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A nonlocal stochastic Cahn-Hilliard equation

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Abstract

In this thesis we discuss the mathematical analysis of the Cahn-Hilliard partial differential equation, which describes the time evolution of the physical phenomenon called *spinodal decomposition*. This phenomenon occurs, for example, when a binary alloy, which is initially very hot, is abruptly cooled down. Throughout this cooling process, the alloy follows various specific stages during which one can observe distinctive geometric patterns. These configurations are approximated by the solution of the Cahn-Hilliard equation. Such equation also applies to other phenomena occurring in fields such as tumor growth, population dynamics, image impainting.

We first describe the spinodal decomposition phenomenon and provide a concise physical motivation of the structure of the equation. We then move to a detailed mathematical analysis of some known results associated with both the deterministic and stochastic versions of the Cahn-Hilliard equation. Finally, we prove some new theorems of existence, uniqueness and measurability of suitable solutions to a specific stochastic extension of a nonlocal Cahn-Hilliard equation.

Sommario

In questa tesi trattiamo l'analisi matematica dell'equazione alle derivate parziali di Cahn-Hilliard, la quale descrive l'evoluzione temporale del fenomeno fisico chiamato *decomposizione spinodale*. Questo fenomeno si verifica, per esempio, quando una lega metallica binaria, che inizialmente è molto calda, viene bruscamente raffreddata. Nel corso di questo processo di raffreddamento, la lega passa attraverso alcuni stadi durante i quali si possono osservare configurazioni geometriche caratteristiche. Queste configurazioni vengono approssimate dalla soluzione dell'equazione di Cahn-Hilliard. Tale equazione si applica anche ad altri fenomeni attinenti a campi come la diffusione dei tumori, la dinamica delle popolazioni, la ricostruzione di immagini.

Per prima cosa descriviamo il fenomeno della decomposizione spinodale e forniamo una breve motivazione fisica della struttura dell'equazione. Quindi passiamo ad una analisi matematica dettagliata di alcuni risultati noti attinenti sia alla versione deterministica dell'equazione di Cahn-Hilliard sia alla versione stocastica della medesima. Infine dimostriamo alcuni teoremi originali sull'esistenza, l'unicità e la misurabilità di appropriate soluzioni per una specifica versione stocastica di una equazione di Cahn-Hilliard in forma non locale.

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Contents

Chapter 1

Introduction

1.1 Physics of the Cahn-Hilliard model

During the last century many reseachers attempted to formulate a mathematical model to properly describe a physical phenomenon commonly referred to as *spinodal decomposition*. This phenomenon is a special type of phase separation process in a two-phase system and it is studied extensively in materials science. In order to analytically describe this phenomenon, a partial differential equation, known as the *Cahn-Hilliard equation*, was proposed by scientist John W. Cahn in 1961 [12] after a joint collaboration with scientist John E. Hilliard.
In Subsection (1.1.1) we illustrate the key qualitative features of the spinodal decomposition phenomenon by means of its basic motivating example. For its complete treatment, the reader may consult [43], [46]. Relying on this example, we illustrate the physical formulation of the deterministic Cahn-Hilliard equation in Subsections (1.1.2)-(1.1.3) and the physical formulation of the stochastic Cahn-Hilliard equation in Section (1.2). In Section (1.3) we give a review of some related applications.
The remaining part of this Introduction and the following chapters constitute the very core of this thesis, which is devoted to a purely mathematical analysis of suitable versions of the Cahn-Hilliard equation. In particular, we present some new results in Chapter 4.

1.1.1 The motivating example

Let us consider a spatial domain $D \subset \mathbb{R}^d$, $d \in \{1; 2; 3\}$, to which a binary alloy is confined. The alloy is composed of two different metallic elements A and B (e.g., aluminum and zinc, gold and platinum, gold and nickel). The alloy's temperature T_0 is initially extremely high, so that the metallic elements A and B are melted together and the material is quite homogeneous. Clearly, at this stage, A and B are not distinguishable at all. After a while the alloy is cooled down abruptly and

it reaches a much lower temperature T_1. During the cooling period, the elements A and B separate from one another and the material becomes inhomogeneous. This phenomenon is most commonly referred to as the *spinodal decomposition* phenomenon.
The phase separation follows a number of different stages: in the first place one can observe a *partial nucleation* (i.e., the appearance of some nucleides of element A or B in some portions of the domain D) or a *total nucleation* (i. e., the division of the entire alloy in nucleides of A or B). The material quickly becomes inhomogeneous, forming a (possibly incomplete) fine-grained structure composed of separate blocks of A and B. Then there is a second stage which occurs more slowly, called *coarsening*, during which the blocks of A and B grow and finally form a set of well-defined spatial domains composed either by A or B. The interface between two spatial domains of A and B is not sharp, but has a non degenerate thickness. In this interface the composition of the alloy changes gradually between A and B. See Figure I.I. The presence of a thick interface

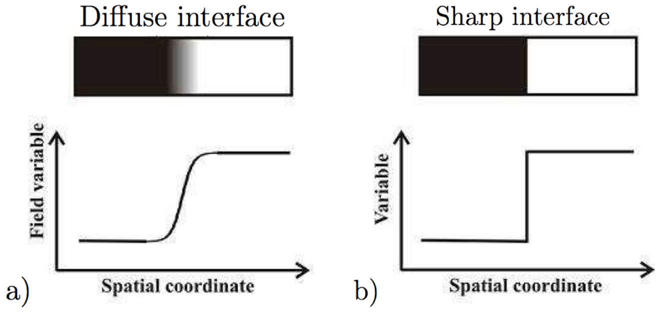

Figure I.I: a) The diffuse interface is the greyscaled region between the black rectangle (metal A) and the white rectangle (metal B). b) A sharp interface (related to other phenomena, **not** to the spinodal decomposition phenomenon) abruptly separates the black and white regions. Image b) has the only purpose of being put in contrast with a).

is distinctive and it characterizes a specific class of phenomena. A graphical description of the progress of the binary alloy system is shown in Figure I.II.

1.1.2 The *local* Cahn-Hilliard equation

In this Subsection we give a concise derivation of the Cahn-Hilliard equation by means of some physical considerations associated with the motivating example described in Subsection (1.1.1). The necessary physical assumptions are specified in the forthcoming Remark (1.1.3). For a complete treatment, the reader is referred to [28] and, since the motivating example is related to thermodynamics,

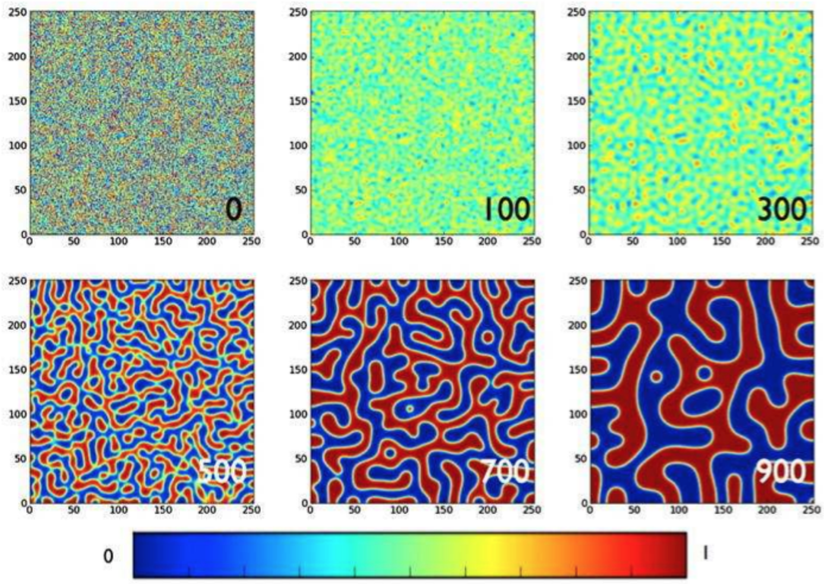

Figure I.II: Evolution of the binary alloy during the cooling process. The spatial domains dominated by metal A are coloured in red, the spatial domains dominated by metal B are coloured in blue. Windows labeled with time units 0/100/300/500 illustrate the nucleation stage, the remaining windows illustrate the coarsening stage. The latter show the distinctive patterns associated to the spinodal decomposition phenomenon.

also to [13], [37], [61] for the basic thermodynamical discussions.

In order to keep track of the advancing binary alloy cooling process and to fully describe it, it is reasonable to monitor the relative concentration of A with respect to B in space and time by means of a (unknown) function

$$\phi = \phi(x,t), \quad x \in D \subset \mathbb{R}^d, \quad t \geq 0,$$
$$\phi \in [0,1]. \tag{1.1}$$

Ideally, the function ϕ approximates, with its "heat map", the patterns shown in Figure I.II. It should take values in $[0,1]$ for physical consistency reasons.

Remark 1.1.1. It is not always possible to guarantee (1.1) in the mathematical analysis of the forthcoming Cahn-Hilliard equation. See [30, p. 405-406].

Remark 1.1.2. Instead of ϕ, we might choose to investigate the unknown u representing the difference between the mass densities of the two components of the alloy. Consequently u should take values in $[-1, 1]$. If we did so, there would be almost no difference in the forthcoming physical derivation of the Cahn-Hilliard equation.

In order to write down a mathematical equation accurately describing the spinodal decomposition process and having $\phi(x,t)$ as approximate solution, J. W. Cahn and J. E. Hilliard introduced in 1958 [13] the following functional of ϕ

$$E[\phi] = \int_D \left\{ \frac{\varepsilon^2}{2} |\nabla \phi|^2 + F(\phi) \right\} \mathrm{d}x. \tag{1.2}$$

$E[\phi]$ represents the *(total) free energy* of the system, also called *Ginzburg-Landau free energy*: this name is derived from the associated superconductivity theory. In the context of statistical mechanics, the square gradient in (1.2) arises from attractive long-range interactions between molecules of the binary alloy. The coefficient ε^2 can be related to the pair correlation function (see [3] and references therein). In fact, the square gradient represents an energy which takes large values if the material is strongly inhomogeneous (we recall that ϕ is the relative concentration of element A, therefore in an inhomogeneous material its gradient's norm is quite large). The parameter ε^2 represents the *interaction distance* and is related to the thickness of the diffusion interface between two different domains dominated by A and B respectively. The second term $F(\phi)$ is an energy associated with the presence of element A in the material: it is often called *Helmholtz free energy density per molecule.*

Remark 1.1.3. The expression of E in (1.2) has been rigorously derived by J. W. Cahn and J. E. Hilliard under a certain number of physical assumptions, among which:

- the molar masses of A and B are independent of composition and pressure.

- the solid solution is free from imperfections.

- the solid solution is isotropic at a given temperature.

- the fluid solution is initially isothermal (with temperature T_0).

We now examine the term $F(\phi)$ which appears in (1.2). As we have said, this term keeps track of the free energy associated with the presence of component A. Many different analitic expressions of $F(\phi)$ have been proposed. We introduce one which is thermodynamically relevant, namely

$$F(\phi) = 2k_B T_c \phi(1-\phi) + k_B T_a \{\phi \ln \phi + (1-\phi)\ln(1-\phi)\}, \tag{1.3}$$

where k_B is the Boltzmann constant, T_c is a specific *critical* temperature and T_a is the absolute temperature of the system. A graphical rapresentation of F

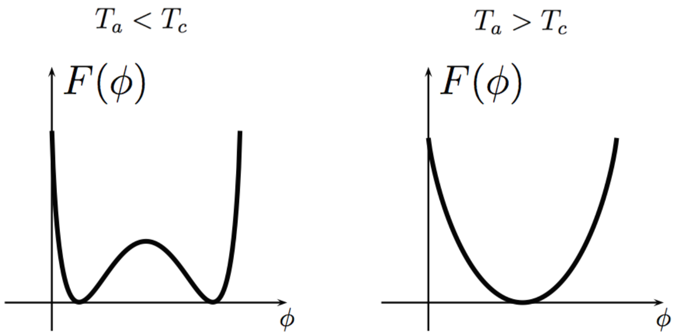

Figure I.III: Free energy F: dependence on T_a.

can be found in Figure I.III. The considerations made in [13] show that (1.3) is a physically well motivated term; because of its expression, it is often called a *logarithmic potential*. However, it is very difficult to deal with and for this reason it is often approximated with more tractable analytical functions, typically polynomials: a standard expression for F is a polynomial *double well potential* such as

$$F(\phi) = \frac{1}{4}\left(\phi^2 - 1\right)^2. \tag{1.4}$$

We will discuss the topic of the choice of the expression F in more detail throughout the whole thesis.

Let's go back to $E[\phi]$. The quantity which is crucial in order to formulate the Cahn-Hilliard equation is not E itself, but its first variation. A physical motivation of this fact resides in the minimization of E. Thanks to (1.2), we can take its Fréchet derivative and define the *chemical potential*

$$\mu := E'[\phi] = -\varepsilon^2 \Delta\phi + F'(\phi) = -\varepsilon^2 \Delta\phi + f(\phi), \tag{1.5}$$

where $f(\phi) := F'(\phi)$. See [49, Appendix] and references therein for a detailed discussion of the chemical potential.

Remark 1.1.4. In the following, with the single word *potential* we refer to F, not to μ.

We now define the *mass flux* of the system as

$$J := -M(\phi)\nabla\mu. \tag{1.6}$$

The term $M(\phi)$ in (1.6), is called *(concentration) mobility coefficient* and it was introduced in [14]. It basically determines the local speed of the mass flux.

Originally it was assigned a polynomial expression in [45] and later on in [16], [15]. These expressions are closely related to the following one[1]

$$M(\phi) \propto \phi(1-\phi). \tag{1.7}$$

Remark 1.1.5. The mobility coefficient appearing in (1.7) presents some problems, since it may assume negative values if $\phi \notin [-1; 1]$. For this reason, it is commonly referred to as a *degenerate* mobility coefficient. To face this problem, the following modification, given in terms of the unknown u, was proposed in [30]

$$\tilde{M}(u) = (1-u^2)^m B(u),$$

where m is positive integer and $B \in \mathcal{C}^1(\mathbb{R})$ is a function such that

$$B(u) \in [b_0, B_0], \quad \text{if } u \in [-1, 1],$$

$$B(u) = 0, \quad \text{if } u \notin [-1, 1],$$

where $B_0 > b_0 > 0$ are given constants. By so doing $\tilde{M}$ does not assume negative values and it is hence more tractable.

In a real physical scenario, if the alloy is confined to D and there is no mass flux in and out of D, the quantity

$$\overline{\phi} = \frac{1}{|D|}\int_D \phi(x,t)\mathrm{d}x \tag{1.8}$$

where $|D|$ denotes the Lebesgue measure of D, is conserved in time. Relation (1.8) implies that the relation

$$\phi_t = -\mathrm{div}(J) \tag{1.9}$$

is a sensible consitutive equation for the mass transport of the system[2]. If we insert (1.6) into (1.9) we obtain the so called *(local) Cahn-Hilliard partial differential equation*

$$\phi_t = -\mathrm{div}\{-M(\phi)\nabla\mu\}. \tag{1.10}$$

This equation was derived by J. W. Cahn in 1961 in [12].

Remark 1.1.6. Equation (1.10) is referred to as the local Cahn-Hilliard differential equation. In this specific setting, the term *local* indicates that each single component of the equation can be computed, for each $x \in D$, by means of some functions and their derivatives evaluated at x only. Therefore, the information

[1] in these works, the expression is given with dependence by the unknown u and it is related to $\tilde{M}(u) \propto 1-u^2$. This relation is analogous to (1.7).

[2] subscript t in ϕ_t denotes the time derivative of ϕ. See Section (1.4).

being used, due to the nature of the (classical) derivatives, refers to an arbitrary small open set contaning x, hence the term local. In the next subsection, we will see that the expression of the free energy E, hence the structure of the equation, can be properly modified with the introduction of typical *nonlocal* elements, such as spatial convolutions.

Remark 1.1.7. One of the most striking advantages of using the Cahn-Hilliard equation for simulating microstructural evolution is the avoidance of the explicit tracking of the diffuse interface, the latter having been introduced in Subsection (1.1.1) and Figure I.I. The concept of diffuse interface has been adopted to model various physical phenomena involving moving interfaces in order to describe the spatial distribution of the entire microstructure of a system.

Equation (1.10) can be written, thanks to (1.5), as

$$\phi_t = -\text{div}\{-M(\phi)\nabla(-\varepsilon^2\Delta\phi + f(\phi))\}. \tag{1.11}$$

Similarly to the case of F, the expression of M can be drastically simplified. The term $M(\phi)$ is often considered to be constant ($M(\phi) = \kappa$), therefore (1.11) simplifies to

$$\phi_t = -\kappa\varepsilon^2\Delta^2\phi + \kappa\Delta f(\phi). \tag{1.12}$$

Equation (1.12) is an evolutional, fourth-order, nonlinear partial differential equation. It is usually endowed with suitable initial and boundary conditions, e.g. Neumann's type conditions

$$\frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \Gamma,$$

$$\frac{\partial\mu}{\partial\nu} = 0 \quad \text{on } \Gamma, \tag{1.13}$$

where $\Gamma = \partial D$ and ν is the unit outer normal to the boundary. Condition (1.13) is quite natural because it implies the conservation of mass in the system. In fact, if we integrate (1.12) over D and we use (1.13), we obtain that the quantity (1.8) (the normalized total mass) is conserved in time, in accordance with the construction of the equation. In this particular case, we may use the invariant quantity $\overline{\phi}$ to better characterize the evolution of the system.

We recall that the phenomenon which is being investigated is called *spinodal decomposition*. We are now able to give a proper motivation for this name.

Let us consider the plot shown in Figure I.IV. In this plot we can see two different curves both relating $\overline{\phi}$ and $\overline{T}$. The dotted blue line is called *spinodal* curve: it is defined by the vanishing of the second derivative of the Helmholtz free energy

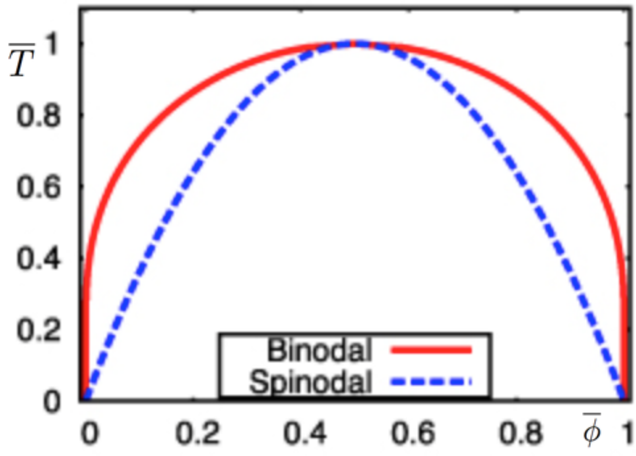

Figure I.IV: Spinodal and binodal curves. Here $\overline{T} := T_1/T_c$.

with respect to ϕ. Such curve gives the name to the physical process we are describing. The continuous red line is called *binodal* curve: it defines the region of composition and temperature in a phase diagram in which a transition occurs from miscibility of the components to conditions where single-phase mixtures are *metastable* or *stable* (i.e., chemical potentials on both phases are equal).
These curves are obtained experimentally (the plot is meant to describe their general behaviour) and the position of $(\overline{\phi}, \overline{T})$ in relation to them determines the evolution of the system during the cooling period. We are able to distinguish two relevant cases:

(i) $(\overline{\phi}, \overline{T})$ lies between the binodal and spinodal curve. In this case the partial nucleation and block growth we have previously described take place, although the Cahn-Hilliard equation (1.11) does not faithfully adhere to the physics of the phenomenon.

(ii) $(\overline{\phi}, \overline{T})$ lies under the spinodal curve. In this case a total nucleation can be observed and (1.11) gives a satisfactory approximation of the true physical model.

Remark 1.1.8. Equation (1.10) is a basic version of a large class of physical equations related to the spinodal decomposition phenomenon. Similarly to what happens with the standard heat equation, one may add to (1.10) a convective term like $\operatorname{div}\{u\phi\}$, where u is a vector field rapresenting a velocity. In addition, (1.10) may be coupled with other relevant equations, such as a Navier-Stokes system, a reaction-diffusion equation or an elasticity equation.

Remark 1.1.9. The expression of the free energy E indicated in (1.2) goes back to 1958. Afterwards, other forms for E were proposed. Some of them mantain the property of having integrals over D in their expressions and are at least *conceptually* similar to (1.2), in the sense that they rapresent comparable physical properties. However, other forms keep track of physical aspects which are completely neglected in the expression of E appearing in (1.2). They often include integrals on $\Gamma = \partial D$. We discuss an alternative form of E in Subsection (1.1.3).

1.1.3 The *nonlocal* Cahn-Hilliard equation

Keeping Remarks (1.1.6) and (1.1.9) in mind, one may define an alternative total free energy $\mathcal{E}$

$$\mathcal{E}[\phi] = \frac{1}{4}\int_{D\times D} J(x-y)(\phi(x)-\phi(y))^2 \mathrm{d}x\mathrm{d}y + \eta \int_D F(\phi(x))\mathrm{d}x \tag{1.14}$$

replacing E in (1.2). Here $J : \mathbb{R}^d \to \mathbb{R}$ is a smooth, positive function such that $J(x) = J(-x)$, $\forall x \in \mathbb{R}^d$. This expression was proposed in [39], [40] and rigorously justified as a macroscopic limit of microscopic phase segregation models with particle conserving dynamics (see also [18]). Therefore the chemical potential (the first variation of $\mathcal{E}$ with respect to ϕ) is

$$\mu(x) = \phi(x)\int_D J(x-y)\mathrm{d}y - (J*\phi)(x) + \eta f(\phi(x)), \tag{1.15}$$

where $*$ indicates the convolution operation over D, namely

$$(J*\phi)(x) = \int_D J(x-y)\phi(y)\mathrm{d}y, \qquad x \in D.$$

In contrast to (1.5), this expression of μ contains a nonlocal term, namely a spatial convolution. The corresponding *nonlocal* Cahn-Hilliard equation with constant mobility

$$\phi_t = m\Delta\mu \tag{1.16}$$

can be derived from idealized microscopic models through suitable limits such as the heat diffusion equation and the Boltzmann equation. In addition, the evolution in the sharp interface limits are the same as those derived from the classical (local) Cahn-Hilliard equation in the corresponding limits. Hence the two models are quite similar for $\varepsilon \approx 0$, see [40].

Remark 1.1.10. The nonlocal Cahn-Hilliard equation is widely regarded as a better mathematical rapresentation of the spinodal decomposition phenomenon than the local equation, the latter being a "local approximation" of the nonlocal one. Nevertheless, due to its integrodifferential nature, it is quite delicate to handle. The reader may consult [8], [9], [22], [34], [35], [44], [55].

A more tractable version of (1.16) can be obtained by adding a convective term, resulting in

$$\phi_t + u \cdot \nabla\phi = m\Delta\mu, \tag{1.17}$$

where u is as in Remark (1.1.8). In this context, u represents a capillarity force called *Korteweg force.*

1.2 The introduction of randomness

In Section (1.1) we have briefly introduced the physical model which led to the formulation of the Cahn-Hilliard differential equation in the case of a binary alloy suddenly quenched to a lower temperature. While this model has been widely accepted, it presents some flaws.
As already highlighted Subsection (1.1.2), page 8, if $(\overline{\phi}, \overline{T})$ lies between the binodal and spinodal curve, equation (1.11) does not exactly adhere to the ongoing physical process. In particular, the equation fails to provide a satisfactory representation of the phenomenon in the early stages of the spinodal decomposition. We now give a concise explanation of the reason behind this: for a complete reference on this subject, see [23].
When formulating their differential equation, Cahn and Hilliard proposed the following formula regulating the mass flux in the binary alloy

$$J := -M(\phi)\nabla\mu = -M(\phi)\nabla E'[\phi]. \tag{1.18}$$

If ww have already mentioned such equation in Section (1.1). If we reasonably assume that the binary alloy reaches a stable solid condition after a long time, we obtain that $\nabla E'[\phi] = 0$ for very large t (ideally $t = \infty$). This means that after a long time there is no mass flux in any point of the domain, namely

$$J = 0, \quad \forall x \in D.$$

However, it is well known that, for the case of a stable, single phase, binary solid solution, atomic movement always takes place: an appreciable flux of solute occurs at equilibrium with clusters or ordered arrangements of atoms continually forming and dissolving, as shown in Figure I.VI. These movements are associated with the quantity $k_B T_a$, where k_B denotes the Boltzmann constant and T_a is the absolute system temperature. For this reason they are called *thermal fluctuations.* In addition, the physical discussion carried out in Subsection (1.1.2) by means of [28] intentionally fails to take into account some other “dynamical” aspects of the solute, such as vibrational, electronic and magnetic properties, as clarified in [28, p. 217].
The thermal fluctuations and these dynamical aspects of the atoms cause collisions between them. These collisions themselves cause further atomic movements. See Figure I.V. This atomic process necessarily has to be represented

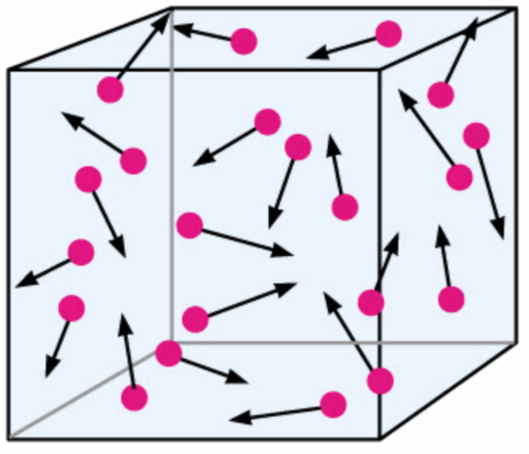

Figure I.V: Sketch of random solute movements, resulting from collisions with the surrounding atoms.

by means of some quasi-random[3] process. The most direct way to keep track of them in the Cahn-Hilliard equation is that of modifying the formulation of the flux J by adding a random term, namely

$$J = -M(\phi)\nabla\mu + w, \tag{1.19}$$

wgere w is a random term[4] which indicates the presence of the thermal fluctuations and dynamical aspects previously discussed.

Let us now rewrite in mathematical language the new elements we have been discussing. Due to the randomness of w, it is unavoidable to introduce a *probability space* $(\Omega, \mathcal{F}, \mathbf{m})$ along with the spatial domain D. As a natural consequence our solution ϕ will be a stochastic process taking values in a proper Hilbert space H, namely

$$\phi = \phi(t, \omega) : [0, T] \times \Omega \to H, \quad \omega \in \Omega, \quad t > 0.$$

We will often drop ω in the notation. We now have to define the stochastic Cahn-Hilliard equation whose solution will be the random process ϕ. Recalling what we have said in Subsection (1.1.2), the Cahn-Hilliard equation is derived via the mass flux equation, thus we simply have to define the random term w. A reasonable, physically-consistent choice for w is that of a random process such

[3]we use the term *quasi-random* instead of *random* because, while the collisions are determistic, they cannot be observed and one must rely on a statistical approach to describe them.

[4]it will formally become a suitable stochastic process

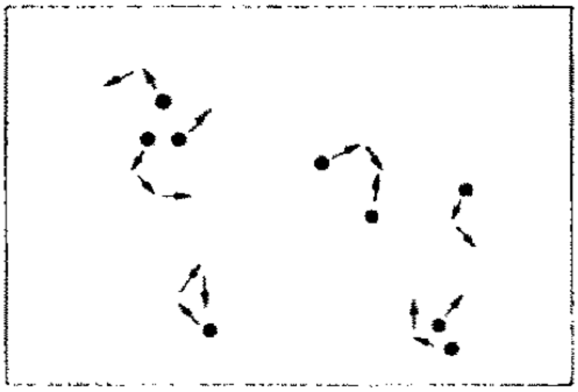

Figure I.VI: Sketch of random paths of single atoms.

that div(w) is a Wiener process[5] defined on a proper infinite-dimensional Hilbert space. Thanks to (1.5), (1.9) and (1.19) we are now able to write down the basic version of the stochastic Cahn-Hilliard equation, namely[6]

$$\mathrm{d}\phi(t) = -\mathrm{div}\{-M(\phi(t))\nabla(-\varepsilon^2\Delta\phi(t) + f(\phi(t)))\}\mathrm{d}t + \mathrm{d}w(t), \tag{1.20}$$

where $w(t)$ is a Q-Wiener process. Equation (1.20), which as interpreted as a *formal* relation, is an improved version of the deterministic one, though it is obviously more difficult to deal with because of the presence of a random term. However, it is more faithful to the physics underlying the model. Of course, (1.20) can take many shapes according to the expression of E, and hence of μ; in addition to that, one may add a non trivial stochastic integrand $\sigma(\phi(t))$ to better describe the thermal fluctuations and the influence of the dynamical aspects in the model, thus obtaining

$$\mathrm{d}\phi(t) = -\mathrm{div}\{-M(\phi(t))\nabla(-\varepsilon^2\Delta\phi(t) + f(\phi(t)))\}\mathrm{d}t + \sigma(\phi(t))\mathrm{d}w(t). \tag{1.21}$$

Remark 1.2.1. In this thesis we restrict ourselves to the study of the stochastic Cahn-Hilliard equation with an additive noise with constant stochastic integrand

[5]we recall that a Q-Wiener process defined on an infinite-dimensional Hilbert space Y is given by the formula

$$w(t) = \sum_{i=1}^{\infty} \sqrt{\alpha_i}\beta_i(t) f_i,$$

where $\alpha_i \downarrow 0$, $\{\beta_i(t), \mathcal{F}_t, \Omega\mathcal{F}, \mathbf{m}\}_{i\in\mathbb{N}}$ is a family of independent real brownian motions adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, $\{f_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of Y, $Q \in L(Y)$ is a positive, symmetric, finite trace operator such that $Qf_i = \alpha_i f_i$, $\forall i \in \mathbb{N}$. The series converges in $L^2((\Omega, \mathcal{F}, \mathbf{m}); \mathcal{C}([0,T]); Y)$ for each $T > 0$. See [62, p. 13, Proposition 2.1.10] for full details.

[6]we will sometimes drop the indication of the dependence on time and simply write ϕ instead of $\phi(t)$.

such as in (1.20), since it is tractable and physically consistent. This is not the only way of introducing some kind of randomness in the problem. One may add, e.g., noise on $\Gamma = \partial D$, stochastic dynamical boundary conditions, Poisson processes instead of Wiener processes, random mobility coefficient or random interaction distance.

1.3 Other phenomena related to the Cahn-Hilliard equation

Although the Cahn-Hilliard equation was originally formulated in relation to the motivating example described in Subsection (1.1.1), subsequently its basic structure, possibly modified or enriched, was applied to a variety of different settings. We now list some relevant cases in which it has been used. We first mention some applications along with the respective leading equations; we then quote some other applications without the leading equations; we finally examine in detail a peculiar and certainly hilarious case study.

I. Examples accompanied by leading equations.

- Population dynamics: this variant was proposed in [51] to describe the proliferation and interaction via adhesion of cells in tumor growth

$$\phi_t = \Delta\big(\ln(1-q)(-\varepsilon^2\Delta\phi) + f(\phi)\big) + \alpha\phi(1-\phi),$$

  where ϕ is the local density of the cells, q, α are parameters associated with adhesion and proliferation.

- Skin Cancer model: see [7]

$$\begin{cases} \phi_t = \operatorname{div}\{\omega\phi(1-\phi)\nabla(-\varepsilon^2\Delta\phi + f(\phi))\} + \phi(\eta-\delta), \\ -\Delta\eta + \phi\eta - \beta(1-\eta) = 0. \end{cases}$$

  Here ϕ represents the volume fraction of a cancerous cellular phase, η is the concentration of a diffusing nutrient, δ is the nutrient consumption rate, ω and β are free parameters.

- Image inpainting: see [10]

$$\phi_t = \Delta\left(-\varepsilon\Delta\phi + \frac{1}{\varepsilon}f(\phi)\right) + \lambda\chi_{E^C}(f-\phi),$$

  where f is a given binary image, ϕ is an inpainted version of f, λ is a real positive coefficient, $E \subset D \subset \mathbb{R}^2$ is the inpainting domain.

- Saturn's rings: (see [69]) this is a simple model for the irregular structure of Saturn's rings:

$$\phi_t = \frac{\partial}{\partial x^2}\left(-\varepsilon\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{\sigma} g(\phi)\right).$$

Here ϕ is the shear in the x direction, σ is a surface density of the viscous incompressible fluid, $g(s) = as^3 + bs|s| + cs$, where $a, c > 0$, $b > -\sqrt{4ac}$.

In all the cases quoted above, the main cores of the equations are some modifications of (1.5), which is used to derive the "standard" Cahn-Hilliard equation.

II. Examples without leading equations.

(a) Multi-phase fluid flows, [11], [52].

(b) Taylor flow in mini/microchannels, [36].

(c) Ternary liquid mixture, [2].

(d) Two-dimensional two-layer channel with sharp topografical features, [73].

(e) Spinodal decomposition with with compisition-dependent heat conductivities, [56].

(f) Phase decomposition and coarsening in solder balls, [1].

(g) Thermal-induced phase separation phenomenon, [68].

(h) Evolution of arbitrary morphologies and complex microstructures such as solidification and solid-state structural phase transformations, [19].

(i) Grain growth, [72].

(j) Meta-stable chemical composition modulations in the spinodal region, [38].

III. Pattern formation in biological systems.

We finally illustrate a striking (and also hilarious) analogy related to the Cahn-Hilliard equation. The reader is first invited to take a close look at the fifth and sixth windows of Figure (I.II). As said, these windows show the final stage of the previously discussed cooling process of a metallic binary alloy. A similar pattern can be observed with the metallic elements A and B replaced by empty spaces and mussels (mussels !!!) respectively. This last sentence may sound peculiar and needs to be clarified. We refer to [54] and to the references therein.

The phenomenon being analysed is the following: let us suppose to place, with a uniform distribution, a considerable amount of mussels on intertidal flats. If the colony is then left free to act on his own will, then mussels beds exhibit spatial self-organization by forming a pattern of regularly spaced clumps. By so doing, they balance optimal protection against predation with optimal access to food. This self-organization process has been attributed to the dependence of the speed of movement on local mussel density. Mussels move at high speed when they occur in low density and decrease their speed of movement once they are included in small clusters. However, when occurring in large and dense clusters, they tend to move faster again, due to food shortage. Mussel pattern formation is a fast process, giving rise to stable patterning within a few hours, and clearly is independent from birth or death processes (Figure I.VII, A and B).

The surprising thing is that the physical modeling underlying the mussel pattern-formation phenomenon adheres to the Cahn-Hilliard binary alloy physical model. As a result, we have a close similarity between the pattern formations at the final stage of the two processes. The reader is hence invited to compare Figure I.II (5th and 6th windows) and Figure I.VII, D.

The reader is also invited to visit the webpage

`https://www.youtube.com/watch?v=u-mEjfBaYks`

for a nice dynamic visualization of the intermadiate stages between Figure I.VII, A and Figure I.VII, B.

1.4 Notation and basic functional setup

In this section we define a few basic mathematical objects, whose notation will be kept throughout the entire thesis.

- The spatial domain where the Cahn-Hilliard equation lives is denoted by D and $D \subset \mathbb{R}^d$, $d \in \{1;2;3\}$. Its boundary is indicated as $\Gamma := \partial D$. The time interval in which we make all our considerations is $[0,T]$, for a given $T > 0$.

- We use the notation L^p (where $p \in [1;+\infty]$) to indicate both $L^p(D)$ and $[L^p(D)]^d$. We use the notation H^s (where $s \geq 0$) to indicate both $H^s(D)$ and $[H^s(D)]^d$. For $s \geq 0$, we denote by H^{-s} the dual space of H^s. For a Banach space W, we denote by $L^p([0,T];W)$, where $p \in [1,+\infty]$, the L^p-space of W-valued, Bochner-integrable functions on $[0,T]$. We denote by $\mathcal{C}([0,T];W)$ (resp. $\mathcal{C}^\beta([0,T];W)$) the set of W-valued continuous (resp. β-Hölder continuous) functions on $[0,T]$, for $\beta \in (0,1)$.

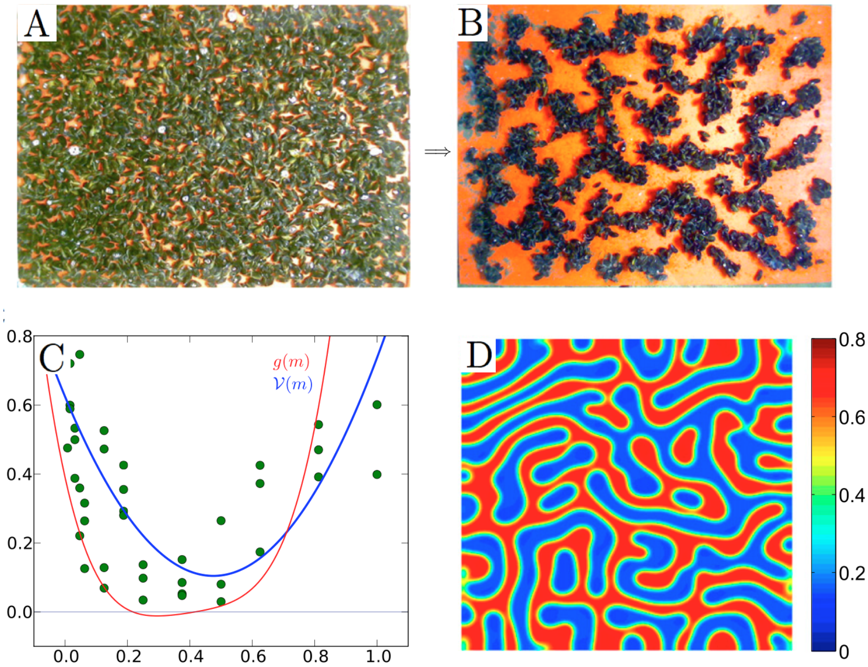

Figure I.VII: (A and B) Mussels that were laid out evenly under controlled conditions on a homogeneous substrate developed spatial patterns similar to "labyrinth-like" after 24 hours (images represent a surface of 60 cm $\times$ 80 cm). (C) Relation between movement speed and density within a series of mussels clusters. The blue line $\mathcal{V}(m)$ describes the rescaled second-order polynomial curve which tries to fit the rescaled speed $g(m)$. The red line depicts the effective diffusion $g(m)$ of mussels as a function of the local densities according to the diffusion-drift theory. The green open circles show the original experiment data. (D) A specific numerical simulation associated with [54, Equation [4]], which is a proposed equation regulating the mussel pattern-formation process, with a nearly uniform initial mussel distribution.

- We define

$$H := L^2, \qquad U := H^1, \qquad V := \left\{ u \in H^2 : \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \right\},$$

$$\mathcal{Q}_T := (0, T) \times D.$$

  We denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the inner product and norm in H, respectively.

- For a Banach space W which is not H, we denote its dual space by W', its inner product by $(\cdot, \cdot)_W$, its duality pairing by $\langle \cdot, \cdot \rangle_{W', W}$, and its norm by $\|\cdot\|_W$.

- For a Banach space Z and a subset $W \subset Z$, we indicate the Borel σ-algebra of W with respect to the topology of Z as $\mathcal{B}_Z(W)$, namely $\mathcal{B}_Z(W) := \mathcal{B}(Z) \cap W$. In addition $\mathcal{B}(Z)$ denotes the natural Borel σ-algebra of Z. The previous definitions are consistent if $W \in \mathcal{B}(Z)$.

- We indifferently use the notations $u_{x_1 \cdots x_m}$ and $\partial^m u / \partial x_1 \cdots \partial x_m$ to indicate the m-th derivative of u with respect to the spatial coordinates $x_1, \cdots, x_m$. For the time derivative we use the notations $\partial u / \partial t$, u_t, $(\mathrm{d}/\mathrm{d}t)u$, u'. For the normal outer derivative we use the notations $\partial_\nu u$, $\partial u / \partial \nu$, u_ν. The meaning of each time derivative (e.g., *classical*, *distributional*, etc...) will be specified when necessary. For a functional $E[u]$, we use the notations $E'[u]$ or $E_u[u]$ to denote its Fréchet derivative with respect to u.

- $(\Omega, \mathcal{F}, \mathbf{m})$ denotes the probability space, i. e. the domain of all random variables. We use bold characters to indicate measures on measurable spaces. We use the bold characters along with a hat $(\hat{\cdot})$ to indicate the *characteristic functionals* of measures. Finally, the symbol $\mathbb{E}$ stands for the *expected value* operator.

- For a given sequence $\{x_n\}_{n \in \mathbb{N}}$ (also denoted by $\{x_n\}$ or x_n) taking values in an abstract set X, we may use the compact notation x_n to indicate the entire sequence or one of its subsequences: namely, we do not (always) relabel subsequences in order to avoid too heavy a notation.

- When integrating on a set X (different from the probability space Ω), we will often simplify the differential notation and write $\int_X f$, $\int_X f(x)$ or $\int_X f \mu(\mathrm{d}x)$ instead of $\int_X f(x) \mu(\mathrm{d}x)$, where μ is a measure on a suitable σ-algebra of X. If X is D or $[0, T]$ and μ is the Lebesgue meausure we will simply write $\mathrm{d}x$ instead of $\mu(\mathrm{d}x)$ to denote the differential.

- We use the concise notations a.e. and a.s. to say *almost everywhere* and *almost surely*. These terms are obviously related to measure theory. We will mainly use the notation a.e. when referring to measures on spatial or time domains and the notation a.s. when referring to probability measures on a probability space.

- When referring to constant quantities whose exact value is irrelevant, we may often share the same symbol for more than one of these objects. Typically, but not always, these constant are indicated by C_α, α being an index.

More specialized funtional and probabilistic tools will be specified when necessary.

1.5 Plan of the work

This thesis has two aims. Firstly, it is meant to provide the reader with a detailed analysis of a restricted number of articles dealing with the deterministic and stochastic Cahn-Hilliard differential equation. These articles deal with a number of relevant topics whose analysis is crucial for a mathematical discussion of some basics concepts associated with the equation. Such concepts range from the possible definitions of solutions, theorems of existence and uniqueness of solutions and long time behaviour, to the introduction of critical boundary conditions and analysis of systems of coupled equations, one of which is the Cahn-Hilliard equation.

Secondly, this thesis provides a new contribution to the subject. An extensive analysis of a stochastic extension for a specific version of the deterministic *nonlocal* Cahn-Hilliard equation is carried out using some general ideas from two of the articles presented in this thesis. Some *ad hoc* results for this specific problem are proved. This second part is the most important one: for this reason every single statement, auxiliary lemma and theorem is proved in detail. We will discuss this specific part in the forthcoming description of Chapter 4 and in Chapter 4 itself.

We now go through the contents of the subsequent chapters.

Chapter 2. This chapter is devoted to the deterministic Cahn-Hilliard equation. After a quick review of the literature on the subject, we focus our attention on some works. The first one [31] is presented without proofs. This article studies aspects of the spatial monodimensional ($d = 1$) Cahn-Hilliard equation. In particular, global existence and long time behaviour are mentioned.

The second work [20] states similar results for the multidimensional case.

Finally, we study a nonlocal Cahn-Hilliard equation presented in [21]. This article is exposed in detailed way, since the main original contribution of this thesis (contained in Chapter 4) is the study of a stochastic version of the nonlocal Cahn-Hilliard that appears in the coupled Cahn-Hilliard-Navier-Stokes system studied in the paper.

Chapter 3. This Chapter is the twin brother of Chapter 2. It deals with the stochastic Cahn-Hilliard equation. After a brief review of the literature on the subject, we turn to the analysis of two specific papers.

The first one [29] provides two different types of solution for a given Cahn-Hilliard stochastic equation and shows results of existence and uniqueness by means of stochastic tools. Some ideas described in this article are used in Chapter 4 in a suitably modified form in order to study our new stochastic version of the nonlocal equation appearing in [21].

The second paper [24] rewrites a similar Cahn-Hilliard equation as a system of infinitely many deterministic evolutional differential equations. The stochasticity is basically reshaped and handled by means of deterministic arguments.

Chapter 4. This is the main original part of the thesis and hence we provide an extended outline of its contents, in contrast with the concise description of we have given Chapters 2 and 3.

In Chapter 4 we *formally* study the stochastic partial differential equation

$$(SCHE)\begin{cases} \mathrm{d}\phi = (-u\cdot\nabla\phi + \Delta\mu)\mathrm{d}t + \mathrm{d}w, & (1.22a)\\ \mu = a\phi - J*\phi + F'(\phi), & (1.22b)\\ \dfrac{\partial\mu}{\partial\nu} = 0 \quad \text{on } \Gamma\times(0,T), & \\ \phi(0) = \phi_0 \in U, & \end{cases}$$

for a given velocity field u, a specific *Helmholtz free energy* F, a regular H-valued Wiener process w, a regular *kernel* J. Problem ($SCHE$) is characterised by (1.22a)-(1.22b), which are stochastic extensions of [21, p. 429, (1.9)-(1.10)]. (1.22a) is a nonlocal equation due to the presence of the spatial convolution $J*\phi$ in the definition (1.22b) of the chemical potential μ.

Our approach to the analysis of problem (1.22) is a *variational* approach: we do not look for regular solutions of (1.22) (e.g., in the sense given in [62, p. 73, Definition 4.2.1.]) because of the complexity of the equation. Instead, we define a proper *test function* space $\mathscr{V}$, two suitable function spaces $\mathscr{U}$, $\mathscr{Z}$, and give two different definitions of *solution* to Problem (1.22).

Definition (A)**.** *A* ***weak statistical*** *solution (or simply a* ***weak*** *solution) to Problem* (1.22) *is a probability measure* $\mathbf{P}$ *(concentrated) on* $\mathcal{B}_{\mathscr{Z}}(\mathscr{U})$ *which, for every* $\xi \in H^\varepsilon$ *and* $v\in\mathscr{V}$*, satisfies*

$$\int_{\mathscr{Z}} \exp\left\{i\langle\phi(0),\xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} + iC(\phi,v)\right\}\mathbf{P}(\mathrm{d}\phi) = \hat{\Xi}(\xi)\hat{\mathbf{N}}(v), \tag{1.23}$$

where $\varepsilon\in(0,1/4)$ *and*

$$C(\phi,v) := -(\phi_0, v(0)) - \int_0^T\int_D \phi u\cdot\nabla v + \\ -\int_0^T\int_D (a\phi+\phi^3-\phi-J*\phi)\Delta v - \int_0^T\left(\phi,\frac{\partial v}{\partial t}\right).$$

Here Ξ indicates the distribution of the random variable ϕ_0 on H and $\mathbf{N}$ indicates the distribution of the white noise $\partial w/\partial t$. The symbol $\hat{}$ denotes the *characteristic functional operator*, see Section (1.4). Definition (A) and the expression of C are motivated by the nature of $\mathscr{V}$.

Definition (B). *A process $\phi = \phi(t, x, \omega)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbf{m})$ is a **strong** solution to problem* (1.22) *if*

1. *ϕ satisfies*
$$\mathscr{D}(\phi(\omega)) = \left\{\phi_0(\omega), \frac{\partial w}{\partial t}(\omega)\right\} \qquad \mathbf{m} - a.s. \tag{1.24}$$
where $C(u) \in \mathscr{V}'$, $\langle C(u), v\rangle_{\mathscr{V}',\mathscr{V}} := C(u, v)$ and $\mathscr{D} : \mathscr{X} \to H^{-\varepsilon} \times \mathscr{V}' : u \mapsto \{u(0), C(u)\}$.

2. *the mapping $\omega \mapsto \phi(\omega)$ is a random variable from $(\Omega, \mathcal{F})$ to $(\mathscr{U}_1, \mathcal{B}_{\mathscr{X}}(\mathscr{U}_1))$. Here $\mathscr{U}_1$ is a regular subset of $\mathscr{U}$.*

We now state the main results of existence and uniqueness we have proved for Problem (1.22). We also provide a sketch of their proofs, which are carried out in detail in Chapter 4.

Theorem (T1). *Let $d \leq 3$. Let w be a H-valued Q-Wiener process and let w, u, Q, J, F, ϕ_0 and $\{e_j\}_{j\in\mathbb{N}}$ satisfy some suitable properties (which will be stated in Chapter 4). Let ϕ_0 be a U-valued random variable such that*
$$\mathbb{E}\left[\|\phi_0\|_U^2 + \int_D \frac{\phi_0^4}{4} - \int_D \frac{\phi_0^2}{2}\right] < +\infty.$$
Then problem (1.22) *admits a weak statistical solution in the sense of Definition (A).*

Outline of the Proof of Theorem (T1). We build a Galerkin scheme (indexed by a positive integer m) for Problem (1.22) and we show the existence and uniqueness of a solution ϕ_m to an approximated, m-dimensional version of (1.22). We then use the Itö formula to obtain some uniform[7] estimates of a suitable norm of ϕ_m. These estimates are used to deduce that ϕ_m is defined on $[0, T]$ (a.s.) for each m. We then deduce some further estimates which imply that $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$ (the family of the distributions of ϕ_m on $\mathscr{X}$) is uniformly concentrated on $\mathscr{U}$. Since the injection $\mathscr{U} \hookrightarrow \mathscr{X}$ is compact, we use Prohorov's Theorem to deduce the existence of a weakly convergent (not relabeled) subsequence $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$ to a probability measure $\mathbf{P}$ on $\mathscr{X}$. We then exploit a passage to the limit and a density argument on some suitable expressions satisfied by $\mathbb{P}_m$ and deduce that $\mathbf{P}$ is a weak statistical solution in the sense of Definition (A). □

Theorem (T2). *Let ϕ_1, $\phi_2 \in \mathscr{U}_1$ be two strong solutions for problem* (1.22) *(for the same $\phi_0 \in U$, i. e. $\phi_1(0) = \phi_2(0) = \phi_0$) in the sense of Definition (B). Then*
$$\phi_1(t) = \phi_2(t) \quad \text{in } U', \quad \text{for a.e. } t \in [0, T].$$

[7] with respect to $m \in \mathbb{N}$.

Outline of the Proof of Theorem (T2). This Proof is a suitable application of some parts of [29, pp. 1195-1198, Proof of Theorem 8.1.] and [32, Section 4, Proof of Proposition 5]. This proof is carried out by means of purely deterministic arguments; this is possible because the stochastic noise is additive and has a constant stochastic integrand (the identity operator). □

Theorem (T3)**.** *Let $d \leq 3$. Let w be a H-valued Q-Wiener process and let w, u, Q, J, F, ϕ_0 and $\{e_j\}_{j\in\mathbb{N}}$ satisfy the same properties as in Theorem (A). Let ϕ_0 be a U-valued random variable such that*

$$\mathbb{E}\left[\|\phi_0\|_U^2 + \int_D \frac{\phi_0^4}{4} - \int_D \frac{\phi_0^2}{2}\right] < +\infty.$$

Then problem (1.22) *admits a unique strong solution (in the sense that two strong solutions coincide for all $\omega \in \Omega$ except for a set of* $\mathbf{m}$*-measure zero).*

Outline of the Proof of Theorem (T3). Let $\mathbf{P}$ be the weak statistical solution built in Theorem (A). We first build a set X which is the countable union of $\mathscr{X}$-compact sets $C_j \subset \mathscr{U}_1$ and such that $\mathbf{P}(X) = 1$. We define the mapping $\mathscr{D}_1$ as the restriction of $\mathscr{D}$ to X. The mapping $\mathscr{D}$ is bijective on $\mathscr{U}_1$ because of Theorem (T2). Hence $\mathscr{D}_1$ is one-to-one. Thanks to the regularity of X, $\mathscr{D}_1^{-1}$ has some measurability properties which are used to build a strong solution ϕ by applying $\mathscr{D}_1^{-1}$ on the trajectories of $(\phi_0, \partial w/\partial t)$, (a.s.). The uniqueness is a consequence of the injectivity of the restriction of $\mathscr{D}$ to $\mathscr{U}_1$. We also show that $\mathbf{P}$ is the distribution of ϕ. □

The previous Theorem also uses this auxiliary lemma.

Lemma (L1)**.** *With the notation of Theorem (T1), we have*

$$\int_{\mathscr{X}} \|\nabla\mu(\phi)\|^2_{L^2([0,T];H)}\mathbf{P}(\mathrm{d}\phi) \leq C_4, \tag{1.25}$$

$$\int_{\mathscr{X}} \|\phi\|^2_{L^2([0,T];U)}\mathbf{P}(\mathrm{d}\phi) \leq C_1, \tag{1.26}$$

where $\mu(\phi) := a\phi + \phi^3 - \phi - J * \phi$. The integrands are meant to assume the value $+\infty$ whenever $\nabla\mu(\phi) \notin L^2([0,T];H)$ or $\phi \notin L^2([0,T];U)$. Constants C_1, C_4 derive from the Proof of Theorem (A).

Outline of the Proof of Lemma (L1). It is an application of Portmanteau Theorem applied to suitable finite-dimensional approximations of the integrands of (1.25) and (1.26). Some estimates from the Proof of Theorem (A) are exploited. □

1.6 New results

We now provide, for each chapter, a complete list of the parts of this thesis in which we provide an original contribution.

Chapter 2. The proof of Theorem (2.2.8) is a reduced and modified version of [21, p. 432, Proof of Theorem 1.]. We added a few comments and footnotes here and there.

Chapter 3. The spaces $\mathscr{U}$ and $\mathscr{V}$ of [29] have been modified for our purposes. In [29, p. 1183, Proof of Theorem 5.2.] the estimate concerning the Hölder continuity of ϕ_m has been reshaped and some computations have been expanded. The proof of Theorem (3.2.9) is a modified version of [29, p. 1187, Proof of Theorem 6.2.]. The proof of Theorem (3.2.20) is a expansion of [24, p. 249, Proofs of Proposition 2.1., Theorem 2.1., Theorem 2.2.]. In particular, we show that the function called v is a solution to (3.38) in the sense of distributions. In addition, we prove the homogeneous boundary conditions for v and Δv in the sense of (3.47), (3.48).

Chapter 4. This chapter is completely original as previously mentioned.

Chapter 2

Deterministic Cahn-Hilliard equations

The discussions made throughout the Introduction show that we cannot simply refer to the Cahn-Hilliard equation, since it presents a lot of variants. In fact, the following aspects of the equation have be specified:

(a) the equation may be *local* or *nonlocal*.

(b) the equation may be deterministic or stochastic.

(c) the mobility coefficient M may be *degenerate*, constant or none of the above.

(d) the Helmholtz free energy F may have a *logarithmic* expression, a polynomial expression or none of the above.

(e) the equation may present additional terms (e.g., a convective term) or may be coupled with other equations (e.g., the Navier-Stokes equation).

In this chapter we study specific versions of the deterministic Cahn-Hilliard equation (i.e., specific choices of (a)-(c)-(d)-(e)). In particular we will discuss topics such as:

(i) theorems of existence and uniqueness of the solution $\phi(x,t)$

(ii) different types of solution.

(iii) asymptotic behaviour of ϕ.

(iv) introduction of dynamic boundary conditions.

(v) coupling of the Cahn-Hilliard equation with other differential equations, e.g., the Navier-Stokes equation.

In Section (2.1) we provide a brief historical summary of the main analytical results which have been proved for the deterministic Cahn-Hilliard equation. In Section (2.2) we focus on a set of selected results. Our goal is to give an overview of the main analytical theorems and to describe the relevant features which arise when dealing with issues (i)-(v). We will put more emphasis on some articles rather than on others. We follow the chronological order of appearance of the works.

2.1 A brief history

Reference [31] is one of the first mathematical articles dealing with the deterministic Cahn-Hilliard equation. The setting is a simple one: the equation is local, the mobility is constant, the potential is a fourth-order polynomial, the boundary condition are of standard Neumann type. The potential depends on a specific constant parameter. According to the nature of this parameter and of the mobility coefficient, the authors either prove the global existence of a classical solution and some longtime behaviour properties, or the *blow up* of the solution in finite time. The argument uses classic Sobolev space analysis.
In [50], a local Cahn-Hilliard equation with constant mobility, logarithmic potential and physical constraints is studied. These physical constraints are in accordance with Remark (1.1.1). The topics of existence, uniqueness and asymptotic behaviour of solutions are argued with subdifferential operator techniques.
A subsequent reference related to [50] is [26]; in particular, topics such as global attractors and their fractal and Hausdorff dimensions are added to the analysis.
We now move to [30], in which an existence result for the local Cahn-Hilliard equation with a *degenerate* mobility coefficient and irregular potential is presented. In particular, the mobility is allowed to vanish when the scaled concentration u assumes the values ± 1, and it is shown that the solution is bounded by 1 in magnitude.
In article [63], the dynamic boundary conditions are introducted for a local Cahn-Hilliard equation with constant mobility and regular potential. A pertubation element governed by a parameter $\varepsilon \in (0, 1)$ is added to the equation, which is then decoupled into a Cahn-Hilliard/heat equation system with dynamic boundary conditions. The equations are solved seperately. Suitable estimates are shown and used to exploit a passage to the limit with respect to ε. Existence and uniqueness of a solution for the original equation is recovered.
In [35], a simplified Cahn-Hilliard model of phase separation for two-phase systems is given. The model is derived from a free energy with a *nonlocal* interacting term. Using the free energy as a Lyapunov functional, the asymptotic state of the system is investigated and characterized by a variational principle.
In [9], the authors study the existence, uniqueness and continuous dependence

on initial data of the solution to a *nonlocal* Cahn-Hilliard equation. A nonlinear Poincaré inequality is applied to show the existence of an absorbing set in each constant mass affine space.

We now turn to the most recent references. Work [41] presents a complicated version of the Cahn-Hilliard equation: highly *irregular* potentials[1] and dynamic boundary conditions are jointly considered. Well-posedness theorems are obtained by means of many *ad hoc* preliminary results and the exploitation of Galerkin and noise parameter-dependent schemes.

Article [5] must be quoted even though it deals with a *generalized* Cahn-Hilliard equation with forcing terms: this setting goes far beyond the physical context discussed in the Introduction and we won't provide any further detail.

We finally mention works [21], [32]. These are some of the most recent contributions to the study of the nonlocal Cahn-Hilliard-Navier-Stokes system. In [21] the global existence of a weak solution for spatial dimensions $d \in \{2; 3\}$ is proved. In the case $d = 2$, under suitable assumptions on the free energy F, an energy identity and a dissipative estimate are shown. Work [32] is a prosecution of the previous one. Relying on the energy identity previously mentioned, the authors define, following J. M. Ball's approach, a generalized semiflow which has a global attractor. The existence of a connected global attractor for the convective nonlocal Cahn-Hilliard equation with a given velocity field is proved for $d \in \{2; 3\}$. Finally, it is shown that any weak solution fulfilling the energy inequality also satisfies a dissipative estimate. Hence the existence of the trajectory attractor with a time dependent external force is established for $d \in \{2; 3\}$.

2.2 The Cahn-Hilliard differential equation. Some deterministic results

Here we mention in some details some meaningful theoretical results on the deterministic Cahn-Hilliard equation. The first and the oldest is concerned with the one dimensional case and polynomial potential. The second and more recent is related to singular potentials. The third and fourth are related to the analysis of the nonlocal equation. These are just a few examples and we remind that the theoretical aspects of the deterministic Cahn-Hilliard equation have been analysed in many papers (see also Introduction).

2.2.1 Well-posedness results in spatial dimension $d = 1$

One of the first papers concerning the theoretical aspects of the Cahn-Hilliard equation appeared in 1986, see [31]. In this paper Elliott and Zheng prove,

[1] possibly more problematic than logarithmic ones.

in particular, some results of well-posedness for a unidimensional Cahn-Hilliard equation with constant mobility and polynomial potential.
The equation being analysed is

$$\begin{cases} \phi_t = -\varepsilon^2 \dfrac{\partial^4 \phi}{\partial x^4} + \dfrac{\partial^2 f(\phi)}{\partial x^2}, \quad x \in D = (0, L), \quad t \in (0, T), \\ \dfrac{\partial}{\partial x}\phi(0,t) = \dfrac{\partial}{\partial x}\phi(L,t) = \dfrac{\partial^3}{\partial x^3}\phi(0,t) = \dfrac{\partial^3}{\partial x^3}\phi(L,t) = 0, \quad t \in (0,T), \\ \phi(x,0) = \phi_0(x), \quad x \in D, \end{cases} \tag{2.1}$$

where

$$f(\phi) = -\phi + \gamma_1 \phi^2 + \gamma_2 \phi^3,$$

and $\varepsilon^2, \gamma_1, \gamma_2$ are given constants. We are ready to state a preliminary theorem (see [31, p. 342, Theorem 2.1.]).

Theorem 2.2.1. *Let $\gamma_2 > 0$. For any intial condition $\phi_0 \in V$ there exists a unique global solution of* (2.1) *belonging to $H^{4,1}(\mathcal{Q}_T)$. Morever, if $\phi_0 \in H^6 \cap V$ and $\partial^2\phi_0/\partial x^2 \in V$, then the solution is a classical one.*

Morever, if we add some additional conditions, we may also get information about the asymptotic behaviour of the (unique) solution of (2.1). More precisely, we have the following theorem (see [31, p. 345, Theorem 2.2.]).

Theorem 2.2.2. *Let the hypothesis of Theorem* (2.2.1) *be true. If $\varepsilon^2 > L^2/\pi^2$ and $\|\phi_0\|_{H^2}$ is sufficiently small, then the unique global solution ϕ of* (2.1) *satisfies*

$$\lim_{t\to+\infty} \|\phi - M\|_{L^\infty} = \lim_{t\to+\infty} \|\phi_x\|_{L^\infty} = \lim_{t\to+\infty} \|\phi_{xx}\|_H = 0.$$

where $M = L^{-1}\int_0^L \phi_0(x)dx = L^{-1}\int_0^L \phi(x,t)dx$.

Remark 2.2.3. Well-posedness and regularity issues for the multidimensional case ($d \in \{2;3\}$) when F has a polynomial growth are treated, for instance, in [57]. There, also the large time behaviour of the corresponding dynamical system is analyzed.

2.2.2 Well-posedness results in spatial dimensions $d \in \{2;3\}$

We quote a classical well-posedness result in the multidimensional case. See, for instance, [20] and the references therein. In the setting of such work, the potential F is assumed to be singular, i.e., defined on a bounded interval with infinite derivative at the endpoints, namely

$$\lim_{s\to\pm1} f(s) = \pm\infty, \quad \lim_{s\to\pm1} f'(s) = +\infty, \tag{2.2}$$

where $f = F'$. This assumption allows F to have the logarithmic expression shown in (1.3). Such expression is the most physically relevant one, as anticipated in the Introduction. See also [26]. We now define the following space

$$\begin{aligned} D_{m_0} \quad := \quad & \Big\{ q \in H^2,\ \frac{\partial q}{\partial \nu} = 0 \text{ on } \Gamma,\ \|q\|_{L^\infty} \leq 1,\ \frac{1}{|D|}\int_D q = m_0, \\ & f(q) \in H,\ \Delta^2 q - \Delta f(q) \in H^{-1} \Big\}, \end{aligned}$$

where m_0 is such that $|m_0| \leq 1 - \eta$, where $\eta \in (0,1)$ is given.

Theorem 2.2.4. *Let us consider the problem*

$$\begin{cases} \phi_t + \Delta^2 \phi - \Delta f(\phi) = 0, & (2.3a) \\ \dfrac{\partial \phi}{\partial \nu} = \dfrac{\partial \mu}{\partial \nu} = 0 \quad \text{on } \Gamma, & (2.3b) \\ \phi(0,x) = \phi_0(x) \quad \text{on } D. & (2.3c) \end{cases}$$

where $\phi_0 \in D_{m_0}$, $\mu := -\Delta\phi + f(\phi)$, and f satisfies (2.2). *Then problem* (2.3) *has a unique solution $\phi \in L^\infty([0,T]; D_{m_0}) \cap \mathcal{C}([0,T]; H^{-1})$. In addition, $\|\phi(t)\|_{L^\infty} \leq 1$ for almost every $t \in [0,T]$.*

2.2.3 A nonlocal Cahn-Hilliard equation

We now discuss in detail the properties of a weak solution to a *nonlocal* Cahn-Hilliard equation (see [21]). More precisely, let the free energy $\mathcal{E}[\phi]$ be defined as in (1.14). Then its first variation is $\mu = a\phi - J * \phi + \eta F'(\phi)$, where $*$ denotes the convolution operator over D and

$$a(x) = \int_D J(x-y)\mathrm{d}y, \quad x \in D.$$

For the sake of semplicity we take $\eta = 1$ and we consider the problem

$$(NL) \begin{cases} \phi_t + u \cdot \nabla \phi = \Delta \mu, \\ \mu = a\phi - J * \phi + F'(\phi), & (2.4a) \\ \dfrac{\partial \mu}{\partial \nu} = 0 \qquad \text{on } \partial D \times (0,T), \\ \phi(0) = \phi_0 \quad \text{ in } D. \end{cases}$$

where $D \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a bounded domain with regular boundary, and $u \in L^\infty([0,T]; L^\infty) \cap U$, $\mathrm{div}(u) = 0$, $u = 0$ on Γ.

Remark 2.2.5. In the original work [21], system (2.4) also contains the Navier-Stokes equation, hence u is an unknown. We focus our analysis of the Cahn-Hilliard equation only.

Physical hypothesis. We will assume the following physical hypothesis to hold[2]:

(H1) $J \in W^{1,1}(\mathbb{R}^d)$, $J(x) = J(-x)$, $a(x) := \int_D J(x-y)\mathrm{d}y \geq 0$, a.e. $x \in D$.

(H3) $F \in \mathcal{C}^{2,1}_{loc}(\mathbb{R})$ and there exists $c_0 > 0$ such that

$$F''(s) + a(x) \geq c_0, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in D.$$

(H4) There exist $c_1 \geq \frac{1}{2}\|J\|_{L^1(\mathbb{R}^d)}$ and $c_2 \in \mathbb{R}$ such that

$$F(s) \geq c_1 s^2 - c_2, \quad \forall s \in \mathbb{R}.$$

(H5) There exist $c_3 > 0$, $c_4 \geq 0$ and $p \in (1,2]$ such that

$$|F'(s)|^p \leq c_3|F(s)| + c_4, \quad \forall s \in \mathbb{R}.$$

Remark 2.2.6. Assumption (H3) implies that F can be expressed as follows

$$F(s) = G(s) - \frac{a^*}{2}s^2, \quad a^* = \|a\|_{L^\infty}, \tag{2.5}$$

with $G \in \mathcal{C}^{2,1}(\mathbb{R})$ strictly convex, since $G'' \geq c_0$ in $\mathbb{R}$.

Weak solution per problem (NL). We shall now define what we mean by *weak solution* for problem (NL).

Definition 2.2.7. *Let $\phi_0 \in H$ be given. Then ϕ is said to be a weak solution to problem (NL) on $[0,T]$ with initial data ϕ_0 if*

(i) ϕ, μ satisfy

$$\begin{cases} \phi \in L^\infty([0,T];H) \cap L^2([0,T];U), & \text{(2.6a)} \\ \phi_t \in L^{4/3}([0,T];U'), \ \textit{if } d = 3, & \text{(2.6b)} \\ \phi_t \in L^{2-\delta}([0,T];U'), \ \forall \delta \in (0,1) \ \textit{if } d = 2, & \text{(2.6c)} \\ \mu := a\phi - J * \phi + F'(\phi) \in L^2([0,T];U). & \text{(2.6d)} \end{cases}$$

(ii) setting

$$\rho := a\phi + F'(\phi), \tag{2.7}$$

[2] their numbering has been kept from the original article.

we have, for every $\psi \in U$ and for almost every $t \in (0,T)$

$$\langle \phi_t, \psi \rangle + (\nabla \rho, \nabla \psi) = \int_D (u \cdot \nabla \psi)\phi + \int_D (\nabla J * \phi) \cdot \nabla \psi. \tag{2.8}$$

(iii) The initial conditions hold in the following weak sense

$$(\phi(t), \chi) \to (\phi_0, \chi) \text{ as } t \to 0, \quad \forall \chi \in U. \tag{2.9}$$

We are now ready to state and prove the main theorem of this Subsection.

Theorem 2.2.8. *Let $\phi_0 \in H$ such that $F(\phi_0) \in L^1$. Suppose that hypothesis* (H1)/(H3)/(H4)/(H5) *are satisfied. Suppose that $u \in L^\infty([0,T];L^\infty) \cap U$,* $\operatorname{div}(u) = 0$*, $u = 0$ on Γ. Then, for every $T > 0$, there exists a weak solution ϕ to problem (NL) on $[0,T]$ corresponding to initial data ϕ_0. Moreover ϕ_t satisfies*

$$\begin{cases} \phi_t \in L^\infty([0,T];V_s'), \text{ if } 1 < p < \dfrac{d}{d-1}, \ s = \dfrac{(4-d)p + 2d}{2p}, \\ \phi_t \in L^\infty([0,T];V_s') \cap L^r([0,T];V'_{\frac{d+2}{d}}), \text{ if } p = \dfrac{d}{d-1}, \ s > \dfrac{d+2}{d}, \ r \geq 2, \\ \phi_t \in L^{2p/(2p-3)}([0,T];V_s'), \text{ if } d = 3, \ 3/2 < p \leq 2, \ s = \dfrac{p+6}{2p}. \end{cases}$$

Remark 2.2.9. The proof of Theorem (2.2.8) is a reduced and adapted version of [21, p. 432, Theorem 1].

Proof of Theorem (2.2.8). *Step 1: Galerkin approximation.* We first assume that $\phi_0 \in V$. We define the Neumann operator

$$B : \mathcal{D}(B) = V \to H : v \mapsto -\Delta v + v.$$

Let $\{\psi_j\}_{j\geq 1}$ be the eigenfunctions of B forming a Galerkin base[3] in U. For each $n \in \mathbb{N}$ we define the n-dimensional subspace $\Psi_n = \operatorname{span}\{\psi_1, \cdots, \psi_n\}$. Let P_n be the orthogonal projectors on Ψ_n in H. We look for functions of the form

$$\phi_n(t) = \sum_{k=1}^n b_k^{(n)}(t)\psi_k, \quad \mu_n(t) = \sum_{k=1}^n c_k^{(n)}(t)\psi_k,$$

solving the following approximating problem[4]

$$\begin{cases} (\phi_n', \psi) + (\nabla \rho(\cdot, \phi_n), \nabla \psi) = \\ \qquad \displaystyle\int_D (u \cdot \nabla \psi)\phi_n + \int_D (\nabla J * \phi_n) \cdot \nabla \psi, & (2.11a) \\ \rho(\cdot, \phi_n) := a(\cdot)\phi_n + F'(\phi_n), & (2.11b) \\ \mu_n = P_n(\rho(\cdot, \phi_n) - J * \phi_n), & (2.11c) \\ \phi_n(0) = \phi_{0n}, & (2.11d) \end{cases}$$

[3] i. e. $\{\psi_j\}_{j\geq 1}$ is an orthonormal base in H and an orthogonal base in U.

[4] primes denote the derivatives with respect to time.

for every $\psi \in \Psi_n$, where $\phi_{0n} = P_n\phi_0$. Problem (2.11) is a system of ordinary differential equations in the n unknowns $b_i^{(n)}$. Since $F' \in \mathcal{C}^{1,1}_{loc}$ thanks to (H3), we conclude by the Cauchy-Lipschitz theorem that there exists $T_n^* \in (0, +\infty)$ such that the system has a unique maximal solution $b^{(n)} = (b_1^{(n)}, \cdots, b_n^{(n)})$ on $[0, T_n^*)$ and $b^{(n)} \in \mathcal{C}^1([0, T_n^*; \mathbb{R}^n)$.

Step 2: Bounds. We will now derive some *a priori* estimates on ϕ_n, μ_n in order to prove that $T_n^* \geq T$ for every $n \geq 1$. By using μ_n as a test function in (2.11a) we obtain

$$(\phi_n', \mu_n) + (\nabla\rho(\cdot, \phi_n), \nabla\mu_n) = \int_D (u \cdot \nabla\mu_n)\phi_n + \int_D (\nabla J * \phi_n) \cdot \nabla\mu_n,$$

Since $J(x) = J(-x)$ we have $(\phi, J * \psi) = (\psi, J * \phi)$, hence

$$\begin{aligned} & (\phi_n', \mu_n) = (\phi_n', a\phi_n + F'(\phi_n) - J * \phi_n) \\ = & \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|\sqrt{a}\phi_n\|^2 + \int_D F(\phi_n) - \frac{1}{2}(\phi_n, J * \phi_n)\right] \\ = & \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{4}\int_D\int_D J(x-y)(\phi_n(x) - \phi_n(y))^2 \mathrm{d}x\mathrm{d}y + \int_D F(\phi_n)\right]. \end{aligned} \tag{2.12}$$

Moreover, since $\partial\mu_n/\partial\nu = 0$ on Γ, we have

$$(\nabla\rho(\cdot, \phi_n), \nabla\mu_n) = (-\rho(\cdot, \phi_n), \Delta\mu_n) = (-\rho_n, \Delta\mu_n) = (\nabla\rho_n, \nabla\mu_n),$$

where $\rho_n = P_n\rho(\cdot, \phi_n) = \mu_n + P_n(J * \phi_n)$. Moreover, we observe that

$$\|\nabla(P_n(J * \phi_n))\| \leq \|B^{1/2}P_n(J * \phi_n)\| \leq \|\nabla J * \phi_n\| + \|J * \phi_n\| \leq \|J\|_{W^{1,1}}\|\phi_n\|, \tag{2.13}$$

and that, by means of (H4), we obtain

$$\begin{aligned} & \frac{1}{2}\int_D\int_D J(x-y)(\phi_n(x) - \phi_n(y))^2 \mathrm{d}x\mathrm{d}y + 2\int_D F(\phi_n) \\ = & \|\sqrt{a}\phi_n\|^2 + 2\int_D F(\phi_n) - (\phi_n, J * \phi_n) \\ \geq & \int_D (a + 2c_1 - \|J\|_{L^1})\phi_n^2 - 2c_2|D| \ \geq \ \alpha\|\phi_n\|^2 - c, \end{aligned} \tag{2.14}$$

where $\alpha = 2c_1 - \|J\|_{L^1} > 0$. If we integrate (2.12) with respect to time between 0 and $t \in (0, T_n^*)$, using (2.13), (2.14), and the fact that $b^{(n)} \in \mathcal{C}^1([0, T_n^*); \mathbb{R}^n)$ we obtain the following inequality for every $t \in (0, T_n^*)$

$$\begin{aligned}
& \alpha\|\phi_n(t)\|^2 + C'\int_0^t \|\nabla\mu_n\|^2 \mathrm{d}\tau \\
\leq\ & c\|J\|^2_{W^{1,1}}\int_0^t \|\phi_n\|^2\mathrm{d}\tau + \frac{1}{2}\int_D\int_D J(x-y)(\phi_{0n}(x)-\phi_{0n}(y))^2\mathrm{d}x\mathrm{d}y \\
+\ & 2\int_D F(\phi_{0n}) \leq M + c\int_0^t \|\phi_n\|^2\mathrm{d}\tau, \qquad \forall t \in [0, T_n^*),
\end{aligned} \tag{2.15}$$

where C', c may depend on J, $|D|$ and M may depend on ϕ_0, F, c, T. In (2.15) we have used the fact that $\phi_0 \in \mathcal{D}(B)$, hence $\phi_{0n} \to \phi_0$ in H^2 ($\{\psi_n\}_{n\geq 1}$ is a orthogonal basis in $\mathcal{D}(B)$) and therefore in L^∞ ($d = 2, 3$). Moreover we observe that $\|\phi_n(t)\| = |b^{(n)}(t)|$, therefore we can apply Gronwall Lemma to (2.15) and get that $T_n^* \geq T$ for every $n \geq 1$. It follows that problem (2.11) has a unique solution defined on $(0, T)$ and the following estimates hold

$$\|\phi_n\|_{L^\infty([0,T];H)} \leq N, \tag{2.16}$$

$$\|\nabla\mu_n\|_{L^2([0,T];H)} \leq N, \tag{2.17}$$

where N might depend on T, J, F, ϕ_0. We recall the inequality $\|x * y\|_{L^p} \leq \|x\|_{L^1}\|y\|_{L^p}$, valid for each $x \in L^1$, $y \in L^p$. Since $\partial\Delta\phi_n/\partial\nu = 0$ and $\nabla a(x) = \int_D \nabla J(x-y)\mathrm{d}y$, using (H3), integration by parts, Young inequality and defining $k := (2/c_0)\|\nabla J\|^2_{L^1(\mathbb{R}^d)}$, we can write

$$\begin{aligned}
(\nabla\mu_n, \nabla\phi_n) &= (-\Delta\phi_n, \mu_n) = (-\Delta\phi_n, P_n(a\phi_n + F'(\phi_n) - J * \phi_n)) \\
&= (-\Delta\phi_n, a\phi_n + F'(\phi_n) - J * \phi_n) \\
&= (\nabla\phi_n, a\nabla\phi_n + \phi_n\nabla a + F''(\phi_n)\nabla\phi_n - \nabla J * \phi_n) \\
&\geq c_0\|\nabla\phi_n\|^2 - 2\|\nabla J\|_{L^1(\mathbb{R}^d)}\|\nabla\phi_n\|\|\phi_n\| \\
&\geq \frac{c_0}{2}\|\nabla\phi_n\|^2 - k\|\phi_n\|^2.
\end{aligned} \tag{2.18}$$

Young inequality also yields

$$(\nabla\mu_n, \nabla\phi_n) \leq \frac{c_0}{4}\|\nabla\phi_n\|^2 + \frac{1}{c_0}\|\nabla\mu_n\|^2,$$

which implies, in combination with (2.18), that

$$\|\nabla\mu_n\|^2 \geq \frac{c_0^2}{4}\|\nabla\phi_n\|^2 - c\|\phi_n\|^2. \tag{2.19}$$

Inequalities (2.16), (2.17) and (2.19) yield

$$\|\phi_n\|_{L^2([0,T];U)} \leq N. \tag{2.20}$$

We shall now derive an estimate for μ_n in $L^2([0,T];U)$. Since $|x| \leq |x|^p+1$, $\forall x \in \mathbb{R}$, thanks to (H5) we have that $|F'(s)| \leq c|F(s)| + c$ for every $s \in \mathbb{R}$ and we obtain

$$\left|\int_D \mu_n\right| = |(\mu_n, 1)| = |(F'(\phi_n), 1)| \leq \int_D |F'(\phi_n)| \leq \int_D c|F(\phi_n)| + c \leq N. \quad (2.21)$$

We have used the identity[5] $(P_n(-J * \phi_n + a\phi_n), 1) = 0$ and the uniform bound $\|F(\phi_n)\|_{L^\infty([0,T];L^1)} \leq N$, which can be derived integrating (2.12) in time over $(0,T)$ and using (2.14), (2.16), (2.17). Hence, by means of Poincaré-Wirtinger inequality, from (2.17) and (2.21) we get

$$\|\mu_n\|_{L^2([0,T];U)} \leq N. \quad (2.22)$$

From (H5) we obtain

$$\|\rho(\cdot, \phi_n)\|_{L^p} \leq (c\|a\|_{L^\infty}\|\phi_n\| + \|F'(\phi_n)\|_{L^p}) \leq c\left(\left(\int_D |F(\phi_n)|\right)^{1/p} + 1\right) \leq N, \quad (2.23)$$

and hence we get

$$\|\rho(\cdot, \phi_n)\|_{L^\infty([0,T];L^p)} \leq N. \quad (2.24)$$

We now need to estimate ϕ_n' in a suitable space. We investigate an estimate for the sequence of ϕ_n' taking values in V_s', for a suitable s. More precisely, we take a Gelfand triple $V_s \hookrightarrow L^2 \hookrightarrow V_s'$, where $s \geq 2$ is such that, if $\psi \in V_s$, then $\Delta\psi \in H^{s-2} \hookrightarrow L^{p'}$, with p' being the conjugate of p. Since $H^{s-2} \hookrightarrow L^{p^*}$, where $p^* = 2d/(d+4-2s)$, it is sufficient to take

$$s \geq \frac{(4-d)p + 2d}{2p}$$

If $\psi \in V_s$, we can decompose it as $\psi = \psi_1 + \psi_2$, where

$$\psi_1 = P_n\psi = \sum_{k=1}^{n} (\psi, \psi_k)\psi_k \in \Psi_n$$

and

$$\psi_2 = (I - P_n)\psi = \sum_{k=n+1}^{\infty} (\psi, \psi_k)\psi_k \in \Psi_n^\perp.$$

We notice that ψ_1 and ψ_2 are orthogonal in V_r, $0 \leq r \leq s$. Thanks to (2.24) we can write

$$|(\nabla\rho(\cdot, \phi_n), \nabla\psi_1)| = |(\rho(\cdot, \phi_n), \Delta\psi_1)| \leq N\|\Delta\psi_1\|_{L^{p'}} \leq N\|\psi_1\|_{V_s} \leq N\|\psi\|_{V_s}. \quad (2.25)$$

[5] since the eigenvalues of the Neumann operator B are equal or greater than 1, we can choose $\psi_1 = 1$, so that $P_n(1) = 1$, $\forall n$.

We also notice that

$$\left|\int_D (\nabla J * \phi_n)\cdot\nabla\psi_1\right| \leq c\|\nabla J\|_{L^1(\mathbb{R}^d)}\|\psi_n\|\|\psi\|_{V_s} \leq N\|\psi\|_{V_s}. \tag{2.26}$$

We now notice that $\nabla\psi_1 \in H^{s-1}$. Therefore, if $1 < p < d/(d-1)$ and $s = ((4-d)p+2d)/2p$ or $p = d/(d-1)$ and $s > ((4-d)p+2d)/2p = (d+2)/2$, due to the embedding $H^{s-1} \hookrightarrow L^\infty$, we have

$$\left|\int_D (u\cdot\nabla\psi_1)\phi_n\right| \leq c\|u\|_{L^\infty([0,T];L^\infty)}\|\phi_n\|\|\psi\|_{V_s} \leq c\|u\|_{L^\infty([0,T];L^\infty)}\|\psi\|_{V_s}. \tag{2.27}$$

If $p = d/(d-1)$ and $s = ((4-d)p+2d)/2p = (d+2)/2$ due to the embedding $H^{s-1} \hookrightarrow L^q$ for every $1 \leq q < +\infty$ and the interpolation results in the L^p spaces, we have, for every $r \geq 2$

$$\begin{aligned}\left|\int_D (u\cdot\nabla\psi_1)\phi_n\right| &\leq c\|u\|_{L^\infty([0,T];L^\infty)}\|\psi\|_{V_s}\|\phi_n\|_{L^{2r/(r-1)}} \\ &\leq c\|u\|_{L^\infty([0,T];L^\infty)}\psi\|_{V_s}\|\phi_n\|^{(r-2)/r}\|\phi_n\|^{2/r}_{L^4} \leq C''\|\psi\|_{V_s}\|\phi_n\|^{2/r}_U. \end{aligned}\tag{2.28}$$

Finally, in the case $d = 3$, when $3/2 < p \leq 2$ and $s = ((4-d)p+2d)/2p = (p+6)/2p$, due to the embedding $H^{s-1} \hookrightarrow L^{3p/(2p-3)}$, we obtain

$$\begin{aligned}\left|\int_D (u\cdot\nabla\psi_1)\phi_n\right| &\leq c\|u\|_{L^\infty([0,T];L^\infty)}\|\psi\|_{V_s}\|\phi_n\|_{L^{6p/(6-p)}} \\ &\leq C'''\|\psi\|_{V_s}\|\phi_n\|^{(3-p)/p}\|\phi_n\|^{(2p-3)/p}_{L^6} \leq C'''\|\psi\|_{V_s}\|\phi_n\|^{(2p-3)/p}_U. \end{aligned}\tag{2.29}$$

Taking into account (2.25)-(2.29) we deduce from (2.11a) that

$$\|\phi_n'\|_{L^\infty([0,T];V_s')} \leq L, \quad \text{if } 1 < p < d/(d-1), \quad s = \frac{(4-d)p+2d}{2p}, \tag{2.30}$$

$$\|\phi_n'\|_{L^\infty([0,T];V_s')\cap L^r([0,T];V'_{\frac{d+2}{d}})} \leq L, \quad \text{if } p = d/(d-1), \quad s > \frac{d+2}{d}, \quad r \geq 2, \tag{2.31}$$

$$\|\phi_n'\|_{L^{2p/(2p-3)}([0,T];V_s')} \leq L, \quad \text{if } d = 3, \quad 3/2 < p \leq 2, \quad s = \frac{p+6}{2p}, \tag{2.32}$$

where L is a constant depending from T, u, J, F, ϕ_0.

Step 3: Existence of suitable limits. By putting together (2.17), (2.20), (2.22), (2.24), (2.30)-(2.32) and on the account of the compact embedding

$$L^2([0,T];U) \cap H^1([0,T];V_s') \overset{c}{\hookrightarrow} L^2([0,T];H),$$

we deduce that there exist

$$\phi \in L^\infty([0,T];H) \cap L^2([0,T];U), \tag{2.33}$$

$$\mu \in L^2([0,T];U), \tag{2.34}$$

$$\rho \in L^\infty([0,T];L^p), \tag{2.35}$$

with

$$\phi_t \in L^\infty([0,T];V_s'), \quad \text{if } 1 < p < d/(d-1), \quad s = \frac{(4-d)p+2d}{2p},$$

$$\phi_t \in L^\infty([0,T];V_s') \cap L^r([0,T];V'_{\frac{d+2}{d}}), \quad \text{if } p = d/(d-1), \quad s > \frac{d+2}{d}, \quad r \geq 2,$$

$$\phi_t \in L^{2p/(2p-3)}([0,T];V_s'), \quad \text{if } d=3, \quad 3/2 < p \leq 2, \quad s = \frac{p+6}{2p}, \tag{2.36}$$

such that, for a not relabeled subsequence, we have

$$\phi_n \overset{*}{\rightharpoonup} \phi \quad \text{in } L^\infty([0,T];H), \tag{2.37}$$

$$\phi_n \rightharpoonup \phi \quad \text{in } L^2([0,T];U), \tag{2.38}$$

$$\phi_n \to \phi \quad \text{in } L^2([0,T];H), \text{ a.e. in } D \times (0,T), \tag{2.39}$$

$$\mu_n \rightharpoonup \mu \quad \text{in } L^2([0,T];U), \tag{2.40}$$

$$\rho(\cdot,\phi_n) \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty([0,T];L^p), \tag{2.41}$$

and

$$\phi_n' \overset{*}{\rightharpoonup} \phi_t \quad \text{in } L^\infty([0,T];V_s'), \tag{2.42}$$

if $1 < p < d/(d-1)$, with $s = \{(4-d)p+2d\}/(2p)$,

$$\phi_n' \overset{*}{\rightharpoonup} \phi_t \quad \text{in } L^\infty([0,T];V_s'), \quad \phi_n' \rightharpoonup \phi_t \quad \text{in } L^r([0,T];V'_{\frac{d+2}{d}}), \tag{2.43}$$

if $p = d/(d-1)$, with $s > \frac{d+2}{d}$ and $r \geq 2$,

$$\phi_n' \rightharpoonup \phi_t \quad \text{in } L^{2p/(2p-3)}([0,T];V_s'), \tag{2.44}$$

if $d = 3$, $3/2 < p \leq 2$, and $s = \frac{p+6}{2p}$.

Step 4: Passage to limit and existence of a weak solution. We now show that ϕ is a weak solution to Problem (NL), i.e., ϕ, μ and ρ satisfy (2.4a), (2.7), (2.9). From the pointwise convergence in (2.39) we have $\rho(\cdot,\phi_n) \to a\phi + F'(\phi)$ almost everywhere in $D\times(0,T)$ and therefore from (2.41), the fact that $L^{p'}([0,T];L^{p'}) \hookrightarrow$

$L^1([0,T];L^{p'})$ and [60, Proposition 3.19], we deduce that $\rho = a\phi + F'(\phi)$, i.e. (2.7). Moreover, since $\mu_k = P_k(\rho(\cdot,\phi_k) - J*\phi_k)$, we have, for every $v \in \Psi_n$ and every $k \geq n$ (n is fixed)

$$\int_0^T (\mu_k, v)\chi(t)\mathrm{d}t = \int_0^T (\rho(\cdot,\phi_k) - J*\phi_k, v)\chi(t)\mathrm{d}t, \quad \forall \chi \in \mathcal{C}_0^\infty(0,T).$$

By passing to the limit as $k \to +\infty$ in the previous identity and using (2.39), (2.40) (which implies in particular $J*\phi_k \to J*\phi$ in $L^2([0,T];U)$) and (2.41), because of the density of $\{\Psi_n\}_{n\geq 1}$ in H we get $\mu = \rho - J*\phi = a\phi + F'(\phi) - J*\phi$, i. e. (2.4a). In particular, $\rho \in L^2([0,T];U)$.
We can also pass to the limit in (2.11a), to recover (2.8). We multiply (2.11a) by χ, where $\chi \in \mathcal{C}_0^\infty(0,T)$ and integrate between 0 and T. If we pass to limit, we write the term $(\nabla\rho(\cdot,\phi_n), \nabla\psi)$ as $(\rho(\cdot,\phi_n), -\Delta\psi)$ and we use (2.41). The limit equation

$$\begin{aligned} &\int_0^T \langle \phi_t, \psi\rangle \chi \mathrm{d}t + \int_0^T (\nabla\rho, \nabla\psi)\chi \mathrm{d}t \\ = \ &\int_0^T\int_D (u\cdot\nabla\psi)\phi\chi \mathrm{d}x\mathrm{d}t + \int_0^T\int_D (\nabla J*\phi)\nabla\psi\chi\mathrm{d}x\mathrm{d}t, \end{aligned}$$

holds for every $\psi \in \Psi_n$ (for fixed n) and every $\chi \in \mathcal{C}_0^\infty(0,T)$. Since each of the temporal functions appearing in (2.8) (with $\psi \in \Psi_n$ for some n) belong to $L^q(0,T)$ for some $q > 1$, because of the density of $\mathcal{C}_0^\infty(0,T)$ in $L^{q'}(0,T)$, because of the density of $\{\Psi_n\}_{n\geq 1}$ in V_s, we conclude that ϕ, μ and ρ satisfy[6] (2.8) for every $\psi \in V_s$. In addition to that, (2.8) can be written as follows

$$\langle \phi_t, \psi\rangle = -(\nabla\mu, \nabla\psi) + (u, \phi\nabla\psi). \tag{2.45}$$

If the case $d = 3$ we have

$$|(u, \phi\nabla\psi)| \leq N^{1/2}\|\nabla u\|\|\phi\|_U^{1/2}\|\nabla\psi\|, \tag{2.46}$$

while, in the case $d = 2$ we have

$$|(u, \phi\nabla\psi)| \leq N^{2(1-\delta)/(2-\delta)}\|\nabla u\|\|\phi\|_U^{\delta/(2-\delta)}\|\nabla\psi\|, \tag{2.47}$$

for every $\delta \in (0,1)$. From (2.46) and (2.47) we deduce that $\phi_t(t)$ can be continuosly extended to U for almost every $t > 0$ (see footnote (6)) and from these equations and (2.45) we also infer that

$$\phi_t \in L^{4/3}([0,T];U'), \quad \text{if } d = 3; \qquad \phi_t \in L^{2-\delta}([0,T];U'), \quad \forall\delta \in (0,1), \text{ if } d = 2.$$

[6] (2.8) is satisfied for every $t \in C \subset [0,T]$ such that $[0,T]\setminus C$ has Lebesgue measure 0. This happens because these equations are linear in ψ and because V_s is separable.

Hence (2.6b), (2.6c) hold and (2.45), (2.8) are satisfied for every $\psi \in U$.
Finally, if we integrate (2.11a), between 0 and t and we pass to the limit for $n \to +\infty$ we get (2.9).

Step 5: General initial conditions. Let us now assume $u_0 \in G_{div}$ and $\phi_0 \in H$ such that $F(\phi_0) \in L^1$. For every $k \in \mathbb{N}$ we define $\phi_{0k} \in \mathcal{D}(B)$ as

$$\phi_{0k} := \left(I + \frac{1}{k}B\right)^{-1}\phi_0.$$

From the maximal monotone operators theory we deduce that $\phi_{0k} \to \phi_0$ in H. Let ϕ_k be the weak solution corresponding to initial data ϕ_{0k} constructed by the Faedo-Galerkin scheme described in *Steps 1-4* of this proof. Recalling (2.5), we can write

$$\int_D F(\phi_{0k}) = \int_D G(\phi_{0k}) - \frac{a^*}{2}\|\phi_{0k}\|^2. \tag{2.48}$$

We now multiply the equation $\phi_{0k} - \phi_0 = -\frac{1}{k}B\phi_{0k}$ by $g(\phi_{0k})$, where $g = G'$. We get

$$\begin{aligned}\int_D g(\phi_{0k})(\phi_{0k} - \phi_0) &= -\frac{1}{k}\int_D g(\phi_{0k})B\phi_{0k}\\ = \quad -\frac{1}{k}\int_D g'(\phi_{0k})\|\nabla\phi_{0k}\|^2 &- \frac{1}{k}\int_D g(\phi_{0k})\phi_{0k} \le 0,\end{aligned} \tag{2.49}$$

since g is monotone nondecreasing and we can suppose, without loss of generality, that $g(0) = 0$. Due to the convexity of G we have

$$\int_D G(\phi_{0k}) \le \int_D G(\phi_0) + \int_D g(\phi_{0k})(\phi_{0k} - \phi_0) \le \int_D G(\phi_0). \tag{2.50}$$

Hence, thanks to (2.48), (2.50), written for each ϕ_k, by means of (H4) and of Gronwall lemma, we deduce the estimates (2.16)-(2.17) for the sequences ϕ_k and $\nabla\mu_k$ respectively. By taking the H-inner product of $\nabla\mu_k = \nabla(a\phi_k - J * \phi_k + F'(\phi_k))$ and $\nabla\phi_k$ in H, using (H3), we obtain the estimates of $\nabla\phi_k$ from $\nabla\mu_k$ (see (2.19)) and therefore we get estimate (2.20) for ϕ_k. Moreover we can argue as we have done in the previous steps of the proof to get (2.22), (2.24) for μ_k and $\rho(\cdot, \phi_k)$, and (2.30)-(2.32) for ϕ_k'. Using a compactness argument we deduce the existence of three functions ϕ, μ and ρ satisfying (2.33)-(2.36) and such that the convergences (2.37)-(2.41) hold. It is now immediate to see that ϕ is a weak solution for $\phi_0 \in H$ such that $F(\phi_0) \in L^1$. □

Remark 2.2.10. The original nonlocal Cahn-Hilliard equation does not contain the convective term $u \cdot \nabla\phi$. For the sake of completeness, we quote a classical result well-posedness result for a nonlocal Cahn-Hilliard equation without the convective term. See [9].

Theorem 2.2.11. *Let us consider the problem*

$$\begin{cases} \phi_t = \Delta\left(\phi(x)\displaystyle\int_D J(x-y)\mathrm{d}y - \int_D J(x-y)\phi(y)\,dy + f(\phi)\right) \quad \text{in } D, & (2.51a) \\ \dfrac{\partial\left(\phi(x)\int_D J(x-y)\mathrm{d}y - \int_D J(x-y)\phi(y)\,dy + f(\phi)\right)}{\partial\nu} = 0 \quad \text{on } \Gamma, & (2.51b) \\ \phi(0,x) = \phi_0(x) \quad \text{on } D. & (2.51c) \end{cases}$$

We assume that

- $\int_D J(x-y)\mathrm{d}y \in \mathcal{C}^{2+\beta}(\overline{D})$, $f \in \mathcal{C}^{2+\beta}(\mathbb{R})$ *for some* $\beta > 0$.
- *There exist* $c_1 > 0$, $c_2 > 0$, $r > 0$ *such that*

$$\int_D J(x-y)\mathrm{d}y + f'(\phi) \geq c_1 + c_2|\phi|^{2r}$$

- Γ *is of class* $\mathcal{C}^{2+\beta}$.
- $\phi_0 \in \mathcal{C}^{2+\beta,(2+\beta)/2}$ *and satifies*

$$\frac{\partial\left(\phi_0(x)\int_D J(x-y)\mathrm{d}y - \int_D J(x-y)\phi_0(y)\mathrm{d}y + f(\phi_0)\right)}{\partial\nu} = 0 \quad \text{on } \Gamma.$$

Then Problem (2.51) *has a unique solution* $\phi \in \mathcal{C}^{2+\beta,(2+\beta)/2}(\mathcal{Q}_T)$. *In addition, the following dependence on initial data holds*

$$\sup_{0\leq t\leq T}\int_D |\phi_1 - \phi_2| \leq C\int_D |\phi_{0,1} - \phi_{0,2}|$$

where C *depends on* T *only. The subscripts* $1, 2$ *have obvious meaning.*

Chapter 3

Stochastic Cahn-Hilliard equations

This chapter is meant to be the specular version of the previous one. We are now interested in the stochastic Cahn-Hilliard equation. In this case, along with topics (i)-(v) listed at page 23, one may investigate some further aspects such as:

(vi) existence and uniqueness of invariant measures for the transition semigroup and Dirichlet form.

(vii) strong Feller property and irreducibility of the transition semigroup.

(viii) differentiable properties (in the sense of Malliavin calculus) of solutions.

In Section (3.1) we provide a brief historical summary of the main analytical results which have been proved for the stochastic Cahn-Hilliard equation. In Section (3.2) we focus on a set of selected results concerning the stochastic Cahn-Hilliard equation and we analyze them. We follow the chronological order of appearance of the articles.

3.1 A brief history

First of all, we specify that the literature associated with the stochastic version of the Cahn-Hilliard equation is more restricted than the one dealing with the deterministic equation. Nevertheless a considerable amount of results have been proved in the last decades.

The majority of the relevant works dealing with this topic is devoted to the analysis of local stochastic versions of the Cahn-Hilliard equation with constant mobility, regular potential and standard boundary condition (e.g., homogenous Neumann boundary condition). In [29], the authors enstablish the existence of a *weak statistical* solution and existence, uniqueness and measurability of a *strong* solution for a local Cahn-Hilliard equation with additive constant stochastic noise

(see Remark (1.2.1)) by means of a variational approach, a Galerkin scheme and a Prohorov compactness argument. In this case, the free energy is a polynomial of fourth grade, and no improvement can be done without major modifications of the reasoning.
In [24], an analogous equation having an arbitrary even-grade polynomial as free energy is examined by means of a pathwise approach in which, for each $\omega \in \Omega$, a single determinisitic differential equation is solved. A Galerkin scheme and the *stochastic convolution* operator are exploited. Existence and uniqueness of a *classical* solution is proved, along with existence and uniqueness of an invariant measure for the transition semigroup. This approach is completely different from the one argued in [29].
In [17], an equation with noise having a nonlinear diffusion coefficient[1] is considered. The author proves existence and uniqueness of a *classical* solution, proves that this solution is differentiable in the sense of the Malliavian calculus, and, under some further assumptions, proves that the law of the solution is absolutely continuous with respect to Lebesgue measure.
In [27], a stochastic Cahn-Hilliard equation with reflection on a portion of Γ and with the constraint of conservation of the space average is considered. Existence and uniqueness of a strong solution is shown for all continuous nonnegative initial conditions using a method based on infinite-dimensional integration by parts formulae. Detailed information on the associated invariant measure and Dirichlet form is provided.
In [4], the authors study a generalized stochastic Cahn-Hilliard equation with multiplicative white noise posed on bounded spatial convex domains, with piecewise smooth boundary, and an additive time dependent white noise term in the chemical potential. The equation is presented in a weak stochastic integral formulation and the existence of solutions is carried out distinguishing the case $d \in \{1; 2\}$ and $d = 3$. The analysis is based on stochastic integral calculus, the galerkin approximation scheme and the asymptotic spectral properties of the Neumann Laplacian operator. Existence is also derived for some non-convex cases when the boundary Γ is smooth.

3.2 A local stochastic Cahn-Hilliard equation with regular potential

We now carefully study two articles previously mentioned in Section (3.1). These works deal with a stochastic local Cahn-Hilliard equation with constant mobility and regular potential. They follow two completely different approaches and it is worth analyzing both of them.

[1] similar to (1.21).

3.2.1 Probabilistic approach

One of the first articles concerning the stochastic Cahn-Hilliard equation was published in 1991 by N. Elezović and A. Mikelić (see [29]). In this work the authors choose the Helmholtz free energy to be

$$F(\phi) = \frac{1}{4}\phi^4 - \frac{\beta}{2}\phi^2, \qquad \beta > 0.$$

The random term is introduced in the right side of (1.12), which becomes

$$\phi_t = -\gamma\Delta^2\phi + \Delta f(\phi) + \theta. \tag{3.1}$$

Here $\gamma > 0$, $f(\phi) = F'(\phi) = \phi^3 - \beta\phi$ and θ is a *white noise* type process with the following covariance

$$\mathbb{E}[\theta(t,x)\theta(s,y)] = -2M\delta(t-s)k(x,y).$$

Here $M > 0$ is a given constant and k is a sufficiently smooth function. The domain D is supposed to have a smooth boundary $\Gamma \in \mathcal{C}^\infty$.
In our case θ will be the time derivative of Wiener process w. We will remind the reader with the necessary definitions later. In particular, we will give two different definitions of solution (*weak* and *strong* solutions) for problem (3.1) and we will prove existence for both type of solution and uniqueness and measurability for the strong one.

Remark 3.2.1. We present a slight modification of the arguments contained in [29]: we do that in order to complete certain proofs, in which a few details are missing. However, these modifications leave intact the nature of the results exposed in the original article.

We shall use the Gelfand triple $V \overset{c}{\hookrightarrow} H \hookrightarrow V'$. On H we consider the symmetric operator $\Delta^2 + I$ associated with the bilinear form

$$a(u,v) = \int_D \Delta u \Delta v + \int_D uv.$$

The spectral theory implies that there exists an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H made of eigenvectors of $\Delta^2 + I$ with eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$, $\lambda \to +\infty$. The $\{e_i\}_{i\in\mathbb{N}}$ belong to V and they are also eigenvectors of the operator $-\Delta + I$ corresponding to

$$b(u,v) = \int_D \nabla u \nabla v + \int_D uv.$$

Their eigenvalues $\{\mu_i\}_{i\in\mathbb{N}}$ satisfy the formula $\lambda_i - 1 = (\mu_i - 1)^2$ for every i.
We now introduce the following time dependent function spaces

$$\mathscr{U} := L^2([0,T];V) \cap L^\infty([0,T];H) \cap \mathcal{C}^{1/4}([0,T];V'),$$

$$\mathscr{Z} := L^2([0,T]; H_E^{2-\varepsilon}) \cap \mathcal{C}([0,T]; H^{-\varepsilon}), \qquad \varepsilon \in (0, 1/4).$$

Note that we have changed the exponent of the Hölder-continuity in the definition of $\mathscr{U}$. (it is 2/5 in [29, p. 1172, (2.9)].) We will also need the following lemma, whose proof is identical to that in [29, p. 1173, Lemmas 2.3. and 2.4.].

Lemma 3.2.2. *The space $\mathscr{U}$ is compactly embedded in $\mathscr{Z}$, which is continuously embedded in $L^4(\mathcal{Q}_T)$.*

We will also need the Hilbert space

$$\mathscr{V} := \left\{ u \in L^2([0,T]; H^3) : \frac{\partial u}{\partial \nu} = 0,\ \frac{\partial u}{\partial t} \in L^2([0,T]; H),\ u(T) = 0 \right\},$$

which makes part of the Gelfand triple $\mathscr{V} \hookrightarrow L^2([0,T]; H) \hookrightarrow \mathscr{V}'$. We also define the nonlinear operator $\mathscr{A} : L^2([0,T]; V) \to L^2([0,T]; V')$ as

$$\int_0^T \langle \mathscr{A}(\phi), \xi \rangle_{V',V} := \gamma \int_0^T \int_D \Delta\phi \Delta\xi - \int_0^T \int_D f(\phi)\Delta\xi, \qquad \forall \xi \in L^2([0,T]; V),$$

and the nonlinear operator[2]

$$\mathscr{R}(\phi) : \mathscr{U}^{(1)} \to L^{5/3}([0,T]; V') : \phi \mapsto \phi_t + \mathscr{A}(\phi),$$

where ϕ_t is a *formal* time derivative.
Given a Banach space X, we call $\mathcal{C}_w([0,T]; X)$ the space of all functions $u \in \mathcal{C}([0,T]; X)$ such that

$$\sup_{0 \le t_1 < t_2 \le T} \frac{\|u(t_1) - u(t_2)\|_X}{h_T(t_1 - t_2)} < +\infty, \quad h_T(t) := \left\{ 2t \ln \frac{T}{t} \right\}^{1/2}.$$

It also holds that $\mathcal{C}_w([0,T]; X) \subset \mathcal{C}^\kappa([0,T]; X)$ for $0 \le \kappa < 1/2$. Finally, let $V_m = \operatorname{span}\{e_1, \cdots, e_m\}$ and let π_m be the H-orthogonal projector on V_m.

Introduction of the Wiener Process. We introduce a Q-Wiener process $w = w(t) = w(t, x, \omega)$, $t \in [0,T]$, $x \in D$, $\omega \in \Omega$, where $Q : H \to H$ is a Hilbert-Schmidt operator defined by

$$(Qu)(x) := \int_D k(x,y) u(y) \mathrm{d}y, \qquad u \in H.$$

[2]note that $\mathscr{R}$ can be extended to a continuous operator $\mathscr{R} : \mathscr{Z} \to \mathscr{V}'$ using the formula

$$\int_0^T \langle \mathscr{R}(\phi), v \rangle_{V',V} = -\int_D \phi_0 v(0) - \int_0^T \int_D \phi v_t - \gamma \int_0^T \int_D \nabla\phi \nabla(\Delta v) - \int_0^T \int_D f(\phi) \Delta v, \quad \forall v \in \mathscr{V}.$$

Here k is a sufficiently smooth kernel function such that Q is symmetric, positive definite and nuclear[3]. We will also need the finite dimensional Wiener processes

$$w_m(t) := \pi_m w(t) = \sum_{j=1}^{m} w_m^j(t) e_j, \tag{3.2}$$

where w_m^j are real-valued Wiener processes satisfying

$$\mathbb{E}[|w_m^j(t)|^2] = t(Qe_j, e_j)_H.$$

We indicate with $\mathbf{W}$, $\mathbf{W}_m$ the probability measures on $L^2([0,T];H)$ associated with w and w_m.
For our purposes, we define the distributional time derivative w_t of a $L^2([0,T];H)$-valued Wiener process w as the element of $\mathscr{V}'$ such that

$$\langle w_t, v\rangle_{\mathscr{V}',\mathscr{V}} := -\left(w, v_t\right)_{L^2([0,T];H)}. \tag{3.3}$$

Let $\mathbf{N}$ denote the probability measure on $\mathscr{V}'$ associated with w_t. Then we have the relation

$$\hat{\mathbf{N}}(v) = \hat{\mathbf{W}}\left(-\frac{\partial v}{\partial t}\right), \qquad \forall v \in \mathscr{V}.$$

The stochastic Cahn-Hilliard equation. The problem being studied is the following

$$(SCHE)\begin{cases} \phi_t = -\gamma\Delta^2\phi + \Delta f(\phi) + w_t, & (3.4a)\\ \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial\Delta\phi}{\partial\nu} = 0 \quad \text{on } \Gamma, & (3.4b)\\ \phi(0,x) = \phi_0(x) \quad \text{on } D, \ \text{(a.s.)}. & (3.4c)\end{cases}$$

Here ϕ_0 is an H-valued random variable such that $\mathbb{E}[\|\phi_0\|_H^4] < +\infty$. We shall denote by Ξ its corresponding measure. We also assume that w and ϕ_0 are independent and that the mass conservation occurs, namely

$$\int_D \phi(t) = \int_D \phi(0), \quad \forall t \in [0,T].$$

We now give two different definitions of solution for Problem $(SCHE)$.

Definition 3.2.3. *A* ***weak statistical*** *solution of the Cahn-Hilliard equation is a measure* $\mathbf{P}$ *(concentrated) on the* σ*-algebra* $\mathcal{B}_{\mathscr{X}}(\mathscr{U})$ *which, for every* $\xi \in H^\varepsilon$ *and* $v \in \mathscr{V}$*, satisfies*

$$\int_{\mathscr{X}} \exp\left\{ i\langle\phi(0),\xi\rangle_{H^{-\varepsilon},H^\varepsilon} + i\int_0^T \langle\mathscr{R}(\phi), v\rangle_{V',V}\right\}\mathbf{P}(\mathrm{d}\phi) = \hat{\Xi}(\xi)\hat{\mathbf{N}}(v). \tag{3.5}$$

[3] a nuclear operator also has finite trace, namely $\mathrm{tr}(Q) = \sum_{j=1}^{\infty}(Qe_j, e_j)_H < +\infty$.

Remark 3.2.4. If we define the operator $\mathscr{D} : \mathscr{X} \to H^{-\varepsilon} \times \mathscr{V}' : \phi \mapsto \{\phi(0), \mathscr{R}(\phi)\}$ we see that (3.5) is equivalent to

$$\mathbf{P}(\mathscr{D}^{-1}(C)) = (\Xi \times \mathbf{N})(C), \qquad \forall C \in \mathcal{B}(H^{-\varepsilon} \times \mathscr{V}'). \tag{3.6}$$

Relation (3.6) does not guarantee uniqueness for weak solutions of the Cahn-Hilliard equation since $\mathscr{D}$ may not be one-to-one.

We give another definition.

Definition 3.2.5. *A strong solution of the Cahn-Hilliard equation is a stochastic process $\phi(t)$ such that*

(i) ϕ satisfies

$$\mathscr{D}(\phi(\omega)) = \left\{\phi_0(\omega), \frac{\partial w}{\partial t}(\omega)\right\}, \qquad a.s.$$

(ii) The random variable $\omega \mapsto \phi(t, x, \omega)$ is measurable from $(\Omega, \mathcal{F})$ to $(\mathscr{U}, \mathcal{B}_{\mathscr{X}}(\mathscr{U}))$.

Galerkin approximation. We now consider a finite-dimensional approximation of (3.4a)

$$(\phi_m)_t = -\gamma\Delta^2\phi_m + \pi_m[\Delta f(\phi_m)] + (w_m)_t, \tag{3.7}$$

In fact (3.7) is a compact notation that indicates the following finite-dimensional stochastic equation

$$\mathrm{d}\phi_m = \left\{-\gamma\Delta^2\phi_m + \pi_m[\Delta f(\phi_m)]\right\} \mathrm{d}t + \mathrm{d}w_m. \tag{3.8}$$

We also remind the following Theorem.

Theorem 3.2.6. *Let w_m be given by* (3.2) *and let $X \hookrightarrow H \hookrightarrow X'$ be a Gelfand triple. Then*

$$\mathbb{E}\left[\|w_m\|^2_{\mathcal{C}_w([0,T];X')}\right] \leq C(T)\mathrm{tr}(Q).$$

In the following we need the fact the $\mathscr{U}$ is a borel set in $\mathscr{X}$ (see [59] and [66, p. 119]) to assure the measurability of the various norms we will encounter. We are now ready to state the following[4]

Theorem 3.2.7. *Let ϕ_0 satisfy*

$$\mathbb{E}\left[\|\phi_0\|^4_H\right] < +\infty. \tag{3.9}$$

Then (3.8) *has a weak statistical solution $\mathbf{P}_m$ concentrated on $\mathscr{U}$ which satisfies*

$$\int_{\mathscr{X}} \|\phi\|^2_{L^2([0,T];V)} \mathbf{P}_m(\mathrm{d}\phi) < C_1\left[1 + \mathbb{E}\left[\|\phi_0\|^2_H\right]\right], \tag{3.10}$$

[4] Proof of Theorem (3.2.7) is a modification of [29, p. 1183-1187, Proof of Theorem 5.2.].

$$\int_{\mathscr{X}} \|\phi\|^2_{L^\infty([0,T];H)} \mathbf{P}_m(\mathrm{d}\phi) < C_2 \left[1 + \mathbb{E}\left[\|\phi_0\|_H^2\right]\right], \tag{3.11}$$

$$\int_{\mathscr{X}} \|\phi\|_{\mathcal{C}^{1/4}([0,T];V')} \mathbf{P}_m(\mathrm{d}\phi) < C_3 \left[1 + \mathbb{E}\left[\|\phi_0\|_H^4\right]\right]. \tag{3.12}$$

Proof. Step 1: Existence of a solution to (3.8). Let $\phi_m(t) = \sum_{j=1}^m c_m^j(t) e_j$. We take the scalar product in H of (3.8) with $e_1, \cdots, e_m$, use the boundary conditions and rewrite (3.8) as a differential stochastic equation in $\mathbb{R}^m$, namely

$$c_m(t) = c_m(0) + \int_0^t \underbrace{-Dc_m - G(c_m)}_{b(c_m)} \mathrm{d}t + \int_0^t \mathrm{d}w_m, \tag{3.13}$$

where $c_m = (c_m^1, \cdots, c_m^m) \in \mathbb{R}^m$ and $w_m = (w_m^1, \cdots, w_m^m)$ is a Wiener process in $\mathbb{R}^m$ with covariance matrix Q_m. Here D is a diagonal matrix and $[G(c_m)]_i = -\int_D f(\phi_m)\Delta e_i = -\int_D (\phi_m^3 - \beta\phi_m)\Delta e_i$. It can be easily seen that G and hence b are locally Lipschitz functions. By using the theory of finite-dimensional stochastic differential equations, we deduce that, for every initial condition $\pi_m \phi_0$, there exists a unique solution $c_m(t)$ of the system (3.8) (see [47, Theorem 3.1]). The solution is defined, in principle, up to some random moment $\zeta_m(\omega)$. We will show that $\zeta_m(\omega) \geq T$ a.s.

Step 2: Estimates. We now look for some proper estimates of ϕ_m. If we apply the Itö formula with the function $F(t, c_m(t)) = |c_m(t)|^{2+2\alpha} = \|\phi_m(t)\|_H^{2+2\alpha}$, for some $\alpha \geq 0$, we get

$$\begin{aligned}
& |c_m(t)|^{2+2\alpha} = |c_m(0)|^{2+2\alpha} + \int_0^t \langle 2(1+\alpha)|c_m(s)|^{2\alpha} c_m(s) I \mathrm{d}w_m(s)\rangle \\
+ & \int_0^t \left\{ \langle 2(1+\alpha)|c_m(s)|^{2\alpha} c_m(s), b(c_m(s))\rangle + \frac{1}{2}\mathrm{tr}(\mathrm{Hes}(|c_m(s)|^{2+2\alpha})Q_m) \right\} \mathrm{d}s \\
= & |c_m(0)|^{2+2\alpha} + \int_0^t \langle 2(1+\alpha)|c_m(s)|^{2\alpha} c_m(s), I \mathrm{d}w_m(s)\rangle \\
+ & \int_0^t \langle 2(1+\alpha)|c_m(s)|^{2\alpha} c_m(s), b(c_m(s))\rangle \mathrm{d}s \\
+ & \int_0^t \left\{ \mathbf{1}_{\alpha>0} 2\alpha(1+\alpha)|c_m(s)|^{2\alpha-2} \langle Q_m c_m(s), c_m(s)\rangle \right\} \mathrm{d}s \\
+ & \int_0^t \left\{ (1+\alpha)|c_m(s)|^{2\alpha} \mathrm{tr}(Q_m) \right\} \mathrm{d}s,
\end{aligned} \tag{3.14}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^m$. We now observe that, thanks to integration by parts, Young inequality and the spectral properties of the family $\{e_i\}_{i\in\mathbb{N}}$ we have

$$\begin{aligned}(\Delta\phi_m(t), f(\phi_m(t)))_H &= -\int_D (3\phi_m^2(t)-\beta)|\nabla\phi_m(t)|^2 \le \beta\int_D |\nabla\phi_m(t)|^2 \\ &\le \beta\|\phi_m(t)\|_H\|\phi_m(t)\|_V \le \frac{\gamma}{2}\|\phi_m(t)\|_V^2 + \frac{\beta^2}{2\gamma}\|\phi_m(t)\|_H^2.\end{aligned}$$

Moreover $(Q_m c_m(t), c_m(t)) \le \|Q_m\||c_m(t)|^2 \le \|Q\|\|\phi_m(t)\|_H^2$. Let us now define

$$\tau_N(\omega) := \begin{cases} \inf\{\tau > 0 : \|\phi_m(\tau,\cdot,\omega)\|_H \ge N\}, \\ +\infty, \quad \text{if } \|\phi_m(\tau,\cdot,\omega)\|_H < N, \quad \forall\tau > 0.\end{cases}$$

Using the previous relations and recalling that the expectation of a stochastic integral is equal to 0 and that $Cz^{2\alpha}+z^{2+2\alpha} \le 1+2Cz^{2+2\alpha}$ if $z \ge 0$, for a suitable constant $C = C(\alpha) > 0$, we obtain from (3.14)

$$\begin{aligned}&\mathbb{E}\left[\|\phi_m(t\wedge\tau_N)\|_H^{2+2\alpha}\right] + \gamma(1+\alpha)\mathbb{E}\left[\int_0^{t\wedge\tau_N} \|\phi_m\|_H^{2\alpha}\|\phi_m\|_V^2\right] \\ \le\ & \mathbb{E}\left[\|\phi_m(0)\|_H^{2+2\alpha}\right] + C\,\mathbb{E}\left[\int_0^{t\wedge\tau_N} \|\phi_m(s)\|_H^{2\alpha}(\|\phi_m(s)\|_H^2 + C_1)\mathrm{d}s\right] \\ \le\ & \mathbb{E}\left[\|\phi_m(0)\|_H^{2+2\alpha}\right] + C\int_0^{t\wedge\tau_N} \left(1+\mathbb{E}\left[\|\phi_m(s)\|_H^{2+2\alpha}\right]\right)\mathrm{d}s. \qquad (3.15)\end{aligned}$$

Let us prove that $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$, the family of the distributions of $\{\phi_m\}_{m\in\mathbb{N}}$, is concentrated on $\mathscr{U}$.
By the Gronwall inequality it follows that

$$\mathbb{E}\left[\|\phi_m(t\wedge\tau_N)\|_H^{2+2\alpha}\right] \le C(T)\left(1+\mathbb{E}\left[\|\phi_m(0)\|_H^{2+2\alpha}\right]\right). \qquad (3.16)$$

The previous inequality implies that $\zeta(\omega) = T$ (a.s.), see [33, p. 352] for a similar discussion. In addition, for every $N\in\mathbb{N}$ there exists a process ϕ_m^N which satisfies (3.4) for all $0 < s < t\wedge\tau_N$. In addition to that, $t\wedge\tau_N \uparrow t$ a.s. and $\phi_m^N(t) \to u_m(t)$ a.s.
Hence, for $\alpha = 0$, we apply Fatou lemma twice and get

$$\begin{aligned}&\mathbb{E}\left[\|\phi_m\|_{L^2([0,T];H)}^2\right] \le \mathbb{E}\left[\int_0^T \liminf_N \int_D (\phi_m^N)^2\right] = \int_0^T \mathbb{E}\left[\liminf_N \|\phi_m^N\|_H^2\right] \\ \le\ & \int_0^T \liminf_N \mathbb{E}\left[\|\phi_m^N\|_H^2\right] \le C_1(T)\left(1+\mathbb{E}\left[\|\phi(0)\|_H^2\right]\right), \qquad (3.17)\end{aligned}$$

and similarly, for $\alpha > 0$, we get

$$\mathbb{E}\left[\|\phi_m\|_{L^{2+2\alpha}([0,T];H)}^{2+2\alpha}\right] \le C(T)\left(1+\mathbb{E}\left[\|\phi_0\|_H^{2+2\alpha}\right]\right). \qquad (3.18)$$

In addition, using the simple estimate $\|\nabla\phi_m\|_H^4 \leq \|\phi\|_H^2\|\phi\|_V^2$, picking $\alpha = 1$, using the monotone convergence theorem in (3.15) and recalling the hypothesis on ϕ_0, we obtain

$$\mathbb{E}\left[\|\nabla\phi_m\|^4_{L^4([0,T];H)}\right] \leq C(T)\left(1 + \mathbb{E}\left[\|\phi_0\|_H^4\right]\right). \tag{3.19}$$

Moreover $\phi_m \in \mathcal{C}([0,T];\mathcal{C}^\infty(\overline{D}))$. By putting $\alpha = 0$, we deduce from (3.14) that

$$\|\phi_m(t)\|_H^2 \leq \|\phi_m(0)\|_H^2 + C\int_0^t \left(1 + \|\phi_m(s)\|_H^2\right) \mathrm{d}s + 2\int_0^t \langle c_m(s), I\mathrm{d}w_m(s)\rangle,$$

which implies

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}\|\phi_m(t)\|_H^2\right] &\leq \mathbb{E}\left[\|\phi_m(0)\|_H^2\right] + C\left(T + \mathbb{E}\left[\|\phi_m\|^2_{L^2([0,T];H)}\right]\right)\\ &+ 2\mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t \langle c_m, I\mathrm{d}w_m\rangle\right].\end{aligned} \tag{3.20}$$

By means of the Doob submartingale inequality we obtain

$$\begin{aligned}\left\{\mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t \langle c_m, I\mathrm{d}w_m\rangle\right]\right\}^2 &\leq \mathbb{E}\left[\sup_{0\leq t\leq T}\left\{\int_0^t \langle c_m, I\mathrm{d}w_m\rangle\right\}^2\right]\\ \leq\ & 4\,\mathbb{E}\left[\left\{\int_0^T \langle c_m, I\mathrm{d}w_m\rangle\right\}^2\right] = 4\,\mathbb{E}\left[\int_0^T \langle Q_m c_m, c_m\rangle\right]\\ \leq\ & 4\|Q\|\mathbb{E}\left[\|\phi_m\|^2_{L^2([0,T];H)}\right].\end{aligned} \tag{3.21}$$

Hence we obtain (3.11) from (3.9), (3.20) and (3.21).
Taking $\alpha = 0$ again, if we do not remove the term with the V-norm of ϕ_m in the estimation of the right hand side of (3.14) and we apply Fatou lemma as we have done in (3.17), we get

$$\begin{aligned}&\mathbb{E}\left[\|\phi_m(t)\|_H^2\right] + \frac{\gamma}{2}\mathbb{E}\left[\int_0^t \|\phi_m(s)\|_V^2\mathrm{d}s\right]\\ \leq\ & C\mathbb{E}\left[\int_0^t \|\phi_m(s)\|_H^2\mathrm{d}s\right] + C_1 t + \mathbb{E}\left[\|\phi_m(0)\|_H^2\right].\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}\left[\|\phi_m\|^2_{L^2([0,T];V)}\right] &= \mathbb{E}\left[\int_0^T \|\phi_m(s)\|_V^2\mathrm{d}s\right] \leq C'\int_0^T \mathbb{E}\left[\|\phi_m(s)\|_H^2\right]\mathrm{d}s\\ \leq\ & +C_1' t + C_2' + C(T)\left[1 + \mathbb{E}\left[\|\phi_m(0)\|_H^2\right]\right] \leq C(T)\left[1 + \mathbb{E}\left[\|\phi(0)\|_H^2\right]\right],\end{aligned}$$

so that $\mathbf{P}_m$ satisfies (3.10). To prove (3.12), we write, for $0 \le t_1 < t_2 \le T$ and for every $v \in V$

$$\begin{aligned} & |\langle \phi_m(t_2) - \phi_m(t_1), v\rangle_{V',V}| \le \left| \int_{t_1}^{t_2} (-\gamma\Delta\phi_m + f(\phi_m), v) \right| \\ + \quad & |\langle w_m(t_2) - w_m(t_1), v\rangle_{V',V}| \le \left\{ \int_{t_1}^{t_2} (\gamma\|\Delta\phi_m\| + \|f(\phi_m)\|) \right. \\ + \quad & \left. \|w_m(t_2) - w_m(t_1)\|_{V'} \right\} \|v\|_V. \end{aligned} \tag{3.22}$$

We now observe that, thanks to the embedding $U \hookrightarrow L^6$, we have

$$\int_{t_1}^{t_2} \|\phi_m^3\|^{4/3} = \int_{t_1}^{t_2} \|\phi_m\|_{L^6}^4 \le C \int_{t_1}^{t_2} \big(\|\nabla\phi_m\|^4 + \|\phi_m\|^4\big),$$

and using Theorem (3.2.6), (3.19) and (3.18) with $\alpha = 1$ we deduce (3.12). Hence $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$ is thus concentrated on $\mathscr{U}$.

Step 3: Final properties. Since $\phi_m(0)$ and $w_m(t)$ are independent random variables, we have

$$\begin{aligned} \mathbb{E}\left[\exp\left\{ i\langle\phi_m(0), \xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} + i\int_0^T \langle \mathscr{R}(\phi_m), v\rangle_{V',V} \right\}\right] \quad &= \quad \hat{\Xi}(\xi)\hat{\mathbf{W}}_m\left(-\frac{\partial v}{\partial t}\right), \\ & \quad\quad \forall \xi \in V_m,\ \forall v \in \mathscr{V}_m, \end{aligned}$$

where

$$\mathscr{V}_m := \{u \in \mathscr{V} : \exists v : u(t) = \pi_m v(t), \text{ for a.e. } t \in [0,T]\}.$$

Thus all assertions in the theorem are proved. □

We now turn to prove the existence of a weak solution for the infinite-dimensional original problem. We use a compactness argument.

Theorem 3.2.8. *The family $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$ of Theorem (3.2.7) is relatively compact in $\mathscr{Z}$.*

Proof. We will check the hypothesis of the Phokhorov's theorem. It is sufficient to show that $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$ is uniformly concentrated on some compact set in $\mathscr{Z}$. Let $\varepsilon > 0$ be arbitrary. Since $\mathscr{U} \overset{c}{\hookrightarrow} \mathscr{Z}$ (see Lemma (3.2.2)), the set $K_\rho(\mathscr{U}) := \{u \in \mathscr{U} : \|u\|_{\mathscr{U}} \le \rho\}$ is relatively compact in $\mathscr{Z}$. Inequalities (3.10)-(3.12) imply that $\int_{\mathscr{Z}} \|u\|_{\mathscr{U}} \mathbf{P}_m(\mathrm{d}u) < C$. Hence, we have

$$\mathbf{P}_m(\mathscr{Z} \setminus K_{\rho(\mathscr{U})}) = \int_{\mathscr{Z}\setminus K_\rho(\mathscr{U})} \mathbf{P}_m(\mathrm{d}u) \le \frac{1}{\rho}\int_{\mathscr{Z}} \|u\|_{\mathscr{U}} \mathbf{P}_m(\mathrm{d}u) \le \frac{C}{\rho} \le \varepsilon,$$

for ρ large enough. Thus there exists a (not relabeled) subsequence $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$ which converges weakly to some probability measure $\mathbf{P}$ on $\mathscr{Z}$. □

We can now state a crucial theorem.

Theorem 3.2.9. *Let the random variable $\phi_0 \in H$ satisfy $\mathbb{E}[\|\phi_0\|_H^4] < +\infty$. Then* (3.4) *has a weak statistical solution $\mathbf{P}$ concentrated on $\mathscr{U}$. Moreover, $\mathbf{P}$ satisfies*

$$\int_{\mathscr{Z}} \|\phi\|^2_{L^2([0,T];V)} \mathbf{P}(d\phi) < C_1 \left[1 + \mathbb{E}\left[\|\phi_0\|_H^2\right]\right], \tag{3.23}$$

$$\int_{\mathscr{Z}} \|\phi\|^2_{L^\infty([0,T];H)} \mathbf{P}(d\phi) < C_2 \left[1 + \mathbb{E}\left[\|\phi_0\|_H^2\right]\right], \tag{3.24}$$

$$\int_{\mathscr{Z}} \|\phi\|^{5/3}_{\mathcal{C}^{2/5}([0,T];V')} \mathbf{P}(d\phi) < C_1 \left[1 + \mathbb{E}\left[\|\phi_0\|_H^3\right]\right], \tag{3.25}$$

and there exists a set $X \subset \mathscr{U}$, X closed in the topology of $\mathscr{Z}$, $X \in \mathcal{B}_{\mathscr{Z}}(\mathscr{U})$ such that $\mathbf{P}(X) = 1$.

Proof. Let $r \in \mathbb{N}$. For every $m \geq r$ it holds from Theorem (3.2.7) that

$$\int_{\mathscr{Z}} \exp\left\{ i\langle \phi_m(0), \xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} + i\int_0^T \langle \mathscr{R}(\phi_m), v\rangle_{V',V} \right\} \mathbf{P}_m(\mathrm{d}\phi) = \hat{\Xi}(\xi)\hat{\mathbf{W}}_m\left(-\frac{\partial v}{\partial t}\right), \quad \forall \xi \in V_r, \forall v \in \mathscr{V}_r.$$

Since the functional $\phi \mapsto \int_0^T \langle \mathscr{R}(\phi(t)), v(t)\rangle_{V',V}$ is continuous on $\mathscr{Z}$ and $\mathbf{P}_m$ converges weakly to $\mathbf{P}$ we have that (3.5) holds for each $\xi \in V_r$, $v \in \mathscr{V}_r$; hence, by a density argument (see [59, p. 830 and Lemma 3.2]), (3.5) holds for every $\xi \in H^\varepsilon$ and $v \in \mathscr{V}$.
To prove inequalities (3.23)-(3.25), we observe that the following statement is true thanks to the Fatou Lemma:
Let $\Phi(u), \Phi_r(u)$, $u \in \mathscr{U}$, $r \in \mathbb{N}$ be nonnegative functionals such that $\Phi_r(u)$ is continuous and bounded on $\mathscr{Z}$, $\Phi_r(u) \to \Phi(u)$ as $r \to +\infty$. If $\int \Phi_r(u)\mathbf{P}_m(du) \leq C$ $\forall m > r$ and $\mathbf{P}_m \to \mathbf{P}$ weakly on $\mathscr{Z}$, then $\int \Phi(u)\mathbf{P}(du) \leq C$.
For $\Phi(u) = \|u\|^2_{L^2([0,T];V)}$, we put $\Phi_r(u) := \min\{\Phi(\pi_r u); r\}$ and the hypothesis of the previous statement are met. Similiarly, for $\Phi(u) = \|u\|^2_{L^\infty([0,T];H)}$, we put once again $\Phi_r(u) := \min\{\Phi(\pi_r u); r\}$. Finally, if $\Phi(u) = \|u\|^{5/3}_{\mathcal{C}^{2/5}([0,T];V')}$, we put

$$\Phi_r(u) := \min\left\{ \sup_{\substack{1/r<|t_1-t_2|<1 \\ 0\leq t_1<t_2\leq T}} \left(\frac{\|u(t_1)-u(t_2)\|_{V'}}{|t_1-t_2|^{2/5}}\right); r \right\}^{5/3},$$

and the estimates (3.23)-(3.25) follow, hence $\mathbf{P}(\mathscr{U}) = 1$. We omit the proof of the existence of X. $\square$

We now turn to build the preliminary steps which will lead to prove the existence of a strong solution to Problem (3.4). We need two ingredients:

(i) the existence of a weak solution $\mathbf{P}$ on an appropriate space.

(ii) the injectivity of the operator $\mathscr{R}(\phi) := \partial\phi/\partial t + \mathscr{A}(\phi)$.

To prove (ii) we need to reinforce the assumption on the initial condition u_0. We will assume that

$$\phi_0 \in U \qquad \text{(a.s.)}.$$

We introduce the following Banach space

$$\mathscr{U}_w := L^2([0,T];V) \cap L^\infty([0,T];U) \cap \mathcal{C}_w([0,T];U'),$$

and we prove the following

Theorem 3.2.10. *Let* $\phi_0 \in U$ *satisfy* $\mathbb{E}\left[\|\phi_0\|_H^4\right] < +\infty$ *and* $\mathbb{E}\left[E[\phi_0]\right] < +\infty$, *where* E *is the free energy introduced in* (1.2). *Then the weak solution of* (3.8) *is concentrated on* $\mathscr{U}_w$ *and the following estimates hold uniformly*[5] *on* m*:*

$$\int_{\mathscr{X}} \|\phi\|^2_{L^2([0,T];V)} \mathbf{P}_m(\mathrm{d}\phi) \leq C_1 \mathbb{E}\left[\|\phi_0\|_H^2\right], \tag{3.26}$$

$$\int_{\mathscr{X}} \|\phi\|^2_{L^\infty([0,T];U)} \mathbf{P}_m(\mathrm{d}\phi) \leq C_2 \left[1 + \mathbb{E}\left[E[\phi_0]\right]\right], \tag{3.27}$$

$$\int_{\mathscr{X}} \|\phi\|^2_{\mathcal{C}_w([0,T];U')} \mathbf{P}_m(\mathrm{d}\phi) \leq C_3 \left[1 + \mathbb{E}\left[E[\phi_0]\right]\right]. \tag{3.28}$$

Proof. The existence of $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$ is proved in Theorem (3.2.7), along with (3.26). Let us prove (3.27). Let us denote

$$z_m := -\gamma\Delta\phi_m + \pi_m\left[f(\phi_m)\right].$$

If we apply Itô Rule[6] to the function $F(t, \phi_m(t)) := E[\phi_m(t)]$, we get

$$\begin{aligned}
& \frac{\gamma}{2}\int_D |\nabla\phi_m(t)|^2 + \frac{1}{4}\int_D \phi_m^4(t) + \int_0^t\!\!\int_D |\nabla z_m|^2 \\
= \;& E\left[\phi_m(0)\right] + \left(\frac{\gamma}{2} - \beta\right)\mathrm{tr}(Q_m)t + \frac{\beta}{2}\int_D \phi_m^2(t) \\
+ \;& \frac{3}{2}\int_0^t\!\!\int_D \phi_m^2 \sum_{i,j=1}^m Q_{ij}e_ie_j + \int_0^t\!\!\int_D z_m \mathrm{d}w_m.
\end{aligned} \tag{3.29}$$

[5]Unlike the constants in Theorem (3.2.7), the following constants C_1, C_2, C_3 might depend on u_0. This is of absolutely no harm because we are not interested in the dependence of the solution on the initial data. The same happens in Theorem (3.2.14).

[6]from now on, w_m resume its meaning given in (3.2) and Q_m indicates its covariance operator whose matrix with respect to the basis $\{e_i\}$ is indicated by $[Q_{ij}]_{i,j}$. Note that, since F is defined in U, we must apply Itô Formula computing the Fréchet derivatives with respect to the U-norm, and similarly for the trace.

Let us estimate the right hand side in (3.29). Since Q_m is definite positive, we have

$$\sup_{0\le t\le T}\int_0^t\!\!\int_D \phi_m^2\sum_{i,j=1}^m Q_{ij}e_ie_j \le \int_0^T \sup_{x\in D}\phi_m^2\sum_{i,j=1}^m Q_{ij}\int_D e_ie_j$$
$$= \|\phi_m\|^2_{L^2([0,T];L^\infty)}\sum_{i=1}^m Q_{ii} \le \|\phi_m\|^2_{L^2([0,T];V)}\mathrm{tr}(Q). \tag{3.30}$$

In order to estimate the last term in (3.29), we act as in the proof of Theorem (3.2.7), computation (3.21), and we get

$$\mathbb{E}\left[\sup_{0\le t\le T}\int_0^t\!\!\int_D z_m^0 \mathrm{d}w_m\right] \le 2\|Q\|^{1/2}\left\{\mathbb{E}\left[\int_0^T \|z_m^0\|_H^2\right]\right\}^{1/2},$$

where $z_m^0 := z_m - |D|^{-1}\int_D z_m$. Since $\|z_m^0\|_H^2 \le \|z_m^0\|_U^2 \le C_P\|\nabla z_m\|_H^2$, we use Young inequality and we get

$$\mathbb{E}\left[\sup_{0\le t\le T}\int_0^t\!\!\int_D z_m \mathrm{d}w_m\right] \le C\left\{\mathbb{E}\left[\int_0^T \|\nabla z_m\|_H^2\right]\right\}^{1/2} \le \frac{1}{2}\mathbb{E}\left[\int_0^T \|\nabla z_m\|_H^2\right] + C'. \tag{3.31}$$

From (3.29)-(3.31) it follows that

$$\frac{\gamma}{2}\mathbb{E}\left[\sup_{0\le t\le T}\|\nabla\phi_m(t)\|_H^2\right] + \frac{1}{2}\mathbb{E}\left[\int_0^T \|\nabla z_m\|_H^2\right]$$
$$\le \left|\frac{\gamma}{2}-\beta\right|\mathrm{tr}(Q)T + \mathbb{E}\left[E[\phi_m(0)]\right] + \frac{\beta}{2}\mathbb{E}\left[\sup_{0\le t\le T}\|\phi_m(t)\|_H^2\right]$$
$$+ \frac{3}{2}\mathrm{tr}(Q)\|\phi_m\|^2_{L^2([0,T];V)} + C''.$$

Hence, by means of (3.10) and (3.11), we obtain (3.27).
To prove (3.28) we introduce $v_m = \phi_m - w_m$ and arguing as in the proof of Theorem (3.2.7) we get

$$\mathbb{E}\left[\|v_m\|^2_{\mathcal{C}^{1/2}([0,T];U')}\right] \le \mathbb{E}\left[\int_0^T \|(v_m)_t\|^2_{U'}\right] \le \mathbb{E}\left[\int_0^T \|\nabla z_m\|_H^2\right] \le C\left[1+\mathbb{E}\left[E[\phi_0]\right]\right],$$

because of the previous inequalities. Since $\mathcal{C}^{1/2}([0,T];U') \hookrightarrow \mathcal{C}_w([0,T];U')$, we deduce (3.28) as in Theorem (3.2.7). We may now proceed as in the last part of Theorem (3.2.7). For $0<\kappa<1/2$ it holds that $\mathcal{C}_w([0,T];U') \hookrightarrow \mathcal{C}^\kappa([0,T];V')$ and so $\mathscr{U}_w \hookrightarrow \mathscr{U}$. It follows that $\mathscr{U}_w \overset{c}{\hookrightarrow} \mathscr{Z}$ and that $\mathscr{R}:\mathscr{U}_w \to \mathscr{V}'$. Notice that $\mathscr{U}_w$ is a Borel set of $\mathscr{Z}$. By a compactness argument we deduce that there exists a (not relabeled) subsequence $\mathbf{P}_m$ weakly convergent to a probability measure $\mathbf{P}$. We deduce the remainig the properties of $\mathbf{P}$ as in Theorem (3.2.9). $\square$

We can summarize what we have obtained in the following

Theorem 3.2.11. *Let the random variable* $\phi_0 \in U$ *satisfy* $\mathbb{E}\left[\|\phi_0\|_H^4\right] < +\infty$ *and* $\mathbb{E}\left[E[\phi_0]\right)] < +\infty$. *Then Cahn-Hilliard equation* (3.4) *has a weak statistical solution* $\mathbf{P}$ *concentrated on* $\mathscr{U}_w$. *Moreover,* $\mathbf{P}$ *satisfies*

$$\int_{\mathscr{Z}} \|\phi\|_{\mathscr{U}_w}^2 \mathbf{P}(d\phi) \leq C\left[1 + \mathbb{E}\left[E[\phi_0]\right]\right], \tag{3.32}$$

and there exists a set $X \subset \mathscr{U}_w$, X *closed in the topology of* $\mathscr{Z}$, $X \in \mathcal{B}_{\mathscr{Z}}(\mathscr{U}_w)$ *such that* $\mathbf{P}(X) = 1$.

Thus we have achieved item (i), pag. 49. We now state a uniqueness theorem which guarantees item (ii), pag. 49. We omit the proof, which is technical but uninteresting. The attentive reader may consult [29, pages 1195-1198, Theorem 8.1.].

Theorem 3.2.12. *Let* $d \in \{2; 3\}$ *and* ϕ_1, $\phi_2 \in \mathscr{U}_w$ *satisfy* (3.4) *for the same* $u_0 \in U$ *and* $\partial w/\partial t \in \mathscr{V}'$. *Then* $\phi_1(t) = \phi_2(t)$ *in* U', *a.e. on* $[0, T]$.

Therefore we have an immediate

Corollary 3.2.13. *The operator* $\mathscr{D} : \mathscr{U}_w \to H^{-\varepsilon} \times \mathscr{V}'$ *is injective.*

We are now ready to introduce our last

Theorem 3.2.14. *Let* $d \in \{2; 3\}$, $\phi_0 \in U$ *(a.s.) and* $\mathbb{E}\left[\|\phi_0\|_H^4\right] < +\infty$, $\mathbb{E}\left[E[\phi_0]\right] < +\infty$. *Then the Cahn-Hilliard stochastic equation has a* ***strong*** *solution* ϕ, *with trajectories in* $\mathscr{U}_w$. *In addition* ϕ *satisfies*

$$\mathbb{E}\left[\|\phi\|_{\mathscr{U}_w}^2\right] \leq C\left[1 + \mathbb{E}\left[E[\phi_0]\right]\right]. \tag{3.33}$$

Furthermore, such solution is unique (in the sense that two strong solutions coincide a.s.) and its distribution coincides with the weak statistical solution $\mathbf{P}$ *of the Cahn-Hilliard equation with the same initial data.*

Proof. Let X be the set constructed in Theorem (3.2.11), $\mathbf{P}(X) = 1$. Since $\mathscr{Z}$ is a Polish space, for each $m \in \mathbb{N}$ there exists a compact set $K_m \subset \mathscr{Z}$, $\mathbf{P}(K_m) > 1 - 1/m$. Let $\tilde{K}_m := K_m \cap X \in \mathcal{B}_{\mathscr{Z}}(\mathscr{U}_w)$ and $\mathscr{U}_0 := \cup_{m=1}^{\infty} \tilde{K}_m$. Then $\mathscr{U}_0 \in \mathcal{B}_{\mathscr{Z}}(\mathscr{U}_w)$ and $\mathbf{P}(\mathscr{U}_0) = 1$. $K_m \cap X$ is a compact set in $\mathscr{Z}$ and $\mathscr{D} : \mathscr{Z} \to \mathscr{Y} := \{H^{-\varepsilon}, \mathscr{V}'\}$ is continuous. Hence $\mathscr{D}(K_m \cap X)$ is compact in $\mathscr{Y}$ and $\mathscr{F} := \mathscr{D}(\mathscr{U}_0) = \cup_{m=1}^{\infty} \mathscr{D}(K_m \cap X) \in \mathcal{B}(\mathscr{Y})$. We have $\mathbf{P}(\mathscr{D}^{-1}(\mathscr{F})) \geq \mathbf{P}(\mathscr{U}_0) = 1$. Denote $\mathscr{U}_1 := (\mathscr{D}^{-1}(\mathscr{F})) \cap \mathscr{U}_0 \in \mathcal{B}_{\mathscr{Z}}(\mathscr{U}_w)$. It follows that $\mathscr{D}(\mathscr{U}_1) = \mathscr{F}$. Let $\mathscr{D}_1 := \mathscr{D}|_{\mathscr{U}_1}$. By Corollary (3.2.13), $\mathscr{D}_1$ is bijective. Furthermore $\mathscr{D}_1^{-1} : \mathscr{F} \to \mathscr{U}_1$

is a measurable mapping from $(\mathscr{F}, \mathcal{B}_{\mathscr{Y}}(\mathscr{F}))$ to $(\mathscr{U}_1, \mathcal{B}_{\mathscr{X}}(\mathscr{U}_1))$ (in analogy with [33, p. 369]). Denote

$$\Omega_1 := \left\{ \omega \in \Omega : \left\{ \phi_0(\omega), \frac{\partial w}{\partial t}(\omega) \right\} \in \mathscr{F} \right\}.$$

Because of the measurability of u_0 and w_t, we have $\Omega_1 \in \mathcal{F}$ and $\mathbf{m}(\Omega_1) = \mathbf{P}(\mathscr{D}^{-1}(\mathscr{F})) = 1$. We define

$$\phi(\omega) := \begin{cases} \mathscr{D}_1^{-1}\left(\left\{ \phi_0(\omega), \dfrac{\partial w}{\partial t}(\omega) \right\} \right), & \text{if } \omega \in \Omega_1, \\ 0, & \text{otherwise.} \end{cases}$$

The injectivity of the operator $\mathscr{D} : \mathscr{U}_w \to H^{-\varepsilon} \times \mathscr{V}'$ implies the uniqueness of a strong solution. The measurability of $\mathscr{D}_1^{-1}$ and the properties of $\mathscr{U}_1$ imply that ϕ is measurable from $(\Omega, \mathcal{F})$ to $(\mathscr{U}_w, \mathcal{B}_{\mathscr{X}}(\mathscr{U}_w))$, hence the requirements of Definition (3.2.5) are satisfied. It is also obvious that in this context the weak statistical solution of the Cahn-Hilliard is unique. It is also easy to see that the distribution of ϕ is a weak statistical solution, hence it coincides with $\mathbf{P}$ and consequentially (3.33) holds. □

3.2.2 Pathwise deterministic approach

Reference [24] certainly is among the basic and most important articles in this field. In this work G. Da Prato and A. Debussche prove existence and uniqueness of a solution of (1.20) in the most "simple" setting: we have the following ingredients[7]

(a) The problem being analysed is

$$\begin{cases} \mathrm{d}\phi + (\Delta^2\phi - \Delta f(\phi))\,\mathrm{d}t = \mathrm{d}w, \\ u(0) = u_0. \end{cases} \tag{3.34}$$

(b) The domain $D \subset \mathbb{R}^d$ is $D = \times_{i=1}^{d}[0, L_i]$, $L_i > 0$ for $i = 1, \cdots, d$. Here $d \in \{2; 3\}$.

(c) The boundary conditions are

$$\phi_\nu = (\Delta\phi)_\nu = 0 \quad \text{on } \Gamma. \tag{3.35}$$

[7] the reader must be warned: even if these preliminaries provide strong hypothesis, the proof of uniqueness and existence is anyway complicated.

(d) The free energy derivative f has a simple polynomial form, namely

$$f(x) = \sum_{k=1}^{2p-1} a_k x^k, \quad a_{2p-1} > 0, \quad p \in \mathbb{N}.$$

(e) Let[8] $\tilde{H} = \{u \in L^2 : m(u) := \frac{1}{|D|} \int_D u = 0\}$. We define the linear unbounded operator with dense domain

$$A : \mathcal{D}(A) = V \subset H \to \tilde{H} : u \mapsto Au := -\Delta u,$$

where $\mathcal{D}(A) := V$. The operator A is self-adjoint, positive and has compact resolvent. It possesses a basis of eigenvectors[9] $\{e_i\}_{i\in\mathbb{N}}$ which is orthonormal in H. The respective sequence of eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ diverges

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to +\infty.$$

Later we will also need the Hilbert spaces defined by means of the real powers A^s of the operator[10] A (see [71]). For $s \geq 0$, the domain of $A^{s/2}$ is

$$V_s = \left\{ u = \sum_{j=0}^{\infty} u_j e_j : \sum_{j=1}^{\infty} \lambda_j^s u_j^2 < +\infty \right\}.$$

For $s < 0$, the domain of $A^{s/2}$ is the completion of H with respect to the norm

$$\left\{ \sum_{j=1}^{\infty} \lambda_j^s (\cdot, u_j)^2 \right\}^{1/2}.$$

We may define a seminorm $|\cdot|_s$, an inner product $(\cdot,\cdot)_s$ and a norm $\|\cdot\|_s$ in V_s in the following way

$$|u|_s := \|A^{s/2}u\|, \quad (u,v)_s := (A^{s/2}u, A^{s/2}v), \quad \|u\|_s := \left(|u|_s^2 + m^2(u)\right)^{1/2}.$$

(f) w is a Q-Wiener process with values in H, such that $f_i = e_i, \forall i \in \mathbb{N}$ and $\mathrm{Tr}[AQ] < +\infty$.

---

[8]here $|D|$ denotes the Lebesgue measure of D.

[9]thanks to the simple structure of D, we recall that the eigenvalues of A form a trigonometric basis of H, namely

$$e_0(x) = |D|^{-1/2}, \quad e_i(x) \propto \cos(i_1 x_1/L_1)\cos(i_2 x_2/L_2)\cos(i_3 x_3/L_3), \quad i \in \mathbb{N}^3,$$

They also are a Schauder basis of L^p, where $p \in (1, +\infty)$.

[10]the reader must notice that in general A^s and Δ^s are not the same operator on H^s when $s \in \mathbb{N}$. In fact, A^s is simmetric while Δ^s is not.

After these preliminary considerations, we recall some properties of the solution of the auxiliary problem

$$\begin{cases} \mathrm{d}\phi + A^2\phi\,\mathrm{d}t = \mathrm{d}w, \\ u(0) = 0. \end{cases} \tag{3.36}$$

We have the following theorem (see [25]).

Theorem 3.2.15. *The linear equation* (3.36) *has a unique solution, called* ***stochastic convolution****, given by*

$$W_A(t) = \int_0^t e^{-(t-s)A^2}\,\mathrm{d}w(s). \tag{3.37}$$

The process W_A has many regularity properties[11], which we are going to summarize in the following Lemma (see [24, Propositions 1.1., 1.2., 1.3.]).

Lemma 3.2.16. *Let W_A be defined as in* (3.37). *Then*

(i) W_A has a version which is α-Hölder continuous with respect to $(t,x) \in [0,+\infty) \times D$ for any $\alpha \in [0,1/8)$.

(ii) ∇W_A has a version which is α-Hölder continuous with respect to $(t,x) \in [0,+\infty) \times D$ for any $\alpha \in [0,1/2)$.

(iii) Let P_m be the orthogonal projection operator on $\mathrm{Span}\{e_0, \cdots, e_m\}$. *If we define $W_A^m := P_m W_A$ then, for any $r \in [1,+\infty)$ we have*

$$\lim_{m\to\infty} \mathbb{E}\left[\|W_A - W_A^m\|^r_{L^r([0,T]\times D)}\right] = 0,$$

$$\lim_{m\to\infty} \mathbb{E}\left[\|\nabla W_A - \nabla W_A^m\|^r_{L^r([0,T]\times D)}\right] = 0.$$

Remark 3.2.17. Item (i) of Lemma (3.2.16) is a slight modification of [24, Proposition 1.1].

Remark 3.2.18. We have introduced the random process $W_A(t)$ and its properties for the following reason: the solve problem (3.34), it is preferable to write an equivalent set of infinitely many deterministic differential equations[12]; in order to do so, it is crucial to introduce the translated unknown process

$$v(t) = \phi(t) - W_A(t),$$

[11] the reader is referred to [25] for the definition and the properties of W_A.

[12] in Theorem (3.2.20) we will solve a differential equation for each $\omega \in B$, $\mathbf{m}(B) = 1$.

where ϕ is a solution of (3.34). It can been easily seen that $v(t)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} + A^2 v + Af(v + W_A) = 0, \\ v(0) = u_0. \end{cases} \tag{3.38}$$

for $\omega \in \Omega$ (a.s.). Expression (3.38) may seem a litte weird if seen as a stochastic differential equation, due to the presence of an *actual* time derivative of the unknown v. However the definition of v contains W_A in a way such that the Itö integral in (3.34) is "erased".

In order to prove existence and uniqueness of the solution of (3.38), and thus of (3.34), we need the following auxiliary

Lemma 3.2.19. *Let u be a function in $L^2([0,T];V_1) \cap L^{2p}([0,T] \times D)$ such that*

$$A^{-1}\frac{\mathrm{d}(u - m(u))}{\mathrm{d}t} \in L^2([0,T], V_{-1}) + L^{2p/(2p-1)}([0,T] \times D),$$

and such that $t \mapsto m(u(t))$ is continuous. Then $u \in \mathcal{C}([0,T], V_{-1})$ and the following equality holds

$$\left(A^{-1}\frac{\mathrm{d}(u - m(u))}{\mathrm{d}t}, u - m(u)\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_{-1}^2.$$

We are now ready to state and prove the main uniqueness and existence theorem.

Theorem 3.2.20. *Let* $\mathrm{Tr}[AQ] < \infty$*. If u_0 is $\mathcal{F}_0$-measurable and belongs to H $\mathbf{m}$-a.s., then* (3.34) *admits a unique solution $u \in \mathcal{C}([0,T];V_{-1}) \cap L^\infty([0,T];H)$ $\mathbf{m}$-a.s. Morever $t \mapsto u(t)$ is continuous with respect to t with values in H $\mathbf{m}$-a.s.*

Proof. Step 1: Uniqueness. As we have anticipated in Remark (3.2.18), we consider the deterministic problems

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + A^2 u + Af(u + g) = 0, \\ u(0) = u_0, \end{cases} \qquad \begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} + A^2 v + Af(v + h) = 0, \\ v(0) = v_0, \end{cases}$$

where $u, v \in L^2([0,T];V_1) \cap L^{2p}([0,T] \times D)$, $u_0, v_0 \in V_{-1}$, $g, h \in L^{2p}([0,T] \times D)$ and $\partial g/\partial \nu = \partial h/\partial \nu = 0$ on Γ. Let $z = u - v$. Then z satisfies

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} + A^2 z + A(f(u + g) - f(v + h)) = 0, \\ z(0) = u_0 - v_0. \end{cases}$$

Thanks to the homogeneous Neumann's boundary conditions, we deduce $m(z(t)) = m(u_0 - v_0)$. Moreover, recalling hypothesis (d) at page 53, since $z \in L^2([0,T], V_1)$, we have that

$$A^{-1}\frac{\mathrm{d}z}{\mathrm{d}t} = \underbrace{-Az}_{\in L^2([0,T];V_{-1})} - \underbrace{(f(u+g) - f(v+h))}_{\in L^{2p/(2p-1)}([0,T]\times D)}$$

and therefore we may take the H-inner product with $z - m(z)$ in the last equality, apply Lemma (3.2.19) and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z|_{-1}^2 + |z|_1^2 + (f(u+g) - f(v+h), z - m(z)) = 0. \tag{3.39}$$

We act on the last scalar product as follows

$$\begin{aligned}(f(u+g) - f(v+h), z - m(z)) &= \int_D (f(u+g) - f(v+h))(z + g - h)\mathrm{d}x \\ &- \int_D (f(u+g) - f(v+h))(m(z) + g - h)\mathrm{d}x.\end{aligned}$$

Since f' has even degree, there is $C_1 > 0$ such that $f'(x) \geq -C_1$, $\forall x \in \mathbb{R}$. Moreover, by means of Lagrange theorem, Hölder inequality and the elementary inequality

$$\left(\sum_{i=1}^m y_i\right)^2 \leq m \sum_{i=1}^m y_i^2, \qquad \forall y = (y_1, \cdots, y_m) \in \mathbb{R}^m,$$

we deduce

$$\begin{aligned}\int_D (f(u+g) - f(v+h))(z+g-h)\mathrm{d}x &\geq -C_1 \int_D |z+g-h|^2 \mathrm{d}x \\ &\geq -3C_1 \left(|z|_0^2 + |D| m^2(z) + |G|^{(p-1)/p} \|g-h\|_{L^{2p}}^2\right).\end{aligned} \tag{3.40}$$

We may also find $C_2 > 0$ such that $f(x) \leq 2a_{2p-1}|x|^{2p-1} + C_2$. Therefore

$$\begin{aligned}&\int_D (f(u+g) - f(v+h))(m(z) + g - h)\mathrm{d}x \\ \leq\ & 2a_{2p-1} \int_D (|u+g|^{2p-1} + |v+h|^{2p-1})(|g-h| + |m(z)|)\mathrm{d}x \\ +\ & C_2 \int_D (|g-h| + |m(z)|)\mathrm{d}x \\ \leq\ & 2a_{2p-1}\left(\|u+g\|_{L^{2p}}^{2p-1} + \|v+h\|_{L^{2p}}^{2p-1}\right)\left(\|g-h\|_{L^{2p}} + |D|^{1/2}|m(z)|\right) \\ +\ & C_2(|D||m(z)| + |D|^{(2p-1)/2p}\|g-h\|_{L^{2p}}).\end{aligned} \tag{3.41}$$

Putting together (3.39), (3.40), (3.41) we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z|_{-1}^2 + |z|_1^2 \le 3C_1|z|_0^2 + C_3 m^2(z)\\ + \quad & C_4\left(\|u\|_{L^{2p}}^{2p-1} + \|v\|_{L^{2p}}^{2p-1} + \|g\|_{L^{2p}}^{2p-1} + \|h\|_{L^{2p}}^{2p-1} + 1\right)\\ \times \quad & \left(\|g-h\|_{L^{2p}} + |m(z)|\right),\end{aligned}$$

where C_3, C_4 depend on D, p, f. Using a simple interpolatory property of the $\{V_s\}$ spaces and Young's inequality we obtain

$$|z|_0^2 \le |z|_{-1}|z|_1 \le \frac{1}{3C_1}|z|_1^2 + \frac{3C_1}{4}|z|_{-1}^2,$$

and we deduce

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}|z|_{-1}^2 \le \frac{3C_1}{2}|z|_{-1}^2 + 2C_3 m^2(z)\\ + \quad & 2C_4\left(\|u\|_{L^{2p}}^{2p-1} + \|v\|_{L^{2p}}^{2p-1} + \|g\|_{L^{2p}}^{2p-1} + \|h\|_{L^{2p}}^{2p-1} + 1\right)\\ \times \quad & \left(\|g-h\|_{L^{2p}} + |m(z)|\right).\end{aligned}$$

By means of Gronwall Lemma and Hölder inequality we finally get

$$\begin{aligned}&|z(t)|_{-1}^2 \le e^{\frac{3C_1}{2}t}\left(|z(0)|_{-1}^2 + \frac{4C_3}{3C_1}m^2(z(0))\right)\\ + \quad & 2C_4\Big(\|u\|_{L^{2p}([0,T]\times D)}^{2p-1} + \|v\|_{L^{2p}([0,T]\times D))}^{2p-1} + \|g\|_{L^{2p}([0,T]\times D))}^{2p-1}\\ + \quad & \|h\|_{L^{2p}([0,T]\times D))}^{2p-1} + T^{(2p-1)/2p}\Big)\\ \times \quad & (\|g-h\|_{L^{2p}([0,T]\times D))} + T^{1/2p}|m(z(0))|).\end{aligned} \tag{3.42}$$

$W_A(t) \in L^{2p}([0,T]\times D)$ $\mathbf{m}$-a.s. thanks to Lemma (3.2.16). Hence, if we take $g = h = W_A(t)$ and $v_0 = u_0$, the uniqueness of the solution of (3.34) follows from (3.42).

Step 2: Galerkin approximation, estimates in V_{-1} and passage to the limit. Since u_0 belongs to H for every $\omega \in B$, $\mathbf{m}(B) = 1$, u_0 also belongs to V_{-1} for every $\omega \in B$. We will now construct a deterministic solution of (3.38) for almost every ω by means of a Galerkin method.
Let $\omega \in B$ be fixed. Let P_m and W_A^m be defined as in Lemma (3.2.16). For each $m \in \mathbb{N}$ we consider the Galerkin approximation

$$\begin{cases}\dfrac{\mathrm{d}v_m}{\mathrm{d}t} + A^2 v_m + P_m A f(v_m + W_A^m) = 0,\\ v_m(0) = P_m u_0,\end{cases} \tag{3.43}$$

where v_m takes values in Span$\{e_0, \cdots, e_m\}$. Since P_m is a finite-rank operator, it can be shown that (3.43) possesses a unique $\mathcal{F}$-measurable solution defined on $[0, +\infty)$. If we take the V_{-1}-semiscalar product of (3.43) with v_m, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_m|_{-1}^2 + |v_m|_1^2 + (P_m f(v_m + W_A^m), v_m - m(v_m)) = 0. \tag{3.44}$$

Since P_m is self-adjoint, we have

$$\begin{aligned} & (P_m f(v_m + W_A^m), v_m - m(v_m)) = (f(v_m + W_A^m), v_m - m(v_m)) \\ = \; & (f(v_m + W_A^m), v_m + W_A^m) - (f(v_m + W_A^m), m(v_m) + W_A^m). \end{aligned}$$

Take $C_5, C_6 > 0$ such that

$$xf(x) \geq \frac{1}{2}a_{2p-1}x^{2p} - \frac{C_5}{|G|}, \qquad |f(x)| \leq 2a_{2p-1}x^{2p-1} + C_6, \qquad \forall x \in \mathbb{R}.$$

We have $(f(v_m + W_A^m), v_m + W_A^m) \geq \frac{1}{2}a_{2p-1}\|v_m + W_A^m\|_{L^{2p}}^{2p} - C_5$ and, by Hölder's inequality

$$\begin{aligned} & (f(v_m + W_A^m), m(v_m) + W_A^m) \\ \leq \; & 2a_{2p-1}\int_D |v_m + W_A^m|^{2p-1}(|m(v_m)| + |W_A^m|)\mathrm{d}x + C_6\int_D (|m(v_m)| + |W_A^m|)\mathrm{d}x \\ \leq \; & 2a_{2p-1}\|v_m + W_A^m\|_{L^{2p}}^{2p-1}\left(|D|^{1/2p}|m(v_m)| + \|W_A^m\|_{L^{2p}}\right) \\ + \; & C_6\left(|D||m(v_m)| + G^{(2p-1)/2p}\|W_A^m\|_{L^{2p}}\right) \\ \leq \; & \frac{1}{4}a_{2p-1}\|v_m + W_A^m\|_{L^{2p}}^{2p} + C_7\left(|m(v_m)|^{2p} + \|W_A^m\|_{L^{2p}}^{2p} + 1\right), \end{aligned}$$

where C_7 depends again on f, G, p. We deduce from (3.44) and the previous computations that there exists $C_8 > 0$ such that

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_m|_{-1}^2 + |v_m|_1^2 + \frac{1}{4}a_{2p-1}\|v_m + W_A^m\|_{L^{2p}}^{2p} \\ & \leq C_8 + C_7\left(|m(v_m)|^{2p} + \|W_A^m\|_{L^{2p}}^{2p}\right). \end{aligned} \tag{3.45}$$

By taking the H-scalar product of (3.43) with e_0 we get that $m(v_m(t)) = m(u_0), \forall m \in \mathbb{N}, \forall t \in [0, T]$. Using this fact, Lemma (3.2.16) and integrating (3.45) with respect to time we deduce that $\{v_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty([0, T]; V_{-1})$, $L^{2p}([0, T] \times D)$, $L^2([0, T]; V_1)$. These estimates imply that $\{Av_m\}_{m\in\mathbb{N}}$ is bounded in $L^2([0, T]; V_{-1})$ and $\{f(v_m + W_A^m)\}_{m\in\mathbb{N}}$ is bounded in $L^{2p/(2p-1)}([0, T] \times D)$. Since $L^2([0, T], V_{-1})$ and $L^{2p/(2p-1)}([0, T] \times D)$ are embedded in $L^{2p/(2p-1)}([0, T]; V_{-2})$, we deduce that $\{\mathrm{d}v_m/\mathrm{d}t\}_{m\in\mathbb{N}}$ is bounded in $L^{2p/(2p-1)}([0, T]; V_{-4})$. Since

$$V_1 \overset{c}{\hookrightarrow} V_0 \hookrightarrow V_{-4},$$

the Aubin compactness theorem implies that there exists a (not relabeled) subsequence $\{v_m\}_{m\in\mathbb{N}}$ such that $v_m \to v$ in $L^2([0,T];V_0)$, where $v \in L^2([0,T],V_1) \cap L^{2p}([0,T]\times D)$. We notice that $\mathrm{d}v_m/\mathrm{d}t \rightharpoonup \mathrm{d}v/\mathrm{d}t$ in $L^2([0,T];V_{-4})$. Moreover, by taking another (not relabeled) subsequence $\{v_{m_k}\}_{k\in\mathbb{N}}$ it is easy to see from what we have said that $f(v_m + W_A^m) \rightharpoonup f(v+W_A)$ in $L^{2p/(2p-1)}([0,T]\times D)$.

Step 3: Solution to (3.38). We show that v is a solution to (3.38) in the sense of the distributions[13]. Let $\varphi \in \mathcal{C}_0^\infty(D)$. Since $\Delta\varphi \in \mathcal{C}_0^\infty(D) \subset L^{2p}(D)$, we have that $P_m\Delta\varphi \to \Delta\varphi$ in L^{2p} and that φ may be written as $\varphi = A^{-4}\xi + \varphi_0$, where $\xi \in \mathcal{C}_0^\infty(D)$ and φ_0 is costant. Since P_m is self-adjoint and $m\,(\mathrm{d}v_m/\mathrm{d}t) = 0$, we obtain $m\,(\mathrm{d}v/\mathrm{d}t) = 0$ thanks to the weak convergence of the sequence $\{\mathrm{d}v_m/\mathrm{d}t\}_{m\in\mathbb{N}}$. We can integrate (3.43) on $[0,t]\times D$ for every $t\in[0,T]$ and get[14]

$$\begin{aligned}
0 &= \int_0^t\!\!\int_D \frac{\mathrm{d}v_m}{\mathrm{d}t}\varphi + \int_0^t\!\!\int_D A^2 v_m\varphi - \int_0^t\!\!\int_D AP_m f(v_m+W_A^m)\varphi \\
&= \int_0^t\!\!\int_D A^{-2}\frac{\mathrm{d}v_m}{\mathrm{d}t}A^{-2}\xi + \int_0^t\!\!\int_D v_m\Delta^2\varphi - \int_0^t\!\!\int_D P_m f(v_m+W_A^m)\Delta\varphi \\
&= \int_0^t\!\!\int_D A^{-2}\frac{\mathrm{d}v_m}{\mathrm{d}t}A^{-2}\xi + \int_0^t\!\!\int_D v_m\Delta^2\varphi - \int_0^t\!\!\int_D f(v_m+W_A^m)P_m\Delta\varphi \\
&\to \int_0^t\!\!\int_D \frac{\mathrm{d}v}{\mathrm{d}t}(\varphi-\varphi_0) + \int_0^t\!\!\int_D v\Delta^2\varphi - \int_0^t\!\!\int_D f(v+W_A)\Delta\varphi \\
&= \int_0^t\!\!\int_D \frac{\mathrm{d}v}{\mathrm{d}t}\varphi + \int_0^t\!\!\int_D \Delta^2 v\,\varphi - \int_0^t\!\!\int_D \Delta f(v+W_A)\varphi, \qquad (3.46)
\end{aligned}$$

as $m\to+\infty$. We are done. Moreover, taking suitable limits in the Galerkin scheme, it is easy to see that

$$A^{-1}v \in L^2([0,T];V_{-1}) + L^{2p/(2p-1)}([0,T]\times D),$$

so $v\in\mathcal{C}([0,T];V_{-1})$ because of Lemma (3.2.19). In addition, it is easy to see that v is a solution of (3.38). From *Step 1* we know that (3.38) has a unique solution, therefore the entire sequence $\{v_m\}_{m\in\mathbb{N}}$ converges to v, so v is $\mathcal{F}$-measurable being pointwise limit of measurable functions. We will verify that v satisfies (3.35) in *Step 5.*

Step 4: v H-norm properties. Let $\omega\in B$. From *Step 2* we know that $v_m\in H$ for almost every $t\in[0,T]$. For such t we take the scalar product of (3.38) with v_m and we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_m\|^2 + |v_m|_2^2 + (Af(v_m+W_A^m),v_m) = 0.$$

[13]therefore we do not require v and $f(v+W_A)$ to take values in $V_4 = D(A^2)$ and $V_2 = D(A^1)$ respectively as one may think when reading (3.38).

[14]for every $\varphi\in\mathcal{C}_0^\infty(D)$ it is clear that $A^n\varphi = \Delta^n\varphi$ for all $n\in\mathbb{N}$.

Integrating by parts we obtain

$$(Af(v_m + W_A^m), v_m) = \int_D f'(v_m + W_A^m)|\nabla v_m|^2 + \int_D f'(v_m + W_A^m)\nabla W_A^m \nabla v_m.$$

We take $C_9, C_{10} > 0$ such that

$$f'(x) \geq \frac{2p-1}{2} a_{2p-1} x^{2p-2} - C_9 \qquad |f'(x)| \leq 2(2p-1)a_{2p-1}x^{2p-2} + C_{10}.$$

Thanks to Hölder's and Young's inequalities, we obtain

$$\begin{aligned}
&(Af(v_m + W_A^m), v_m) \geq \frac{2p-1}{2} a_{2p-1} \int_D |v_m + W_A^m|^{2p-2} |\nabla v_m|^2 \mathrm{d}x \\
- \;& C_9 \int_D |\nabla v_m|^2 \mathrm{d}x - 2(2p-1)a_{2p-1} \int_D |v_m + W_A^m|^{2p-2} |\nabla v_m||\nabla W_A^m| \mathrm{d}x \\
- \;& C_{10} \int_D |\nabla v_m||\nabla W_A^m| \mathrm{d}x \\
\geq \;& \frac{1}{4}(2p-1)a_{2p-1} \int_D |v_m + W_A^m|^{2p-2} |\nabla v_m|^2 \mathrm{d}x - 2C_{10} \int_D |\nabla v_m|^2 \mathrm{d}x \\
- \;& C_{11} \left(\int_D |v_m + W_A^m|^{2p} \mathrm{d}x + \int_D |\nabla W_A^m|^{2p} \mathrm{d}x + \int_D |\nabla W_A^m|^2 \mathrm{d}x \right).
\end{aligned}$$

We deduce

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_m\|^2 + |v_m|_2^2 + \frac{1}{4}(2p-1)a_{2p-1} \int_D |v_m + W_A^m|^{2p-2} |\nabla v_m|^2 \mathrm{d}x \\
\leq \;& 2C_9 \int_D |\nabla v_m|^2 \mathrm{d}x \\
+ \;& C_{11} \left(\int_D |v_m + W_A^m|^{2p} \mathrm{d}x + \int_D |\nabla W_A^m|^{2p} \mathrm{d}x + \int_D |\nabla W_A^m|^2 \mathrm{d}x \right).
\end{aligned}$$

We notice that

$$\int_D |\nabla v_m|^2 \mathrm{d}x = \|v_m\|_1^2.$$

After integration in time we obtain

$$\begin{aligned}
&\frac{1}{2}\|v_m\|^2 + \int_0^t |v_m|_2^2 \mathrm{d}s + \frac{2p-1}{4} a_{2p-1} \int_0^t \int_D |v_m + W_A^m|^{2p-2} |\nabla v_m|^2 \mathrm{d}x \mathrm{d}s \\
\leq \;& 2C_9 \int_0^t \|v_m\|_1^2 \mathrm{d}s + C_{11} \int_0^t \left(\|v_m + W_A^m\|_{L^{2p}}^{2p} + \|\nabla W_A^m\|_{L^{2p}}^{2p} + \|\nabla W_A^m\|_{L^2}^2 \right) \mathrm{d}s.
\end{aligned}$$

From *Step 2* we know that $\{v_m + W_A^m\}_{m\in\mathbb{N}}$ is bounded in $L^{2p}([0,T] \times D)$ and $\{v_m\}_{m\in\mathbb{N}}$ is bounded in $L^2([0,T]; V_1)$. Furthermore, $\{\nabla W_A^m\}_{m\in\mathbb{N}}$ is bounded in

$L^{2p}([0,T] \times D)$ and $L^2([0,T] \times D)$ thanks to Lemma (3.2.16). Thus $\{v_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty([0,T];H)$ and $L^2([0,T];V_2)$: as a straightforward consequence, we deduce that $v \in L^\infty([0,T];H) \cap L^2([0,T];V_2)$. We now prove that $t \mapsto v(t)$ is strongly continuous with values in H. We have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|v_m\|^2\right| &= 2\left||v_m|_2^2 + 2\int_D f'(v_m + W_A^m)\nabla v_m \mathrm{d}x\right| \\ &\leq 2|v_m|_2^2 + 2\int_D |f'(v_m + W_A^m)\nabla(v_m + W_A^m)\nabla v_m|\mathrm{d}x.\end{aligned}$$

Thanks to the estimates proved above, we integrate on $[0,T]$ and get

$$\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t}\|v_m\|^2\right| \mathrm{d}s \leq C_{12},$$

which implies, because of the lower semicontinuity of the norm, that

$$\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2\right| \mathrm{d}s \leq C_{12}.$$

We have shown that $\partial\|v\|^2/\partial t$ belongs to $L^1(0,T)$. It follows that $t \mapsto \|v(t)\|^2$ is continuous. Moreover, since we know from *Step 2* that $v \in \mathcal{C}([0,T];V_{-1})$, the mapping $t \mapsto v(t)$ is weakly continuous with values in H. The last assertions combined give the strong continuity of v with values in H.

Step 5: Boundary conditions. Finally, it remains to prove that v satisfies (3.35). The approximate solutions v_m satisfy (3.35). We can therefore write, for every $\psi \in U$ and every $t \in [0,T]$

$$\begin{aligned}\underbrace{\int_0^t\!\!\int_D \Delta v_m \psi}_{\downarrow \text{ for } m\to+\infty} &= -\underbrace{\int_0^t\!\!\int_D \nabla v_m \nabla\psi}_{\downarrow \text{ for } m\to+\infty}. \\ \int_0^t\!\!\int_D \Delta v \psi &= -\int_0^t\!\!\int_D \nabla v \nabla \psi. \end{aligned} \tag{3.47}$$

so $\partial_\nu v = 0$ almost everywhere on $\Gamma \times [0,T]$. On the other hand we can only say that, since $\Delta v \in H$, $\partial_\nu(\Delta v) = 0$ in the following weak sense: for every $\psi \in \mathcal{D}(A) \cap V_8$, we can act as we've done in (3.46) to get

$$\int_0^t\!\!\int_D \frac{\mathrm{d}v}{\mathrm{d}t}\psi + \int_0^t\!\!\int_D \Delta v \Delta\psi - \int_0^t\!\!\int_D \Delta f(v + W_A)\psi = 0, \tag{3.48}$$

for every $t \in [0,T]$. □

Chapter 4

A nonlocal stochastic Cahn-Hilliard equation

Chapters 2 and 3 of this thesis both provide a review of some articles dealing with the deterministic Cahn-Hilliard equation (Chapter 2) and the stochastic Cahn-Hilliard equation (Chapter 3). They illustrate in a detailed and accessible way certain proofs contained in the quoted articles. This fourth chapter, on the other hand, is the creative part of the thesis: it contains the new results we have obtained on a stochastic extension of a nonlocal Cahn-Hilliard equation.
We first introduce a suitable analytical setting and make same mathematical and physical assumptions. We then show, in a *variational* context, the existence of a *weak statistical* solution for this problem. Finally we prove existence, uniqueness and measurability of a *strong* solution. We use ideas from [21], [29], [33]. We refer the reader to Section (1.4) for a very concise review of the basic notation and the functional setup.

4.1 Abstract definition of the Problem

We *formally* study the stochastic partial differential equation

$$(SNL)\begin{cases} \mathrm{d}\phi = (-u\cdot\nabla\phi + \Delta\mu)\mathrm{d}t + \mathrm{d}w, & (4.1a)\\ \mu = a\phi - J*\phi + F'(\phi), & (4.1b)\\ \dfrac{\partial\mu}{\partial\nu} = 0 \quad \text{on } \Gamma\times(0,T), & (4.1c)\\ \phi(0) = \phi_0 \in U, \end{cases}$$

where w, u, J, a, F, ϕ_0 are mathematical objects whose nature will be specified in the following section. The symbol $*$ in (4.1b) denotes the convolution operator over D, namely

$$(J * \phi)(x) := \int_D J(x-y)\phi(y)\mathrm{d}y, \quad \forall x \in D.$$

Remark 4.1.1. The reader is invited to remark the strong analogy between Problem (4.1) and Problem (2.4).

4.2 Hypotheses

We now give a precise meaning to the elements componing Problem (4.1). More precisely, we work under the following mathematical and physical assumptions[1].

(i) u is a given velocity field satisfying $u \in L^\infty([0,T] \times D)$, $\mathrm{div}(u) = 0$ in D, $u = 0$ on Γ.

(ii) $w = w(t)$, $t \in [0,T]$ is a H-valued Q-Wiener process[2] defined on the probability space $(\Omega, \mathcal{F}, \mathbf{m})$, where $Q : H \to H$ is a continuous, symmetric, definite positive, finite trace linear operator.

(iii) J is a kernel function[3] satisfying the following properties:

$$J \in W^{1,1}(\mathbb{R}^d), \quad J(x) = J(-x), \quad \forall x \in \mathbb{R}^d$$

$$a(x) := \int_D J(x-y)\mathrm{d}y \geq 0 \quad \text{for a.e. } x \in D.$$

(iv) We choose the density of potential energy[4] F to be

$$F(s) = \frac{s^4}{4} - \frac{s^2}{2},$$

so that $F'(s) = s^3 - s$. Obviously $F \in \mathcal{C}^{2,1}_{loc}(\mathbb{R})$. We assume that there exists $c_0 > 0$ such that

$$F''(s) + a(x) \geq c_0, \quad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in D. \tag{4.2}$$

It is straightforward to verify that there exist $c_1 > (1/2)\|J\|_{L^1(\mathbb{R}^d)}$, $c_2 \in \mathbb{R}$, $c_3 > 0$, $c_4 \geq 0$, $p \in (1,2]$ such that

$$F(s) \geq c_1 s^2 - c_2, \quad \forall s \in \mathbb{R},$$

[1] Hypotheses (i),(iii),(iv) are physically consistent hypotheses. The remaining ones are dictated by the forthcoming mathematical analysis of the problem.

[2] in the sense of [62, p. 13, Definition 2.1.9.].

[3] which must not be confused with the mass flux defined at page 5.

[4] previously called *Helmholtz free energy*, see page 4 and (1.2).

$$|F'(s)|^p \leq c_3|F(s)| + c_4, \quad \forall s \in \mathbb{R}.$$

The properties we have just listed for F and J are exactly [21, p. 431, hypotheses (H1)/(H3)/(H4)/(H5)].

(v) H has a orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ made of the eigenvectors of the operator $A : \mathcal{D}(A) = V : v \mapsto (-\Delta + I)v$ with associated eigenvalues $\{\mu_i\}_{i\in\mathbb{N}}$, i.e., $Ae_i = \mu_i e_i$, $\forall i \in \mathbb{N}$. We remind that $\mu_i \geq 1, \forall i \in \mathbb{N}$ and $e_i \to +\infty$.
The space V is endowed with the norm

$$\|v\|_V := \left(\|v\|^2 + \|\Delta v\|^2\right)^{1/2}_{,} \qquad \forall v \in V.$$

Due to the regularity of D, such norm is equivalent to the standard H^2-norm. It follows that $\{e_i\}_{i\in\mathbb{N}}$ is an orthogonal basis in V. In addition, $\{e_i\}_{i\in\mathbb{N}}$ is also an orthogonal basis in U.
We also indentify H with its dual space by means of Riesz isomorphism and hence use the continuous injections

$$V \hookrightarrow H^{\varepsilon} \hookrightarrow H \equiv H' \hookrightarrow H^{-\varepsilon} \hookrightarrow V'.$$

(vi) $D \subset \mathbb{R}^d$, $d \in \{2; 3\}$, is regular enough to apply [65, p. 470, Teorema 8.5.] and [42, p. 1285, Theorem 1]. As consequences we have that U is compactly embedded in L^4, that $\{e_i\}_{i\in\mathbb{N}} \subset \mathcal{C}^\infty(\overline{D})$, and that

$$\|e_i\|_{L^\infty} \leq C(D)(\mu_i - 1)^{(d-1)/4}.$$

D is also regular enough such that H is complactly embedded in the interpolation space $[H, V']_\theta = H^{-2\theta}$, $\theta \in (0, 1/8)$. For the details on such embedding, the reader is referred to [53, pages 99-103].

(vii) $\{e_i\}_{i\in\mathbb{N}}$ are eigenvectors for Q as well, namely there is sequence of nonnegative real numbers $\{\vartheta_i\}_{i\in\mathbb{N}}$ such that $Qe_i = \vartheta_i e_i$ for each $i \in \mathbb{N}$. In addition we require that

$$K(Q) := \sum_{i=1}^{\infty} (\mu_i - 1)^{(d-1)/2} \vartheta_i < +\infty. \tag{4.3}$$

(viii) ϕ_0 is a U-valued random variable which is independent of w. In addition we define the *white noise* $\partial w/\partial t$ as the distributional time derivative of the Wiener process w. Namely, $\partial w/\partial t$ is an element of $\mathscr{V}'$ such that

$$\left\langle \frac{\partial w}{\partial t}, v \right\rangle_{\mathscr{V}',\mathscr{V}} := -\left(w, \frac{\partial v}{\partial t}\right)_{L^2([0,T];H)}, \qquad \forall v \in \mathscr{V}. \tag{4.4}$$

The function space $\mathscr{V}$ will be specified in the Section (4.3). We require $\partial w/\partial t$ to be a $\mathscr{V}'$-valued, measurable random variable. We will discuss the relation between $\partial w/\partial t$ and ϕ_0 in the forthcoming Remark (4.3.3).

Remark 4.2.1. Due to the nature of w, equation (4.1) is a stochastic infinite-dimensional differential equation. It can interpreted, with the due careful analogies and generalisations, as in [62, p. 73, Definition 4.2.1.] or [25, Chapter 7].

Remark 4.2.2. Condition (4.3) is stronger than requiring Q to have finite trace. However, it enables us not to make any harmful assumption on the geometry of Γ. The geometry of Γ, in fact, affects many interpolation results.
We could have replaced condition (4.3) with a condition of uniform boundness of the family $\{e_i\}_{i\in\mathbb{N}}$ in L^∞; by so doing, however, we would have been forced to require additional conditions on the geometry of Γ and, consequentially, we would have had to check the validity of some interpolation results. The latter approach is very tough and hence unadvisable.

4.3 Existence of a *weak statistical* solution

In this section we prove the existence of a *weak statistical* solution (whose definition is in analogy to the one given in [29, p. 1181, Definition 5.1.]) for Problem (4.1).
We first introduce some function spaces for a given time $T > 0$.

$$\mathscr{U} := L^2([0,T];U) \cap L^\infty([0,T];H) \cap \mathcal{C}^{2/5}([0,T];V') \cap L^4([0,T];L^4),$$

$$\mathscr{Z} := L^{p'}([0,T];L^4) \cap \mathcal{C}([0,T];H^{-\varepsilon}), \quad p' \in (3,4), \quad \varepsilon \in (0,1/4).$$

Because of the compatible nature of the Banach spaces componing the definition $\mathscr{U}$ and $\mathscr{Z}$, we can define their norms in the following trivial way

$$\|v\|_{\mathscr{U}} := \|v\|_{L^2([0,T];U)} + \|v\|_{L^\infty([0,T];H)} + \|v\|_{\mathcal{C}^{2/5}([0,T];V')} + \|v\|_{L^4([0,T];L^4)}, \ \forall v \in \mathscr{U},$$

$$\|v\|_{\mathscr{Z}} := \|v\|_{L^{p'}([0,T];L^4)} + \|v\|_{\mathcal{C}([0,T];H^{-\varepsilon})}, \ \forall v \in \mathscr{Z}.$$

We state and prove a preliminary result.

Theorem 4.3.1. *$\mathscr{U}$ is compactly embedded in $\mathscr{Z}$.*

Proof. Let $\mathcal{F}$ be a bounded set in $\mathscr{U}$. If we apply [67, p. 86, Theorem 6] with $X = U$, $B = L^4$, $Y = V'$, $q = 4$, $p = p'$, we deduce that $\mathcal{F}$ is relatively compact in $L^{p'}([0,T];L^4)$. If we apply [67, p. 84, Theorem 5] with $X = H$, $B = [H,V']_{\varepsilon/2} = H^{-\varepsilon}$ ($\varepsilon \in (0,1/4)$), $Y = V'$, $p = \infty$, we deduce that $\mathcal{F}$ is relatively compact in $\mathcal{C}([0,T];H^{-\varepsilon})$, hence the conclusion. □

We define

$$V_m := \operatorname{span}\{e_1, \cdots, e_m\},$$

$$\mathscr{V}_m := \Big\{ v \in L^2([0,T];V) \cap L^{q'}([0,T];W^{2,4}) \cap H^2([0,T];H) : \\ v(t) \in V_m \text{ for a.e. } t \in [0,T],\ v(T) = 0 \Big\},$$

where

$$W^{2,4} := \big\{ v \in L^4 : D^\alpha u \in L^4 \text{ for each multi-index } \alpha \text{ such that } |\alpha| \leq 2 \big\}$$

is a classical Sobolev space and

$$\frac{2}{3} + \frac{1}{p'} + \frac{1}{q'} = 1. \tag{4.5}$$

Remark 4.3.2. We remind that $\cup_{r=1}^{\infty} V_r$ is dense in H^ε.

We now define a proper *test function space* for our problem as

$$\mathscr{V} := \text{ completion of } \bigcup_{m=1}^{\infty} \mathscr{V}_m \text{ with respect to the norm } \|\cdot\|_{\mathscr{V}}, \text{ where} \\ \|v\|_{\mathscr{V}}^2 := \|v\|_{L^2([0,T];V)}^2 + \|v\|_{L^{q'}([0,T];W^{2,4})}^2 + \|v\|_{H^2([0,T];H)}^2.$$

We point out that $L^2([0,T];V)$ is continuously embedded in $L^{q'}([0,T];W^{2,4})$, hence the definition on $\mathscr{V}$ is in fact redundant. However, we leave it as it is for the clarity of future computations.

Remark 4.3.3. The reader can now fully understand the definition of the white noise $\partial w/\partial t$ given in (viii), page 64. In addition, the operator

$$\mathcal{A} : L^2([0,T];H) \to \mathscr{V}' : w \mapsto \frac{\partial w}{\partial t},$$

with the time derivative defined as in (4.4), is continuous, hence measurable. Using [48, p. 65, Theorem 10.1., item (c)] we deduce that ϕ_0 and $\partial w/\partial t$ are independent, hence the mapping

$$\omega \mapsto \left\{ \phi_0(\omega), \frac{\partial w}{\partial t}(\omega) \right\}$$

is a random variable from $(\Omega, \mathcal{F})$ to $(H^{-\varepsilon} \times \mathscr{V}', \mathcal{B}(H^{-\varepsilon} \times \mathscr{V}'))$.

Remark 4.3.4. The test function space $\mathscr{V}$ is reflexive and separable. To see this, let us consider the space

$$\mathscr{W} := L^2([0,T];V) \cap L^{q'}([0,T];W^{2,4}) \cap H^2([0,T];H).$$

The space $\mathscr{W}$ is clearly reflexive and separable, being the intersection of reflexive and separable banach spaces. Since $\mathscr{V}$ is a closed subspace of $\mathscr{W}$, it is reflexive and separable.

We now specify what we mean by *weak statistical* solution.

Definition 4.3.5. *A* ***weak statistical*** *solution (or simply a* ***weak*** *solution) to Problem* (4.1) *is a probability measure* $\mathbf{P}$ *(concentrated) on* $\mathcal{B}_{\mathscr{X}}(\mathscr{U})$ *which, for every* $\xi \in H^{\varepsilon}$ *and* $v \in \mathscr{V}$*, satisfies*

$$\int_{\mathscr{X}} \exp\left\{i\langle\phi(0),\xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} + iC(\phi,v)\right\}\mathbf{P}(\mathrm{d}\phi) = \hat{\Xi}(\xi)\hat{\mathbf{N}}(v), \tag{4.6}$$

where

$$C(\phi,v) := -(\phi_0, v(0)) - \int_0^T\int_D \phi u\cdot\nabla v + \\ - \int_0^T\int_D (a\phi + \phi^3 - \phi - J*\phi)\Delta v - \int_0^T\left(\phi, \frac{\partial v}{\partial t}\right).$$

Here Ξ indicates the distribution of the random variable ϕ_0 on H. $\hat{\mathbf{N}}$ is a functional on $\mathscr{V}$ defined by

$$\hat{\mathbf{N}}(v) := \hat{\mathbf{W}}\left(-\frac{\partial v}{\partial t}\right), \quad \forall v\in\mathscr{V},$$

where $\mathbf{W}$ is the distribution of w. The symbol $\hat{\cdot}$ over $\mathbf{W}$ and Ξ indicates the characteristic functional operator[5].

Remark 4.3.6. A weak solution is defined (more precisely, concentrated) on the σ-algebra $\mathcal{B}_{\mathscr{X}}(\mathscr{U})$, since $\mathscr{U}$ is not separable. For a similar discussion, see [29, p. 1181]. Such solution may not be unique. In addition, we warn the reader not to confuse $\phi(0)$ and ϕ_0 in Definition (4.3.5).

Remark 4.3.7. We here give a *formal*[6] derivation of Definition (4.3.5). If ϕ is a solution of the infinite dimensional stochastic equation (4.1a)-(4.1b) (e.g., in

[5]given a separable Banach space Y and a probability measure ν defined on $(Y,\mathcal{B}(Y))$, the *characteristic functional* of ν is a $\mathbb{C}$-valued functional with domain Y', defined as

$$\hat{\nu}(f) := \int_Y \exp\left\{i\langle f,y\rangle_{Y',Y}\right\}\nu(\mathrm{d}y), \qquad \forall f\in Y'.$$

If X is a Y-valued random variable, the *characteristic functional* of X is the characteristic functional of the law of X, namely

$$\hat{\mathcal{L}}_X(f) := \int_Y \exp\left\{i\langle f,y\rangle_{Y',Y}\right\}\mathcal{L}_X(\mathrm{d}y) = \mathbb{E}\left[\exp\{i\langle f,X\rangle_{Y',Y}\}\right], \qquad \forall f\in Y',$$

where $\mathcal{L}_X$ denotes the law of X. If Y is a separable Hilbert space, we adapt the previous definitions by replacing the duality $\langle\cdot,\cdot\rangle_{Y',Y}$ with the Y-inner product $(\cdot,\cdot)_Y$ and by considering $f\in Y$ instead of $f\in Y'$.

[6]hence it may lack of rigors.

the sense of [62, p. 73, Definition 4.2.1.]), then we may compute, by means of the Itö formula[7], the stochastic differential equation which is satisfied by the real functional $B(\phi) := (\phi, v)$, for each $v \in \cup_{m=1}^{\infty} \mathscr{V}_m$. We obtain

$$\begin{aligned} 0 &= (\phi_0, v(0)) - \int_0^T \int_D vu \cdot \nabla\phi + \\ &+ \int_0^T \int_D (\Delta\mu) v + \int_0^T \left(\phi, \frac{\partial v}{\partial t}\right) - \int_0^T \left(\frac{\partial v}{\partial t}, w\right). \end{aligned}$$

If we recall the definition of white noise, condition (4.1c) and we observe that $(v, u \cdot \nabla\phi) = -(\phi, u \cdot \nabla v)$ thanks to the properties of u listed in (i), page 63, we derive

$$C(\phi, v) = \left\langle \frac{\partial w}{\partial t}, v \right\rangle_{\mathscr{V}', \mathscr{V}}. \tag{4.7}$$

Relation (4.7) can be extended by a *formal* density argument (with respect to norm $\|\cdot\|_{\mathscr{V}}$) to all $v \in \mathscr{V}$, provided that ϕ is sufficiently regular.
If we add the term $\langle \phi(0), \xi \rangle_{H^{-\varepsilon}, H^{\varepsilon}}$ to both sides of (4.7), we multiply by i, apply the exponential function, take to expected value and use the independence of w and ϕ_0 we see that (4.6) holds with $\mathbf{P}$ being the distribution of ϕ. Obviously we may generalize (4.6) omitting the requirement that $\mathbf{P}$ is the distribution of a given process, thus obtaining Definition (4.3.5). We emphasize the fact that

[7] we here recall the Itö formula. Let H be a Hilbert space; let φ be a H-valued, $[0,T]$-Bochner integrable, predictable process; let Φ be a L_2^0-valued process stochastically integrable in $[0,T]$; let $X(0)$ be a $\mathcal{F}_0$-measurable H-valued random variable. Then the process X

$$X(t) := X(0) + \int_0^t \varphi(s) \mathrm{d}s + \int_0^t \Phi(s) \mathrm{d}w(s)$$

is well defined. Assume that a function $F : [0,T] \times H \to \mathbb{R}$ and its partial Fréchet derivatives F_t, F_x, F_{xx} are uniformly continuous on bounded subsets of $[0,T] \times H$. Then the following (Itö) formula holds a.s. for all $t \in [0,T]$

$$\begin{aligned} F(t, X(t)) &= F(0, X(0)) + \int_0^t (F_x(s, X(s)), \Phi(s) \mathrm{d}w(s)) \\ &+ \int_0^t \{F_t(s, X(s)) + (F_x(s, X(s)), \varphi(s))\} \mathrm{d}s \\ &+ \int_0^t \left\{ \frac{1}{2} \mathrm{tr} \left[F_{xx}(s, X(s)) \left(\Phi(s) Q^{1/2}\right) \left(\Phi(s) Q^{1/2}\right)^* \right] \right\} \mathrm{d}s \end{aligned}$$

where Q is the covariance of the (H-valued) Wiener process w. For the meaning of the stochastic integral in the previous relation we refer the reader to [62, p. 36, Lemma 2.4.2.]. For a detailed discussion on the Itö formula we refer the reader to [25, p. 105, Paragraph 4.5]. For the sake of clarity, in the following we apply a slightly different version of such formula, valid for each time interval $[0, \zeta_m(\omega))$.

relation (4.7) also plays an important role in the forthcoming Definition (4.4.1), aside from the fact that it justifies the expression of $C(\phi, v)$.

Remark 4.3.8. It can be seen that the real-valued functional $\phi \mapsto C(\phi, v)$ is continuous on $\mathscr{X}$ for each fixed $v \in \mathscr{V}$. To show this, let ϕ_n be a sequence such that $\phi_n \to \phi$ in $\mathscr{X}$. Because of the convergence in $L^{p'}([0,T]; L^4)$, it is straightforward to deduce the convergence of all the elements componing C which are linear in ϕ. We only have to treat the nonlinearity ϕ^3 with a little bit of care. In fact, reminding (4.5) and using Hölder inequality in space and time, we obtain

$$\begin{aligned}
&\left| \int_0^T \int_D (\phi_n^3 - \phi^3)\Delta v \right| \le \frac{3}{2} \int_0^T \int_D |\phi_n - \phi|(\phi_n^2 + \phi^2)|\Delta v| \\
\le\ & \frac{3\sqrt{2}}{2} \int_0^T \|\phi_n - \phi\|_{L^4} (\|\phi_n\|_{L^4}^2 + \|\phi\|_{L^4}^2) \|\Delta v\|_{L^4} \\
\le\ & 3\|\phi_n - \phi\|_{L^{p'}([0,T];L^4)} \left[\|\phi_n\|^2_{L^3([0,T];L^4)} + \|\phi\|^2_{L^3([0,T];L^4)} \right] \|\Delta v\|_{L^{q'}([0,T];L^4)} \\
\le\ & 3\|\phi_n - \phi\|_{L^{p'}([0,T];L^4)} \left[\|\phi_n\|^2_{L^3([0,T];L^4)} + \|\phi\|^2_{L^3([0,T];L^4)} \right] \|v\|_{\mathscr{V}}. \qquad (4.8)
\end{aligned}$$

We deduce

$$\left| \int_0^T \int_D (\phi_n^3 - \phi^3)\Delta v \right| \to 0$$

as $n \to +\infty$, hence the conclusion. The definition of $\mathscr{V}$ and property (4.8) also allow us to define $C(\phi) \in \mathscr{V}'$ as

$$\langle C(\phi), v \rangle_{\mathscr{V}', \mathscr{V}} := C(\phi, v)$$

for every $\phi \in \mathscr{X}$.

Remark 4.3.9. It is straightforward to deduce two facts from Remark (4.3.8).

(a) For each $\xi \in H^\varepsilon$ and each $v \in \mathscr{V}$, the functional

$$\phi \mapsto \exp\left\{ i\langle \phi(0), \xi \rangle_{H^{-\varepsilon}, H^\varepsilon} + iC(\phi, v) \right\}$$

is continuous on $\mathscr{X}$, since $\langle \phi_n(0), \xi \rangle_{H^{-\varepsilon}, H^\varepsilon} \to \langle \phi(0), \xi \rangle_{H^{-\varepsilon}, H^\varepsilon}$ (thanks to the convergence in $\mathcal{C}([0,T]; H^{-\varepsilon})$) and since $C(\phi_n, v) \to C(\phi, v)$, for any sequence $\phi_n \to \phi$ in $\mathscr{X}$.

(b) The mapping

$$\mathscr{D} : \mathscr{X} \to H^{-\varepsilon} \times \mathscr{V}' : \phi \mapsto \{\phi(0), C(\phi)\}$$

is continuous as a mapping from $(\mathscr{X}, \mathcal{B}(\mathscr{X}))$ to $(H^{-\varepsilon} \times \mathscr{V}', \mathcal{B}(H^{-\varepsilon} \times \mathscr{V}'))$. This is consequence of the nature of $\| \cdot \|_{\mathscr{V}}$ and of (4.8).

In the forthcoming sections, we will need the following equivalent characterization on (4.6).

Lemma 4.3.10. *Equality* (4.6) *implies that*

$$\mathbf{P}(\mathscr{D}^{-1}(C)) = (\Xi \times \mathbf{N})(C), \quad \forall C \in \mathcal{B}(H^{-\varepsilon} \times \mathscr{V}'). \tag{4.9}$$

The left hand side of equality (4.9) *is well defined since* $\mathscr{D}$ *is continuous, as stated in Remark* (4.3.9).

Proof. Let $\xi \in H^{\varepsilon}$, $v \in \mathscr{V}$ be arbitrarily fixed. We rely on the theory of *pushforward measures*[8]. In accordance with the notation introduced in Definition (4.3.5) and footnote 8, we set $X_1 := \mathscr{X}$, $X_2 := H^{-\varepsilon} \times \mathscr{V}'$, $f := \mathscr{D}$, $\nu := \mathbf{P}$, and

$$g : X_2 \to \mathbb{C} : \{x, y\} \mapsto \exp\left\{i\langle x, \xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} + i\langle y, v\rangle_{\mathscr{V}',\mathscr{V}}\right\}.$$

If we use the integral equality stated in footnote 8 and the definition of weak statistical solution, we deduce

$$\begin{aligned}
&\int_{\mathscr{X}} \exp\left\{i\langle \phi(0), \xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} + iC(\phi, v)\right\} \mathbf{P}(\mathrm{d}\phi) = \int_{X_1} (g \circ f)\, \mathrm{d}\nu = \int_{X_2} g\, \mathrm{d}\nu_* \\
=\ &\int_{H^{-\varepsilon}\times\mathscr{V}'} \exp\left\{i\langle x, \xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} + i\langle y, v\rangle_{\mathscr{V}',\mathscr{V}}\right\} \mathbf{P}_*(\mathrm{d}\{x, y\}) = \hat{\Xi}(\xi)\hat{\mathbf{N}}(v) \\
=\ &\int_{H^{-\varepsilon}\times\mathscr{V}'} \exp\left\{i\langle x, \xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} + i\langle y, v\rangle_{\mathscr{V}',\mathscr{V}}\right\} (\Xi \times \mathbf{N})\, (\mathrm{d}\{x, y\}).
\end{aligned} \tag{4.10}$$

Remarks (4.3.2), (4.3.4) imply that H^{ε} and $\mathscr{V}$ are reflexive and separable, thus $H^{\varepsilon} \times \mathscr{V}$ is reflexive and separable. Since ξ and v are arbitrarily chosen in H^{ε} and $\mathscr{V}$, the reflexivity of H^{ε}, $\mathscr{V}$ and relation (4.10) imply that

$$\int_{H^{-\varepsilon}\times\mathscr{V}'} \exp\left\{iL(\{x, y\})\right\} \mathbf{P}_*(\mathrm{d}\{x, y\}) = \int_{H^{-\varepsilon}\times\mathscr{V}'} \exp\left\{iL(\{x, y\})\right\} (\Xi \times \mathbf{N})(\mathrm{d}\{x, y\}) \tag{4.11}$$

[8] we recall the definition of pushforward measure and a related integral property. Let $(X_1, \mathcal{F}_1)$, $(X_2, \mathcal{F}_2)$ be two measurable spaces. Let $f : X_1 \to X_2$ be a measurable function and let ν be a probability measure on $(X_1, \mathcal{F}_1)$. The *pushforward measure* of ν associated to f is a probability measure ν_* on $(X_2, \mathcal{F}_2)$ defined as

$$\nu_*(C) := \nu\left(f^{-1}(C)\right), \quad \forall C \in \mathcal{F}_2.$$

The following fact holds: let $g : X_2 \to \mathbb{C}$ be a measurable function. Then g is integrable on X_2 with respect to ν_* if and only if $g \circ f$ is integrable on X_1 with respect to μ. In this case, the integrals coincide, i.e.

$$\int_{X_2} g\, \mathrm{d}\nu_* = \int_{X_1} (g \circ f)\, \mathrm{d}\nu.$$

for each $L \in (H^{-\varepsilon} \times \mathscr{V}')'$. See [70, Proposition 1.1] for the characterization of the reflexivity and separability properties of the product of two banach spaces. Since $H^{\varepsilon} \times \mathscr{V}$ is reflexive and separable, it follows that $(H^{\varepsilon} \times \mathscr{V})'$ is separable. Since there is an isometric isomorphism between $(H^{\varepsilon} \times \mathscr{V})'$ and $H^{-\varepsilon} \times \mathscr{V}'$, it follows that $H^{-\varepsilon} \times \mathscr{V}'$ is separable. We can hence apply [6, p. 28, Proposition 4.15.] and deduce that $\mathbf{P}_* \equiv (\Xi \times \mathbf{N})$, i.e. (4.9). $\square$

We can now state and prove the main theorem of this section.

Theorem 4.3.11. *Let $d \leq 3$. Let w be a H-valued Q-Wiener process and let w, u, Q, J, F, ϕ_0 and $\{e_j\}_{j\in\mathbb{N}}$ satisfy the properties* (i)-(viii) *previously listed (page 63 and following). Let ϕ_0 be a U-valued random variable satisfying*

$$\mathbb{E}\left[\|\phi_0\|_U^2 + \int_D \frac{\phi_0^4}{4} - \int_D \frac{\phi_0^2}{2}\right] < +\infty.$$

Then problem (4.1) *admits a weak statistical solution in the sense of Definition* (4.3.5).

Remark 4.3.12. In the hypotheses of the previous Theorem we do not require $\mathbb{E}\left[\int_D \phi_0^4/4\right] < +\infty$ but only $\mathbb{E}\left[\int_D \phi_0^4/4 - \int_D \phi_0^2/2\right] < +\infty$. This requirement is motivated by [29, p. 1190, (7.6)]: such computation will be used in the forthcoming proof.

Proof of Theorem (4.3.11). *Step 1: Galerkin Approximation of Problem* (4.1). For each $m \in \mathbb{N}$, we denote by π_m the H-orthogonal projection operator on V_m. More precisely, we use the extended operator

$$\pi_m : L^1 \to V_m : v \mapsto \sum_{j=1}^{m} (v, e_j)e_j. \tag{4.12}$$

The previous expression is well defined thanks to the regularity of the family $\{e_i\}_{i\in\mathbb{N}}$; in addition, it permits to apply such projector to functions not belonging to H. This will be useful, e.g., in the forthcoming Lemma (4.3.14).
We are ready to write down a Galerkin approximation scheme for problem (4.1). For each $m \in \mathbb{N}$, we look for a stochastic process $\phi_m = \sum_{j=1}^{m} c_j(t)e_j(x)$ such that

$$\begin{aligned} \mathrm{d}\phi_m &= \pi_m(-u \cdot \nabla\phi_m + \Delta\mu_m)\mathrm{d}t + \mathrm{d}w_m, \\ \mu_m &:= \pi_m(a\phi_m + \phi_m^3 - \phi_m - J * \phi_m), \\ \phi_m(0) &= \pi_m\phi_0, \end{aligned} \tag{4.13}$$

where $w_m := \pi_m w$ is a V_m-valued Wiener process. If we take the H-inner product of (4.13) with $e_1, \cdots, e_m$ we see that the resulting $\mathbb{R}^m$-valued stochastic differential equation has a locally lipschitz deterministic integrand, therefore for each

$m \in \mathbb{N}$ problem (4.13) admits a solution, which is in principle defined up to some random variable ζ_m, hence for all $t \in [0, \zeta_m(\omega))$. See [47, Theorem 3.1.] for full details. We will show in the forthcoming *Step 3* that the solutions ϕ_m exist (a.s.) for every $t \in [0, T]$.

Step 2: Some inequalities for the family $\{\phi_m\}_{m\in\mathbb{N}}$. We apply the Ito formula to the functional $F(\phi_m(t)) := \|\phi_m(t)\|^2$ and obtain[9] the following equality for each $t \in [0, \zeta(\omega) \wedge T)$

$$
\begin{aligned}
\|\phi_m(t)\|^2 &= \|\phi_m(0)\|^2 + 2\int_0^t (\phi_m(s), -u \cdot \nabla\phi_m(s) + \Delta\mu_m(s))\mathrm{d}s \\
&+ \int_0^t \mathrm{tr}(Q_m) + 2\int_0^t (\phi_m(s), \mathrm{d}w_m(s)).
\end{aligned}
$$

Since $\mathrm{div}(u) = 0$ and $u = 0$ on Γ we have $(\phi_m, u \cdot \nabla\phi_m) = 0$. If we use estimate [21, p. 436, (4.15)] we deduce

$$
\begin{aligned}
&\|\phi_m(t)\|^2 + \int_0^t \frac{c_0}{2}\|\nabla\phi_m(s)\|^2\mathrm{d}s \le \|\phi_m(0)\|^2 + \int_0^t k\|\phi_m(s)\|^2\mathrm{d}s \\
&+ \; C_1 + 2\int_0^t (\phi_m(s), \mathrm{d}w_m(s)), \qquad (4.14)
\end{aligned}
$$

where $k = (2/c_0)\|J\|^2_{L^1(\mathbb{R}^d)}$, c_0 being the positive costant from (4.2).

Step 3: Time domain of $\{\phi_m\}_{m\in\mathbb{N}}$ *and further inequalities.* We now prove that

$$
\zeta_m(\omega) \ge T, \quad \text{a.s. in } \Omega, \ \forall m \in \mathbb{N}. \qquad (4.15)
$$

which implies that ϕ_m is defined on $[0, T]$ (a.s.). We use some ideas from [64, p. 132, Proof of Theorem 12.1], which however cannot be applied directly since condition [64, p. 132, (12.3)] is not satisfied for our finite-dimensional stochastic differential equations.

Let us fix $m \in \mathbb{N}$. If we consider the $\mathbb{R}^m$-valued stochastic differential equation associated with ϕ_m, we see that its determistic integrand b_m and its stochastic integrand σ_m are locally lipschitz. In addition, they depend on time exclusively by means of ϕ_m. For each $N \in \mathbb{N}$, we may define a sequence $(b_{m,N}, \sigma_{m,N})$, agreeing with (b_m, σ_m) on $\{(s, x) : x \in \mathbb{R}^m, \ s \ge 0, \ |x| \le N\}$, such that $(b_{m,N}, \sigma_{m,N})$ are *globally* lipschitz. As a consequence, [64, p. 128, Theorem 11.2] guarantees that there is a unique solution $\phi_{m,N}$ associated to (b_N, σ_N) and defined on $[0, +\infty)$ (a.s.). Finally, we define a sequence of $\mathbb{R}^+$-valued stopping times as follows

$$
\tau_N := \inf\{\tau > 0 : \|\phi_{m,N}(\tau)\| \ge N\} \wedge N
$$

[9] In the following, Q_m denotes the covariance operator of w_m and $[Q]_{ij}$, $i, j \in \{1, \cdots, m\}$, denote the entries of its matrix representation with respect to the basis $\{e_1, \cdots, e_m\}$.

The sequence $\{\tau_N\}_{N\in\mathbb{N}}$ is obviously increasing. Moreover [64, p. 131, Corollary 11.10] implies that

$$\phi_m = \phi_{m,N} \qquad \text{on } [0,\tau_N]. \tag{4.16}$$

Therefore for each $t \in [0,T]$ we have

$$\begin{aligned}\mathbb{E}\left[\|\phi_m(t\wedge\tau_N)\|^2\right] + \frac{c_0}{2}\mathbb{E}\left[\int_0^{t\wedge\tau_N}\|\nabla\phi_m(s)\|^2\right] \le \mathbb{E}\left[\|\phi_0\|^2\right]\\ + \ k\mathbb{E}\left[\int_0^{t\wedge\tau_N}\|\phi_m(s)\|^2\right] + C_2,\end{aligned} \tag{4.17}$$

which implies

$$\begin{aligned}\mathbb{E}\left[\|\phi_m(t\wedge\tau_N)\|^2\right] \le C_2\\ + \ \mathbb{E}\left[\|\phi_0\|^2\right] + k\int_0^t \mathbb{E}\left[\|\phi_m(s\wedge\tau_N)\|^2\right].\end{aligned}$$

Gronwall inequality consequently gives

$$\mathbb{E}\left[\|\phi_m(t\wedge\tau_N)\|^2\right] \le C_3, \qquad \text{for } t\in[0,T]. \tag{4.18}$$

We notice that the previous relation holds for every $T>0$. Hence we may take $K=2T$, use Markov inequality (see [48, p. 29, Corollary 5.2 (a)]) and recall (4.16) to deduce that, for $N>K$, we have

$$\begin{aligned}\mathbf{m}(\tau_N<K) \le \mathbf{m}(\|\phi_{m,N}(\tau_N\wedge K)\|\ge N) \le \mathbf{m}(\|\phi_m(\tau_N\wedge K)\|\ge N)\\ \le \ \frac{\mathbb{E}\left[\|\phi_m(K\wedge\tau_N)\|^2\right]}{N^2} \le \frac{C(K,\phi_0)}{N^2} \to 0\end{aligned} \tag{4.19}$$

for $N\to+\infty$. Computation (4.19) clearly implies that $\mathbf{m}(\sup_N\tau_N>T)=1$. Hence (4.15) holds. In addition

$$\phi_m \in \mathcal{C}([0,T];\mathcal{C}^\infty(\overline{D})), \qquad m\in\mathbb{N}. \tag{4.20}$$

Step 4: Main estimates for the family $\{\phi_m\}_{m\in\mathbb{N}}$. The processes $\phi_m^N(t) := \phi_m(t\wedge\tau_N)$ are non anticipating and $\phi_m^{N+1}(s)=\phi_m^N(s)$ for all $0<s<t\wedge\tau_N$ (a.s.). Therefore $t\wedge\tau_N\uparrow t$ (a.s.) and $\phi_m(t)=\lim_{N\to+\infty}\phi_m^N(t)$ (a.s.). We can therefore use Fatou Lemma twice (both on the spatial domain and on the probability space) to deduce that

$$\begin{aligned}\mathbb{E}\left[\|\phi_m\|^2_{L^2([0,T];H)}\right] = \mathbb{E}\left[\int_0^T\|\phi_m\|^2\right] \le \mathbb{E}\left[\int_0^T\liminf_N\|\phi_m^N\|^2\right]\\ = \ \int_0^T\mathbb{E}\left[\liminf_N\|\phi_m^N\|^2\right] \le \int_0^T\liminf_N\mathbb{E}\left[\|\phi_m^N\|^2\right] \le C_4,\end{aligned} \tag{4.21}$$

where we have used (4.18). We apply the monotone convergence theorem in (4.17) and we conclude that

$$\mathbb{E}\left[\|\phi_m\|^2_{L^2([0,T];U)}\right] \leq C_5. \tag{4.22}$$

We now take the superior extreme in (4.14) for $0 \leq t \leq T$ and get

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq t\leq T}\|\phi_m(t)\|^2\right] &\leq \mathbb{E}\left[\|\phi_m(0)\|^2\right] + C_1 \\ + \quad k\mathbb{E}\left[\|\phi_m\|^2_{L^2([0,T];H)}\right] &+ 2\mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t (\phi_m(s), \mathrm{d}w_m(s))\right] \end{aligned} \tag{4.23}$$

We estimate the last term of the right hand side of (4.23) by means of Doob submartingale inequality, obtaining

$$\begin{aligned} \left\{\mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t (\phi_m(s), \mathrm{d}w_m(s))\right]\right\}^2 &\leq \mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t (\phi_m(s), \mathrm{d}w_m(s))\right]^2 \\ \leq \quad 4\mathbb{E}\left[\left\{\int_0^T (\phi_m(s), \mathrm{d}w_m(s))\right\}^2\right] &= 4\mathbb{E}\left[\int_0^T (Q_m\phi_m, \phi_m)\right] \\ \leq \quad 4\|Q\|\mathbb{E}\left[\int_0^T \|\phi_m\|^2\right] &\leq 4\|Q\|C_6, \end{aligned}$$

hence we deduce

$$\mathbb{E}\left[\|\phi_m\|^2_{L^\infty([0,T];H)}\right] \leq C_7. \tag{4.24}$$

We now apply the Itö formula once again to the functional

$$Z(\phi) : U \to \mathbb{R} : \phi \mapsto \int_D \left\{a\frac{\phi^2}{2} + \frac{\phi^4}{4} - \frac{\phi^2}{2}\right\} - \frac{1}{2}(J * \phi, \phi).$$

We defined Z such that its Fréchet derivative is $Z_\phi(\phi) = a\phi + \phi^3 - \phi - J * \phi$, hence $\pi_m(Z_\phi(\phi)) = \mu_m$. Thus we obtain, recalling the hypotheses on Q, u and the family $\{e_i\}_{i\in\mathbb{N}}$, that

$$\begin{aligned} Z(\phi_m(t)) &= Z(\phi_m(0)) + \int_0^t (\mu_m, -u\cdot\nabla\phi_m + \Delta\mu_m) \\ + \quad &\int_0^t (Z_\phi(\phi_m(s)), \mathrm{d}w_m(s))_U \\ + \quad &\frac{1}{2}\left(\int_0^t\!\!\int_D (3\phi_m^2 + a - 1)\sum_{i,j=1}^m Q_{ij}e_ie_j - \int_0^t \sum_{i=1}^m (J * Q_me_i, e_i)\right) \end{aligned}$$

$$
\begin{aligned}
&\le Z(\phi_m(0)) + \int_0^t \big(\|u\|_{L^\infty([0,T]\times\Omega)}\|\nabla\mu_m(s)\|\|\phi_m(s)\| - \|\nabla\mu_m(s)\|^2\big)\\
+ \quad & \int_0^t (Z_\phi(\phi_m(s)), \mathrm{d}w_m(s))_U\\
+ \quad & C_8 \mathrm{tr}(Q)\int_0^t \big[\|\phi_m(s)\|^2 + \|a\|_{L^\infty} + \|J\|_{L^1(\mathbb{R}^d)}\big],
\end{aligned}
$$

which, thanks to Young inequality, implies

$$
\begin{aligned}
& Z(\phi_m(t)) + \frac{1}{2}\int_0^t \|\nabla\mu_m(s)\|^2 \le Z(\phi_m(0))\\
+ \quad & \int_0^t (Z_\phi(\phi_m(s)), \mathrm{d}w_m(s))_U + C(T,u,Q)\int_0^t \|\phi_m(s)\|^2\\
+ \quad & C_9(K(Q))\int_0^t \big[\|\phi_m(s)\|^2 + \|a\|_{L^\infty} + \|J\|_{L^1(\mathbb{R}^d)}\big]. \qquad (4.25)
\end{aligned}
$$

If we define

$$
\tau_N^1 = \begin{cases} \inf\{\tau > 0 : \|Z_\phi(\phi_m(\tau))\|_U \ge N\} & \text{if } \exists \tau > 0 : \|Z_\phi(\phi_m(\tau))\|_{H^1} \ge N,\\ +\infty \quad \text{if } \|Z_\phi(\phi_m(\tau))\|_U < N, \quad \forall \tau > 0, \end{cases}
$$

we may act on (4.25) and τ_N^1 similarly to the computations previously done with τ_N. We deduce

$$
\mathbb{E}\left[Z(\phi_m(t\wedge\tau_N^1))\right] + \frac{1}{2}\mathbb{E}\left[\int_0^{t\wedge\tau_N^1} \|\nabla\mu_m(s)\|^2\right] \le \mathbb{E}\left[Z(\phi_m(0))\right] + C_{10} \qquad (4.26)
$$

In addition, the regularity of the trajectories of ϕ_m highlighted in (4.20) imply that τ_N^1 definitely coincides with T, (a.s.). Computation [29, p. 1190, (7.6)] permits to estimate $\mathbb{E}\left[Z(\phi_m(0))\right]$ *uniformly* in $m \in \mathbb{N}$. In addition we can rely on estimates (4.22),(4.24) and get

$$
\mathbb{E}\left[\|\phi_m(t\wedge\tau_N^1)\|_{L^4}^4\right] \le C_{11} \qquad (4.27)
$$

We replicate the application of Fatou Lemma as we have done in (4.21) and obtain[10]

$$
\begin{aligned}
& \mathbb{E}\left[\|\phi_m\|_{L^4([0,T];L^4)}^4\right] = \mathbb{E}\left[\int_0^T \|\phi_m\|_{L^4}^4\right] \le \mathbb{E}\left[\int_0^T \liminf_N \|\phi_m^N\|_{L^4}^4\right]\\
= \quad & \int_0^T \mathbb{E}\left[\liminf_N \|\phi_m^N\|_{L^4}^4\right] \le \int_0^T \liminf_N \mathbb{E}\left[\|\phi_m^N\|_{L^4}^4\right] \le C_{12}, \qquad (4.28)
\end{aligned}
$$

[10] in (4.26), we denote $\phi_m^N(t) := \phi_m(t\wedge\tau_N^1)$, with no reference to previous similar notations.

where we have used (4.27). We can apply the monotone convergence theorem in (4.26) by passing to limit with respect to $N \to +\infty$ and deduce

$$\mathbb{E}\left[\|\nabla\mu_m\|^2_{L^2([0,T];H)}\right] \leq C_{13}. \tag{4.29}$$

Using the Gelfand triple $V \hookrightarrow H \hookrightarrow V'$, we recall [29, p. 1179, Theorem 4.2.] and we write, for every $v \in V$ and for every $0 \leq t_1 < t_2 \leq T$

$$\begin{aligned} & |\langle \phi_m(t_2) - \phi_m(t_1), v\rangle_{V',V}| = |(\phi_m(t_2) - \phi_m(t_1), v)| \\ = & \left|\int_{t_1}^{t_2} (-u\cdot\nabla\phi_m + \Delta\mu_m, v) + (w_m(t_2) - w_m(t_1), v)\right| \\ \leq & \left(\left[\int_{t_1}^{t_2} \|u\|_{L^\infty([0,T]\times D)}\|\nabla\phi_m\| + \|\nabla\mu_m\|\right] + |t_2 - t_1|^{\frac{2}{5}}\|w_m\|_{\mathcal{C}^{\frac{2}{5}}([0,T];V')}\right)\|v\|_V \\ \leq & C_6(u)|t_2 - t_1|^{\frac{2}{5}} \\ \times & \left[\|\phi_m\|_{L^2([0,T];U)} + \|\nabla\mu_m\|_{L^2([0,T];H)} + \|w_m\|_{\mathcal{C}^{\frac{2}{5}}([0,T];V')}\right]\|v\|_V. \end{aligned} \tag{4.30}$$

In addition, (4.24) and (4.20) imply

$$\mathbb{E}\left[\|\phi_m\|^2_{\mathcal{C}([0,T];H)}\right] \leq C_7. \tag{4.31}$$

The combination of (4.30)-(4.31) allows us to deduce

$$\mathbb{E}\left[\|\phi_m\|_{\mathcal{C}^{2/5}([0,T];V')}\right] \leq C_{14}. \tag{4.32}$$

The constants $C_1, \cdots, C_{14}$ are independent of m but may depend on ϕ_0, T, u, J, Q. Inequalities (4.22), (4.24), (4.28), (4.32) imply that $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$, the family of the distributions of $\{\phi_m\}_{m\in\mathbb{N}}$ on $\mathscr{X}$, is *uniformly* concentrated on $\mathscr{U}$.

Step 5: Existence of a weak limit. Since $\mathscr{U}$ is compactly embedded in $\mathscr{X}$ as proved in Theorem (4.3.1), we can use a compactness argument by means of Prohorov Theorem. Estimates (4.22), (4.24), (4.28), (4.32), the definition of the $\mathscr{U}$-norm and the L^p embeddings imply that

$$\int_{\mathscr{X}} \|\phi\|_{\mathscr{U}}\mathbf{P}_m(\mathrm{d}\phi) \leq C_{15}, \quad \forall m \in \mathbb{N}, \tag{4.33}$$

where C_{15} is independent of m. In addition we have that the sets $G_n(\mathscr{U})$ defined as follows

$$\begin{aligned} G_n(\mathscr{U}) \;\; := \;\; & \{v \in \mathscr{U} : \; \|v\|_{L^2([0,T];U)} \leq n, \, \|v\|_{L^\infty([0,T];H)} \leq n, \\ & \|v\|_{L^4([0,T];L^4)} \leq n, \, \|v\|_{\mathcal{C}^{2/5}([0,T];V')} \leq n\} \end{aligned}$$

are compact in $\mathscr{X}$, see the first part of the forthcoming *Step 1* of *Proof of Theorem* (4.4.7), footnote (14). Thanks to (4.33), we deduce

$$\begin{aligned} \mathbf{P}_m(\mathscr{X} \setminus G_n(\mathscr{U})) &= \int_{\mathscr{X}\setminus G_n(\mathscr{U})} \mathbf{P}_m(\mathrm{d}\phi) = \int_{\mathscr{U}\setminus G_n(\mathscr{U})} \mathbf{P}_m(\mathrm{d}\phi) \\ \leq \ & \frac{1}{n}\int_{\mathscr{U}\setminus G_n(\mathscr{U})} \|\phi\|_{\mathscr{U}} \mathbf{P}_m(\mathrm{d}\phi) \leq \frac{1}{n}\int_{\mathscr{X}} \|\phi\|_{\mathscr{U}} \mathbf{P}_m(\mathrm{d}\phi) \leq \frac{C_{15}}{n} \leq \varepsilon, \end{aligned}$$

where the last inequality holds if n is large enough. We have verified the validity of the Prohorov theorem hypotheses. Hence we deduce that there is a (not relabeled) subsequence $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$ and a probablity $\mathbf{P}$ defined on $(\mathscr{X}, \mathcal{B}(\mathscr{X}))$ such that $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$ weakly converges to $\mathbf{P}$.

Step 6: Passage to the limit. The probablity $\mathbf{P}_m$ satisfies

$$\int_{\mathscr{X}} \exp\left\{i\langle\phi(0),\xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} + iC(\phi,v)\right\} \mathbf{P}_m(\mathrm{d}\phi) = \hat{\Xi}(\xi)\hat{\mathbf{W}}_m\Big(-\frac{\partial v}{\partial t}\Big), \\ \forall \xi \in V_r,\ \ \forall v \in \mathscr{V}_r,\ \ m \geq r, \qquad (4.34)$$

where $\hat{\mathbf{W}}_m$ denotes the characteristic functional of w_m. Equality (4.34) can be proved as follows: we initially replicate the computations we have done in Remark (4.3.7). Let $r \leq m$ be positive integers. For each $v \in \mathscr{V}_r$, we apply the Itö formula to the functional $B(\phi_m) := (\phi_m, v)$ and obtain[11]

$$\begin{aligned} 0 &= (\phi_m(0), v(0)) + \int_0^T\int_D \pi_m(-u\cdot\nabla\phi_m + \Delta\mu_m)v \\ &+ \int_0^T \Big(\phi_m, \frac{\partial v}{\partial t}\Big) - \int_0^T \Big(\frac{\partial v}{\partial t}, w_m\Big). \end{aligned} \qquad (4.35)$$

Since v takes values in V_m, we can integrate by parts and rewrite (4.35) as

$$\begin{aligned} 0 &= (\phi_0, v(0)) + \int_0^T\int_D \{-uv\cdot\nabla\phi_m + \mu\Delta v\} \\ &+ \int_0^T \Big(\phi_m, \frac{\partial v}{\partial t}\Big) - \int_0^T \Big(\frac{\partial v}{\partial t}, w_m\Big). \end{aligned}$$

Hence we can rearrange the terms and add the term $\langle\phi_m(0),\xi\rangle_{H^{-\varepsilon},H^{\varepsilon}}$ in the last equality to obtain

$$\langle\phi_m(0),\xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} + C(\phi_m, v) = \langle\phi_m(0),\xi\rangle_{H^{-\varepsilon},H^{\varepsilon}} - \int_0^T \Big(w_m, \frac{\partial v}{\partial t}\Big),$$

[11]unlike the computations made in Remark (4.3.7), we here need to be rigorous: the function $B(\phi_m(t),t) := (\phi_m(t), v(t))$ satisfies the hypotheses of the application of the Itö formula because, in particular, $v \in H^2([0,T];H)$.

which, since $\xi \in V_m$ and $H^\varepsilon \hookrightarrow H \hookrightarrow H^{-\varepsilon}$, becomes

$$\langle \phi_m(0), \xi \rangle_{H^{-\varepsilon},H^\varepsilon} + C(\phi_m, v) = (\phi_0, \xi) - \int_0^T \left(w_m, \frac{\partial v}{\partial t} \right). \tag{4.36}$$

We observe that w_m and $\phi_m(0)$ are independent random variables because of the independence of w and ϕ_0. We multiply (4.36) by i, apply the exponential function, take the expected value, use the independence of w_m and $\phi_m(0)$. Hence we deduce (4.34).
In Remark (4.3.8) we have seen that $\phi \mapsto C(\phi, v)$ is continuous on $\mathscr{X}$ for every $v \in \mathscr{V}$. In addition, we recall that $w_m \to w$ in $L^2((\Omega, \mathcal{F}, \mathbf{m}); \mathcal{C}([0,T]; H))$ (see [62, p. 13, Proposition 2.1.10]). Therefore we can rely on the weak convergence of the sequence $\{\mathbf{P}_m\}_{m\in\mathbb{N}}$, hence take the limit for $m \to +\infty$ in (4.34) and deduce that (4.6) holds for $\mathbf{P}$ and $\xi \in \cup_{r=1}^\infty V_r$, $v \in \cup_{r=1}^\infty \mathscr{V}_r$. By the density of $\cup_{r=1}^\infty V_r$ in H^ε (see Remark (4.3.2)) and the density of $\cup_{r=1}^\infty \mathscr{V}_r$ in $\mathscr{V}$, by means of Lebesgue dominated theorem[12], we deduce that $\mathbf{P}$ satisfies (4.6) for each $\xi \in H^\varepsilon$ and $v \in \mathscr{V}$. To prove that $\mathbf{P}$ is a weak solution to Problem (4.1) in the sense of Definition (4.3.5), it remains to show that $\mathbf{P}$ is concentrated on $\mathcal{B}_{\mathscr{X}}(\mathscr{U})$.

Step 7: $\mathbf{P}$ *is concentrated on* $\mathcal{B}_{\mathscr{X}}(\mathscr{U})$. It is straightforward to notice that

$$\mathscr{U} = \bigcup_{n=1}^{\infty} G_n(\mathscr{U}). \tag{4.37}$$

It follows from (4.37) that $\mathscr{U} \in \mathcal{B}_{\mathscr{X}}(\mathscr{U})$. Moreover, as already said in *Step 5*, $G_n(\mathscr{U})$ is compact in $\mathscr{X}$ and hence closed in $\mathscr{X}$. As a consequence of *Step 5*, for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $\mathbf{P}_m(G_n(\mathscr{U})) \geq 1 - \varepsilon$ for each $m \in \mathbb{N}$. We may use the Portmanteau Theorem to deduce that $\mathbf{P}(G_n(\mathscr{U})) \geq 1 - \varepsilon$. Hence $\mathbf{P}(\mathscr{U}) = 1$.

The proof is complete. □

Remark 4.3.13. Estimate (4.29) gives a uniform estimate on some H-projections μ_m of the chemical potential μ. In the following section, however, we need to act on the "full" chemical potential μ. To this purpose, we need the following lemma, which shows some further regularity of the weak solution $\mathbf{P}$ we have built in *Proof of Theorem* (4.3.11).

Lemma 4.3.14. *With the notation of Theorem* (4.3.11)*, we have*

$$\int_{\mathscr{X}} \|\nabla \mu(\phi)\|^2_{L^2([0,T];H)} \mathbf{P}(\mathrm{d}\phi) \leq C_{13}, \tag{4.38}$$

[12] which can be applied since trivially $|\exp\{i\alpha\}| \leq 1, \ \forall \alpha \in \mathbb{R}$.

$$\int_{\mathscr{X}} \|\phi\|^2_{L^2([0,T];U)} \mathbf{P}(\mathrm{d}\phi) \leq C_5, \tag{4.39}$$

where $\mu(\phi) := a\phi + \phi^3 - \phi - J * \phi$. The integrands are meant to assume the value $+\infty$ whenever $\nabla\mu(\phi) \notin L^2([0,T];H)$ or $\phi \notin L^2([0,T];U)$.

Proof. Estimate (4.29) implies

$$\int_{\mathscr{X}} \|\nabla\mu_r(\phi)\|^2_{L^2([0,T];H)} \mathbf{P}_m(\mathrm{d}\phi) \leq C_{13}, \quad \text{for } r \leq m. \tag{4.40}$$

For each $r \in \mathbb{N}$, we define

$$\Phi_r : \mathscr{X} \to \mathbb{R} \cup \{+\infty\} : \phi \mapsto \|\nabla\mu_r(\phi)\|^2_{L^2([0,T];H)}.$$

We use the definition of π_r given in *Proof of Theorem* (4.3.11), *Step 1*, since $\mu(\phi)$ takes values in $L^{4/3}$, a.e. $t \in [0,T]$, for $\phi \in \mathscr{X}$. The functional Φ_r is lower semicontinuous in $\mathscr{X}$. To prove this, let $\phi_n \to \phi$ in $\mathscr{X}$. Then, for any (not relabeled) subsequence, there is another (not relabeled subsequence) ϕ_n such that $\phi_n(t) \to \phi(t)$ in L^4 for a.e. $t \in [0,T]$. By taking the H-inner product of $\phi_n(t) - \phi(t)$ with $e_1, \cdots, e_m$ and using Hölder inequality along with the regularity of $e_1, \cdots, e_m$, we deduce that $\mu_r(\phi_n)(t) \to \mu_r(\phi)(t)$ in H. Hence, relying on the equivalence of norms of finite-dimensional vector spaces, we deduce that $\nabla\mu_r(\phi_n)(t) \to \nabla\mu_r(\phi)(t)$ in H for a.e. $t \in [0,T]$. Since the original subsequence ϕ_n was arbitrary, we may apply Fatou Lemma to deduce that Φ_r is lower semicontinuous in $\mathscr{X}$.
In addition, Φ_r, being nonnegative, is trivially bounded from below. Hence we may apply Portmanteau Theorem and write

$$\int_{\mathscr{X}} \Phi_r(\phi)\mathbf{P}(\mathrm{d}\phi) \leq \liminf_m \int_{\mathscr{X}} \Phi_r(\phi)\mathbf{P}_m(\mathrm{d}\phi) \leq C_{13}, \tag{4.41}$$

where the last inequality follows from (4.40). But now, thanks to the monotonicity of H-norm under the projection on growing subspaces, we may use the monotone convergence Theorem in (4.41) to deduce

$$\int_{\mathscr{X}} \|\nabla\mu(\phi)\|^2_{L^2([0,T];H)} \mathbf{P}(\mathrm{d}\phi) \leq C_{13},$$

i.e., (4.38). The proof of (4.39) is similar. It suffices to consider $\Phi_r(\phi) = \|\pi_r\phi\|_{L^2([0,T];U)}$. A finite-dimensional argument identical to the one used before shows that Φ_r is lower semicontinuous in $\mathscr{X}$ and bounded from below. Hence we once again apply Portmanteau theorem and get

$$\int_{\mathscr{X}} \Phi_r(\phi)\mathbf{P}(\mathrm{d}\phi) \leq \liminf_m \int_{\mathscr{X}} \Phi_r(\phi)\mathbf{P}_m(\mathrm{d}\phi) \leq C_5, \tag{4.42}$$

where the last inequality follows from (4.22). Thanks to the monotonicity of H-norm under the projection on growing subspaces, we replicate the application of the monotone convergence Theorem in (4.42) and deduce

$$\int_{\mathscr{X}} \|\phi\|^2_{L^2([0,T];U)} \mathbf{P}(\mathrm{d}\phi) \leq C_5.$$

The proof is complete. □

Remark 4.3.15. In the previous lemma, we have implicitly used the fact that, for every $x \in L^{4/3}$:
- the series $\{\|\nabla(\pi_r x)\|\}_{r\in\mathbb{N}}$ converges to $\|\nabla x\|$ if $x \in U$, and diverges to $+\infty$ if $\nabla x \notin H$.
- the series $\{\|\pi_r x\|_U\}_{r\in\mathbb{N}}$ converges to $\|x\|_U$ if $x \in U$, and diverges to $+\infty$ if $x \notin U$.

Remark 4.3.16. Let $\mathbf{P}$ be the weak solution built in Theorem (4.3.11). Lemma (4.3.14) and Remark (4.3.15) imply that

$$\mathbf{P}(\{v : \nabla\mu(v) \in L^2([0,T];H)\}) = 1,$$
$$\mathbf{P}(\{v : \nabla(av - v) \in L^2([0,T];H)\}) = 1,$$
$$\mathbf{P}(\{v : \nabla(J * v) \in L^2([0,T];H)\}) = 1.$$

Hence, recalling *Proof of Theorem* (4.3.11), *Step 6*, we deduce that the set

$$\big\{v \in \mathscr{U} :\ \nabla(av - v) \in L^2([0,T];H),\\ \nabla(J * v) \in L^2([0,T];H),\ 3v^2\nabla v \in L^2([0,T];H)\big\}$$

has $\mathbf{P}$-probability one. This fact will be useful in the next section.

4.4 Existence and uniqueness of a *strong* solution

We now proceed to prove existence, uniqueness and measurability of a *strong* solution for Problem (4.1).
Estimate (4.29) suggests the definition of a space more regular than $\mathscr{U}$, namely

$$\mathscr{U}_1 := \{v \in \mathscr{U} :\ \nabla\mu(v) \in L^2([0,T];H)\},$$

where as usual $\mu(v) := av + v^3 - v - J * v$. We provide the reader with the necessary definition.

Definition 4.4.1. *A process $\phi = \phi(t,x,\omega)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbf{m})$ is a **strong** solution for problem* (4.1) *if*

1. *ϕ satisfies*
$$\mathscr{D}(\phi(\omega)) = \left\{\phi_0(\omega), \frac{\partial w}{\partial t}(\omega)\right\} \qquad \mathbf{m} - a.s. \tag{4.43}$$
We recall from the previous section that $C(\phi) \in \mathscr{V}'$, $\langle C(\phi), v\rangle_{\mathscr{V}',\mathscr{V}} := C(\phi, v)$ if $\phi \in \mathscr{Z}$ and $\mathscr{D} : \mathscr{Z} \to H^{-\varepsilon} \times \mathscr{V}' : \phi \mapsto \{\phi(0), C(\phi)\}$.

2. *the mapping $\omega \mapsto \phi(\omega)$ is a random variable from $(\Omega, \mathcal{F})$ to $(\mathscr{U}_1, \mathcal{B}_{\mathscr{Z}}(\mathscr{U}_1))$.*

Remark 4.4.2. Our definition of strong solution to Problem (4.1) is in fact a little weaker than the definition of *classical strong* solution for a stochastic partial differential equation. We have [62, p. 73, Definition 4.2.1.] in mind. As a matter of fact, our solution is a *variational* solution, hence it is always joined by a *test function $v \in \mathscr{V}$*, while in the classical contest ([62, p. 73, Definition 4.2.1.] again) there is no need of test functions whatsoever. In addition, even though there is no restriction on the final positive time T (we did not have to make any assumptions up to now and the same will happen in the remaining part of this chapter), our solution satisfies a relation which involves the *entire* time interval $[0, T]$ instead of any arbitrary interval $[0, t]$, for $t \in [0, T]$.
The reason for which we gave the previous definition of strong solution is motivated by the relative lack of regularity of the equation (lack of sublinear growth conditions in particular) which forbids to apply the classical theorems of existence and uniqueness of a *classical strong* solution. We have [62, p. 75, Theorem 4.2.4.] in mind.

We now state and prove an auxiliary result which will be used to deduce the uniqueness of a strong solution.

Theorem 4.4.3. *Let ϕ_1, ϕ_2 be two strong solutions for problem* (4.1) *(for the same $\phi_0 \in U$, i.e. $\phi_1(0) = \phi_2(0) = \phi_0$) in the sense of Definition* (4.4.1)*. Then*
$$\phi_1(t) = \phi_2(t) \quad \text{in } U', \quad \text{for a.e. } t \in [0, T].$$

Remark 4.4.4. The Proof of Theorem (4.4.3) will show that it suffices to take $\phi_0 \in H$. However, the stronger condition $\phi_0 \in U$ will be required in the conclusive and most important theorem of this section, so it is natural to require it straight away.

Remark 4.4.5. The Proof of Theorem (4.4.3) is argued by means of purely deterministic arguments. The stochastic noise of our version of Cahn-Hilliard equation is additive and its stochastic integrand is constant (the identity operator); hence, when we subtract the expressions associated with two strong solutions the stochastic noise vanishes from the computations. If the stochastic were not constant but, e.g., were as in (1.21), the proof of the uniqueness would be significantly more complicated and we would be forced to rely on a neatly different theory: in fact, the stochasticity could not be removed. An even worse scenario would occur if the noise wasn't additive.

Proof of Theorem (4.4.3). Let $r := \phi_1 - \phi_2$. In analogy with [29, p. 1195, computations (8.1)-(8.3)], we can define, for every $\xi \in V_j$ and $t \in [0,T]$, the sequence

$$v_n(s) := \begin{cases} (t-s)\xi, & \text{if } s \in \left[0, t - 1/(4n^2)\right], \\ \left(n(s - [t + 1/(4n^2)])\right)^2 \xi, & \text{if } s \in [t - 1/(4n^2),\, t + 1/(4n^2)], \\ 0, & \text{if } s \in (t + 1/(4n^2), T]. \end{cases}$$

Here j is an arbitrary nonnegative integer. As a result, $v_n \in \mathscr{V}$ and

$$\frac{\partial v_n}{\partial s} = \begin{cases} -\xi, & \text{if } s \in [0, t - 1/(4n^2)], \\ 2n^2(s - [t + 1/(4n^2)])\xi, & \text{if } s \in [t - 1/(4n^2),\, t + 1/(4n^2)], \\ 0, & \text{if } s \in (t + 1/(4n^2), T]. \end{cases}$$

We now evaluate $C(\phi_1, v_n) - C(\phi_2, v_n)$. We can write

$$\begin{aligned} 0 &= -\int_0^T \int_D ru \cdot \nabla v_n - \int_0^T \int_D (\mu_1 - \mu_2)\Delta v_n - \int_0^T \left(r, \frac{\partial v_n}{\partial s}\right) \\ &= -\int_0^{t-1/(4n^2)} \int_D ru \cdot (t-s)\nabla\xi - \int_0^{t-1/(4n^2)} \int_D (\mu_1 - \mu_2)(t-s)\Delta\xi \\ &- \int_{t-1/(4n^2)}^{t+1/(4n^2)} \int_D ru \cdot (n(s - [t - 1/(4n^2)]))^2 \nabla\xi \\ &- \int_{t-1/(4n^2)}^{t+1/(4n^2)} \int_D (\mu_1 - \mu_2)(n(s - [t - 1/(4n^2)]))^2 \Delta\xi \\ &+ \int_0^{t-1/(4n^2)} (r, \xi) - \int_{t-1/(4n^2)}^{t+1/(4n^2)} (r, 2n^2(s - [t + 1/(4n^2)])\xi) \end{aligned} \tag{4.44}$$

where $\mu_1 := \mu(\phi_1)$, $\mu_2 := \mu(\phi_2)$. We may use the Lebesgue dominated theorem in (4.44) and deduce

$$-\int_0^t \int_D ru \cdot (t-s)\nabla\xi + \int_0^t \int_D \nabla(\mu_1 - \mu_2) \cdot (t-s)\nabla\xi + \int_0^t (r, \xi) = 0 \tag{4.45}$$

where $\mu_1 := \mu(\phi_1)$, $\mu_2 := \mu(\phi_2)$. We have also used integration by parts to treat the term $(\mu_1 - \mu_2, \Delta v)$. If we differentiate the last equality in (4.45) with respect to t, we obtain

$$\langle r(t), \xi \rangle_{V',V} - \int_0^t \int_D ru \cdot \nabla\xi + \int_0^t \int_D (\nabla\mu_1 - \nabla\mu_2) \cdot \nabla\xi = 0, \qquad \forall \xi \in V_j.$$

Since j is an arbitrary nonnegative integer, we may use the density of $\cup_{j=1}^{\infty} V_j$ in V and deduce that

$$\langle r(t), \xi\rangle_{V',V} - \int_0^t\int_D ru\cdot\nabla\xi + \int_0^t\int_D (\nabla\mu_1 - \nabla\mu_2)\cdot\nabla\xi = 0, \qquad \forall\xi\in V.$$

Thanks to the density of V in U, the Gelfand theory and the definition of $\mathscr{U}_1$ we get

$$\langle r(t), \xi\rangle_{U',U} - \int_0^t\int_D ru\cdot\nabla\xi + \int_0^t\int_D (\nabla\mu_1 - \nabla\mu_2)\cdot\nabla\xi = 0, \quad \forall\xi\in U. \tag{4.46}$$

Equation (4.46) implies that $\partial r/\partial t \in L^2([0,T];U')$ since, for every $\xi\in U$ and $\phi\in\mathcal{C}_0^\infty[0,T]$

$$\begin{aligned}\int_0^T \langle r(t),\xi\rangle_{U',U}\phi'(t) &= \int_0^T\left[\int_0^t (ru,\nabla\xi) - \int_0^t (\nabla\mu_1-\nabla\mu_2,\nabla\xi)\right]\phi'(t)\\ = \ -\int_0^T \left[(ru,\nabla\xi) - (\nabla\mu_1-\nabla\mu_2,\nabla\xi)\right]\phi(t) &= -\int_0^T \left\langle \frac{\partial r}{\partial t},\xi\right\rangle_{U',U}\phi(t),\end{aligned}$$

where we have defined $\langle \partial r/\partial t, \xi\rangle_{U',U} := (ru - (\nabla\mu_1 - \nabla\mu_2), \nabla\xi)$. Hence equation (4.46) leads to

$$\left\langle \frac{\partial r}{\partial t},\xi\right\rangle_{U',U} + (\nabla\mu_1-\nabla\mu_2,\nabla\xi) = (ru,\nabla\xi), \quad \xi\in U. \tag{4.47}$$

We can therefore act as in [32, Section 4, Proposition 5][13], of which we reproduce only the computations we need. Since $r(0) = \phi_1(0) - \phi_2(0) = 0$, it is clear that $(r(t),1) = 0$. We consider the operator

$$B : \mathcal{D}(B) = V \to \tilde{H} : u \mapsto -\Delta u,$$

where $\tilde{H} := \{u\in H : (u,1) = 0\}$. Then, if we take $\xi = B^{-1}r(t)\in\mathcal{D}(B)$, for almost every $t\in[0,T]$, (4.47) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|B^{-1/2}r\|^2 + 2(\mu_1-\mu_2,r) = 2(u, r\nabla B^{-1}r).$$

If we apply Lagrange theorem to F'' (F is regular enough to do so, recall hyphothesis (iv) of Section (4.2)) and use (4.2) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|B^{-1/2}r\|^2 + 2c_0\|r\|^2 \le 2(J*r,r) + C\|u\|_{L^\infty([0,T]\times D)}\|r\|\|B^{-1/2}r\|. \tag{4.48}$$

[13] we do not need certain hyphotesis contained in the statement of this proposition because we are arguing only the uniqueness argument. In particular, there is no additional requirement upon the spatial dimension d.

Moreover, assumption (iii) and Young inequality imply

$$|(J * r, r)| \leq \|B^{-1/2}(J * r)\| \|B^{-1/2} r\| \leq \frac{c_0}{4}\|r\|^2 + C'\|B^{-1/2} r\|^2. \tag{4.49}$$

If we combine (4.48)-(4.49) and use once again Young inequality to control the last term of the righthand side of (4.48) we obtain that $r \equiv 0$. □

As an immediate consequence we get the following.

Corollary 4.4.6. *The (restricted) operator* $\mathscr{D}|_{\mathscr{U}_1} : \mathscr{U}_1 \to H^{-\varepsilon} \times \mathscr{V}' : \phi \mapsto \{\phi(0), C(\phi)\}$ *is injective.*

We can now state and prove our conclusive theorem. We will use many facts which have been proved in Section (4.3); the theorem constitutes itself a "bridge" between this section and the previous one.

Theorem 4.4.7. *Let* $d \leq 3$. *Let* w *be a* H*-valued* Q*-Wiener process and let* w, u, Q, J, F, ϕ_0 *and* $\{e_j\}_{j\in\mathbb{N}}$ *satisfy properties* (i)-(viii) *previously listed at page 63 and following. Let* ϕ_0 *be a* U*-valued random variable such that*

$$\mathbb{E}\left[\|\phi_0\|_U^2 + \int_D \frac{\phi_0^4}{4} - \int_D \frac{\phi_0^2}{2}\right] < +\infty.$$

Then problem (4.1) *admits a unique strong solution (in the sense that two strong solutions coincide for all* $\omega \in \Omega$ *except for a set of* $\mathbf{m}$*-measure zero).*

Proof. The hypothesis of Theorem (4.3.11) are satisfied, thus we have the weak solution $\mathbf{P}$ built in *Proof of Theorem* (4.3.11).

Step 1: Costruction of suitable $\mathscr{X}$*-compact sets.* Let us consider the countable family of the sets

$$C_j := \Big\{ v \in \mathscr{X} : \|v\|_{L^2([0,T];U)} \leq j,\ \|v\|_{L^\infty([0,T];H)} \leq j,\ \|v\|_{L^4([0,T];L^4)} \leq j,$$
$$\|v\|_{\mathcal{C}^{2/5}([0,T];V')} \leq j,\ \|\nabla(J * v)\|_{L^2([0,T];H)} \leq j,\ \|3v^2\nabla v\|_{L^2([0,T];H)} \leq j,$$
$$\|\nabla(av - v)\|_{L^2([0,T];H)} \leq j. \Big\},$$

indexed by $j \in \mathbb{N}$. We show that C_j is a compact set in $\mathscr{X}$. Let v_n be an arbitrary sequence in C_j. Because of the compact injection $\mathscr{U} \hookrightarrow \mathscr{X}$, there is a (not relabeled) subsequence $v_n \to v$ in $\mathscr{X}$. We show that $v \in C_j$. Since $v_n \to v$ in $\mathcal{C}([0,T]; H^{-\varepsilon})$, we deduce $v_n \to v$ in $\mathcal{C}([0,T]; V')$. It is also obvious that

$$\|v_n\|_{\mathcal{C}([0,T];V')} + \frac{\|v_n(t_1) - v_n(t_2)\|_{V'}}{|t_1 - t_2|^{2/5}} \leq j, \quad \forall n \in \mathbb{N}, \quad \forall t_1, t_2 : 0 \leq t_1 < t_2 \leq T.$$

If, in the previous inequality, we first take the limit for $n \to +\infty$ and then the extreme superior for all $0 \le t_1 < t_2 \le T$ we deduce

$$\|v\|_{\mathcal{C}^{2/5}([0,T];V')} \le j.$$

Since v_n is bounded in $L^2([0,T];U)$ and $L^4([0,T];L^4)$, we deduce, using [60, p. 75, Esercizio 54], that v is the weak limit both in $L^2([0,T];U)$ and $L^4([0,T];L^4)$ for a (not relabeled) subsequence v_n. Hence

$$\|v\|_{L^2([0,T];U)} \le j,$$

$$\|v\|_{L^4([0,T];L^4)} \le j.$$

We may use a similar argument for the weak convergences in $L^p([0,T];H)$ and deduce

$$\begin{aligned}\|v\|_{L^\infty([0,T];H)} &= \lim_{p\to+\infty} \|v\|_{L^p([0,T];H)} \le \lim_{p\to+\infty} \{\liminf_n \|v_n\|_{L^p([0,T];H)}\} \\ &\le \lim_{p\to+\infty} T^{1/p} j = j.\end{aligned}$$

It follows that $v \in G_j(\mathscr{U})$[14].
Once again, we rely on the $[0,T]$-almost sure convergence in L^4 previously noticed to apply Fatou lemma and deduce

$$\|\nabla(J * v)\|_{L^2([0,T];H)} = \|\nabla(J) * v\|_{L^2([0,T];H)} \le j.$$

Since $v_n \rightharpoonup v$ in $L^2([0,T];U)$, we deduce that

$$\|\nabla(av - v)\|_{L^2([0,T];H)} \le j,$$

thanks to the regularity of a. We now turn to the last and most delicate term appearing in the definition of C_j. Let us consider a (not relabeled) subsequence $v_n \rightharpoonup v$ in $L^2([0,T];U)$ such that $v_n(t) \to v(t)$ in L^4 for a.e. $t \in [0,T]$. For each $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we define some suitable *truncated* functions as follows

$$z_n^k := \min\{3v_n^2; k\},$$

$$z^k := \min\{3v^2; k\}.$$

The sequence $\{z_n^k\}_{n\in\mathbb{N}}$ is clearly bounded in $L^\infty([0,T];L^\infty)$. Using the time almost sure convergence $v_n(t) \to v(t)$ in L^4 and Hölder inequality we deduce

[14] the computations done so far in this proof show that $G_j(\mathscr{U})$ is a compact set in $\mathscr{X}$. This fact is used in *Proof of Theorem* (4.3.11), *Steps 5 and 7.*

that, for a.e. $t \in [0,T]$

$$
\begin{aligned}
\|z_n^k - z^k\|^2 &= \int_D \left|\min\{3v_n^2; k\} - \min\{3v^2; k\}\right|^2 \\
\leq\ & 9 \int_D |v_n^2 - v^2|^2 = 9 \int_D |v_n - v|^2 |v_n + v|^2 \\
\leq\ & 36 \|v_n - v\|_{L^4}^2 \left\{ \|v_n\|_{L^4}^2 + \|v\|_{L^4}^2 \right\} \to 0
\end{aligned}
\tag{4.50}
$$

as $n \to +\infty$. In computations (4.50) we have also used to elementary inequalities

$$
|\min\{a; c\} - \min\{b; c\}| \leq |a - b|, \quad \forall a, b, c \in [0, +\infty),
$$

$$
(a + b)^p \leq a^p + b^p, \quad \forall a, b \in [0, +\infty),\ p \in (0, 1).
$$

In addition, $\|z_n^k - z^k\|^2 \leq |D| k^2$ for every $t \in [0,T]$ and every $n \in \mathbb{N}$. Hence the Lebesgue dominated convergence theorem implies that $z_n^k \to z^k$ in $L^2([0,T]; H)$ as $n \to +\infty$. Moreover, $\{z_n^k \nabla v_n\}_{n \in \mathbb{N}}$ is bounded in $L^2([0,T]; H)$, hence a (not relabeled) subsequence $z_n^k \nabla v_n \rightharpoonup h$ in $L^2([0,T]; H)$ as $n \to +\infty$. In addition $\nabla v_n \rightharpoonup \nabla v$ in $L^2([0,T]; H)$ since $v_n \rightharpoonup v$ in $L^2([0,T]; U)$. Hence, for each $\ell \in L^\infty([0,T]; L^\infty)$, we deduce

$$
\begin{aligned}
&\left| (z_n^k \nabla v_n - z^k \nabla v\, ,\, \ell)_{L^2([0,T];H)} \right| \leq \left| \int_0^T (z_n^k - z^k) \nabla v_n \cdot \ell \right| \\
+\ & \left| \int_0^T z^k (\nabla v_n - \nabla v) \cdot \ell \right| \leq \|\ell\|_{L^\infty([0,T];L^\infty)} \|z_n^k - z^k\|_{L^2([0,T];H)} \|\nabla v_n\|_{L^2([0,T];H)} \\
+\ & \left| \int_0^T z^k (\nabla v_n - \nabla v) \cdot \ell \right| \to 0,
\end{aligned}
$$

as $n \to +\infty$. Because of the density of $L^\infty([0,T]; L^\infty)$ in $L^2([0,T]; H)$, we deduce $h = z^k \nabla v$. We can hence rely on the lower semicontinuity property for weakly convergent sequences and deduce

$$
\begin{aligned}
\int_0^T \|z^k \nabla v\|^2 &\leq \left[\liminf_n \left(\int_0^T \|z_n^k \nabla v_n\|^2 \right)^{1/2} \right]^2 \\
\leq\ & \left[\liminf_n \left(\int_0^T \|3 v_n^2 \nabla v_n\|^2 \right)^{1/2} \right]^2 \leq j^2.
\end{aligned}
\tag{4.51}
$$

For a.e. $t \in [0,T]$ we have that $z^k \to 3v^2$ a.e. in D for $k \to +\infty$. Hence we deduce, applying Fatou lemma in space and time and using (4.51)

$$
\int_0^T \|3v^2 \nabla v\|^2 \leq \int_0^T \liminf_k \|z^k \nabla v\|^2 \leq \liminf_k \int_0^T \|z^k \nabla v\|^2 \leq j^2
$$

and consequently

$$\|3v^2\nabla v\|_{L^2([0,T];H)} = \left(\int_0^T \|3v^2\nabla v\|^2\right)^{1/2} \leq j.$$

We conclude that C_j is compact in $\mathscr{X}$.

Step 2: Costruction of a suitable restriction of $\mathscr{D}$. Let $X := \cup_{j\in\mathbb{N}} C_j \in \mathcal{B}_{\mathscr{X}}(\mathscr{U})$. Thanks to Remark (4.3.16) we deduce that $\mathbf{P}(X) = 1$.
Since any $v \in C_j$ takes values in U, it is easy to verify that $\nabla(v^3) = 3v^2\nabla v$ for a.e. $t \in [0,T]$. This fact, along with *Step 1*, implies that $C_j \subset \mathscr{U}_1$, $\forall j \in \mathbb{N}$.
Since X is a countable union of $\mathscr{X}$-compact sets contained in $\mathscr{U}_1$ and $\mathscr{D} : \mathscr{X} \to H^{-\varepsilon} \times \mathscr{V}' =: \mathscr{Y}$ is continuous as observed in Remark (4.3.9), it follows that $X \in \mathcal{B}_{\mathscr{X}}(\mathscr{U}_1)$ and that $\mathscr{F} := \mathscr{D}(X) \in \mathcal{B}(\mathscr{Y})$. In addition

$$(\Xi \times \mathbf{N})(\mathscr{F}) = \mathbf{P}(\mathscr{D}^{-1}(\mathscr{F})) \geq \mathbf{P}(X) = 1, \tag{4.52}$$

where we have used (4.9). Let $\mathscr{D}_1 : X \to \mathscr{F}$ be the restriction $\mathscr{D}|_X$. Since $X \subset \mathscr{U}_1$, Corollary (4.4.6) implies that $\mathscr{D}_1$ is one-to-one.

Step 3: Measurability of $\mathscr{D}_1^{-1}$. The mapping $\mathscr{D}_1^{-1} : (\mathscr{F}, \mathcal{B}_{\mathscr{Y}}(\mathscr{F})) \to (X, \mathcal{B}_{\mathscr{X}}(X))$ is measurable. To prove this, we only need to show that $\mathscr{D}(B) \in \mathcal{B}_{\mathscr{Y}}(\mathscr{F})$ for every B closed set in X in the topology of $\mathscr{X}$, i.e. $B = B_1 \cap X$ for B_1 closed set in $\mathscr{X}$. Since $X = \cup_{j\in\mathbb{N}} C_j$, we have

$$\begin{aligned} \mathscr{D}(B) &= \mathscr{D}(B_1 \cap X) = \mathscr{D}(B_1 \cap (\cup_{j\in\mathbb{N}} C_j)) \\ &= \mathscr{D}(\cup_{j\in\mathbb{N}}(B_1 \cap C_j)) = \cup_{j\in\mathbb{N}} \mathscr{D}(B_1 \cap C_j) \in \mathcal{B}_{\mathscr{Y}}(\mathscr{F}), \end{aligned}$$

since $B_1 \cap C_j$ is compact for each $j \in \mathbb{N}$, $\mathscr{D} : \mathscr{X} \to \mathscr{Y}$ is continuous and hence $\mathscr{D}(B_1 \cap C_j) \in \mathcal{B}_{\mathscr{Y}}(\mathscr{F})$.

Step 4: Construction of the unique strong solution ϕ. We now denote

$$\Omega_1 := \left\{\omega \in \Omega : \left\{\phi_0(\omega), \frac{\partial w}{\partial t}(\omega)\right\} \in \mathscr{F}\right\}.$$

Thanks to the measurability of ϕ_0 and $\partial w/\partial t$ and to Remark (4.3.3), we have $\Omega_1 \in \mathcal{F}$. Relying on (4.52), we get $\mathbf{m}(\Omega_1) = (\Xi \times \mathbf{N})(\mathscr{F}) = 1$. We finally define

$$\phi(\omega) := \begin{cases} \mathscr{D}_1^{-1}\left(\left\{\phi_0(\omega), \dfrac{\partial w}{\partial t}(\omega)\right\}\right), & \text{if } \omega \in \Omega_1, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that ϕ satisfies (4.43). Since $X \in \mathcal{B}_{\mathscr{Z}}(\mathscr{U}_1)$, for each $G \in \mathcal{B}_{\mathscr{Z}}(\mathscr{U}_1)$ we have $G \cap X \in \mathcal{B}_{\mathscr{Z}}(X)$. In addition, we get

$$\begin{aligned}
& \{\omega \in \Omega : \phi(\omega) \in G\} \\
= \; & \left\{\omega \in \Omega_1 : \mathscr{D}_1^{-1}\left(\left\{\phi_0(\omega), \frac{\partial w}{\partial t}(\omega)\right\}\right) \in G \cap X\right\} \cup \left\{\omega \in \Omega_1^C : 0 \in G\right\} \\
= \; & \underbrace{\left\{\omega \in \Omega_1 : \left\{\phi_0(\omega), \frac{\partial w}{\partial t}(\omega)\right\} \in \mathscr{D}_1(G \cap X)\right\}}_{\Omega_2} \cup \underbrace{\left\{\omega \in \Omega_1^C : 0 \in G\right\}}_{\Omega_3} \in \mathcal{F}.
\end{aligned}$$

We have used the bijectivity of $\mathscr{D}_1^{-1}$ in the second equality, the measurability of $\mathscr{D}_1^{-1}$ to see that $\mathscr{D}_1(G \cap X) \in \mathcal{B}_{\mathscr{Y}}(\mathscr{F})$, the measurability of the random variable $\omega \mapsto \{\phi_0(\omega), \partial w/\partial t(\omega)\}$ to deduce that $\Omega_2 \in \mathcal{F}$ and the fact that Ω_3 is either $\emptyset$ or Ω_1^C, hence $\Omega_3 \in \mathcal{F}$ in both cases.
Hence, ϕ is measurable as a random variable from $(\Omega, \mathcal{F})$ to $(\mathscr{U}_1, \mathcal{B}_{\mathscr{Z}}(\mathscr{U}_1))$ and is then a strong solution in the sense of Definition (4.4.1). In addition, the injectivity of $\mathscr{D} : \mathscr{U}_1 \to \mathscr{Y}$ also implies the uniqueness of a strong solution.

Step 5: Distribution of ϕ. It is also clear, from the computations made in Remark (4.3.7), that the distribution of a strong solution is a weak solution. In addition, $\mathbf{P}$ is the distribution of ϕ: to prove this, let $\mathbf{P}_1$ be another weak solution concentrated on $\mathcal{B}_{\mathscr{Z}}(X)$[15]. We rely on the bijectivity of $\mathscr{D}_1$ and write, for any given subset $C \in \mathcal{B}_{\mathscr{Z}}(X)$,

$$\begin{aligned}
\mathbf{P}_1(C) &= \mathbf{P}_1(\mathscr{D}_1^{-1}(\mathscr{D}_1(C))) = \mathbf{P}_1(\mathscr{D}^{-1}(\mathscr{D}_1(C))) = (\Xi \times \mathbf{N})(\mathscr{D}_1(C)) \\
&= \mathbf{P}(\mathscr{D}^{-1}(\mathscr{D}_1(C))) = \mathbf{P}(\mathscr{D}_1^{-1}(\mathscr{D}_1(C))) = \mathbf{P}(C).
\end{aligned}$$

Hence $\mathbf{P} = \mathbf{P}_1$. Since the distribution of ϕ clearly is a weak solution concentrated on $\mathcal{B}_{\mathscr{Z}}(X)$, we deduce that $\mathbf{P}$ is the distribution of ϕ.

The proof is complete. $\square$

Remark 4.4.8. We point out that *Proof of Theorem* (4.4.7), *Step 5* can be seen as a result of *partial* uniqueness for a weak solution, since the requirement upon the σ-algebra $\mathcal{B}_{\mathscr{Z}}(X)$ restricts the set of probability measures in which we look for a weak solution. We do not investigate the uniqueness of a weak solution concentrated on the σ-algebra $\mathcal{B}_{\mathscr{Z}}(\mathscr{U})$ since, in such a context, we cannot rely on the bijectivity of $\mathscr{D}$.

Remark 4.4.9. The spatial dimension requirement $d \leq 3$ is needed only in *Proof of Theorem* (4.3.1) to guarantee the compact embedding $U \hookrightarrow L^4$ and in *Proof*

[15] note that we require this other weak solution to be concentrated on $\mathcal{B}_{\mathscr{Z}}(X)$ instead of the σ-algebra $\mathcal{B}_{\mathscr{Z}}(\mathscr{U})$ appearing in Definition (4.3.5).

of Theorem (4.4.7), *Step 2*, to compute the distributional gradient of v^3, for any $v \in U$.

Remark 4.4.10. Apparently it does not seem straightforward to relax the growth restriction on F (i.e. to consider a higher-order polynomial growth) we have used to prove Theorems (4.3.11) and (4.4.7).

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List of Figures