leq (1-\gamma)||v||^2_{\mathcal{H}_2} + \frac{1}{(1-\gamma)}||v||^2_{\mathcal{H}_0}, \tag{3.9}$$

where $\gamma \in (0, 1)$. In order to use the hypothesis on the nonlinear term, we set $\gamma = \frac{\delta}{1+\delta}$. Using last inequality and (**H2**), we get

$$\frac{1}{2}\frac{d}{dt}||\phi||^2_{\mathcal{H}_0} + \gamma||\phi||^2_{\mathcal{H}_2} \leq K_1|\Omega|. \tag{3.10}$$

In particular, thanks to Poincaré inequality, we have

$$\frac{d}{dt}||\phi||^2_{\mathcal{H}_0} + \frac{2\gamma}{C_p}||\phi||^2_{\mathcal{H}_0} \leq 2K_1|\Omega|.$$

Applying the Gronwall Lemma, we obtain the following dissipative estimate

$$||\phi(t)||_{\mathcal{H}_0}^2 \leq ||\phi_0||_{\mathcal{H}_0}^2 e^{-\frac{2\gamma}{C_p}t} + \frac{K_1|\Omega|C_p}{2\gamma}[1 - e^{-\frac{2\gamma}{C_p}t}], \qquad \forall t \geq 0.$$

Setting $\rho_0 = \sqrt{\frac{K_1|\Omega|C_p}{2\gamma}}$, we can deduce (3.7). Now we fix $r > 0$, $\rho_1 > \rho_0$ and $t_0 = \frac{C_p}{2\gamma}\ln(\frac{R^2}{\rho_1^2 - \rho_0^2})$. Thanks to Lemma 3.7, we have

$$||\phi(t)||_{\mathcal{H}_0} \leq \rho_1, \quad \forall t \geq t_0.$$

Integrating in time (3.10) from t to $t + r$, where $t \geq t_0$, we get

$$||\phi(t+r)||_{\mathcal{H}_0}^2 + \frac{2\gamma}{C_p}\int_t^{t+r} ||\phi(s)||_{\mathcal{H}_2}^2 \, ds \leq 2K_1|\Omega|r + ||\phi(t)||_{\mathcal{H}_0}^2.$$

Thus we can conclude that

$$\int_t^{t+r} ||\phi(s)||_{\mathcal{H}_2}^2 \, ds \leq \frac{C_p}{2\gamma}[2K_1|\Omega|r + \rho_1^2].$$

□

Lemma 3.2.4. *Let $\phi(t)$ be the weak solution to problem* (2.14) *such that $||\phi_0||_{\mathcal{H}_0} \leq R$. Then there holds*

$$||\phi(t)||_{\mathcal{H}_2} \leq Q(\rho_1), \quad \forall t \geq t_0(R, \rho_1) + r. \tag{3.11}$$

where $r > 0$ and Q is a positive monotone function which depends on r, f, Ω but it is independent of R.

Proof. Let $\rho_1 > \rho_0$ and $t_0 = t_0(R, \rho_1)$ be as in Lemma 3.2.3. We take $v = \phi_t$ as test function in (2.2.1) and we obtain

$$||\phi_t||_{\mathcal{H}_0}^2 + \frac{d}{dt}\left\{\frac{1}{2}||\phi||_{\mathcal{H}_2}^2 - ||\phi||_{\mathcal{H}_1}^2 + \int_\Omega F(\phi) \, dx\right\} = 0. \tag{3.12}$$

We define the energy functional

$$\Lambda(t) = ||\phi||_{\mathcal{H}_2}^2 - 2||\phi||_{\mathcal{H}_1}^2 + 2\int_\Omega F(\phi) \, dx, \tag{3.13}$$

and we have

$$\frac{d}{dt}\Lambda(t) \leq 0.$$

Using (3.9) with $\gamma = \frac{\delta}{2+\delta}$ and (2.1) for the nonlinear term, we get

$$\begin{aligned}\Lambda(t) &\geq \gamma||\phi||_{\mathcal{H}_2}^2 - \frac{2+\delta}{4}||\phi||_{\mathcal{H}_0}^2 + 2\int_\Omega F(\phi) \, dx \\ &\geq \gamma||\phi||_{\mathcal{H}_2}^2 - 2K_2|\Omega|.\end{aligned}$$

Let us fix $r > 0$ and let be $t \geq t_0(R, \rho_1)$. We can deduce from Lemma 3.2.3 that $||\phi||_{L^\infty((t,t+r);\mathcal{H}_0)} \leq \rho_1$ as well as $||\phi||_{L^2((t,t+r);\mathcal{H}_2)} \leq Q(r, \rho_1)$. These estimates imply that $||\phi||_{L^\infty((t,t+r);L^\infty(\Omega))} \leq Q(r, \rho_1)$. So we are able to state that

$$\int_t^{t+r} \Lambda(s)\, ds \leq \int_t^{t+r} ||\phi(s)||^2_{\mathcal{H}_2}\, ds + 2\int_t^{t+r}\int_\Omega F(\phi(s))\, dx\, ds \leq Q(r, \rho_1) =: a_3.$$

Using the uniform Gronwall Lemma, we obtain

$$||\phi(t+r)||^2_{\mathcal{H}_2} \leq \frac{1}{\gamma}\left[\frac{a_3}{r} + 2K_2|\Omega|\right], \quad \forall t \geq t_0(R, \rho_1).$$

□

Corollary 3.2.5. *Under the above assumptions, then we have*

$$\int_s^\infty ||\phi_t||^2_{\mathcal{H}_0}\, dt \leq Q(\rho_1), \tag{3.14}$$

where $s = t_0(R, \rho_1) + r$.

Proof. We consider (3.12)

$$||\phi_t||^2_{\mathcal{H}_0} + \frac{d}{dt}\left\{\frac{1}{2}||\phi||^2_{\mathcal{H}_2} - ||\phi||^2_{\mathcal{H}_1} + \int_\Omega F(\phi)\, dx\right\} = 0.$$

Integrating from $s = t_0(R, \rho_1) + r$ to t, we get

$$\begin{aligned}\int_s^t ||\phi_t||^2_{\mathcal{H}_0}\, dt &\leq \frac{1}{2}||\phi(s)||^2_{\mathcal{H}_2} + \int_\Omega F(\phi(s))dx + ||\phi(t)||_{\mathcal{H}_1} \\ &\leq Q(||\phi(s)||_{\mathcal{H}_2}) + ||\phi(t)||_{\mathcal{H}_1},\end{aligned}$$

where Q is a generic positive monotone function. Letting $t \nearrow \infty$ and using the existence of an absorbing set in V, we obtain

$$\int_s^\infty ||\phi_t||^2_H dt \leq Q(\rho_1). \tag{3.15}$$

□

Proof of Theorem 3.2.1. We can assert from Lemma 3.2.4 that

$$\mathcal{X}^0 = B_{\mathcal{H}_2}(0, Q(\rho_1))$$

is a compact absorbing set for $S(t)$ on $\mathcal{H}_0$. The conclusion follows from Theorem 3.1.13. □

3.3 Modified Swift-Hohenberg equation

According to the Theorem 2.3.4, we define the strongly continuous semigroup $S_\sigma(t)$ associated to the problem (2.14) such that

$$S_\sigma(t) : \mathcal{E}_0 \to \mathcal{E}_0, \qquad S_\sigma(t)(\phi_0, \phi_1) = (\phi(t), \phi_t(t)), \quad \forall t \geq 0,$$

where (ϕ, ϕ_t) is the unique solution to problem (2.14) with initial data in $\mathcal{E}_0$. We study the dynamical system $(\mathcal{E}_0, S_\sigma(t))$ in this section.

The first step towards the existence of the global attractor for the semigroup $S_\sigma(t)$ is to prove the dissipative nature of the dynamical system $(\mathcal{E}_0, S_\sigma(t))$. We report a technical result which we will need in the course of our analysis (see [3] for the proof).

Lemma 3.3.1. *Let X be a Banach space and let $\mathcal{C} \subset C([0,\infty), X)$. Let $\Phi : X \to [0,\infty)$ be given such that $\Phi(v(0)) \leq c$, for some $c > 0$ and every $v \in \mathcal{C}$. In addition, assume that for every $v \in \mathcal{C}$ the function $t \mapsto \Phi(v(t))$ be continuously differentiable and satisfy the differential inequality*

$$\frac{d}{dt}\Phi(v(t)) + k||v(t)||_X^2 \leq \omega, \tag{3.16}$$

for some $\omega \geq 0$ and $k \geq 0$ independent of $v \in \mathcal{C}$. Then for every $\delta > 0$ there exists $t_\delta > 0$ such that

$$\Phi(v(t)) \leq \sup_{v \in X}\{\Phi(v) : k||v||_X^2 \leq \omega + \delta\}, \qquad \forall t \geq t_\delta = \frac{c}{\delta}. \tag{3.17}$$

Lemma 3.3.2. *Let (ϕ, ϕ_t) be the weak solution to problem (2.14). Then $\mathcal{V}^0 = B_{\mathcal{E}_0}(0, C)$ is an absorbing set for $(\mathcal{E}_0, S_\sigma(t))$, where C is a positive constant depending on σ, f and Ω but it is independent of the norm of the initial data. Moreover, if the initial data (ϕ_0, ϕ_1) is such that $||(\phi_0, \phi_1)||_{\mathcal{E}_0} \leq R$, the following estimate holds*

$$||(\phi, \phi_t)||_{\mathcal{E}_0}^2 \leq Q(R)e^{-t} + C, \qquad \forall t \geq 0. \tag{3.18}$$

Proof. Let us test (2.14) by ϕ_t

$$\frac{d}{dt}\left\{\frac{\sigma}{2}||\phi_t||_{\mathcal{H}_0}^2 + \frac{1}{2}a(\phi,\phi) + \int_\Omega F(\phi)dx\right\} + ||\phi_t||_{\mathcal{H}_0}^2 = 0 \tag{3.19}$$

and by ϕ

$$< \sigma\phi_{tt}, \phi >_{\mathcal{H}_2^*, \mathcal{H}_2} + \frac{1}{2}\frac{d}{dt}||\phi||_{\mathcal{H}_0}^2 + a(\phi,\phi) + (f(\phi), \phi) = 0. \tag{3.20}$$

We recall the next formula

$$< \phi_{tt}, \phi >_{\mathcal{H}_2^*, \mathcal{H}_2} = \frac{d}{dt}(\phi_t, \phi) - ||\phi_t||_{\mathcal{H}_0}^2.$$

Let ε be a small positive constant, we add (3.19) to ε(3.20) and we obtain

$$\frac{d}{dt}\left\{\frac{\sigma}{2}||\phi_t||^2_{\mathcal{H}_0}+\frac{1}{2}a(\phi,\phi)+\int_\Omega F(\phi)dx+\frac{\varepsilon}{2}||\phi||^2_{\mathcal{H}_0}+\sigma\varepsilon(\phi_t,\phi)\right\}$$
$$+||\phi_t||^2_{\mathcal{H}_0}-\sigma\varepsilon||\phi_t||^2_{\mathcal{H}_0}+\varepsilon a(\phi,\phi)+\varepsilon(f(\phi),\phi)=0.$$

We can rewrite last equality as follows

$$\frac{d}{dt}\mathcal{Y}(t)+\mathcal{D}(t)=0 \tag{3.21}$$

where

$$\mathcal{Y}(t)=\frac{\sigma}{2}||\phi_t||^2_{\mathcal{H}_0}+\frac{1}{2}a(\phi,\phi)+\int_\Omega F(\phi)dx+\frac{\varepsilon}{2}||\phi||^2_{\mathcal{H}_0}+\sigma\varepsilon(\phi_t,\phi),$$
$$\mathcal{D}(t)=||\phi_t||^2_{\mathcal{H}_0}-\sigma\varepsilon||\phi_t||^2_{\mathcal{H}_0}+\varepsilon a(\phi,\phi)+\varepsilon(f(\phi),\phi).$$

Using (2.12) with $\gamma=\frac{\delta}{2+\delta}$, the hypothesis on the nonlinear term and the Young inequality, we have

$$\begin{aligned}\mathcal{Y}(t)&\geq\frac{\sigma}{2}||\phi_t||^2_{\mathcal{H}_0}+\frac{1}{2}||\phi||^2_{\mathcal{H}_2}-\frac{1-\gamma}{2}||\phi||^2_{\mathcal{H}_2}-\frac{1}{2(1-\gamma)}||\phi||^2_{\mathcal{H}_0}+\int_\Omega F(\phi)dx\\&\quad+\frac{\varepsilon}{2}||\phi||^2_{\mathcal{H}_0}-\frac{\sigma}{4}||\phi_t||^2_{\mathcal{H}_0}-\varepsilon^2\sigma||\phi||^2_{\mathcal{H}_0}\\&\geq\frac{\sigma}{4}||\phi_t||^2_{\mathcal{H}_0}+\frac{\gamma}{2}||\phi||^2_{\mathcal{H}_2}-\frac{2+\delta}{4}||\phi||^2_{\mathcal{H}_0}+(\frac{\varepsilon}{2}-\varepsilon^2\sigma)||\phi||^2_H\\&\quad+\frac{2+\delta}{4}||\phi||^2_{\mathcal{H}_0}-K_2|\Omega|.\end{aligned}$$

Setting $\varepsilon\in\left(0,\frac{1}{2\sigma}\right)$, we get

$$\mathcal{Y}(t)\geq\frac{\sigma}{4}||\phi_t||^2_H+\frac{\gamma}{2}||\phi||^2_V-K_2|\Omega|. \tag{3.22}$$

From the choice of ε, the hypothesis (**H2**) and (2.4) with $\gamma=\frac{\delta}{1+\delta}$, we compute the other term in (3.21) as follows

$$\begin{aligned}\mathcal{D}(t)&\geq\frac{1}{2}||\phi_t||^2_{\mathcal{H}_0}+\varepsilon||\phi||^2_{\mathcal{H}_2}-2\varepsilon||\phi||^2_{\mathcal{H}_1}+\varepsilon(f(\phi),\phi)\\&\geq\frac{1}{2}||\phi_t||^2_{\mathcal{H}_0}+\varepsilon||\phi||^2_{\mathcal{H}_2}-\varepsilon(1-\gamma)||\phi||^2_{\mathcal{H}_2}-\frac{\varepsilon}{(1-\gamma)}||\phi||^2_{\mathcal{H}_0}+\varepsilon(f(\phi),\phi)\\&\geq\frac{1}{2}||\phi_t||^2_{\mathcal{H}_0}+\varepsilon\gamma||\phi||^2_{\mathcal{H}_2}-\frac{\varepsilon}{(1-\gamma)}||\phi||^2_{\mathcal{H}_0}+\varepsilon(1+\delta)||\phi||^2_{\mathcal{H}_0}-K_1|\Omega|\\&=\frac{1}{2}||\phi_t||^2_{\mathcal{H}_0}+\varepsilon\gamma||\phi||^2_{\mathcal{H}_2}-K_1|\Omega|.\end{aligned}$$

Thus combining last inequality with (3.21), we get

$$\frac{d}{dt}\mathcal{Y}(t)+\frac{1}{2}||\phi_t||^2_{\mathcal{H}_0}+\varepsilon\gamma||\phi||^2_{\mathcal{H}_2}\leq K_1|\Omega|. \tag{3.23}$$

In order to use Lemma 3.3.1, we consider $\mathcal{C}$ as the set of the solution trajectories to problem (2.14) with initial data (ϕ_0, ϕ_1) such that $||(\phi_0, \phi_1)||_{\mathcal{E}_0} \leq R$. Let $(v, w) \in \mathcal{E}_0$. We define the following positive functional

$$\Phi(v, w) = \frac{\sigma}{2}||w||^2_{\mathcal{H}_0} + \frac{1}{2}a(v, v) + \int_\Omega F(v)dx + \frac{\varepsilon}{2}||v||^2_{\mathcal{H}_0} + \sigma\varepsilon(w, v) + K_2|\Omega|.$$

Using the Sobolev embedding $\mathcal{H}_2 \overset{c}{\hookrightarrow} L^\infty$, we can deduce an upper bound estimate of $\Phi(\phi(0), \phi_t(0))$ for a generic couple $(\phi, \phi_t) \in \mathcal{C}$

$$\begin{aligned}\Phi(\phi(0), \phi_t(0)) &= \Phi(\phi_0, \phi_1) = \mathcal{Y}(0) + K_2|\Omega| \\ &\leq \frac{\sigma}{2}||\phi_1||^2_{\mathcal{H}_0} + \frac{1}{2}||\phi_0||^2_{\mathcal{H}_2} + Q(||\phi_0||_{\mathcal{H}_2}) + \frac{\varepsilon}{2}||\phi_0||^2_{\mathcal{H}_0} \\ &\quad + K_2|\Omega| + \frac{\sigma\varepsilon}{2}||\phi_1||^2_{\mathcal{H}_0} + \frac{\sigma\varepsilon}{2}||\phi_0||^2_{\mathcal{H}_0} \\ &\leq Q(||(\phi_0, \phi_1)||_{\mathcal{E}_0}) \leq Q(R).\end{aligned}$$

We set $C_1 = \min\{\frac{1}{2}, \varepsilon\gamma\}$ and we rewrite (3.23) in terms of Φ as follows

$$\frac{d}{dt}\Phi(\phi(t), \phi_t(t)) + C_1||(\phi(t), \phi_t(t))||^2_{\mathcal{E}_0} \leq K_1|\Omega|. \tag{3.24}$$

Applying Lemma 3.3.1, for every $\eta > 0$, we obtain

$$\Phi(\phi(t), \phi_t(t)) \leq \sup_{(v,w)\in\mathcal{E}_0} \left\{\Phi(v, w) : ||(v, w)||^2_{\mathcal{E}_0} \leq \frac{K_1\Omega + \eta}{C_1}\right\}, \quad \forall t \geq \frac{Q(R)}{\eta}.$$

Thanks to the above estimates and setting $C_2 = \min\{\frac{\sigma}{4}, \frac{\gamma}{2}\}$, we can conclude that

$$||(\phi, \phi_t)||^2_{\mathcal{E}_0} \leq \frac{1}{C_2}\left(Q\left(\sqrt{\frac{K_1\Omega + \eta}{C_1}}\right) + K_2|\Omega|\right) =: C_3, \quad \forall t \geq \frac{Q(R)}{\eta}. \tag{3.25}$$

We recall the energy estimate (2.23)

$$||(\phi, \phi_t)||^2_{\mathcal{E}_0} \leq C_4 + Q(R), \qquad t \geq 0 \tag{3.26}$$

where $C_4 = \frac{K_2|\Omega|}{\min\{\sigma, \frac{\gamma}{2}\}}$. Collecting the above estimates together we get

$$||(\phi, \phi_t)||_{\mathcal{E}_0} \leq Q(R)e^{-t} + C, \qquad t \geq 0 \tag{3.27}$$

where C is a positive constant depending on σ, Ω, f and Ω but it is independent of the norm of the initial data. □

Now we split the solution into two parts as follows

$$(\phi(t), \phi_t(t)) = (\phi^l(t), \phi^l_t(t)) + (\phi^o(t), \phi^o_t(t)), \tag{3.28}$$

such that

$$\begin{cases} \sigma\phi^l_{tt} + \phi^l_t + \Delta^2\phi^l + \Delta\phi^l + k\phi^l = 0 \\ \phi^l(0) = \phi_0 \qquad \phi^l_t(0) = \phi_1 \end{cases} \tag{3.29}$$

and

$$\begin{cases} \sigma\phi^o_{tt} + \phi^o_t + \Delta^2\phi^o + \Delta\phi^o + k\phi^o + f(\phi) - k\phi = 0 \\ \phi^o(0) = 0 \qquad \phi^o_t(0) = 0. \end{cases} \tag{3.30}$$

where $k > 0$ is a large fixed constant to be further determined.

Lemma 3.3.3. *There exists k such that $(\phi^l(t), \phi^l_t(t))$ fulfils the following inequality*

$$||(\phi^l(t), \phi^l_t(t))||^2_{\mathcal{E}_0} \le C||(\phi_0, \phi_1)||^2_{\mathcal{E}_0} e^{-\varepsilon t}, \qquad \forall t \ge 0, \tag{3.31}$$

where C, ε may depend on σ, f, Ω but they are independent of the initial data ϕ_0, ϕ_1.

Proof. We observe that the existence and the uniqueness of the solution to the linear problem (3.29) directly follow from the Theorem 2.3.4. We proceed analogously to the proof of Lemma 3.3.2 so, testing by $\phi^l_t + \varepsilon\phi^l$, we obtain

$$\frac{d}{dt}\left\{\frac{\sigma}{2}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{1}{2}a(\phi^l_t, \phi^l) + \frac{k+\varepsilon}{2}||\phi^l||^2_{\mathcal{H}_0}\right\} + ||\phi^l_t||^2_{\mathcal{H}_0}$$
$$+\sigma\varepsilon < \phi^l_{tt}, \phi^l >_* + \varepsilon a(\phi^l, \phi^l) + k\varepsilon||\phi^l||^2_{\mathcal{H}_0} = 0.$$

It is equivalent to

$$\frac{d}{dt}\mathcal{Y}^l(t) + \mathcal{D}^l(t) = 0, \tag{3.32}$$

where

$$\mathcal{Y}^l(t) = \frac{\sigma}{2}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{1}{2}a(\phi^l_t, \phi^l_t) + \frac{k+\varepsilon}{2}||\phi^l||^2_{\mathcal{H}_0} + \sigma\varepsilon(\phi^l_t, \phi^l),$$
$$\mathcal{D}^l(t) = (1 - \sigma\varepsilon)||\phi^l_t||^2_{\mathcal{H}_0} + \varepsilon a(\phi^l, \phi^l) + k\varepsilon||\phi^l||^2_{\mathcal{H}_0}.$$

Using (2.4) and the Young inequality, we get

$$\mathcal{Y}^l(t) \ge \frac{\sigma}{4}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\phi^l||^2_{\mathcal{H}_2} + (\frac{k+\varepsilon}{2} - \frac{1}{4} - \varepsilon^2\sigma)||\phi^l||^2_{\mathcal{H}_0}.$$

Therefore, choosing $\varepsilon \in \left(0, \frac{1}{2\sigma}\right)$ and $k > \frac{1}{2}$, we have

$$\mathcal{Y}^l(t) \ge \frac{\sigma}{4}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\phi^l||^2_{\mathcal{H}_2}.$$

We compute the other term $D^l(t)$ as follows

$$\mathcal{D}^l(t) \ge \frac{1}{2}||\phi^l_t||^2_{\mathcal{H}_0} + \varepsilon a(\phi^l, \phi^l) + k\varepsilon||\phi^l||^2_{\mathcal{H}_0}$$
$$\ge C_1\mathcal{Y}^l(t) + (k\varepsilon - C_1\frac{k+\varepsilon}{2})||\phi^l||^2_{\mathcal{H}_0} - C_1\sigma\varepsilon(\phi^l_t, \phi^l),$$

where $C_1 = 2\varepsilon$. Dividing $\mathcal{D}^l(t)$ into two equal parts and using Cauchy-Schwarz and Young inequalities, we obtain

$$\begin{aligned}\mathcal{D}^l(t) &\geq \frac{1}{2}C_1\mathcal{Y}^l(t) + (k\varepsilon - C_1\frac{k+\varepsilon}{2})||\phi^l||^2_{\mathcal{H}_0} - C_1\sigma\varepsilon(\phi^l_t, \phi^l) \\ &+ \frac{1}{4}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}a(\phi^l, \phi^l) + \frac{k\varepsilon}{2}||\phi^l||^2_{\mathcal{H}_0} \\ &\geq \frac{1}{2}C_1\mathcal{Y}^l(t) + \frac{\varepsilon}{4}||\phi^l||^2_{\mathcal{H}_2} + (\frac{3}{2}k\varepsilon - C_1\frac{k+\varepsilon}{2} - \frac{C_2\sigma^2\varepsilon^2}{4} - \frac{\varepsilon}{4})||\phi^l||^2_{\mathcal{H}_0}.\end{aligned}$$

For k large enough, we have

$$\mathcal{D}^l(t) \geq \varepsilon\mathcal{Y}^l(t) + \frac{\varepsilon}{4}||\phi^l||^2_{\mathcal{H}_2}. \tag{3.33}$$

Substituting (3.33) in (3.32), we get

$$\frac{d}{dt}\mathcal{Y}^l(t) + \varepsilon\mathcal{Y}^l(t) \leq 0. \tag{3.34}$$

Applying the Gronwall Lemma, we can conclude that

$$\mathcal{Y}^l(t) \leq \mathcal{Y}^l(0)e^{-\varepsilon t}.$$

By the definition of $\mathcal{Y}^l(t)$, it follows that

$$||\phi^l_t(t)||^2_{\mathcal{H}_0} + ||\phi^l(t)||^2_{\mathcal{H}_2} \leq C\{||\phi_1||^2_{\mathcal{H}_0} + ||\phi_0||^2_{\mathcal{H}_2}\}e^{-\varepsilon t}, \qquad \forall t \geq 0,$$

where C may depend on σ, f, Ω but it is independent of the initial data ϕ_0, ϕ_1. □

Lemma 3.3.4. *Let* $(\phi_0, \phi_1) \in \mathcal{E}_0$ *such that* $||(\phi_0, \phi_1)||_{\mathcal{E}_0} \leq R$. *Then the following inequality holds*

$$||(\phi^o(t), \phi^o_t(t))||_{\mathcal{E}_1} \leq Q(R), \qquad \forall t \geq 0. \tag{3.35}$$

Proof. The existence and the uniqueness of the solution (ϕ^l, ϕ^l_t) imply the existence and the uniqueness of the solution to the homogeneous problem (3.30). By comparison, we can assert that

$$||\phi^o||^2_{\mathcal{H}_2} + ||\phi^o_t||^2_{\mathcal{H}_0} \leq Q(R), \qquad \forall t \geq 0. \tag{3.36}$$

Testing the equation in ϕ^o by $-\Delta\phi^o_t - \varepsilon\Delta\phi^o$, where ε is a sufficiently small positive constant, we have

$$\frac{d}{dt}\mathcal{Y}^o(t) + \mathcal{D}^o(t) = \mathcal{R}^o(t), \tag{3.37}$$

where

$$\mathcal{Y}^o(t) = \frac{\sigma}{2}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{2}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - ||\Delta\phi^o||^2_{\mathcal{H}_0}$$
$$+ \varepsilon\sigma(\nabla\phi_t^o, \nabla\phi^o) + \frac{\varepsilon}{2}||\nabla\phi^o||^2_{\mathcal{H}_0},$$
$$\mathcal{D}^o(t) = ||\nabla\phi_t^o||^2_{\mathcal{H}_0} - \varepsilon\sigma||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - 2\varepsilon||\Delta\phi^o||^2_{\mathcal{H}_0},$$
$$\mathcal{R}^o(t) = (f(\phi) - k\phi + k\phi^o, \Delta\phi_t^o) + \varepsilon(f(\phi) - k\phi + k\phi^o, \Delta\phi^o).$$

Thanks to the Green formula and the Navier boundary conditions, we have the following estimate in $\mathcal{H}_3$

$$||\Delta v||^2_{\mathcal{H}_0} \leq ||\nabla\Delta v||_{\mathcal{H}_0}||\nabla v||_{\mathcal{H}_0} \leq \frac{1}{4}||\nabla\Delta v||^2_{\mathcal{H}_0} + ||\nabla v||^2_{\mathcal{H}_0}. \tag{3.38}$$

We can infer from (3.38) and Young inequality that

$$\mathcal{Y}^o(t) \geq \frac{\sigma}{2}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - ||\nabla\phi^o||^2_{\mathcal{H}_0} - \frac{\sigma}{4}||\nabla\phi_t^o||^0_{\mathcal{H}}$$
$$- \sigma\varepsilon^2||\nabla\phi^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\nabla\phi^o||^2_{\mathcal{H}_0}$$
$$\geq \frac{\sigma}{4}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - ||\nabla\phi^o||^2_{\mathcal{H}_0} + \varepsilon(\frac{1}{2} - \sigma\varepsilon)||\nabla\phi^o||^2_{\mathcal{H}_0}.$$

Setting $\varepsilon \in (0, \frac{1}{2\sigma})$, we have

$$\mathcal{Y}^o(t) \geq \frac{\sigma}{4}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - ||\nabla\phi^o||^2_{\mathcal{H}_0}$$
$$= \frac{\sigma}{4}||\phi_t^o||^2_{\mathcal{H}_1} + \frac{1}{4}||\phi^o||^2_{\mathcal{H}_3} - ||\phi^o||^2_{\mathcal{H}_1}. \tag{3.39}$$

For $\varepsilon \in (0, \frac{1}{2\sigma})$, we deduce

$$\mathcal{D}^o(t) \geq \frac{1}{2}||\nabla\phi^o(t)||^2_{\mathcal{H}_0} + \varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - 2\varepsilon||\Delta\phi^o||^2_{\mathcal{H}_0}$$
$$\geq \varepsilon\mathcal{Y}^o(t) + (\frac{1}{2} - \frac{\varepsilon\sigma}{2})||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - \varepsilon||\Delta\phi^o||^2_{\mathcal{H}_0}$$
$$- \varepsilon^2\sigma(\nabla\phi_t^o, \nabla\phi^o) - \frac{\varepsilon^2}{2}||\nabla\phi||^2_{\mathcal{H}_0}$$
$$\geq \varepsilon\mathcal{Y}^o(t) + \frac{1}{4}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - \varepsilon||\nabla\phi^o||^2_{\mathcal{H}_0}$$
$$- \frac{1}{8}||\nabla\phi_t^o||^2_{\mathcal{H}_0} - 2\varepsilon^4\sigma^2||\nabla\phi^o||^2_{\mathcal{H}_0} - \frac{\varepsilon^2}{2}||\nabla\phi^o||^2_{\mathcal{H}_0}$$
$$\geq \varepsilon\mathcal{Y}^o(t) + \frac{1}{8}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - C_1||\nabla\phi^o||^2_{\mathcal{H}_0}$$
$$= \varepsilon\mathcal{Y}^o(t) + \frac{1}{8}||\phi_t^o||^2_{\mathcal{H}_1} + \frac{\varepsilon}{4}||\phi^o||^2_{\mathcal{H}_3} - C_1||\phi^o||^2_{\mathcal{H}_1}. \tag{3.40}$$

On the other hand, using standard argument, we get

$$
\begin{aligned}
\mathcal{R}^o(t) &= (f'(\phi)\nabla\phi - k\nabla\phi + k\nabla\phi^o, \nabla\phi_t^o) + \varepsilon(f(\phi) - k\phi + k\phi^o, \Delta\phi^o) \\
&\leq \frac{1}{8}||\nabla\phi_t^o||_{\mathcal{H}_0}^2 + \frac{\varepsilon}{4}||\Delta\phi^o||_{\mathcal{H}_0}^2 + 2||f'(\phi)\nabla\phi - k\nabla\phi + k\nabla\phi^o||_{\mathcal{H}_0}^2 \\
&\quad + \varepsilon||f(\phi) - k\phi + k\phi^o||_{\mathcal{H}_0}^2 \\
&\leq \frac{1}{8}||\phi_t^o||_{\mathcal{H}_1}^2 + \frac{\varepsilon}{4}||\phi^o||_{\mathcal{H}_2}^2 + Q(R). \qquad (3.41)
\end{aligned}
$$

Combining (3.40) and (3.41) with (3.37), we obtain

$$\frac{d}{dt}\mathcal{Y}^o(t) + \varepsilon\mathcal{Y}^o(t) \leq Q(R).$$

Applying the Gronwall Lemma, we have

$$\mathcal{Y}^o(t) \leq \mathcal{Y}^o(0)e^{-\varepsilon t} + Q(R)(1 - e^{-\varepsilon t}) \leq Q(R), \qquad (3.42)$$

and, using (3.39), we conclude that

$$||\phi_t^o||_{\mathcal{H}_1}^2 + ||\phi^o||_{\mathcal{H}_3}^2 \leq Q(R). \qquad (3.43)$$

where Q also depends on σ, f, Ω. □

We state our main result of this section for the modified Swift-Hohenberg equation.

Theorem 3.3.5. *For each $\sigma > 0$, the dynamical system $(\mathcal{E}_0, S_\sigma(t))$ has the global attractor $\mathcal{A}_\sigma$, which is connected and bounded in $\mathcal{E}_1$.*

Proof. Let us fix $\sigma > 0$. Thanks to Lemma 3.3.2, $\mathcal{V}^0 = B_{\mathcal{E}_0}(0, C)$ is an absorbing set for $(\mathcal{E}_0, S_\sigma(t))$. Lemma 3.3.2 also implies that the trajectories, which start from the absorbing set $\mathcal{V}^0$, are uniformly bounded in $\mathcal{E}_0$. Combining the split (3.28), Lemmas 3.3.3, 3.3.4 with the absorbing set $\mathcal{V}^0$ and using Theorem 3.1.14, we conclude that there exists the global attractor $\mathcal{A}_\sigma$, which is bounded in $\mathcal{E}_1$. Since the semigroup $S_\sigma(t)$ satisfies the property **S3**, we also have that $\mathcal{A}_\sigma$ is connected. □

Under the hypothesis stated in the second chapter for the nonlinear term, we formulate and prove a more fine result of regularity for the global attractor $\mathcal{A}_\sigma$.

Lemma 3.3.6. *Let $(\phi_0, \phi_1) \in \mathcal{E}_0$ be initial data such that $||(\phi_0, \phi_1)||_{\mathcal{E}_0} \leq R$. Then the following inequality holds*

$$||(\phi^o(t), \phi_t^o(t))||_{\mathcal{E}_2} \leq Q(R), \qquad \forall t \geq 0. \qquad (3.44)$$

Proof. We recall that there holds

$$||\phi^o||^2_{\mathcal{H}_2} + ||\phi^o_t||^2_{\mathcal{H}_0} \le Q(R), \qquad \forall t \ge 0. \tag{3.45}$$

Testing the equation in ϕ^o by $\Delta^2\phi^o_t + \varepsilon\Delta^2\phi^o$, where ε is a sufficiently small positive constant, we have

$$\frac{d}{dt}\mathcal{Y}^o(t) + \mathcal{D}^o(t) = \mathcal{R}^o(t), \tag{3.46}$$

where

$$\begin{aligned}
\mathcal{Y}^o(t) &= \frac{\sigma}{2}||\Delta\phi^o_t||^2_{\mathcal{H}_0} + \frac{1}{2}||\Delta^2\phi^o||^2_{\mathcal{H}_0} - ||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\quad + \varepsilon\sigma(\Delta\phi^o_t, \Delta\phi^o) + \frac{\varepsilon}{2}||\Delta\phi^o||^2_{\mathcal{H}_0}, \\
\mathcal{D}^o(t) &= ||\Delta\phi^o_t||^2_{\mathcal{H}_0} - \varepsilon\sigma||\Delta\phi^o||^2_{\mathcal{H}_0} + \varepsilon||\Delta^2\phi^o||^2_{\mathcal{H}_0} - 2\varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0}, \\
\mathcal{R}^o(t) &= -(f(\phi) - k\phi + k\phi^o, \Delta^2\phi_t) - \varepsilon(f(\phi) - k\phi + k\phi^o, \Delta^2\phi).
\end{aligned}$$

Thanks to the Green formula and the Navier boundary conditions, we have the following estimate in $\mathcal{H}_4$

$$||\nabla\Delta v||^2_{\mathcal{H}_0} \le ||\Delta^2 v||_{\mathcal{H}_0}||\Delta v||_{\mathcal{H}_0} \le \frac{1}{4}||\Delta^2 v||^2_{\mathcal{H}_0} + ||\Delta v||^2_{\mathcal{H}_0}. \tag{3.47}$$

Using (3.47) and standard arguments, we get

$$\begin{aligned}
\mathcal{Y}^o(t) &\ge \frac{\sigma}{2}||\Delta\phi^o_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0} - ||\Delta\phi^o||^2_{\mathcal{H}_0} - \frac{\sigma}{4}||\Delta\phi^o_t||^0_{\mathcal{H}} \\
&\quad - \sigma\varepsilon^2||\Delta\phi^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\ge \frac{\sigma}{4}||\Delta\phi^o_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0} - ||\Delta\phi^o||^2_{\mathcal{H}_0} + \varepsilon(\frac{1}{2} - \sigma\varepsilon)||\Delta\phi^o||^2_{\mathcal{H}_0}.
\end{aligned}$$

Choosing $\varepsilon \in (0, \frac{1}{2\sigma})$, we have

$$\begin{aligned}
\mathcal{Y}^o(t) &\ge \frac{\sigma}{4}||\Delta\phi^o_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0} - ||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&= \frac{\sigma}{4}||\phi^o_t||^2_{\mathcal{H}_2} + \frac{1}{4}||\phi^o||^2_{\mathcal{H}_4} - ||\phi^o||^2_{\mathcal{H}_2}.
\end{aligned} \tag{3.48}$$

We estimate the other terms as follows

$$\begin{aligned}
\mathcal{D}^o(t) &\ge \frac{1}{2}||\Delta\phi^o(t)||^2_{\mathcal{H}_0} + \varepsilon||\Delta^2\phi^o||^2_{\mathcal{H}_0} - 2\varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\ge \varepsilon\mathcal{Y}^o(t) + (\frac{1}{2} - \frac{\varepsilon\sigma}{2})||\Delta\phi^o_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0} - ||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\quad - \frac{1}{8}||\Delta\phi^o_t||^2_{\mathcal{H}_0} - 2\varepsilon^4\sigma^2||\Delta\phi^o||^2_{\mathcal{H}_0} - \frac{\varepsilon^2}{2}||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\ge \varepsilon\mathcal{Y}^o(t) + \frac{1}{8}||\Delta\phi^o_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0} - C_1||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\ge \varepsilon\mathcal{Y}^o(t) + \frac{1}{8}||\phi^o_t||^2_{\mathcal{H}_2} + \frac{\varepsilon}{4}||\phi^o||^2_{\mathcal{H}_4} - C_1||\phi^o||^2_{\mathcal{H}_2},
\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}^o(t) &= -(f''(\phi)\Delta\phi - k\Delta\phi + k\Delta\phi^o, \Delta\phi_t^o) - \varepsilon(f(\phi) - k\phi + k\phi^o, \Delta^2\phi^o)\\ &\leq \frac{1}{8}||\Delta\phi_t^o||_{\mathcal{H}_0}^2 + \frac{\varepsilon}{4}||\Delta^2\phi^o||_{\mathcal{H}_0}^2 + 2||f''(\phi)\Delta\phi - k\Delta\phi + k\Delta\phi^o||_{\mathcal{H}_0}^2\\ &\quad + \varepsilon||f(\phi) - k\phi + k\phi^o||_{\mathcal{H}_0}^2\\ &\leq \frac{1}{8}||\phi_t^o||_{\mathcal{H}_2}^2 + \frac{\varepsilon}{4}||\phi^o||_{\mathcal{H}_4}^2 + Q(R).\end{aligned}$$

Combining these inequalities together, we obtain

$$\frac{d}{dt}\mathcal{Y}^o(t) + \varepsilon\mathcal{Y}^o(t) \leq Q(R). \tag{3.49}$$

Applying the Gronwall Lemma, we conclude that

$$\mathcal{Y}^o(t) \leq \mathcal{Y}^o(0)e^{-\varepsilon t} + Q(R)(1 - e^{-\varepsilon t}) \leq Q(R). \tag{3.50}$$

This implies that

$$||\phi_t^o||_{\mathcal{H}_2}^2 + ||\phi^o||_{\mathcal{H}_4}^2 \leq Q(R), \tag{3.51}$$

where Q also depends on σ, f, Ω. □

Corollary 3.3.7. *The global attractor $\mathcal{A}_\sigma$ is a bounded set in $\mathcal{E}_2$.*

Proof. This result directly follows from Lemma 3.3.6. Indeed, it is sufficient to substitute Lemma 3.3.4 with 3.3.6 in the proof of Theorem 3.3.5. □

4 Exponential attractors

A second type of attractor set plays a central role in this chapter. We will briefly explain the notion of exponential attractor, its properties and a general method to prove the existence of this set in the framework of dissipative PDEs. Next, applying this strategy, we will examine the particular case of the Swift-Hohenberg and the modified Swift-Hohenberg equations.

4.1 Construction of exponential attractors

In order to overcome the drawbacks presented about the global attractor, A. Eden, C. Foias, B. Nicolaenko and R. Temam proposed a new object, called exponential attractor (see [5]).

Definition 4.1.1. A compact set $\mathcal{M} \subset X$ is an exponential attractor for $(X, S(t))$ if

E.1 $\mathcal{M}$ has finite fractal dimension, $dim_X \mathcal{M} < +\infty$,

E.2 $\mathcal{M}$ is positively invariant,

E.3 $\mathcal{M}$ attracts exponentially the bounded subsets of X, namely

$$\forall B \subset X \text{ bounded} \quad dist_X(S(t)B, \mathcal{M}) \leq Q(||B||_X)e^{-\omega t}, \quad \forall t \geq 0,$$

where the positive constant ω and monotonic function Q are independent of B.

The exponential attractor, if it exists, contains the global attractor and it can be used to prove the finite (fractal) dimensionality of the global attractor. The main feature of this set is the control of the rate of attraction of the trajectories which starts from a generic bounded set of the phase space. Moreover, it enjoys more properties of robustness than the global attractor as we will see in the next chapter. Instead, a negative aspect of the exponential attractor is the lack of uniqueness. The first method to construct an exponential attractor consisted in a *fractal expansion* of the global attractor, making use of the Zorn's Lemma. A control of this expansion at each step was performed by the squeezing property, which essentially used an orthogonal projector with finite rank, so it was valid only in the Hilbert setting (see [5]). Then, a new construction was provided in Banach spaces (see [6], [7] and references given there).

Now we explain a general strategy to prove the existence of an exponential attractor for equations with parabolic or damped hyperbolic nature (see [22] for more details).

Construction of a more regular set
The first step is to try out the existence of a positively invariant absorbing set $\mathcal{B}$, which is also more regular, in terms of spatial derivatives, than a generic absorbing set of X. Next, we confine the dynamics of the semigroup into this set and we verify some sufficient conditions to guarantee the existence of an exponential attractor for trajectories departing from $\mathcal{B}$. We observe how the existence of these regular sets is usually a consequence of the estimates already proved for the global attractor (see [14]).

Discrete dynamical system
Let X and X_1 be two Banach spaces with X_1 compactly embedded into X. Let Σ: $\mathcal{B} \to \mathcal{B}$ be a map. We consider the discrete dynamical system generated by the iterations of Σ, that is

$$\Sigma(0) = \mathbb{I}, \ \Sigma(n) = \Sigma \circ ... \circ \Sigma n \text{ times}, \ \ n \in \mathbb{N}.$$

Definition 4.1.2. A compact set $\mathcal{M}^d$(in the topology of X) is an exponential attractor for Σ on $\mathcal{B}$ if

D.1 $\mathcal{M}^d$ has finite fractal dimension,

D.2 $\mathcal{M}^d$ is positively invariant,

D.3 $dist_X(\Sigma(n)\mathcal{B}, \mathcal{M}^d) \leq \alpha e^{-\beta n}$, where α, β are positive constants only depend on $\mathcal{B}$.

We introduce two relevant properties of the maps concerned.

Definition 4.1.3. A nonlinear map Σ enjoys the smoothing property if

$$||\Sigma x - \Sigma y||_{X_1} \leq C||x - y||_X, \quad \forall x, y \in \mathcal{B}. \tag{4.1}$$

Definition 4.1.4. A nonlinear map Σ satisfies the asymptotic smoothing property if

$$\Sigma = \Sigma_0 + \Sigma_1, \tag{4.2}$$

where

$$||\Sigma_0 x - \Sigma_0 y||_X \leq \lambda ||x - y||_X, \quad \forall x, y \in \mathcal{B},\ \alpha < \frac{1}{2}, \tag{4.3}$$

and

$$||\Sigma_1 x - \Sigma_1 y||_{X_1} \leq \Lambda ||x - y||_X, \quad \forall x, y \in \mathcal{B}. \tag{4.4}$$

The following Theorem is the key to construct an exponential attractor for the semigroup $S(t)$ (see [7] for the proof).

Theorem 4.1.5. *If the map Σ satisfies the smoothing property or the asymptotic smoothing property on $\mathcal{B}$ then the discrete dynamical system generated by the iterations of Σ possesses an exponentially attractor $\mathcal{M}^d \subset \mathcal{B}$.*

In application to PDEs, the nonlinear map Σ is the continuous semigroup at a time t^* such that $S(t^*)$ fulfils one of sufficient conditions to apply (4.1.5).

Continuous exponential attractor

As final step, we consider the following set

$$\mathcal{M} = \bigcup_{t \in [t^*, 2t^*]} S(t)\mathcal{M}^d. \tag{4.5}$$

If the semigroup $S(\cdot)$ is Lipschitz (or Hölder) continuous on $[t^*, 2t^*] \times B$, then $\mathcal{M}$ is an exponential attractor for the dynamical system $(S(t), \mathcal{B})$. However, at this level, $\mathcal{M}$ attracts the bounded subsets of $\mathcal{B}$ and not of the whole phace space. This problem is being overcome introducing the transitivity property of the exponential attraction (see [10] for the proof).

Lemma 4.1.6. *Let (X, d) be a metric space and $S(t)$ be a semigroup acting on X such that*

$$d(S(t)x, S(t)y) \leq c_1 e^{\alpha_1 t} d(x, y), \quad t \leq 0, x, y \in X \tag{4.6}$$

for some positive constants c_1 and α_1. Assume that there exist B_1, B_2, B_3 such that

$$dist_H(S(t)B_1, B_2) \leq c_2 e^{-\alpha_2 t}, \quad t \geq 0, \alpha_2 > 0 \tag{4.7}$$

and

$$dist_H(S(t)B_2, B_3) \leq c_3 e^{-\alpha_3 t}, \quad t \geq 0, \alpha_3 > 0, \tag{4.8}$$

then

$$dist_H(S(t)B_1, B_3) \leq c_4 e^{-\alpha_4 t}, \quad t \geq 0, \alpha_2 > 0 \tag{4.9}$$

where $c_4 := c_1 c_2 + c_3$ and $\alpha_4 := \frac{\alpha_2 \alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}$.

Applying Lemma 4.1.6, it is possible to conclude that $\mathcal{M}$ is the exponential attractor for $S(t)$ on X.

4.2 Swift-Hohenberg equation

We start with the construction of a more regular set where we will confine the dynamic of the semigroup $S(t)$. From Lemma 3.2.3, we can assert that $\mathcal{X}^0 = \mathcal{B}_{\mathcal{H}_0}(0, \rho_1)$, where $\rho_1 > \rho_0$, is an absorbing set for the dynamical system $(\mathcal{H}_0, S(t))$. Let $t_{\mathcal{X}^0}$ be such that $S(t)\mathcal{X}^0 \subset \mathcal{X}^0$, for $\forall t \geq t_{\mathcal{X}^0}$. We define the positively invariant bounded absorbing set

$$\mathcal{X}^1 = \overline{\bigcup_{t \geq t_{\mathcal{X}^0}} S(t)\mathcal{X}^0}^{\mathcal{H}_0}. \tag{4.10}$$

Thanks to the Lemma 3.2.4, we also state that $\mathcal{B}_{\mathcal{H}_2}(0, Q(\rho_1))$ absorbs $\mathcal{X}^1$. Now setting

$$\mathcal{X}^2 = \mathcal{X}^1 \cap \mathcal{B}_{\mathcal{H}_2}(0, Q(\rho_1)), \tag{4.11}$$

we have that $\mathcal{X}^2$ fulfils the following properties

$$\begin{cases} \exists\ t_{\mathcal{X}^2} :\ S(t)\mathcal{X}^2 \subset \mathcal{X}^2, & \forall t \geq t_{\mathcal{X}^2}, \\ ||S(t)u||_{\mathcal{H}_2} \leq Q(\rho_1), & \forall u \in \mathcal{X}^2, \forall t \geq 0. \end{cases} \tag{4.12}$$

We observe that the second property easily follows from energy estimate (2.13). Moreover, due to the absorbing and invariance properties of $\mathcal{X}^2$, it is clear that $\mathcal{X}^2$ exponentially attracts every bounded set of $\mathcal{H}_0$.

We now show the smoothing property for a suitable map acting from $\mathcal{X}^2$ into $\mathcal{X}^2$.

Lemma 4.2.1. *There exists $t^* > 0$ such that the map $S(t^*) : \mathcal{X}^2 \to \mathcal{X}^2$ satisfies*

$$||S(t^*)u_1 - S(t^*)u_2||_{\mathcal{H}_2} \leq C||u_1 - u_2||_{\mathcal{H}_0}, \qquad \forall u_1, u_2 \in \mathcal{X}^2, \tag{4.13}$$

where C is a positive constant depending on ρ_1, t^* and Ω.

Proof. Let $\phi_1(t), \phi_2(t)$ be two solutions respectively with initial data ϕ_{01}, ϕ_{02} in $\mathcal{X}^2$. We consider $\phi(t) = \phi_1(t) - \phi_2(t)$ and the corresponding equation

$$\phi_t + \Delta^2\phi + 2\Delta\phi + f(\phi_1) - f(\phi_2) = 0. \tag{4.14}$$

Testing by $t\Delta^2\phi$, we obtain

$$\int_\Omega t\phi_t\Delta^2\phi dx + \int_\Omega t(\Delta^2\phi)^2 dx + \int_\Omega 2t\Delta\phi\Delta^2\phi + \int_\Omega (f(\phi_1) - f(\phi_2))t\Delta^2\phi\, dx = 0.$$

Using integration by parts, we have

$$t\int_\Omega \Delta\phi_t\Delta\phi dx + t||\Delta^2\phi||_H^2 - 2t||\nabla\Delta\phi||_H^2 + \int_\Omega (f(\phi_1) - f(\phi_2))t\Delta^2\phi\, dx = 0.$$

We observe that there holds the following equality

$$t\int_\Omega \Delta\phi_t\Delta\phi dx = \frac{t}{2}\frac{d}{dt}||\Delta\phi||_H^2 = \frac{1}{2}\frac{d}{dt}(t||\Delta\phi||_H^2) - \frac{1}{2}||\Delta\phi||_H^2. \tag{4.15}$$

We deduce from (4.15) and Cauchy-Schwarz inequality that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(t||\Delta\phi||_H^2) + t||\Delta^2\phi||_H^2 - 2t||\nabla\Delta\phi||_H^2 \\ \leq \frac{1}{2}||\Delta\phi||_H^2 + \frac{t}{2}||f(\phi_1)-f(\phi_2)||_{\mathcal{H}_0}^2 + \frac{t}{2}||\Delta^2\phi||_{\mathcal{H}_0}^2.\end{aligned} \tag{4.16}$$

Thanks to integration by parts and Navier boundary conditions, we have

$$2||\nabla\Delta\phi||_{\mathcal{H}_0}^2 \leq \frac{1}{2}||\Delta^2\phi||_{\mathcal{H}_0}^2 + 2||\Delta\phi||_{\mathcal{H}_0}^2. \tag{4.17}$$

From (4.16) and (4.17), we get

$$\frac{1}{2}\frac{d}{dt}(t||\Delta\phi||_H^2) \leq (\frac{t}{2}+2)||\Delta\phi||_{\mathcal{H}_0}^2 + \frac{t}{2}||\int_0^1 f'(\tau\phi_1+(1-\tau)\phi_2)\phi\, d\tau||_{\mathcal{H}_0}^2.$$

Using the Sobolev embedding $\mathcal{H}_2 \hookrightarrow L^\infty(\Omega)$ and the properties of $\mathcal{X}^2$, we conclude that

$$\frac{1}{2}\frac{d}{dt}(t||\Delta\phi||_H^2) \leq \frac{t+4}{2}||\Delta\phi||_{\mathcal{H}_0}^2 + C_1\frac{t}{2}||\phi||_{\mathcal{H}_0}^2,$$

where C_1 is a positive constant depending on ρ_1. Integrating from 0 to T, we obtain

$$T||\Delta\phi(T)||_{\mathcal{H}_0}^2 \leq (T+4)\int_0^T ||\Delta\phi||_{\mathcal{H}_0}^2 + C_2T\int_0^T ||\Delta\phi||_{\mathcal{H}_0}^2.$$

Using the continuity from initial data (2.2), we have

$$T||\Delta\phi(T)||_{\mathcal{H}_0}^2 \leq C_3e^{C_4T}||\phi_{01}-\phi_{02}||_{\mathcal{H}_0}^2, \tag{4.18}$$

where C_3, C_4 depend on ρ_1, T and Ω. □

The next result is a direct consequence of Lemma 4.2.1 and Theorem 4.1.5.

Proposition 4.2.2. *There exists an exponential attractor $\mathcal{M}^d$ for the discrete dynamical system generated by the iteration of $\Sigma = S(t^*)$ on $\mathcal{X}^2$.*

Next, we prove a regularity property for the semigroup $S(t)$ acting on $\mathcal{X}_2$ endowed with the $\mathcal{H}_0$-topology.

Lemma 4.2.3. *The map $(t,u) \mapsto S(t)u : [t^*, 2t^*] \times \mathcal{X}^2 \to \mathcal{X}^2$ is $\frac{1}{2}$-Hölder continuous in time and Lipschitz continuous in the initial data.*

Proof. Let u_1, u_2 be in $\mathcal{X}^2$ and $t^* \leq \tau \leq t \leq 2t^*$. Thanks to (2.2), we get

$$\begin{aligned}||S(t)u_1 - S(\tau)u_2||_{\mathcal{H}_0} &\leq ||S(t)u_1 - S(t)u_2||_{\mathcal{H}_0} + ||S(t)u_2 - S(\tau)u_2||_{\mathcal{H}_0} \\ &\leq C||u_1 - u_2||_{\mathcal{H}_0} + ||\phi_2(t) - \phi_2(\tau)||_{\mathcal{H}_0}.\end{aligned}$$

In order to estimate the second term in the right-hand side, we observe that the trajectories, which start from $\mathcal{X}^2$, satisfy the following property

$$||\phi_i||_{\mathcal{H}_2} \leq C, \qquad ||\phi_{i,t}||_{\mathcal{H}_2^*} \leq C, \tag{4.19}$$

where C only depends on ρ_1. Then we have

$$\begin{aligned}||\phi_2(t) - \phi_2(\tau)||^2_{\mathcal{H}_0} &\leq ||\phi_2(t) - \phi_2(\tau)||_{\mathcal{H}_2}||\phi_2(t) - \phi_2(\tau)||_{\mathcal{H}_2^*} \\ &\leq C||\int_\tau^t \phi_{2,t}(s)\ ds||_{\mathcal{H}_2^*} \\ &\leq C\int_\tau^t ||\phi_{2,t}(s)||_{\mathcal{H}_2^*}\ ds \leq C^2|t-\tau|.\end{aligned}$$

We conclude that

$$||S(t)u_1 - S(\tau)u_2||_{\mathcal{H}_0} \leq C||z_1 - z_2||_{\mathcal{H}_0} + C|t-\tau|^{\frac{1}{2}}, \tag{4.20}$$

where C depends on t^* and $\mathcal{X}_2$. □

Now we formulate our main result of this section.

Theorem 4.2.4. *The dynamical system* $(\mathcal{H}_0, S(t))$ *has an exponential attractor* $\mathcal{M} \subset \mathcal{X}^2$.

Proof. Thanks to Proposition 4.2.2 and following the general strategy, we define

$$\mathcal{M} = \bigcup_{t\in[t^*,2t^*]} S(t)\mathcal{M}^d. \tag{4.21}$$

The positive invariance, the compactness property and the finite fractal dimension of $\mathcal{M}$ follow from the properties of $\mathcal{M}^d$ and Lemma 4.2.3. Using the discrete exponential attraction property of $\mathcal{M}^d$, we have that $\forall \mathcal{B} \subset \mathcal{X}^2$

$$\exists K, \omega: \quad dist_{\mathcal{H}_0}(S(t)\mathcal{B}, \mathcal{M}) \leq Ke^{-\omega t}, \qquad \forall t \geq 0. \tag{4.22}$$

We deduce that the exponential attraction property holds for every $\mathcal{B} \subset \mathcal{H}_0$ from (2.2), properties of $\mathcal{X}^2$ and Lemma 4.1.6. □

4.3 Modified Swift-Hohenberg equation

In the previous chapter we have proved the existence of an absorbing set $\mathcal{V}^0 = B_{\mathcal{E}_0}(0, C)$. Thanks to Lemma 3.3.2 and by definition of absorbing set, we have

$$\begin{cases} \exists\ t_{\mathcal{V}^0} :\ S_\sigma(t)\mathcal{V}^0 \subset \mathcal{V}^0, & \forall t \geq t_{\mathcal{V}^0}, \\ ||S_\sigma(t)(u,v)||_{\mathcal{E}_0} \leq Q(C), & \forall (u,v) \in \mathcal{V}^0, \forall t \geq 0. \end{cases} \tag{4.23}$$

We define a set which contains at least all the states of the dynamic that starts from $\mathcal{V}^0$ as follows

$$\mathcal{W}^0 = \overline{\bigcup_{t\in[0,t_{\mathcal{V}^0}]} S_\sigma(t)\mathcal{V}^0}^{\mathcal{E}_0}. \tag{4.24}$$

We observe that the following properties hold

$$S_\sigma(t)\mathcal{W}^0 \subset \mathcal{W}^0, \quad ||S_\sigma(t)(u,v)||_{\mathcal{E}_0} \leq Q(C) \quad \forall (u,v) \in \mathcal{W}^0,\ \forall t \geq 0.$$

Let $\mathcal{B}$ be a bounded set of $\mathcal{W}^0$. We infer from Lemmas 3.3.3 and 3.3.6 that $\forall (u_1, u_2) \in \mathcal{B}$ we also have

$$S_\sigma(t)(u_1,u_2) = (\phi(t), \phi_t(t)) = (\phi^l(t), \phi^l_t(t)) + (\phi^o(t), \phi^o_t(t)),$$

$$||(\phi^l(t), \phi^l_t(t))||_{\mathcal{E}_0} \leq Q(C)e^{-\varepsilon t}, \quad ||(\phi^o(t), \phi^o_t(t))||_{\mathcal{E}_2} \leq Q(C) =: R,\ \forall t \geq 0.$$

We can conclude that the set

$$\mathcal{V}^1 = B_{\mathcal{E}_2}(0, R) \cap \mathcal{W}^0 \tag{4.25}$$

exponentially attracts any bounded set $\mathcal{B}$ of $\mathcal{W}^0$. Observing that $\mathcal{V}^0 \subset \mathcal{W}^0$, we have that $\mathcal{W}^0$ exponentially attracts any bounded set of $\mathcal{E}_0$. Thus, using Lemma 4.1.6, we also obtain that $\mathcal{V}^1$ exponentially attracts any bounded set of $\mathcal{E}_0$ with respect to the $\mathcal{E}_0$-metric.

To get a more regular absorbing set, we prove the high-order dissipative estimate in $\mathcal{E}_2$.

Lemma 4.3.1. *Let $\mathcal{B}$ be a bounded set of $\mathcal{E}_2 \cap \mathcal{W}^0$. Assume that (ϕ, ϕ_t) is the solution to problem* (2.14) *with initial data $(\phi_0, \phi_1) \in \mathcal{B}$. Then the following estimate holds*

$$||(\phi(t), \phi_t(t))||^2_{\mathcal{E}_2} \leq Q(\mathcal{B})e^{-\varepsilon t} + R_1, \qquad \forall t \geq 0, \tag{4.26}$$

where $R_1 = Q(R)$.

Proof. Lemma 3.3.2 allows us to state that

$$||(\phi, \phi_t)||_{\mathcal{E}_0} \leq Q(R) + \sqrt{C}, \qquad \forall t \geq 0. \tag{4.27}$$

Testing with $\Delta^2\phi_t$, we obtain

$$\begin{aligned} &\frac{\sigma}{2}\frac{d}{dt}||\Delta\phi_t||^2_{\mathcal{H}_0} + ||\Delta\phi_t||^2_{\mathcal{H}_0} + \frac{1}{2}\frac{d}{dt}||\Delta^2\phi||^2_{\mathcal{H}_0} \\ &\qquad - \frac{d}{dt}||\nabla\Delta\phi||^2_{\mathcal{H}_0} + (f(\phi), \Delta^2\phi_t) = 0. \end{aligned} \tag{4.28}$$

Testing with $\Delta^2\phi$, we have

$$\begin{aligned} &\int_\Omega \sigma\phi_{tt}\Delta^2\phi \; dx + \frac{1}{2}\frac{d}{dt}||\Delta\phi||^2_{\mathcal{H}_0} + ||\Delta^2\phi||_{\mathcal{H}_0} \\ &\qquad - 2||\nabla\Delta\phi||^2_{\mathcal{H}_0} + (f(\phi), \Delta^2\phi) = 0. \end{aligned} \tag{4.29}$$

We recall that holds

$$\int_\Omega \sigma\phi_{tt}\Delta^2\phi \; dx = \sigma\frac{d}{dt}(\Delta\phi_t, \Delta\phi) - \sigma||\Delta\phi_t||^2_{\mathcal{H}_0}. \tag{4.30}$$

Using (4.30) and adding (4.28) to ε(4.29), where ε is a positive constant, we get

$$\frac{d}{dt}\mathcal{Y}(t) + \mathcal{D}(t) = \mathcal{R}(t), \tag{4.31}$$

where

$$\begin{aligned} \mathcal{Y}(t) &= \frac{\sigma}{2}||\Delta\phi_t||^2_{\mathcal{H}_0} + \frac{1}{2}||\Delta^2\phi||^2_{\mathcal{H}_0} - ||\nabla\Delta\phi||^2_{\mathcal{H}_0} + \varepsilon\sigma(\Delta\phi_t, \Delta\phi) + \frac{\varepsilon}{2}||\Delta\phi||^2_{\mathcal{H}_0}, \\ \mathcal{D}(t) &= ||\Delta\phi_t||^2_{\mathcal{H}_0} - \sigma\varepsilon||\Delta\phi_t||^2_{\mathcal{H}_0} + \varepsilon||\Delta^2\phi||^2_{\mathcal{H}_0} - 2\varepsilon||\nabla\Delta\phi||^2_{\mathcal{H}_0}, \\ \mathcal{R}(t) &= -(f(\phi), \Delta^2\phi_t) - \varepsilon(f(\phi), \Delta^2\phi). \end{aligned}$$

We observe that holds the following inequalities

$$\varepsilon\sigma|(\Delta\phi_t, \Delta\phi)| \leq \frac{\sigma}{4}||\Delta\phi_t||^2_{\mathcal{H}_0} + \varepsilon^2\sigma||\Delta\phi||^2_{\mathcal{H}_0}, \tag{4.32}$$

$$||\nabla\Delta\phi||^2_{\mathcal{H}_0} \leq \frac{1}{4}||\Delta^2\phi||^2_{\mathcal{H}_0} + ||\Delta\phi||^2_{\mathcal{H}_0}. \tag{4.33}$$

Combining (4.32) and (4.33) with the expression of $\mathcal{Y}(t)$, we have

$$\mathcal{Y}(t) \geq \frac{\sigma}{4}||\Delta\phi_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\Delta^2\phi||^2_{\mathcal{H}_0} - ||\Delta\phi||^2_{\mathcal{H}_0} + (\frac{\varepsilon}{2} - \varepsilon^2\sigma)||\Delta\phi||^2_{\mathcal{H}_0}.$$

Taking $\varepsilon \in (0, \frac{1}{2\sigma})$, we can conclude that

$$\begin{aligned} \mathcal{Y}(t) &\geq \frac{\sigma}{4}||\Delta\phi_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\Delta^2\phi||^2_{\mathcal{H}_0} - ||\Delta\phi||^2_{\mathcal{H}_0} \\ &\geq \frac{\sigma}{4}||\phi_t||^2_{\mathcal{H}_2} + \frac{1}{4}||\phi||^2_{\mathcal{H}_4} - ||\phi||^2_{\mathcal{H}_2}. \end{aligned}$$

Thanks to the form of $\mathcal{Y}(t)$ and the choice of ε, we deduce

$$\begin{aligned}\mathcal{D}(t) &\geq \frac{1}{2}||\Delta\phi_t||^2_{\mathcal{H}_0} + \varepsilon||\Delta^2\phi||^2_{\mathcal{H}_0} - 2\varepsilon||\nabla\Delta\phi||^2_{\mathcal{H}_0} \\ &\geq \varepsilon\mathcal{Y}(t) + (\frac{1}{2} - \frac{\varepsilon\sigma}{2})||\Delta\phi_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\Delta^2\phi||^2_{\mathcal{H}_0} - \varepsilon||\nabla\Delta\phi||^2_{\mathcal{H}_0} \\ &\quad - \varepsilon^2\sigma(\Delta\phi_t, \Delta\phi) - \frac{\varepsilon^2}{2}||\Delta\phi||^2_{\mathcal{H}_0}.\end{aligned}$$

Using (4.32) and (4.33), we obtain

$$\begin{aligned}\mathcal{D}(t) &\geq \varepsilon\mathcal{Y}(t) + \frac{1}{8}||\Delta\phi_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta^2\phi||^2_{\mathcal{H}_0} - C||\Delta\phi||^2_{\mathcal{H}_0} \\ &= \varepsilon\mathcal{Y}(t) + \frac{1}{8}||\phi_t||^2_{\mathcal{H}_2} + \frac{\varepsilon}{4}||\phi||^2_{\mathcal{H}_4} - C||\phi||^2_{\mathcal{H}_2},\end{aligned}$$

where C depends on σ. We infer from the Sobolev embedding $\mathcal{H}_2 \hookrightarrow L^\infty(\Omega)$ and the hypothesis on the nonlinear term that

$$\begin{aligned}\mathcal{R}(t) &\leq ||f''(\phi)\Delta\phi||_{\mathcal{H}_0}||\Delta\phi_t||_{\mathcal{H}_0} + \varepsilon||f(\phi)||_{\mathcal{H}_0}||\Delta^2\phi||_{\mathcal{H}_0} \\ &\leq \frac{1}{8}||\Delta\phi_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta^2\phi||^2_{\mathcal{H}_0} + Q(||\phi||_{\mathcal{H}_2}) \\ &= \frac{1}{8}||\phi_t||^2_{\mathcal{H}_2} + \frac{\varepsilon}{4}||\phi||^2_{\mathcal{H}_4} + Q(R).\end{aligned}$$

Collecting these inequalities together, we have

$$\frac{d}{dt}\mathcal{Y}(t) + \varepsilon\mathcal{Y}(t) \leq Q(R). \tag{4.34}$$

Using the Gronwall Lemma, we can assert that

$$\mathcal{Y}(t) \leq \mathcal{Y}(0)e^{-\varepsilon t} + \int_0^t Q(R)e^{-\varepsilon(t-s)}\ ds.$$

Then, we infer from the lower bound of $\mathcal{Y}(t)$ that

$$||\phi_t||^2_{\mathcal{H}_2} + ||\phi||^2_{\mathcal{H}_4} \leq Q(B)e^{-\varepsilon t} + Q(R). \tag{4.35}$$

□

Lemma 4.3.1 allows us to state that $\mathcal{V}^2 = B_{\mathcal{E}_2}(0, R_2) \cap \mathcal{W}^0$, where $R_2 > R_1$, is an absorbing set for any bounded set of $\mathcal{E}_2 \cap \mathcal{W}^0$. In particular $\mathcal{V}^2$ absorbs $\mathcal{V}^1$, so $\mathcal{V}^2$ exponentially attracts any bounded set of $\mathcal{E}_0$ for the transitivity property of the exponential attraction. Moreover, by definition of absorbing set and Lemma 4.3.1, we conclude that $\mathcal{V}_2$ fulfils

$$\begin{cases}\exists\ t_{\mathcal{V}^2} :\ S_\sigma(t)\mathcal{V}^2 \subset \mathcal{V}^2, & \forall t \geq t_{\mathcal{V}^2}, \\ ||S_\sigma(t)(u,v)||_{\mathcal{E}_2} \leq Q(R_1), & \forall (u,v) \in \mathcal{V}^2, \forall t \geq 0.\end{cases} \tag{4.36}$$

Now we proceed to prove the asymptotic smoothing property for a suitable discrete dynamical system $S_\sigma(t^*)$, where t^* is sufficiently large.
Let $(\phi_{01}, \phi_{11}), (\phi_{02}, \phi_{12})$ be in $\mathcal{V}^2$. We consider the corresponding solution defined by the semigroup operator $S_\sigma(t)$. We denote the difference of the two trajectories as follows

$$\phi(t) = \phi_1(t) - \phi_2(t), \quad \phi_t(t) = \phi_{1,t}(t) - \phi_{2,t}(t), \qquad \forall t \geq 0,$$
$$\phi_0 = \phi_{01} - \phi_{02}, \quad \phi_1 = \phi_{11} - \phi_{12}.$$

The couple $(\phi(t), \phi_t(t))$ satisfies the following equation

$$\sigma\phi_{tt} + \phi_t + \Delta^2\phi + 2\Delta\phi + f(\phi_1) - f(\phi_2) = 0.$$

We consider the split

$$(\phi, \phi_t) = (\phi^l, \phi^l_t) + (\phi^o, \phi^o_t), \tag{4.37}$$

such that

$$\begin{cases} \sigma\phi^l_{tt} + \phi^l_t + \Delta^2\phi^l + 2\Delta\phi^l + k\phi^l = 0 \\ \phi^l(0) = \phi_{0,} \qquad \phi^l_t(0) = \phi_1 \end{cases} \tag{4.38}$$

and

$$\begin{cases} \sigma\phi^o_{tt} + \phi^o_t + \Delta^2\phi^o + 2\Delta\phi^o + k\phi^o + f(\phi_1) - f(\phi_2) - k\phi = 0 \\ \phi^o(0) = 0 \qquad \phi^o_t(0) = 0. \end{cases} \tag{4.39}$$

By means Lemma 3.3.3 and by observing the problem for (ϕ^l, ϕ^l_t), we can assert that there exists $k > 0$ such that

$$||(\phi^l, \phi^l_t)(t)||^2_{\mathcal{E}_0} \leq C||(\phi_0, \phi_1)||^2_{\mathcal{E}_0} e^{-\varepsilon t}, \qquad \forall t \geq 0, \tag{4.40}$$

where C, ε are independent of the norm of the initial data.

Lemma 4.3.2. $(\phi^o(t), \phi^o_t(t))$ *fulfils the following estimate*

$$||(\phi^o(t), \phi^o_t(t))||^2_{\mathcal{E}_1} \leq C(t)||(\phi_0, \phi_1)||^2_{\mathcal{E}_0}, \forall t \geq 0. \tag{4.41}$$

Proof. Testing the equation (4.39) by $-\Delta\phi^o_t - \varepsilon\Delta\phi^o$, where ε is a positive small constant, we get

$$\frac{d}{dt}\mathcal{Y}^o(t) + \mathcal{D}^o(t) = \mathcal{R}^o(t), \tag{4.42}$$

where

$$\begin{aligned}
\mathcal{Y}^o(t) &= \frac{\sigma}{2}||\nabla\phi^o_t||^2_{\mathcal{H}_0} + \frac{1}{2}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - ||\Delta\phi^o||^2_{\mathcal{H}_0} + \varepsilon\sigma(\nabla\phi^o_t, \nabla\phi^o) \\
&\quad + \frac{\varepsilon}{2}||\nabla\phi^o||^2_{\mathcal{H}_0} + \frac{k}{2}||\nabla\phi^o||^2_{\mathcal{H}_0}, \\
\mathcal{D}^o(t) &= ||\nabla\phi^o_t||^2_{\mathcal{H}_0} - \varepsilon\sigma||\nabla\phi^o||^2_{\mathcal{H}_0} + \varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\quad - 2\varepsilon||\Delta\phi^o||^2_{\mathcal{H}_0} + \varepsilon k||\nabla\phi^o||^2_{\mathcal{H}_0}, \\
\mathcal{R}^o(t) &= (f(\phi_1) - f(\phi_2) - k\phi, \Delta\phi^o_t) + \varepsilon(f(\phi_1) - f(\phi_2) - k\phi, \Delta\phi^o).
\end{aligned}$$

We recall the following inequality

$$\varepsilon\sigma(\nabla\phi_t^o, \nabla\phi^o) \geq -\frac{\sigma}{4}||\nabla\phi_t^o||^2_{\mathcal{H}_0} - \sigma\varepsilon^2||\nabla\phi||^2_{\mathcal{H}_0}. \tag{4.43}$$

Using (3.38) and (4.43), we have

$$\begin{aligned}\mathcal{Y}^o(t) \geq \frac{\sigma}{4}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - ||\nabla\phi^o||^2_{\mathcal{H}_0} \\ - \sigma\varepsilon^2||\nabla\phi||^2_{\mathcal{H}_0} + \frac{k+\varepsilon}{2}||\nabla\phi||^2_{\mathcal{H}_0}.\end{aligned}$$

Thus, taking $\varepsilon \in (0, \frac{1}{2\sigma})$ and $k > 2$, we obtain

$$\mathcal{Y}^o(t) \geq \frac{\sigma}{4}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} \geq C_1||(\phi^o, \phi_t^o)||^2_{\mathcal{E}_1}, \tag{4.44}$$

where $C_1 = \min\{\frac{\sigma}{4}, \frac{1}{4}\}$. On the other hand, we deduce that

$$\begin{aligned}\mathcal{D}^o(t) &\geq \frac{1}{2}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - 2\varepsilon||\Delta\phi^o||^2_{\mathcal{H}_0} + \varepsilon k||\nabla\phi^o||^2_{\mathcal{H}_0} \\ &\geq \varepsilon\mathcal{Y}^o(t) + \left(\frac{1}{2} - \frac{\sigma\varepsilon}{2}\right)||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - \varepsilon||\Delta\phi^o||^2_{\mathcal{H}_0} \\ &\quad - \sigma\varepsilon^2(\nabla\phi_t^o, \nabla\phi^o) + \left(\frac{\varepsilon k}{2} - \frac{\varepsilon^2}{2}\right)||\nabla\phi^o||^2_{\mathcal{H}_0} \\ &\geq \varepsilon\mathcal{Y}^o(t) + \frac{1}{8}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} \\ &\quad + \left(\frac{\varepsilon k - 2\varepsilon - \varepsilon^2}{2} - 2\sigma^2\varepsilon^4\right)||\nabla\phi^o||^2_{\mathcal{H}_0}.\end{aligned}$$

Setting k large enough, it is follows that

$$\begin{aligned}\mathcal{D}^o(t) &\geq \varepsilon\mathcal{Y}^o(t) + \frac{1}{8}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} \\ &\geq \varepsilon\mathcal{Y}^o(t) + \frac{1}{8}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4C_2}||\Delta\phi^o||^2_{\mathcal{H}_0}.\end{aligned}$$

Using standard arguments, we also have

$$\begin{aligned}\mathcal{R}^o(t) &\leq |(f'(\phi_1)\nabla\phi_1 - f'(\phi_2)\nabla\phi_2 - k\nabla\phi, \nabla\phi_t^o)| \\ &\quad + \varepsilon|(f(\phi_1) - f(\phi_1) - k\phi, \Delta\phi^o)| \\ &\leq \frac{1}{8}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4C_2}||\Delta\phi^o||^2_{\mathcal{H}_0} + 4||f'(\phi_1)\nabla\phi_1 - f'(\phi_2)\nabla\phi_2||^2_{\mathcal{H}_0} \\ &\quad + 2k^2||\nabla\phi||^2_{\mathcal{H}_0} + \frac{2C_2}{\varepsilon}||f(\phi_1) - f(\phi_2)||^2_{\mathcal{H}_0} + 2k^2||\phi||^2_{\mathcal{H}_0}\end{aligned}$$

Using the Sobolev embedding $\mathcal{H}_2 \hookrightarrow L^\infty(\Omega)$, we proceed term by term in the following way

$$
\begin{aligned}
&||f'(\phi_1)\nabla\phi_1 - f'(\phi_2)\nabla\phi_2||^2_{\mathcal{H}_0} \\
&\leq 2||f'(\phi_1)\nabla\phi||^2_{\mathcal{H}_0} + 2||\left(f'(\phi_1) - f'(\phi_2)\right)\nabla\phi_2||^2_{\mathcal{H}_0} \\
&\leq C(||\phi_1||_{\mathcal{H}_2})||\nabla\phi||^2_{\mathcal{H}_0} + C_3||\int_0^1 f''(\tau\phi_1 + (1-\tau)\phi_2)\phi \, d\tau||^2_{L^\infty(\Omega)}||\nabla\phi_2||^2_{\mathcal{H}_0} \\
&\leq C(||\phi_1||_{\mathcal{H}_2})||\phi||^2_{\mathcal{H}_1} + C(||\phi_1||_{\mathcal{H}_2}, ||\phi_2||_{\mathcal{H}_2})||\phi||^2_{\mathcal{H}_2},
\end{aligned}
$$

$$
\begin{aligned}
||f(\phi_1) - f(\phi_2)||^2_{\mathcal{H}_0} &\leq ||\int_0^1 f'(\tau\phi_1 + (1-\tau)\phi_2)\phi \, d\tau||^2_{\mathcal{H}_0} \\
&\leq C(||\phi_1||_{\mathcal{H}_2}, ||\phi_2||_{\mathcal{H}_2})||\phi||^2_{\mathcal{H}_0}.
\end{aligned}
$$

Collecting these estimates together, we get

$$
\frac{d}{dt}\mathcal{Y}^o(t) + \varepsilon\mathcal{Y}^o(t) \leq C(||\phi_1||_{\mathcal{H}_2}, ||\phi_2||_{\mathcal{H}_2})||\phi||^2_{\mathcal{H}_2}. \tag{4.45}
$$

Integrating (4.45) with respect to time and using the Lipschitz continuity estimate (2.21), we obtain

$$
\begin{aligned}
\mathcal{Y}^o(t) &\leq C(||\phi_1||_{\mathcal{H}_2}, ||\phi_2||_{\mathcal{H}_2}) \int_0^t ||\phi||^2_{\mathcal{H}_2} \, ds \\
&\leq C(t)\{||\phi_0||^2_{\mathcal{H}_2} + ||\phi_1||^2_{\mathcal{H}_0}\}.
\end{aligned}
$$

Thanks to (4.44), we can conclude that

$$
||(\phi^o, \phi^o_t)||^2_{\mathcal{E}_1} \leq C(t)||(\phi_0, \phi_1)||^2_{\mathcal{E}_0}. \tag{4.46}
$$

□

We infer from (4.40) and (4.41) that there exists $t^* \geq t_{\mathcal{V}^2}$ such that

$$
||(\phi^l(t^*), \phi^l_t(t^*))||_{\mathcal{E}_0} \leq \lambda||(\phi_0, \phi_1)||_{\mathcal{E}_0}, \tag{4.47}
$$

$$
||(\phi^o(t^*), \phi^o_t(t^*))||_{\mathcal{E}_1} \leq \Lambda||(\phi_0, \phi_1)||_{\mathcal{E}_0}, \tag{4.48}
$$

where $\lambda \in (0, \frac{1}{2})$. As a consequence, from the choice of t^* and Theorem 4.1.5, we deduce

Proposition 4.3.3. *There exists an exponential attractor $\mathcal{M}^d_\sigma$ for the discrete dynamical system generated by the iterations of $S_\sigma(t^*)$ on $\mathcal{V}^2$.*

The next result states a Lipschitz dependence of the semigroup $S_\sigma(t)$ with respect to t and to the initial data.

Lemma 4.3.4. *The map $(t,(u,v)) \mapsto S_\sigma(t)(u,v) : [t^*, 2t^*] \times \mathcal{V}^2 \to \mathcal{V}^2$ is Lipschitz continuous in time and in the initial data, when $\mathcal{V}^2$ is endowed with the $\mathcal{E}_0$-topology.*

Proof. Let $t^* \le \tau \le t \le 2t^*$, $(u_1,v_1),(u_2,v_2) \in \mathcal{V}^2$. Using (2.21), we have

$$\begin{aligned}||S_\sigma(t)(u_1,v_1) - S_\sigma(\tau)(u_2,v_2)||^2_{\mathcal{E}_0} &\le 2||S_\sigma(t)(u_1,v_1) - S_\sigma(t)(u_2,v_2)||^2_{\mathcal{E}_0} \\ &\quad + 2||S_\sigma(t)(u_2,v_2) - S_\sigma(\tau)(u_2,v_2)||^2_{\mathcal{E}_0} \\ &\le C||(u_1,v_1) - (u_2,v_2)||^2_{\mathcal{E}_0} \\ &\quad + 2||S_\sigma(t)(u_2,v_2) - S_\sigma(\tau)(u_2,v_2)||^2_{\mathcal{E}_0},\end{aligned}$$

where C is a positive constant depending on $\mathcal{V}^2, t^*$. We recall that the trajectories starting from $\mathcal{V}^2$ satisfy

$$||\phi_{i,t}||_{\mathcal{H}_2} \le C, \qquad ||\phi_{i,tt}||_{\mathcal{H}_0} \le C,$$

where C only depends on $\mathcal{V}^2$. Then we have

$$\begin{aligned}&||S_\sigma(t)(u_2,v_2) - S_\sigma(\tau)(u_2,v_2)||^2_{\mathcal{E}_0} \\ &\qquad = ||\phi_2(t) - \phi_2(\tau)||^2_{\mathcal{H}_2} + ||\phi_{2,t}(t) - \phi_{2,t}(\tau)||^2_{\mathcal{H}_0} \\ &\qquad = ||\int_\tau^t \phi_{2,t}(s)\ ds||^2_{\mathcal{H}_2} + ||\int_\tau^t \phi_{2,tt}(s)\ ds||^2_{\mathcal{H}_0} \\ &\qquad \le \left(\int_\tau^t ||\phi_{2,t}(s)||_{\mathcal{H}_2}\ ds\right)^2 + \left(\int_\tau^t ||\phi_{2,tt}(s)||_{\mathcal{H}_0}\ ds\right)^2 \\ &\qquad \le C|t-\tau|^2.\end{aligned}$$

Collecting the above estimates together, we conclude that

$$||S_\sigma(t)(u_1,v_1) - S_\sigma(\tau)(u_2,v_2)||_{\mathcal{E}_0} \le C||(u_1,v_1) - (u_2,v_2)||_{\mathcal{E}_0} + C|t-\tau|.$$

□

Now we state the main result of this section.

Theorem 4.3.5. *For each $\sigma > 0$, the dynamical system $(\mathcal{E}_0, S_\sigma(t))$ possesses an exponential attractor $\mathcal{M}_\sigma \subset \mathcal{V}^2$.*

Proof. We define the following set

$$\mathcal{M}_\sigma := \bigcup_{t \in [t^*, 2t^*]} S_\sigma(t)\mathcal{M}^d_\sigma. \tag{4.49}$$

Thanks to the positive invariance of $\mathcal{M}^d_\sigma$, it follows that $\mathcal{M}_\sigma$ is positive invariance. The compactness property of $\mathcal{M}_\sigma$ is a consequence of the compactness of $\mathcal{M}^d_\sigma$ and Lemma 4.3.4. Moreover, $\mathcal{M}_\sigma$ is the image of a Lipschitz function on $[t^*, 2t^*] \times \mathcal{M}^d_\sigma \to \mathcal{M}^d_\sigma$, so we have

$$dim_{\mathcal{E}_0}\mathcal{M}_\sigma \le dim_{\mathcal{E}_0}([t^*, 2t^*] \times \mathcal{M}^d_\sigma) \le dim_{\mathcal{E}_0}\mathcal{M}^d_\sigma + 1,$$

because Lipschitz functions preserves the fractal dimension. The discrete exponential attraction property of $\mathcal{M}_\sigma^d$ imply that $\forall \mathcal{B} \subset \mathcal{V}^2$

$$\exists K, \omega : \quad dist_{\mathcal{E}_0}(S_\sigma(t)\mathcal{B}, \mathcal{M}_\sigma) \leq Ke^{-\omega t}, \ \forall t \geq 0. \tag{4.50}$$

We infer that $\mathcal{M}_\sigma$ fulfills the exponential attraction property for every bounded set of $\mathcal{E}_0$ from (4.50), properties of $\mathcal{V}^2$ and Lemma 4.1.6. $\square$

5 Robust families of attractors

In the third and fourth chapters the long-time behavior of the equations concerned, in the sense of global and exponential attractors, has been studied separately. In this chapter we change our point of view and we want to highlight the relationship between the two models in the limiting case $\sigma \to 0$. In particular, we will read the equation with the inertial term as a singular perturbation of the original equation. Then we will prove the upper semicontinuity of the global attractor and the existence of a robust family of exponential attractors with respect to the parameter σ.

5.1 Preliminaries

In this chapter, for any $\sigma \in (0, \sigma_0]$, we work with the Hilbert spaces

$$\mathcal{E}_0^\sigma = \mathcal{H}_2 \times \sqrt{\sigma}\mathcal{H}_0, \quad \mathcal{E}_1^\sigma = \mathcal{H}_3 \times \sqrt{\sigma}\mathcal{H}_1, \quad \mathcal{E}_2^\sigma = \mathcal{H}_4 \times \sqrt{\sigma}\mathcal{H}_2, \tag{5.1}$$

endowed with the following norms

$$||(u,v)||_{\mathcal{E}_0^\sigma} = \left(||u||^2_{\mathcal{H}_2} + \sigma||v||^2_{\mathcal{H}_0}\right)^{\frac{1}{2}}, \quad ||(u,v)||_{\mathcal{E}_1^\sigma} = \left(||u||^2_{\mathcal{H}_3} + \sigma||v||^2_{\mathcal{H}_1}\right)^{\frac{1}{2}},$$

$$||(u,v)||_{\mathcal{E}_2^\sigma} = \left(||u||^2_{\mathcal{H}_4} + \sigma||v||^2_{\mathcal{H}_2}\right)^{\frac{1}{2}}.$$

For $\sigma = 0$, we set the second component of the above Hilbert spaces equal to $\{0\}$. We observe how these spaces are the right setting in order to compare the behavior of the solutions when $\sigma \to 0$. Indeed, as we will see, we are able to prove dissipative estimates which are independent of σ. We also note that if $\sigma_1 > \sigma_2$, the closed ball $B_{\mathcal{E}_i^{\sigma_1}}(0,R) \subseteq B_{\mathcal{E}_i^{\sigma_2}}(0,R)$, for $i = 0,1,2$.
Now we state the well-posedness result of the modified Swift-Hohenebrg equation in this framework of phase spaces. We note that the following Theorem may be proved in much the same way as Theorem 2.3.4.

Theorem 5.1.1. *For any $\sigma \in (0, \sigma_0]$ and initial data $(\phi_0, \phi_1) \in \mathcal{E}_0^\sigma$, the equation* (2.14) *has a unique weak solution such that*

$$\phi \in C([0,T]; \mathcal{H}_2) \cap C^1([0,T]; \mathcal{H}_0^*)$$

satisfying $\forall t \in [0,T]$

$$||(\phi_1 - \phi_2, \phi_{1,t} - \phi_{2,t})(t)||^2_{\mathcal{E}_0^\sigma} \leq ||(\phi_{10} - \phi_{20}, \phi_{11} - \phi_{21})||^2_{\mathcal{E}_0^\sigma} C_1 e^{C_2 t}, \quad (5.2)$$

where ϕ_1 and ϕ_2 are weak solutions to (2.14) *respectively with initial data (ϕ_{10}, ϕ_{11}) and (ϕ_{20}, ϕ_{21}). C_1, C_2 are positive constants depending on the norm of the initial data as well as on Ω and f, but independent of σ and t.*

Thanks to Theorem 5.1.1, we can define the strongly continuous semigroup for the modified Swift-Hohenberg equation: for any $\sigma \in (0, \sigma_0]$

$$S_\sigma(t) : \mathcal{E}_0^\sigma \to \mathcal{E}_0^\sigma, \qquad S_\sigma(t)(\phi_0, \phi_1) = (\phi(t), \phi_t(t)), \qquad \forall t \geq 0, \quad (5.3)$$

where (ϕ, ϕ_t) is the unique solution to problem (2.14) with initial data in $\mathcal{E}_0^\sigma$. Using Theorem 2.2.1, we also set for $\sigma = 0$

$$S_0(t) : \mathcal{E}_0^0 \to \mathcal{E}_0^0 \qquad S_0(t)(\phi_0) = \phi(t), \qquad \forall t \geq 0, \quad (5.4)$$

where ϕ is the unique solution to problem (2.1) with initial data in $\mathcal{E}_0^0$. In the next result we prove the dissipative nature of the dynamical system $(\mathcal{E}_0^\sigma)$.

Lemma 5.1.2. *For any $\sigma \in (0, \sigma_0]$, assume that (ϕ, ϕ_t) is the weak solution to problem* (2.14)*. Then $\mathcal{V}_0^\sigma = B_{\mathcal{E}_0^\sigma}(0, R_0)$ is an absorbing set for $(\mathcal{E}_0^\sigma, S_\sigma(t))$, where R_0 is a positive constant depending on σ_0, Ω and f but independent of the initial data and σ. In addition, if the initial data (ϕ_0, ϕ_1) is such that $||(\phi_0, \phi_1)||_{\mathcal{E}_0^\sigma} \leq R$, the following estimate holds*

$$||(\phi, \phi_t)||^2_{\mathcal{E}_0^\sigma} \leq Q(R) e^{-t} + R_0, \qquad \forall t \geq 0. \quad (5.5)$$

Proof. Testing (2.14) by $\phi_t + \varepsilon\phi$, where ε is a small positive constant, we have

$$\frac{d}{dt}\left\{\frac{\sigma}{2}||\phi_t||^2_{\mathcal{H}_0} + \frac{1}{2}a(\phi,\phi) + \int_\Omega F(\phi)dx + \frac{\varepsilon}{2}||\phi||^2_{\mathcal{H}_0} + \sigma\varepsilon(\phi_t, \phi)\right\}$$
$$+ ||\phi_t||^2_{\mathcal{H}_0} - \sigma\varepsilon||\phi_t||^2_{\mathcal{H}_0} + \varepsilon a(\phi,\phi) + \varepsilon(f(\phi), \phi) = 0.$$

We rewrite last equality as follows

$$\frac{d}{dt}\mathcal{Y}(t) + \mathcal{D}(t) = 0, \quad (5.6)$$

where

$$\mathcal{Y}(t) = \frac{\sigma}{2}||\phi_t||^2_{\mathcal{H}_0} + \frac{1}{2}a(\phi,\phi) + \int_\Omega F(\phi)dx + \frac{\varepsilon}{2}||\phi||^2_{\mathcal{H}_0} + \sigma\varepsilon(\phi_t, \phi),$$
$$\mathcal{D}(t) = ||\phi_t||^2_{\mathcal{H}_0} - \sigma\varepsilon||\phi_t||^2_{\mathcal{H}_0} + \varepsilon a(\phi,\phi) + \varepsilon(f(\phi), \phi).$$

Thanks to (2.4) with $\gamma = \frac{\delta}{1+\delta}$, the hypothesis on the nonlinear term and the Young inequality, we obtain

$$\begin{aligned}
\mathcal{Y}(t) &\geq \frac{\sigma}{2}||\phi_t||^2_{\mathcal{H}_0} + \frac{1}{2}||\phi||^2_{\mathcal{H}_2} - \frac{1-\gamma}{2}||\phi||^2_{\mathcal{H}_2} - \frac{1}{2(1-\gamma)}||\phi||^2_{\mathcal{H}_0} + \int_\Omega F(\phi)dx \\
&\quad + \frac{\varepsilon}{2}||\phi||^2_{\mathcal{H}_0} - \frac{\sigma}{4}||\phi_t||^2_{\mathcal{H}_0} - \varepsilon^2\sigma||\phi||^2_{\mathcal{H}_0} \\
&\geq \frac{\sigma}{4}||\phi_t||^2_{\mathcal{H}_0} + \frac{\gamma}{2}||\phi||^2_{\mathcal{H}_2} - \frac{2+\delta}{4}||\phi||^2_{\mathcal{H}_0} + \left(\frac{\varepsilon}{2} - \varepsilon^2\sigma\right)||\phi||^2_H \\
&\quad + \frac{2+\delta}{4}||\phi||^2_{\mathcal{H}_0} - K_2|\Omega|.
\end{aligned}$$

Setting $\varepsilon \in \left(0, \frac{1}{2\sigma_0}\right)$, we get

$$\mathcal{Y}(t) \geq \frac{\sigma}{4}||\phi_t||^2_H + \frac{\gamma}{2}||\phi||^2_V - K_2|\Omega|. \tag{5.7}$$

Using the choice of ε, (2.4) with $\gamma = \frac{\delta}{1+\delta}$ and (**H2**), we compute the other term in (5.6) as follows

$$\begin{aligned}
\mathcal{D}(t) &\geq \frac{1}{2}||\phi_t||^2_{\mathcal{H}_0} + \varepsilon||\phi||^2_{\mathcal{H}_2} - 2\varepsilon||\phi||^2_{\mathcal{H}_1} + \varepsilon(f(\phi), \phi) \\
&\geq \frac{1}{2}||\phi_t||^2_{\mathcal{H}_0} + \varepsilon||\phi||^2_{\mathcal{H}_2} - \varepsilon(1-\gamma)||\phi||^2_{\mathcal{H}_2} - \frac{\varepsilon}{(1-\gamma)}||\phi||^2_{\mathcal{H}_0} + \varepsilon(f(\phi), \phi) \\
&\geq \frac{1}{2}||\phi_t||^2_{\mathcal{H}_0} + \varepsilon\gamma||\phi||^2_{\mathcal{H}_2} - \frac{\varepsilon}{(1-\gamma)}||\phi||^2_{\mathcal{H}_0} + \varepsilon(1+\delta)||\phi||^2_{\mathcal{H}_0} - K_1|\Omega| \\
&= \frac{1}{2}||\phi_t||^2_{\mathcal{H}_0} + \varepsilon\gamma||\phi||^2_{\mathcal{H}_2} - K_1|\Omega|.
\end{aligned} \tag{5.8}$$

Thus combining (5.8) with (5.6), we have

$$\frac{d}{dt}\mathcal{Y}(t) + \frac{1}{2}||\phi_t||^2_{\mathcal{H}_0} + \varepsilon\gamma||\phi||^2_{\mathcal{H}_2} \leq K_1|\Omega|. \tag{5.9}$$

We consider $\mathcal{C}$ as the set of the solution trajectories to the problem (2.14) with initial data (ϕ_0, ϕ_1) such that $||(\phi_0, \phi_1)||_{\mathcal{E}_0} \leq R$. Let $(v, w) \in \mathcal{E}_0^\sigma$, we define the following positive functional

$$\Phi(v, w) = \frac{\sigma}{2}||w||^2_{\mathcal{H}_0} + \frac{1}{2}a(v, v) + \int_\Omega F(v)dx + \frac{\varepsilon}{2}||v||^2_{\mathcal{H}_0} + \sigma\varepsilon(w, v) + K_2|\Omega|.$$

Using the Sobolev embedding $\mathcal{H}_2 \hookrightarrow L^\infty(\Omega)$, we can infer an upper bound estimate of $\Phi(\phi(0), \phi_t(0))$ for a generic couple $(\phi, \phi_t) \in \mathcal{C}$. Indeed, we have

$$\begin{aligned}
\Phi(\phi(0), \phi_t(0)) &= \Phi(\phi_0, \phi_1) = \mathcal{Y}(0) + K_2|\Omega| \\
&\leq \frac{\sigma}{2}||\phi_1||^2_{\mathcal{H}_0} + \frac{1}{2}||\phi_0||^2_{\mathcal{H}_2} + Q(||\phi_0||_{\mathcal{H}_2}) + \frac{\varepsilon}{2}||\phi_0||^2_{\mathcal{H}_0} + K_2|\Omega| \\
&\quad + \frac{\sigma\varepsilon}{2}||\phi_1||^2_{\mathcal{H}_0} + \frac{\sigma\varepsilon}{2}||\phi_0||^2_{\mathcal{H}_0} \leq Q(||(\phi_0, \phi_1)||_{\mathcal{E}_0^\sigma}) \leq Q(R).
\end{aligned}$$

Setting $C_1 = \min\{\frac{1}{2\sigma_0}, \varepsilon\gamma\}$ and rewriting (5.9) in terms of Φ, we get

$$\frac{d}{dt}\Phi(\phi(t), \phi_t(t)) + C_1||(\phi(t), \phi_t(t))||^2_{\mathcal{E}_0^\sigma} \leq K_1|\Omega|.$$

Applying Lemma 3.3.1, for every $\eta > 0$ we obtain

$$\Phi(\phi(t), \phi_t(t)) \leq \sup_{(v,w)\in\mathcal{E}_0^\sigma}\{\Phi(v, w) : ||(v, w)||^2_{\mathcal{E}_0^\sigma} \leq \frac{K_1\Omega + \eta}{C_1}\}, \quad \forall t \geq \frac{Q(R)}{\eta}.$$

Setting $C_2 = \min\{\frac{1}{4}, \frac{\gamma}{2}\}$, we infer from the above estimates that

$$||(\phi, \phi_t)||^2_{\mathcal{E}_0^\sigma} \leq \frac{1}{C_2}\left(Q\left(\sqrt{\frac{K_1\Omega + \eta}{C_1}}\right) + K_2|\Omega|\right) =: C_3, \quad \forall t \geq \frac{Q(R)}{\eta}. \tag{5.10}$$

As in the proof of Theorem 2.3.4, an easy computations shows that the following energy estimates holds

$$||(\phi, \phi_t)||^2_{\mathcal{E}_0^\sigma} \leq C_4 + Q(R), \qquad \forall t \geq 0. \tag{5.11}$$

where $C_4 = \frac{2K_2|\Omega|}{\gamma}$. Collecting the above estimates together, we get

$$||(\phi, \phi_t)||_{\mathcal{E}_0} \leq Q(R)e^{-t} + R_0, \qquad \forall t \geq 0. \tag{5.12}$$

where R_0 is a positive constant depending on σ_0, Ω, f and Ω but it is independent of the norm of the initial data and σ. □

Thanks to the Lemma 5.1.2, we can conclude that

$$\mathcal{Y}_0^\sigma = \overline{\bigcup_{t\in[0,t_{\mathcal{V}0}]} S_\sigma(t)\mathcal{V}_0^\sigma}^{\mathcal{E}_0^\sigma}. \tag{5.13}$$

is a positively invariant absorbing set, where $t_{\mathcal{V}_0^\sigma}$ is such that $S_\sigma(t)\mathcal{V}_0^\sigma \subset \mathcal{V}_0^\sigma$, for $\forall t \geq t_{\mathcal{V}_0^\sigma}$. We observe that $t_{\mathcal{V}_0^\sigma}$ is independent of σ. Moreover, the following properties hold

$$S_\sigma(t)\mathcal{Y}_0^\sigma \subset \mathcal{Y}_0^\sigma, \quad ||S_\sigma(t)(u, v)||_{\mathcal{E}_0^\sigma} \leq Q(R_0), \quad \forall (u, v) \in \mathcal{Y}_0^\sigma,\ \forall t \geq 0.$$

where $Q(R_0)$ may depend on σ_0 but is independent of σ.

Now we define some phase spaces using $\mathcal{Y}_0^\sigma$

$$\mathcal{Y}_1^\sigma = \mathcal{E}_1^\sigma \cap \mathcal{Y}_0^\sigma, \qquad \mathcal{Y}_2^\sigma = \mathcal{E}_2^\sigma \cap \mathcal{Y}_0^\sigma \tag{5.14}$$

and we prove the high-order dissipative estimates in these spaces.

Lemma 5.1.3. *Let $(\phi_0, \phi_1) \in \mathcal{Y}_1^\sigma$. Then the solution (ϕ, ϕ_t) satisfies the following inequalities*

$$||(\phi(t), \phi_t(t))||^2_{\mathcal{E}_1^\sigma} \leq Q(||(\phi_0, \phi_1)||_{\mathcal{E}_1^\sigma}) e^{-\varepsilon t} + \kappa_1, \qquad \forall t \geq 0, \tag{5.15}$$

$$\sup_{t\geq 0} \int_t^{t+1} ||\nabla \phi_t(s)||^2_{\mathcal{H}_0}\, ds \leq \kappa_2 + Q(||(\phi_0, \phi_1)||_{\mathcal{E}_1^\sigma}), \tag{5.16}$$

where κ_1, κ_2 and ε are positive constants which depend on R_0 and σ_0, but are independent of the norm of the initial data and σ.

Proof. Testing 2.14 by $-\Delta\phi_t - \varepsilon\Delta\phi$, where $\varepsilon > 0$ is a constant to be further determined, we obtain

$$\frac{d}{dt}\mathcal{Z}(t) + \mathcal{D}(t) = \mathcal{R}(t), \tag{5.17}$$

where

$$\mathcal{Z}(t) = \frac{\sigma}{2}||\nabla\phi_t||^2_{\mathcal{H}_0} + \frac{1}{2}||\nabla\Delta\phi||^2_{\mathcal{H}_0} - ||\Delta\phi||^2_{\mathcal{H}_0} + \sigma\varepsilon(\Delta\phi_t, \Delta\phi) + \frac{\varepsilon}{2}||\nabla\phi||^2_{\mathcal{H}_0},$$
$$\mathcal{D}(t) = ||\nabla\phi_t||^2_{\mathcal{H}_0} - \sigma\varepsilon||\nabla\phi_t||^2_{\mathcal{H}_0} + \varepsilon||\nabla\Delta\phi||^2_{\mathcal{H}_0} - 2\varepsilon||\Delta\phi||^2_{\mathcal{H}_0},$$
$$\mathcal{R}(t) = (f(\phi), \Delta\phi_t) + \varepsilon(f(\phi), \Delta\phi).$$

For $\varepsilon \in (0, \frac{1}{2\sigma_0})$, we get

$$\begin{aligned}\mathcal{Z}(t) &\geq \frac{\sigma}{4}||\nabla\phi_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\nabla\Delta\phi||^2_{\mathcal{H}_0} - ||\nabla\phi||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\nabla\phi||^2_{\mathcal{H}_0} - \sigma\varepsilon||\nabla\phi||^2_{\mathcal{H}_0}\\ &\geq \frac{\sigma}{4}||\nabla\phi_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\nabla\Delta\phi||^2_{\mathcal{H}_0} - ||\nabla\phi||^2_{\mathcal{H}_0}.\end{aligned}$$

In a standard way, we deduce

$$\begin{aligned}\mathcal{D}(t) &\geq \frac{1}{2}||\nabla\phi_t||^2_{\mathcal{H}_0} + \varepsilon||\nabla\Delta\phi||^2_{\mathcal{H}_0} - 2\varepsilon||\Delta\phi||^2_{\mathcal{H}_0}\\ &\geq \varepsilon\mathcal{Z}(t) + \frac{1}{4}||\nabla\phi_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\nabla\Delta\phi||^2_{\mathcal{H}_0} - \varepsilon||\Delta\phi||^2_{\mathcal{H}_0}\\ &\quad - \sigma\varepsilon^2(\nabla\phi_t, \nabla\phi) - \frac{\varepsilon^2}{2}||\nabla\phi||^2_{\mathcal{H}_0}\\ &\geq \varepsilon\mathcal{Z}(t) + \frac{1}{8}||\nabla\phi_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\nabla\Delta\phi||^2_{\mathcal{H}_0} - \varepsilon\left(1 + 2\sigma^2\varepsilon^3 + \frac{\varepsilon}{2}\right)||\nabla\phi||^2_{\mathcal{H}_0}\\ &\geq \varepsilon\mathcal{Z}(t) + \frac{1}{8}||\nabla\phi_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\nabla\Delta\phi||^2_{\mathcal{H}_0} - \varepsilon\left(1 + 2\sigma_0^2\varepsilon^3 + \frac{\varepsilon}{2}\right)||\nabla\phi||^2_{\mathcal{H}_0}.\end{aligned}$$

We also estimate $\mathcal{R}(t)$ term by term as follows

$$\begin{aligned}\int_\Omega f(\phi)\Delta\phi_t\, dx &\leq ||\nabla f(\phi)||_{\mathcal{H}_0}||\nabla\phi_t||_{\mathcal{H}_0} \leq ||f'(\phi)||_{L^\infty(\Omega)}||\nabla\phi||_{\mathcal{H}_0}||\nabla\phi_t||_{\mathcal{H}_0}\\ &\leq Q(||\phi||_{\mathcal{H}_2}) + \frac{1}{16}||\nabla\phi_t||^2_{\mathcal{H}_0},\\ \varepsilon\int_\Omega f(\phi)\Delta\phi\, dx &\leq \frac{\varepsilon}{2}||f(\phi)||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\Delta\phi||^2_{\mathcal{H}_0}\\ &\leq Q(||\phi||_{\mathcal{H}_2}) + \frac{\varepsilon}{4}||\nabla\Delta\phi||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\nabla\phi||^2_{\mathcal{H}_0}.\end{aligned}$$

Collecting the above inequalities together, we have

$$\frac{d}{dt}\mathcal{Z}(t) + \varepsilon\mathcal{Z}(t) + \frac{1}{16}||\nabla\phi_t||^2_{\mathcal{H}_0} \leq Q(||\phi||_{\mathcal{H}_2}), \tag{5.18}$$

where Q is a positive monotone function which may depend on σ_0. In particular, applying the Gronwall Lemma and using the estimate from below for $\mathcal{Z}(t)$, we obtain

$$\begin{aligned}\sigma||\phi_t||^2_{\mathcal{H}_1} + ||\phi||^2_{\mathcal{H}_3} \leq\ & Q(||\phi_0||_{\mathcal{H}_3}, ||\phi_t||_{\mathcal{H}_1})e^{-\varepsilon t} \\ &+ \int_0^t Q(||\phi||_{\mathcal{H}_2})e^{-\varepsilon(t-s)}\ ds + ||\phi||^2_{\mathcal{H}_1}.\end{aligned}$$

Since $(\phi_0, \phi_1) \in \mathcal{Y}_1^\sigma$, we can conclude that

$$||(\phi, \phi_t)||^2_{\mathcal{E}_1^\sigma} \leq Q(||(\phi_0, \phi_1)||_{\mathcal{E}_1^\sigma})e^{-\varepsilon t} + \kappa_1, \tag{5.19}$$

where κ_1 depends on R_0. On the other hand, we have

$$\mathcal{Z}(t+1) + \varepsilon\int_t^{t+1} \mathcal{Z}(s)\ ds + \frac{1}{16}\int_t^{t+1} ||\nabla\phi_t||^2_{\mathcal{H}_0}\ ds \leq C_1 + \mathcal{Z}(t).$$

where C_1 depends only on R_0 and f. Thanks to the above inequalities, we get

$$\int_t^{t+1} ||\nabla\phi_t||^2_{\mathcal{H}_0}\ ds \leq C_2 + Q(||(\phi_0, \phi_1)||_{\mathcal{E}_1^\sigma}).$$

Passing to the supremum in $t \geq 0$ and setting $\kappa_2 = C_2$, the proof is complete. □

In the same matter we can see the next result.

Lemma 5.1.4. *Let $(\phi_0, \phi_1) \in \mathcal{Y}_2^\sigma$. Then the solution (ϕ, ϕ_t) satisfies the following inequalities*

$$||(\phi(t), \phi_t(t))||^2_{\mathcal{E}_2^\sigma} \leq Q(||(\phi_0, \phi_1)||_{\mathcal{E}_2^\sigma})e^{-\varepsilon t} + \kappa_3, \qquad \forall t \geq 0, \tag{5.20}$$

$$\sup_{t\geq 0}\int_t^{t+1} ||\Delta\phi_t(s)||^2_{\mathcal{H}_0}\ ds \leq \kappa_4 + Q(||(\phi_0, \phi_1)||_{\mathcal{E}_2^\sigma}), \tag{5.21}$$

where κ_3, κ_4 and ε are positive constants which depend on R_0 and σ_0, but are independent of the norm of the initial data and σ.

As a consequence, we obtain a result for ϕ_{tt}.

Corollary 5.1.5. *Let $(\phi_0, \phi_1) \in \mathcal{Y}_2^\sigma$. Then the following estimate holds*

$$\sup_{t\geq 0}\int_t^{t+1} \sigma||\phi_{tt}(s)||^2_{\mathcal{H}_0}\ ds \leq Q(||(\phi_0, \phi_1)||_{\mathcal{E}_2^\sigma}). \tag{5.22}$$

Proof. We consider ϕ_{tt} as a test function in (2.14)

$$\sigma||\phi_{tt}||^2_{\mathcal{H}_0} + \frac{1}{2}\frac{d}{dt}||\phi_t||^2_{\mathcal{H}_0} + (\Delta^2\phi, \phi_{tt}) + (\Delta\phi, \phi_{tt}) + (f(\phi), \phi_{tt}) = 0.$$

Using the integration by parts we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\Big\{||\phi_t||^2_{\mathcal{H}_0} + (\Delta\phi, \Delta\phi_t) - 2(\nabla\phi, \nabla\phi_t) + (f(\phi), \phi_t)\Big\} + \sigma||\phi_{tt}||^2_{\mathcal{H}_0}& \\ = ||\Delta\phi_t||^2_{\mathcal{H}_0} - 2||\nabla\phi_t||^2_{\mathcal{H}_0} + (f'(\phi)\phi_t, \phi_t)& \\ \leq ||\Delta\phi_t||^2_{\mathcal{H}_0} + ||f'(\phi)||_{L^\infty(\Omega)}||\phi_t||^2_{\mathcal{H}_0}.&\end{aligned}$$

Integrating in time from t to $t+1$ and using the above Lemmas, we get

$$\int_t^{t+1} \sigma||\phi_{tt}||^2_{\mathcal{H}_0}\, ds \leq Q(||(\phi_0, \phi_1)||_{\mathcal{E}_2^\sigma}), \tag{5.23}$$

where Q is a positive monotone function which may depend on f, σ_0 and Ω. Passing to the supremum in $t \geq 0$, the proof is complete. □

Thanks to Lemmas 5.1.3 and 5.1.4, there exist R_1 and R_2, with $R_1 > \sqrt{\kappa_1}$ and $R_2 > \sqrt{\kappa_3}$, such that

$$\mathcal{V}_1^\sigma = B_{\mathcal{E}_1^\sigma}(0, R_1) \cap \mathcal{Y}_0^\sigma, \quad \mathcal{V}_2^\sigma = B_{\mathcal{E}_2^\sigma}(0, R_2) \cap \mathcal{Y}_0^\sigma$$

are absorbing sets respectively for $(\mathcal{Y}_1^\sigma, S_\sigma(t))$ and $(\mathcal{Y}_2^\sigma, S_\sigma(t))$.

Now we follow the same strategy of the previous chapter to prove the existence of a regular set which exponentially attracts the bounded sets of $\mathcal{Y}_0^\sigma$. In order to do this, we split the semigroup $S_\sigma(t)$ into two parts

$$(\phi(t), \phi_t(t)) = (\phi^l(t), \phi^l_t(t)) + (\phi^o(t), \phi^o_t(t)), \tag{5.24}$$

such that

$$\begin{cases}\sigma\phi^l_{tt} + \phi^l_t + \Delta^2\phi^l + 2\Delta\phi^l + k\phi^l = 0\\ \phi^l(0) = \phi_0, \qquad \phi^l_t(0) = \phi_1\end{cases} \tag{5.25}$$

and

$$\begin{cases}\sigma\phi^o_{tt} + \phi^o_t + \Delta^2\phi^o + 2\Delta\phi^o + k\phi^o + f(\phi) - k\phi = 0\\ \phi^o(0) = 0 \qquad \phi^o_t(0) = 0,\end{cases} \tag{5.26}$$

where $k > 0$ is a large fixed constant.

Lemma 5.1.6. *There exists $k > 0$ such that (ϕ^l, ϕ^l_t) fulfils the following inequality*

$$||(\phi^l(t), \phi^l_t(t))||^2_{\mathcal{E}_0^\sigma} \leq C||(\phi_0, \phi_1)||^2_{\mathcal{E}_0^\sigma} e^{-\varepsilon t}, \quad \forall t \geq 0, \tag{5.27}$$

where C, ε may depend on σ_0, f, Ω but they are independent of the initial data ϕ_0, ϕ_1 and σ.

Proof. Testing (5.25) by $\phi^l_t + \varepsilon\phi^l$, we have

$$\frac{d}{dt}\mathcal{Z}^l(t) + \mathcal{D}^l(t) = 0, \tag{5.28}$$

where

$$\mathcal{Z}^l(t) = \frac{\sigma}{2}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{1}{2}a(\phi^l_t, \phi^l_t) + \frac{k+\varepsilon}{2}||\phi^l||^2_{\mathcal{H}_0} + \sigma\varepsilon(\phi^l_t, \phi^l),$$
$$\mathcal{D}^l(t) = (1-\sigma\varepsilon)||\phi^l_t||^2_{\mathcal{H}_0} + \varepsilon a(\phi^l, \phi^l) + k\varepsilon||\phi^l||^2_{\mathcal{H}_0}.$$

Using (2.4) with $\gamma = \frac{1}{2}$ and the Young inequality, we get

$$\mathcal{Z}^l(t) \geq \frac{\sigma}{4}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\phi^l||^2_{\mathcal{H}_2} + \left(\frac{k+\varepsilon}{2} - 1 - \varepsilon^2\sigma\right)||\phi^l||^2_{\mathcal{H}_0}.$$

Therefore choosing $\varepsilon \in \left(0, \frac{1}{2\sigma_0}\right)$ and $k > 2$, we have

$$\mathcal{Z}^l(t) \geq \frac{\sigma}{4}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\phi^l||^2_{\mathcal{H}_2}.$$

Using standard arguments, we estimate the remainder term $\mathcal{D}^l(t)$ as follows

$$\begin{aligned}
\mathcal{D}^l(t) &\geq \frac{1}{2}||\phi^l_t||^2_{\mathcal{H}_0} + \varepsilon a(\phi^l, \phi^l) + k\varepsilon||\phi^l||^2_{\mathcal{H}_0} \\
&\geq \varepsilon\mathcal{Z}^l(t) + \frac{1}{4}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\phi^l||^2_{\mathcal{H}_2} - \varepsilon||\phi^l||^2_{\mathcal{H}_0} + \frac{\varepsilon k}{2}||\phi^l||^2_{\mathcal{H}_0} \\
&\quad - \frac{1}{8}||\phi^l_t||^2_{\mathcal{H}_0} - 2\sigma^2\varepsilon^4||\phi^l||^2_{\mathcal{H}_0} - \frac{\varepsilon^2}{2}||\phi^l||^2_{\mathcal{H}_0} \\
&\geq \varepsilon\mathcal{Z}^l(t) + \frac{1}{8}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\phi^l||^2_{\mathcal{H}_2} + \varepsilon\left(\frac{k}{2} - 1 - 2\sigma^2\varepsilon^3 - \frac{\varepsilon}{2}\right)||\phi^l||^2_{\mathcal{H}_0}.
\end{aligned}$$

For $k \geq 2 + \varepsilon + 4\sigma_0^2\varepsilon^3$, we obtain

$$\frac{d}{dt}\mathcal{Z}^l(t) + \varepsilon\mathcal{Z}^l(t) + \frac{1}{8}||\phi^l_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\phi^l||^2_{\mathcal{H}_2} \leq 0. \tag{5.29}$$

In particular, applying the Gronwall Lemma, we have

$$\mathcal{Z}^l(t) \leq \mathcal{Z}^l(0)e^{-\varepsilon t}.$$

Since k is large enough, we can conclude that

$$||(\phi^l, \phi^l_t)||^2_{\mathcal{Y}^\sigma_0} \leq 4\mathcal{Z}^l(0)e^{-\varepsilon t} \leq C||(\phi_0, \phi_1)||^2_{\mathcal{E}^\sigma_0}e^{-\varepsilon t}.$$

□

Lemma 5.1.7. *For any $(\phi_0, \phi_1) \in \mathcal{E}^\sigma_0$, the following inequality holds*

$$||(\phi^o(t), \phi^o_t(t))||_{\mathcal{E}^\sigma_1} \leq Q(R_0) = \kappa_5, \qquad \forall t \geq 0. \tag{5.30}$$

Proof. The existence and the uniqueness of the solution (ϕ^l, ϕ_t^l) imply the existence and the uniqueness of the solution to the homogeneous problem (5.26). By comparison, we can assert that

$$||(\phi^o, \phi_t^o)||^2_{\mathcal{E}_0^\sigma} \leq Q(R_0), \qquad \forall t \geq 0. \tag{5.31}$$

Testing the equation in ϕ^o by $-\Delta\phi_t^o - \varepsilon\Delta\phi^o$, where ε is a sufficiently positive constant, we have

$$\frac{d}{dt}\mathcal{Z}^o(t) + \mathcal{D}^o(t) = \mathcal{R}^o(t), \tag{5.32}$$

where

$$\begin{aligned}
\mathcal{Z}^o(t) &= \frac{\sigma}{2}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{2}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - ||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\quad + \varepsilon\sigma(\nabla\phi_t^o, \nabla\phi^o) + \frac{\varepsilon}{2}||\nabla\phi^o||^2_{\mathcal{H}_0}, \\
\mathcal{D}^o(t) &= ||\nabla\phi_t^o||^2_{\mathcal{H}_0} - \varepsilon\sigma||\nabla\phi^o||^2_{\mathcal{H}_0} + \varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - 2\varepsilon||\Delta\phi^o||^2_{\mathcal{H}_0}, \\
\mathcal{R}^o(t) &= (f(\phi) - k\phi + k\phi^o, \Delta\phi_t^o) + \varepsilon(f(\phi) - k\phi + k\phi^o, \Delta\phi^o).
\end{aligned}$$

Using Cauchy-Schwarz, Young and (3.38) inequalities, we get

$$\mathcal{Z}^o(t) \geq \frac{\sigma}{4}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - ||\nabla\phi^o||^2_{\mathcal{H}_0} + \varepsilon(\frac{1}{2} - \sigma\varepsilon)||\nabla\phi^o||^2_{\mathcal{H}_0}.$$

Setting $\varepsilon \in (0, \frac{1}{2\sigma_0})$, we have

$$\begin{aligned}
\mathcal{Z}^o(t) &\geq \frac{\sigma}{4}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - ||\nabla\phi^o||^2_{\mathcal{H}_0} \\
&= \frac{\sigma}{4}||\phi_t^o||^2_{\mathcal{H}_1} + \frac{1}{4}||\phi^o||^2_{\mathcal{H}_3} - ||\phi^o||^2_{\mathcal{H}_1}.
\end{aligned}$$

From the choice of ε, we also deduce

$$\begin{aligned}
\mathcal{D}^o(t) &\geq \frac{1}{2}||\nabla\phi^o(t)||^2_{\mathcal{H}_0} + \varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - 2\varepsilon||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\geq \varepsilon\mathcal{Z}^o(t) + (\frac{1}{2} - \frac{\varepsilon\sigma}{2})||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - \varepsilon||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\quad - \varepsilon^2\sigma(\nabla\phi_t^o, \nabla\phi^o) - \frac{\varepsilon^2}{2}||\nabla\phi||^2_{\mathcal{H}_0} \\
&\geq \varepsilon\mathcal{Z}^o(t) + \frac{1}{4}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} - ||\nabla\phi^o||^2_{\mathcal{H}_0} \\
&\quad - \frac{1}{8}||\nabla\phi_t^o||^2_{\mathcal{H}_0} - 2\varepsilon^4\sigma^2||\nabla\phi^o||^2_{\mathcal{H}_0} - \frac{\varepsilon^2}{2}||\nabla\phi^o||^2_{\mathcal{H}_0} \\
&\geq \varepsilon\mathcal{Z}^o(t) + \frac{1}{8}||\phi_t^o||^2_{\mathcal{H}_1} + \frac{\varepsilon}{4}||\phi^o||^2_{\mathcal{H}_3} - C_1||\phi^o||^2_{\mathcal{H}_1},
\end{aligned}$$

where C_1 only depends on ε and σ_0.

$$
\begin{aligned}
\mathcal{R}^o(t) &= -(f'(\phi)\nabla\phi - k\nabla\phi + k\nabla\phi^o, \nabla\phi_t^o) + \varepsilon(f(\phi) - k\phi + k\phi^o, \Delta\phi^o) \\
&\leq \frac{1}{8}||\nabla\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta\phi^o||^2_{\mathcal{H}_0} + 2||f'(\phi)\nabla\phi - k\nabla\phi + k\nabla\phi^o||^2_{\mathcal{H}_0} \\
&\quad + \varepsilon||f(\phi) - k\phi + k\phi^o||^2_{\mathcal{H}_0} \\
&\leq \frac{1}{8}||\phi_t^o||^2_{\mathcal{H}_1} + \frac{\varepsilon}{4}||\phi^o||^2_{\mathcal{H}_2} + Q(R_0).
\end{aligned}
$$

Combining these inequalities we obtain

$$
\frac{d}{dt}\mathcal{Z}^o(t) + \varepsilon\mathcal{Z}^o(t) \leq Q(R_0). \tag{5.33}
$$

Applying the Gronwall Lemma we conclude that

$$
\mathcal{Z}^o(t) \leq \mathcal{Z}^o(0)e^{-\varepsilon t} + Q(R_0)(1 - e^{-\varepsilon t}) \leq Q(R_0).
$$

This implies that

$$
||(\phi^o, \phi_t^o)||^2_{\mathcal{E}_1^\sigma} \leq Q(R_0),
$$

where Q also depends on σ_0, f, Ω. □

We infer from Lemmas 5.1.6 and 5.1.7 that the set

$$
\mathcal{W}_1^\sigma = B_{\mathcal{E}_1^\sigma}(0, \kappa_5) \cap \mathcal{Y}_0^\sigma \tag{5.34}
$$

exponentially attracts any bounded set $\mathcal{B}$ of $\mathcal{Y}_0^\sigma$. Observing that $\mathcal{V}_0^\sigma \subset \mathcal{Y}_0^\sigma$, we have that $\mathcal{Y}_0^\sigma$ exponentially attracts any bounded set of $\mathcal{E}_0^\sigma$. Thus, using Lemma 4.1.6, we conclude that $\mathcal{W}_1^\sigma$ exponentially attracts any bounded set of $\mathcal{E}_0^\sigma$ with respect to the $\mathcal{E}_0^\sigma$-metric. We recall that $\mathcal{V}_1^\sigma$ is an absorbing set in $\mathcal{Y}_1^\sigma$, then in particular $\mathcal{V}_1^\sigma$ absorbs $\mathcal{W}_1^\sigma$, so $\mathcal{V}_1^\sigma$ exponentially attracts any bounded set of $\mathcal{E}_0^\sigma$ for the transitivity property of the exponential attraction. Moreover, by definition of absorbing set and Lemma 5.1.3, we conclude that the following properties hold

$$
\begin{cases}
\exists\, t_{\mathcal{V}_1^\sigma} : \; S_\sigma(t)\mathcal{V}_1^\sigma \subset \mathcal{V}_1^\sigma, & \forall t \geq t_{\mathcal{V}_1^\sigma}, \\
||S_\sigma(t)(u,v)||_{\mathcal{E}_1^\sigma} \leq Q(\kappa_1), & \forall (u,v) \in \mathcal{V}_1^\sigma, \forall t \geq 0.
\end{cases} \tag{5.35}
$$

With the same argument, we can prove the next result.

Lemma 5.1.8. *Under the assumption of Lemma* 5.1.7*, the following inequality holds*

$$
||(\phi^o(t), \phi_t^o(t))||_{\mathcal{E}_2^\sigma} \leq Q(R_0) = \kappa_6, \qquad \forall t \geq 0. \tag{5.36}
$$

In conclusion, the set $\mathcal{V}_2^\sigma$ absorbs $\mathcal{W}_2^\sigma = B_{\mathcal{E}_2^\sigma}(0,\kappa_6) \cap \mathcal{Y}_0^\sigma$, which exponentially absorbs any bounded set of $\mathcal{E}_0^\sigma$ with respect to the $\mathcal{E}_0^\sigma$-metric. Then, $\mathcal{V}_2^\sigma$ exponentially attracts any bounded set of $\mathcal{E}_0^\sigma$ and we have

$$\begin{cases} \exists\ t_{\mathcal{V}_2^\sigma} :\ S_\sigma(t)\mathcal{V}_2^\sigma \subset \mathcal{V}_2^\sigma, & \forall t \geq t_{\mathcal{V}_2^\sigma}, \\ ||S_\sigma(t)(u,v)||_{\mathcal{E}_2^\sigma} \leq Q(\kappa_3), & \forall (u,v) \in \mathcal{V}_2^\sigma, \forall t \geq 0. \end{cases} \tag{5.37}$$

Now we consider two couples of initial data (ϕ_{01},ϕ_{11}) and (ϕ_{02},ϕ_{12}) in $\mathcal{B}_{\mathcal{E}_2^\sigma}(0,\rho) \cap \mathcal{Y}_0^\sigma$ and their respectively solutions $(\phi_1,\phi_{t,1})$ and $(\phi_2,\phi_{t,2})$. We set their difference

$$(\phi(t),\phi_t(t)) = (\phi_1(t),\phi_{t,1}(t)) - (\phi_2(t),\phi_{t,2}(t)), \tag{5.38}$$

which satisfies

$$\begin{cases} \sigma\phi_{tt} + \phi_t + \Delta^2\phi + 2\Delta\phi + f(\phi_1) - f(\phi_2) = 0 & \text{in } \Omega \times (0,T) \\ \phi = \Delta\phi = 0 & \text{on } \partial\Omega \times (0,T) \\ \phi(0) = \phi_{01} - \phi_{02} = \phi_0 & \text{in } \Omega \\ \phi_t(0) = \phi_{11} - \phi_{12} = \phi_1 & \text{in } \Omega. \end{cases} \tag{5.39}$$

We consider the following split

$$(\phi(t),\phi_t(t)) = (\phi^l(t),\phi_t^l(t)) + (\phi^o(t),\phi_t^o(t)), \tag{5.40}$$

such that

$$\begin{cases} \sigma\phi_{tt}^l + \phi_t^l + \Delta^2\phi^l + 2\Delta\phi^l + k\phi^l = 0 \\ \phi^l(0) = \phi_0, \qquad \phi_t^l(0) = \phi_1, \end{cases} \tag{5.41}$$

and

$$\begin{cases} \sigma\phi_{tt}^o + \phi_t^o + \Delta^2\phi^o + 2\Delta\phi^o + k\phi^o + f(\phi_1) - f(\phi_2) - k\phi = 0 \\ \phi^o(0) = 0, \qquad \phi_t^o(0) = 0. \end{cases} \tag{5.42}$$

By observing the problem for (ϕ^l,ϕ_t^l), we can infer from Lemma 5.1.6 that there holds

$$||(\phi^l(t),\phi_t^l(t))||_{\mathcal{E}_0^\sigma}^2 \leq C||(\phi_0,\phi_1)||_{\mathcal{E}_0^\sigma}^2 e^{-\varepsilon t}, \qquad \forall t \geq 0. \tag{5.43}$$

where C and ε are independent of the norm of the initial data and σ.

Lemma 5.1.9. *Under the above assumptions, we have*

$$||(\phi^o(t),\phi_t^o(t))||_{\mathcal{E}_1}^2 \leq C(t)||(\phi_0,\phi_1)||_{\mathcal{E}_0}^2, \qquad \forall t \geq 0. \tag{5.44}$$

Proof. Testing the equation (5.42) by $\Delta^2\phi_t^o + \varepsilon\Delta^2\phi^o$, where ε is a positive small constant, we get

$$\frac{d}{dt}\mathcal{Z}^o(t) + \mathcal{D}^o(t) = \mathcal{R}^o(t), \tag{5.45}$$

where

$$\begin{aligned}
\mathcal{Z}^o(t) &= \frac{\sigma}{2}||\Delta\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{2}||\Delta^2\phi^o||^2_{\mathcal{H}_0} - ||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} + \varepsilon\sigma(\Delta\phi_t^o, \Delta\phi^o) \\
&\quad + \frac{\varepsilon}{2}||\Delta\phi^o||^2_{\mathcal{H}_0} + \frac{k}{2}||\Delta\phi^o||^2_{\mathcal{H}_0}, \\
\mathcal{D}^o(t) &= ||\Delta\phi_t^o||^2_{\mathcal{H}_0} - \varepsilon\sigma||\Delta\phi^o||^2_{\mathcal{H}_0} + \varepsilon||\Delta^2\phi^o||^2_{\mathcal{H}_0} - 2\varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\quad + \varepsilon k||\Delta\phi^o||^2_{\mathcal{H}_0}, \\
\mathcal{R}^o(t) &= -(f(\phi_1) - f(\phi_2) - k\phi, \Delta^2\phi_t^o) - \varepsilon(f(\phi_1) - f(\phi_2) - k\phi, \Delta^2\phi^o).
\end{aligned}$$

Using (3.38) and (4.43), we have

$$\begin{aligned}
\mathcal{Z}^o(t) \geq \frac{\sigma}{4}||\Delta\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0} - ||\Delta\phi^o||^2_{\mathcal{H}_0} \\
- \sigma\varepsilon^2||\Delta\phi||^2_{\mathcal{H}_0} + \frac{k+\varepsilon}{2}||\Delta\phi||^2_{\mathcal{H}_0}.
\end{aligned}$$

Thus, taking $\varepsilon \in (0, \frac{1}{2\sigma_0})$ and $k > 2$, we obtain

$$\mathcal{Y}^o(t) \geq \frac{\sigma}{4}||\Delta\phi_t^o||^2_{\mathcal{H}_0} + \frac{1}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0} = \frac{1}{4}||(\phi^o, \phi_t^o)||^2_{\mathcal{E}_2^\sigma}. \tag{5.46}$$

Consequently, in a standard way, we estimate $\mathcal{D}(t)$ as follows

$$\begin{aligned}
\mathcal{D}^o(t) &\geq \frac{1}{2}||\Delta\phi_t^o||^2_{\mathcal{H}_0} + \varepsilon||\Delta^2\phi^o||^2_{\mathcal{H}_0} - 2\varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} + \varepsilon k||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\geq \varepsilon\mathcal{Y}^o(t) + \left(\frac{1}{2} - \frac{\sigma\varepsilon}{2}\right)||\Delta\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\Delta^2\phi^o||^2_{\mathcal{H}_0} - \varepsilon||\nabla\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\quad - \sigma\varepsilon^2(\Delta\phi_t^o, \Delta\phi^o) + \left(\frac{\varepsilon k}{2} - \frac{\varepsilon^2}{2}\right)||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\geq \varepsilon\mathcal{Y}^o(t) + \frac{1}{4}||\Delta\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0} - \varepsilon||\Delta\phi^o||^2_{\mathcal{H}_0} - \frac{1}{8}||\Delta\phi_t^o||^2_{\mathcal{H}_0} \\
&\quad - 2\sigma^2\varepsilon^4||\Delta\phi_t^o||^2_{\mathcal{H}_0} + \left(\frac{\varepsilon k}{2} - \frac{\varepsilon^2}{2}\right)||\Delta\phi^o||^2_{\mathcal{H}_0} \\
&\geq \varepsilon\mathcal{Y}^o(t) + \frac{1}{8}||\Delta\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0} \\
&\quad + \left(\frac{\varepsilon k}{2} - \varepsilon - \frac{\varepsilon^2}{2} - 2\sigma^2\varepsilon^4\right)||\Delta\phi^o||^2_{\mathcal{H}_0}.
\end{aligned}$$

So, with k large enough, we can conclude that

$$\mathcal{D}^o(t) \geq \varepsilon\mathcal{Y}^o(t) + \frac{1}{8}||\Delta\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0}.$$

Moreover, applying Cauchy-Schwarz and Young inequalities, we deduce

$$\begin{aligned}\mathcal{R}^o(t) &\leq |(f''(\phi_1)\Delta\phi_1 - f''(\phi_2)\Delta\phi_2 - k\Delta\phi, \Delta\phi_t^o)| \\ &\quad + \varepsilon|(f(\phi_1) - f(\phi_2) - k\phi, \Delta^2\phi^o)| \\ &\leq \frac{1}{8}||\Delta\phi_t^o||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta^2\phi^o||^2_{\mathcal{H}_0} + 4||f''(\phi_1)\Delta\phi_1 - f''(\phi_2)\Delta\phi_2||^2_{\mathcal{H}_0} \\ &\quad + 4k^2||\Delta\phi||^2_{\mathcal{H}_0} + \frac{4}{\varepsilon}||f(\phi_1) - f(\phi_2)||^2_{\mathcal{H}_0} + 4k^2||\phi||^2_{\mathcal{H}_0}.\end{aligned}$$

Using the Sobolev embedding $\mathcal{H}_2 \hookrightarrow L^\infty(\Omega)$, we proceed term by term in the following way

$$\begin{aligned}&||f''(\phi_1)\Delta\phi_1 - f''(\phi_2)\Delta\phi_2||^2_{\mathcal{H}_0} \\ &\leq 2||f''(\phi_1)\Delta\phi||^2_{\mathcal{H}_0} + 2||\left(f''(\phi_1) - f''(\phi_2)\right)\Delta\phi_2||^2_{\mathcal{H}_0} \\ &\leq Q(R_0)||\Delta\phi||^2_{\mathcal{H}_0} + 2||\int_0^1 f'''(\tau\phi_1 + (1-\tau)\phi_2)\phi \, d\tau||^2_{L^\infty(\Omega)}||\Delta\phi_2||^2_{\mathcal{H}_0} \\ &\leq Q(R_0)||\Delta\phi||^2_{\mathcal{H}_0} + Q(R_0)||\Delta\phi||^2_{\mathcal{H}_0},\end{aligned}$$

$$\begin{aligned}||f(\phi_1) - f(\phi_2)||^2_{\mathcal{H}_0} &\leq ||\int_0^1 f'(\tau\phi_1 + (1-\tau)\phi_2)\phi \, d\tau||^2_{\mathcal{H}_0} \\ &\leq Q(R_0)||\phi||^2_{\mathcal{H}_0}.\end{aligned}$$

Collecting these estimates together, we get

$$\frac{d}{dt}\mathcal{Y}^o(t) + \varepsilon\mathcal{Y}^o(t) \leq Q(R_0)||\phi||^2_{\mathcal{H}_2}. \tag{5.47}$$

Integrating (5.47) with respect to time and using the Lipschitz continuity estimate (5.2), we have

$$\begin{aligned}\mathcal{Y}^o(t) &\leq Q(R_0)\int_0^t ||\phi(s)||^2_{\mathcal{H}_2} \, ds \\ &\leq Q(R_0, t)\{||\phi_0||^2_{\mathcal{H}_2} + \sigma||\phi_1||^2_{\mathcal{H}_0}\}.\end{aligned}$$

Thanks to (5.46) we can conclude that

$$||(\phi^o, \phi_t^o)||^2_{\mathcal{E}_2^\sigma} \leq C(t)||(\phi_0, \phi_1)||^2_{\mathcal{E}_0^\sigma}. \tag{5.48}$$

□

We can summarize the above results as follows

$$\begin{aligned}S_\sigma(t)(\phi_{01}, \phi_{11}) - S_\sigma(t)(\phi_{02}, \phi_{12}) &= L_\sigma(t)((\phi_{01}, \phi_{11}), (\phi_{02}, \phi_{12})) \\ &\quad + K_\sigma(t)((\phi_{01}, \phi_{11}), (\phi_{02}, \phi_{12})),\end{aligned}$$

such that

$$||L_\sigma(t)((\phi_{01}, \phi_{11}), (\phi_{02}, \phi_{12}))||_{\mathcal{E}_0^\sigma} \leq C||(\phi_{01} - \phi_{02}, \phi_{11} - \phi_{12})||_{\mathcal{E}_0^\sigma} e^{-\varepsilon t},$$
$$||K_\sigma(t)((\phi_{01}, \phi_{11}), (\phi_{02}, \phi_{12}))||_{\mathcal{E}_2^\sigma} \leq C(t)||(\phi_{01} - \phi_{02}, \phi_{11} - \phi_{12})||_{\mathcal{E}_0^\sigma},$$

with ε and $C > 0$ independent of σ.

Lemma 5.1.10. *Let us fix $t^* > 0$. Then the map $(t,(u,v)) \mapsto S_\sigma(t)(u,v) : [t^*, 2t^*] \times \mathcal{V}_2^\sigma \to \mathcal{V}_2^\sigma$ is Lipschitz continuous, when $\mathcal{V}_2^\sigma$ is endowed with the $\mathcal{E}_0^\sigma$-topology.*

Proof. Let us consider $t^* \leq \tau \leq t \leq 2t^*$, $(u_1, v_1), (u_2, v_2) \in \mathcal{V}_2^\sigma$. Using (5.2) we have

$$
\begin{aligned}
||S_\sigma(t)(u_1,v_1) - S_\sigma(\tau)(u_2,v_2)||^2_{\mathcal{E}_0^\sigma} &\leq 2||S_\sigma(t)(u_1,v_1) - S_\sigma(t)(u_2,v_2)||^2_{\mathcal{E}_0^\sigma} \\
&\quad + 2||S_\sigma(t)(u_2,v_2) - S_\sigma(\tau)(u_2,v_2)||^2_{\mathcal{E}_0} \\
&\leq C||(u_1,v_1) - (u_2,v_2)||^2_{\mathcal{E}_0^\sigma} \\
&\quad + 2||S_\sigma(t)(u_2,v_2) - S_\sigma(\tau)(u_2,v_2)||^2_{\mathcal{E}_0^\sigma},
\end{aligned}
$$

where C is a positive constant depending on $\mathcal{V}_2^\sigma, t^*$. We recall that the trajectories starting from $\mathcal{V}_2^\sigma$ satisfy

$$
||\phi_{i,t}||_{\mathcal{H}_2} \leq C, \qquad ||\phi_{i,tt}||_{\mathcal{H}_0} \leq C,
$$

where C depends on $\mathcal{V}_2^\sigma$ but is independent of σ.

$$
\begin{aligned}
&||S_\sigma(t)(u_2,v_2) - S_\sigma(\tau)(u_2,v_2)||^2_{\mathcal{E}_0^\sigma} \\
&\qquad = ||\phi_2(t) - \phi_2(\tau)||^2_{\mathcal{H}_2} + \sigma||\phi_{2,t}(t) - \phi_{2,t}(\tau)||^2_{\mathcal{H}_0} \\
&\qquad = ||\int_\tau^t \phi_{2,t}(s)\ ds||^2_{\mathcal{H}_2} + \sigma||\int_\tau^t \phi_{2,tt}(s)\ ds||^2_{\mathcal{H}_0} \\
&\qquad \leq \left(\int_\tau^t ||\phi_{2,t}(s)||_{\mathcal{H}_2}\ ds\right)^2 + \sigma\left(\int_\tau^t ||\phi_{2,tt}(s)||_{\mathcal{H}_0}\ ds\right)^2 \\
&\qquad \leq C^2(1+\sigma_0)|t-\tau|^2.
\end{aligned}
$$

Collecting the above estimates together, we conclude that

$$
\begin{aligned}
||S_\sigma(t)(u_1,v_1) - S_\sigma(\tau)(u_2,v_2)||_{\mathcal{E}_0^\sigma} &\leq C_1||(u_1,v_1) - (u_2,v_2)||_{\mathcal{E}_0} \\
&\quad + C_2|t-\tau|, \qquad (5.49)
\end{aligned}
$$

where C_1,C_2 are positive constant depending on $\mathcal{V}_2^\sigma$, t^*. □

5.2 Upper semicontinuity of the global attractor

The aim of this section is to investigate a result of stability for the global attractor $\mathcal{A}_\sigma$ with respect to $\mathcal{A}$ when the parameter σ goes to 0. We recall that the dynamical system $(\mathcal{H}_0, S(t))$ possesses a global attractor $\mathcal{A}$, which is bounded in $\mathcal{H}_2$. Furthermore, for each $\sigma \in (0, \sigma_0]$, the semigroup $S_\sigma(t)$ admits a global attractor $\mathcal{A}_\sigma$ in $\mathcal{E}_0$. Thanks to the Corollary (3.3.7), this set is also bounded in $\mathcal{E}_2$.

Lemma 5.2.1. *The global attractor $\mathcal{A}$ is a bounded set in $\mathcal{H}_4$.*

Proof. Since $\mathcal{A}$ is bounded in $\mathcal{H}_2$, using the invariance property of the global attractor, we have

$$||\phi(t)||_{\mathcal{H}_2} \leq R, \qquad \forall t \geq 0, \tag{5.50}$$

where ϕ is the strong solution to problem (2.1) with initial data $\phi_0 \in \mathcal{A}$. Multiplying by $\Delta^2\phi$ in (2.1) and integrating on Ω, we get

$$\frac{1}{2}\frac{d}{dt}||\Delta\phi||^2_{\mathcal{H}_0} + ||\phi||^2_{\mathcal{H}_4} - 2||\phi||^2_{\mathcal{H}_3} + (f(\phi), \Delta^2\phi) = 0.$$

Thanks to (5.50), (3.47 and the Sobolev embedding $\mathcal{H}_2 \hookrightarrow L^\infty(\Omega)$, we deduce that

$$\frac{d}{dt}||\Delta\phi||^2_{\mathcal{H}_0} + ||\phi||^2_{\mathcal{H}_4} \leq Q(R).$$

Let us fix $r > 0$. Integrating from t to $t + r$, we obtain

$$\int_t^{t+r} ||\phi(s)||^2_{\mathcal{H}_4}\, ds \leq ||\phi(t)||^2_{\mathcal{H}_2} + rQ(R) \leq Q(R, r). \tag{5.51}$$

Now we test (2.1) with $\Delta^2\phi_t$ and we have

$$||\Delta\phi_t||^2_{\mathcal{H}_0} + \frac{d}{dt}\Lambda(t) + (f(\phi), \Delta^2\phi_t) = 0,$$

where $\Lambda(t) = \frac{1}{2}||\phi||^2_{\mathcal{H}_4} - ||\phi||^2_{\mathcal{H}_3}$.
We can infer from (5.50), (3.47 and the Sobolev embedding $\mathcal{H}_2 \hookrightarrow L^\infty(\Omega)$ that

$$\Lambda(t) \geq \frac{1}{4}||\phi||^2_{\mathcal{H}_4} - Q(R) \tag{5.52}$$

and

$$||\Delta\phi_t||^2_{\mathcal{H}_0} + \frac{d}{dt}\Lambda(t) \leq Q(R). \tag{5.53}$$

Thanks to (5.51), (5.52) and (5.53), we can apply the uniform Gronwall Lemma and we conclude that

$$||\phi(t + r)||^2_{\mathcal{H}_4} \leq Q(R, r), \qquad \forall t \geq 0. \tag{5.54}$$

From the invariance property of $\mathcal{A}$, it is immediate that $\mathcal{A}$ is bounded in $\mathcal{H}_4$. □

Lemma 5.2.1 allows us to define the set

$$\mathcal{A}_0 = \{(\phi, \psi) : \phi \in \mathcal{A}, \psi = -\Delta^2\phi - \Delta\phi - f(\phi)\}, \tag{5.55}$$

which is the lifting in $\mathcal{E}_0$ of the global attractor $\mathcal{A}$ of the equation (2.1).

We now proceed our analysis with regularity properties concerning $\mathcal{A}_\sigma$. As a consequence of Lemma 5.1.2 and the invariance of the global attractor, we have the following results.

Corollary 5.2.2. *There exists a positive constant C such that, for any $\sigma \in (0, \sigma_0]$,*

$$\forall(\phi_0, \phi_1) \in \mathcal{A}_\sigma, \qquad ||(\phi_0, \phi_1)||_{\mathcal{E}_0^\sigma} \le C. \tag{5.56}$$

Lemma 5.2.3. *Let (ϕ_0, ϕ_1) be such that $||(\phi_0, \phi_1)||_{\mathcal{E}_0^\sigma} \le \rho_0$. Then we have*

$$\int_0^\infty ||\phi_t(s)||^2_{\mathcal{H}_0} \, ds \le Q(\rho_0). \tag{5.57}$$

Proof. Let us define the following functional

$$\Lambda_\sigma(u, v) = \frac{\sigma}{2}||v||^2_{\mathcal{H}_0} + \frac{1}{2}||\Delta u||^2_{\mathcal{H}_0} - ||\nabla u||^2_{\mathcal{H}_0} + \int_\Omega F(u) \, dx. \tag{5.58}$$

Testing the equation (2.14) by ϕ_t and integrating from 0 to t, we get

$$\int_0^t ||\phi_t(s)||^2_{\mathcal{H}_0} \, ds = \Lambda_\sigma(\phi_0, \phi_1) - \Lambda_\sigma(\phi, \phi_t) \le \Lambda_\sigma(\phi_0, \phi_1) + K_2|\Omega|. \tag{5.59}$$

In particular, this imply that

$$\int_0^t ||\phi_t(s)||^2_{\mathcal{H}_0} \, ds \le Q(\rho_0), \qquad \forall t \ge 0. \tag{5.60}$$

□

Lemma 5.2.4. *Let (ϕ_0, ϕ_1) be the initial data such that $||(\phi_0, \phi_1)||_{\mathcal{E}_0^\sigma} \le \rho_0$ and $||(\phi_0, \phi_1)||_{\mathcal{E}_2^\sigma} \le \rho_1$, then the following estimate holds, for all $t \ge 0$,*

$$\sigma||\phi_{tt}||^2_{\mathcal{H}_0} + ||\Delta\phi_t||^2_{\mathcal{H}_0} + ||\Delta^2\phi||^2_{\mathcal{H}_0} \le Q(\rho_0) + \frac{Q(\rho_0, \rho_1)}{\sigma^2} e^{-\varepsilon t}. \tag{5.61}$$

Proof. First of all,we can infer from Lemma 5.1.2 that

$$||(\phi, \phi_t)||^2_{\mathcal{E}_0^\sigma} \le Q(\rho_0)e^{-t} + R_0, \qquad t \ge 0. \tag{5.62}$$

We consider the following problem

$$\begin{cases} \sigma\psi_{tt} + \psi_t + \Delta^2\psi + 2\Delta\psi + 2\psi = -\left(f'(\phi) + 2\right)\phi_t & \text{in } \Omega \times (0, T) \\ \psi = \Delta\psi = 0 & \text{on } \partial\Omega \times (0, T) \\ \psi(0) = \phi_1 & \text{in } \Omega \\ \psi_t(0) = \frac{1}{\sigma}(-f(\phi_0) - \Delta^2\phi_0 - 2\Delta\phi_0 - \phi_1) & \text{in } \Omega. \end{cases}$$

Since $\psi(0) \in \mathcal{H}_2$, $\psi_t(0) \in \mathcal{H}_0$ and the right-hand side of the equation belongs to $L^2(0, +\infty; \mathcal{H}_0)$, there is a unique weak solution $(w, w_t) \in C([0, +\infty), \mathcal{E}_0^\sigma)$ and $w(t) = \phi_t(t)$.
Testing the previous equation by $\psi_t + \varepsilon\psi$, we obtain

$$\frac{d}{dt}\mathcal{Z}(t) + \mathcal{D}(t) = \mathcal{R}(t), \tag{5.63}$$

where

$$
\begin{aligned}
\mathcal{Z}(t) &= \frac{\sigma}{2}||\psi_t||^2_{\mathcal{H}_0} + \frac{1}{2}||\Delta\psi||^2_{\mathcal{H}_0} - ||\nabla\psi||^2_{\mathcal{H}_0} \\
&\quad + ||\psi||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\psi||^2_{\mathcal{H}_0} + \varepsilon\sigma(\psi_t, \psi), \\
\mathcal{D}(t) &= (1-\varepsilon\sigma)\,||\psi_t||^2_{\mathcal{H}_0} + \varepsilon||\Delta\psi||^2_{\mathcal{H}_0} - 2\varepsilon||\nabla\psi||^2_{\mathcal{H}_0} + 2\varepsilon||\psi||^2_{\mathcal{H}_0}, \\
\mathcal{R}(t) &= ((-f'(\phi)+2)\phi_t, \psi_t) + \varepsilon\left((-f'(\phi)+2)\phi_t, \psi\right).
\end{aligned}
$$

In a standard way, we get

$$
\mathcal{Z}(t) \geq \frac{\sigma}{4}||\psi_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\Delta\psi||^2_{\mathcal{H}_0} + \left(\frac{\varepsilon}{2} - \varepsilon^2\sigma\right)||\psi||^2_{\mathcal{H}_0}.
$$

Taking $\varepsilon \in (0, \frac{1}{2\sigma_0})$, we have

$$
\mathcal{Z}(t) \geq \frac{\sigma}{4}||\psi_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\Delta\psi||^2_{\mathcal{H}_0} \tag{5.64}
$$

We proceed with the other terms in (5.63) as follows

$$
\begin{aligned}
\mathcal{D}(t) &\geq \frac{1}{2}||\psi_t||^2_{\mathcal{H}_0} + \varepsilon||\Delta\psi||^2_{\mathcal{H}_0} - 2\varepsilon||\nabla\psi||^2_{\mathcal{H}_0} + 2\varepsilon||\psi||^2_{\mathcal{H}_0} \\
&\geq \varepsilon\mathcal{Z}(t) + \left(\frac{1}{2} - \frac{\varepsilon\sigma}{2}\right)||\psi_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{2}||\Delta\psi||^2_{\mathcal{H}_0} - \varepsilon||\nabla\psi||^2_{\mathcal{H}_0} + \varepsilon||\psi||^2_{\mathcal{H}_0} \\
&\quad - \varepsilon^2\sigma(\psi_t, \psi) - \frac{\varepsilon^2}{2}||\psi||^2_{\mathcal{H}_0} \\
&\geq \varepsilon\mathcal{Z}(t) + \frac{1}{8}||\psi_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta\psi||^2_{\mathcal{H}_0} - (2\varepsilon\sigma_0^2 + \frac{\varepsilon^2}{2})||\psi||^2_{\mathcal{H}_0}, \\
\mathcal{R}(t) &\leq \frac{1}{8}||\psi_t||^2_{\mathcal{H}_0} + \frac{\varepsilon}{4}||\Delta\psi||^2_{\mathcal{H}_0} + 2||f'(\phi)+2||^2_{L^\infty(\Omega)}||\phi_t||^2_{\mathcal{H}_0} + \\
&\quad + C_1\varepsilon||f'(\phi)+2||^2_{L^\infty(\Omega)}||\phi_t||^2_{\mathcal{H}_0}.
\end{aligned}
$$

Collecting these estimates together and using (5.62), we get

$$
\frac{d}{dt}\mathcal{Z}(t) + \varepsilon\mathcal{Z}(t) \leq Q(\rho_0)||\phi_t||^2_{\mathcal{H}_0}. \tag{5.65}
$$

In particular, applying the Gronwall Lemma and Lemma 5.2.3, yields that

$$
\begin{aligned}
\mathcal{Z}(t) &\leq \mathcal{Z}(0)e^{-\varepsilon t} + \int_0^t Q(\rho_0)||\phi_t||^2_{\mathcal{H}_0}e^{\varepsilon(s-t)}\, ds \\
&\leq \mathcal{Z}(0)e^{-\varepsilon t} + Q(\rho_0).
\end{aligned}
$$

Consequently, we obtain from (5.64)

$$
\sigma||\psi_t||^2_{\mathcal{H}_0} + ||\Delta\psi||^2_{\mathcal{H}_0} \leq 4\mathcal{Z}(0)e^{-\varepsilon t} + Q(\rho_0). \tag{5.66}
$$

On the other hand, using standard argument, we deduce

$$\begin{aligned}\mathcal{Z}(0) &\leq \sigma||\psi_t(0)||^2_{\mathcal{H}_0} + C_2||\Delta\phi_1||^2_{\mathcal{H}_0}\\ &\leq \sigma||\frac{1}{\sigma}(-f(\phi_0) - \Delta^2\phi_0 - 2\Delta\phi_0 - \phi_1)||^2_{\mathcal{H}_0} + C_2||\Delta\phi_1||^2_{\mathcal{H}_0}\\ &\leq \frac{C_2}{\sigma}\left\{||f(\phi_0)||^2_{\mathcal{H}_0} + ||\Delta^2\phi_0||^2_{\mathcal{H}_0} + ||\Delta\phi_0||^2_{\mathcal{H}_0} + ||\phi_1||^2_{\mathcal{H}_0} + \sigma||\Delta\phi_1||^2_{\mathcal{H}_0}\right\}\\ &\leq \frac{C_2}{\sigma}\left\{Q(\rho_1) + ||\phi_1||^2_{\mathcal{H}_0}\right\}\\ &\leq \frac{C_2}{\sigma}\left\{Q(\rho_1) + \frac{1}{\sigma}Q(\rho_0)\right\}.\end{aligned}$$

We can infer directly from the problem in ψ that

$$\begin{aligned}||\Delta^2\phi||^2_{\mathcal{H}_0} &\leq C_3\left\{||f(\phi)||^2_{\mathcal{H}_0} + ||\Delta\phi||^2_{\mathcal{H}_0} + ||\phi_t||^2_{\mathcal{H}_0} + \sigma^2||\phi_{tt}||^2_{\mathcal{H}_0}\right\}\\ &\leq C_3\left\{Q(R_0) + (\sigma + C_4)\left[4\mathcal{Z}(0)e^{-\varepsilon t} + Q(R_0)\right]\right\}.\end{aligned}$$

In conclusion, we can assert that

$$\begin{aligned}\sigma||\phi_{tt}||^2_{\mathcal{H}_0} + ||\Delta\phi_t||^2_{\mathcal{H}_0} + ||\Delta^2\phi||^2_{\mathcal{H}_0} &\leq Q(\rho_0) + C^{(5)}\mathcal{Z}(0)e^{-\varepsilon t}\\ &\leq Q(\rho_0) + \frac{Q(\rho_0, \rho_1)}{\sigma^2}e^{-\varepsilon t}.\end{aligned}$$

□

Now we can state an immediate consequence of Lemma 5.2.4, based on the invariance property of the global attractor.

Corollary 5.2.5. *For any sigma* $\in (0, \sigma_0]$*, the global attractor* $\mathcal{A}_\sigma$ *is uniformly bounded in* $\mathcal{E}_2^{\sigma_0}$*, namely*

$$\forall(\phi_0, \phi_1) \in \mathcal{A}_\sigma, \qquad ||(\phi_0, \phi_1)||_{\mathcal{E}_2^{\sigma_0}} \leq C, \tag{5.67}$$

where C *is independent of* σ*. Moreover, for any orbit* $\Phi(t) = (\phi, \phi_t)(t)$ *of* 2.14 *with* $\Phi(\mathbb{R}) \subset \mathcal{A}_\sigma$*, we have*

$$\sqrt{\sigma}||\phi_{tt}(t)||_{\mathcal{H}_0} \leq C, \qquad \forall t \in \mathbb{R}. \tag{5.68}$$

Following the method introduced in [16] and using the regularity properties for which:

i. there exists a bounded set $\mathcal{B}_1$ in $\mathcal{E}_2^{\sigma_0}$ such that

$$\bigcup_{\sigma\in(0,\sigma_0]} \mathcal{A}_\sigma \cup \mathcal{A}_0 \subset \mathcal{B}_1; \tag{5.69}$$

ii. for any $\sigma \in (0, \sigma_0]$ and for any orbit $\Phi(t) = (\phi, \phi_t)(t)$ of 2.14 with $\Phi(\mathbb{R}) \subset \mathcal{A}_\sigma$, the following estimate holds

$$\sqrt{\sigma}||\phi_{tt}(t)||_{\mathcal{H}_0} \leq C, \qquad \forall t \in \mathbb{R}; \tag{5.70}$$

we formulate our stability result.

Theorem 5.2.6. *The global attractor $\mathcal{A}_\sigma$ is upper semicontinuous at zero with respect to $\mathcal{A}_0$ as $\sigma \to 0$, namely*

$$dist_{\mathcal{E}_0}(\mathcal{A}_\sigma, \mathcal{A}_0) \to 0, \qquad \sigma \to 0. \tag{5.71}$$

5.3 A robust family of exponential attractors

The main result of this section is the following Theorem which states the existence of a family of exponential attractors fulfilling a Hölder continuous property with respect to the parameter of the singular perturbation, σ.

Theorem 5.3.1. *Let σ_0 be a real positive fixed value. For any $\sigma \in [0, \sigma_0]$, then there exists an exponential attractor $\mathcal{M}_\sigma$ for the semigroup $S_\sigma(t)$ on the phace space $\mathcal{E}_0^\sigma$, which satisfies the following properties:*

(P1) *$\mathcal{M}_\sigma$ is positively invariant and bounded set in $\mathcal{E}_2^\sigma$ and $\mathcal{E}_0^{\sigma_0}$ with bounds independent of σ.*

(P2) *The rate of attraction is uniformly exponential, i.e. for every $\mathcal{B}$ bounded set in $\mathcal{E}_0^\sigma$*

$$dist_{\mathcal{E}_0^\sigma}(S_\sigma(t)\mathcal{B}, \mathcal{M}_\sigma) \leq Q(||\mathcal{B}||_{\mathcal{E}_0^\sigma})e^{-\omega t}, \quad \forall t \geq 0. \tag{5.72}$$

(P3) *The fractal dimension of $\mathcal{M}_\sigma$ is uniform bounded,*

$$dim_{\mathcal{E}_0^\sigma} \mathcal{M}_\sigma \leq C, \tag{5.73}$$

where C is independent of σ.

(P4) *The map $\sigma \to \mathcal{M}_\sigma$ is Hölder continuous in σ with exponent $\frac{1}{4}$, namely*

$$dist^{sym}_{\mathcal{E}_0^{\sigma_1}}(\mathcal{M}_{\sigma_1}, \mathcal{M}_{\sigma_2}) \leq C(\sigma_1 - \sigma_2)^{\frac{1}{4}}, \quad 0 \leq \sigma_2 < \sigma_1 \leq \sigma_0. \tag{5.74}$$

The idea to prove Theorem 5.3.1 involves a discrete semigroup as in the previous chapter. Now we also need a control about the difference between solutions associated to different semigroups in terms of the perturbation. We will apply the strategy introduced in [21] and adopted to study the singularly perturbed damped wave equation, which is a standard model in the study of a singularly perturbed dynamical system. We also refer to [15], in which this construction is adapted to a phase-field model with high regularity gap between the phase function ϕ and its time derivative ϕ_t.

The following abstract Theorem is the key to constructing a robust family of exponential attractors (see [6]).

Theorem 5.3.2. *Let H and H_1 be two Banach spaces with H_1 compactly embedded into H, and let P be a closed subset of H bounded in H_1. For every $\varepsilon \in [0,1]$, assume that there exists a δ-neighbourhood $\mathcal{O}_\delta(P)$ ($\delta > 0$) of the set P in H_1 and a family of maps $\widehat{S}_\varepsilon : \mathcal{O}_\delta(P) \to P$ satisfying the following conditions:*

(C1) For every $x_1, x_2 \in \mathcal{O}_\delta(P)$,

$$\widehat{S}_\varepsilon x_1 - \widehat{S}_\varepsilon x_2 = \mathcal{L}_\varepsilon(x_1,x_2) + \mathcal{K}_\varepsilon(x_1,x_2), \tag{5.75}$$

where

$$||\mathcal{L}_\varepsilon(x_1,x_2)||_H \leq \theta ||x_1 - x_2||_H,$$
$$||\mathcal{K}_\varepsilon(x_1,x_2)||_{H_1} \leq C||x_1 - x_2||_H,$$

with $\theta < \frac{1}{2}, C > 0$ independent of ε.

(C2) The family $\widehat{S}_\varepsilon$ is uniformly Hölder continuous with respect to ε, that is,

$$\sup_{x \in \mathcal{O}_\delta(P)} ||\widehat{S}_{\varepsilon_1} x - \widehat{S}_{\varepsilon_2} x||_H \leq C|\varepsilon_1 - \varepsilon_2|^\theta,$$

with $\theta < \frac{1}{2}, C > 0$ independent of ε.

Then, there exists a family of closed sets $\widehat{\mathcal{M}}^d_\varepsilon \subset P$, positively invariant for $\widehat{S}_\varepsilon$, such that

$$dist_H(\widehat{S}^n_\varepsilon P, \widehat{\mathcal{M}}^d_\varepsilon) \leq Ce^{-\omega n},$$
$$dim_H \widehat{\mathcal{M}}^d_\varepsilon \leq C,$$
$$dist^{sym}_H(\widehat{\mathcal{M}}^d_{\varepsilon_1}, \widehat{\mathcal{M}}^d_{\varepsilon_2}) \leq C|\varepsilon_1 - \varepsilon_2|^\theta,$$

where $\widehat{S}^n_\varepsilon (n \in \mathbb{N})$ is the family of discrete semigroups generated by the iterations of $\hat{S}_\varepsilon$.

We observe that the same thesis of Theorem 5.3.2 is also true if we replace $\widehat{\mathcal{M}}^d_\varepsilon$ with $\widehat{\mathcal{M}}^d_{\varepsilon,1} = \widehat{S}_\varepsilon \widehat{\mathcal{M}}^d_\varepsilon$. This abstract result provides a criterion to obtain a family of exponential attractors, which continuously depends on the perturbation parameter ε, when we work with a family of maps defined in the same Banach spaces. Conversely, in our setting, for any $\sigma \in (0, \sigma_0]$ the semigroup $S_\sigma(t)$ acts on $\mathcal{E}^\sigma_0$, which explicitly depends on σ. In order to overcome this problem, in [21] a scaling argument is introduced and it allows us to apply the Theorem 5.3.2 in the case of singular perturbations.

For any $\sigma \in [0, \sigma_0]$, the scaling operator $\mathcal{T}_\sigma$ is such that

$$\mathcal{T}_\sigma : \mathcal{Y}^\sigma_i \to \mathcal{Y}^{\sigma_0}_i, \quad \mathcal{T}_\sigma(u,v) = (\phi, \sqrt{\sigma\sigma_0^{-1}}\phi_t), \quad i = 0,1,2. \tag{5.76}$$

We observe that

$$(u,v) \in \mathcal{V}_2^{\sigma} \quad \Rightarrow \quad \mathcal{T}_\sigma(u,v) \in \mathcal{V}_2^{\sigma_0}. \tag{5.77}$$

The rescaled semigroup is the map $\widehat{S}_\sigma(t)$: $\mathcal{Y}_i^{\sigma_0} \to \mathcal{Y}_i^{\sigma_0}$ defined by

$$\widehat{S}_\sigma(t)(u,v) = \begin{cases} \mathcal{T}_\sigma S_\sigma(t) \mathcal{T}_\sigma^{-1}(u,v), & \sigma \in (0,\sigma_0], \\ S_0(t)(u,v), & \sigma = 0. \end{cases}$$

Now we proceed to prove two important preliminary results.

Lemma 5.3.3. *For any $\sigma \in (0,\sigma_0]$ and initial data $(\phi_0,\phi_1) \in \mathcal{Y}_2^{\sigma}$ such that $||(\phi_0,\phi_1)||_{\mathcal{E}_2^{\sigma}} \le \rho$, there holds*

$$||(\phi,\phi_t)(t)||_{\mathcal{E}_0^{\sigma_0}} = ||S_\sigma(t)(\phi_0,\phi_1)||_{\mathcal{E}_0^{\sigma_0}} \le Q(\rho), \quad \forall t \ge 1. \tag{5.78}$$

Proof. We consider the solution (ϕ,ϕ_t) corresponding to the initial data (ϕ_0,ϕ_1). We set $v(t) = \phi_t$ and we read problem (2.14) as follows

$$\sigma v_t + v = -\Delta^2\phi - 2\Delta\phi - f(\phi). \tag{5.79}$$

Thanks to Lemma 5.1.4, we have

$$||\phi||_{\mathcal{H}_4} \le Q(\rho,\kappa_3) \;\Rightarrow\; ||-\Delta^2\phi - 2\Delta\phi - f(\phi)||_{\mathcal{H}_0} \le Q(\rho,\kappa_3), \quad \forall t \ge 0. \tag{5.80}$$

Now we resolve (5.79) and we obtain

$$v(t) = v(0)e^{-\frac{t}{\sigma}} + \frac{1}{\sigma}e^{-\frac{t}{\sigma}} \int_0^t e^{\frac{s}{\sigma}} \left(-\Delta^2\phi - 2\Delta\phi - f(\phi)\right)\, ds. \tag{5.81}$$

Computing the norm, we get

$$\begin{aligned} ||v(t)||_{\mathcal{H}_0} &\le ||v(0)||_{\mathcal{H}_0} e^{-\frac{t}{\sigma}} + \frac{1}{\sigma}e^{-\frac{t}{\sigma}} \int_0^t e^{\frac{s}{\sigma}} ||\Delta^2\phi + 2\Delta\phi + f(\phi)||_{\mathcal{H}_0}\, ds \\ &\le \sigma^{-\frac{1}{2}} ||(\phi_0,\phi_1)||_{\mathcal{E}_0^{\sigma}} e^{-\frac{1}{\sigma}t} + \sup_{t\ge 0} ||\Delta^2\phi + 2\Delta\phi + f(\phi)||_{\mathcal{H}_0} \\ &\le Q(\rho,\kappa_3), \qquad \forall t \ge 1 \end{aligned}$$

This imply that

$$||\phi_t||_{\mathcal{H}_0} \le Q(\rho), \quad \forall t \ge 1, \tag{5.82}$$

where Q is independent of σ. In conclusion we have the so-called boundary layer estimate

$$||(\phi,\phi_t)||_{\mathcal{E}_0^{\sigma_0}} \le Q(\rho), \qquad \forall t \ge 1, \tag{5.83}$$

where Q may depend on σ_0 and κ_3 but is independent of σ. □

Lemma 5.3.4. *For any $0 \leq \sigma_2 < \sigma_1 \leq \sigma_0$, the following estimate holds for $t \geq 1$*

$$||\widehat{S}_{\sigma_1}(t)(\phi_0,\phi_1) - \widehat{S}_{\sigma_2}(t)(\phi_0,\phi_1)||_{\mathcal{E}_0^{\sigma_0}} \leq Q(\rho)\sqrt{t}e^{Q(\rho)t}\left(\sigma_1-\sigma_2\right)^{\frac{1}{4}}, \tag{5.84}$$

where $(\phi_0,\phi_1) \in \mathcal{Y}_2^{\sigma_0}$ such that $||(\phi_0,\phi_1)||_{\mathcal{E}_2^{\sigma_0}} \leq \rho$.

Proof. We need to consider the following two steps.

Step 1 We fix $0 = \sigma_2 < \sigma_1 \leq \sigma_0$. We call $\phi_0(t)$ the solution to problem (2.1) with $\phi_0(0) = \phi_0$, and we denote $(\phi,\phi_t)(t)$ the solution to problem (2.14) such that $(\phi,\phi_t)(0) = (\phi_0, \sqrt{\sigma_1^{-1}\sigma_0}\phi_1)$. We define $\psi = \phi_0 - \phi$, which fulfils the parabolic equation

$$\begin{cases} \psi_t + \Delta^2\psi + 2\Delta\psi = f(\phi) - f(\phi_0) + \sigma_1\phi_{tt} & \text{in } \Omega\times(0,T) \\ \psi = \Delta\psi = 0 & \text{on } \partial\Omega\times(0,T) \\ \psi(0) = 0 & \text{in } \Omega \end{cases} \tag{5.85}$$

Adding $\pm 2\psi$ to the left-hand side and testing by ψ_t, we obtain

$$||\psi_t||^2_{\mathcal{H}_0} + \frac{d}{dt}\mathcal{Z}(t) = \mathcal{R}(t), \tag{5.86}$$

where

$$\mathcal{Z}(t) = \frac{1}{2}||\Delta\psi||^2_{\mathcal{H}_0} - ||\nabla\psi||^2_{\mathcal{H}_0} + ||\psi||^2_{\mathcal{H}_0},$$
$$\mathcal{R}(t) = (f(\phi) - f(\phi_0), \psi_t) + 2(\psi,\psi_t) + \sigma_1(\phi_{tt},\psi_t).$$

Using standard arguments, we get

$$\mathcal{Z}(t) \geq \frac{1}{4}||\Delta\psi||^2_{\mathcal{H}_0} - ||\psi||^2_{\mathcal{H}_0} + ||\psi||^2_{\mathcal{H}_0} \geq \frac{1}{4}||\Delta\psi||^2_{\mathcal{H}_0},$$

and

$$\begin{aligned} \mathcal{R}(t) &\leq ||f(\phi) - f(\phi_0)||_{\mathcal{H}_0}||\psi_t||_{\mathcal{H}_0} + 2||\psi||_{\mathcal{H}_0}||\psi_t||_{\mathcal{H}_0} + \sigma_1||\phi_{tt}||_{\mathcal{H}_0}||\psi_t||_{\mathcal{H}_0} \\ &\leq \frac{1}{2}||\psi_t||^2_{\mathcal{H}_0} + ||f(\phi) - f(\phi_0)||^2_{\mathcal{H}_0} + 8||\psi||^2_{\mathcal{H}_0} + 2\sigma_1^2||\phi_{tt}||^2_{\mathcal{H}_0} \\ &\leq \frac{1}{2}||\psi_t||^2_{\mathcal{H}_0} + Q(\rho)||\psi||^2_{\mathcal{H}_0} + 2\sigma_1^2||\phi_{tt}||^2_{\mathcal{H}_0} \\ &\leq \frac{1}{2}||\psi_t||^2_{\mathcal{H}_0} + Q(\rho)\mathcal{Z}(t) + 2\sigma_1^2||\phi_{tt}||^2_{\mathcal{H}_0}. \end{aligned}$$

Collecting these estimates together, we have

$$\frac{d}{dt}\mathcal{Z}(t) + \frac{1}{2}||\psi_t||^2_{\mathcal{H}_0} \leq Q(\rho)\mathcal{Z}(t) + 2\sigma_1^2||\phi_{tt}||^2_{\mathcal{H}_0}. \tag{5.87}$$

In particular we can infer from the Gronwall Lemma and Corollary 5.1.5 that

$$\begin{aligned}\mathcal{Z}(t) &\le \int_0^t \frac{1}{2}\sigma_1^2 ||\phi_{tt}(s)||^2_{\mathcal{H}_0} e^{-Q(\rho)(s-t)}\ ds \\ &\le \frac{1}{2}\sigma_1 e^{Q(\rho)t} \int_0^t \sigma_1 ||\phi_{tt}(s)||^2_{\mathcal{H}_0} e^{-Q(\rho)s}\ ds \\ &\le \sigma_1 e^{Q(\rho)t} Q(\rho)t.\end{aligned}$$

Thanks to the above estimate on $Z(t)$, we obtain

$$||\psi||^2_{\mathcal{H}_2} = ||\Delta\psi||^2_{\mathcal{H}_0} \le Q(\rho)te^{Q(\rho)t}\sigma_1. \tag{5.88}$$

Using the boundary layer estimate, we have

$$||\phi_t||_{\mathcal{H}_0} \le C(\sigma_0, \kappa_3), \quad \forall t \ge 1. \tag{5.89}$$

In conclusion we can state for $t \ge 1$

$$\begin{aligned}||\widehat{S}_{\sigma_1}(t)(\phi_0,\phi_1) - \widehat{S}_0(t)(\phi_0,\phi_1)||^2_{\mathcal{E}_0^{\sigma_0}} &= ||(\phi,\phi_t)(t) - (\phi_0,0)(t)||^2_{\mathcal{E}_0^{\sigma_1}} \\ &= ||\Delta\psi||^2_{\mathcal{H}_0} + \sigma_1 ||\phi_t||^2_{\mathcal{H}_0} \\ &\le Q(\rho)te^{Q(\rho)t}\sigma_1.\end{aligned}$$

Step 2 Let us fix $0 < \sigma_2 < \sigma_1 \le \sigma_0$ and let $(\phi_1, \phi_{1,t})$, $(\phi_2, \phi_{2,t})$ be the solution to problem (2.14) respectively with initial data $(\phi_0, \sqrt{\sigma_1^{-1}\sigma_0}\phi_1)$ and $(\phi_0, \sqrt{\sigma_2^{-1}\sigma_0}\phi_1)$. We consider the difference of the two solutions $(\psi, \psi_t) = (\phi_1 - \phi_2, \phi_{1,t} - \phi_{2,t})$, which is the solution to the problem

$$\begin{cases}\sigma_2\psi_{tt} + \psi_t + \Delta^2\psi + 2\Delta\psi = f(\phi_2) - f(\phi_1) - (\sigma_1 - \sigma_2)\phi_{1,tt} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega \times (0,T) \\ \psi = \Delta\psi = 0 & \text{on } \partial\Omega \times (0,T) \\ \psi(0) = 0 & \text{in } \Omega \\ \psi_t(0) = (\sqrt{\sigma_1^{-1}\sigma_0} - \sqrt{\sigma_2^{-1}\sigma_0})\phi_1 & \text{in } \Omega.\end{cases}$$

As in the first case, testing by ψ_t, we have

$$||\psi_t||^2_{\mathcal{H}_0} + \frac{d}{dt}\mathcal{Z}(t) = \mathcal{R}(t), \tag{5.90}$$

where

$$\begin{aligned}\mathcal{Z}(t) &= \frac{\sigma_2}{2}||\psi_t||^2_{\mathcal{H}_0} + \frac{1}{2}||\Delta\psi||^2_{\mathcal{H}_0} - ||\nabla\psi||^2_{\mathcal{H}_0} + ||\psi||^2_{\mathcal{H}_0}, \\ \mathcal{R}(t) &= 2(\psi,\psi_t) + (f(\phi_2) - f(\phi_1), \psi_t) - (\sigma_1 - \sigma_2)(\phi_{1,tt}, \psi_t).\end{aligned}$$

From Lemma 5.1.4, we have in a standard way

$$\mathcal{Z}(t) \geq \frac{\sigma_2}{2}||\psi_t||^2_{\mathcal{H}_0} + ||\Delta\psi||^2_{\mathcal{H}_0},$$

and

$$\begin{aligned}\mathcal{R}(t) &\leq ||\psi_t||^2_{\mathcal{H}_0} + \frac{1}{4}||\psi||^2_{\mathcal{H}_0} + \frac{1}{2}||f(\phi_2) - f(\phi_1)||^2_{\mathcal{H}_0} + (\sigma_1 - \sigma_2)^2||\phi||_{1,tt}||^2_{\mathcal{H}_0} \\ &\leq ||\psi_t||^2_{\mathcal{H}_0} + Q(\rho)||\psi||^2_{\mathcal{H}_0} + (\sigma_1 - \sigma_2)^2||\phi_{1,tt}||^2_{\mathcal{H}_0} \\ &\leq ||\psi_t||^2_{\mathcal{H}_0} + Q(\rho)\mathcal{Z}(t) + (\sigma_1 - \sigma_2)^2||\phi_{1,tt}||^2_{\mathcal{H}_0}.\end{aligned}$$

Combining these estimates together and using the Gronwall Lemma, we obtain

$$\mathcal{Z}(t) \leq e^{Q(\rho)t}\mathcal{Z}(0) + \int_0^t (\sigma_1 - \sigma_2)^2||\phi_{1,tt}||^2_{\mathcal{H}_0} e^{-Q(\rho)(s-t)}\ ds. \tag{5.91}$$

Thanks to Corollary 5.1.5, we can estimate the second term in the right-hand side in (5.91) as follows

$$\begin{aligned}&\int_0^t (\sigma_1 - \sigma_2)^2||\phi_{1,tt}||^2_{\mathcal{H}_0} e^{-Q(\rho)(s-t)}\ ds \\ &\qquad\leq \frac{(\sigma_1 - \sigma_2)^2}{\sigma_1} e^{Q(\rho)t} \int_0^t \sigma_1||\phi_{1,tt}||^2_{\mathcal{H}_0} e^{-Q(\rho)s}\ ds \\ &\qquad\leq \frac{(\sigma_1 - \sigma_2)^2}{\sigma_1} e^{Q(\rho)t} Q(\rho)t.\end{aligned}$$

Using the estimate from below on $\mathcal{Z}(t)$ and observing that $\psi(0) = 0$, we get

$$\begin{aligned}\sigma_2||\psi_t||^2_{\mathcal{H}_0} + ||\psi||^2_{\mathcal{H}_2} &\leq 4e^{Q(\rho)t}\frac{\sigma_2}{2}||\psi_t(0)||^2_{\mathcal{H}_0} + 4Q(\rho)te^{Q(\rho)t}\frac{(\sigma_1 - \sigma_2)^2}{\sigma_1} \\ &\leq e^{Q(\rho)t}\sigma_2\left(\sqrt{\sigma_2^{-1}\sigma_0} - \sqrt{\sigma_1^{-1}\sigma_0}\right)^2 ||\phi_1||^2_{\mathcal{H}_0} \\ &\quad + Q(\rho)te^{Q(\rho)t}\frac{(\sigma_1 - \sigma_2)^2}{\sigma_1} \\ &\leq e^{Q(\rho)t}\sigma_0\left(1 - \sqrt{\sigma_1^{-1}\sigma_2}\right)^2 ||\phi_1||^2_{\mathcal{H}_0} \\ &\quad + Q(\rho)te^{Q(\rho)t}\frac{(\sigma_1 - \sigma_2)^2}{\sigma_1} \\ &\leq e^{Q(\rho)t}\sigma_0\frac{\sigma_1 - \sigma_2}{\sigma_1}||\phi_1||^2_{\mathcal{H}_0} + Q(\rho)te^{Q(\rho)t}\frac{\sigma_1 - \sigma_2}{\sigma_1} \\ &\leq Q(\rho)te^{Q(\rho)t}\frac{\sigma_1 - \sigma_2}{\sigma_1},\end{aligned}$$

where $Q(\rho)$ may depend on σ_0. Applying the boundary layer estimate, we consequently obtain

$$
\begin{aligned}
&||\hat{S}_{\sigma_1}(t)(\phi_0,\phi_1) - \hat{S}_{\sigma_2}(t)(\phi_0,\phi_1)||^2_{\mathcal{E}_0^{\sigma_0}} \\
&= ||\psi(t)||^2_{\mathcal{H}_2} + \sigma_0||\sqrt{\sigma_1\sigma_0^{-1}}\phi_{1,t}(t) - \sqrt{\sigma_2\sigma_0^{-1}}\phi_{2,t}(t)||^2_{\mathcal{H}_0} \\
&= ||\psi(t)||^2_{\mathcal{H}_2} + ||\sqrt{\sigma_1}\phi_{1,t}(t) - \sqrt{\sigma_2}\phi_{2,t}(t)||^2_{\mathcal{H}_0} \\
&= ||\psi(t)||^2_{\mathcal{H}_2} + 2\sigma_2||\psi_t(t)||^2_{\mathcal{H}_0} + 2(\sqrt{\sigma_1} - \sqrt{\sigma_2})^2||\phi_{1,t}||^2_{\mathcal{H}_0} \\
&\leq 2Q(\rho)te^{Q(\rho)t}\frac{\sigma_1 - \sigma_2}{\sigma_1} + Q(\rho)\,(\sigma_1 - \sigma_2) \\
&\leq Q(\rho)te^{Q(\rho)t}\frac{\sigma_1 - \sigma_2}{\sigma_1}, \quad \forall t \geq 1.
\end{aligned}
$$

Using also the information from the first case, we can infer that

$$
\begin{aligned}
||\hat{S}_{\sigma_1}(t)(\phi_0,\phi_1) - \hat{S}_{\sigma_2}(t)(\phi_0,\phi_1)||^2_{\mathcal{E}_0^{\sigma_0}}
&\leq 2||\hat{S}_{\sigma_1}(t)(\phi_0,\phi_1) - \hat{S}_{\sigma_0}(t)(\phi_0,\phi_1)||^2_{\mathcal{E}_0^{\sigma_0}} \\
&\quad + 2||\hat{S}_{\sigma_2}(t)(\phi_0,\phi_1) - \hat{S}_{\sigma_0}(t)(\phi_0,\phi_1)||^2_{\mathcal{E}_0^{\sigma_0}} \\
&\leq Q(\rho)te^{Q(\rho)t}\sigma_1, \quad \forall t \geq 1.
\end{aligned}
$$

An easy computation shows that

$$\min\{\sigma_1, \frac{\sigma_1 - \sigma_2}{\sigma_1}\} \leq (\sigma_1 - \sigma_2)^{\frac{1}{2}},$$

so we can conclude that

$$||\hat{S}_{\sigma_1}(t)(\phi_0,\phi_1) - \hat{S}_{\sigma_2}(t)(\phi_0,\phi_1)||_{\mathcal{E}_0^{\sigma_0}} \leq Q(\rho)\sqrt{t}e^{Q(\rho)t}\,(\sigma_1 - \sigma_2)^{\frac{1}{4}}, \ \forall t \geq 1.$$

□

Proof of Theorem 5.3.1. We divide the proof into three steps.

Family of robust exponential attractors: the discrete case
We set $H = \mathcal{Y}_0^{\sigma_0}$, $H_1 = \mathcal{Y}_2^{\sigma_0}$ and $P = \mathcal{V}_2^{\sigma_0}$. Fixing $\delta > 0$, we consider $\mathcal{O}_\delta(P) = B_{\mathcal{E}_2^{\sigma_0}}(0, R_2 + \delta) \cap \mathcal{Y}_0^{\sigma_0}$. Thanks to the property of the scaling operator (5.77) and to Lemma 5.1.4, we can state that exists $t^* > 0$ such that

$$\widehat{S}_\sigma(t^*) : \mathcal{O}_\delta(P) \to P, \qquad \forall \sigma \in [0, \sigma_0]. \tag{5.92}$$

Moreover, by means of (5.43), (5.44) and Lemma 5.3.4 we have that

(1) For every $(u_1, v_1), (u_2, v_2) \in \mathcal{O}_\delta(P)$,

$$
\begin{aligned}
&\widehat{S}_\sigma(t^*)(u_1, v_1) - \widehat{S}_\sigma(t^*)(u_2, v_2) \\
&\qquad = \widehat{L}_\sigma((u_1, v_1), (u_2, v_2)) + \widehat{K}_\sigma((u_1, v_1), (u_2, v_2)),
\end{aligned}
$$

where $\widehat{L}_\sigma = \mathcal{T}_\sigma L_\sigma(t^*)\mathcal{T}_\sigma^{-1}$, $\widehat{K}_\sigma = \mathcal{T}_\sigma K_\sigma(t^*)\mathcal{T}_\sigma^{-1}$ such that

$$||\widehat{L}_\sigma((u_1,v_1),(u_2,v_2))||_{\mathcal{E}_0^{\sigma_0}} \le \theta||(u_1-u_2,v_1-v_2)||_{\mathcal{E}_0^{\sigma_0}},$$
$$||\widehat{K}_\sigma((u_1,v_1),(u_2,v_2))||_{\mathcal{E}_2^{\sigma_0}} \le C||(u_1-u_2,v_1-v_2)||_{\mathcal{E}_0^{\sigma_0}},$$

with $\theta \le \frac{1}{2}$ and $K > 0$ independent of σ.

(2) For every $0 \le \sigma_2 < \sigma_1 \le \sigma_0$, the family $\widehat{S}_\sigma$ satisfies

$$\sup_{(u_1,v_1)\in\mathcal{O}_\delta(P)} ||\widehat{S}_{\sigma_1}(u_1,v_1) - \widehat{S}_{\sigma_2}(u_1,v_1)||_{\mathcal{E}_0^{\sigma_0}} \le Q(R_2+\delta,t^*)(\sigma_1-\sigma_2)^{\frac{1}{4}}.$$

Applying the abstract Theorem 5.3.2, there exists a family of compact sets $\mathcal{M}_\sigma^d \subset \mathcal{V}_2^\sigma$ positively invariant for $S_\sigma(t^*)$ and uniformly bounded in $\mathcal{E}_0^{\sigma_0}$ such that

$$dist_{\mathcal{E}_0^\sigma}(S_\sigma(t^*)^n\mathcal{V}_2^\sigma, \mathcal{M}_\sigma^d) \le Ce^{-\omega n}, \qquad dim_{\mathcal{E}_0^\sigma}\mathcal{M}_\sigma^d \le C, \tag{5.93}$$

$$dist^{sym}_{\mathcal{E}_0^{\sigma_1}}(\mathcal{M}_{\sigma_1}^d, \mathcal{M}_{\sigma_2}^d) \le C(\sigma_1-\sigma_2)^{\frac{1}{4}}, \quad 0 \le \sigma_2 < \sigma_1 \le \sigma_0. \tag{5.94}$$

Family of robust exponential attractors

In a standard way, as well as in chapter four, we define for any $\sigma \in [0,\sigma_0]$

$$\mathcal{M}_\sigma = \bigcup_{t\in[t^*,2t^*]} S_\sigma(t)\mathcal{M}_\sigma^d. \tag{5.95}$$

Thanks to the properties of $\mathcal{M}_\sigma^d$, joined to the Lipschitz property of the map $(t,(u,v)) \mapsto S_\sigma(t)(u,v)$, we obtain that for any $\sigma \in [0,\sigma_0]$, $\mathcal{M}_\sigma \subset \mathcal{V}_2^\sigma$ is compact, positively invariant,

$$dim_{\mathcal{E}_0^\sigma}\mathcal{M}_\sigma \le dim_{\mathcal{E}_0^\sigma}\mathcal{M}_\sigma^d + 1 \le C+1, \tag{5.96}$$

and for all B bounded in $\mathcal{V}_2^\sigma$

$$dist_{\mathcal{E}_0^\sigma}(S_\sigma(t)B, \mathcal{M}_\sigma) \le Ce^{-\omega t}, \quad \forall t \ge 0. \tag{5.97}$$

Since the family of $\mathcal{M}_\sigma^d$ is bounded in $\mathcal{V}_2^\sigma$, we can infer from the boundary layer estimate (5.78) that also the family of $\mathcal{M}_\sigma$ is uniformly bounded in $\mathcal{E}_0^{\sigma_0}$. Finally, using (5.94), (5.2) and 2.2, we obtain

$$dist^{sym}_{\mathcal{E}_0^{\sigma_1}}(\mathcal{M}_{\sigma_1}, \mathcal{M}_{\sigma_2}) \le C(\sigma_1-\sigma_2)^{\frac{1}{4}}, \quad 0 \le \sigma_2 < \sigma_1 \le \sigma_0. \tag{5.98}$$

Enlarging the basin of attraction

To conclude the proof, we need to ensure that $\mathcal{M}_\sigma$ exponentially attracts any bounded set of $\mathcal{E}_0^\sigma$. This property easily follows from (5.2), the properties of $\mathcal{V}_2^\sigma$, $\mathcal{M}_\sigma$ and the transitivity property of the exponential attraction. □

Conclusions

The present work provides a mathematical analysis of the Swift-Hohenberg and the modified Swift-Hohenberg equations. The classic model was originally introduced by Swift and Hohenberg to describe the effects of thermal fluctuations on the convective instability in Rayleigh-Bénard experiment. Subsequently, this equation has become a paradigm in the study of pattern formation. The Swif-Hohenberg equation has also been recently employed to model the evolution from an unstable to a metastable or stable state in phase transition. In particular, the original model has been enriched with an inertial term (i.e. a second order-time derivative) in order to predict fast degrees of freedom.

A theoretical analysis of qualitative properties of the two models is proposed in this thesis. In the first part we formulate and prove theorems of existence, uniqueness and continuous dependence from initial data. We are able to deal with a more generic nonlinear term than the physical relevant one. In particular, we do not require any conditions to the derivatives of the nonlinear term in addition to the continuity. Furthermore, the two models present an intrinsic difference: the parabolic nature of the Swift-Hohenberg model implies the regularization of the solution in finite-time, which does not appear in the model with the inertial term. The second part is devoted to study the long-time behavior of the solutions following the theory of dissipative dynamical system in infinite dimension. The well-posedness allows us to define the strongly semigroup map in suitable phase spaces of the models concerned. Reading the equation as a dynamical system, we demonstrate the existence of the global attractor and the exponential attractor. Also, we interpret the modified Swift-Hohenberg equation as a singular perturbation of the Swift-Hohenberg equation and we discuss the robustness of such invariant sets with respect to the perturbation parameter. The main results regard the upper semicontinuity of the global attractor and the construction of a family of exponential attractors which are Hölder continuous with respect to σ.

In closing, although we provide a wide dissertation of the main objects to describe the long-time behavior within the theory of dissipative dynamical system in infinite dimension, different ways of investigation may be considered in a future work. We may establish the finite-dimensionality of the global attractor related to the physical parameters or study the convergence of a single solution to an equilibrium point. It would be interesting to consider a nonlocal version of the Swift-Hohenberg equation, which is proposed in recent papers, or the Swift-Hohenberg equation with advection term to predict spiral chaos in Rayleigh-Bénard convection.

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