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Vibration analysis of composite laminated plates and shells using a spectral method

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Summary

Framework: This thesis deals with the free vibration analysis of beams, composite plates and shells with different boundary conditions. A spectral collocation technique is adopted for the numerical solution of the differential problem. Solutions are sought in the form of interpolating polynomials over 1D and 2D grid of Chebyshev-Gauss-Lobatto points. Such approximation is substituted into the equations of motion and the resulting residual is enforced to vanish at the interior collocation points. The same approximation is also put into the set of boundary conditions which are enforced to be satisfied at the boundary points. The implementation of the method is performed in an efficient and elegant matrix notation by using the Chebyshev differentiation matrix and the Kronecker product.

Motivation: In most engineering areas, such as aerospace, naval and automotive industries, the use of plate-and shell-like structural elements having anisotropic properties through the thickness has considerably increased in recent years. This work should be able to give accurate results for structures exhibiting non-classical behavior of transversely anisotropic structures (sandwich plates with soft core, laminated plates and shells). Anisotropic multi-layered structures often posses higher transverse shear and normal flexibility than traditional isotropic one-layer construction. An accurate description of the stress and strain fields of these structures requires theories that are able to describe the so-called zig-zag form of displacement fields in the thickness direction as well as inter-laminar continuous transverse shear and normal stresses. Transverse and in-plane anisotropy of multi-layered structures make it difficult to find closed-form solutions when these structures are subjected to the dynamical loading. Combination of spectral collocation method which is showing very good convergence and having straight forward implementation, and also (CUF) technique which is a power full 3D quasi technique for solving composite plates and shells with considering real behavior through the thickness in structures is so helpful and is a good framework.

Approach: In this thesis, a relatively new modeling technique will be adopted. It is known as Carrera's Unified Formulation (CUF). The proposed technique meets all the requirement introduced thus far since it allows to generate arbitrarily accurate solutions from a large variety of refined theories by properly expanding so-called fundamental nuclei of the mass, stiffness and load matrices. The fundamental nuclei are invariant with respect to the order of theory and thus any theoretical development and software coding is needed when the order is changed.

Novelty: CUF was already implemented in standard finite element [173, 174], Ritz [180, 181] and Galerkin [190] methods based on the weak form and recently, using GDQ [183] method. The aim of this work is to build a power full and general numerical framework into which CUF modeling is associated with spectral methods. The proposed method is based on strong form.

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Chapter 1

Introduction and Literature review

Review of Chebyshev collocation method

The Chebyshev collocation method has been applied to different applications due to its high rate of convergence and accuracy. Gottlieb and Orszag [1] surveyed application of spectral methods for solving hyperbolic and advection-diffusion equations and studied these methods not only on solid mechanics but also on incompressible fluid dynamics. Yagci et.al [2] used a spectral Chebyshev technique for solving linear and nonlinear beam equations, and the method was applied for Euler-Bernoulli and Timoshenko beams, in which the spectral-Chebyshev technique incorporates the boundary conditions into the derivation, thereby enabling the utilization of the solution for any linear boundary conditions without re-derivation. Furthermore, Lin and Jen [3] applied the collocation method on the laminated anisotropic plates in which the solution of the problem is assumed to be a set of Chebyshev polynomials with unknown constants. A set of collocation points, also called Gauss-Lobatto points, are selected to be substituted into those polynomials. Also, Lin and Jen [4] applied the collocation method on non-rectangular anisotropic plates, and the results were verified by comparing them with the finite element method. Moreover, Deshmukh [5] applied the method with quadratic optimization for parameter estimation in nonlinear time varying systems, and the results were accurate. Trefethen [6] obtained the natural frequencies of a clamped square Kirchhoff plates by defining a function that satisfies the boundary conditions in the x and y directions. Luo [7] used the Chebyshev collocation method to solve a wave propagation problem in a three-dimensional cube with all sides constrained with homogeneous Dirichlet boundaries, by stacking two-dimensional (x, y) slice matrices along the z direction. Ehrenstien and Peyret [8] applied the Chebyshev collocation method for solving the unsteady two-dimensional Navier-Stokes equations in vorticity-stream function variables. Furthermore, the method was applied to a double diffusive convection problem concerning the stability of a fluid stratified by salinity and heated from below, and the results showed excellent agreement when compared to other results available in the literature. Deshmukh et.al [9] applied the Chebyshev spectral collocation method to reduce the dimensions of nonlinear delay differential equations with periodic coefficients. Regarding the Chebyshev polynomials, Zhou et al. [10] used Chebyshev polynomials for studying the three-dimensional vibration of thick rectangular plates. Also, Sinha and Butcher [11] used Chebyshev polynomials to study the stability of systems with parametric excitation and structures with time-dependent loads. In these works, the product and integration operational matrices associated with the shifted Chebyshev polynomials are utilized. The advantages of this method are, it is easily employed in a symbolic form and, the number of polynomials may be adjusted to attain convergence. Gourgoulhon [12] implemented basic principles of spectral methods, Legendre and Chebyshev polynomials with their behavior and some examples. Ferreira [13] presented vibration solution of thin composite plates using combination of radial basis function and Pseudospectral method for more accuracy. Karageorghis [14] reported the grid collocation points distribution on the boundary conditions of horizontal and vertical sides in rectangular plate. Several researchers [15, 16, 17] published books about spectral methods for solving differential equations.

Review of Timoshenko beam

Vibration analysis of beams has an important role in applications of mechanical, aerospace and civil engineering. Two models for beams include Euler beams and Timoshenko beams. In the Euler beam model (classical model), the ratio of the beam thickness to its length is relatively small ($\frac{h}{L} \sim \frac{1}{20}$), and thus the transverse shear and the rotary inertia are neglected. The Timoshenko beam, on the other hand, transverse shear and rotary inertia are not neglected as in the classical model. Thomas and Abbas [18] and Friedman and Kosmatka [19] applied the finite element method to obtain the natural frequencies for the Timoshenko beam. Laura and Gutierrez [20] applied the differential quadrature method for the free vibration of non-uniform Timoshenko beams. The free vibration of Timoshenko beams with elastically restrained boundaries were studied by Abbas [21] using the finite element method, where the effects of the translational and rotational springs on the natural frequencies were investigated. Stafford and Giurgiuti [22] applied a semi-analytical approach by assuming converged power series to obtain the eigenvalues for a rotating Timoshenko beam. Yokoyama [23] analyzed in-plane and out-of-plane free vibrations by the finite element technique. Moreover, Lee and Kuo [24] derived the upper bound of the fundamental frequency of a rotating Timoshenko beam with elastically restrained boundaries by Rayleigh's principle. Banerjee [25] applied the dynamic stiffness matrix to obtain the natural frequencies of the rotating Timoshenko beam, where the author showed that other researchers [24, 26] had incorrect results due to omitting the term related to the rotational speed in the governing equations. It was shown that at higher rotational speeds the previous results are not accurate. Recently, Kaya [27] applied the differential transform method to solve the natural frequencies of a uniform Timoshenko beam with constant rotating speed. Lee and Schultz [28] used the Chebyshev Pseudospectral method to solve the natural frequencies of Timoshenko beam for many boundary conditions and thickness ratio. Sprague and Geers [29] studied and compared Spectral, hierarchical and h-type finite elements in the context of their applications to structural vibration and plotted the eigenvalues convergence. Coskun et al. [30] evaluated eigenvalues of Euler-Bernoulli beam using several methods such as adomain decomposition, variational iteration and homotopy perturbation method with constant cross-section and also variable cross section and compared all these methods with exact solution.

Review of classical plate theory

Vibration analysis of plates has an important role in applications of mechanical, aerospace and civil engineering. Studying the free vibration of plates has a wide range of interest in the literature. For the Kirchoff plate, the ratio of the plate thickness to edge lengths is relatively small, and thus the transverse shear and rotary inertia are neglected. The well known paper by Leissa [31] is considered as the main reference for studying the free vibration of rectangular plates, in which exact characteristic equations are obtained when two opposite edges are simply supported, and the Ritz method is applied to the remaining cases. Dickinson and Li [32] used the Rayleigh-Ritz method with simply supported plate functions to solve the free vibration problem. Cupial [33] and Bhat [34] used the Rayleigh-Ritz method with orthogonal polynomials to study the free vibration of composite and isotropic plates respectively. Dozio [35] studied free in-plane vibration analysis of rectangular plates with edge elastic restrained varying linearly and parabolically both against the rotational and translational direction based on the classical theory of plate using Ritz method via trigonometric functions. Li et al. [36] applied two-dimensional Fourier series supplemented with several one- dimensional Fourier series to solve the free vibration problem of plates with general elastic boundary supports. Shu and Du [37] applied the differential quadrature method to solve the free vibration problem. Bert et al. [38] applied the differential quadrature method with δ-technique to solve the static and free vibration problems of anisotropic plates. Due to their importance in the design of structural elements in aerospace, ocean and naval structures, orthotropic plates are widely used in the fields of structural engineering where the use of such structures requires an understanding of the vibration features of orthotropic plates. The free vibration analysis of orthotropic plates was analyzed by many researchers. Hearmon [39] applied the Rayleigh method with characteristic beam functions for the free vibrations of the orthotropic plates. Dickinson and Di Blasio [40] studied the free vibration and buckling of rectangular isotropic and orthotropic plates with various boundary conditions using orthogonal polynomials in the Rayleigh-Ritz method. Rossi et al. [41] analyzed the rectangular orthotropic plates with one or more free edges through the optimized Rayleigh-Ritz method and a pseudo-Fourier expansion, in which the obtained results showed excellent agreement with those obtained by analytical and numerical methods. Gorman and Ding extended the method based on the superposition of appropriate Levy type solutions for the analysis of rectangular plates to the free vibration analysis of clamped orthotropic [42] and free orthotropic plates [43, 44]. Yu and Cleghorn [45] obtained the natural frequencies for orthotropic rectangular plates with combinations of clamped and simply supported boundary conditions by applying the superposition method. Dalaei and Kerr [46] applied the Kantorovich method [47] of reducing a partial differential equation to an ordinary differential equation to obtain natural frequencies of orthotropic thin plates with clamped edges. Sakata et al. [48] analyzed the free vibration of orthotropic rectangular plates by applying the successive reduction of the plate partial differential equations and assuming an approximate solution which satisfies the boundary conditions along one direction while employing the Kantorovich method. Ramkumar et al. [49] studied the free vibration analysis of clamped orthotropic plates through the Lagrange multiplier method. Huang et al. [50] studied the free vibration of orthotropic rectangular plates with variable thickness and general boundary conditions by applying a discrete method in which the Green function is used to establish the characteristic equation of the free vibration. Xing and Liu [51] obtained new exact solutions for free vibrations of thin orthotropic rectangular plates through a novel separation of variables method. Bert and Malik [52] applied the differential quadrature method to study the free vibration analysis of tapered isotropic and orthotropic rectangular plates having two opposite edges simply supported. Sari et al. [53] reported the solution of Kirchhoff plate's vibration using spectral collocation technique with and without damage boundary conditions and compared the results with perturbation method. Zhao and Wei [54] compared the results obtained for thin isotropic plates using discrete singular convolution (DSC) method with exact solution which is called Levy solution in high frequencies and showed that DSC method is so applicable for this analysis. Shu and Wang [55] dealed with the treatment of mixed and non-uniform boundary conditions using generalized differential quadrature method for thin plates. Ashour [56] applied the finite strip transition technique to angle-ply laminated thin plates with edge elastically restrained against rotation and translation and studied the effect of edge conditions on the natural frequencies and mode shapes. Chow et al. [57], Leissa and Narita [58] applied Ritz method using the admissible 2D orthogonal polynomials for solving vibration of composite thin plates considering the effect of material properties, number of layers and fiber orientation. Ng et al. [59] compared DSC-LK method with GDQ method in vibration analysis of thin isotropic plates in many cases such as mixed boundary conditions, edge constrained and non-uniform boundary conditions. Also Wei et al. [60] applied DSC for the same problem. Leissa et al. [61] presented vibration analysis of rectangular thin plates with elastic edge constraints for first time using two methods, one is exact solution and the other one is extension of Ritz method. Karami et al. [62] reported solving the vibration and stability analysis of laminated skew and trapezoidal thin plates using differential quadrature method. Yousefi et al. [63] used a semi analytical method based on the Rayleigh-Ritz method for thin composite laminated plates in various shape such as square, triangular, circular and rectangular by cut off quarter circle. Catanial and sorrentino [64] adopted Ritz method using Trigonometric and polynomial interpolation functions for mapping the shape of the skew and trapezoidal plate with curved edge, annular and sector annular plate and triangular plate. Secgin and Sarigul [65] considered vibration analysis of composite laminated of thin plates using discrete singular convolution for clamped and simply-supported cases.

Review of in-plane vibration

Kim et al. [66] presented the in-plane vibration of circular plates with edge restrained elastically both radial and tangential. Mode shapes are presented by trigonometric functions with a number of nodal diameters in the circumferential direction and mode functions in the radial directions. Singh and Muhammad [67] studied free in-plane vibration of skew and sector annular plate for isotropic plates using energy functional. Plate is defined by four curved boundaries using eight points and natural coordinates to map the geometry. Each displacement nodes has two degrees of freedom. Du et al. [68] considered the free in-plane vibration of isotropic rectangular plates with edge elastically restrained using analytical solution. Dozio [69] solved free vibration of rectangular plates with elastic boundaries using Ritz method with a set of trigonometric functions. Irie et al. [70] considered the in-plane vibration of annular isotropic plates using Runge-Kutta-Gill integration method and assumed that the in-plane displacements are periodic in circumferential direction. Gorman [71] solved in-plane vibration of plates for fully free boundary conditions using superposition method as an analytical solution and compared it with in the literature which is done by Ritz method. Ravari and Forouzan [72] present the vibration analysis of orthotropic circular plate using Helmholtz decomposition method for uncoupling the equations of motion and separation of variable. Wave equations are assumed in a harmonic type. Park [73] studied in-plane vibration analysis of clamped circular plates using kinetic and potential energy and applying Hamilton's principle on them and after that the same as previous author used Helmholts decomposition for solving differential equations and also separation of variable. Liu and Xing [74] solved in-plane vibration of rectangular plates analytically for simply-supported condition at two opposite edges and other type of conditions for other two edges. The characteristic equations for obtaining eigenvalues are solved by Newton-Raphson method. Bashmal et al. [75] reported in-plane modal characteristic of circular annular disk under classical boundary conditions. The boundary characteristic orthogonal polynomials are employed in the RayleighRitz method to obtain the natural frequencies and associated mode shapes. Fantuzzi et al. [76, 77, 78] considered decomposition method for solving composite laminated arbitrary shape plates using differential quadrature method.

Review of first and higher-order theories of the plates

Beams and plates usually are modeled as either Kirchhoff or Mindlin theory. For the Kirchhoff plate, the ratio of the thickness to the lengths is relatively small, and thus the transverse shear and rotary inertia are neglected, unlike in the Mindlin model where this ratio is no longer considered small. Mindlin [79] developed a theory of the transverse vibration of thick plates including the effect of transverse shear and rotary inertia, in which few analytical solutions were obtained for plates with free edges. Levinson [80] obtained an exact solution for free vibrations of a simply supported rectangular plate. Srinivas and Rao [81] obtained analytical solutions for the problem of bending, buckling and vibration of simply supported thick orthotropic plates. Chen and Doong [82] applied the Galerkin method to study the vibration of simply supported rectangular plates initially stressed by a uniform bending stress. Dawe and Roufaeil [83] applied the Rayleigh-Ritz method for the free vibration problem of Mindlin plates by using Timoshenko beam functions as the trial functions. Greimann and Lynn [84] applied the finite element method to study the plate bending with transverse shear deformation. Moreover, Rock and Hinton [85] applied the finite element method to study the transient response and free vibration of both thin and thick plates. Liew et al. [86] used boundary characteristic orthogonal polynomials. Liew et al. [87] applied pb-2 Ritz method to obtain the natural frequencies of skew plates based on shear deformation theory. Chung et al. [88] applied the Rayleigh-Ritz method with Timoshenko beam functions to study the free vibrations of orthotropic Mindlin plates with non-classical boundary conditions where the edges were elastically restrained against rotation. Saha et al. [89] studied the free vibration analysis of Mindlin plates with elastic restraints by a variational method. Many researchers studied the free vibration of annular plates using different techniques. Rao and Prasad [90] derived the frequency equations in explicit form for nine sets of boundary conditions, and they concluded that the shear deformation has a larger effect on the natural frequencies than does the rotary inertia effect. Furthermore, Irie et al. [91] studied the free vibration of Mindlin annular plates of radially varying thickness using the transfer matrix approach in which the governing equations of an annular plate are written as a coupled set of first-order differential equations using the transfer matrix of the plate. Irie et al. [92] obtained the natural frequencies for uniform annular plates under nine combinations of boundary conditions by assuming the displacements in terms of Bessel functions of the first and the second kind. They were then substituted back into the governing equations after applying the boundary conditions to obtain the frequency equation. Moreover, Sinha [93] calculated the natural frequencies of a thick spinning annular disk (modeled as a Mindlin plate) using the Rayleigh-Ritz method with trial functions obtained numerically by an iterative scheme. In a like manner, Nayar et al. [94] applied the finite element method to calculate the buckling loads and the natural frequencies for moderately thick annular plates uniformly compressed at the inner edge. Han and Liew [95] obtained the first five non-symmetric natural frequencies of thick annular plates for different thickness and radii ratios by the differential quadrature method. Civalek and Giirses [96] studied the free vibration analysis of annular Mindlin plates with free edges by the discrete singular convolution method. Regarding circular Mindlin plates, Pardoen [97] studied the vibration and buckling analysis of non-symmetric circular plates using the finite element method. Liew et al. [98] applied the differential quadrature method to study the free vibration problem. Lee and Schultz [99] studied the same problem by the Chebyshev pseudospectral method. Both of these studies showed excellent results. Mindlin plates can have different geometries; for example, there are rectangular, circular, circular annular, triangular, and annular sector plates, and the later type of plate is one of the most widely used structural components in engineering applications. Vibration analysis of annular sector plates is of principal importance in practical design. Many researchers have analyzed the free vibration of the annular sector plates. Guruswamy and Yang [100] developed 24 degrees of freedom sector finite element for the static and dynamic analysis of thick circular plates, where the flexibility of the proposed method was validated by performing free vibration analysis of full clamped sector plates with different sectoral angles and various thicknesses. In the same manner, Cheung and Chan [101] proposed two and three-dimensional finite strips for the analysis of thin and thick sectoral plates, in which the displacement functions for the finite strips are expressed in terms of beam eigenfunctions and polynomial shape functions. In their developed method, the two- dimensional finite strips were derived based on plate bending theory, while three- dimensional finite strips were formulated using three-dimensional elasticity constitutive equations. Srinivasan and Thiruvenkatachari [102] applied the integral equation technique to study the free vibration problem of fully clamped plates. Mizusawa [103] studied the free vibration problem for thick annular sector plates with simply supported straight edges through the semi analytical finite strip method and finite prism method. Xiang et al. [104] applied the Rayleigh-Ritz method for the analysis of the same problem. Liew and Liu [105] obtained the natural frequencies for the annular sector plates by applying the differential quadrature method, in which six different boundary conditions were considered. Lanhe et al. [106] presented vibration analysis of non-symmetric angle-ply laminated composite thick plate based on first-order shear deformation theory using moving least square differential quadrature method. Xiang et al. [107] proposed a nth-order shear deformation theory for solving vibration analysis of general composite laminated plate. This theory can satisfies the zero transverse shear stress boundary conditions on the top and bottom surface of the plate and Reddy's plate theory can be considered as a special case of this theory. Natural frequencies are computed by a mesh less radial point collocation method based on the thin plate spline radial basis function. Liew and Liu [108] presented free vibration analysis of moderately thick annular sector isotropic plates for various boundary conditions, sector angle and thickness ratio based on the Mindlin first-order shear deformation theory using differential quadrature method. And also Liew et al. [109] solved the free vibration analysis of circular Mindlin plates using the same method. Viswanathan and Kim [110] applied point collocation and spline function method for solving non-symmetric laminated thick plate. Tai and Kim [111] proposed a refined plate theory which accounts for parabolic distribution of the transverse shear strains through the plate thickness and satisfies the zero traction boundary conditions on the surfaces of the plate without using shear correction factors. Equations were derived using Hamilton's principle and Navier solution was applied for obtaining closed-form solution. Results are compared with three-dimensional elasticity, first- and third-order shear deformation theory. Gurses et al. [112] presented free vibration analysis of symmetric laminated trapezoidal using first-order shear deformation theory of plate by discrete singular convolution method. Kant and Swaminathan [113, 114] presented a comparison between several theories for thick plates including theory of Kant and Manjunatha (1988), Pandya and Kant (1988), Reddy (1984), Senthilnathan et al. (1987) and Whitney and Pagano (1970). Equation of motion for all displacements were derived using Hamilton's principle and solutions are obtained in closed-form Navier type solution. Karami et al. [115] considered differential quadrature method for solving cross-ply and angle-ply with and without elastic edges for laminated composite thick plates based on Mindlin theory. Wang [116] considered the free vibration analysis of skew composite laminated plates using B-spline Rayleigh-Ritz method for symmetric and antisymmetric, skew and rectangular Mindlin plate for clamped and simply-supported cases. Daia et al. [117] presented a mesh-free formulation for the static and free vibration of composite Mindlin plates via a linearly conforming radial point interpolation method. The radial and polynomial basis functions were employed to construct the shape functions bearing Delta functions property. A strain smoothing stabilization technique for nodal integration was employed to restore comformability and to improve the accuracy and the rate of convergence. Civalek [118] was developed Discrete singular convolution for vibration analysis of moderately thick symmetrically laminated composite plates based on the first-order shear deformation. Regularized Shannon's delta (RSD) kernel was selected as singular convolution. Kant and Mallikarjuna [119] reported a refined higher-order theory for free vibration analysis of non-symmetrically multi-layered plates using a simple c^o finite element formulation and nine-coded Lagrangian element with seven degrees of freedom. The theory accounted for parabolic distribution of the transverse shear strains through the thickness of the plate and rotary inertia effects. Ngo-Cong et al. [120] presented a new effective radial basis function (RBF) collocation technique for the free vibration analysis of laminated plates using Mindlin plate theory. In this method instead of using conventional differential RBF networks, one-dimensional integrated RBF networks (1D-IRBFN) were employed on grid lines to approximate the field variables. Plates can be rectangular or non-rectangular but could simply discretized by means of Cartesian grids. Nguyen-Van et al. [121] presented a method based on a novel four-node quadrilateral element namely MISQ20 within the framework of the first-order shear deformation theory for vibration analysis of composite laminated plates. The element was built by incorporating a strain smoothing method into the bilinear four-node quadrilateral finite element where the strain smoothing operation is based on mesh-free conforming nodal integration. The bending and membrane stiffness matrices were based on the boundaries of smoothing cells while the shear term is evaluated with two by two Gauss quadrature. Liew et al. [122] adopted first shear deformable theory in the moving least squares differential quadrature (MLSDQ) for vibration analysis of composite laminated symmetrically rectangular and circular plates. The transverse deflection and two rotations of the lamina are independently approximated by moving least squares (MLS). The weighting coefficients approximated through the fast computation of the MLS shape functions and their partial derivatives. Asadi and Fariborz [123] considered a higher-order shear deformation theory to obtain the governing equations of composite plates non-symmetrically in various boundary conditions and lay-up under dynamic excitation. The time-harmonic solution leads to an eigenvalue problem and it is converted to a set of homogeneous algebraic equations using differential quadrature method. Liew et al. [124, 125] employed vibration analysis of rectangular laminated Mindlin plates with various boundary conditions. In this method a set of boundary beam characteristic orthogonal polynomials is used as admissible functions in the Ritz minimization procedure. Han and Liew [126] reported the non-symmetric free vibration of moderately thick annular plates using differential quadrature method for various combination of boundary conditions. Sari [127] presented vibration analysis of isotropic rectangular and annular Mindlin plates with and without damaged boundary conditions using spectral collocation technique. Khdeira and Reddy [128] presented a second-order shear deformation theory for vibration analysis of generally composite laminated plates using generalized Levy type solution in conjunction with the state space concept. The exact analytical solutions were obtained for thick and moderately thick plates as well as thin plates and strip plates. Hosseini-Hashemi [129] studied free vibration of moderately thick plates with several internal line support. Potential and kinetic energy are based on Mindlin plate theory. Ritz method assumed in two-dimensional polynomial functions as admissible displacement functions. Zamani et al. [130] reported vibration analysis of moderately thick trapezoidal, skew and triangular symmetrically laminated Mindlin plates using generalized differential quadrature method. Effect of various parameters such as geometry, thickness, boundary conditions and lay-up configuration was considered on the natural frequencies. Naginoa et al [131] considered three-dimensional analysis of thin and thick isotropic rectangular plates using B-spline Ritz method based on the theory of elasticity. The geometry boundary conditions were numerically satisfied by the method of artificial spring and the proposed method formulated by a triplicated series of B-spline functions as amplitude displacement components. Zhoua et al. [132] studied free vibration analysis of isotropic sector annular plates using Chebyshev-Ritz method based on three-dimensional theory. Sharma et al. [133] studied free vibration analysis of non-symmetric composite laminated sector annular plate using analytical method and Chebyshev polynomials based on first-order shear deformation theory.

Review of shells

Mochida et al. [134] presented vibration analysis of isotropic thin doubly curved shallow shells of rectangular platform using the Superposition-Galerkin method for seven sets of boundary conditions. Monterubbio [135] presented the Rayleigh-Ritz and penalty function method for solving free vibration of isotropic thin shallow shells of rectangular platform with spherical, cylindrical and hyperbolic paraboloidal geometries. Liew and Lim [136] considered the vibration analysis of thin doubly-curved shallow shells of rectangular platform in many boundary conditions and with many Gaussian curvatures. The pb-2 Ritz energy based approach along with in-plane and transverse deflections assumed in the form of a product of mathematically complete two-dimensional orthogonal polynomials and a basic function, was employed to model of vibration. Ruotolo [137] compared Donnell's, Love's, Sander's and Flugge's thin shell theories in the evaluation of natural frequency of cylindrical shells based on Kirchhoff Hypothesis. He used energy function for soling the equation of motion and levy solution for satisfaction of boundaries. Civalek [138, 139] presented free vibration analysis of laminated conical and cylindrical shell using discrete singular convolution approach. This study carried out using Love's first approximation thin shell theory. Free vibration of isotropic cylindrical shell and annular plates are solved as special cases. The effects of circumferential wave number and number of layers on natural frequencies were considered. Lim et al. [140] presented the vibration analysis of composite shallow conical shells including the effects of pretwist using energy method for symmetric, non-symmetric and various number of layers. Kabir [141, 142, 143] presented an analytical solution to the boundary value problem for free vibration analysis of antisymmetric angle-ply laminated cylindrical panels. The solution is based on a boundary-continuous double Fourier series. Equations including rotary and in-plane inertias. The characteristic equations of the panel are defined by five highly coupled second and third-order partial differential equations in five unknowns. Noseir and Reddy [144] presented Donnell shear deformation type theory and Donnell's classical theory for vibration and stability of cross-ply laminated circular cylindrical shells using analytical solution based on Levy-type. Soldatos [145] reported vibration analysis of composite laminated thin cylindrical shallow shell panels based on thin hypothesis theory of shell and compared four types of theory of shell including Flugge, Sanders, Love and Donnell. Soldatos [146] considered vibration analysis of laminated shells either circular or non-circular using refined shear deformation theory. This theory accounts for parabolic variation of transverse shear strains and it is capable of satisfying zero shear traction boundary conditions at the external shell surfaces, and make no use of transverse shear correction factor. Shu [147, 148] applied generalized differential quadrature method for vibration analysis of isotropic and composite laminated conical shells based on Love's first approximation thin shell theory. The displacement fields were expressed as product of unknown functions along the axial direction and Fourier functions along the circumferential direction. The same author [149] considered vibration analysis of laminated cylindrical shells using the same method and based on the same theory. Ganapathi et al. [150, 151] characterized free vibration of thick laminated composite non-circular shells using higher-order theory. The formulation accounts for the variation of the in-plane and transverse displacements through the thickness, abrupt discontinuity in slope of the in-plane displacements at the interfaces, and includes in-plane, rotary inertia terms, and also the inertia contributions due to the coupling between the different order displacement terms. The energy method and finite element procedure used for solving the governing equations. Leissa et al. [152] considered vibration analysis of circular cylindrical cantilevered thin shallow shells of rectangular platform using Ritz method. Leissa [153] presented a closed-form solution for vibration analysis of shallow shells and studied the effects of shallowness on the natural frequencies. Soldatos and Hadjigeorgiou [154] studied three-dimensional vibration analysis of isotropic cylindrical thick shells and panels according to Levy solution. Asadi and Qatu [155, 156] presented the vibration analysis of generally moderately thick deep composite laminated cylindrical shells with various lay-up using differential quadrature method based on the first-order shear deformation theory. Hosseini-Hashemi and Fadaee [157] presented an exact closed-form procedure using a new auxiliary and potential functions for free vibration analysis of moderately thick spherical shell panels based on the first-order shear deformation theory. The strain-displacements relations of both Donnell and Sanders theories were used and compared with 3D finite element analysis. Shell has two opposite simply supported (Levy-type). The effects of various stretching-bending coupling on the frequency parameters were discussed. Hosseini-Hashemi et al. [158] dealed with closed-form solution for in-plane and out-plane free vibration of moderately thick laminated transversely isotropic spherical shell panels based on Sanders theory without any approximation. The governing equations of motion and boundary conditions were derived using Hamilton's principle. The highly coupled governing equations were recast to uncoupled equations introducing four potential functions. According to the proposed analytical solution both Levy and Navier type solutions were developed for this problem. Qatu [159, 160] reviewed many papers 1989-2009 for many shell theories and shell problems. Liew et al. [161] studied three-dimensional vibration analysis of thin and thick doubly curved shell panels of rectangular platform in various boundary conditions using p-Ritz method. Chen and Ding [162] also used three-dimensional vibration analysis for spherical isotropic shells using a state-space method. They introduced three displacement functions and two stress functions for derivation of two independent state equations with varying coefficients. Taylor expansion theorem employed to obtain solutions to the two state equations and relationships between the state variables at the upper and lower surfaces of each lamina. Liew and Lim [163] presented vibration analysis of thick rectangular shallow shells using Ritz method based on a refined first-order theory. Displacement formulation followed the first-order shear deformation but Lame parameters for the transverse shear strain through shell thickness was considered and they multiplied by in-plane displacement of mid surface of the shell. Chern and Chao [164] studied three-dimensional composite shallow spherical, cylindrical, plates and saddle (hyperbolic) panels in rectangular platform using Ritz method and Levy solution. Yu et al. [165] studied the free vibration analysis of thin shallow shells based on Donnell Mushtary Vlasov theory analytically using generalized Navier solution. Considering different boundary conditions leads to different analytical solutions. Buchanan and Rich [166] reported the vibration analysis of thick isotropic spherical and toroidal shells using nine-node Lagrangian finite elements. Viola et al. [167, 168] investigated static vibration analysis of doubly-curved composite laminated shells and panels based on the 2D Higher-order shear deformation theory using generalized differential quadrature method. The HSDT is based on nine parameters kinematic hypothesis in a unified form. Strains and stresses are corrected after the recovery to satisfy the top and bottom boundary conditions of the laminated composite shells or panels. They compared the results for plotting the displacements and stresses distribution through the thickness in FSDT case and reported the frequencies for various types of doubly-curved composite shells such as conical, elliptical, catenoidal and toroidal in many kinds of boundary conditions. Tornabene [169] applied generalized differential quadrature method on the composite laminated doubly-curved shells and panels such as toroidal shell, parabolic panel, elliptic panel, cycloidal shell, catenary panel, hyperbolic panel and shell based on the first-order shear deformation theory. He produced the geometries of shells and panels through the revolution of curved line about one axis. Bardell et al.[170] considered free vibration analysis of isotropic conical shells using hirarcial finite element based on thin shallow shells theory. Cheung et al. [171] considered free vibration analysis of conical shells using spline finite strip method. Zhoa et al. [172] considered free vibration analysis of conical shells using 2D Ritz method.

Review of CUF technique

Carrera [173, 174] proposed a new technique for vibration analysis of plates and shells which is suitable for every thickness such as thin and thick and suitable for any kinds of lay-up for laminated composite structures. He presented this procedure based on weak form using finite element method, stiffness and mass matrices were derived by integration. He satisfied the simply supported boundary conditions using Navier solution. He presented also his method [175, 176, 177, 178] in strong form using finite element and Navier solution. Demasi [179] applied Carrera's Unified Formulation technique for vibration analysis of generally composite plates based on Reissner's mixed variational theorem

(RMVT) which allows one to assume two independent fields for displacements and transverse stress variables. The inter-laminar continuous transverse shear and normal stress fields, so called C_z^0-requirements can be satisfied. Results were validated by NASTRAN software. Dozio [180, 181] applied CUF model on the rectangular and quadrilateral, isotropic and composite plates respectively. He considered many boundary conditions and skew angles. He used Trigonometric Ritz formulation for implementing this model. The same author [182] presented vibration analysis for annular isotropic plates using CUF model and Trigonometric Ritz formulation. Tornabene et al. [183] presented a general formulation of a 2D Higher-order equivalent single layer theory for free vibration of thin and thick doubly-curved composite laminated shells and panels using generalized differential quadrature method with different curvature, geometry and boundary conditions. General displacement field was based on the Carrera's Unified Formulation (CUF), including the stretching and zig-zag effects. The order of the expansion along the thickness direction is taken as a free parameter. The fundamental operator can be used not only for equivalent single layer approach, but also for layer-wise approach. Ferreira et al. [184, 185, 186] applied CUF model on the laminated shells using radial basis functions collocation according to a sinusoidal shear deformation theory (SSDT). Displacements were assumed in sinusoidal forms and used Navier solution for satisfaction of the boundary conditions. Carrera [187, 188] presented some solutions for thickness locking because in some boundary conditions considering thin plates and shells, natural frequencies are not very accurate. For overcoming this problem by using new coefficients in the Nuclei, quasi-3D analysis recovers to the Mindlin plate and accurate results. Carrera [189] considered the effects of transverse normal stress σ_{zz} on vibration of multilayered structures using Reissner's mixed theorem. The evaluations of transverse stress effects have been conducted by comparing constant, linear and higher-order distributions of transverse displacement components in the plate thickness directions. Fazzolari and Carrera [190] applied Ritz, Galerkin and generalized Galerkin method on the CUF model and compared all of them for nonlinear analysis vibration analysis in two problem, one is nonlinear strains considered in Von-karman theory and the other to exact nonlinear strains.

Chapter 2

Implementation of Method

2.1 Spectral Methods

This method is based on the Chebyshev spectral collocation technique for directly solving high-order ordinary differential equations (ODEs). Consider the PDE with boundary condition

$$\begin{aligned} Lu(x) &= s(x) \qquad & x \in U \subset R^d \\ Bu(x) &= 0 \qquad & x \in \partial U \end{aligned} \tag{2.1}$$

where L and B are linear differential operators.

The answer is a function $\bar{u}$ which satisfies the boundary condition and makes the residual $R := L\bar{u} - s$. Search for solutions $\bar{u}$ in a finite-dimensional sub-space $\mathcal{P}_N$ of some Hilbert space $\mathcal{W}$ (typically a L^2 space) [12, 17].

Expansion functions = trial functions: basis of $\mathcal{P}_N$: $(\phi_0, ..., \phi_N)$, $\bar{u}$ is expanded in terms of the trial functions: $\bar{u}(x) = \sum_{n=0}^{N} \tilde{u}_n \phi_n(x)$

Test functions: family of functions $(\mathcal{X}_0, ..., \mathcal{X}_N)$ to define the smallness of the residual R, by means of the Hilbert space scalar product:

$$\forall n \in \{0, ..., N\}, \qquad (\mathcal{X}_n, R) = 0 \tag{2.2}$$

Classification according to the trial functions ϕ_n:

Finite difference: trial functions = overlapping local polynomials of low-order
Finite element: trial functions = local smooth functions (polynomial of fixed degree which are non-zero only on sub domains of U)
Spectral methods : trial functions = complete family of smooth global functions

Classification spectral methods according to the test functions χ_n:

Galerkin method: test functions = trial functions: $\chi_n = \phi_n$ and each ϕ_n satisfies the boundary condition : $B\phi_n(y) = 0$.
Tau method: (Lanczos 1938) test functions = (most of) trial functions: $\chi_n = \phi_n$ but the ϕ_n does not satisfy the boundary conditions; the latter are enforced by an additional set of equations.
Collocation or pseudospectral method: test functions = delta functions at special points,

called collocation points: $\chi_n = \delta(x - x_n)$.

Collocation method

The integral of the weighted residual over the domain of the problem is set equal to zero where the weighting function is the same as one of the comparison functions used in the series solution. The weighting functions are spatial Dirac delta functions. Thus, for a one-dimensional eigenvalue problem, an approximate solution is assumed in the form of a linear sum of trial functions $\phi_i(x)$ as

$$\bar{\phi}^{(n)}(x) = \sum_{i=1}^{n} c_i \phi_i(x) \tag{2.3}$$

where c_i are unknown coefficient and the $\phi_i(x)$ are the trial functions. Depending on the nature of the trial functions used, The collocation method may be classified in one of the following three types:

1. Boundary methods: used when the functions $\phi_i(x)$ satisfy the governing differential equation over the domain but not all the boundary conditions of the problem.

2. Interior method: used when the functions $\phi_i(x)$ satisfy all the boundary conditions but not the governing differential equation of the problem.

3. Mixed method: used when the functions $\phi_i(x)$ do not satisfy either the governing differential equation or the boundary conditions of the problem.

When the integral of the weighted residual is set equal to zero, the collocation method yields,

$$\int_0^l \delta(x - x_i) R(\bar{\phi}^{(n)}(x)) dx = 0, \qquad i = 1, 2, ..., n \tag{2.4}$$

where δ is the Dirac delta function and $x_i, i = 1, 2, ..., n$ are the known collocation points where the residual is specified to be equal to zero. Due to the sampling property of the Dirac delta function, above equation require no integration and hence can be expressed

$$R(\bar{\phi}^{(n)}(x)) = 0, \qquad i = 1, 2, ..., n \tag{2.5}$$

This amounts to setting the residue at $x_1, x_2, ..., x_n$ equal to zero. Above equation denote a system of n homogeneous algebraic equations in the unknown coefficients $c_1, c_2, ..., c_n$ and the parameter λ. In fact, they represent the algebraic eigenvalue problem of order n. It can be seen that the selection of the collocation points $x_1, x_2, ..., x_n$ is important in obtaining a well-conditioned system of equations and a convergent solutions. The location of the collocation points should be selected as evenly as possible in the domain and/or boundary of the system to avoid ill-conditioning of the resulting equations.

The main advantage of the collocation method is simplicity. Evaluation of the stiffness and mass coefficients involves no integration. The main disadvantage of the method is that the stiffness and mass matrices, are not symmetric although the system is conservative. Hence, the solution of the non-symmetric eigenvalue problem is not simple. In

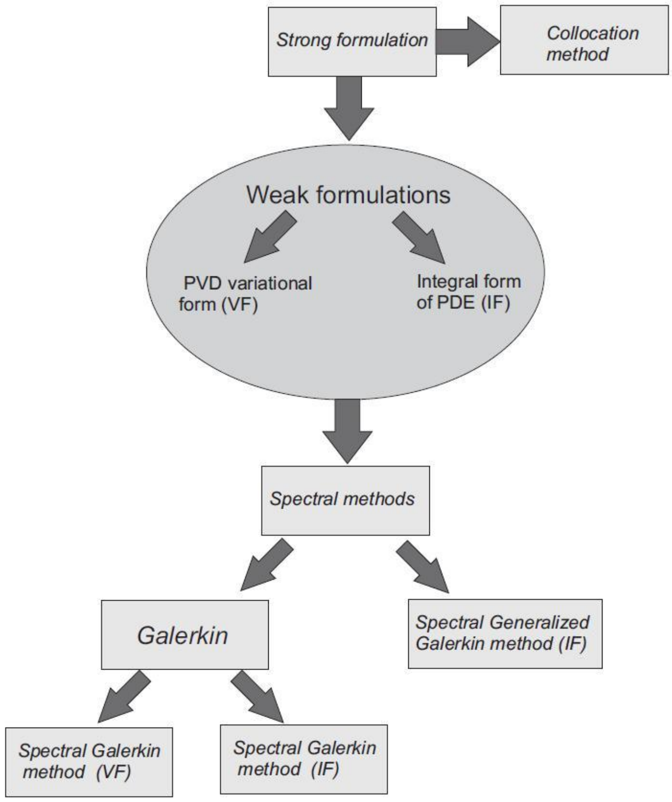

Figure 2.1: Reference scheme of spectral methods

general we need to find both right and left eigenvectors of the system in order to find the system response.

(Figure 2.1) shows how to deal with the differential problems through spectral methods in strong and weak forms.

Solving PDE with Tau method:

Since $\phi_n = \chi_n$, but the ϕ_n does not satisfy the boundary condition: $B\phi_n(y) \neq 0$. Let (g_p) be an orthonormal basis of $M+1 < N+1$ functions on the boundary ∂U and let us expand $B\phi_n(y)$ upon it.

$$B\phi_n(y) = \sum_{p=0}^{m} b_{pn} g_p(y) \tag{2.6}$$

The boundary condition then becomes

$$Bu(y) = 0 \Longleftrightarrow \sum_{k=0}^{N} \sum_{p=0}^{M} \tilde{u}_k b_{pk} g_p(y) = 0 \tag{2.7}$$

hence the $M+1$ conditions:

$$\sum_{k=0}^{N} b_{pk} \tilde{u}_k = 0 \qquad\qquad 0 \leq p \leq M \tag{2.8}$$

The system of linear equations for the $N+1$ coefficients $\tilde{u}_n$ is then taken to be the $N-M$ first raws of the Galerkin system plus the $M+1$ equations above:

$$\begin{aligned} \sum_{k=0}^{N} L_{nk}\tilde{u}_k &= (\phi_n, s) & \quad 0 \le n \le N-M-1 \\ \sum_{k=0}^{N} b_{pk}\tilde{u}_k &= 0 & \quad 0 \le p \le M \end{aligned} \tag{2.9}$$

The solution $(\tilde{u}_k)$ of this system gives rise to a function $\tilde{u} = \sum_{k=0}^{N} \tilde{u}_k \phi_k$ such that

$$L\tilde{u}(x) = s(x) + \sum_{p=0}^{M} \tau_p \phi_{N-M+p}(x) \tag{2.10}$$

Solving a PDE with a Pseudospectral (collocation) method

Since $\chi_n(x) = \delta(x - x_n)$, where the (x_n) constitute the collocation points. The smallness condition for the residual reads, for all $n \in 0, ..., N$,

$$\begin{aligned} (\mathcal{X}_n, R) = 0 \iff & (\delta(x-x_n), R) = 0 \iff R(x_n) = 0 \iff Lu(x_n) = s(x_n) \\ \iff & \sum_{k=0}^{N} L\phi_k(x_n)\tilde{u}_k = s(x_n) \end{aligned} \tag{2.11}$$

The boundary condition is imposed as in the tau method. One then drops $M+1$ raws in the above linear system and solve the system

$$\begin{aligned} \sum_{k=0}^{N} L\phi(x_n)\tilde{u}_k &= s(x_n) & \quad 0 \le n \le N-M-1 \\ \sum_{k=0}^{N} b_{pk}\tilde{u}_k &= 0 & \quad 0 \le p \le M \end{aligned} \tag{2.12}$$

There are two choices for trial functions ϕ_n, Periodic problem: ϕ_n = trigonometric polynomials (Fourier series) Non-periodic problem: ϕ_n = orthogonal polynomials.

2.2 Discretization of 1D problem

Chebyshev polynomial of the first kind is defined by

$$T_{k+1} = 2xT_k(x) - T_{k-1}(x) \tag{2.13}$$

where $T_0(x) = 1$ and $T_1(x) = x$, and relation between differentiation of two polynomials is as

$$2T_k(x) = \frac{1}{k+1}T'_{k+1}(x) - \frac{1}{k-1}T'_{k-1}(x) \tag{2.14}$$

Some properties of the Chebyshev polynomials are as

$$| T_k(m) | \leq 1, \qquad -1 \leq x \leq 1, \tag{2.15}$$

$$T_k(\pm 1) = (\pm 1)^k, \tag{2.16}$$

$$| T'_k(m) | \leq k^2, \qquad -1 \leq x \leq 1, \tag{2.17}$$

$$T'_k(\pm 1) = (\pm 1)^{(k+1)} k^2, \tag{2.18}$$

$$\int_{-1}^{1} T_k^2(x) \frac{dx}{\sqrt{(1-x^2)}} = c_k \frac{\pi}{2}, \tag{2.19}$$

where

$$c_k = \begin{cases} 2 & i = 0 \quad \text{or} \quad N \\ 1 & \text{otherwise} \end{cases} \tag{2.20}$$

Eigenfunctions of the singular Sturm-Liouville problem

$$\frac{d}{dx}(\sqrt{1-x^2}\frac{dT_n}{dx}) = -\frac{n^2}{\sqrt{1-x^2}} T_n(x) \tag{2.21}$$

are orthogonal family in the Hilbert space $L^2_\omega[-1,1]$, equipped by the weight $w(x) = (1-x^2)^{-\frac{1}{2}}$.

$$(f,g) := \int_{-1}^{1} f(x)g(x)w(x)dx \tag{2.22}$$

Let w be a Jacobi weight and let ϕ_k be the corresponding system of orthogonal polynomials. Gauss-type quadrature formula are in the form of

$$\sum_{j=0}^{N} f(x_j)w_j \sim \int_{-1}^{1} f(x)w(x)dx \tag{2.23}$$

formula (2.23) allows for the approximate computation of the Jacobi coefficients of a continuous function u defined in the interval $[1,1]$. The coefficients

$$\tilde{u}_k = \frac{1}{\gamma_k}(u, \phi_k)_n \qquad k = 0, ..., N \tag{2.24}$$

are called the discrete Jacobi coefficients of u. Since

$$u(x_j) = \sum_{k=0}^{N} \tilde{u}_k \phi_k(x_j) \qquad k = 0, ..., N \tag{2.25}$$

equations (2.24) and (2.25) enable one to transform between physical space $u(x_j)$ and transform space $\tilde{u}_k$. Chebyshev expansion of function $u \in L^2_\omega(-1,1)$ is

$$u(x) = \sum_{k=0}^{\infty} \hat{u_k} T_k(x), \qquad \hat{u_k} = \frac{2}{\pi c_k}\int_{-1}^{1} u(x)T_k(x)w(x)dx \tag{2.26}$$

The derivative of function u expanded in Chebyshev polynomials according to above equation can be represented formally as

$$u' = \sum_{k=0}^{\infty} \hat{u}_k^{(1)} T_k \tag{2.27}$$

From eq. (2.14) one has

$$2k\hat{u}_k = c_{k-1}\hat{u}^{(1)}_{k-1} - \hat{u}^{(1)}_{k+1} \qquad k \geq 1 \tag{2.28}$$

or, equivalently

$$c_k\hat{u}^{(1)}_k = \hat{u}^{(1)}_{k+2} + 2(k+1)\hat{u}_{k+1} \qquad k \geq 0 \tag{2.29}$$

This yields,

$$\hat{u}^{(1)}_k = \frac{2}{c_k}\sum_{\substack{p=k+1 \\ p+k\ odd}}^{\infty} p\hat{u}_p \qquad k \geq 0 \tag{2.30}$$

Chebyshev-Gauss-Lobatto (CGL) points which are used for construction of differentiation matrix for solving differential equations are as

$$\xi_i = \cos\left(i\frac{\pi}{N}\right), \qquad i = 0, 1, ..., N \tag{2.31}$$

The characteristic Lagrange polynomials for the Chebyshev-Gauss-Lobatto points are expressed as

$$\phi_l(x) = \frac{(-1)^{l+1}(1-x^2)\acute{T}_N(x)}{c_l N^2(x-x_l)} \tag{2.32}$$

By eq. (2.25), the polynomial

$$I_N u = \sum_{k=0}^{N} \tilde{u}_k\phi_k \tag{2.33}$$

satisfies

$$I_N u(x_j) = u(x_j), \qquad 0 \leq j \leq N \tag{2.34}$$

Another expression of $I_N u$ is

$$I_N u = \sum_{l=0}^{N} u(x_l)\phi_l \tag{2.35}$$

where ϕ_l denoted the lth characteristic Lagrange polynomial relative to the given set of nodes, i.e., the unique polynomial that satisfies

$$\phi_l \in P_N(-1,1), \qquad \phi_l(x_j) = \delta_{jl} \qquad \text{for} \quad j = 0, ..., N \tag{2.36}$$

Differentiation in physical space is accomplished by replacing truncation by interpolation. Given a set of $(N+1)$ Gaussian nodes in $[1,1]$, the polynomial

$$D_N u = (I_N u)' \tag{2.37}$$

is called the Jacobi interpolation derivative of u relative to the chosen set of quadrature nodes. The physical values of the interpolation derivative can be expressed as linear combinations of the physical values of the function, i.e.,

$$(D_N u)(x_j) = \sum_{l=0}^{N}(D_N)_{jl}u(x_l), \qquad j = 0, ..., N \tag{2.38}$$

By eq. (2.35), the coefficients are given by $(D_N)_{jl} = \phi_l'(x_j)$ they form the entries of the first-derivative interpolation matrix D_N.

Differentiation matrix is as $(N+1)\times(N+1)$ order. The first derivative matrix (Gottlieb, Hussaini and Orszag (1984)) is

$$(D_N^{(1)})_{jl} = \begin{cases} \frac{c_j}{c_l}\frac{(-1)^{(j+l)}}{x_j - x_l} & j \neq l \\ -\frac{x_l}{2(1-x_l^2)} & 1 \leq j = l \leq N-1 \\ \frac{2N^2+1}{6} & j = l = 0 \\ -\frac{2N^2+1}{6} & j = l = N \end{cases}$$

$$\mathbf{D}_N^{(2)} = (\mathbf{D}_N^{(1)})^2 \qquad \mathbf{D}_N^{(3)} = (\mathbf{D}_N^{(1)})^3 \qquad \mathbf{D}_N^{(4)} = (\mathbf{D}_N^{(1)})^4 \tag{2.39}$$

2.3 Euler-Bernoulli Beam

Perpendicular surface to the mid surface before and after deformation remains normal and strains except ϵ_{xx} all are zero and shear deformation is neglected. Pure bending moment is just analyzed in this theory. Displacements (u, w) for Euler-Bernoulli theory of beam (see Figure 2.2) are assumed as

$$\begin{aligned} u &= u_0 - z\frac{dw_0}{dx} \\ w &= w_0 \end{aligned} \tag{2.40}$$

Strain-displacement relation is as follows

$$\epsilon_{xx} = \frac{du}{dx} = \frac{du_0}{dx} - z\frac{d^2w_0}{dx^2} \tag{2.41}$$

Deriving the equilibrium equations using Newton's second law of motion are as follow

$$\begin{aligned} &\sum F_x = -\rho A\frac{d^2u_0}{dt^2} && \rightarrow && -\frac{dN}{dx} - f(x) = -\rho A\frac{d^2u_0}{dt^2} \\ &\sum F_z = -\rho A\frac{d^2w_0}{dt^2} && \rightarrow && -\frac{dV}{dx} - q(x) = -\rho A\frac{d^2w_0}{dt^2} \\ &\sum M_y = 0 && \rightarrow && V = \frac{dM}{dx} \end{aligned} \tag{2.42}$$

where

$$\begin{aligned} N(x) &= \int_A \sigma_{xx}dA = E\frac{du_0}{dx}\int_A dA = EA\frac{du_0}{dx} \\ M(x) &= \int_A \sigma_{xx}z dA = E\int_A \left(\frac{du_0}{dx} - z\frac{d^2w_0}{dx^2}\right)z dA = -EI\frac{d^2w_0}{dx^2} \end{aligned} \tag{2.43}$$

Assuming (E) modulus elasticity, (I) inertia moment and (A) cross section of the beam are constant by changing x. After substitution eqs. (2.43) into eqs. (2.42) yield,

$$\begin{aligned} EA\frac{d^2u_0}{dx^2} &= f(x) + \rho A\frac{d^2u_0}{dt^2} \quad \text{(Rod problem)} \\ EI\frac{d^4w_0}{dx^4} &= q(x) - \rho A\frac{d^2w_0}{dt^2} \quad \text{(Euler-Bernoulli beam)} \end{aligned} \tag{2.44}$$

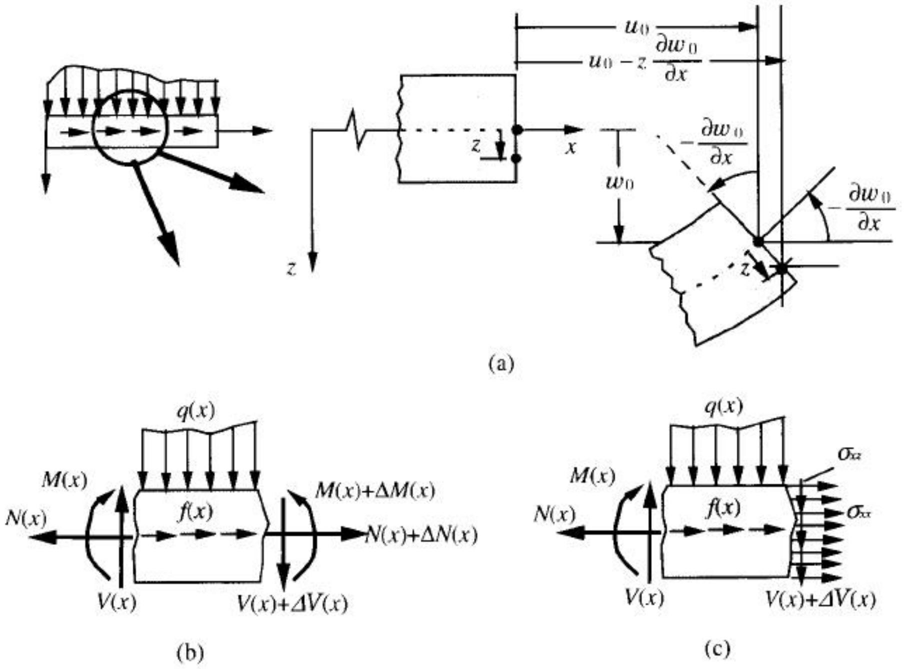

Figure 2.2: Bending of beams. (a) Kinematics of deformation of Euler-Bernoulli beam, (b) Equilibrium of a beam element, (c) Definitions of stress resultants

where $f(x)$ is axial force correspond to the rod problem and $q(x)$ is shear force correspond to the Euler-Bernoulli beam, which both are zero in the free vibration analysis.

Euler-Bernoulli beam's displacement is assumed in the form of

$$w(x,t) = w(x)e^{iwt} \tag{2.45}$$

After substitution above equation into the second part of eq. (2.44) and neglecting q, yields

$$EI\frac{d^4w}{dx^4} = \omega^2 \rho A w \tag{2.46}$$

Since the range of displacement is given by $x \in [0, L]$, where L is the length of the beam, one non-dimensional variable is introduced as:

$$X = \frac{x}{L} \in [0,1] \tag{2.47}$$

Chebyshev-Gauss-Lobatto points in this range are

$$\xi_i = \frac{1}{2}\Big(1 - \cos\Big(i\frac{\pi}{N}\Big)\Big), \qquad i = 0, 1, ..., N \tag{2.48}$$

Thus, differentiation matrix based on these CGL points are as

$$(D_N)_{00} = \frac{2N^2+1}{3} \qquad (D_N)_{NN} = -\frac{2N^2+1}{3}$$
$$(D_N)_{jj} = -\frac{x_j}{(1-x_j^2)} \qquad j = 1, ..., N-1$$

$$(D_N)_{ij} = -\frac{c_i}{c_j}\frac{2(-1)^{(i+j)}}{x_i - x_j} \qquad i \neq j \quad j = 1, ..., N-1$$
$$c_i = 2 \quad (i = 0, N) \qquad \text{otherwise} \quad c_i = 1 \tag{2.49}$$

Equation of motion is rewritten as

$$\mathcal{L}w = \lambda^2 w \tag{2.50}$$

where $\lambda^2 = L^4\omega^2\rho A/EI$ and $\mathcal{L} = \frac{\partial^4}{\partial X^4}$.

Displacement implementation of Euler-Bernoulli beam in spectral collocation method is expressed as

$$\mathbf{u}(x) = \sum_{j=0}^{N} \mathbf{u}_j \phi_j(x) \rightarrow \frac{\partial \mathbf{u}}{\partial x} = \sum_{j=0}^{N} \mathbf{u}_j \frac{\partial \phi_j}{\partial \zeta}(\zeta_i) = \sum_{j=0}^{N} D^{(1)}_{(i,j)} \mathbf{u}_j \tag{2.51}$$

where $\mathbf{u}_j$ which includes displacements is the vector of all unknown values at the grid points and $\mathbf{u}_j = \mathbf{u}^N(\zeta_j)$, $\mathbf{u}^N$ is vector of displacement of the problem and $\phi_l(l = 0, ..., N)$ is the Lagrange interpolating polynomial corresponding to the Chebyshev-Gauss-Lobatto (CGL) points, thus $\phi_j(\zeta_i) = \delta_{ij}$ for $(i = 0, ...N)$.

Vector of displacements at grid points ($w \in R^{(N+1)\times 1}$) are

$$w = \{w_0,\ w_1,\ ...\ w_N\} \tag{2.52}$$

Equation of motion should be satisfied for interior points $(i = 3, ..., N-1)$. $Z_I \in R^{(N-3)\times(N+1)}$ is a matrix correspond to interior points.

$$Z_I = \begin{bmatrix} e_3^T \\ e_4^T \\ \vdots \\ e_{N-1}^T \end{bmatrix} \tag{2.53}$$

where $e_i \in R^{(N+1)\times 1}$ is the ith unit vector. This vector is zero in all entries expect for the ith entry at which it is equal to 1. Matrix $Z_B \in R^{4\times(N+1)}$ corresponding to the border point and a point before the border point.

$$Z_B = \begin{bmatrix} e_1^T \\ e_2^T \\ e_N^T \\ e_{(N+1)}^T \end{bmatrix} \tag{2.54}$$

Displacements and their derivatives at the CGL points are expressed as

$$w = I, \quad \frac{\partial w}{\partial X} = D^{(1)}, \quad \frac{\partial^2 w}{\partial X^2} = D^{(2)}$$
$$\frac{\partial^3 w}{\partial X^3} = D^{(3)}, \quad \frac{\partial^4 w}{\partial X^4} = D^{(4)} \tag{2.55}$$

where I is identity matrix by order $(N+1)$ and D is a matrix for implementing differentiation with respect to the length based on the CGL points, thus eq. (2.50) can be written as

$$D^{(4)} = \lambda^2 I \tag{2.56}$$

The equation of motion for interior points can be written as

$$(L_B s_B + L_I s_I) = \lambda^2 s_I \tag{2.57}$$

where

$$L_B = \left[\begin{array}{c} Z_I K Z_B^T \end{array} \right] \qquad L_I = \left[\begin{array}{c} Z_I K Z_I^T \end{array} \right] \tag{2.58}$$

and K is left hand side of eq. (2.56).

$$s_I = \left\{ \begin{array}{c} w_I \end{array} \right\} \qquad s_B = \left\{ \begin{array}{c} w_B \end{array} \right\} \tag{2.59}$$

where s_I and s_B denote interior and boundary points, respectively.

$$\left[\begin{array}{c} \mathcal{B}_{\bar{w}} \\ \mathcal{B}_{\hat{w}} \end{array} \right] \left\{ \begin{array}{c} w \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \qquad (each\ \ end) \tag{2.60}$$

$\mathcal{B}_{\bar{w}}$ is applied on the border point at each end and $\mathcal{B}_{\hat{w}}$ is applied on the point before the border point at each end.

Analysis of problem is summarized in the form below as

$$\left[\begin{array}{c} \mathcal{B}_w \end{array} \right] \left\{ \begin{array}{c} w \end{array} \right\} = \left\{ \begin{array}{c} 0 \end{array} \right\} \tag{2.61}$$

where $\mathcal{B}_w$ is a matrix which is included $\mathcal{B}_{\bar{w}}$ and $\mathcal{B}_{\hat{w}}$ and are correspond to the displacement variable w.

The boundary equations can be written as

$$B_B s_B + B_I s_I = 0 \rightarrow s_B = -B_B^{-1} B_I s_I \tag{2.62}$$

where

$$B_B = \left[\begin{array}{c} B_w Z_B^T \end{array} \right], \qquad B_I = \left[\begin{array}{c} B_w Z_I^T \end{array} \right] \tag{2.63}$$

Substitution of equation (3.62) to (2.57) results into

$$(L_I - L_B B_B^{-1} B_I) s_I = \lambda^2 s_I \tag{2.64}$$

Boundary conditions:

Clamped: $w = 0, \frac{\partial w}{\partial x} = 0$
Pined: $w = 0, \frac{\partial^2 w}{\partial x^2} = 0$
Sliding: $\frac{\partial w}{\partial x} = 0, \frac{\partial^3 w}{\partial x^3} = 0$
Free: $\frac{\partial^2 w}{\partial x^2} = 0, \frac{\partial^3 w}{\partial x^3} = 0$

Boundary conditions implementation at (ζ_0)

Clamped

$$\mathcal{B}_{\bar{w}} = e_1^T I \qquad \mathcal{B}_{\hat{w}} = e_1^T D^{(1)} \tag{2.65}$$

Pined

$$\mathcal{B}_{\bar{w}} = e_1^T I \qquad \mathcal{B}_{\hat{w}} = e_1^T D^{(2)} \tag{2.66}$$

Sliding

$$\mathcal{B}_{\bar{w}} = e_1^T D^{(1)} \qquad \mathcal{B}_{\hat{w}} = e_1^T D^{(3)} \tag{2.67}$$

Free

$$\mathcal{B}_{\bar{w}} = e_1^T D^{(2)} \qquad \mathcal{B}_{\hat{w}} = e_1^T D^{(3)} \tag{2.68}$$

Describing boundary conditions at (ζ_N) is similar, and it is easily done by replacing e_1 with e_{N+1}.

Table 2.1: Non-dimensional natural frequencies of the Euler-Bernoulli beam, $\lambda = \omega L^2\sqrt{\frac{M}{EI}}$

BCs	Method	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
C-C	Present	4.73004	7.85320	10.9956	14.1371	17.2787	20.4203
	[28]	4.73004	7.85320	10.9956	14.1372	17.2788	20.4204
P-P	Present	3.14159	6.28318	9.42477	12.5663	15.7079	18.8495
	[28]	3.14159	6.28318	9.42477	12.5664	15.7080	18.8496
P-S	Present	1.57079	4.71238	7.85398	10.9955	14.1371	17.2787
	[28]	1.57080	4.71239	7.85398	10.9956	14.1373	17.2788
C-P	Present	3.92660	7.06858	10.2101	13.3517	16.4933	19.6349
	[30]	3.92660	7.06858	10.2101			
C-F	Present	1.87510	4.69409	7.85475	10.9955	14.1371	17.2787
	[30]	1.87510	4.69409	7.85475			
C-S	Present	2.36502	5.49780	8.63937	11.7809	14.9225	18.0641
	[30]	2.36502	5.49780	8.63938			

For Euler-Bernoulli beam the eigenvalues in some boundary conditions are the same. C-C, C-P, C-S and P-P are the same as F-F, P-F, F-S and S-S without considering rigid mode or modes, respectively. Among 10 types of boundary conditions which fourth of them are similar and fourth of them has rigid mode or modes. F-F has two rigid modes and S-S, F-S and P-F has one rigid mode.

2.4 Rod problem

Axial behavior of all beam theories is called rod problem. According to the first part of eq. (2.44) equation of motion for rod problem is as

$$\frac{\partial}{\partial x}\left[E(x)A(x)\frac{\partial u(x,t)}{\partial x}\right] + f(x,t) = \rho(x)A(x)\frac{\partial^2 u(x,t)}{\partial t^2} \tag{2.69}$$

If the young modulus and cross area are constant by neglecting the axial force, equation of motion becomes

$$EA\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = \rho A\frac{\partial^2 u(x,t)}{\partial t^2} \tag{2.70}$$

Displacement of the rod problem is assumed in the form of

$$u(x,t) = u(x)e^{i\omega t} \tag{2.71}$$

Substituting the exponential transformation on time into the equation of motion, yields

$$E(x)A(x)u(x,t)'' = -\rho(x)A(x)\omega^2 u(x,t) \tag{2.72}$$

For non-dimensional equation of motion with respect to the $x \in [0, L]$, where L is the length of the rod, introduce the non-dimensional parameter as follows

$$X = \frac{x}{L} \in [0,1] \tag{2.73}$$

The equation of motion becomes

$$E(x)A(x)\frac{4}{L^2}u(x,t)'' = -\rho(x)A(x)\omega^2 u(x,t) \tag{2.74}$$

where $(')$ denotes the differentiation with respect to X.

Assuming E and A are constant, yields

$$-\mathcal{L}u = \lambda^2 u \tag{2.75}$$

where $\mathcal{L} = \frac{\partial^2}{\partial X^2}$, $\lambda^2 = \rho L^2\omega^2/4E$.

Vector of displacements at grid points ($u \in R^{(N+1)\times 1}$) are

$$u = \{u_0,\ u_1,\ ...\ u_N\} \tag{2.76}$$

Equation of motion should be satisfied for interior points ($i = 2, ..., N$). $Z_I \in R^{(N-1)\times(N+1)}$ is a matrix correspond to interior points.

$$Z_I = \begin{bmatrix} e_2^T \\ e_3^T \\ \vdots \\ e_N^T \end{bmatrix} \tag{2.77}$$

where $e_i \in R^{(N+1)\times 1}$ is the ith unit vector. This vector is zero in all entries expect for the ith entry at which it is equal to 1. Similar matrix $Z_B \in R^{2\times(N+1)}$ corresponding to the border point.

$$Z_B = \begin{bmatrix} e_1^T \\ e_{(N+1)}^T \end{bmatrix} \tag{2.78}$$

Displacements and their derivatives at the CGL points are expressed as

$$u = I, \qquad \frac{\partial u}{\partial X} = D^{(1)}, \qquad \frac{\partial^2 u}{\partial X^2} = D^{(2)} \tag{2.79}$$

where I is identity matrix by order $(N+1)$ and D is a matrix for implementing differentiation with respect to the length based on the CGL points, thus eq. (2.75) can be written as

$$-D^{(2)} = \lambda^2 I \tag{2.80}$$

The equation of motion for interior points can be written as

$$(L_B s_B + L_I s_I) = \lambda^2 s_I \tag{2.81}$$

where

$$L_B = \begin{bmatrix} Z_I K Z_B^T \end{bmatrix} \qquad L_I = \begin{bmatrix} Z_I K Z_I^T \end{bmatrix} \tag{2.82}$$

and K is left hand side of eq. (2.80).

$$s_I = \left\{ u_I \right\} \qquad s_B = \left\{ u_B \right\} \tag{2.83}$$

where s_I and s_B denote interior and boundary points, respectively.

$$\begin{bmatrix} \mathcal{B}_u \end{bmatrix} \left\{ u \right\} = \left\{ 0 \right\} \qquad\qquad (each\ \ end) \tag{2.84}$$

$\mathcal{B}_u$ is applied on the first point at each end.

The boundary equations can be written as

$$B_B s_B + B_I s_I = 0 \rightarrow s_B = -B_B^{-1} B_I s_I \tag{2.85}$$

where

$$B_B = \begin{bmatrix} B_u Z_B^T \end{bmatrix}, \qquad B_I = \begin{bmatrix} B_u Z_I^T \end{bmatrix} \tag{2.86}$$

Substitution of equation (2.85) to (2.81) results into

$$(L_I - L_B B_B^{-1} B_I) s_I = \lambda^2 s_I \tag{2.87}$$

Boundary conditions

Clamped: u=0
Free: $\frac{\partial u}{\partial x} = 0$

Boundary conditions implementation at (ζ_0)

Clamped

$$\mathcal{B}_u = e_1^T I \tag{2.88}$$

Free

$$\mathcal{B}_u = e_1^T D^{(1)} \tag{2.89}$$

At (ζ_N) describing boundary conditions are similar, and it is easily done by replacing e_1 with e_{N+1}.

Table 2.2: Non-dimensional natural frequencies of the rod problem, $\lambda = \omega L^2\sqrt{\frac{M}{EI}}$

BCs	Author	ω_1	ω_2	ω_3	ω_4
C-C	Present	3.141592	6.283183	9.424777	12.566370
	Exact	π	2π	3π	4π
C-F	Present	1.570796	4.712388	7.853981	10.995574
F-F	Present	3.141592	6.283185	9.424777	12.566370

Table (2.1) is shown natural frequencies of Euler-Bernoulli beam for many boundary conditions and results are compared with pseudospectral method using tau method. Table (2.2) is shown C-C boundary condition of the rod problem which the results are similar to P-P of Euler-Bernoulli beam. Exact solution in this case is $\pi, 2\pi, 3\pi, ...$ which is exactly similar to the above table. C-F and F-F boundary conditions of the rod problem also are shown in this table, results are similar to P-S and S-S of Euler-Bernoulli beam, respectively. The F-F case has one rigid mode.

2.5 Timoshenko Beam

Transverse normals do not remain perpendicular to the mid surface after deformation and this theory of beam includes pure bending and shear deformation. Displacements (u, w) of Timoshenko theory of beam (see Figure 2.3) are assumed as

$$\begin{aligned} u &= u_0 + z\left(\frac{\partial u}{\partial z}\right)_{z=0} = u_0 + z\theta \\ w &= w_0 \end{aligned} \tag{2.90}$$

After using principle of virtual displacements, equilibrium equations are as follow

$$\begin{aligned} \frac{\partial N_{xx}}{\partial x} + f &= \rho A \frac{\partial^2 u_0}{\partial t^2} \qquad \text{(Rod problem)} \\ \frac{\partial Q_x}{\partial x} + q &= \rho A \frac{\partial^2 w_0}{\partial t^2} \qquad \text{(Timoshenko beam)} \\ \frac{\partial M_{xx}}{\partial x} + Q_x &= \rho I \frac{\partial^2 \theta}{\partial t^2} \qquad \text{(Timoshenko beam)} \end{aligned} \tag{2.91}$$

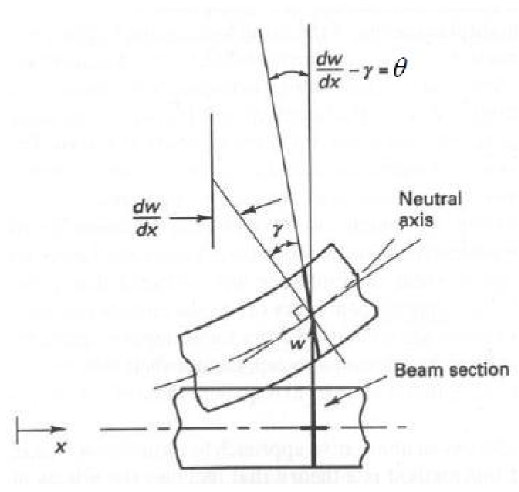

Figure 2.3: Timoshenko beam

where

$$
\begin{aligned}
Q_x &= K\int_A \sigma_{xz} dA = KG\int_A \gamma_{xz} dA = KG\int_A (-\theta + \frac{\partial w_0}{\partial x}) dA = KGA(-\theta + \frac{\partial w_0}{\partial x}) \\
M_{xx} &= \int_A \sigma_{xx} z dA = E\int_A (\frac{\partial u_0}{\partial x} + z\frac{\partial \theta}{\partial x}) z dA = EI\frac{\partial \theta}{\partial x}
\end{aligned} \quad (2.92)
$$

Substituting of eqs. (2.92) into the second and third parts of eqs. (2.91) and elimination of time with exponential transformation, yield governing equations of Timoshenko beam as follow

$$
\begin{aligned}
-EI\frac{\partial^2 \theta}{\partial x^2} + KGA\theta - KGA\frac{\partial w_0}{\partial x} &= \rho I \omega^2 \theta \\
KGA\frac{\partial \theta}{\partial x} - KGA\frac{\partial^2 w_0}{\partial x^2} &= \rho A \omega^2 w_0
\end{aligned} \quad (2.93)
$$

where $I = bh^3/12$ is moment of area, $A = bh$ is cross sectional area of beam, h is thickness of beam, b is width, ρ is density, K is shear coefficients and it is assumed 5/6, M is bending moment, Q is shear force and G is shear modulus and equal to $1/2(1+\nu)$.

Applying non-dimensional parameter $z = \frac{x}{L} \in [0, 1]$, spectral collocation method and considering the parameters $\chi = 2(1+\nu)/K$, $\mu = I/AL^2$, equation of motion on matrix form can be written as

$$
\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \theta \\ w \end{Bmatrix} = \lambda^2 \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{Bmatrix} \theta \\ w \end{Bmatrix} \quad (2.94)
$$

where

$$
\begin{aligned}
K_{11} &= 1/\chi\mu^2(-\chi\mu\frac{4}{L^2}\frac{\partial^2}{\partial x^2} + 1) = 1/\chi\mu^2(-\chi\mu\frac{4}{L^2}D^{(2)} + I) \\
K_{12} &= -1/\chi\mu^2\frac{2}{L}\frac{\partial}{\partial x} = -1/\chi\mu^2\frac{2}{L}D^{(1)} \\
K_{21} &= 1/\chi\mu\frac{2}{L}\frac{\partial}{\partial x} = 1/\chi\mu\frac{2}{L}D^{(1)} \\
K_{22} &= -1/\chi\mu\frac{4}{L^2}\frac{\partial^2}{\partial x^2} = -1/\chi\mu\frac{4}{L^2}D^{(2)}
\end{aligned} \quad (2.95)
$$

where $\lambda^2 = \rho A L^4 \omega^2 / EI$ and matrix I is identity matrix by order $(N+1)$, matrix $\mathbf{0}$ is zeros matrix by order $(N+1)$.

Vector of displacements at grid points ($w \in R^{(N+1)\times 1}$) are

$$w = \{w_0,\ \ w_1,\ \ ...\ \ w_N\} \tag{2.96}$$

Equation of motion should be satisfied for interior points ($i = 2, ..., N$). $Z_I \in R^{(N-1)\times(N+1)}$ is a matrix correspond to interior points.

$$Z_I = \begin{bmatrix} e_2^T \\ e_3^T \\ \vdots \\ e_N^T \end{bmatrix} \tag{2.97}$$

where $e_i \in R^{(N+1)\times 1}$ is the ith unit vector. This vector is zero in all entries expect for the ith entry at which it is equal to 1. Similar matrix $Z_B \in R^{2\times(N+1)}$ corresponding to the boundary points.

$$Z_B = \begin{bmatrix} e_1^T \\ e_{N+1}^T \end{bmatrix} \tag{2.98}$$

The equation of motion for interior points can be written as

$$(L_B s_B + L_I s_I) = \lambda^2 s_I \tag{2.99}$$

where

$$L_B = \begin{bmatrix} Z_I K_{11} Z_B^T & Z_I K_{12} Z_B^T \\ Z_I K_{21} Z_B^T & Z_I K_{22} Z_B^T \end{bmatrix} \qquad L_I = \begin{bmatrix} Z_I K_{11} Z_I^T & Z_I K_{12} Z_I^T \\ Z_I K_{21} Z_I^T & Z_I K_{22} Z_I^T \end{bmatrix} \tag{2.100}$$

and interior and boundary points are

$$s_I = \begin{Bmatrix} \theta_I \\ w_I \end{Bmatrix} \qquad s_B = \begin{Bmatrix} \theta_B \\ w_B \end{Bmatrix} \tag{2.101}$$

For other end ($\zeta = 1$) describing boundary conditions are similar to the end ($\zeta = 0$) and it is easily done by replacing e_1 with e_{N+1}.

$$\begin{bmatrix} \mathcal{B}_{\theta\theta} & \mathcal{B}_{\theta w} \\ \mathcal{B}_{w\theta} & \mathcal{B}_{ww} \end{bmatrix} \begin{Bmatrix} \theta \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (each\ \ end) \tag{2.102}$$

Analysis of problem is summarized in the form below as

$$\begin{bmatrix} \mathcal{B}_\theta & \mathcal{B}_w \end{bmatrix} \begin{Bmatrix} \theta \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.103}$$

where $\mathcal{B}_\theta$ is a matrix which is include $B_{\theta\theta}$ and $B_{w\theta}$ and are correspond to rotation variable θ, $\mathcal{B}_w$ also is a matrix which is include $B_{\theta w}$ and B_{ww} and are correspond to displacement variable w.

The boundary equations can be written as

$$B_B s_B + B_I s_I = 0 \rightarrow s_B = -B_B^{-1} B_I s_I \tag{2.104}$$

where

$$B_B = \left[\begin{array}{cc} B_\theta Z_B^T & B_w Z_B^T \end{array} \right], \qquad B_I = \left[\begin{array}{cc} B_\theta Z_I^T & B_w Z_I^T \end{array} \right] \tag{2.105}$$

Substitution of equation (2.104) to (2.99) results into

$$(L_I - L_B B_B^{-1} B_I) s_I = \lambda^2 s_I \tag{2.106}$$

Boundary conditions:

Clamped:	$\theta = 0, w = 0$
Pined:	$M = 0, w = 0$
Sliding:	$\theta = 0, N = 0$
Free:	$M = 0, N = 0$

Boundary condition implementation at (ζ_0)

Clamped

$$B_{\theta\theta} = e_1^T I, \qquad B_{\theta w} = e_1^T \mathbf{0}, \qquad B_{w\theta} = e_1^T \mathbf{0}, \qquad B_{ww} = e_1^T I \tag{2.107}$$

Pined

$$B_{\theta\theta} = e_1^T D^{(1)}, \qquad B_{\theta w} = e_1^T \mathbf{0}, \qquad B_{w\theta} = e_1^T \mathbf{0}, \qquad B_{ww} = e_1^T I \tag{2.108}$$

Sliding

$$B_{\theta\theta} = e_1^T I, \qquad B_{\theta w} = e_1^T \mathbf{0}, \qquad B_{w\theta} = -e_1^T I, \qquad B_{ww} = e_1^T D^{(1)} \tag{2.109}$$

Free

$$B_{\theta\theta} = e_1^T D^{(1)}, \qquad B_{\theta w} = e_1^T \mathbf{0}, \qquad B_{w\theta} = -e_1^T I, \qquad B_{ww} = e_1^T D^{(1)} \tag{2.110}$$

Describing boundary conditions at (ζ_N) is similar, and it is easily done by replacing e_1 with e_{N+1}.

Table (2.3) is shown natural frequencies of Timoshenko beam which is applied on many boundaries and result are compared with pseudospectral method. As seen with increasing the ratio of thickness with respect to length of the beam, frequencies are decreased. And also in table (2.4) other boundary conditions for Timoshenko beam are tabulated.

Among all the boundary conditions for Timoshenko beam theory, for SS and PF cases there is one rigid mode and for FF there are two rigid modes.

Table 2.3: Non-dimensional natural frequencies of the Timoshenko beam, $\lambda = \omega L^2 \sqrt{\frac{M}{EI}}$

BCs	h/L	Author	ω_1	ω_2	ω_3	ω_4
C-C	0.1	Present	4.57954	7.33121	9.85611	12.1453
		[28]	4.57955	7.33122	9.85611	12.1454
	0.2	Present	4.24201	6.41793	8.28531	9.90372
		[28]	4.24201	6.41794	8.28532	9.90372
P-P	0.1	Present	3.11568	6.09066	8.84051	11.3431
		[28]	3.11568	6.09066	8.84052	11.3431
	0.2	Present	3.04533	5.67155	7.83951	9.65709
		[28]	3.04533	5.67155	7.83952	9.65709
P-S	0.1	Present	1.56749	4.62768	7.49632	10.1223
		[28]	1.56749	4.62769	7.49632	10.1223
	0.2	Present	1.55784	4.42025	6.80658	8.78524
		[28]	1.55784	4.42026	6.80658	8.78525
F-F	0.1	Present	4.64849	7.49718	10.1254	12.5076
		[28]	4.64849	7.49719	10.1255	12.5076
	0.2	Present	4.44957	6.80257	8.77286	10.4093
		[28]	4.44958	6.80257	8.77287	10.4094

Table 2.4: Non-dimensional natural frequencies of the Timoshenko beam, $\lambda = \omega L^2 \sqrt{\frac{M}{EI}}$

BCs	h/L	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
SS	0.1	3.1157	6.0907	8.8405	11.343	13.613	15.679	17.570	19.314
	0.2	3.0453	5.6716	7.8395	9.6571	11.222	12.602	13.444	13.843
CF	0.1	1.8677	4.5724	7.4154	9.9873	12.322	14.445	16.388	18.176
	0.2	1.8466	4.2853	6.6113	8.5186	10.158	11.572	12.782	13.349
CP	0.1	3.8518	6.7306	9.3659	11.758	13.933	15.920	17.750	19.446
	0.2	3.6656	6.0727	8.0744	9.7862	11.286	12.623	13.141	13.784
CS	0.1	2.3450	5.3201	8.0795	10.591	12.871	14.948	16.853	18.613
	0.2	2.2898	4.9281	7.1162	8.9607	10.559	11.973	13.229	13.461
PF	0.1	3.8770	6.8020	9.4906	11.932	14.147	16.163	18.010	19.711
	0.2	3.7486	6.2538	8.3179	10.048	11.522	12.750	13.148	13.643
FS	0.1	2.3544	5.3666	8.1775	10.741	13.066	15.178	17.106	18.877
	0.2	2.3242	5.0627	7.3341	9.2187	10.814	12.173	13.193	13.567

Chapter 3

Plates

3.1 Discretization of 2D problem

A grid of Chebyshev-Gauss-Lobatto (CGL) points (ζ_i, η_j), $(i, j = 0, ..., N)$ over two-dimensional problem such as plate or shells is defined as (see Figure 3.1)

$$\xi_i = \tfrac{1}{2}\Big(1 - \cos\left(i\tfrac{\pi}{N}\right)\Big)$$
$$\eta_j = \tfrac{1}{2}\Big(1 - \cos\left(j\tfrac{\pi}{N}\right)\Big) \tag{3.1}$$

The discrete solution of the problem is sought in the form of tensor product of one-dimensional expansions as follows

$$\hat{\mathbf{u}}^N(\xi, \eta) = \mathcal{I}_N\hat{\mathbf{u}} = \sum_{m=0}^{N}\sum_{n=0}^{N}\hat{\mathbf{u}}_{(m,n)}L_m(\xi)L_n(\eta) \tag{3.2}$$

where $\hat{\mathbf{u}}_{mn}$ is the vector of unknown values at the grid points and $\hat{\mathbf{u}}_{mn} = \hat{\mathbf{u}}^N(\zeta_m, \eta_n)$, $\hat{\mathbf{u}}^N$ is vector of displacements and rotations of the problem and $L_l(l = 0, ..., N)$ is the Lagrange interpolating polynomial corresponding to the given set of CGL points, thus $L_m(\zeta_i) = \delta_{mi}$ and $L_n(\eta_i) = \delta_{nj}$ for $(i, j = 0, ...N)$.

Approximation should satisfy the differential equation of the problem along with the boundary conditions at the CGL points. The resulting residual is enforced to vanish at the interior collocation points

$$-\mathcal{L}\hat{\mathbf{u}}^N{}_{\xi=\xi_i,\eta=\eta_j} = \lambda^2\hat{\mathbf{u}}^N{}_{\xi=\xi_i,\eta=\eta_j}$$

$$-L_N\hat{\mathbf{u}}^N = \lambda^2\hat{\mathbf{u}}_{(i,j)} \tag{3.3}$$

where L_N is the pseudo-spectral operator obtained by approximating each derivative by the corresponding interpolation derivative

$$D_N\hat{\mathbf{u}} = \mathcal{D}\left(\mathcal{I}_N\hat{\mathbf{u}}\right) \tag{3.4}$$

The remaining equations are prescribed by enforcing the satisfaction of the boundary conditions.

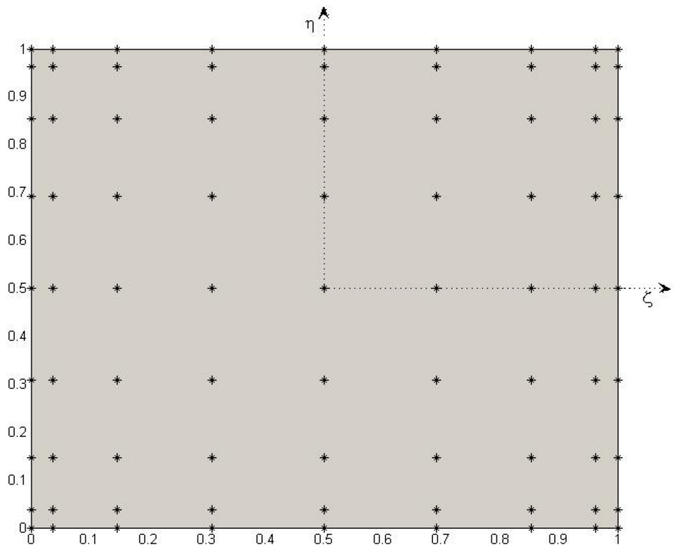

Figure 3.1: Two dimensional grid of (CGL) points on the computational domain(N=8)

The interpolation derivatives are given by:

$$\frac{\partial^2 \hat{\mathbf{u}}^N}{\partial \xi^2}_{\xi=\xi_i,\eta=\eta_j} = \sum_{m=0}^{N} \hat{\mathbf{u}}_{mj} \frac{\partial^2 L_m}{\partial \xi^2}(\xi_i) = \sum_{m=0}^{N} D^{(2)}_{N(i,m)} \hat{\mathbf{u}}_{(m,j)} = (D^2_{(N)} \otimes I_{(N)})$$

$$\frac{\partial^2 \hat{\mathbf{u}}^N}{\partial \eta^2}_{\xi=\xi_i,\eta=\eta_j} = \sum_{n=0}^{N} \hat{\mathbf{u}}_{in} \frac{\partial^2 L_n}{\partial \eta^2}(\eta_j) = \sum_{n=0}^{N} D^{(2)}_{N(j,n)} \hat{\mathbf{u}}_{(i,n)} = (I_{(N)} \otimes D^2_{(N)})$$

$$\frac{\partial^2 \hat{\mathbf{u}}^N}{\partial \xi \partial \eta}_{\xi=\xi_i,\eta=\eta_j} = \sum_{m=0}^{N} \sum_{n=0}^{N} \hat{\mathbf{u}}_{mn} \frac{\partial L_m}{\partial \xi}(\xi_i) \frac{\partial L_n}{\partial \eta}(\eta_j)$$

$$= \sum_{m=0}^{N} \sum_{n=0}^{N} D^{(1)}_{N(i,m)} D^{(1)}_{N(j,n)} \hat{\mathbf{u}}_{(m,n)} = (D_{(N)} \otimes D_{(N)}) \quad (3.5)$$

The following interpolation derivative matrices are introduced as

$$D^{(1)}_N = \begin{bmatrix} D^{(1)}_N(0,0) & D^{(1)}_N(0,1) & \cdots & D^{(1)}_N(0,N) \\ D^{(1)}_N(1,0) & D^{(1)}_N(1,1) & \cdots & D^{(1)}_N(1,N) \\ \vdots & \vdots & \vdots & \vdots \\ D^{(1)}_N(N,0) & D^{(1)}_N(N,1) & \cdots & D^{(1)}_N(N,N) \end{bmatrix} \quad (3.6)$$

Chebyshev differentiation matrix with CGL points is defined by

$$(D_N)_{00} = \frac{2N^2+1}{3} \qquad (D_N)_{NN} = -\frac{2N^2+1}{3}$$

$$(D_N)_{jj} = -\frac{x_j}{(1-x_j^2)} \qquad j = 1, ..., N-1$$

$$(D_N)_{ij} = -\frac{c_i}{c_j} \frac{2(-1)^{(i+j)}}{x_i - x_j} \qquad i \neq j \quad j = 1, ..., N-1$$

$$c_i = 2 \quad (i = 0, N) \qquad \text{otherwise} \quad c_i = 1 \quad (3.7)$$

The Kronecker product $\otimes$ in two-dimensional domain is as follows

$$
\begin{aligned}
&\frac{\partial^n}{\partial x^n} = (D_N^{(n)} \otimes I_N) \qquad &&\frac{\partial^n}{\partial y^n} = (I_N \otimes D_N^{(n)}) \qquad n = 1,2,3,4\\
&\frac{\partial^2}{\partial x \partial y} = (D_N^{(1)} \otimes D_N^{(1)}) \qquad &&\frac{\partial^4}{\partial x^2 \partial y^2} = (D_N^{(2)} \otimes D_N^{(2)})
\end{aligned} \tag{3.8}
$$

where I is the identity matrix by $(N+1)$ order $D^{(1)}, D^{(2)}, D^{(3)}, D^{(4)}$ are the first-, second-, third- and fourth-order derivatives of the interpolation at the CGL nodes, respectively. Kronecker product tensor works as follows

$$
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{N\times N} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{N\times N} = \begin{bmatrix} a & b & | & 2a & 2b \\ c & d & | & 2c & 2d \\ -- & -- & -- & -- & -- \\ 3a & 3b & | & 4a & 4b \\ 3c & 3d & | & 4c & 4d \end{bmatrix}_{N^2\times N^2} \tag{3.9}
$$

3.2 Classical plate theory

Consider a plate with length a, width b, thickness h and number of layers N_l which can be isotropic or orthotropic is defined in the orthogonal coordinates (x_1^k, x_2^k, x_3^k) of the kth lamina oriented at an angle θ_k. Take xy-plane of the problem in the undeformed mid plane Ω_0 of the laminate. The z-axis is taken positive downward from the mid plane. The kth layer is located between the points $z = z_k$ and $z = z_{k+1}$ in the thickness direction.

Assumption of theory of thin plates which is called classical plate theory (CLPT) or Kirchhoff plate theory are similar to Euler-Bernoulli beam theory in two-dimension. Assumptions are in the following

1- Thickness of the plate is small compared to other dimensions.
2- The mid plane of the plate does n't have any in-plane deformation after loading.
3- The displacement components of the mid surface of the plate are small compared to the thickness.
4- Transverse shear deformation is neglected. Perpendicular surface to the mid surface before and after deformation remains normal and leads to $\epsilon_{xz} = \epsilon_{yz} = 0$.
5- The transverse normal strain ϵ_{zz} under transverse loading can be neglected and also σ_{zz} is small and is negligible compared with the other stress components.

3.2.1 Out-of-plane vibration of rectangular plates

In the Kirchhoff hypothesis displacements (u, v, w) are assumed as (see Figure 3.2)

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) - z\frac{\partial w_0}{\partial x}\\
v(x,y,z,t) &= v_0(x,y,t) - z\frac{\partial w_0}{\partial y}\\
w(x,y,z,t) &= w_0(x,y,t)
\end{aligned} \tag{3.10}
$$

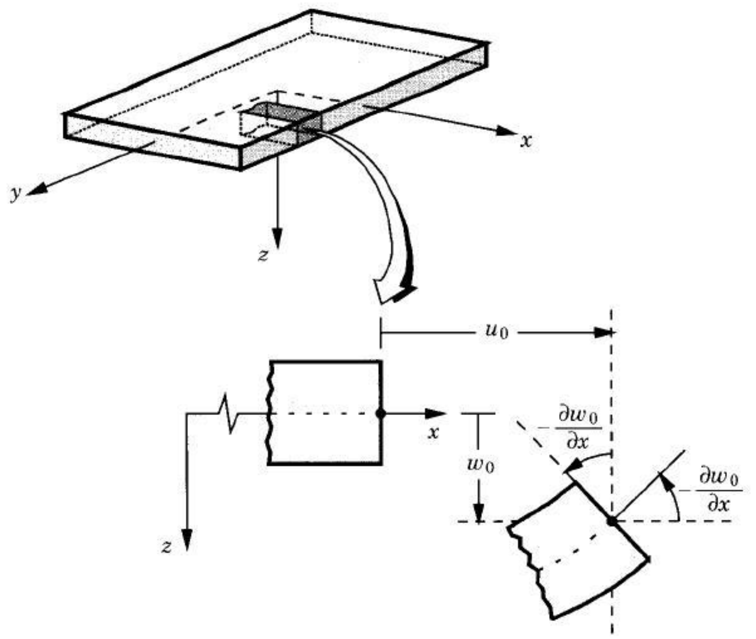

Figure 3.2: Undeformed and deformed geometry in thickness direction under Kirchhoff hypothesis

where u_0, v_0, w_0 are displacements of the mid surface. x, y are in-plane coordinates and z is the thickness direction.

The linear three-dimensional strain-displacement relations are written in below as

$$\begin{aligned}
\epsilon_{xx} &= \frac{\partial u}{\partial x} \\
\epsilon_{xy} &= \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\
\epsilon_{yy} &= \frac{\partial v}{\partial y} \\
\epsilon_{xz} &= \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \\
\epsilon_{yz} &= \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \\
\epsilon_{zz} &= \frac{\partial w}{\partial z}
\end{aligned} \tag{3.11}$$

where the strains $(\epsilon_{xx}, \epsilon_{yy}, \gamma_{xy})$ are linear through the thickness, considering assumptions of Kirchhoff plate theory while $\frac{\partial u_0}{\partial z} = -\frac{\partial w_0}{\partial x}$ and $\frac{\partial v_0}{\partial z} = -\frac{\partial w_0}{\partial y}$ the transverse shear strains $(\epsilon_{xz}, \epsilon_{yz})$ are zero and also $\epsilon_{zz} = 0$ in this theory of laminated plates.

Substituting eq. (3.10) into the eq. (3.11), strain-displacement relations yield as

$$\left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{array} \right\} = \left\{ \begin{array}{c} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{array} \right\} + z \left\{ \begin{array}{c} \epsilon_{xx}^{(1)} \\ \epsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{array} \right\} + \left\{ \begin{array}{c} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{array} \right\} \tag{3.12}$$

where $\epsilon_{xx}^{(0)}, \epsilon_{yy}^{(0)}, \gamma_{xy}^{(0)}$ are the membrane strains and $\epsilon_{xx}^{(1)}, \epsilon_{yy}^{(1)}, \gamma_{xy}^{(1)}$ are the flexural (bending) strains which is known as the curvature.

Based on the principle of virtual work, dynamic equation is written as follows

$$\int_0^t (\delta U + \delta V - \delta K)dt = 0 \tag{3.13}$$

where δU is virtual strain energy, δV is virtual work done by applied forces and the virtual kinetic energy δK are given by

$$\begin{aligned}
\delta U &= \int_{\Omega_0} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx}\delta\epsilon_{xx} + \sigma_{yy}\delta\epsilon_{yy} + \sigma_{xy}\delta\gamma_{xy} + \sigma_{xz}\delta\gamma_{xz} + \sigma_{yz}\delta\gamma_{yz} + \sigma_{zz}\delta\epsilon_{zz})dzdxdy \\
\delta V &= -\int_{\Omega_0} \Big(q_b(x,y)\delta w_0(x,y,\frac{h}{2}) + q_t(x,y)\delta w_0(x,y,-\frac{h}{2})\Big) dxdy \\
&\quad - \int_{\Gamma_\sigma} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{nn}\delta u_n + \sigma_{ns}\delta u_s + \sigma_{nz}\delta u_z)dzds \\
\delta K &= \int_{\Omega_0} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_0 (u\delta u + v\delta v + w\delta w)dzdxdy
\end{aligned} \tag{3.14}$$

Above equations are general form of principle of virtual work. Virtual strain energy for classical plate theory just has first three terms, First-order shear deformation theory has first five terms and complete form of δU is used for three-dimensional theory of plates. $\sigma_{nn}, \sigma_{ns}, \sigma_{nz}$ are specified stress components on the portion of Γ_σ. q_b and q_t are distributed forces on the bottom and top surfaces of plate, respectively.

Substituting eqs. (3.10) and (3.12) into the eq. (3.14) and integration by part, Euler-lagrange equations of the theory are obtained by setting the coefficients of $\delta u_0, \delta v_0$, and δw_0 equal to zero separately as follow

$$\begin{aligned}
\delta u_0 : &\quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2}(\frac{\partial w_0}{\partial x}) \\
\delta v_0 : &\quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2}(\frac{\partial w_0}{\partial y}) \\
\delta w_0 : &\quad \frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}) \\
&\quad + I_1 \frac{\partial^2}{\partial t^2}(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y})
\end{aligned} \tag{3.15}$$

where q is transverse force, M_{xx}, M_{yy}, M_{xy} are moments resultants and N_{xx}, N_{yy}, N_{xy} are force resultants as

$$\begin{aligned}
\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(1)} \\ \epsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \\
\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} &= \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(1)} \\ \epsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}
\end{aligned} \tag{3.16}$$

and I_0, I_1, I_2 are mass moment of inertia as

$$I_i = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \tilde{C}_{ij}^{(k)} (1, z, z^2) dz \qquad i = 0, 1, 2 \tag{3.17}$$

Stress resultants are as

$$\left\{ \begin{array}{c} N_{\alpha\beta} \\ M_{\alpha\beta} \end{array} \right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} \left\{ \begin{array}{c} 1 \\ z \end{array} \right\} dz \tag{3.18}$$

A_{ij}, B_{ij}, D_{ij} are extensional stiffness, bending-extensional coupling stiffness and bending stiffness, respectively as

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \tilde{C}_{ij}^{(k)} (1, z, z^2) dz \qquad (i, j) = 1, 2, 6 \tag{3.19}$$

where

$$\tilde{C}_{ij} = T' C_{ij} T = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{16} \\ \tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{26} \\ \tilde{C}_{16} & \tilde{C}_{26} & \tilde{C}_{66} \end{bmatrix} \tag{3.20}$$

is the elastic coefficients of the kth lamina taking into account the angle of orthotropy $\theta^{(k)}$ in each layer. C_{ij} is the engineering parameters as

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \tag{3.21}$$

where

$$\begin{aligned} C_{11} &= \frac{E_1}{1 - \nu_{12}\nu_{21}} \\ C_{12} &= \frac{\nu_{12} E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21} E_1}{1 - \nu_{12}\nu_{21}} \\ C_{22} &= \frac{E_2}{1 - \nu_{12}\nu_{21}}, C_{66} = G_{12} \end{aligned} \tag{3.22}$$

and also E_1, E_2 are Young's moduli in first and second material coordinates, ν_{ij} are poisson's ratio, defined as the ratio of transverse strains in the j direction to the axial strain in the ith direction when stressed in the ith direction and G_{ij} are shear moduli in the $x - y$ plane.

$$T = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 \\ \sin^2\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.23}$$

is transformation matrix. Complete matrix and relations which is for three-dimensional theory is expressed by eqs. (5.9) to (5.12). In Kirchhoff hypothesis C_{ij} and T matrices

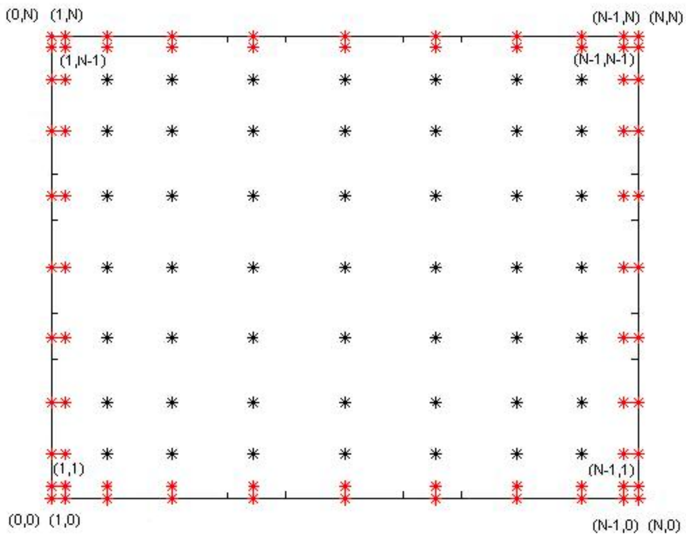

Figure 3.3: Interior and boundary points of the classical theory

are 3×3 matrix, in first-order shear deformation theory are 5×5 and finally for three-dimensional theory are 6×6.

$$
\begin{aligned}
V_x &= \frac{\partial M_{xx}}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} \\
V_y &= 2\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y}
\end{aligned}
\tag{3.24}
$$

are transverse force resultants. Complete form of the above equation is mentioned in equation (4.10) for shells structure.

Substituting eq. (3.12) and second part of eq. (3.16) into the third part of eq. (3.15), yields the governing equation for out-of-plane vibration of Kirchhoff plate as

$$
\begin{aligned}
&D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} \\
&+D_{22}\frac{\partial^4 w}{\partial y^4} = I_0\frac{\partial^2 w}{\partial t^2} - I_2\frac{\partial^2}{\partial t^2}(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})
\end{aligned}
\tag{3.25}
$$

Vector of displacements at grid points ($w \in R^{(N+1)^2 \times 1}$) are

$$
w = \{w_{(0,0)} \ w_{(0,1)} \ ... \ w_{(0,N)} \ w_{(1,0)} \ w_{(1,1)} \ ... \ w_{(1,N)} \ ... \ ... \ w_{(N,0)} \ w_{(N,1)} \ ... \ w_{(N,N)}\}
\tag{3.26}
$$

Equation of motion should be satisfied only for interior points ($i, j = 2, ..., N-2$). $Z_I \in R^{(N-3)^2 \times (N+1)^2}$ is a matrix correspond to interior points.

$$Z_I = \begin{bmatrix} e^T_{2(N+1)+3} \\ \vdots \\ e^T_{3(N+1)-2} \\ e^T_{3(N+1)+3} \\ \vdots \\ \vdots \\ e^T_{(N-2)(N+1)-2} \\ e^T_{(N-2)(N+1)+3} \\ \vdots \\ e^T_{(N-1)(N+1)-2} \end{bmatrix} \tag{3.27}$$

where $e_i \in R^{(N+1)^2 \times 1}$ is the ith unit vector. This vector is zero in all entries expect for the ith entry at which it is equal to 1. Matrices $Z_{B_{\bar{w}}} \in R^{4N \times (N+1)^2}$ and $Z_{B_{\hat{w}}} \in R^{(4N-8) \times (N+1)^2}$ corresponding to the border points and set of points before the border points, respectively. (see Figure 3.3)

$$Z_B^{CLPT} = \begin{bmatrix} Z_{B_{\bar{w}}} \\ Z_{B_{\hat{w}}} \end{bmatrix} \tag{3.28}$$

where

$$Z_{B_{\bar{w}}} = \begin{bmatrix} e^T_1 \\ e^T_{(N+1)+1} \\ e^T_{2(N+1)+1} \\ \vdots \\ e^T_{(N)(N+1)+1} \\ e^T_{(N)(N+1)+2} \\ \vdots \\ e^T_{(N)(N+1)+N} \\ e^T_{(N+1)} \\ e^T_{2(N+1)} \\ \vdots \\ e^T_{(N+1)(N+1)} \\ e^T_2 \\ \vdots \\ e^T_N \end{bmatrix} \qquad Z_{B_{\hat{w}}} = \begin{bmatrix} e^T_{(N+1)+2} \\ e^T_{2(N+1)+2} \\ \vdots \\ e^T_{(N-1)(N+1)+2} \\ e^T_{(N-1)(N+1)+3} \\ \vdots \\ e^T_{(N-1)(N+1)+N-1} \\ e^T_{2(N+1)-1} \\ e^T_{3(N+1)-1} \\ \vdots \\ e^T_{(N)(N+1)-1} \\ e^T_{(N+1)+3} \\ e^T_{(N+1)+4} \\ \vdots \\ e^T_{2(N+1)-2} \end{bmatrix} \tag{3.29}$$

The equation of motion for interior points can be written as

$$L_R^{-1}(L_B s_B + L_I s_I) = -\lambda^2 s_I \tag{3.30}$$

where

$$L_B = \left[\ Z_I K Z_B^T\ \right] \qquad L_I = \left[\ Z_I K Z_I^T\ \right] \qquad L_R = \left[\ Z_I M Z_I^T\ \right] \tag{3.31}$$

K is the left hand side of eq. (3.25) and also M is the right hand side of the this equation which the second term is neglected because it is very negligible in compared with the first term in the right hand side and L_R is what does we have in the right.

$$s_I = \left\{\ w_I\ \right\} \qquad s_B = \left\{\ w_B\ \right\} \tag{3.32}$$

where s_I and s_B are displacement vector variable of interior and boundary points, respectively.

Displacements and their derivatives at the CGL points along $\zeta = 0$ are expressed as

$$\begin{aligned} &w = (e_1^T \otimes I)w && w_{\zeta\eta} = (e_1^T D^{(1)} \otimes D^{(1)})w \\ &w_\zeta = (e_1^T D^{(1)} \otimes I)w && w_{\zeta\zeta} = (e_1^T D^{(2)} \otimes I)w \\ &w_\eta = (e_1^T \otimes D^{(1)})w && w_{\eta\eta} = (e_1^T \otimes D^{(2)})w \\ &w_{\zeta\zeta\eta} = (e_1^T D^{(2)} \otimes D^{(1)})w && w_{\zeta\eta\eta} = (e_1^T D^{(1)} \otimes D^{(2)})w \\ &w_{\zeta\zeta\zeta\zeta} = (e_1^T D^{(4)} \otimes I)w && w_{\eta\eta\eta\eta} = (e_1^T \otimes D^{(4)})w \end{aligned} \tag{3.33}$$

Similarly, for edge $\eta = 0$ it is written as

$$\begin{aligned} &w = (I \otimes e_1^T)w && w_{\zeta\eta} = (D^{(1)} \otimes e_1^T D^{(1)})w \\ &w_\zeta = (D^{(1)} \otimes e_1^T)w && w_{\zeta\zeta} = (D^{(2)} \otimes e_1^T)w \\ &w_\eta = (I \otimes e_1^T D^{(1)})w && w_{\eta\eta} = (I \otimes e_1^T D^{(2)})w \\ &w_{\zeta\zeta\eta} = (D^{(2)} \otimes e_1^T D^{(1)})w && w_{\zeta\eta\eta} = (D^{(1)} \otimes e_1^T D^{(2)})w \\ &w_{\zeta\zeta\zeta\zeta} = (D^{(4)} \otimes e_1^T)w && w_{\eta\eta\eta\eta} = (I \otimes e_1^T D^{(4)})w \end{aligned} \tag{3.34}$$

For the edges $(\zeta, \eta = 1)$ describing boundary conditions and equation of motion are similar to the previous formula and it is easily done by replacing e_1 with e_{N+1}. For each edge, there are two boundary conditions, $B_{\bar{w}}$ is applied on $\bar{w}$ and $B_{\hat{w}}$ is applied on $\hat{w}$.

$$\begin{bmatrix} \mathcal{B}_{\bar{w}} \\ \mathcal{B}_{\hat{w}} \end{bmatrix} \left\{\ w\ \right\} = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} \qquad\qquad (each\ \ edge) \tag{3.35}$$

where $\mathcal{B}_{\bar{w}}$ and $\mathcal{B}_{\hat{w}}$ matrices should implemented on four edges correspond to displacement vector w.

The boundary equations can be written as

$$B_B s_B + B_I s_I = 0 \rightarrow s_B = -B_B^{-1} B_I s_I \tag{3.36}$$

where

$$B_B = \left[\ B_w Z_B^T\ \right], \qquad B_I = \left[\ B_w Z_I^T\ \right], \qquad B_w = \begin{bmatrix} \mathcal{B}_{\bar{w}} \\ \mathcal{B}_{\hat{w}} \end{bmatrix} \tag{3.37}$$

B_w is evaluated for all four edges considering which points are included or not. Substitution of equation (3.36) to (3.30) results into

$$L_R^{-1}(L_I - L_B B_B^{-1} B_I)s_I = \lambda^2 s_I \tag{3.38}$$

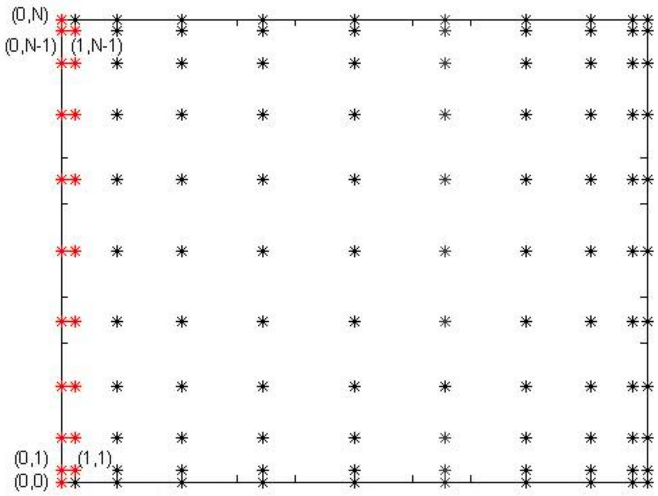

Figure 3.4: Boundary conditions at $\zeta = 0$

Boundary conditions at $\zeta = 0$

Clamped ($w = \frac{\partial w}{\partial x} = 0$)

$$\begin{aligned} \mathcal{B}_{\bar{w}} &= w = (e_1^T \otimes I)w \qquad (i = 0, j = 0, ..., N) \\ \mathcal{B}_{\hat{w}} &= \frac{\partial w}{\partial x} = (e_1^T D^{(1)} \otimes I)w \qquad (i = 1, j = 1, ..., N-1) \end{aligned} \tag{3.39}$$

Simply-supported ($w = M_{xx} = 0$)

$$\begin{aligned} &\mathcal{B}_{\bar{w}} = w = (e_1^T \otimes I)w \qquad (i = 0, j = 0, ..., N) \\ &\mathcal{B}_{\hat{w}} = M_{xx} = \Big(D_{11}(e_1^T D^{(2)} \otimes I) + 2D_{16}(e_1^T D^{(1)} \otimes D^{(1)}) \\ &+D_{12}(e_1^T \otimes D^{(2)})\Big)w \qquad (i = 1, j = 1, ..., N-1) \end{aligned} \tag{3.40}$$

Free ($M_{xx} = V_x = 0$)

$$\begin{aligned} &\mathcal{B}_{\bar{w}} = M_{xx} = \Big(D_{11}(e_1^T D^{(2)} \otimes I) + 2D_{16}(e_1^T D^{(1)} \otimes D^{(1)}) \\ &+D_{12}(e_1^T \otimes D^{(2)})\Big)w \qquad (i = 0, j = 0, ..., N) \\ &\mathcal{B}_{\hat{w}} = V_x = \Big(D_{11}(e_1^T D^{(3)} \otimes I) + 4D_{16}(e_1^T D^{(2)} \otimes D^{(1)}) \\ &+(D_{12} + 4D_{66})(e_1^T D^{(1)} \otimes D^{(2)}) + 2D_{26}(e_1^T \otimes D^{(3)}\Big)w \qquad (i = 1, j = 1, ..., N-1) \end{aligned} \tag{3.41}$$

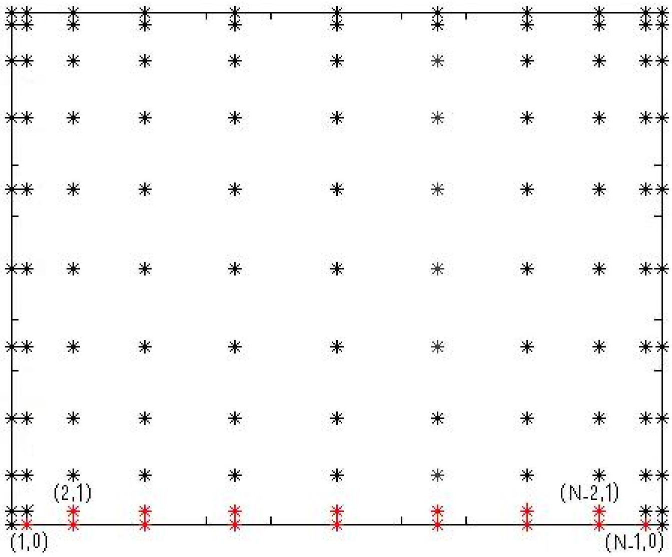

Figure 3.5: Boundary conditions at $\eta = 0$

Boundary conditions at $\eta = 0$

Clamped ($w = \frac{\partial w}{\partial y} = 0$)

$$\mathcal{B}_{\bar{w}} = w = (I \otimes e_1^T)w \qquad (j = 0, i = 1, ..., N-1)$$
$$\mathcal{B}_{\hat{w}} = \frac{\partial w}{\partial y} = (D^{(1)} \otimes e_1^T)w \qquad (j = 1, i = 2, ..., N-2) \tag{3.42}$$

Simply supported ($w = M_{yy} = 0$)

$$\mathcal{B}_{\bar{w}} = w = (I \otimes e_1^T)w \qquad (j = 0, i = 1, ..., N-1)$$

$$\mathcal{B}_{\hat{w}} = M_{yy} = \Big(D_{12}(D^{(2)} \otimes e_1^T) + 2D_{26}(D^{(1)} \otimes e_1^T D^{(1)})$$
$$+D_{22}(I \otimes e_1^T D^{(2)})\Big)w \qquad (j = 1, i = 2, ..., N-2) \tag{3.43}$$

Free ($M_{yy} = V_y = 0$)

$$\mathcal{B}_{\bar{w}} = M_{yy} = \Big(D_{12}(D^{(2)} \otimes e_1^T) + 2D_{26}(D^{(1)} \otimes e_1^T D^{(1)})$$
$$+D_{22}(I \otimes e_1^T D^{(2)})\Big)w \qquad (j = 0, i = 1, ..., N-1)$$

$$\mathcal{B}_{\hat{w}} = V_y = \Big(D_{22}(I \otimes e_1^T D^{(3)}) + 4D_{26}(D^{(1)} \otimes e_1^T D^{(2)})$$
$$+(D_{12} + 4D_{66})(D^{(2)} \otimes e_1^T D^{(1)}) + 2D_{16}(D^{(3)} \otimes e_1^T)\Big)w \qquad (j = 1, i = 2, ..., N-2) \tag{3.44}$$

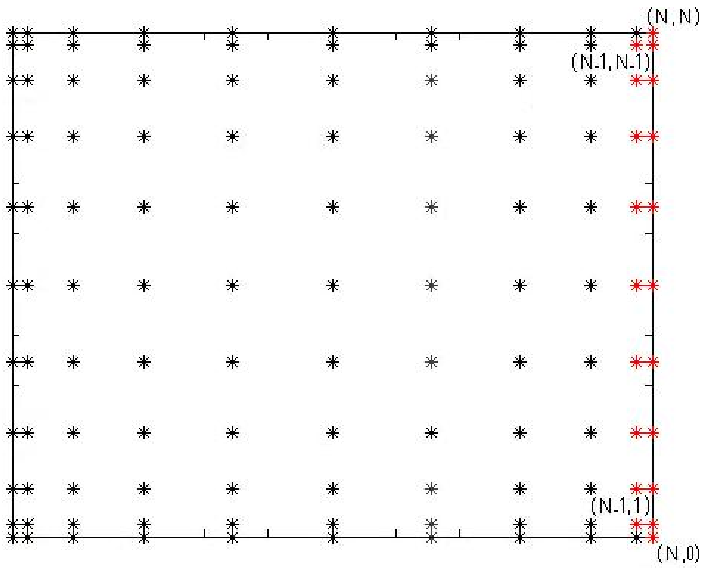

Figure 3.6: Boundary conditions at $\zeta = 1$

At corners for interaction of two free edges $M_{xy} = 0$ just for one point in the angle. It can be assumed that these corners are belong to ($\zeta = 0$ or 1) or ($\eta = 0$ or 1). Thus, if corner is at $\zeta = 0$, boundary condition is written as

$$\Big(D_{16}(e_1^T D^{(2)} \otimes I) + D_{26}(e_1^T \otimes D^{(2)}) \\ +2D_{66}(e_1^T D^{(1)} \otimes D^{(1)})\Big)w = 0 \quad (i = 0, j = 0 \text{ or } N \text{ or } j = 0, N) \tag{3.45}$$

and for corner at $\eta = 0$ boundary condition is

$$\Big(D_{16}(D^{(2)} \otimes e_1^T) + D_{26}(I \otimes e_1^T D^{(2)}) \\ +2D_{66}(D^{(1)} \otimes e_1^T D^{(1)})\Big)w = 0 \quad (j = 0, i = 0 \text{ or } N \text{ or } i = 0, N) \tag{3.46}$$

For describing boundary conditions at $\zeta = 1$ and $\eta = 1$, inside of formula it is enough to replace e_1 with $e_{(N+1)}$ and for points which are imposed by the boundary conditions

$$\begin{array}{ll} \zeta = 0 : \left\{ \begin{array}{l} (i = 0, j = 0, ..., N) \\ (i = 1, j = 1, ..., N-1) \end{array} \right. & \rightarrow \zeta = 1 : \left\{ \begin{array}{l} (i = N, j = 0, ..., N) \\ (i = N-1, j = 1, ..., N-1) \end{array} \right. \\ \eta = 0 : \left\{ \begin{array}{l} (j = 0, i = 1, ..., N-1) \\ (j = 1, i = 2, ..., N-2) \end{array} \right. & \rightarrow \eta = 1 : \left\{ \begin{array}{l} (j = N, i = 1, ..., N-1) \\ (j = N-1, i = 2, ..., N-2) \end{array} \right. \end{array} \tag{3.47}$$

and at corners

$$\begin{array}{l} \zeta = 0 : (i = 0, j = 0 \text{ or } N \text{ or } j = 0, N) \rightarrow \zeta = 1 : (i = N, j = 0 \text{ or } N \text{ or } j = 0, N) \\ \eta = 0 : (j = 0, i = 0 \text{ or } N \text{ or } i = 0, N) \rightarrow \eta = 1 : (j = N, i = 0 \text{ or } N \text{ or } i = 0, N) \end{array} \tag{3.48}$$

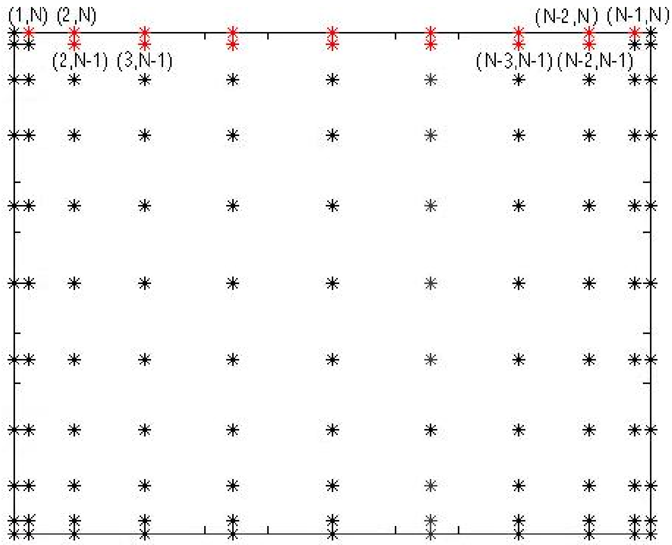

Figure 3.7: Boundary conditions at $\eta = 1$

For isotropic plate $D = \frac{Eh^3}{12(1-\nu^2)}$, $D_{22} = D_{11} = D$, $D_{12} = \nu D, D_{16} = D_{26} = 0$ and $D_{66} = (1-\nu)\frac{D}{2}$.

Displacement assumed as $w(x,y,z,t) = w(x,y,z)e^{i\omega t}$, substituting this assumption into the equation of motion for isotropic plate, yields

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{\rho h \omega^2}{D} w \tag{3.49}$$

Using non-dimensional parameter $\zeta = \frac{x}{a}, \eta = \frac{y}{b}$, yields

$$\frac{\partial^4 w}{\partial \zeta^4} + 2\beta^2 \frac{\partial^4 w}{\partial \zeta^2 \partial \eta^2} + \beta^4 \frac{\partial^4 w}{\partial \eta^4} = -\lambda^2 w \tag{3.50}$$

where $\beta = \frac{a}{b}, \lambda^2 = \frac{\rho h a^4}{D}\omega^2$.

Displacement is assumed in the Lagrangian form as

$$w(\zeta, \eta) = \sum_{m=0}^{M} \sum_{n=0}^{N} a_{mn} T_m(\zeta_i) T_n(\eta_j) \tag{3.51}$$

After substitution the displacement into the equation of motion, yields

$$\sum_{m=0}^{M} \sum_{n=0}^{N} a_{mn}[T_m^{(4)}(\zeta_i)T_n(\eta_j) + 2\beta^2 T_m^{(2)}(\zeta_i)T_n^{(2)}(\eta_j)$$
$$+\beta^4 T_m(\zeta_i)T_n^{(4)}(\eta_j)] = -\lambda^2 \sum_{m=0}^{M} \sum_{n=0}^{N} a_{mn} T_m(\zeta_i) T_n(\eta_j) \tag{3.52}$$

Spectral form using Kronecker product is

$$(D^{(4)} \otimes I) + 2\beta^2 (D^{(2)} \otimes D^{(2)}) + \beta^4 (I \otimes D^{(4)}) = -\lambda^2 (I \otimes I) \tag{3.53}$$

Table 3.1: Non-dimensional natural frequencies of the Kirchhoff plate, $\lambda = \omega a^2 \sqrt{\frac{\rho h}{D}}$

BCs	β	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
CCCC	$\frac{2}{3}$	27.004	41.703	66.124	66.521	79.804	100.81
	1	35.985	73.393	73.393	108.21	131.58	132.20
	$\frac{3}{2}$	60.761	93.833	148.77	149.67	179.56	226.82
SFSF	$\frac{2}{3}$	9.6983	12.981	22.953	39.105	40.356	42.684
	1	9.6313	16.134	36.725	38.944	46.738	70.740
	$\frac{3}{2}$	9.5581	21.619	38.721	54.844	65.792	87.626
SSSS	$\frac{1}{2}$	12.337	19.739	32.076	41.945	49.348	49.348
	1	19.739	49.348	49.348	78.956	98.696	98.696
	2	49.348	78.956	128.30	167.78	197.39	197.39
	5	256.60	286.21	335.56	404.65	493.48	602.04

where left hand side of above equation is K for isotropic plates in eq. (3.31).

Table (3.1) is shown results for fully clamped, fully simply-supported and SFSF boundary conditions in terms of different aspect ratios which are related to the dimensions. Natural frequencies of fully clamped and simply supported boundaries with increasing the aspect ratio, increased except SFSF case, which is behaved in vice versa.

Table 3.2: Non-dimensional natural frequencies of the Classical laminated composite plate $(\beta, -\beta, \beta, -\beta, \beta)$ for E-glass/epoxy, $\lambda = \omega a^2 \sqrt{\frac{\rho h}{D_{11}}}$

BCs	β^o	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
CCCC	0	Present	29.104	50.828	67.286	85.680	87.142	118.56
		[65]	29.087	50.792	67.279	85.629	87.112	118.50
		[192]	29.13	50.82	67.29	85.67	87.14	118.6
	45	Present	28.643	56.342	59.946	86.477	102.996	104.85
		[65]	28.624	56.308	59.917	86.486	102.95	104.81
		[192]	28.68	56.34	59.94	86.48	103.0	104.9
SSSS	0	Present	15.194	33.299	44.418	60.778	64.529	90.301
		[65]	15.171	33.248	44.387	60.682	64.457	90.145
		[193]	15.19	33.30	44.42	60.77	64.53	90.29
	45	Present	16.387	38.374	41.379	64.353	77.941	79.198
		[65]	16.480	38.436	41.478	64.563	77.958	79.223
		[193]	16.40	38.37	41.40	64.41	77.94	79.23

In table (3.2) presented results are compared with two methods, discrete singular convolution method (DSC) [65] and CLPT [192, 193] based on the finite element analysis both for fully clamped and fully simply supported. Results are shown natural frequencies of five layers symmetric composite laminated thin plate in terms of angle orientation. Increasing angle cause to increasing the frequencies.

For some engineering applications, constrained boundary conditions with constant or varying coefficients are mostly used. each edge may be constrained with translational spring which is a constrained coefficient multiply by displacement, or constrained with torsional spring which is a constrained coefficient multiply by differentiation of displacement with respect to perpendicular coordinate. Sign of these coefficients for two opposite edges should be different for equilibrium. For varying constrained coefficients a polynomial in terms of perpendicular to edge with arbitrary order is considered. If all coefficients are zero, boundary condition becomes free condition and if they are infinity boundary condition becomes clamped condition. If torsional coefficients are zero and translational coefficients are infinity, boundary condition becomes simply supported condition.

Constrained boundary conditions with constant coefficients after normalization:

at $\zeta = 0$

$$\begin{aligned} &\left(\frac{\partial^2 w}{\partial x^2} + \nu\beta^2\frac{\partial^2 w}{\partial y^2}\right) - \frac{k_{1T}a}{D}\frac{\partial w}{\partial x} = 0 \\ &\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\beta^2\frac{\partial^2 w}{\partial y^2 \partial x}\right) + \frac{k_{1L}a^3}{D}w = 0 \end{aligned} \tag{3.54}$$

at $\zeta = 1$

$$\begin{aligned} &\left(\frac{\partial^2 w}{\partial x^2} + \nu\beta^2\frac{\partial^2 w}{\partial y^2}\right) + \frac{k_{2T}a}{D}\frac{\partial w}{\partial x} = 0 \\ &\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\beta^2\frac{\partial^2 w}{\partial y^2 \partial x}\right) - \frac{k_{2L}a^3}{D}w = 0 \end{aligned} \tag{3.55}$$

at $\eta = 0$

$$\begin{aligned} &\left(\beta^2\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) - \frac{k_{3T}a\beta}{D}\frac{\partial w}{\partial y} = 0 \\ &\left(\beta^2\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^2 w}{\partial x^2 \partial y}\right) + \frac{k_{3L}a^3}{\beta D}w = 0 \end{aligned} \tag{3.56}$$

and at $\eta = 1$

$$\begin{aligned} &\left(\beta^2\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) + \frac{k_{4T}a\beta}{D}\frac{\partial w}{\partial y} = 0 \\ &\left(\beta^2\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^2 w}{\partial x^2 \partial y}\right) - \frac{k_{4L}a^3}{\beta D}w = 0 \end{aligned} \tag{3.57}$$

where $(k_{1T}, k_{2T}, k_{3T}, k_{4T})$ are torsional and $(k_{1L}, k_{2L}, k_{3L}, k_{4L})$ are translational constrained coefficients and each group of coefficients can be equal with themselves or not.

Table (3.3) is shown results for isotropic thin rectangular plate with fully constrained boundaries and with not fully constrained boundaries. Increasing aspect ratio for fully constrained boundaries cause to increasing the frequency and for other boundaries increasing the elastic coefficients is shown the increasing in the frequencies.

Table 3.3: Non-dimensional natural frequencies of the Kirchhoff plate with constrained boundaries, $\lambda = \omega a^2 \sqrt{\frac{\rho h}{D}}$

BCs	K_L	K_T	c	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
EEEE	100	1000	1	17.509	25.295	25.295	33.901	46.285	46.862
	100	1000	4	30.815	35.655	52.784	95.250	161.34	162.31
CSES	10	0	1	13.931	33.676	42.088	63.297	72.682	90.794
	100	10	1	19.398	40.718	44.810	67.042	81.051	92.521
CFEE	10	100	1	7.9254	12.504	30.250	34.414	37.683	61.224
			2	8.5237	27.599	30.281	57.153	73.321	103.216

Table 3.4: Non-dimensional natural frequencies of the classical plate with elastic support of parabolically varying stiffness along the four edges both parallel and normal to the edges, $\lambda = \frac{\omega a}{2}\sqrt{\frac{\rho(1-\nu^2)}{E}}$

a/b	K	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
1	1	Present	0.9521	0.9521	1.3307	1.5296	1.7866	1.9089
		[69]	0.9521	0.9521	1.3307	1.5296	1.7866	1.9090
	100	Present	1.7563	1.7563	2.1073	2.5532	2.9002	2.9318
		[69]	1.7564	1.7564	2.1074	2.5535	2.9002	2.9319
0.5	1	Present	0.7799	0.8262	0.9561	1.1607	1.2204	1.3465
		[69]	0.7800	0.8262	0.9562	1.1607	1.2205	1.3465
	100	Present	1.1894	1.5767	1.6626	1.7478	1.8865	2.0209
		[69]	1.1895	1.5769	1.6628	1.7481	1.8867	2.0211

Varying constrained coefficients are written as following

$$\begin{aligned} k_{(T,L)}(x) &= k_{0(T,L)}x(a-x) \rightarrow k_{(T,L)}(\zeta) = k_{0(T,L)}diag(\zeta_i - \zeta_i^2) \\ k_{(T,L)}(y) &= k_{0(T,L)}y(b-y) \rightarrow k_{(T,L)}(\eta) = k_{0(T,L)}diag(\eta_j - \eta_j^2) \end{aligned} \quad (3.58)$$

Tables (3.4) and (3.5) are shown results for varying fully elastic support parallel and normal to edges parabolic ally and linearly for isotropic thin plate with two aspect ratios, respectively. As seen with increasing the elastic coefficients, natural frequencies increased.

Bulking of plates under in-plane loads

Since plate is subjected to in-plane compressive load (normal forces) ($\hat{N}_{xx}, \hat{N}_{yy} < 0$) and sufficiently small, and $\hat{N}_{xy} = 0$ (shear forces), plate is stable and remains flat until critical load. This load is called bulking load and plate is not stable at that load. Equilibrium configuration of plate will be an other one considering load deflection behavior. This phenomenon without magnificent deformation of plate is called bifurcation. The load-deflection curve of bulked plates is often bilinear. The magnitude of bulking load depends on geometry and material properties. For bulking analysis under normal in-plane load, it is assumed other thermal and mechanical loads are zero. The governing equation

Table 3.5: Non-dimensional natural frequencies of the classical plate with elastic support of linearly varying stiffness along the four edges both parallel and normal to the edges, $\lambda = \frac{\omega a}{2}\sqrt{\frac{\rho(1-\nu^2)}{E}}$

a/b	K	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
1	1	Present	0.8535	0.8535	1.1783	1.4294	1.7146	1.7404
		[69]	0.8537	0.8537	1.1786	1.4295	1.7149	1.7405
	100	Present	1.7467	1.7467	2.1041	2.5285	2.8934	2.9259
		[69]	1.7468	1.7468	2.1041	2.5289	2.8935	2.9260
0.5	1	Present	0.7064	0.7360	0.8508	1.0830	1.1031	1.2126
		[69]	0.7066	0.7362	0.8509	1.0831	1.1033	1.2128
	100	Present	1.1857	1.5698	1.6545	1.7409	1.8774	2.0134
		[69]	1.1858	1.5700	1.6547	1.7412	1.8776	2.0137

of bulking (under in-plane load) for orthotropic Kirchhoff plate is

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = \hat{N}_{xx}\frac{\partial^2 w}{\partial x^2} + \hat{N}_{yy}\frac{\partial^2 w}{\partial y^2} \tag{3.59}$$

where

$$\hat{N}_{xx} = N_0, \qquad \hat{N}_{yy} = -kN_0, \qquad k = \frac{\hat{N}_{yy}}{\hat{N}_{xx}} \tag{3.60}$$

For $k = 0$, plate in under bulking with uniform compression and with $k = 1$ is under biaxial compression. Governing equation of bulking (under shear in-plane load) for orthotropic Kirchhoff plate is

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = 2\hat{N}_{xy}\frac{\partial^2 w}{\partial x\partial y} \tag{3.61}$$

where $\hat{N}_{xy} = N_0$.

Table 3.6: Non-dimensional natural frequencies of the SCSC Kirchhoff plate subjected to uniform compressive in-plane load at the simply supported edges, $\lambda = \omega a^2\sqrt{\frac{\rho h}{D_{11}}}$

a/b	N_0/N_{cr}	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
1	0.5	Present	21.530	38.709	66.570	84.123	86.301	127.61
		[35]	21.53	38.708	66.571	84.123	86.303	127.61
	0.8	Present	15.452	24.481	64.860	71.090	80.925	119.42
		[35]	15.452	24.479	64.861	71.090	80.926	119.43
0.5	0.5	Present	9.677	21.575	37.934	47.912	86.987	95.974
		[35]	9.677	21.575	37.934	47.913	86.988	95.977
	0.8	Present	6.120	20.231	34.846	45.506	84.030	93.303
		[35]	6.120	20.231	34.846	45.507	84.031	93.305

Table (3.6) is shown natural frequencies for SCSC boundary condition of isotropic thin plate which is subjected to uniform compressive in-plane load. As seen with increasing parameter $\frac{N_0}{N_{cr}}$ which is being in-plane load divided by critical bulking load, natural frequency is decreased. Natural frequencies of thin rectangular plate for fully clamped and fully simply supported subjected to in-plane shear load is shown in table (3.7). In both tables results are compared with Trigonometric Ritz method.

Table 3.7: Non-dimensional natural frequencies of the Square Kirchhoff plate, subjected to in-plane shear load $\frac{N_{xy}a^2}{D_{11}}$, $\lambda = \omega a^2\sqrt{\frac{\rho h}{D_{11}}}$

BCs	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
SSSS	Present	18.867	44.277	53.066	76.636	98.336	100.10
	[35]	18.868	44.277	53.066	76.636	98.337	100.10
CCCC	Present	33.507	63.136	79.618	102.08	130.38	135.63
	[35]	33.508	63.138	79.620	102.09	130.39	135.64

3.2.2 Non-symmetric composite laminated rectangular plates

In the case of Non-symmetric composite laminated plate using classical plate theory, in-plane displacements (u, v) and out-of-plane displacement (w), are coupling together. Because in symmetric case reaction among all layers are in equilibrium but in non-symmetric case residual stress can be vanished using in-plane reaction of the layers. Equation of motion after normalization using these parameters $\zeta = x/a, \eta = y/b, \beta = a/b, \delta = h/a, \gamma = h/b, W = w/h$ can be written as

$$A_{11}\frac{\partial^2 u}{\partial x^2} + A_{12}\beta\frac{\partial^2 v}{\partial x\partial y} + A_{66}(\beta^2\frac{\partial^2 u}{\partial y^2} + \beta\frac{\partial^2 v}{\partial x\partial y}) - B_{11}\delta\frac{\partial^3 W}{\partial x^3} - 3B_{16}\gamma\frac{\partial^2 W}{\partial x^2\partial y}$$
$$-B_{26}\beta^2\gamma\frac{\partial^3 W}{\partial y^3} = I_0\frac{\partial^2 u}{\partial t^2} - I_1 h\frac{\partial^2 W}{\partial x\partial t^2}$$

$$A_{12}\beta\frac{\partial^2 u}{\partial x\partial y} + A_{22}\beta^2\frac{\partial^2 v}{\partial y^2} + A_{66}(\frac{\partial^2 v}{\partial x^2} + \beta\frac{\partial^2 u}{\partial x\partial y}) + B_{11}\gamma\beta^2\frac{\partial^3 W}{\partial y^3} - B_{16}\delta\frac{\partial^2 W}{\partial x^3}$$
$$-3B_{26}\gamma\beta\frac{\partial^3 W}{\partial x\partial y^2} = I_0\frac{\partial^2 v}{\partial t^2} - I_1 h\frac{\partial^2 W}{\partial y\partial t^2}$$

$$B_{11}(\frac{\partial^3 u}{\partial x^3} - \beta^3\frac{\partial^3 v}{\partial y^3}) + B_{16}(3\beta\frac{\partial^3 u}{\partial x^2\partial y} + \frac{\partial^3 v}{\partial x^3}) + B_{26}(3\beta^2\frac{\partial^3 v}{\partial x\partial y^2} + \beta^3\frac{\partial^3 u}{\partial y^3})$$
$$-D_{11}\delta\frac{\partial^4 W}{\partial x^4} - D_{22}\gamma\beta^3\frac{\partial^4 W}{\partial y^4} - (2D_{12} + 4D_{66})\gamma\beta\frac{\partial^4 W}{\partial x^2\partial y^2} + q + \hat{N}_{xx}\frac{\partial^2 W}{\partial x^2}$$
$$+\hat{N}_{yy}\frac{\partial^2 W}{\partial y^2} + 2\hat{N}_{xy}\frac{\partial^2 W}{\partial x\partial y} = I_0 h\frac{\partial^2 W}{\partial t^2} - I_2 h\frac{\partial^2}{\partial t^2}(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2})$$
$$+I_1\frac{\partial^2}{\partial t^2}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \quad (3.62)$$

In non-symmetric case B_{ij} coefficients connect in-plane equations including A_{ij} coefficients and out-of-plane equations with D_{ij} coefficients. Equation (3.62) is non-symmetric for both cross-ply with B_{11} coefficient, and angle-ply with B_{16} and B_{26} coefficients because for non-symmetric cross-ply laminated plates all B_{ij} are zero except B_{11} and B_{22} where $B_{22} = -B_{11}$, and also for non-symmetric angle-ply laminated plates all B_{ij} are zero except B_{16} and B_{26}.

Similar to section (3.2.1) we need L_B, L_I and L_R, in this case they are obtained as following

$$L_B = \begin{bmatrix} Z_I^{FSDT} K_{11} Z_B^{FSDT^T} & Z_I^{FSDT} K_{12} Z_B^{FSDT^T} & Z_I^{FSDT} K_{13} Z_B^{CLPT^T} \\ Z_I^{FSDT} K_{21} Z_B^{FSDT^T} & Z_I^{FSDT} K_{22} Z_B^{FSDT^T} & Z_I^{FSDT} K_{23} Z_B^{CLPT^T} \\ Z_I^{CLPT} K_{31} Z_B^{FSDT^T} & Z_I^{CLPT} K_{32} Z_B^{FSDT^T} & Z_I^{CLPT} K_{33} Z_B^{CLPT^T} \end{bmatrix} \tag{3.63}$$

and

$$L_I = \begin{bmatrix} Z_I^{FSDT} K_{11} Z_I^{FSDT^T} & Z_I^{FSDT} K_{12} Z_I^{FSDT^T} & Z_I^{FSDT} K_{13} Z_I^{CLPT^T} \\ Z_I^{FSDT} K_{21} Z_I^{FSDT^T} & Z_I^{FSDT} K_{22} Z_I^{FSDT^T} & Z_I^{FSDT} K_{23} Z_I^{CLPT^T} \\ Z_I^{CLPT} K_{31} Z_I^{FSDT^T} & Z_I^{CLPT} K_{32} Z_I^{FSDT^T} & Z_I^{CLPT} K_{33} Z_I^{CLPT^T} \end{bmatrix} \tag{3.64}$$

and also L_R is the same as L_I but instead of stiffness components, mass components will be replaced.

These matrices components are presenting methods which are used for solving out-of-plane classical plate theory (section (3.2.1)) and in-plane analysis which is like summarized method where will be mention in section (3.3.1) with two variables instead of three variables in first-order shear deformation theory. Therefore FSDT in equations of this section does not meaning that they are following this theory, it means that computationally they are similar. Because for both in-plane analysis and FSDT theory for each edge there is just one condition applied on each variable for all edges but in CLPT theory there are two conditions applied on w displacement variable. For these reasons L_B and L_I matrix components are combination of interior and boundary displacements vectors of both classical plate theory (CLPT) and first-order shear deformation plate theory (FSDT).

$$s_I = \begin{Bmatrix} u_I^{FSDT} \\ v_I^{FSDT} \\ w_I^{CLPT} \end{Bmatrix} \qquad s_B = \begin{Bmatrix} u_B^{FSDT} \\ v_B^{FSDT} \\ w_B^{CLPT} \end{Bmatrix} \tag{3.65}$$

where s_I and s_B are displacement vector variable of interior and boundary points, respectively. For applying boundary conditions four equations should be applied on each edge as

$$\begin{bmatrix} \mathcal{B}_{uu} & \mathcal{B}_{uv} & \mathcal{B}_{u\bar{w}} \\ \mathcal{B}_{vu} & \mathcal{B}_{vv} & \mathcal{B}_{v\bar{w}} \\ \mathcal{B}_{\bar{w}u} & \mathcal{B}_{\bar{w}v} & \mathcal{B}_{\bar{w}\bar{w}} \\ \mathcal{B}_{\hat{w}u} & \mathcal{B}_{\hat{w}v} & \mathcal{B}_{\hat{w}\hat{w}} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (each\ \ edge) \tag{3.66}$$

The whole set of discretized boundary conditions are expressed in below

$$\begin{bmatrix} B_u & B_v & B_w \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.67}$$

where B_u collects the $\mathcal{B}_{uu}$, $\mathcal{B}_{vu}$, $\mathcal{B}_{\bar{w}u}$ and $\mathcal{B}_{\hat{w}u}$ matrices of the four edges correspond to displacement vector u. B_v puts to gether the $\mathcal{B}_{uv}$, $\mathcal{B}_{vv}$, $\mathcal{B}_{\bar{w}v}$ and $\mathcal{B}_{\hat{w}v}$ matrices of the four edges correspond to displacement vector v and B_w groups the $\mathcal{B}_{u\bar{w}}$, $\mathcal{B}_{v\bar{w}}$, $\mathcal{B}_{\bar{w}\bar{w}}$ and $\mathcal{B}_{\hat{w}\hat{w}}$ matrices of the four edges correspond to displacement vector w. i.e., B_u can be written as

$$Bu = \begin{bmatrix} \mathcal{B}_{uu}^{(0,\eta)}(0:N,:) \\ \mathcal{B}_{vu}^{(0,\eta)}(0:N,:) \\ \mathcal{B}_{\bar{w}u}^{(0,\eta)}(0:N,:) \\ \mathcal{B}_{\hat{w}u}^{(0,\eta)}(1:N-1,:) \\ \\ \mathcal{B}_{uu}^{(\zeta,0)}(1:N-1,:) \\ \mathcal{B}_{vu}^{(\zeta,0)}(1:N-1,:) \\ \mathcal{B}_{\bar{w}u}^{(\zeta,0)}(1:N-1,:) \\ \mathcal{B}_{\hat{w}u}^{(\zeta,0)}(2:N-2,:) \\ \\ \mathcal{B}_{uu}^{(1,\eta)}(0:N,:) \\ \mathcal{B}_{vu}^{(1,\eta)}(0:N,:) \\ \mathcal{B}_{\bar{w}u}^{(1,\eta)}(0:N,:) \\ \mathcal{B}_{\hat{w}u}^{(1,\eta)}(1:N-1,:) \\ \\ \mathcal{B}_{uu}^{(\zeta,1)}(1:N-1,:) \\ \mathcal{B}_{vu}^{(\zeta,1)}(1:N-1,:) \\ \mathcal{B}_{\bar{w}u}^{(\zeta,1)}(1:N-1,:) \\ \mathcal{B}_{\hat{w}u}^{(\zeta,1)}(2:N-2,:) \end{bmatrix} \tag{3.68}$$

First four terms are corresponded to $\zeta = 0$, and second, third and fourth groups are corresponded to $(\eta = 0)$, $(\zeta = 1)$ and $(\eta = 1)$, respectively. As in programming languages zero for rows or columns does n't have meaning, we can add 1 to rows or columns, i.e., $(0 : N, :)$ can be written as $(1 : N + 1, :)$.

B_B and B_I matrices for obtaining natural frequencies are written as

$$\begin{aligned} B_B &= \begin{bmatrix} B_u Z_B^{FSDT^T} & B_v Z_B^{FSDT^T} & B_w Z_B^{CLPT^T} \end{bmatrix} \\ B_I &= \begin{bmatrix} B_u Z_I^{FSDT^T} & B_v Z_I^{FSDT^T} & B_w Z_I^{CLPT^T} \end{bmatrix} \end{aligned} \tag{3.69}$$

Boundary conditions for cross-ply lay-up at $\zeta = 0$

Clamped ($u = v = w = \frac{\partial w}{\partial x} = 0$)
Simply supported ($N_{xx} = v = w = M_{xx}$=0)

$$\begin{aligned} N_{xx} &= A_{11}\frac{\partial u}{\partial x} + A_{12}\beta\frac{\partial v}{\partial y} - B_{11}\delta\frac{\partial^2 w}{\partial x^2} \\ M_{xx} &= D_{11}\delta\frac{\partial^2 w}{\partial x^2} + D_{12}\beta\gamma\frac{\partial^2 w}{\partial y^2} - B_{11}\frac{\partial u}{\partial x} \end{aligned} \tag{3.70}$$

Free ($N_{xx} = N_{xy} = M_{xx} = V_x = 0$)

$$\begin{aligned} N_{xx} &= A_{11}\frac{\partial u}{\partial x} + A_{12}\beta\frac{\partial v}{\partial y} - B_{11}\delta\frac{\partial^2 w}{\partial x^2} \\ N_{xy} &= A_{66}(\beta\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ M_{xx} &= D_{11}\delta\frac{\partial^2 w}{\partial x^2} + D_{12}\beta\gamma\frac{\partial^2 w}{\partial y^2} - B_{11}\frac{\partial u}{\partial x} \\ V_x &= D_{11}\delta\frac{\partial^3 w}{\partial x^3} + (D_{12} + 4D_{66})\beta\gamma\frac{\partial^3 w}{\partial x \partial y^2} - B_{11}\frac{\partial^2 u}{\partial x^2} \end{aligned} \tag{3.71}$$

Boundary conditions for cross-ply lay-up at $\eta = 0$

Clamped ($u = v = w = \frac{\partial w}{\partial y} = 0$)
Simply supported ($u = N_{yy} = w = M_{yy} = 0$)

$$\begin{aligned} N_{yy} &= A_{12}\frac{\partial u}{\partial x} + A_{22}\beta\frac{\partial v}{\partial y} + B_{11}\beta\gamma\frac{\partial^2 w}{\partial y^2} \\ M_{yy} &= D_{12}\delta\frac{\partial^2 w}{\partial x^2} + D_{22}\beta\gamma\frac{\partial^2 w}{\partial y^2} + B_{11}\beta\frac{\partial v}{\partial y} \end{aligned} \tag{3.72}$$

Free ($N_{xy} = N_{yy} = M_{yy} = V_y = 0$)

$$\begin{aligned} N_{yy} &= A_{12}\frac{\partial u}{\partial x} + A_{22}\beta\frac{\partial v}{\partial y} + B_{11}\beta\gamma\frac{\partial^2 w}{\partial y^2} \\ N_{xy} &= A_{66}(\beta\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ M_{yy} &= D_{12}\delta\frac{\partial^2 w}{\partial x^2} + D_{22}\beta\gamma\frac{\partial^2 w}{\partial y^2} + B_{11}\beta\frac{\partial v}{\partial y} \\ V_y &= D_{22}\beta^2\gamma\frac{\partial^3 w}{\partial y^3} + (D_{12} + 4D_{66})\gamma\frac{\partial^3 w}{\partial x^2 \partial y} + B_{11}\beta^2\frac{\partial^2 v}{\partial y^2} \end{aligned} \tag{3.73}$$

Boundary conditions for angle-ply lay-up at $\zeta = 0$

Clamped ($u = v = w = \frac{\partial w}{\partial x} = 0$)
Simply supported ($u = N_{xy} = w = M_{xx} = 0$)

$$N_{xy} = A_{66}(\beta\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) - B_{16}\delta\frac{\partial^2 w}{\partial x^2} - B_{26}\beta\gamma\frac{\partial^2 w}{\partial y^2}$$
$$M_{xx} = D_{11}\delta\frac{\partial^2 w}{\partial x^2} + D_{12}\beta\gamma\frac{\partial^2 w}{\partial y^2} - B_{16}(\beta\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \tag{3.74}$$

Free ($N_{xx} = N_{xy} = M_{xx} = V_x = 0$)

$$N_{xx} = A_{11}\frac{\partial u}{\partial x} + A_{12}\beta\frac{\partial v}{\partial y} - 2B_{16}\gamma\frac{\partial^2 w}{\partial x\partial y}$$
$$N_{xy} = A_{66}(\beta\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) - B_{16}\delta\frac{\partial^2 w}{\partial x^2} - B_{26}\beta\gamma\frac{\partial^2 w}{\partial y^2}$$
$$M_{xx} = D_{11}\delta\frac{\partial^2 w}{\partial x^2} + D_{12}\beta\gamma\frac{\partial^2 w}{\partial y^2} - B_{16}(\beta\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$
$$V_x = D_{11}\delta\frac{\partial^3 w}{\partial x^3} + (D_{12} + 4D_{66})\beta\gamma\frac{\partial^3 w}{\partial x\partial y^2} - B_{16}(\frac{\partial^2 v}{\partial x^2} + 3\beta\frac{\partial^2 u}{\partial x\partial y})$$
$$-2B_{26}\beta^2\frac{\partial^2 v}{\partial y^2} \tag{3.75}$$

Boundary conditions for angle-ply lay-up at $\eta = 0$

Clamped ($u = v = w = \frac{\partial w}{\partial y}$)
Simply supported ($N_{xy} = v = w = M_{yy} = 0$)

$$N_{xy} = A_{66}(\beta\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) - B_{16}\delta\frac{\partial^2 w}{\partial x^2} - B_{26}\beta\gamma\frac{\partial^2 w}{\partial y^2}$$
$$M_{yy} = D_{12}\delta\frac{\partial^2 w}{\partial x^2} + D_{22}\beta\gamma\frac{\partial^2 w}{\partial y^2} - B_{26}(\beta\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \tag{3.76}$$

Free ($N_{xy} = N_{yy} = M_{yy} = V_y = 0$)

$$N_{yy} = A_{12}\frac{\partial u}{\partial x} + A_{22}\beta\frac{\partial v}{\partial y} - 2B_{26}\gamma\frac{\partial^2 w}{\partial x\partial y}$$
$$N_{xy} = A_{66}(\beta\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) - B_{16}\delta\frac{\partial^2 w}{\partial x^2} - B_{26}\beta\gamma\frac{\partial^2 w}{\partial y^2}$$
$$M_{yy} = D_{12}\delta\frac{\partial^2 w}{\partial x^2} + D_{22}\beta\gamma\frac{\partial^2 w}{\partial y^2} - B_{26}(\beta\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$
$$V_y = D_{22}\beta^2\gamma\frac{\partial^3 w}{\partial y^3} + (D_{12} + 4D_{66})\gamma\frac{\partial^3 w}{\partial x^2\partial y} - B_{26}(\beta^2\frac{\partial^2 u}{\partial y^2} + 3\beta\frac{\partial^2 v}{\partial x\partial y})$$
$$-2B_{16}\frac{\partial^2 u}{\partial x^2} \tag{3.77}$$

As more transparency, free boundary condition in angle-ply case for $\zeta = 0$ is explained.

$$\mathcal{B}_{uu} = A_{11}\frac{\partial}{\partial x} \qquad \mathcal{B}_{uv} = A_{12}\beta\frac{\partial}{\partial y} \qquad \mathcal{B}_{u\bar{w}} = -2B_{16}\gamma\frac{\partial^2}{\partial x\partial y}$$
$$\mathcal{B}_{vu} = A_{66}\beta\frac{\partial}{\partial y} \qquad \mathcal{B}_{vv} = A_{66}\frac{\partial}{\partial x} \qquad \mathcal{B}_{v\bar{w}} = -B_{16}\delta\frac{\partial^2}{\partial x^2} - B_{26}\beta\gamma\frac{\partial^2}{\partial y^2}$$
$$\mathcal{B}_{\bar{w}u} = -B_{16}\beta\frac{\partial}{\partial y} \qquad \mathcal{B}_{\bar{w}v} = -B_{16}\frac{\partial}{\partial x} \qquad \mathcal{B}_{\bar{w}\bar{w}} = D_{11}\delta\frac{\partial^2}{\partial x^2} + D_{12}\beta\gamma\frac{\partial^2}{\partial y^2}$$
$$\mathcal{B}_{\hat{w}u} = -3B_{16}\beta\frac{\partial^2}{\partial x\partial y} \qquad \mathcal{B}_{\hat{w}v} = -B_{16}\frac{\partial^2}{\partial x^2} - 2B_{26}\beta^2\frac{\partial^2}{\partial y^2}$$
$$\mathcal{B}_{\hat{w}\hat{w}} = D_{11}\delta\frac{\partial^3}{\partial x^3} + (D_{12} + 4D_{66})\beta\gamma\frac{\partial^3}{\partial x\partial y^2} \tag{3.78}$$

Position of the points which are imposed by boundary conditions in this case is a combination of what is shown in Figs. (3.13) and (3.14) in section (3.3.1) for displacements u and v and Figs. (3.4) and (3.5) in section (3.2.1) for displacement w.

3.2.3 Out-of-plane vibration of annular and circular plates

Consider an annular plate with r radius, inner radius b, outer radius a and $\beta = b/a$ is radius ratio. For annular plate ($b \leq r \leq a$) after normalization with parameter $R = r/a$, range of non-dimensional radius is ($\beta \leq R \leq 1$). In this case which is similar to Euler-Bernoulli beam, w is displacement variable in the thickness direction based on Kirchhoff hypothesis. Collocation points which are distributed on the annular and circular plates are shown in Figs (3.8). Interior points are in black and boundary points are in red color. Figs (3.8) shows distribution of interior and boundary points for annular (left) and circular (right) plate. The location of these points are not really based on Chebyshev location points and this figure is a schematic figure but Figs (3.9) and (3.10) show the real location of collocation points in 1-D dimensional geometry like beam. In the case of annular plate for inner and outer radius two collocation points are needed. For circular plate one collocation point for implementing center point and two points for outer radius are needed.

Governing equation of motion for both annular and circular plates are as

$$\frac{\partial^4 w}{\partial r^4} + \frac{2}{r}\frac{\partial^3 w}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3}\frac{\partial w}{\partial r} = \frac{q(r)}{D}\frac{\partial^2 w}{\partial t^2} \tag{3.79}$$

where D is bending stiffness and $q(r)$ is the axisymmetric load on the plate.

Bending moment and shear force are as

$$M_{rr} = -D(\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}), \quad Q_r = -D\frac{d}{dr}(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}) \tag{3.80}$$

For annular plate $b \leq r \leq a$ after doing dimensionless with $R = r/a$ parameter, $\beta \leq R \leq 1$ and for circular plate which is $0 \leq r \leq a$, yields $\leq R \leq 1$. Equation of motion in matrix

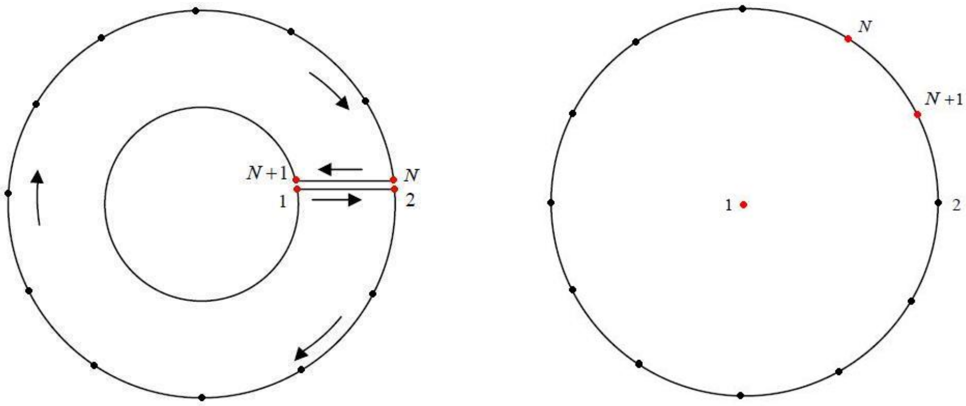

Figure 3.8: 2-D distribution of interior and boundary points of annular and circular plate

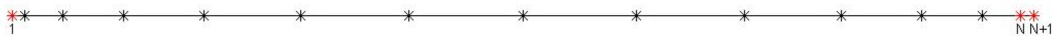

Figure 3.9: 1-D distribution of interior and boundary points of circular plate

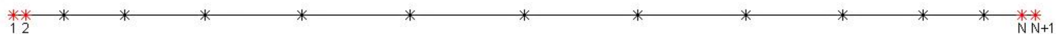

Figure 3.10: 1-D distribution of interior and boundary points of annular plate

form is

$$Kw = -\lambda^2 Mw \tag{3.81}$$

where stiffness and mass matrices after multiplying by suitable power of R avoiding singularity is as

$$\begin{aligned} K &= R^3 \frac{\partial^4}{\partial R^4} + 2R^2 \frac{\partial^3}{\partial R^3} - R \frac{\partial^2}{\partial R^2} + \frac{\partial}{\partial R} \\ M &= \frac{q(Ra)R^3 a^4}{D} \end{aligned} \tag{3.82}$$

and also moment and force resultants are as

$$\begin{aligned} M_{RR} &= R\frac{d^2w}{dR^2} + \nu \frac{dw}{dR} \\ Q_R &= R^2 \frac{d^3w}{dR^3} + R\frac{d^2w}{dR^2} - \frac{dw}{dR} \end{aligned} \tag{3.83}$$

For annular plate equation of motion should be satisfied for interior points ($i = 3, ..., N-1$). $Z_I \in R^{(N-3)\times(N+1)}$ is a matrix correspond to interior points.

$$Z_I = \begin{bmatrix} e_3^T \\ e_4^T \\ \vdots \\ e_{(N-1)}^T \end{bmatrix} \qquad Z_B = \begin{bmatrix} e_1^T \\ e_2^T \\ e_N^T \\ e_{(N+1)}^T \end{bmatrix} \tag{3.84}$$

where $e_i \in R^{(N+1)\times 1}$ is the ith unit vector. This vector is zero in all entries expect for the ith entry at which it is equal to 1. Matrix $Z_B \in R^{4\times(N+1)}$ corresponding to the boundary points.

$$L_B = \left[\ Z_I K Z_B^T\ \right] \qquad L_I = \left[\ Z_I K Z_I^T\ \right] \qquad L_T = \left[\ Z_I M Z_I^T\ \right] \tag{3.85}$$

For both annular and circular geometry, boundary condition on outer radius is as

$$\begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} \left\{\ w\ \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \tag{3.86}$$

where $\mathcal{B}_1$ is applied on the first point and $\mathcal{B}_2$ is applied on the second point of outer radius.

Finally for evaluation of natural frequencies we need

$$B_B = \left[\ B_w Z_B^T\ \right] \qquad B_I = \left[\ B_w Z_I^T\ \right] \tag{3.87}$$

where $B_w = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix}$.

Boundary conditions at $R = 1$

Clamped: $(w = \frac{\partial w}{\partial R} = 0)$

$$\begin{aligned} \mathcal{B}_1 &= e_{(N+1)}^T \\ \mathcal{B}_2 &= e_{(N+1)}^T D^{(1)} \end{aligned} \tag{3.88}$$

Simply supported: $(w = M_{RR} = 0)$

$$\begin{aligned} \mathcal{B}_1 &= e_{(N+1)}^T \\ \mathcal{B}_2 &= R_{(N+1)} I e_{(N+1)}^T D^{(2)} + \nu e_{(N+1)}^T D^{(1)} \end{aligned} \tag{3.89}$$

Free: $(M_{RR} = Q_R = 0)$

$$\begin{aligned} \mathcal{B}_1 &= R_{(N+1)} I e_{(N+1)}^T D^{(2)} + \nu e_{(N+1)}^T D^{(1)} \\ \mathcal{B}_2 &= R_{(N+1)}^2 I e_{(N+1)}^T D^{(3)} + R_{(N+1)} I e_{(N+1)}^T D^{(2)} - e_{(N+1)}^T D^{(1)} \end{aligned} \tag{3.90}$$

Distribution of the collocation points for annular plates are as: $\beta + \frac{(1-\beta)}{2}(1 - cos(\frac{i\pi}{N})$ $(i = 0, 1, ...N)$ and in the case of distribution pattern for circular plate $\beta = 0$. For applying boundary conditions on $R = \beta$ in the case of annular plate the same procedure is done and it is needed to replace $e_{(N+1)}$ with e_1 and also $R_{(N+1)}$ with R_1.

For circular plate which is $(0 \leq R \leq 1)$ equation of motion should be satisfied again for interior points $(i = 2, ..., N-1)$.

$$Z_I = \begin{bmatrix} e_2^T \\ e_3^T \\ \vdots \\ e_{(N-1)}^T \end{bmatrix} \qquad Z_B = \begin{bmatrix} e_1^T \\ e_N^T \\ e_{(N+1)}^T \end{bmatrix} \tag{3.91}$$

$Z_I \in R^{(N-2)\times(N+1)}$ is a matrix correspond to interior points and similar matrix $Z_B \in R^{3\times(N+1)}$ corresponding to the boundary points. $e_i \in R^{(N+1)\times 1}$ is the ith unit vector. Boundary condition on center point for circular plate is $\frac{\partial w}{\partial R}$ or $\mathcal{B}_c = e_1^T D^{(1)}$.

Table 3.8: Non-dimensional natural frequencies of the Kirchhoff circular plate, $\lambda = \omega R^2/\sqrt{\frac{\rho h}{D}}$

BCs	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
C	Present	10.214	39.770	89.103	158.18	247.01	355.57	483.87	631.91
	[99]	10.215	39.771	89.104	158.18	247.01	355.56	483.87	631.91
	[98]	10.216	39.771	89.102	158.18	246.99	355.54	483.82	631.83
S	Present	4.9288	29.716	74.156	138.32	222.22	325.85	449.22	592.33
	[99]	4.977	29.76	74.20	138.34				
	[98]	4.9351	29.720	74.155	138.31	222.21	325.83	499.18	592.27
F	Present	9.0017	38.442	87.750	156.82	245.63	354.19	482.49	630.53
	[99]	9.084	38.55	87.80	157.0	245.9	354.6	483.1	631.0
	[98]	9.0031	38.443	87.749	156.81	245.62	354.17	482.45	630.46

Natural frequencies for thin circular plate with three boundaries such as fully clamped, simply supported and free are shown in table (3.8) and results are compared with pseudospectral [99] and differential quadrature [98] methods, respectively. For fully clamped case the presented results are very close to both methods but for other cases presented results are in a good agreement with results of GDQ method.

3.3 First-order shear deformation plate theory

In first-order shear deformation theory the transverse normals do not remain perpendicular to the mid surface after deformation and it leads to exist $\epsilon_{xz}, \epsilon_{yz}$. Displacement in the thickness direction w is not a function of the thickness coordinate z like thin plate theory. The transverse normal strain ϵ_{zz} under transverse loading can be neglected and still σ_{zz} is small and negligible compared with the other stress components like thin plate theory. Thickness of the plate is not small compared to other dimensions, it's started from h/a equal to 0.1 which is called moderately thick to greater which is called thick and a is one dimension of the plate.

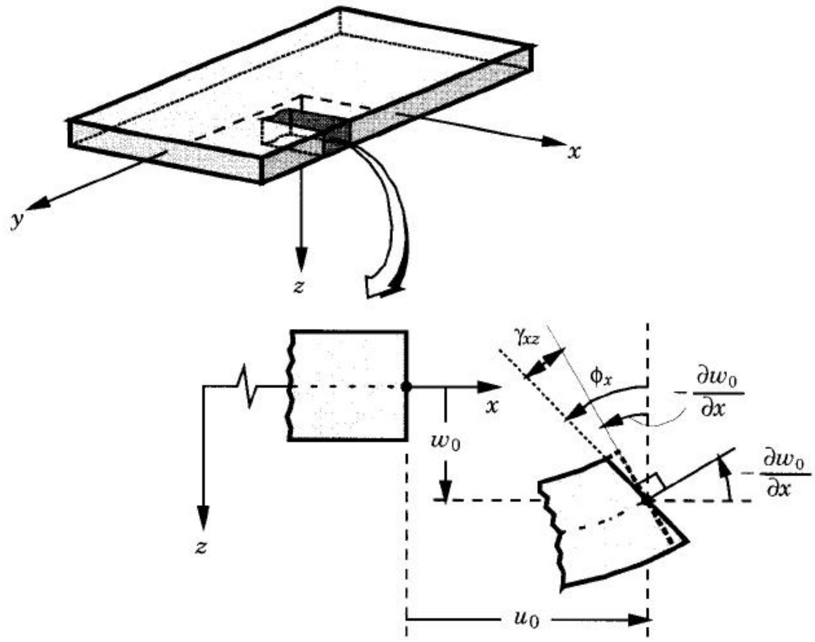

Figure 3.11: Undeformed and deformed in the thickness direction under first-order shear deformation assumption

3.3.1 Out-of-plane vibration of rectangular plates

In the first-order shear deformation theory, displacements (u, v, w) are assumed as (see Figure 3.11)

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t) \\ v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{3.92}$$

where $(u_0, v_0, w_0, \phi_x, \phi_y)$ are unknown functions should be determined. (u_0, v_0, w_0) are the displacements of a point on the plane $z = 0$. It is to be noted that

$$\phi_x = \left(\frac{\partial u}{\partial z}\right)_{z=0}, \qquad \phi_y = \left(\frac{\partial v}{\partial z}\right)_{z=0} \tag{3.93}$$

where ϕ_x, ϕ_y are the rotations of a transverse normal about the y and x axes, respectively. For thin plates when the in-plane characteristic dimension to thickness ratio is on the order 50 or greater, the rotation functions ϕ_x and ϕ_y should approach the respective slopes of the transverse deflection as

$$\phi_x = -\frac{\partial w_0}{\partial x}, \qquad \phi_y = -\frac{\partial w_0}{\partial y} \tag{3.94}$$

Stress resultants which are correspond to generalized displacements $(u_0, v_0, w_0, \phi_x, \phi_y)$ are as follow

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(1)} \\ \epsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

$$\left\{\begin{array}{c} M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right\} = \left[\begin{array}{ccc} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{array}\right]\left\{\begin{array}{c} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{array}\right\} + \left[\begin{array}{ccc} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{array}\right]\left\{\begin{array}{c} \epsilon_{xx}^{(1)} \\ \epsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{array}\right\}$$

$$\left\{\begin{array}{c} Q_x \\ Q_y \end{array}\right\} = \left[\begin{array}{cc} A_{55} & A_{45} \\ A_{45} & A_{44} \end{array}\right]\left\{\begin{array}{c} \gamma_{xz} \\ \gamma_{yz} \end{array}\right\} \tag{3.95}$$

where M_{xx}, M_{yy}, M_{xy} are bending resultants, N_{xx}, N_{yy}, N_{xy} are force resultants, and Q_x, Q_y are transverse force resultants. Stress resultants are as

$$\left\{\begin{array}{c} N_{\alpha\beta} \\ M_{\alpha\beta} \end{array}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta}\left\{\begin{array}{c} 1 \\ z \end{array}\right\} dz \qquad \left\{ Q_\alpha \right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha z}\left\{ 1 \right\} dz \tag{3.96}$$

Stiffness coefficients are as

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \tilde{C}_{ij}^{(k)}(1, z, z^2) dz \qquad (i,j) = 1,2,6$$

$$A_{ij} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \kappa^2 \tilde{C}_{ij}^{(k)}(1, z, z^2) dz \qquad (i,j) = 4,5 \tag{3.97}$$

Substituting eq. (3.92) into the eq. (3.11), strain-displacement relations yield as

$$\left\{\begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{array}\right\} = \left\{\begin{array}{c} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{array}\right\} + z\left\{\begin{array}{c} \epsilon_{xx}^{(1)} \\ \epsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{yz}^{(1)} \end{array}\right\} = \left\{\begin{array}{c} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \\ \frac{\partial w_0}{\partial x} + \phi_x \\ \frac{\partial w_0}{\partial y} + \phi_y \end{array}\right\} + z\left\{\begin{array}{c} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \\ 0 \\ 0 \end{array}\right\} \tag{3.98}$$

where the strains $(\epsilon_{xx}, \epsilon_{yy}, \gamma_{xy})$ are linear through the thickness, while the the transverse shear strains $(\gamma_{xz}, \gamma_{yz})$ are constant through the thickness of the laminate in the first-order laminated plate theory.

Substituting eqs. (3.92) and (3.98) into the eq. (3.14) and integration by part, Euler-lagrange equations of the theory are obtained by setting the coefficients of $\delta u_0, \delta v_0$, δw_0, $\delta\phi_x$ and $\delta\phi_y$ equal to zero separately as follow

$$\delta u_0: \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\frac{\partial^2 u_0}{\partial t^2} + I_1\frac{\partial^2 \phi_x}{\partial t^2}$$

$$\delta v_0: \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0\frac{\partial^2 v_0}{\partial t^2} + I_1\frac{\partial^2 \phi_y}{\partial t^2}$$

$$\delta w_0: \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = I_0\frac{\partial^2 w_0}{\partial t^2}$$

$$\delta \phi_x: \quad \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2\frac{\partial^2 \phi_x}{\partial t^2} + I_1\frac{\partial^2 u_0}{\partial t^2}$$

$$\delta \phi_y: \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2\frac{\partial^2 \phi_y}{\partial t^2} + I_1\frac{\partial^2 v_0}{\partial t^2} \tag{3.99}$$

Since the transverse shear strains are presented as constant through the laminate thickness, it follows that the transverse shear stresses will also be constant. It is well known from elementary theory of homogeneous beams that the transverse shear stress varies parabolically through the beam thickness. For composite laminated beams and plats, the transverse shear stresses vary at least quadratically through layer thickness. The discrepancy between the actual stress state and the constant stress state predicted by the first-order theory if often corrected in computing the transverse shear force resultants (Q_x, Q_y) by multiplying with a parameter κ^2 which is called shear correction factor coefficients.

This amounts to modifying the plate transverse shear stiffness. The factor κ^2 is computed such that the strain energy due to transverse shear stresses in above equations equals the strain energy due to the true transverse stresses predicted by the three-dimensional elasticity theory. Shear correction factor computed in this way: actual shear stress distribution through the thickness of beam with rectangular cross section, width b and height h is

$$\sigma_{xz}^c = \frac{3Q}{2bh}\Big(1 - (\frac{2z}{h})^2\Big) \qquad -h/2 \leq z \leq h/2 \tag{3.100}$$

where Q is transverse shear force. The transverse shear stress in the FSDT thoery is constant and $\sigma_{xz}^f = Q/bh$. The strain energies due to transverse shear stresses in the two theories are

$$\begin{aligned} U_s^c &= \frac{1}{2G_{13}}\int_A (\sigma_{xz}^c)^2 dA = \frac{3Q^2}{5G_{13}bh} \\ U_s^f &= \frac{1}{2G_{13}}\int_A (\sigma_{xz}^f)^2 dA = \frac{Q^2}{2G_{13}bh} \end{aligned} \tag{3.101}$$

Shear correction factor is the ratio of U_s^f to U_s^c which gives $\kappa^2 = 5/6$. The shear correction factor for a general laminate depends on lamina properties and lamination scheme.

Substituting eq. (3.98) and second and third parts of eq. (3.95) into the third, fourth and fifth parts of eq. (3.99), yield out-of-plane equations of FSDT as

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} \phi_x \\ \phi_y \\ w \end{Bmatrix} = -\lambda^2 \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{Bmatrix} \phi_x \\ \phi_y \\ w \end{Bmatrix} \tag{3.102}$$

Vector of displacements at grid points $(\phi_x, \phi_y, w \in R^{(N+1)^2 \times 1})$

$$\begin{aligned} \phi_x &= \{\phi_{x(0,0)} \ \phi_{x(0,1)} \ \cdots \ \phi_{x(0,N)} \ \phi_{x(1,0)} \ \phi_{x(1,1)} \ \cdots \ \phi_{x(1,N)} \ \cdots \ \cdots \ \phi_{x(N,0)} \ \phi_{x(N,1)} \ \cdots \ \phi_{x(N,N)}\} \\ \phi_y &= \{\phi_{y(0,0)} \ \phi_{y(0,1)} \ \cdots \ \phi_{y(0,N)} \ \phi_{y(1,0)} \ \phi_{y(1,1)} \ \cdots \ \phi_{y(1,N)} \ \cdots \ \cdots \ \phi_{y(N,0)} \ \phi_{y(N,1)} \ \cdots \ \phi_{y(N,N)}\} \\ w &= \{w_{(0,0)} \ w_{(0,1)} \ \cdots \ w_{(0,N)} \ w_{(1,0)} \ w_{(1,1)} \ \cdots \ w_{(1,N)} \ \cdots \ \cdots \ w_{(N,0)} \ w_{(N,1)} \ \cdots \ w_{(N,N)}\} \end{aligned} \tag{3.103}$$

Equation of motion should be satisfied for interior points $(i, j = 1, ..., N-1)$.

$$Z_I = \begin{bmatrix} e^T_{(N+1)+2} \\ \vdots \\ e^T_{2(N+1)-1} \\ e^T_{2(N+1)+2} \\ \vdots \\ \vdots \\ e^T_{(N-1)(N+1)-1} \\ e^T_{(N-1)(N+1)+2} \\ \vdots \\ e^T_{N(N+1)-1} \end{bmatrix} \tag{3.104}$$

$Z_I \in R^{(N-1)^2 \times (N+1)^2}$ is a matrix correspond to interior points and $e_i \in R^{(N+1)^2 \times 1}$ is the ith unit vector. This vector is zero in all entries expect for the ith entry at which it is equal to 1, and similar matrix $Z_B \in R^{4N \times (N+1)^2}$ corresponding to the boundary points.

$$Z_B = \begin{bmatrix} e^T_1 \\ e^T_{(N+1)+1} \\ e^T_{2(N+1)+1} \\ \vdots \\ e^T_{(N)(N+1)+1} \\ e^T_{(N)(N+1)+2} \\ \vdots \\ e^T_{(N)(N+1)+N} \\ e^T_{(N+1)} \\ e^T_{2(N+1)} \\ \vdots \\ e^T_{(N+1)(N+1)} \\ e^T_2 \\ \vdots \\ e^T_N \end{bmatrix} \tag{3.105}$$

L_B and L_I matrices are as following

$$L_B = \begin{bmatrix} Z_I K_{11} Z_B^T & Z_I K_{12} Z_B^T & Z_I K_{13} Z_B^T \\ Z_I K_{21} Z_B^T & Z_I K_{22} Z_B^T & Z_I K_{23} Z_B^T \\ Z_I K_{31} Z_B^T & Z_I K_{32} Z_B^T & Z_I K_{33} Z_B^T \end{bmatrix}$$

$$L_I = \begin{bmatrix} Z_I K_{11} Z_I^T & Z_I K_{12} Z_I^T & Z_I K_{13} Z_I^T \\ Z_I K_{21} Z_I^T & Z_I K_{22} Z_I^T & Z_I K_{23} Z_I^T \\ Z_I K_{31} Z_I^T & Z_I K_{32} Z_I^T & Z_I K_{33} Z_I^T \end{bmatrix} \tag{3.106}$$

L_R is obtained by replacing mass components in L_I matrix from above equation.

$$s_I = \begin{Bmatrix} \phi_{xI} \\ \phi_{yI} \\ w_I \end{Bmatrix} \qquad s_B = \begin{Bmatrix} \phi_{xB} \\ \phi_{yB} \\ w_B \end{Bmatrix} \tag{3.107}$$

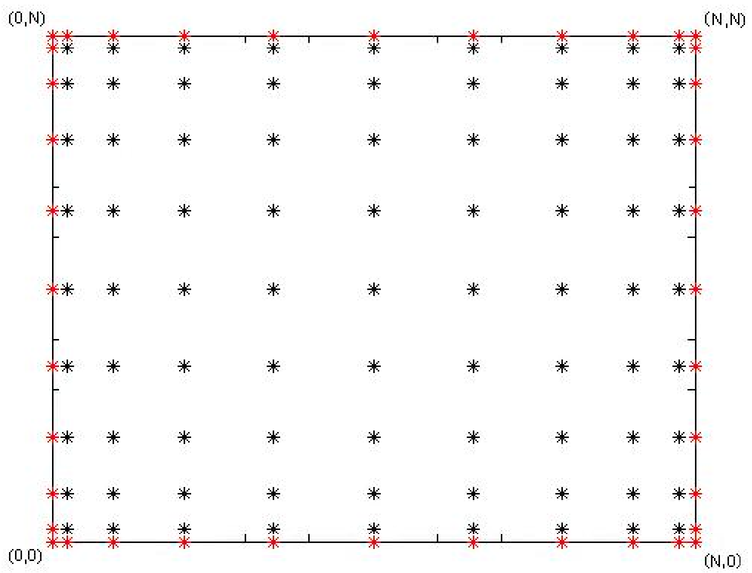

Figure 3.12: Interior and boundary points of the first-order shear deformation theory

where s_I and s_B are displacements vector variables of interior and boundary points, respectively.

The same formulation is applied for satisfaction of boundary conditions. Displacements and their derivatives at the CGL points along $\zeta = 0$ are expressed as

$$\begin{array}{lll} \phi_x = (e_1^T \otimes I)\phi_x & \phi_y = (e_1^T \otimes I)\phi_y & w = (e_1^T \otimes I)w \\ \phi_{x\zeta} = (e_1^T D^{(1)} \otimes I)\phi_x & \phi_{y\zeta} = (e_1^T D^{(1)} \otimes I)\phi_y & w_\zeta = (e_1^T D^{(1)} \otimes I)w \\ \phi_{x\eta} = (e_1^T \otimes D^{(1)})\phi_x & \phi_{y\eta} = (e_1^T \otimes D^{(1)})\phi_y & w_\eta = (e_1^T \otimes D^{(1)})w \end{array} \tag{3.108}$$

Similarly, for edge $\eta = 0$ it is written as

$$\begin{array}{lll} \phi_x = (I \otimes e_1^T)\phi_x & \phi_y = (I \otimes e_1^T)\phi_y & w = (I \otimes e_1^T)w \\ \phi_{x\zeta} = (D^{(1)} \otimes e_1^T)\phi_x & \phi_{y\zeta} = (D^{(1)} \otimes e_1^T)\phi_y & w_\zeta = (D^{(1)} \otimes e_1^T)w \\ \phi_{x\eta} = (I \otimes e_1^T D^{(1)})\phi_x & \phi_{y\eta} = (I \otimes e_1^T D^{(1)})\phi_y & w_\eta = (I \otimes e_1^T D^{(1)})w \end{array} \tag{3.109}$$

For displacements and their derivatives of the edges $(\zeta, \eta = 1)$ it is enough to write similar formula, respectively and replace e_1 with e_{N+1}. Equation (3.110) shows the discretized set of boundary conditions for each edge.

$$\begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{B}_{13} \\ \mathcal{B}_{21} & \mathcal{B}_{22} & \mathcal{B}_{23} \\ \mathcal{B}_{31} & \mathcal{B}_{32} & \mathcal{B}_{33} \end{bmatrix} \left\{ \begin{array}{c} \phi_x \\ \phi_y \\ w \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\} \qquad (each \;\; edge) \tag{3.110}$$

The whole set of discretized boundary conditions are expressed in below

$$\begin{bmatrix} B_{\phi_x} & B_{\phi_y} & B_w \end{bmatrix} \left\{ \begin{array}{c} \phi_x \\ \phi_y \\ w \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\} \tag{3.111}$$

where B_{ϕ_x} collects the $\mathcal{B}_{11}$, $\mathcal{B}_{21}$ and $\mathcal{B}_{31}$ matrices of the four edges which is corresponded to the displacement vector ϕ_x, and B_{ϕ_y} groups the $\mathcal{B}_{12}$, $\mathcal{B}_{22}$ and $\mathcal{B}_{32}$ matrices of the four edges which is corresponded to the displacement vector ϕ_y and finally B_w puts together the $\mathcal{B}_{13}$, $\mathcal{B}_{23}$ and $\mathcal{B}_{33}$ matrices of the four edges which is corresponded to the displacement vector w.

B_B and B_I matrices are as following

$$B_B = \begin{bmatrix} B_{\phi_x} Z_B^T & B_{\phi_y} Z_B^T & B_w Z_B^T \end{bmatrix} \qquad B_I = \begin{bmatrix} B_{\phi_x} Z_I^T & B_{\phi_y} Z_I^T & B_w Z_I^T \end{bmatrix} \tag{3.112}$$

Using below equation for evaluating natural frequencies

$$\left(L_R^{-1}(L_I - L_B B_B^{-1} B_I)\right) s_I = \lambda^2 s_I \tag{3.113}$$

Equation of motion for first-order shear deformation composite laminated plate is

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} \\ \mathbf{K}_{12} & \mathbf{K}_{22} & \mathbf{K}_{23} \\ \mathbf{K}_{31} & \mathbf{K}_{32} & \mathbf{K}_{33} \end{bmatrix} \begin{Bmatrix} \phi_x \\ \phi_y \\ w_0 \end{Bmatrix} = -\lambda^2 \begin{bmatrix} \mathbf{M}_{11} & 0 & 0 \\ 0 & \mathbf{M}_{11} & 0 \\ 0 & 0 & \mathbf{M}_{33} \end{bmatrix} \begin{Bmatrix} \phi_x \\ \phi_y \\ w_0 \end{Bmatrix} \tag{3.114}$$

where stiffness matrix components are

$$\begin{aligned}
\mathbf{K}_{11} &= D_{11}\frac{\partial^2}{\partial x^2} + D_{66}\beta^2\frac{\partial^2}{\partial y^2} + 2D_{16}\beta\frac{\partial^2}{\partial x\partial y} - A_{55} \\
\mathbf{K}_{12} &= (D_{12} + D_{66})\beta\frac{\partial^2}{\partial x\partial y} + D_{16}\frac{\partial^2}{\partial x^2} + D_{26}\beta^2\frac{\partial^2}{\partial y^2} - A_{45} \\
\mathbf{K}_{13} &= -A_{45}\gamma\frac{\partial}{\partial y} - A_{55}\delta\frac{\partial}{\partial x} \\
\mathbf{K}_{22} &= D_{66}\frac{\partial^2}{\partial x^2} + D_{22}\beta^2\frac{\partial^2}{\partial y^2} + 2D_{26}\beta\frac{\partial^2}{\partial x\partial y} - A_{44} \\
\mathbf{K}_{23} &= -A_{44}\gamma\frac{\partial}{\partial y} - A_{45}\delta\frac{\partial}{\partial x} \\
\mathbf{K}_{31} &= A_{45}\frac{\partial}{\partial x} + A_{55}\frac{\partial}{\partial x} \\
\mathbf{K}_{32} &= A_{44}\beta\frac{\partial}{\partial y} + A_{45}\beta\frac{\partial}{\partial x} \\
\mathbf{K}_{33} &= A_{44}\beta\gamma\frac{\partial^2}{\partial y^2} + 2A_{45}\gamma\frac{\partial^2}{\partial x\partial y} + A_{55}\delta\frac{\partial^2}{\partial x^2}
\end{aligned} \tag{3.115}$$

and mass matrix components are as

$$\mathbf{M}_{11} = \frac{\rho h^3}{12} \qquad \mathbf{M}_{33} = \rho h\delta \tag{3.116}$$

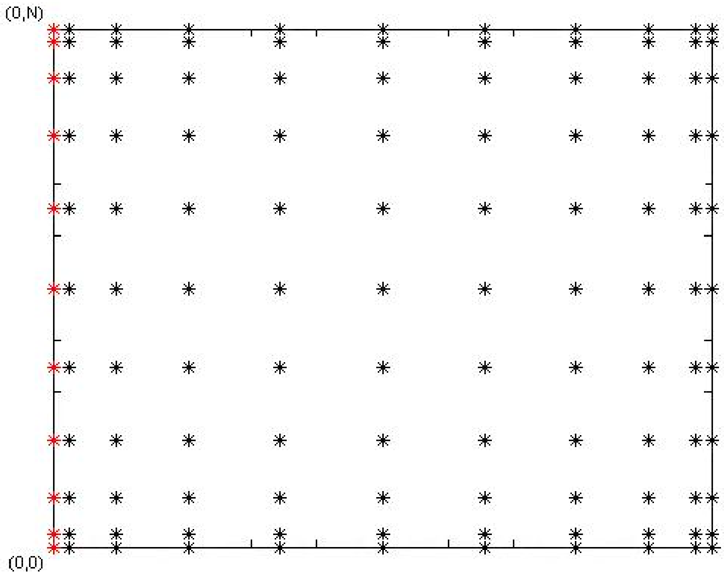

Figure 3.13: Boundary condition at $\zeta = 0$

Boundary conditions at ($\zeta = 0$) $(i = 0, j = 0, ..., N)$

Clamped $(\phi_x = \phi_y = W = 0)$

$$\begin{aligned}
\mathcal{B}_{11} &= (e_1^T \otimes I) \\
\mathcal{B}_{12} &= \mathbf{0} \\
\mathcal{B}_{13} &= \mathbf{0} \\
\mathcal{B}_{21} &= \mathbf{0} \\
\mathcal{B}_{22} &= (e_1^T \otimes I) \\
\mathcal{B}_{23} &= \mathbf{0} \\
\mathcal{B}_{31} &= \mathbf{0} \\
\mathcal{B}_{32} &= \mathbf{0} \\
\mathcal{B}_{33} &= (e_1^T \otimes I)
\end{aligned} \tag{3.117}$$

Simply-supported - type 2 $(M_{xx} = \phi_y = w = 0)$

$$\begin{aligned}
\mathcal{B}_{11} &= D_{11}\frac{\partial}{\partial X} + D_{16}\beta\frac{\partial}{\partial Y} = D_{11}(e_1^T D^{(1)} \otimes I) + D_{16}\beta(e_1^T \otimes D^{(1)}) \\
\mathcal{B}_{12} &= D_{12}\beta\frac{\partial}{\partial Y} + D_{16}\frac{\partial}{\partial X} = D_{12}\beta(e_1^T \otimes D^{(1)}) + D_{16}(e_1^T D^{(1)} \otimes I) \\
\mathcal{B}_{13} &= \mathbf{0} \\
\mathcal{B}_{21} &= \mathbf{0} \\
\mathcal{B}_{22} &= (e_1^T \otimes I)
\end{aligned}$$

$$\mathcal{B}_{23} = \mathbf{0}$$
$$\mathcal{B}_{31} = \mathbf{0}$$
$$\mathcal{B}_{32} = \mathbf{0}$$
$$\mathcal{B}_{33} = (e_1^T \otimes I) \quad (3.118)$$

Simply supported - type 1 ($M_{xx} = M_{xy} = w$)

$$\mathcal{B}_{11} = D_{11}\frac{\partial}{\partial X} + D_{16}\beta\frac{\partial}{\partial Y} = D_{11}(e_1^T D^{(1)} \otimes I) + D_{16}\beta(e_1^T \otimes D^{(1)})$$
$$\mathcal{B}_{12} = D_{12}\beta\frac{\partial}{\partial Y} + D_{16}\frac{\partial}{\partial X} = D_{12}\beta(e_1^T \otimes D^{(1)}) + D_{16}(e_1^T D^{(1)} \otimes I)$$
$$\mathcal{B}_{13} = \mathbf{0}$$
$$\mathcal{B}_{21} = D_{16}\frac{\partial}{\partial X} + D_{66}\beta\frac{\partial}{\partial Y} = D_{16}(e_1^T D^{(1)} \otimes I) + D_{66}\beta(e_1^T \otimes D^{(1)})$$
$$\mathcal{B}_{22} = D_{26}\beta\frac{\partial}{\partial Y} + D_{66}\frac{\partial}{\partial X} = D_{26}\beta(e_1^T \otimes D^{(1)}) + D_{66}(e_1^T D^{(1)} \otimes I)$$
$$\mathcal{B}_{23} = \mathbf{0}$$
$$\mathcal{B}_{31} = \mathbf{0}$$
$$\mathcal{B}_{32} = \mathbf{0}$$
$$\mathcal{B}_{33} = (e_1^T \otimes I) \quad (3.119)$$

Free ($M_{xx} = M_{xy} = Q_x = 0$)

$$\mathcal{B}_{11} = D_{11}\frac{\partial}{\partial X} + D_{16}\beta\frac{\partial}{\partial Y} = D_{11}(e_1^T D^{(1)} \otimes I) + D_{16}\beta(e_1^T \otimes D^{(1)})$$
$$\mathcal{B}_{12} = D_{12}\beta\frac{\partial}{\partial Y} + D_{16}\frac{\partial}{\partial X} = D_{12}\beta(e_1^T \otimes D^{(1)}) + D_{16}(e_1^T D^{(1)} \otimes I)$$
$$\mathcal{B}_{13} = \mathbf{0}$$
$$\mathcal{B}_{21} = D_{16}\frac{\partial}{\partial X} + D_{66}\beta\frac{\partial}{\partial Y} = D_{16}(e_1^T D^{(1)} \otimes I) + D_{66}\beta(e_1^T \otimes D^{(1)})$$
$$\mathcal{B}_{22} = D_{26}\beta\frac{\partial}{\partial Y} + D_{66}\frac{\partial}{\partial X} = D_{26}\beta(e_1^T \otimes D^{(1)}) + D_{66}(e_1^T D^{(1)} \otimes I)$$
$$\mathcal{B}_{23} = \mathbf{0}$$
$$\mathcal{B}_{31} = A_{55} = A_{55}(e_1^T \otimes I)$$
$$\mathcal{B}_{32} = A_{45} = A_{45}(e_1^T \otimes I)$$
$$\mathcal{B}_{33} = \gamma A_{45}\frac{\partial}{\partial Y} + \delta A_{55}\frac{\partial}{\partial X} = \gamma A_{45}(e_1^T \otimes D^{(1)}) + \delta A_{55}e_1^T D^{(1)} \otimes I) \quad (3.120)$$

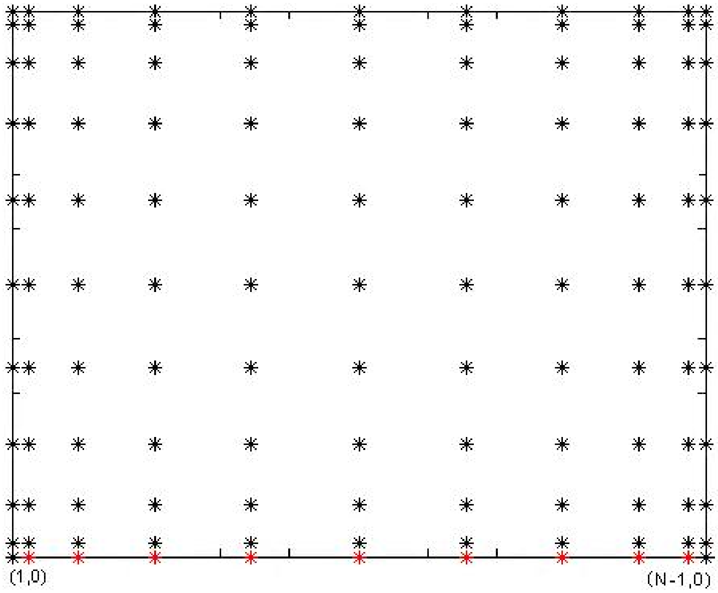

Figure 3.14: Boundary condition at $\eta = 0$

Boundary conditions at ($\eta = 0$) $(j = 0, i = 1, ..., N-1)$

Clamped ($\phi_x = \phi_y = W = 0$)

$$\begin{aligned}
\mathcal{B}_{11} &= (I \otimes e_1^T) \\
\mathcal{B}_{12} &= \mathbf{0} \\
\mathcal{B}_{13} &= \mathbf{0} \\
\mathcal{B}_{21} &= \mathbf{0} \\
\mathcal{B}_{22} &= (I \otimes e_1^T) \\
\mathcal{B}_{23} &= \mathbf{0} \\
\mathcal{B}_{31} &= \mathbf{0} \\
\mathcal{B}_{32} &= \mathbf{0} \\
\mathcal{B}_{33} &= (I \otimes e_1^T)
\end{aligned} \tag{3.121}$$

Simply-supported - type 2 ($\phi_x = M_{yy} = w = 0$)

$$\begin{aligned}
\mathcal{B}_{11} &= (I \otimes e_1^T) \\
\mathcal{B}_{12} &= \mathbf{0} \\
\mathcal{B}_{13} &= \mathbf{0} \\
\mathcal{B}_{21} &= D_{12}\frac{\partial}{\partial X} + D_{26}\beta\frac{\partial}{\partial Y} = D_{12}(D^{(1)} \otimes e_1^T) + D_{26}\beta(I \otimes e_1^T D^{(1)}) \\
\mathcal{B}_{22} &= D_{22}\beta\frac{\partial}{\partial Y} + D_{26}\frac{\partial}{\partial X} = D_{22}(I \otimes e_1^T D^{(1)}) + D_{26}\beta(D^{(1)} \otimes e_1^T)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{B}_{23} &= \mathbf{0} \\
\mathcal{B}_{31} &= \mathbf{0} \\
\mathcal{B}_{32} &= \mathbf{0} \\
\mathcal{B}_{33} &= (I \otimes e_1^T)
\end{aligned}
\tag{3.122}
$$

Simply-supported - type 1 ($M_{xy} = M_{yy} = w = 0$)

$$
\begin{aligned}
\mathcal{B}_{11} &= D_{16}\frac{\partial}{\partial X} + D_{66}\beta\frac{\partial}{\partial Y} = D_{16}(D^{(1)} \otimes e_1^T) + D_{66}\beta(I \otimes e_1^T D^{(1)}) \\
\mathcal{B}_{12} &= D_{26}\beta\frac{\partial}{\partial Y} + D_{66}\frac{\partial}{\partial X} = D_{26}(I \otimes e_1^T D^{(1)}) + D_{66}\beta(D^{(1)} \otimes e_1^T) \\
\mathcal{B}_{13} &= \mathbf{0} \\
\mathcal{B}_{21} &= D_{12}\frac{\partial}{\partial X} + D_{26}\beta\frac{\partial}{\partial Y} = D_{12}(D^{(1)} \otimes e_1^T) + D_{26}\beta(I \otimes e_1^T D^{(1)}) \\
\mathcal{B}_{22} &= D_{22}\beta\frac{\partial}{\partial Y} + D_{26}\frac{\partial}{\partial X} = D_{22}(I \otimes e_1^T D^{(1)}) + D_{26}\beta(D^{(1)} \otimes e_1^T) \\
\mathcal{B}_{23} &= \mathbf{0} \\
\mathcal{B}_{31} &= \mathbf{0} \\
\mathcal{B}_{32} &= \mathbf{0} \\
\mathcal{B}_{33} &= (I \otimes e_1^T)
\end{aligned}
\tag{3.123}
$$

Free ($M_{xy} = M_{yy} = Q_y = 0$)

$$
\begin{aligned}
\mathcal{B}_{11} &= D_{16}\frac{\partial}{\partial X} + D_{66}\beta\frac{\partial}{\partial Y} = D_{16}(D^{(1)} \otimes e_1^T) + D_{66}\beta(I \otimes e_1^T D^{(1)}) \\
\mathcal{B}_{12} &= D_{26}\beta\frac{\partial}{\partial Y} + D_{66}\frac{\partial}{\partial X} = D_{26}(I \otimes e_1^T D^{(1)}) + D_{66}\beta(D^{(1)} \otimes e_1^T) \\
\mathcal{B}_{13} &= \mathbf{0} \\
\mathcal{B}_{21} &= D_{12}\frac{\partial}{\partial X} + D_{26}\beta\frac{\partial}{\partial Y} = D_{12}(D^{(1)} \otimes e_1^T) + D_{26}\beta(I \otimes e_1^T D^{(1)}) \\
\mathcal{B}_{22} &= D_{22}\beta\frac{\partial}{\partial Y} + D_{26}\frac{\partial}{\partial X} = D_{22}(I \otimes e_1^T D^{(1)}) + D_{26}\beta(D^{(1)} \otimes e_1^T) \\
\mathcal{B}_{23} &= \mathbf{0} \\
\mathcal{B}_{31} &= A_{45} = A_{45}(I \otimes e_1^T) \\
\mathcal{B}_{32} &= A_{44} = A_{44}(I \otimes e_1^T) \\
\mathcal{B}_{33} &= \gamma A_{44}\frac{\partial}{\partial Y} + \delta A_{45}\frac{\partial}{\partial X} = \gamma A_{44}(I \otimes e_1^T D^{(1)}) + \delta A_{45}(D^{(1)} \otimes e_1^T)
\end{aligned}
\tag{3.124}
$$

$\mathbf{0}$ is a zeros matrix with order of $(N+1) \times (N+1)^2$. For describing boundary conditions at $\zeta = 1$ and $\eta = 1$, inside of formula it is enough to replace e_1 with $e_{(N+1)}$ and for points which are imposed by boundary conditions

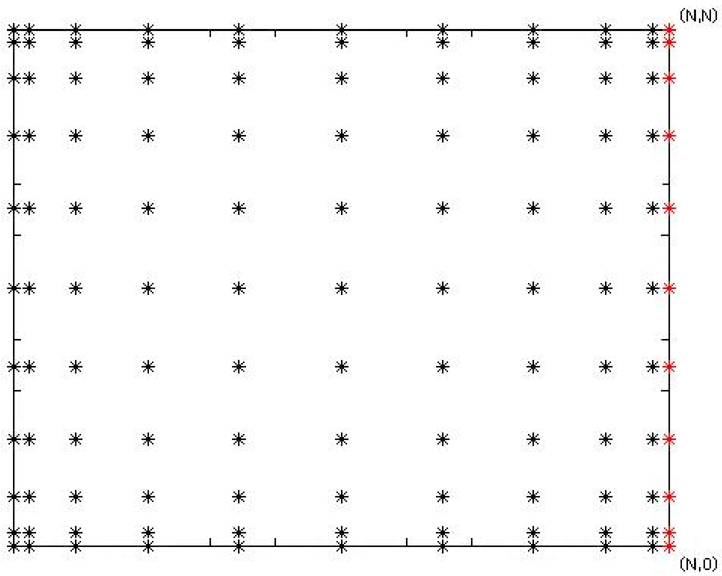

Figure 3.15: Boundary condition at $\zeta = 1$

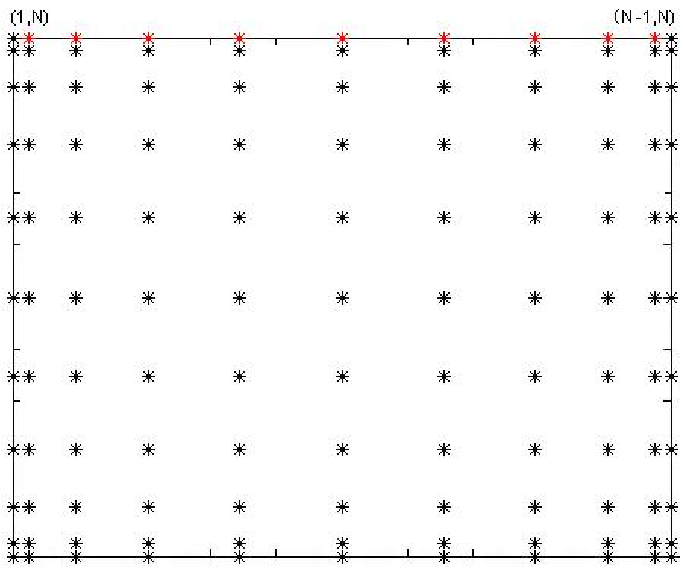

Figure 3.16: Boundary condition at $\eta = 1$

$$\zeta = 0 : (i = 0, j = 0, ..., N) \rightarrow \zeta = 1 : (i = N, j = 0, ..., N)$$
$$\eta = 0 : (j = 0, i = 1, ..., N-1) \rightarrow \eta = 1 : (j = N, i = 1, ..., N-1) \quad (3.125)$$

For corner points at the cross of two free edges, there is no special condition at the angles like classical plate theory and fully free condition is like fully simply-supported condition.

3.3.2 Non-symmetric composite laminated rectangular plate

In the case of Non-symmetric composite laminated plates based on first-order shear deformation theory, in-plane displacements (u, v) and out-of-plane displacement (w, ϕ_x, ϕ_y) are coupling together. Equation of motion in matrix form of differential operator after normalization using these parameters $\zeta = x/a, \eta = y/b, \beta = a/b, \delta = h/a, \gamma = h/b, W = w/h$ can be written as

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & 0 & \mathbf{K}_{14} & \mathbf{K}_{15} \\ \mathbf{K}_{12} & \mathbf{K}_{22} & 0 & \mathbf{K}_{24} & \mathbf{K}_{25} \\ 0 & 0 & \mathbf{K}_{33} & \mathbf{K}_{34} & \mathbf{K}_{35} \\ \mathbf{K}_{14} & \mathbf{K}_{24} & -\mathbf{K}_{34} & \mathbf{K}_{44} & \mathbf{K}_{45} \\ \mathbf{K}_{15} & \mathbf{K}_{25} & -\mathbf{K}_{35} & \mathbf{K}_{45} & \mathbf{K}_{55} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \phi_x \\ \phi_y \end{Bmatrix} = -\lambda^2 \begin{bmatrix} \mathbf{M}_{11} & 0 & 0 & \mathbf{M}_{14} & 0 \\ 0 & \mathbf{M}_{11} & 0 & 0 & \mathbf{M}_{25} \\ 0 & 0 & \mathbf{M}_{11} & 0 & 0 \\ \mathbf{M}_{14} & 0 & 0 & \mathbf{M}_{44} & 0 \\ 0 & \mathbf{M}_{25} & 0 & 0 & \mathbf{M}_{44} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \phi_x \\ \phi_y \end{Bmatrix} \quad (3.126)$$

where stiffness components are as

$$\begin{aligned}
\mathbf{K}_{11} &= A_{11}\frac{\partial^2}{\partial x^2} + A_{66}\frac{\partial^2}{\partial y^2} & \mathbf{K}_{12} &= (A_{12} + A_{66})\frac{\partial^2}{\partial x \partial y} \\
\mathbf{K}_{14} &= B_{11}\frac{\partial^2}{\partial x^2} + 2B_{16}\frac{\partial^2}{\partial x \partial y} & \mathbf{K}_{15} &= B_{16}\frac{\partial^2}{\partial x^2} + B_{26}\frac{\partial^2}{\partial y^2} \\
\mathbf{K}_{22} &= A_{66}\frac{\partial^2}{\partial x^2} + A_{22}\frac{\partial^2}{\partial y^2} & \mathbf{K}_{24} &= B_{16}\frac{\partial^2}{\partial x^2} + B_{26}\frac{\partial^2}{\partial y^2} \\
\mathbf{K}_{25} &= -B_{11}\frac{\partial^2}{\partial y^2} + 2B_{26}\frac{\partial^2}{\partial x \partial y} & \mathbf{K}_{33} &= A_{55}\frac{\partial^2}{\partial x^2} + A_{44}\frac{\partial^2}{\partial y^2} \\
\mathbf{K}_{34} &= A_{55}\frac{\partial}{\partial x} & \mathbf{K}_{35} &= A_{44}\frac{\partial}{\partial y} \\
\mathbf{K}_{44} &= D_{11}\frac{\partial^2}{\partial x^2} + D_{66}\frac{\partial^2}{\partial y^2} - A_{55} & \mathbf{K}_{45} &= (D_{12} + D_{66})\frac{\partial^2}{\partial x \partial y} \\
\mathbf{K}_{55} &= D_{66}\frac{\partial^2}{\partial x^2} + D_{22}\frac{\partial^2}{\partial y^2} - A_{44}
\end{aligned} \quad (3.127)$$

and mass components are as

$$\mathbf{M}_{11} = I_0 \quad \mathbf{M}_{44} = I_2 \quad \mathbf{M}_{14} = I_1 \quad \mathbf{M}_{25} = I_1 \quad (3.128)$$

where B_{11} is for cross-ply non-symmetric and B_{16}, B_{26} are for angle-ply non-symmetric composite laminated plate.

$$(I_0, I_1, I_2) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \rho(1, z, z^2) dz \quad (3.129)$$

where A_{ij} is extensional stiffness, D_{ij} is bending stiffness, B_{ij} is bending-extensional coupling stiffness and I_0, I_1, I_2 are the inertia terms. Hook's law for the first-order shear deformation isotropic plate is as

$$\sigma = C\epsilon \rightarrow \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{23} \end{Bmatrix} \tag{3.130}$$

For composite laminated plate, matrix of coefficients can be obtained as following

$$\tilde{C} = TCT' = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{16} & 0 & 0 \\ \tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{26} & 0 & 0 \\ \tilde{C}_{16} & \tilde{C}_{26} & \tilde{C}_{66} & 0 & 0 \\ 0 & 0 & 0 & \tilde{C}_{55} & \tilde{C}_{45} \\ 0 & 0 & 0 & \tilde{C}_{45} & \tilde{C}_{44} \end{bmatrix} \tag{3.131}$$

where transformation matrix is as

$$T = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta & 0 & 0 \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta & 0 & 0 \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{3.132}$$

$\tilde{C}_{ij}^{(k)}$ are the reduced elastic coefficients of the kth lamina taking into account the angle of orthotropy $\theta^{(k)}$ in each layer.

$$\begin{aligned} &C_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad C_{12} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}, \quad C_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} \\ &C_{66} = G_{12}, \quad C_{55} = G_{13}, \quad C_{44} = G_{23} \end{aligned} \tag{3.133}$$

C_{ij} are the engineering parameters.

Using these parameters $\zeta = x/a\,(0 \le \zeta \le 1), \eta = y/b\,(0 \le \eta \le 1), W = w_0/h, \delta = w/a, \gamma = w/b, \beta = a/b$ for doing dimensionless.

At $\zeta = 0, 1$

$$\begin{aligned} &N_{xx} \mp K_u u_0 = 0 \quad N_{xy} \mp K_v v_0 = 0 \\ &Q_x \mp K_w w_0 = 0 \quad M_{xx} \mp K_{\phi_x}\phi_x = 0 \quad M_{xy} \mp K_{\phi_y}\phi_y = 0 \end{aligned} \tag{3.134}$$

At $\eta = 0, 1$

$$\begin{aligned} &N_{xy} \mp K_u u_0 = 0 \quad N_{yy} \mp K_v v_0 = 0 \\ &Q_y \mp K_w w_0 = 0 \quad M_{xy} \mp K_{\phi_x}\phi_x = 0 \quad M_{yy} \mp K_{\phi_y}\phi_y = 0 \end{aligned} \tag{3.135}$$

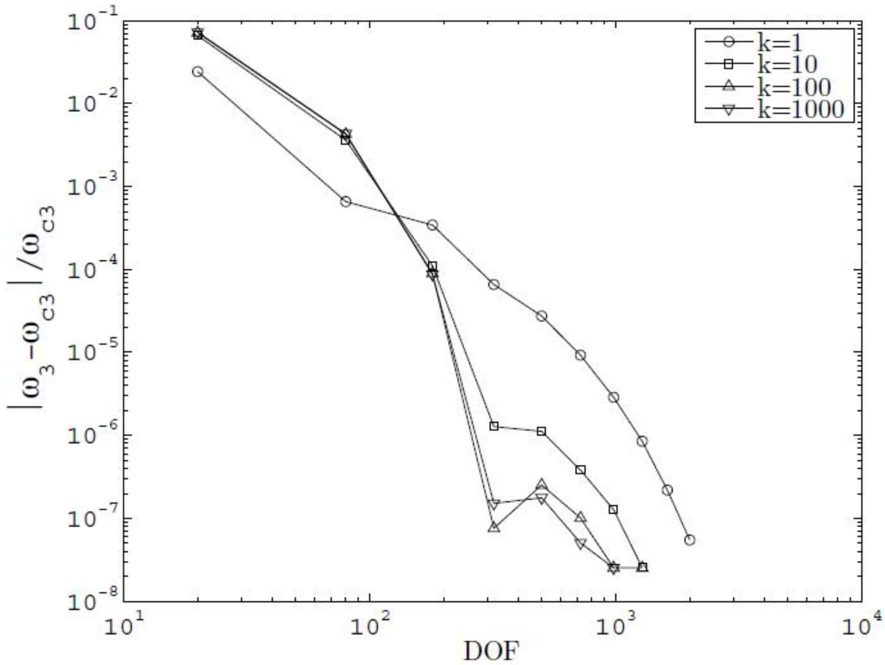

Figure 3.17: Error of natural frequencies for fully elastic restrained 4 layers cross-ply composite Mindlin plate

Constraint's coefficients for all boundary conditions and all edges are as

$$[K]_C = \begin{bmatrix} 1e12 & 1e12 & 1e12 & 1e12 \\ 1e12 & 1e12 & 1e12 & 1e12 \\ 1e12 & 1e12 & 1e12 & 1e12 \\ 1e12 & 1e12 & 1e12 & 1e12 \\ 1e12 & 1e12 & 1e12 & 1e12 \end{bmatrix} \qquad [K]_F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K]_{S1} = \begin{bmatrix} 0 & 1e12 & 0 & 1e12 \\ 1e12 & 0 & 1e12 & 0 \\ 1e12 & 1e12 & 1e12 & 1e12 \\ 0 & 1e12 & 0 & 1e12 \\ 1e12 & 0 & 1e12 & 0 \end{bmatrix} \qquad [K]_{S2} = \begin{bmatrix} 1e12 & 0 & 1e12 & 0 \\ 0 & 1e12 & 0 & 1e12 \\ 1e12 & 1e12 & 1e12 & 1e12 \\ 0 & 1e12 & 0 & 1e12 \\ 1e12 & 0 & 1e12 & 0 \end{bmatrix} \tag{3.136}$$

which rows are corresponded to displacements variables and columns are related to the edges $\zeta = 0, \eta = 0, \zeta = 1$ and $\eta = 1$, respectively.

Solving non-symmetric first-order shear deformation composite plates with all displacements $u_0, v_0, w_0, \phi_x, \phi_y$ coupled, is similar to out-of-plane vibration of first-order composite laminated plate with variables. All the procedures are the same and just equation of motion and boundary conditions matrices is 5×5 instead of 3×3.

Fig (3.17) shows convergence error of the third natural frequencies of non-symmetric cross-ply composite laminated first-order shear deformation theory with four layers with respect to the converged value. As seen, convergence is so fast and with increasing the

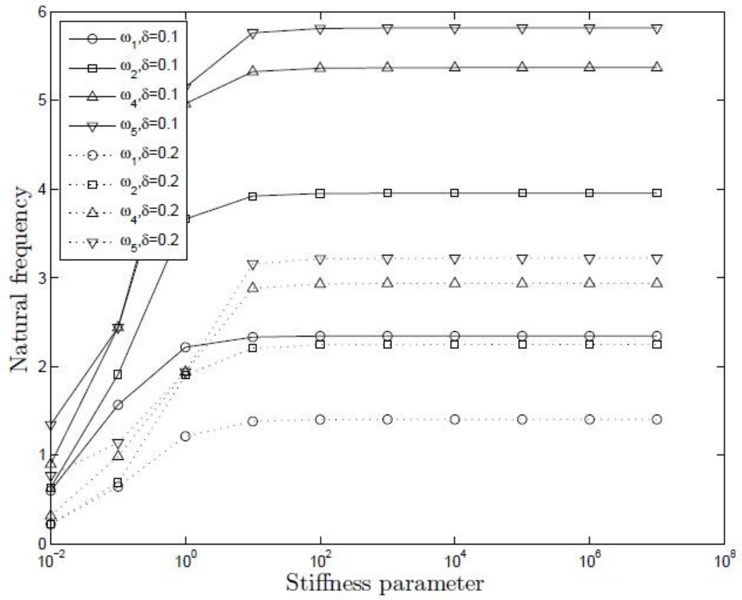

Figure 3.18: Effect of variation of the stiffness parameter on natural frequencies with different thickness on angle-ply non-symmetric laminated Mindlin plate

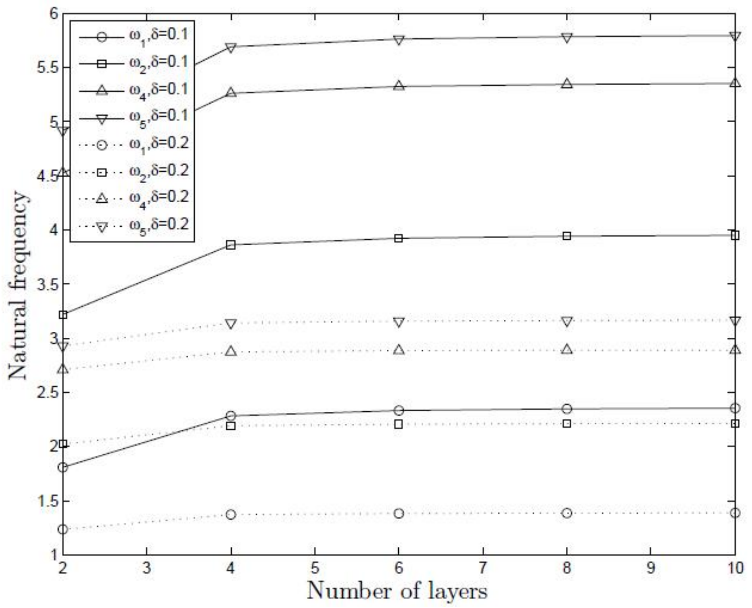

Figure 3.19: Effect of variation of the number of layers on natural frequencies with different thickness on angle-ply non-symmetric laminated Mindlin plate

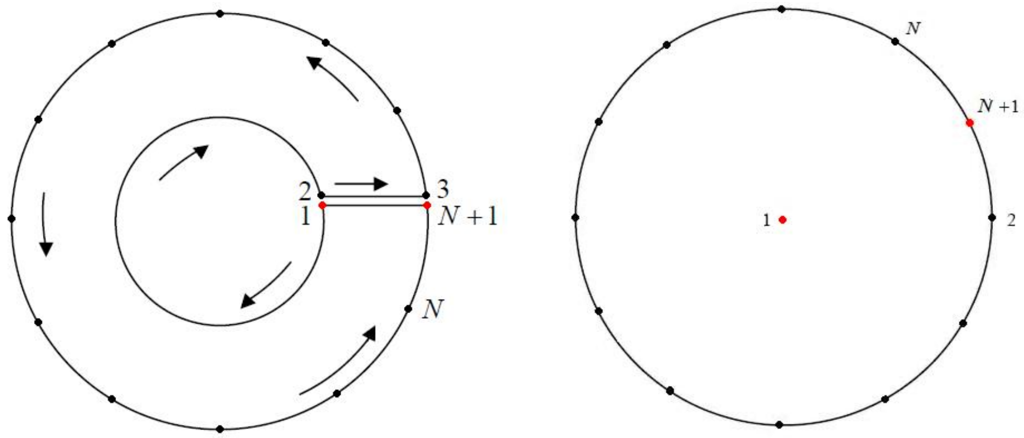

Figure 3.20: 2-D Distribution of interior and boundary points of circular and annular plate

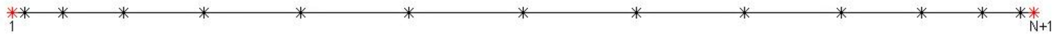

Figure 3.21: 1-D distribution of interior and boundary points of circular and annular plate

stiffness parameter K it becomes much more fast. If the value of all stiffness parameters are the same, increasing this parameter means the boundary condition is fully clamped and the convergence is fine. But lower quantities of the stiffness parameter means constrained boundary conditions and so the convergence is not as fast as clamped boundary condition. For the stiffness parameter values greater than $K = 100$, there is no any significant change in the graph of variation of natural frequencies with respect to this parameter as seen in Fig (3.18). In this figure, first, second, fourth and fifth natural frequencies with two thickness ratio of angle-ply anti-symmetric first-order shear deformation composite laminated are shown. Also with increasing the thickness ratio, natural frequencies become smaller but remain in the similar behavior. Effect of changing the number of layers on natural frequencies are shown in Fig (3.19). In this composite laminated plate which is similar to previous composite in lay-up, with number of layers bigger than six one can not see remarkable change in variation of natural frequencies with respect to number of layers.

3.3.3 Out-of-plane vibration of annular and circular plates

Consider an annular plate with r radius, inner radius b, outer radius a and $\beta = b/a$ is radius ratio. For annular plate ($b \leq r \leq a$) after doing dimensionless with the parameter $R = r/a$, range of non-dimensional radius is ($\beta \leq R \leq 1$). This case is similar to Timoshenko beam and w is displacement variable in the thickness direction and ψ_r is rotation. Collocation points which are distributed on the annular and circular plates are shown in Figs. (3.20), interior points are in black and boundary points are in red color, annular plate (left) and circular plate (right). The location of these points are not really based on Chebyshev location points and this figure is a schematic figure but Fig (3.21) shows the real location of collocation points in 1-D dimensional geometry like beam. For annular plate on inner radius one collocation points is needed and also for outer radius, and for circular plate case one point is corresponded to the center point and one point is needed for outer radius. But it should be considered that in this problem there are two variables and boundary has four points.

The Governing equations of motion for both annular and circular plate is as

$$\begin{aligned} &\frac{\partial M_r}{\partial r} + \frac{1}{r}M_r - Q_r = \frac{\rho h^3}{12}\frac{\partial^2 \psi_r}{\partial t^2} \\ &\frac{\partial Q_r}{\partial r} + \frac{1}{r}Q_r = \rho h \frac{\partial^2 w}{\partial t^2} \end{aligned} \tag{3.137}$$

where bending moment and shear force are

$$\begin{aligned} M_r &= D\Big(\frac{\partial \psi_r}{\partial r} + \frac{\nu}{r}\psi_r\Big) \\ Q_r &= \kappa^2 G h(\psi_r + \frac{\partial w}{\partial r}) \end{aligned} \tag{3.138}$$

After substitution and normalization with these parameters

$$R = \frac{r}{a}, W = \frac{w}{a}, \delta = \frac{h}{a}, \beta = \frac{b}{a} \tag{3.139}$$

Equation of motion in matrix form is as

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{Bmatrix} \psi_r \\ W \end{Bmatrix} = -\lambda^2 \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix} \begin{Bmatrix} \psi_r \\ W \end{Bmatrix} \tag{3.140}$$

where stiffness components are as

$$\begin{aligned} K_{11} &= (R^2 \frac{\partial^2}{\partial R^2} + R\frac{\partial}{\partial R} - 1) - \frac{6\kappa^2(1-\nu)R^2}{\delta^2} \\ K_{12} &= -\frac{6\kappa^2(1-\nu)R^2}{\delta^2}\frac{\partial}{\partial R} \\ K_{21} &= R\frac{\partial}{\partial R} + 1 \\ K_{22} &= R\frac{\partial^2}{\partial R^2} + \frac{\partial}{\partial R} \end{aligned} \tag{3.141}$$

and mass components are as

$$M_{11} = R^2 \qquad M_{22} = \frac{2R}{(1-\nu)\kappa^2} \tag{3.142}$$

and also non-dimensional moment and force resultant are

$$\begin{aligned} M_R &= D\Big(R\frac{\partial \psi_R}{\partial R} + \nu\psi_R\Big) \\ Q_R &= \kappa^2 G h(\psi_R + \frac{\partial w}{\partial R}) \end{aligned} \tag{3.143}$$

Avoiding singularity, it is better to multiply each formula by the biggest power of radius.

For annular and circular plate equation of motion should be satisfied for interior points, $Z_I \in R^{(N-1)\times(N+1)}$ is a matrix correspond to interior points.

$$Z_I = \begin{bmatrix} e_2^T \\ e_3^T \\ \vdots \\ e_N^T \end{bmatrix} \qquad Z_B = \begin{bmatrix} e_1^T \\ e_{(N+1)}^T \end{bmatrix} \tag{3.144}$$

where $e_i \in R^{(N+1)\times 1}$ is the ith unit vector. This vector is zero in all entries expect for the ith entry at which it is equal to 1, and matrix $Z_B \in R^{2\times(N+1)}$ corresponding to the boundary points.

$$L_B = \begin{bmatrix} Z_I K_{11} Z_B^T & Z_I K_{12} Z_B^T \\ Z_I K_{21} Z_B^T & Z_I K_{22} Z_B^T \end{bmatrix} \qquad L_I = \begin{bmatrix} Z_I K_{11} Z_I^T & Z_I K_{12} Z_I^T \\ Z_I K_{21} Z_I^T & Z_I K_{22} Z_I^T \end{bmatrix} \tag{3.145}$$

For both circular and annular geometry boundary conditions are implemented as

$$\begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{bmatrix} \begin{Bmatrix} \psi_r \\ W \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.146}$$

The whole set of discretized boundary conditions are expressed in below

$$\begin{bmatrix} B_{\psi_r} & B_w \end{bmatrix} \begin{Bmatrix} \psi_r \\ W \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.147}$$

where B_{ψ_r} collects the $\mathcal{B}_{11}$, $\mathcal{B}_{21}$ matrices which are corresponded to the displacement vector ψ_r and B_W groups the $\mathcal{B}_{12}$, $\mathcal{B}_{22}$ matrices which are corresponded to the displacement vector W of the inner and outer radius in annular plate, also center point and outer radius for circular plate, respectively.

$$B_B = \begin{bmatrix} B_{\psi_r} Z_B^T & B_W Z_B^T \end{bmatrix} \qquad B_I = \begin{bmatrix} B_{\psi_r} Z_I^T & B_W Z_I^T \end{bmatrix} \tag{3.148}$$

Boundary conditions for annular and circular plates on outer radius are similar.

Boundary conditions at $R = 1$

Clamped: $(\psi_r = 0, W = 0)$

$$\begin{aligned} \mathcal{B}_{11} &= e_{(N+1)}^T, \mathcal{B}_{12} = 0 \\ \mathcal{B}_{21} &= 0, \mathcal{B}_{22} = e_{(N+1)}^T \end{aligned} \tag{3.149}$$

Simply-supported: $(M_R = 0, W = 0)$

$$\mathcal{B}_{11} = e^T_{(N+1)}(RD^{(1)}) + \nu e^T_{(N+1)}, \mathcal{B}_{12} = 0$$
$$\mathcal{B}_{21} = 0, \mathcal{B}_{22} = e^T_{(N+1)} \tag{3.150}$$

Free: $(M_R = 0, Q_R = 0)$

$$\mathcal{B}_{11} = e^T_{(N+1)}(RD^{(1)}) + \nu e^T_{(N+1)}, \mathcal{B}_{12} = 0$$
$$\mathcal{B}_{21} = \kappa^2 Ghe^T_{(N+1)}, \mathcal{B}_{22} = \kappa^2 Ghe^T_{(N+1)}D^{(1)} \tag{3.151}$$

Collocation points distribution for annular plates are as: $\beta + \frac{(1-\beta)}{2}(1 - cos(\frac{i\pi}{N})$ $(i = 0, 1, ...N)$. For applying boundary conditions at $R = \beta$, it is enough to replace $e^T_{(N+1)}$ with e_1.

Boundary conditions for circular plate which is $(0 \leq R \leq 1)$ at center point is as

$$\psi_r = 0, \psi_r + \frac{\partial W}{\partial R} = 0 \quad at \quad R = 0 \tag{3.152}$$

or in spectral form is as

$$\mathcal{B}_{11} = e^T_1, \mathcal{B}_{12} = 0$$
$$\mathcal{B}_{21} = e^T_1, \mathcal{B}_{22} = e^T_1 D^{(1)} \tag{3.153}$$

and the collocation points distribution for circular plates are: $\frac{1}{2}(1 - cos(\frac{j\pi}{N}))$ $(j = 0, 1, ...N)$.

3.3.4 Out-of-plane vibration of sector annular plates

Sector annular plate geometry (see Figure 3.22) is like rectangular plate and all the procedure of solving is the same. Both of them has two rotations and one displacement. In this case ψ_r, ψ_θ, w are unknown variables. Governing equations for sector annular plate are following as

$$\frac{\partial M_r}{\partial r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} + \frac{1}{r}(M_r - M_\theta) - Q_r = \frac{\rho h^3}{12}\frac{\partial^2 \psi_r}{\partial t^2}$$
$$\frac{\partial M_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{2}{r}M_{r\theta} - Q_\theta = \frac{\rho h^3}{12}\frac{\partial^2 \psi_\theta}{\partial t^2}$$
$$\frac{\partial Q_r}{\partial r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} + \frac{1}{r}Q_r = \rho h \frac{\partial^2 w}{\partial t^2} \tag{3.154}$$

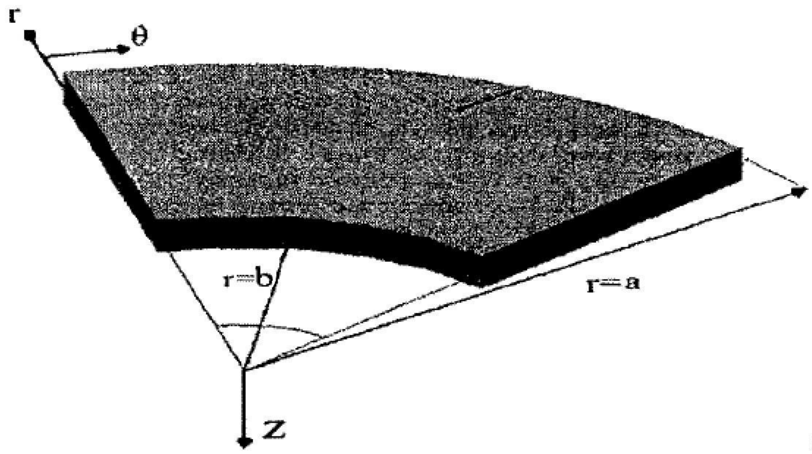

Figure 3.22: Sector annular plate geometry

where M_r, M_θ and $M_{R\theta}$ are bending moments, and Q_r and Q_θ are shear forces in radial and circumferential directions as

$$
\begin{aligned}
M_{rr} &= D\Big(\frac{\partial \psi_r}{\partial r} + \frac{\nu}{r}(\psi_r + \frac{\partial \psi_\theta}{\partial \theta})\Big)\\
M_{\theta\theta} &= D\Big(\frac{1}{r}(\psi_r + \frac{\partial \psi_\theta}{\partial \theta}) + \nu\frac{\partial \psi_r}{\partial r}\Big)\\
M_{r\theta} &= \frac{D}{2}(1-\nu)\Big(\frac{1}{r}(\frac{\partial \psi_r}{\partial \theta} - \psi_\theta) + \frac{\partial \psi_\theta}{\partial r}\Big)\\
Q_r &= \kappa^2 Gh(\psi_r + \frac{\partial w}{\partial r})\\
Q_\theta &= \kappa^2 Gh(\psi_\theta + \frac{1}{r}\frac{\partial w}{\partial \theta})
\end{aligned}
\tag{3.155}
$$

Circumferential direction θ can be non-dimensional using $\Theta = \theta/\alpha$ and so $0 \le \Theta \le 1$, α is angle of the sector annular plate. Radius direction also be non-dimensional using $R = r/a$ and so $\beta \le R \le 1$. Substituting eq. (3.155) into eq. (3.154), components of eq. (3.114) are as

$$
\begin{aligned}
K_{11} &= (R^2DR^{(2)} \otimes I) + (RDR^{(1)} \otimes I) - \Big((I \otimes I) + F(R^2 \otimes I)\Big) + \frac{(1-\nu)}{2\alpha^2}(I \otimes D^{(2)})\\
K_{12} &= \frac{(1+\nu)}{2\alpha}(RDR^{(1)} \otimes D^{(1)}) - \frac{3-\nu}{2\alpha}(I \otimes D^{(1)})\\
K_{13} &= -F(R^2DR^{(1)} \otimes I)\\
K_{21} &= \frac{1+\nu}{2\alpha}(RDR^{(1)} \otimes D^{(1)}) + \frac{3-\nu}{2\alpha}(I \otimes D^{(1)})\\
K_{22} &= \frac{1}{\alpha^2}(I \otimes D^{(2)}) + \frac{1-\nu}{2}(R^2DR^{(2)} \otimes I) + \frac{1-\nu}{2}(RDR^{(1)} \otimes I)\\
&\quad -\Big(\frac{1-\nu}{2}(I \otimes I) + F(R^2 \otimes I)\Big)\\
K_{23} &= -\frac{F}{\alpha}(R \otimes D^{(1)})
\end{aligned}
$$

$$K_{31} = (R^2 DR^{(1)} \otimes I) + (R \otimes I)$$
$$K_{32} = \frac{1}{\alpha}(R \otimes D^{(1)})$$
$$K_{33} = (R^2 DR^{(2)} \otimes I) + (RDR^{(1)} \otimes I) + \frac{1}{\alpha^2}(I \otimes D^{(2)}) \tag{3.156}$$

and

$$M_{11} = (R^2 \otimes I) \qquad M_{22} = (R^2 \otimes I) \qquad M_{33} = \frac{2}{\kappa(1-\nu)}(R^2 \otimes I) \tag{3.157}$$

where $F = 6\kappa^2(1-\nu)/\delta^2$, $\delta = h/a$ and κ^2 is shear correction factor. $DR^{(1)}, DR^{(2)}$ are first and second differentiation with respect to radial direction and $D^{(1)}, D^{(2)}$ are first and second differentiation with respect to circumferential direction, respectively.

Boundary conditions: at $\Theta = 0$

Clamped ($\psi_R = \psi_\Theta = W = 0$)

$$\mathcal{B}_{11} = (e_1^T \otimes I) \qquad \mathcal{B}_{22} = (e_1^T \otimes I) \qquad \mathcal{B}_{33} = (e_1^T \otimes I) \tag{3.158}$$

Simply-supported ($\psi_R = M_\Theta = W = 0$)

$$\mathcal{B}_{11} = (e_1^T \otimes I) \qquad \mathcal{B}_{21} = \nu(e_1^T DRR1 \otimes I) + (e_1^T \otimes I)$$
$$\mathcal{B}_{22} = (1/\alpha)(e_1^T \otimes D1) \qquad \mathcal{B}_{33} = (e_1^T \otimes I) \tag{3.159}$$

Free ($M_{R\Theta} = M_\Theta = Q_\Theta = 0$)

$$\mathcal{B}_{11} = (1/\alpha)(e_1^T \otimes D1) \qquad \mathcal{B}_{12} = -(e_1^T \otimes I) + (e_1^T DRR1 \otimes I)$$
$$\mathcal{B}_{21} = \nu(e_1^T DRR1 \otimes I) + (e_1^T \otimes I) \qquad \mathcal{B}_{22} = (1/\alpha)(e_1^T \otimes D1)$$
$$\mathcal{B}_{32} = (e_1^T diag(R) \otimes I) \qquad \mathcal{B}_{33} = (1/\alpha)(e_1^T \otimes D^{(1)}) \tag{3.160}$$

Boundary conditions at $R = \beta$

Clamped ($\psi_R = \psi_\Theta = W = 0$)

$$\mathcal{B}_{11} = (I \otimes e_1^T) \qquad \mathcal{B}_{22} = (I \otimes e_1^T) \qquad \mathcal{B}_{33} = (I \otimes e_1^T) \tag{3.161}$$

Simply-supported ($M_R = \psi_\Theta = W = 0$)

$$\mathcal{B}_{11} = (DRR1 \otimes e_1^T) + \nu(I \otimes e_1^T) \qquad \mathcal{B}_{12} = \nu(I \otimes e_1^T D^{(1)})$$
$$\mathcal{B}_{22} = (I \otimes e_1^T) \qquad \mathcal{B}_{33} = (I \otimes e_1^T) \tag{3.162}$$

Free ($M_R = M_{R\Theta} = Q_R = 0$)

$$\mathcal{B}_{11} = (DRR1 \otimes e_1^T) + \nu(I \otimes e_1^T) \qquad \mathcal{B}_{12} = \nu(I \otimes e_1^T D^{(1)})$$
$$\mathcal{B}_{21} = (1/\alpha)(I \otimes e_1^T D1) \qquad \mathcal{B}_{22} = -(I \otimes e_1^T) + (DRR1 \otimes e_1^T)$$
$$\mathcal{B}_{31} = (I \otimes e_1^T) \qquad \mathcal{B}_{33} = (DR^{(1)} \otimes e_1^T) \tag{3.163}$$

$DRR1 = DR^{(1)} \times diag(R)$. Other unwritten components are zero by order $((N+1) \times (N+1)^2)$. Applying boundary conditions on $R = 1$ and $\Theta = 1$ is similar to rectangular plate and it is done by replacing e_1^T with $e_{(N+1)}^T$.

3.4 In plane vibration

In plane vibration consider the in-plane stresses $(\sigma_{\alpha\alpha}, \sigma_{\beta\beta}, \sigma_{\alpha\beta})$ which α and β coordinates can be of rectangular, circular and annular plate. In plane vibration is not considered the same as out of plane vibration but it is important in some engineering applications such as seismic, etc. In this type of problem displacement in the thickness direction of plate is not considered and the equations of motion correspond to in-plane problem are similar in all plate's theory. All vibration problem in this section has two coupled equations with respect to u and v such as following

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} = -\lambda^2 \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} \tag{3.164}$$

and matrix of boundary conditions for each edge is as

$$\begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.165}$$

Above equations are solved like section (3.3.1) with one displacement variable less and two boundary condition for each edge.

3.4.1 Annular and circular plates

The governing equations of motion are as:

$$\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial \theta} + \frac{1}{r}(N_r - N_\theta) = \rho h \frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial N_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{2}{r}N_{r\theta} = \rho h \frac{\partial^2 v}{\partial t^2} \tag{3.166}$$

Using below equations and after substitution

$$\begin{Bmatrix} N_{rr} \\ N_{r\theta} \\ N_{\theta\theta} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{Bmatrix} dz \qquad \begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \gamma_{r\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{1}{r}(u + \frac{\partial v}{\partial \theta}) \\ \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \end{Bmatrix} \tag{3.167}$$

yield

$$
\begin{aligned}
&\frac{E_r}{1-\nu_r\nu_\theta}\Big(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\nu_\theta}{r}\frac{\partial^2 v}{\partial r\partial\theta}\Big)-\frac{E_\theta}{1-\nu_r\nu_\theta}\Big(\frac{u}{r^2}+\frac{1}{r^2}\frac{\partial v}{\partial\theta}\Big)\\
&+\frac{E}{2(1+\nu)}\Big(\frac{1}{r}\frac{\partial^2 v}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial v}{\partial\theta}\Big)=\rho\ddot{u}\\
&\frac{E}{1-\nu_r\nu_\theta}\Big(\frac{\nu_r}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial u}{\partial\theta}\Big)\\
&+\frac{E}{2(1+\nu)}\Big(\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta}-\frac{v}{r^2}\Big)=\rho\ddot{v}
\end{aligned}
\tag{3.168}
$$

Using above equations, stiffness components are easily obtained and also boundary conditions are the same as in-plane vibration of sector annular plate in section (3.4.5).

3.4.2 Rectangular plates

Matrix components of equation of motion for cross-ply and angle-ply laminated rectangular plate is

$$
\begin{aligned}
K_{11}&=A_{11}(D^{(2)}\otimes I)+2A_{16}(D^{(1)}\otimes D^{(1)})+A_{66}(I\otimes D^{(2)})\\
K_{12}&=(A_{12}+A_{66})(D^{(1)}\otimes D^{(1)})+A_{16}(D^{(2)}\otimes I)+A_{26}(I\otimes D^{(2)})=K_{21}\\
K_{22}&=2A_{26}(D^{(1)}\otimes D^{(1)})+A_{66}(D^{(2)}\otimes I)+A_{22}(I\otimes D^{(2)})
\end{aligned}
\tag{3.169}
$$

and

$$
M_{11}=\rho h(I\otimes I)\qquad M_{22}=\rho h(I\otimes I)\tag{3.170}
$$

Boundary condition at $\zeta = 0$

Clamped ($u = v = 0$)

$$
\mathcal{B}_{11}=(e_1^T\otimes I)\qquad \mathcal{B}_{22}=(e_1^T\otimes I)\tag{3.171}
$$

Simply-supported - type 1 ($N_{xx} = v = 0$)

$$
\begin{aligned}
\mathcal{B}_{11}&=A_{11}(e_1^TD^{(1)}\otimes I)+A_{16}(e_1^T\otimes D^{(1)})\\
\mathcal{B}_{12}&=A_{12}(e_1^T\otimes D^{(1)})+A_{16}(e_1^TD^{(1)}\otimes I)\qquad \mathcal{B}_{22}=(e_1^T\otimes I)
\end{aligned}
\tag{3.172}
$$

Simply-supported - type 2 ($u = N_{xy} = 0$)

$$\mathcal{B}_{11} = (e_1^T \otimes I) \qquad \mathcal{B}_{21} = A_{16}(e_1^T D^{(1)} \otimes I) + A_{66}(e_1^T \otimes D^{(1)})$$
$$\mathcal{B}_{22} = A_{26}(e_1^T \otimes D^{(1)}) + A_{66}(e_1^T D^{(1)} \otimes I) \tag{3.173}$$

Free ($N_{xx} = N_{xy} = 0$)

$$\mathcal{B}_{11} = A_{11}(e_1^T D^{(1)} \otimes I) + A_{16}(e_1^T \otimes D^{(1)}) \qquad \mathcal{B}_{12} = A_{12}(e_1^T \otimes D^{(1)}) + A_{16}(e_1^T D^{(1)} \otimes I)$$
$$\mathcal{B}_{21} = A_{16}(e_1^T D^{(1)} \otimes I) + A_{66}(e_1^T \otimes D^{(1)}) \qquad \mathcal{B}_{22} = A_{26}(e_1^T \otimes D^{(1)}) + A_{66}(e_1^T D^{(1)} \otimes I) \tag{3.174}$$

Boundary condition at $\eta = 0$

Clamped ($u = v = 0$):

$$\mathcal{B}_{11} = (I \otimes e_1^T) \qquad \mathcal{B}_{22} = (I \otimes e_1^T) \tag{3.175}$$

Simply-supported - type 1 ($u = N_{yy} = 0$)

$$\mathcal{B}_{11} = (I \otimes e_1^T) \qquad \mathcal{B}_{21} = A_{12}(D \otimes e_1^T) + A_{26}(I \otimes e_1^T D^{(1)})$$
$$\mathcal{B}_{22} = A_{22}(I \otimes e_1^T D^{(1)}) + A_{26}(D \otimes e_1^T) \tag{3.176}$$

Simply-supported - type 2 ($N_{xy} = v = 0$)

$$\mathcal{B}_{11} = A_{16}(D \otimes e_1^T) + A_{66}(I \otimes e_1^T D^{(1)})$$
$$\mathcal{B}_{12} = A_{26}(I \otimes e_1^T D^{(1)}) + A_{66}(D \otimes e_1^T) \quad \mathcal{B}_{22} = (I \otimes e_1^T) \tag{3.177}$$

Free at ($N_{xy} = N_{yy} = 0$)

$$\mathcal{B}_{11} = A_{12}(D \otimes e_1^T) + A_{26}(I \otimes e_1^T D^{(1)}) \quad \mathcal{B}_{12} = A_{22}(I \otimes e_1^T D^{(1)}) + A_{26}(D \otimes e_1^T)$$
$$\mathcal{B}_{21} = A_{16}(D \otimes e_1^T) + A_{66}(I \otimes e_1^T D^{(1)}) \quad \mathcal{B}_{22} = A_{26}(I \otimes e_1^T D^{(1)}) + A_{66}(D \otimes e_1^T) \tag{3.178}$$

BCs for elastic restrained orthotropic plates in both normal and parallel to the edges:

At $x = \mp 1$

$$A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} + A_{16}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \mp K_n u = 0$$
$$A_{16}\frac{\partial u}{\partial x} + A_{26}\frac{\partial v}{\partial y} + A_{66}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \mp K_p v = 0 \tag{3.179}$$

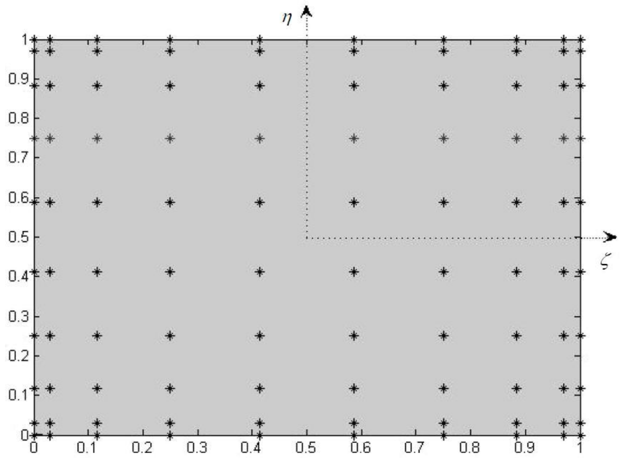

Figure 3.23: 2-D node distribution with $N = 9$

At $y = \mp 1$

$$A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} + A_{26}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \mp K_n v = 0$$
$$A_{16}\frac{\partial u}{\partial x} + A_{26}\frac{\partial v}{\partial y} + A_{66}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \mp K_p u = 0 \qquad (3.180)$$

where K_n, K_p are constrained coefficients normal and parallel to edges, respectively.

Varying elastic restrained:

According to the different varying elastic coefficients of boundaries such as linear and parabolic, distribution of the elastic restrained along the edges are assumed as $k = k_{n,p}(1 - x^2)$ for x side and $k = k_{n,p}(1 - y^2)$ for y side.

Thus, the parabolic variation of elastic boundary conditions are as follow

at $x = \mp 1$

$$A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} + A_{16}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \mp K_n diag(1 - y^2)u = 0$$
$$A_{16}\frac{\partial u}{\partial x} + A_{26}\frac{\partial v}{\partial y} + A_{66}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \mp K_p diag(1 - y^2)v = 0 \qquad (3.181)$$

at $y = \mp 1$

$$A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} + A_{26}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \mp K_n diag(1 - x^2)v = 0$$
$$A_{16}\frac{\partial u}{\partial x} + A_{26}\frac{\partial v}{\partial y} + A_{66}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \mp K_p diag(1 - x^2)u = 0 \qquad (3.182)$$

3.4.3 Rectangular plates with mixed boundary conditions

The number of points which are used discretization the boundary conditions in the mixed boundary case should be equal, although the lengths of the simply supported, clamped or

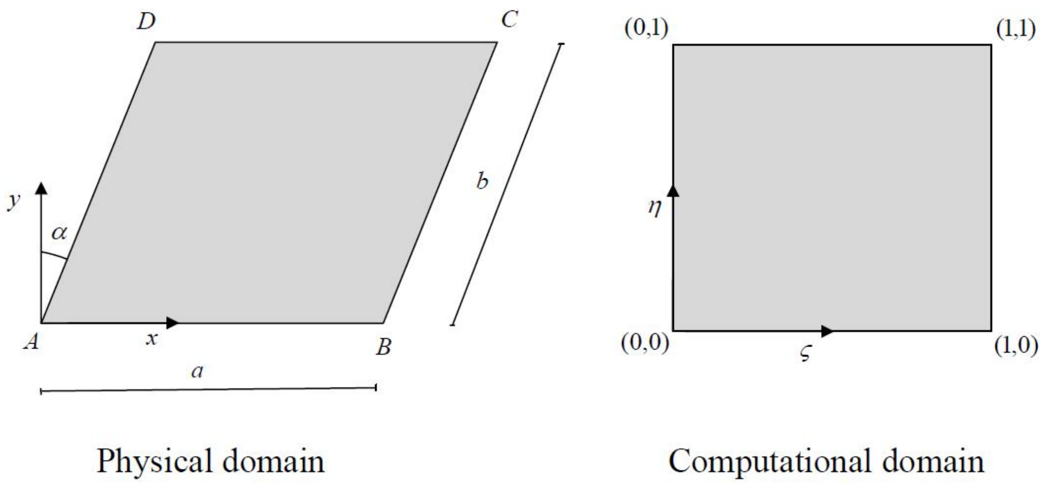

Figure 3.24: Skew plate

free portions are not equal. For example if one edge is divided in two parts implementing the boundary condition including the first $0.5(N+1)$ rows of the first boundary condition and the last $0.5(N+1)$ rows of the second boundary condition. Number of collocation points should be odd, thus N should be even like (Figure 3.1) and not the same as (Figure 3.23). Because with odd number of collocation points, one point locates in the middle and the other points are in the right and left hand side of this point, symmetrically. During increasing the number of collocation points, location of center point which is being 0.5 is constant and is at the center and location of other points are changed symmetrically and convergence of the problem is fine. With even number of collocation points there is no center point and location of all points are changing during increasing N and there is no good convergence. This problem becomes critical when each edge is divided into more than two parts. In this case even with odd number of collocation points, there is no any symmetry. The convergence in these cases are not fine and graphs of convergence don't reach to converged value and has fluctuation behavior. Overcoming this problem, domain decomposition technique is advised. It means that with vertical and horizontal lines, division points of all edges are connected. i.e. with three division in each edge, there are nine rectangular parts which should be solved separately and continuity conditions should apply among all subparts [76, 77, 78]. Continuity conditions include be equal in displacements, moments and forces.

3.4.4 Skew plates

Consider a skew plate (see Figure 3.24) with length a and width b and skew angle α with respect to y in the Cartesian coordinate. For generality and convenience, the present formulation is expressed in dimensionless coordinates using the following relationships

$$\xi = \frac{1}{a}(x - y\tan\alpha)$$
$$\eta = \frac{1}{b}(y\sec\alpha) \tag{3.183}$$

The skew plate in the $(x-y)$ physical domain is mapped in to a square plate in the computational $(\xi-\eta)$ domain $(0\leq\xi,\eta\leq 1)$ by using the following coordinate mapping

$$\begin{aligned} x &= x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)x_i \\ y &= y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)y_i \end{aligned} \tag{3.184}$$

where $N_i = (1+\xi_i\xi)(1+\eta_i\eta)$ is shape function. physical domain coordinates are as

$$\begin{aligned} &(x_1,y_1) = (0,0) && (x_2,y_2) = (a,0), \\ &(x_3,y_3) = (a+b\sin\alpha, b\cos\alpha) && (x_4,y_4) = (b\sin\alpha, b\cos\alpha) \end{aligned} \tag{3.185}$$

and also computational domain coordinates are

$$\begin{aligned} &(\xi_1,\eta_1) = (0,0), && (\xi_2,\eta_2) = (1,0), \\ &(\xi_3,\eta_3) = (1,1), && (\xi_4,\eta_4) = (0,1) \end{aligned} \tag{3.186}$$

The derivatives of variables with respect to non-dimensional variables (ξ,η) are written as

$$\begin{aligned} (.)_{/\xi} &= x_{/\xi}(.)_{/x} + y_{/\xi}(.)_{/y} \\ (.)_{/\eta} &= x_{/\eta}(.)_{/x} + y_{/\eta}(.)_{/y} \\ (.)_{/\xi\xi} &= [x_{/\xi}(.)_{/x} + y_{/\xi}(.)_{/y}]_{/\xi} \\ (.)_{/\xi\eta} &= [x_{/\eta}(.)_{/x} + y_{/\eta}(.)_{/y}]_{/\xi} \\ (.)_{/\eta\eta} &= [x_{/\eta}(.)_{/x} + y_{/\eta}(.)_{/y}]_{/\eta} \end{aligned} \tag{3.187}$$

and in matrix form are as

$$\left\{\begin{array}{c} (.)_{/\xi} \\ (.)_{/\eta} \end{array}\right\} = J_{11}\left\{\begin{array}{c} (.)_{/x} \\ (.)_{/y} \end{array}\right\} \quad \left\{\begin{array}{c} (.)_{/\xi\xi} \\ (.)_{/\eta\eta} \\ (.)_{/\xi\eta} \end{array}\right\} = J_{22}\left\{\begin{array}{c} (.)_{/xx} \\ (.)_{/yy} \\ (.)_{/xy} \end{array}\right\} + J_{21}\left\{\begin{array}{c} (.)_{/x} \\ (.)_{/y} \end{array}\right\} \tag{3.188}$$

where

$$\begin{aligned} J_{11} &= \left[\begin{array}{cc} x_{/\xi} & y_{/\xi} \\ x_{/\eta} & y_{/\eta} \end{array}\right], \quad J_{21} = \left[\begin{array}{cc} x_{/\xi\xi} & y_{/\xi\xi} \\ x_{/\eta\eta} & y_{/\eta\eta} \\ x_{/\xi\eta} & y_{/\xi\eta} \end{array}\right] \\ J_{22} &= \left[\begin{array}{ccc} x_{/\xi}^2 & y_{/\xi}^2 & 2x_{/\xi}y_{/\xi} \\ x_{/\eta}^2 & y_{/\eta}^2 & 2x_{/\eta}y_{/\eta} \\ x_{/\xi}x_{/\eta} & y_{/\xi}y_{/\eta} & x_{/\eta}y_{/\xi} + x_{/\xi}y_{/\eta} \end{array}\right] \end{aligned} \tag{3.189}$$

Inverse relations of the matrix transformation are in the following as

$$\begin{aligned} \left\{\begin{array}{c} (.)_{/x} \\ (.)_{/y} \end{array}\right\} &= J_{11}^{-1}\left\{\begin{array}{c} (.)_{/\xi} \\ (.)_{/\eta} \end{array}\right\} \\ \left\{\begin{array}{c} (.)_{/xx} \\ (.)_{/yy} \\ (.)_{/xy} \end{array}\right\} &= J_{22}^{-1}\left\{\begin{array}{c} (.)_{/\xi\xi} \\ (.)_{/\eta\eta} \\ (.)_{/\xi\eta} \end{array}\right\} - J_{22}^{-1}J_{21}J_{11}^{-1}\left\{\begin{array}{c} (.)_{/\xi} \\ (.)_{/\eta} \end{array}\right\} \end{aligned} \tag{3.190}$$

where

$$J_{11}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{1}{a}\tan(\alpha) & \frac{1}{b}\frac{1}{\cos(\alpha)} \end{bmatrix}, \qquad J_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$J_{22}^{-1} = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ \frac{1}{a^2}\tan^2(\alpha) & \frac{1}{b^2}\frac{1}{\cos^2(\alpha)} & -\frac{2}{ab}\frac{\tan(\alpha)}{\cos(\alpha)} \\ -\frac{1}{a^2}\tan(\alpha) & 0 & \frac{1}{ab}\frac{1}{\cos(\alpha)} \end{bmatrix} \tag{3.191}$$

For skew plates all components of matrix J_{21} are zero but for quadrilateral plates they are not all zero. Matrix components of equation of motion are as

$$\begin{aligned} K_{11} &= (A_{11} + A_{66}\tan^2\alpha)\frac{\partial^2}{\partial\xi^2} + A_{66}\phi^2\sec^2\alpha\frac{\partial^2}{\partial\eta^2} - 2A_{66}\phi\frac{\tan\alpha}{\cos\alpha}\frac{\partial^2}{\partial\xi\partial\eta} \\ K_{12} &= (A_{12} + A_{12})[\phi\sec\alpha\frac{\partial^2}{\partial\xi\partial\eta} - \tan\alpha\frac{\partial^2}{\partial\xi^2}] \\ K_{21} &= K_{12} \\ K_{22} &= (A_{66} + A_{22}\tan^2\alpha)\frac{\partial^2}{\partial\xi^2} + A_{22}\phi^2\sec^2\alpha\frac{\partial^2}{\partial\eta^2} - 2A_{22}\phi\frac{\tan\alpha}{\cos\alpha}\frac{\partial^2}{\partial\xi\partial\eta} \end{aligned} \tag{3.192}$$

where $A_{11} = A_{22} = E/(1-\nu^2), A_{12} = \nu E/(1-\nu^2), A_{66} = E/2(1+\nu), \phi = a/b$.

After applying spectral collocation method on the operators of equations of motion, yield

$$\begin{aligned} K_{11} &= (A_{11} + A_{66}\tan^2\alpha)(D2\otimes I) + A_{66}\phi^2\sec^2\alpha(I\otimes D2) - 2A_{66}\phi\frac{\tan\alpha}{\cos\alpha}(D1\otimes D1) \\ K_{12} &= (A_{12} + A_{66})\Big(\phi\sec\alpha(D1\otimes D1) - \tan\alpha(D2\otimes I)\Big) \\ K_{21} &= K_{12} \\ K_{22} &= (A_{66} + A_{22}\tan^2\alpha)(D2\otimes I) + A_{22}\phi^2\sec^2\alpha(I\otimes D2) - 2A_{22}\phi\frac{\tan\alpha}{\cos\alpha}(D1\otimes D1) \end{aligned} \tag{3.193}$$

and

$$M_{11} = \rho h(I\otimes I) \qquad M_{22} = \rho h(I\otimes I) \tag{3.194}$$

where $\lambda^2 = \rho a^2\omega^2(1-\nu^2)/4E$.

Boundary conditions at $\zeta = 0$

Clamped ($u = v = 0$)

$$\mathcal{B}_{11} = n_x(e_1^T\otimes I), \mathcal{B}_{12} = n_y(e_1^T\otimes I), \mathcal{B}_{21} = n_y(e_1^T\otimes I), \mathcal{B}_{22} = -n_x(e_1^T\otimes I) \tag{3.195}$$

Free ($N_{xx} = N_{xy} = 0$)

$$\begin{aligned}
\mathcal{B}_{11} &= [(A_{11}{n_x}^2 + A_{12}{n_y}^2) - 2A_{66}n_xn_y\tan\alpha](e_1^TD^{(1)}\otimes I) + 2\phi\sec\alpha A_{66}n_xn_y(e_1^T\otimes D^{(1)})\\
\mathcal{B}_{12} &= [-\tan\alpha(A_{12}{n_x}^2 + A_{22}{n_y}^2) + 2A_{66}n_xn_y](e_1^TD^{(1)}\otimes I)\\
&\qquad\qquad\qquad\qquad +\phi\sec\alpha(A_{12}{n_x}^2 + A_{22}{n_y}^2)(e_1^T\otimes D^{(1)})\\
\mathcal{B}_{21} &= [(A_{22} - A_{11})n_xn_y - \tan\alpha A_{66}({n_x}^2 - {n_y}^2)](e_1^TD^{(1)}\otimes I)\\
&\qquad\qquad\qquad\qquad +\phi\sec\alpha A_{66}({n_x}^2 - {n_y}^2)(e_1^T\otimes D^{(1)})\\
\mathcal{B}_{22} &= [-\tan\alpha(A_{22} - A_{12})n_xn_y + A_{66}({n_x}^2 - {n_y}^2)](e_1^TD^{(1)}\otimes I)\\
&\qquad\qquad\qquad\qquad +\phi\sec\alpha(A_{22} - A_{12})n_xn_y(e_1^T\otimes D^{(1)}) \qquad (3.196)
\end{aligned}$$

Simply-supported - type 1 ($v = N_{xx} = 0$)

$$\mathcal{B}_{11} = n_y(e_1^T\otimes I), \mathcal{B}_{12} = -n_x(e_1^T\otimes I), \mathcal{B}_{21} = \mathcal{B}_{11}^{free}, \mathcal{B}_{22} = \mathcal{B}_{12}^{free} \qquad (3.197)$$

Simply-supported - type 2 ($u = N_{xy} = 0$)

$$\mathcal{B}_{11} = n_x(e_1^T\otimes I), \mathcal{B}_{12} = n_y(e_1^T\otimes I), \mathcal{B}_{21} = \mathcal{B}_{21}^{free}, \mathcal{B}_{22} = \mathcal{B}_{22}^{free} \qquad (3.198)$$

Boundary conditions at $\eta = 0$

Clamped ($u = v = 0$)

$$\mathcal{B}_{11} = n_x(I\otimes e_1^T), \mathcal{B}_{12} = n_y(I\otimes e_1^T), \mathcal{B}_{21} = n_y(I\otimes e_1^T), \mathcal{B}_{22} = -n_x(I\otimes e_1^T) \qquad (3.199)$$

Free ($N_{xx} = N_{xy} = 0$)

$$\begin{aligned}
\mathcal{B}_{11} &= [(A_{11}{n_x}^2 + A_{12}{n_y}^2) - 2A_{66}n_xn_y\tan\alpha](D^{(1)}\otimes e_1^T) + 2\phi\sec\alpha A_{66}n_xn_y(I\otimes e_1^TD^{(1)})\\
\mathcal{B}_{12} &= [-\tan\alpha(A_{12}{n_x}^2 + A_{22}{n_y}^2) + 2A_{66}n_xn_y](D^{(1)}\otimes e_1^T)\\
&\qquad\qquad\qquad\qquad +\phi\sec\alpha(A_{12}{n_x}^2 + A_{22}{n_y}^2)(I\otimes e_1^TD^{(1)})\\
\mathcal{B}_{21} &= [(A_{22} - A_{11})n_xn_y - \tan\alpha A_{66}({n_x}^2 - {n_y}^2)](D^{(1)}\otimes e_1^T)\\
&\qquad\qquad\qquad\qquad +\phi\sec\alpha A_{66}({n_x}^2 - {n_y}^2)(I\otimes e_1^TD^{(1)})\\
\mathcal{B}_{22} &= [-\tan\alpha(A_{22} - A_{12})n_xn_y + A_{66}({n_x}^2 - {n_y}^2)](D^{(1)}\otimes e_1^T)\\
&\qquad\qquad\qquad\qquad +\phi\sec\alpha(A_{22} - A_{12})n_xn_y(I\otimes e_1^TD^{(1)}) \qquad (3.200)
\end{aligned}$$

Simply-supported - type 1 ($v = N_{xx} = 0$)

$$\mathcal{B}_{11} = n_y(I \otimes e_1^T), \mathcal{B}_{12} = -n_x(I \otimes e_1^T), \mathcal{B}_{21} = \mathcal{B}_{11}^{free}, \mathcal{B}_{22} = \mathcal{B}_{12}^{free} \tag{3.201}$$

Simply-supported - type 2 ($u = N_{xy} = 0$)

$$\mathcal{B}_{11} = n_x(I \otimes e_1^T), \mathcal{B}_{12} = n_y(I \otimes e_1^T), \mathcal{B}_{21} = \mathcal{B}_{21}^{free}, \mathcal{B}_{22} = \mathcal{B}_{22}^{free} \tag{3.202}$$

3.4.5 Sector annular plates

Consider a sector annular plate with inner radius a and outer radius b with angle α and aspect ratio β which is a/b. The governing equations in the matrix form of differential operator are

$$\begin{aligned}
K_{11} &= R^2\frac{\partial^2}{\partial R^2} + R\frac{\partial}{\partial R} - 1 + \frac{1-\nu}{2\alpha^2}\frac{\partial^2}{\partial \theta^2} \\
K_{12} &= \frac{1+\nu}{2\alpha}R\frac{\partial^2}{\partial R \partial \theta} - \frac{3-\nu}{2\alpha}\frac{\partial}{\partial \theta} \\
K_{21} &= \frac{1+\nu}{2\alpha}R\frac{\partial^2}{\partial R \partial \theta} + \frac{3-\nu}{2\alpha}\frac{\partial}{\partial \theta} \\
K_{22} &= \frac{(1-\nu)}{2}R^2\frac{\partial^2}{\partial R^2} + \frac{1-\nu}{2}R\frac{\partial}{\partial R} + \frac{1}{\alpha^2}\frac{\partial^2}{\partial \theta^2} - \frac{1-\nu}{2}
\end{aligned} \tag{3.203}$$

Distribution of the collocation points in the computational domain of sector annular plate is in two direction, one is in radial direction which is in the range of $(\beta, 1)$, and the other one is in circumferential direction which is in the range of $(0, 1)$. In-plane forces are as

$$\begin{aligned}
N_{RR} &= \frac{Eh}{(1-\nu^2)}\Big(\frac{\partial u}{\partial R} + \frac{\nu}{R}(u + \frac{\partial v}{\partial \theta})\Big) \\
N_{\theta\theta} &= \frac{Eh}{(1-\nu^2)}\Big(\frac{1}{R}(u + \frac{\partial v}{\partial \theta}) + \nu\frac{\partial u}{\partial R}\Big) \\
N_{R\theta} &= \frac{Eh}{2(1+\nu)}\Big(\frac{1}{R}(\frac{\partial u}{\partial \theta} - v) + \frac{\partial v}{\partial R}\Big)
\end{aligned} \tag{3.204}$$

After applying spectral collocation method, yield

$$\begin{aligned}
K_{11} &= (R^2DR^{(2)} \otimes I) + (RDR^{(1)} \otimes I) - (I \otimes I) + \frac{1-\nu}{2\alpha^2}(I \otimes D2) \\
K_{12} &= \frac{1+\nu}{2\alpha}(RDR^{(1)} \otimes D1) - \frac{3-\nu}{2\alpha}(I \otimes D1) \\
K_{21} &= K_{12} \\
K_{22} &= \frac{(1-\nu)}{2}(R^2DR^{(2)} \otimes I) + \frac{1-\nu}{2}(RDR^{(1)} \otimes I) + \frac{1}{\alpha^2}(I \otimes D2) - \frac{1-\nu}{2}(I \otimes I)
\end{aligned} \tag{3.205}$$

and

$$M_{11} = \rho h(R^2 \otimes I) \qquad M_{22} = \rho h(R^2 \otimes I) \tag{3.206}$$

Boundary conditions at $R = \beta$

Clamped ($u = v = 0$):

$$\mathcal{B}_{11} = (e_1^T \otimes I) \qquad \mathcal{B}_{22} = (e_1^T \otimes I) \tag{3.207}$$

Simply-Supported ($N_{RR} = v = 0$)

$$\mathcal{B}_{11} = (e_1^T DRR1 \otimes I) + \nu(e_1^T \otimes I) \quad \mathcal{B}_{12} = \frac{\nu}{\alpha}(e_1^T \otimes D^{(1)}) \quad \mathcal{B}_{22} = e_1^T \tag{3.208}$$

Free ($N_{RR} = N_{R\theta} = 0$)

$$\begin{aligned} &\mathcal{B}_{11} = (e_1^T DRR1 \otimes I) + \nu(e_1^T \otimes I) \quad \mathcal{B}_{12} = \frac{\nu}{\alpha}(e_1^T \otimes D^{(1)}) \\ &\mathcal{B}_{21} = \frac{1}{\alpha}(e_1^T \otimes D^{(1)}) \qquad \mathcal{B}_{22} = -(e_1^T \otimes I) + (e_1^T RDR^{(1)} \otimes I) \end{aligned} \tag{3.209}$$

Boundary condition at $\Theta = 0$

Clamped ($u = v = 0$)

$$\mathcal{B}_{11} = (I \otimes e_1^T) \qquad \mathcal{B}_{22} = (I \otimes e_1^T) \tag{3.210}$$

Simply-Supported ($u = N_{\theta\theta} = 0$)

$$\begin{aligned} &\mathcal{B}_{11} = e_1^T \qquad \mathcal{B}_{21} = (I \otimes e_1^T) + \nu(RDR1 \otimes e_1^T) \\ &\mathcal{B}_{22} = \frac{1}{\alpha}(I \otimes e_1^T D^{(1)}) \end{aligned} \tag{3.211}$$

Free ($N_{R\theta} = N_{\theta\theta} = 0$)

$$\begin{aligned} &\mathcal{B}_{11} = \frac{1}{\alpha}(I \otimes e_1^T D^{(1)}) \qquad \mathcal{B}_{12} = -(I \otimes e_1^T) + (RDR1 \otimes e_1^T) \\ &\mathcal{B}_{21} = (I \otimes e_1^T) + \nu(RDR1 \otimes e_1^T) \qquad \mathcal{B}_{22} = \frac{1}{\alpha}(I \otimes e_1^T D^{(1)}) \end{aligned} \tag{3.212}$$

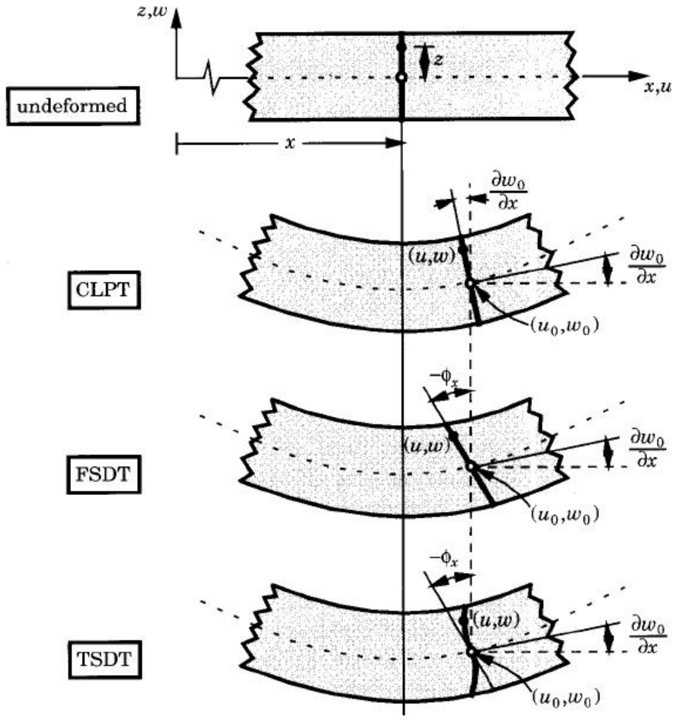

Figure 3.25: Undeformed and deformed in the thickness direction under third-order shear deformation assumption

3.5 Third-order shear deformation plate theory

The third-order shear deformation plate theory (TSDT) which is known to Reddy plate theory is similar to classical and first-order plate theories, except that we relax the assumption for straightness and normality of a transverse normal after deformation by expanding the displacements (u, v, w) as cubic functions of the thickness coordinate (see Figure 3.25). Consider the displacements field as

$$\begin{aligned} u &= u_0 + z\phi_x + z^2\theta_x + z^3\chi_x \\ v &= v_0 + z\phi_y + z^2\theta_y + z^3\chi_y \\ w &= w_0 \end{aligned} \tag{3.213}$$

where (ϕ_x, ϕ_y), (θ_x, θ_y) and (χ,χ_y) are functions to be determined. Clearly we have

$$\begin{array}{lll} u_0 = u(x, y, 0, t) & v_0 = v(x, y, 0, t) & w_0 = w(x, y, 0, t) \\ \phi_x = \left(\dfrac{\partial u}{\partial z}\right)_{z=0} & \phi_y = \left(\dfrac{\partial v}{\partial z}\right)_{z=0} & 2\theta_x = \left(\dfrac{\partial^2 u}{\partial z^2}\right)_{z=0} \\ 2\theta_y = \left(\dfrac{\partial^2 v}{\partial z^2}\right)_{z=0} & 6\chi_x = \left(\dfrac{\partial^3 u}{\partial z^3}\right)_{z=0} & 6\chi_y = \left(\dfrac{\partial^3 v}{\partial z^3}\right)_{z=0} \end{array} \tag{3.214}$$

There are nine dependent unknowns, and the theory derived using the displacement field will result in nine second-order partial differential equations. The number of dependent unknowns can be reduced by imposing certain conditions. Suppose that we wish to impose traction-free boundary conditions on the top an bottom faces of the laminate

$$\sigma_{xz}(x, y, \pm h, t) = 0 \qquad \sigma_{yz}(x, y, \pm h, t) = 0 \tag{3.215}$$

Expressing the above conditions in terms of strains, we have

$$\begin{aligned} 0 &= \sigma_{xz}(x,y,\pm h,t) = Q_{55}\gamma_{xz}(x,y,\pm h,t) + Q_{45}\gamma_{yz}(x,y,\pm h,t) \\ 0 &= \sigma_{yz}(x,y,\pm h,t) = Q_{45}\gamma_{xz}(x,y,\pm h,t) + Q_{44}\gamma_{yz}(x,y,\pm h,t) \end{aligned} \tag{3.216}$$

which in turn requires, for arbitrary $C_{ij}(i,j=4,5)$

$$\begin{aligned} 0 &= \gamma_{xz}(x,y,\pm h,t) = Q_x + \frac{\partial w_0}{\partial x} + \Big(2z\theta_x + 3z^2\chi_x\Big)_{z=\pm h/2} \\ 0 &= \gamma_{yz}(x,y,\pm h,t) = Q_y + \frac{\partial w_0}{\partial y} + \Big(2z\theta_y + 3z^2\chi_y\Big)_{z=\pm h/2} \end{aligned} \tag{3.217}$$

Thus, we have

$$\begin{aligned} Q_x + \frac{\partial w_0}{\partial x} + \Big(-h\theta_x + 3h^2/4\chi_x\Big) = 0, &\qquad Q_x + \frac{\partial w_0}{\partial x} + \Big(h\theta_x + 3h^2/4\chi_x\Big) = 0 \\ Q_y + \frac{\partial w_0}{\partial y} + \Big(-h\theta_y + 3h^2/4\chi_y\Big) = 0, &\qquad Q_y + \frac{\partial w_0}{\partial y} + \Big(h\theta_y + 3h^2/4\chi_y\Big) = 0 \end{aligned} \tag{3.218}$$

or

$$\chi_x = -4/3h^2(Q_x + \frac{\partial w_0}{\partial x}), \quad \theta_x = 0, \quad \chi_y = -4/3h^2(Q_y + \frac{\partial w_0}{\partial y}), \quad \theta_y = 0 \tag{3.219}$$

The supposed displacements field now can be expressed in terms of $u_0, v_0, w_0, \phi_x, \phi_y$ as

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t) - c_1 z^3(\phi_x + \frac{\partial w_0}{\partial x}) \\ v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t) - c_1 z^3(\phi_y + \frac{\partial w_0}{\partial y}) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{3.220}$$

where ϕ_x, ϕ_y is rotation of normal to the middle surface with respect to thickness direction. The equations of motion of the first-order theory can be derived from the present third-order theory by setting $c_1 = 0$, and classical theory also can be derived by replacing ϕ_x with $-\frac{\partial w_0}{\partial x}$ and also ϕ_y with $-\frac{\partial w_0}{\partial y}$, and similar to first-order theory $\epsilon_{zz} = 0$.

Strain-displacement relations are as

$$\begin{aligned} \left\{\begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{array}\right\} &= \left\{\begin{array}{c} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \epsilon_{xy}^{(0)} \end{array}\right\} + z\left\{\begin{array}{c} \epsilon_{xx}^{(1)} \\ \epsilon_{yy}^{(1)} \\ \epsilon_{xy}^{(1)} \end{array}\right\} + z^3\left\{\begin{array}{c} \epsilon_{xx}^{(3)} \\ \epsilon_{yy}^{(3)} \\ \epsilon_{xy}^{(3)} \end{array}\right\} \\ &= \left\{\begin{array}{c} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{array}\right\} + z\left\{\begin{array}{c} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{array}\right\} - c_1 z^3\left\{\begin{array}{c} \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w_0}{\partial x \partial y} \end{array}\right\} \\ \left\{\begin{array}{c} \gamma_{xz} \\ \gamma_{yz} \end{array}\right\} &= \left\{\begin{array}{c} \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{array}\right\} + z^2\left\{\begin{array}{c} \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{array}\right\} \\ &= \left\{\begin{array}{c} \phi_x + \frac{\partial w_0}{\partial x} \\ \phi_y + \frac{\partial w_0}{\partial y} \end{array}\right\} - c_2 z^2\left\{\begin{array}{c} \phi_x + \frac{\partial w_0}{\partial x} \\ \phi_y + \frac{\partial w_0}{\partial y} \end{array}\right\} \end{aligned} \tag{3.221}$$

where $c_1 = 4/3h^2$ and $c_2 = 3c_1$.

The stress resultants are related to the strains by the relations

$$\left\{\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \\ \\ M_{xx} \\ M_{yy} \\ M_{xy} \\ \\ P_{xx} \\ P_{yy} \\ P_{xy} \end{array}\right\} = \left[\begin{array}{ccccccccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & E_{11} & E_{12} & E_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & E_{12} & E_{22} & E_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & E_{16} & E_{26} & E_{66} \\ \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ \\ E_{11} & E_{12} & E_{16} & F_{11} & F_{12} & F_{16} & H_{11} & H_{12} & H_{16} \\ E_{12} & E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{26} \\ E_{16} & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{array}\right] \left\{\begin{array}{c} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \epsilon_{xy}^{(0)} \\ \\ \epsilon_{xx}^{(1)} \\ \epsilon_{yy}^{(1)} \\ \epsilon_{xy}^{(1)} \\ \\ \epsilon_{xx}^{(3)} \\ \epsilon_{yy}^{(3)} \\ \epsilon_{xy}^{(3)} \end{array}\right\}$$

$$\left\{\begin{array}{c} Q_x \\ Q_y \\ \\ R_x \\ R_y \end{array}\right\} = \left[\begin{array}{cccc} A_{55} & A_{45} & D_{55} & D_{45} \\ A_{45} & A_{44} & D_{45} & D_{44} \\ \\ D_{55} & D_{45} & F_{55} & F_{45} \\ D_{45} & D_{44} & F_{45} & F_{44} \end{array}\right] \left\{\begin{array}{c} \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \\ \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{array}\right\} \tag{3.222}$$

For in-plane or out-of-plane analysis separately all B_{ij} and E_{ij} coefficients should be zero and so the in-plane equations are as

$$\left\{\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \end{array}\right\} = \left[\begin{array}{ccc} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{array}\right] \left\{\begin{array}{c} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \epsilon_{xy}^{(0)} \end{array}\right\} \tag{3.223}$$

and also out-of-plane equation of motion are as

$$\left\{\begin{array}{c} \{M\} \\ \{P\} \end{array}\right\} = \left[\begin{array}{cc} [\ D\ ] & [\ F\ ] \\ [\ F\ ] & [\ H\ ] \end{array}\right] \left\{\begin{array}{c} \{\epsilon^{(1)}\} \\ \{\epsilon^{(3)}\} \end{array}\right\}$$

$$\left\{\begin{array}{c} \{Q\} \\ \{R\} \end{array}\right\} = \left[\begin{array}{cc} [\ A\ ] & [\ D\ ] \\ [\ D\ ] & [\ F\ ] \end{array}\right] \left\{\begin{array}{c} \{\gamma^{(0)}\} \\ \{\gamma^{(2)}\} \end{array}\right\} \tag{3.224}$$

where elastic coefficients are as

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \tilde{C}_{ij}(1, z, z^2, z^3, z^4, z^6) dz \quad (i,j) = 1, 2, 6$$

$$(A_{ij}, D_{ij}, F_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \tilde{C}_{ij}(1, z^2, z^4) dz \quad (i,j) = 4, 5 \tag{3.225}$$

Substituting eqs. (3.220) and (3.221) into the eq. (3.14) and integration by part, Euler-lagrange equations of the theory are obtained by setting the coefficients of $\delta u_0, \delta v_0$, and δw_0 equal to zero separately as follow

$$
\begin{aligned}
\delta u_0: &\quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u} + J_1 \ddot{\phi}_x + c_1 I_3 \frac{\partial \ddot{w}_0}{\partial x} \\
\delta v_0: &\quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \ddot{v} + J_1 \ddot{\phi}_y + c_1 I_3 \frac{\partial \ddot{w}_0}{\partial y} \\
\delta w_0: &\quad \frac{\partial \bar{Q}_x}{\partial x} + \frac{\partial \bar{Q}_y}{\partial y} + c_1 (\frac{\partial P_{xx}}{\partial x^2} + 2\frac{\partial P_{xy}}{\partial x \partial y} + \frac{\partial P_{yy}}{\partial y^2}) + q = I_0 \ddot{w}_0 - c_1^2 I_6 (\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2}) \\
&\quad + c_1 \Big(I_3 (\frac{\ddot{u}}{\partial x} + \frac{\ddot{v}}{\partial y}) + I_4 (\frac{\ddot{\phi}_x}{\partial x} + \frac{\ddot{\phi}_y}{\partial y}) \Big) \\
\delta \phi_x: &\quad \frac{\partial \bar{M}_{xx}}{\partial x} + \frac{\partial \bar{M}_{xy}}{\partial y} - \bar{Q}_x = J_1 \ddot{u} + K_2 \ddot{\phi}_x - c_1 J_4 \frac{\partial \ddot{w}_0}{\partial x} \\
\delta \phi_y: &\quad \frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_{yy}}{\partial y} - \bar{Q}_y = J_1 \ddot{v} + K_2 \ddot{\phi}_y - c_1 J_4 \frac{\partial \ddot{w}_0}{\partial y}
\end{aligned}
\tag{3.226}
$$

where

$$
\begin{aligned}
&\left\{ \begin{array}{c} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{array} \right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} \left\{ \begin{array}{c} 1 \\ z \\ z^3 \end{array} \right\} dz, \qquad \left\{ \begin{array}{c} Q_\alpha \\ R_\alpha \end{array} \right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha z} \left\{ \begin{array}{c} 1 \\ z^2 \end{array} \right\} dz \\
&\bar{M}_{\alpha\beta} = M_{\alpha\beta} - c_1 P_{\alpha\beta} (\alpha, \beta = 1, 2, 6), \qquad \bar{Q}_\alpha = Q_\alpha - c_1 R_\alpha (\alpha = 4, 5) \\
&J_i = I_i - c_1 I_{i+2}, \qquad K_2 = I_2 - 2c_1 I_4 + c_1^2 I_6 \\
&I_i = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \rho^{(k)}(z)^i dz \quad (i = 0, 1, ..., 6)
\end{aligned}
\tag{3.227}
$$

The governing equations of motion for out-of-plane symmetric composite laminated based on Reddy plate theory in matrix form is as

$$
\left[\begin{array}{ccc} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \mathbf{K}_{23} \\ \mathbf{K}_{31} & \mathbf{K}_{32} & \mathbf{K}_{33} \end{array} \right] \left\{ \begin{array}{c} \phi_x \\ \phi_y \\ w_0 \end{array} \right\} = -\lambda^2 \left[\begin{array}{ccc} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ -\mathbf{M}_{12} & \mathbf{M}_{22} & 0 \\ -\mathbf{M}_{13} & 0 & \mathbf{M}_{22} \end{array} \right] \left\{ \begin{array}{c} \phi_x \\ \phi_y \\ w_0 \end{array} \right\}
\tag{3.228}
$$

Stiffness and mass components are as

$$
\begin{aligned}
\mathbf{K}_{11} = &\Big(A_{45} - 6c_1 D_{45} - 9c_1^2 F_{45} \Big) \frac{\partial^2}{\partial x^2} + \Big(A_{55} - 6c_1 D_{55} - 9c_1^2 F_{55} \\
&+ A_{44} - 6c_1 D_{44} + 9c_1^2 F_{44} \Big) \frac{\partial^2}{\partial x \partial y} + \Big(A_{45} - 6c_1 D_{45} - 3c_1 F_{45} \Big) \frac{\partial^2}{\partial y^2} \\
&- c_1^2 \Big(H_{11} \frac{\partial^4}{\partial x^4} + 2(H_{12} + 2H_{66}) \frac{\partial^4}{\partial x^2 \partial y^2} + H_{22} \frac{\partial^4}{\partial y^4} + 4H_{16} \frac{\partial^4}{\partial x^3 \partial y} \\
&+ 4H_{26} \frac{\partial^4}{\partial x \partial y^3} \Big) \\
\mathbf{K}_{12} = &\Big(A_{45} - 6c_1 D_{45} + 9c_1^2 F_{45} \Big) \frac{\partial}{\partial x} + \Big(A_{44} - 6c_1 D_{44} + 9c_1^2 F_{44} \Big) \frac{\partial}{\partial y} \\
&+ c_1 \Big((F_{11} - c_1 H_{11}) \frac{\partial^3}{\partial x^3} + (F_{26} - c_1 H_{26}) \frac{\partial^3}{\partial y^3} + (2F_{66} - 2c_1 H_{66}
\end{aligned}
$$

$$+F_{12}-c_1H_{12})\frac{\partial^3}{\partial x\partial y^2}+3(F_{16}-c_1H_{16})\frac{\partial^3}{\partial x^2\partial y}\Big)$$

$$\mathbf{K}_{13}=\Big(A_{55}-6c_1D_{55}+9c_1^2F_{55}\Big)\frac{\partial}{\partial x}+\Big(A_{45}-6c_1D_{45}+9c_1^2F_{45}\Big)\frac{\partial}{\partial y}$$
$$+c_1\Big((F_{16}-c_1H_{16})\frac{\partial^3}{\partial x^3}+(F_{22}-c_1H_{22})\frac{\partial^3}{\partial y^3}+2(F_{26}-c_1H_{26})\frac{\partial^3}{\partial x\partial y^2}$$
$$+(F_{12}-c_1H_{12}+2F_{66}-2c_1H_{66})\frac{\partial^3}{\partial x^2\partial y}\Big)$$

$$\mathbf{K}_{21}=-c_1\Big((F_{11}-c_1H_{11})\frac{\partial^3}{\partial x^3}+(2F_{66}-c_1H_{66}+F_{12}-c_1H_{12})\frac{\partial^3}{\partial x\partial y^2}$$
$$+3(F_{16}-c_1H_{16})\frac{\partial^3}{\partial x^2\partial y}+(F_{26}-c_1H_{26})\frac{\partial^3}{\partial y^3}\Big)+\Big(-A_{45}+6c_1D_{45}$$
$$-9c_1^2F_{45}\Big)\frac{\partial}{\partial x}+\Big(-A_{55}+6c_1D_{55}-9c_1^2F_{55}\Big)\frac{\partial}{\partial y}$$

$$\mathbf{K}_{22}=\Big((D_{11}-c_1F_{11})-c_1(F_{11}-c_1H_{11})\Big)\frac{\partial^2}{\partial x^2}+2\Big((D_{16}-c_1F_{16})$$
$$-c_1(F_{16}-c_1H_{16})\Big)\frac{\partial^2}{\partial x\partial y}+\Big((D_{66}-c_1F_{66})-c_1(F_{66}-c_1H_{66})\Big)\frac{\partial^2}{\partial y^2}$$
$$+\Big(-A_{45}+6c_1D_{45}-9c_1^2F_{45}\Big)$$

$$\mathbf{K}_{23}=\Big((D_{16}-c_1F_{16})-c_1(F_{16}-c_1H_{16})\Big)\frac{\partial^2}{\partial x^2}+\Big((D_{12}-c_1F_{12})$$
$$-c_1(F_{12}-c_1H_{12})+(D_{66}-c_1F_{66})-c_1(F_{66}-c_1H_{66})\Big)\frac{\partial^2}{\partial x\partial y}$$
$$+\Big((D_{26}-c_1F_{26})-c_1(F_{26}-c_1H_{26})\Big)\frac{\partial^2}{\partial y^2}+\Big(-A_{55}+6c_1D_{55}-9c_1^2F_{55}\Big)$$

$$\mathbf{K}_{31}=-c_1\Big((F_{16}-c_1H_{16})\frac{\partial^3}{\partial x^3}+3(F_{26}-c_1H_{26})\frac{\partial^3}{\partial x\partial y^2}+(F_{12}-c_1H_{12}$$
$$+2F_{66}-2c_1H_{66})\frac{\partial^3}{\partial x^2\partial y}+(F_{22}-c_1H_{22})\frac{\partial^3}{\partial y^3}\Big)+\Big(-A_{44}+6c_1D_{44}$$
$$-9c_1^2F_{44}\Big)\frac{\partial}{\partial x}+\Big(-A_{45}+6c_1D_{45}-9c_1^2F_{45}\Big)\frac{\partial}{\partial y}$$

$$\mathbf{K}_{32}=\Big((D_{16}-c_1F_{16})-c_1(F_{16}-c_1H_{16})\Big)\frac{\partial^2}{\partial x^2}+\Big((D_{66}-c_1F_{66})$$
$$-c_1(F_{66}-c_1H_{66})+(D_{12}-c_1F_{12})-c_1(F_{12}-c_1H_{12})\Big)\frac{\partial^2}{\partial x\partial y}$$
$$+\Big((D_{26}-c_1F_{26})-c_1(F_{26}-c_1H_{26})\Big)\frac{\partial^2}{\partial y^2}+\Big(-A_{44}+6c_1D_{44}-9c_1^2F_{44}\Big)$$

$$\mathbf{K}_{33}=\Big((D_{66}-c_1F_{66})-c_1(F_{66}-c_1H_{66})\Big)\frac{\partial^2}{\partial x^2}+2\Big((D_{26}-c_1F_{26})$$
$$-c_1(F_{26}-c_1H_{26})\Big)\frac{\partial^2}{\partial x\partial y}+\Big((D_{22}-c_1F_{22})-c_1(F_{22}-c_1H_{22})\Big)\frac{\partial^2}{\partial y^2}$$
$$+\Big(-A_{45}+6c_1D_{45}-9c_1^2F_{45}\Big)$$

and

$$\mathbf{M}_{11} = I_0 - c_1^2 I_6 (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \quad \mathbf{M}_{22} = K_2 \quad \mathbf{M}_{12} = c_1 J_4 \frac{\partial}{\partial x} \quad \mathbf{M}_{13} = c_1 J_4 \frac{\partial}{\partial y} \tag{3.229}$$

where $\lambda = a/b$ is aspect ratio, h is plate thickness and ω is natural frequency.

Solving out-of-plane vibration of third-order shear deformation symmetry composite plate follows the solving procedure of first-order shear deformation symmetry composite plate with the same distribution of collocation points for interior and boundary points.

Boundary conditions at $\zeta = 0$

Clamped ($\phi_x = \phi_y = w = 0$)

$$\mathcal{B}_{11} = (e_1^T \otimes I) \qquad \mathcal{B}_{22} = (e_1^T \otimes I) \qquad \mathcal{B}_{33} = (e_1^T \otimes I) \tag{3.230}$$

Simply-supported ($M_{xx} = \phi_y = w = 0$)

$$\begin{aligned} \mathcal{B}_{11} &= D_{11}(e_1^T D^{(1)} \otimes I) - c_1(e_1^T D^{(1)} \otimes I) \\ \mathcal{B}_{12} &= \beta D_{12}(e_1^T \otimes D^{(1)}) - c_1 F_{12}(e_1^T \otimes D^{(1)}) \\ \mathcal{B}_{13} &= -c_1 F_{11} \delta (e_1^T D^{(2)} \otimes I) - c_1 F_{12} \gamma (e_1^T \otimes D^{(2)}) \\ \mathcal{B}_{22} &= (e_1^T \otimes I) \\ \mathcal{B}_{33} &= (e_1^T \otimes I) \end{aligned} \tag{3.231}$$

Boundary conditions at $\eta = 0$

Clamped ($\phi_x = \phi_y = w = 0$)

$$\mathcal{B}_{11} = (I \otimes e_1^T) \qquad \mathcal{B}_{22} = (I \otimes e_1^T) \qquad \mathcal{B}_{33} = (I \otimes e_1^T) \tag{3.232}$$

Simply-supported ($\phi_x = M_{yy} = w = 0$)

$$\begin{aligned} \mathcal{B}_{11} &= (I \otimes e_1^T) \\ \mathcal{B}_{21} &= D_{12}(D^{(1)} \otimes e_1^T) - c_1 F_{12}(D^{(1)} \otimes e_1^T) \\ \mathcal{B}_{22} &= \beta D_{22}(I \otimes e_1^T D^{(2)}) - c_1 F_{22}(I \otimes e_1^T D^{(1)}) \\ \mathcal{B}_{23} &= -c_1 F_{12} \delta (D^{(2)} \otimes e_1^T) - c_1 F_{22} \gamma (I \otimes e_1^T D^{(2)}) \\ \mathcal{B}_{33} &= (I \otimes e_1^T) \end{aligned} \tag{3.233}$$

For corner points at interaction of two free edges, there is no special condition at the angles like classical plate theory and similar to first-order shear deformation theory of plate fully free condition is like fully simply-supported boundary condition.

Table (3.9) is shown natural frequencies for annular thin plate for many boundary conditions in terms of β which inner radius is divided by total radius. Increasing β for all boundaries cause to growing the natural frequencies and results compared with GDQ method.

Results for isotropic thick plate and considering many boundary conditions are tabulated in table (3.10). Presented results are compared with the results obtained with boundary characteristics orthogonal polynomials.

Table (3.11) is shown natural frequency of non-symmetric composite laminated moderately thick plate in the case of fully clamped and fully simply supported and results are compared with Ritz method.

Table (3.12) is shown natural frequency of symmetric composite laminated moderately thick plate with constrained boundaries in terms of different stiffness coefficients and results are compared with GDQ method. For constant elastic restrained coefficients with increasing the thickness of plate, natural frequencies decreased and with in the case of constant thickness, natural frequencies increased as elastic restrained coefficients is increased.

In table (3.13) results for non-symmetric composite laminated thick and thin plates for simply supported condition are evaluated for both cross-ply and angle-ply laminated and are compared with Navier solution. Similar to previous table with increasing the thickness of plate, natural frequency decreased.

Natural frequencies with all elastic restrained boundaries and not all edges constrained for non-symmetric both cross-ply and angle-ply lay-up are tabulated in tables (3.14) and (3.15), respectively. For all boundaries,lay-up and thickness, increasing the constrained coefficients cause to increasing the natural frequencies and for constant stiffness coefficients with increasing the thickness, frequencies decreased.

Table (3.16) is shown frequencies of non-symmetric composite laminated thick plate in the case of fully free condition for both cross-ply and angle-ply laminated. Frequencies increased as the number of layers of composite is increased but as seen after six layers there is no any significant change on frequencies.

Results for isotropic thick circular plate of all boundaries are tabulated in table (3.17). Results are compared with pseudospectral [99] and also differential quadrature [98] method. Similar to previous understanding with increasing the thickness, frequencies decreased.

In table (3.18) results for isotropic annular thick plate and all boundary conditions are tabulated and compared with GDQ method and the same results tell us frequency increased as thickness is decreased.

Results for isotropic sector annular thick plate for all boundaries are shown in table (3.19) and compared with GDQ method. As seen they are in a good agreement.

Results for in-plane vibration of rectangular plate for many boundary condition are tabulated in tables (3.20) and (3.21) and compared with Trigonometric Ritz [69] method and analytical solution [68], respectively.

Results for isotropic thick skew plate subjected to fully clamped and fully simply-supported are tabulated in table (3.22) and compared with differential quadrature method and with increasing the skew angle, natural frequencies increased.

Table (3.23) is included results of in-plane isotropic skew plate for fully clamped, CCCF and CFFF which are compared with modified Ritz method and with increasing the skew angle, natural frequencies get increased.

Table (3.24) and (3.25) are included results for isotropic in-plane skew plate for many boundary conditions. Generally, natural frequencies with increasing of skew angle, increased but in the case of simply supported type-1, fundamental frequency decreased, and also for fully free most of them but not all of the frequencies decreased. For the cases of S1CS1C and S2FS2F also fundamental frequency get decreased and for S1CS1F case second frequency get decreased. But in some modes it is oscillated, it means that at first it is decreasing and after that it is increasing or vice-versa.

Results for isotropic in-plane sector annular plate for fully clamped, CCCF and CFCF conditions are shown in table (3.26) and results are compared with modified Ritz method and also with increasing the angle of sector annular plate, natural frequencies decreased.

Table 3.9: Non-dimensional natural frequencies of the Kirchhoff annular plate, $\lambda = \omega R^2\sqrt{\frac{\rho h}{D}}$

BCs	Author	β	ω_1	ω_2	ω_3	ω_4	ω_5
CC	Present	0.1	27.281	75.366	148.21	245.48	367.18
	[98]		27.280	75.364	148.21	245.34	367.14
	Present	0.5	89.251	246.34	483.22	799.02	1193.8
	[98]		89.248	246.33	483.16	798.89	1193.5
CS	Present	0.1	17.789	60.144	126.88	218.06	333.65
	[98]		17.789	60.143	126.88	218.05	333.63
	Present	0.5	59.820	198.05	415.16	711.23	1086.3
	[98]		59.819	198.04	415.12	711.12	1086.0
SC	Present	0.1	22.701	65.639	132.90	224.42	340.28
	[98]		22.701	65.538	132.89	224.41	340.26
	Present	0.5	63.973	202.07	419.23	715.33	1090.4
	[98]		63.972	202.06	419.19	715.22	1090.1
SS	Present	0.1	14.485	51.781	112.99	198.45	308.23
	[98]		14.485	51.781	112.99	198.44	308.21
	Present	0.5	40.043	158.64	356.09	632.46	987.79
	[98]		40.043	158.64	356.06	632.39	987.60
FF	Present	0.1	8.7745	38.236	89.026	162.85	260.49
	[98]		8.7745	38.235	89.025	162.85	260.48
	Present	0.5	9.3135	92.308	249.39	486.20	801.96
	[98]		9.3135	92.306	249.37	486.16	801.85
CF	Present	0.1	4.2374	25.262	73.901	146.69	243.96
	[98]		4.2374	25.262	73.899	146.69	243.94
	Present	0.5	13.024	85.033	243.69	480.45	796.25
	[98]		13.024	85.031	243.68	480.40	796.12
FC	Present	0.1	10.159	39.521	90.445	164.31	261.98
	[98]		10.159	39.521	90.443	164.30	261.96
	Present	0.5	17.715	93.847	252.19	488.94	804.73
	[98]		17.714	93.845	252.18	488.89	804.59
SF	Present	0.1	3.4497	20.889	64.202	131.40	222.91
	[98]		3.4497	20.889	64.201	131.40	222.90
	Present	0.5	4.1210	61.009	199.35	416.47	712.56
	[98]		4.1210	61.008	199.34	416.44	712.47
FS	Present	0.1	4.8532	29.438	74.823	142.84	234.51
	[98]		4.8533	29.438	74.822	142.84	234.50
	Present	0.5	5.0768	65.842	203.84	420.90	716.94
	[98]		5.0769	65.841	203.83	420.87	716.85

Table 3.10: Non-dimensional natural frequencies of the square isotropic Mindlin plate, $\lambda = \omega a^2\sqrt{\frac{\rho h}{D}}$

BCs	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
SSSS	Present	1.9310	4.6048	4.6048	7.0637	8.6048	8.6048	10.792	10.792
	[125]	1.931	4.605	4.605	7.064	8.605	8.605	10.792	10.792
CCCC	Present	3.2918	6.2755	6.2755	8.7924	10.355	10.454	12.522	12.522
	[125]	3.292	6.276	6.276	8.792	10.356	10.455	12.523	12.523
SCSC	Present	2.6996	4.9710	5.9899	7.9722	8.7866	10.248	11.332	12.022
	[125]	2.7	4.971	5.990	7.972	8.787	10.249	11.333	12.023
CFCF	Present	2.0883	2.4313	3.9012	5.3304	5.7711	6.9288	7.2916	9.6027
	[125]	2.088	2.431	3.901	5.331	5.771	6.929	7.292	9.603
SSFF	Present	0.3328	1.6773	1.8743	3.5568	4.7184	4.9451	6.4716	6.6308
	[125]	0.333	1.677	1.874	3.557	4.718	4.945	6.472	6.631
CFFF	Present	0.3475	0.8163	2.0343	2.5824	2.8595	4.8110	5.4769	5.7717
	[125]	0.348	0.816	2.034	2.582	2.860	4.811	5.477	5.772
FFFF	Present	1.2882	1.9190	2.3626	3.2325	3.2325	5.6043	5.6043	5.6397

Table 3.11: Non-dimensional natural frequencies of the non-symmetric composite laminated Mindlin plate, $h/a = 0.1, \lambda = \omega a^2 h/\pi^2\sqrt{\frac{\rho}{E_2}}$

Lay-up	BCs	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
$(0/90)_2$	CCCC	Present	2.3946	3.9504	3.9558	5.0663	5.9660	5.9771	6.7673
		[116]	2.3947	3.9532	3.9532	5.0665	5.9680	5.9757	6.7679
	SSSS1	Present	1.5118	2.4652	2.4652	3.3890	3.3917	4.5677	4.9285
		[116]	1.5119	2.4656	2.4656	3.3904	3.3904	4.5679	4.9312
$(45/\text{-}45)_2$	CCCC	Present	2.2963	3.8894	3.8942	5.3041	5.7445	5.8193	6.9602
		[116]	2.2964	3.8919	3.8919	5.3042	5.7449	5.8196	6.9647
	SSSS1	Present	1.9169	3.4859	3.5148	3.5297	5.1349	5.4749	5.5249
		[116]	1.9171	3.4869	3.5290	3.5290	5.1351	5.4942	5.5251

Table 3.12: Non-dimensional natural frequencies of the Mindlin composite plate $(0, 90, ...)$ with elastic edges, $\lambda = \omega a^2\sqrt{\rho h/D_{11}}$

BCs	Layers	K	δ	Author	ω_1	ω_2	ω_3	ω_4	ω_5
$(E_{RT})_4$	9	100	0.2	Present	1.1235	1.7134	1.7635	2.1908	2.5322
				[115]	1.1236	1.7134	1.7636	2.1909	2.5323
		100	0.001	Present	1.8095	2.5702	2.7234	3.3060	4.6721
				[115]	1.8096	2.5703	2.7234	3.3061	4.6721
$(\text{S-}E_R)_2$	5	10	0.1	Present	10.867	19.381	22.796	27.992	30.177
				[115]	10.867	19.382	22.796	27.992	30.178
		100	0.1	Present	11.141	19.656	22.922	28.176	30.356
				[115]	11.142	19.657	22.923	28.176	30.357

Table 3.13: Non-dimensional natural frequencies of the Mindlin laminated composite plate, simply supported, $E_1/E_2 = 40, \lambda = \omega a^2/h\sqrt{\rho/E_2}$

Lay-up	h/a	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0/90	0.1	Present	10.464	24.236	24.236	25.393	25.712	35.237
		[111]	10.473					
	0.01	Present	11.298	31.428	31.439	45.080	67.225	67.286
		[111]	11.299					
45/-45	0.1	Present	13.027	26.786	26.962	34.224	41.190	44.370
		[111]	13.043					
	0.01	Present	14.618	33.726	33.731	58.239	62.699	62.699
		[111]	14.618					

Table 3.14: Non-dimensional natural frequencies of non-symmetric composite laminated Mindlin plate $(0/90)_3$, $\lambda = \omega a^2/\pi^2 h\sqrt{\frac{\rho}{E_2}}$

Lay-up	K	δ	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
EEEE	10	0.1	2.4286	3.9756	3.9756	5.0844	5.9750	5.9820	6.7734	6.7734
		0.2	1.4048	2.2107	2.2107	2.7991	3.2203	3.2215	3.6512	3.6512
	100	0.1	2.4414	4.0065	4.0065	5.1266	6.0274	6.0343	6.8330	6.8330
		0.2	1.4256	2.2489	2.2489	2.8487	3.2755	3.2768	3.7147	3.7147
ESES	10	0.1	2.0403	2.4242	3.6865	3.7602	4.8499	4.8686	5.8328	5.8424
		0.2	1.1932	1.2834	2.1413	2.1883	2.3890	2.7818	3.1756	3.2125
	100	0.1	2.0493	2.4612	3.7046	3.7894	4.9013	4.9225	5.8688	5.8936
		0.2	1.2285	1.2997	2.1768	2.2195	2.4570	2.8262	3.2289	3.2619
ECEC	10	0.1	2.4357	3.9801	4.0055	5.1079	5.9813	6.0339	6.7911	6.8221
		0.2	1.4164	2.2181	2.2459	2.8268	3.2260	3.2773	3.6725	3.7008
	100	0.1	2.4421	4.0069	4.0095	5.1289	6.0294	6.0381	6.8348	6.8379
		0.2	1.4268	2.2497	2.2525	2.8515	3.2766	3.2820	3.7168	3.7197
EFEF	10	0.1	1.7131	1.7369	2.3491	2.9520	3.5749	3.6120	4.3622	4.7446
		0.2	0.9927	1.0086	1.1551	1.9722	1.9810	2.0049	2.3428	2.6332
	100	0.1	1.7221	1.7460	2.4000	2.9577	3.6047	3.6414	4.3868	4.8056
		0.2	1.0074	1.0227	1.1961	1.9885	2.0076	2.0385	2.3984	2.6592

Table 3.15: Non-dimensional natural frequencies of non-symmetric composite laminated Mindlin plate $(45/-45)_3$, $\lambda = \omega a^2/\pi^2 h\sqrt{\frac{\rho}{E_2}}$

Lay-up	K	δ	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
EEEE	10	0.1	2.3322	3.9227	3.9227	5.3242	5.7618	5.8386	6.9677	6.9677
		0.2	1.3822	2.2071	2.2071	2.8852	3.1587	3.1916	3.6988	3.6990
	100	0.1	2.3445	3.9507	3.9507	5.3641	5.8117	5.8867	7.0241	7.0241
		0.2	1.4022	2.2441	2.2441	2.9322	3.2148	3.2464	3.7608	3.7608
ESES	10	0.1	2.1249	3.6225	3.8993	5.2450	5.5886	5.7999	6.8611	6.9729
		0.2	1.3224	2.1724	2.2144	2.8932	3.1743	3.1966	3.6346	3.7182
	100	0.1	2.1334	3.6326	3.9215	5.2673	5.5965	5.8432	6.8839	7.0110
		0.2	1.3348	2.1823	2.2436	2.9178	3.2031	3.2240	3.7394	3.7606
EFEF	10	0.1	1.5511	1.9653	3.0876	3.2807	3.8148	4.8872	5.0096	5.2616
		0.2	0.9373	1.2193	1.8760	1.9636	2.1732	2.8280	2.8830	2.9345
	100	0.1	1.5578	1.9741	3.0972	3.3024	3.8353	4.8959	5.0314	5.3048
		0.2	0.9496	1.2310	1.9075	1.9739	2.2000	2.8528	2.9602	2.9865
ECEC	10	0.1	2.3390	3.9301	3.9464	5.3464	5.7843	5.8706	6.9891	7.0091
		0.2	1.3933	2.2161	2.2394	2.9115	3.1770	3.2352	3.7213	3.7457
	100	0.1	2.3451	3.9515	3.9531	5.3663	5.8145	5.8895	7.0263	7.0283
		0.2	1.4033	2.2450	2.2474	2.9349	3.2178	3.2497	3.7631	3.7656

Table 3.16: Non-dimensional natural frequencies of fully-free non-symmetric composite laminated Mindlin plate, $\lambda = \omega a^2/\pi^2 h\sqrt{\frac{\rho}{E_2}}$

Lay-up	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
(0/90)	4.8784	15.032	15.139	17.334	17.334	26.271	30.907	35.490
$(0/90)_2$	4.9049	22.315	22.451	23.721	23.721	31.134	34.123	46.179
$(0/90)_3$	4.9068	23.225	23.363	24.549	24.549	31.146	35.192	46.232
$(0/90)_4$	4.9074	23.523	23.662	24.822	24.822	31.149	35.546	46.248
$(0/90)_5$	4.9076	23.658	23.797	24.945	24.945	31.151	35.707	46.256
(45/-45)	7.5760	10.175	13.514	19.945	19.945	27.977	27.977	30.864
$(45/\text{-}45)_2$	7.7868	14.823	19.538	25.587	25.588	34.414	36.036	36.382
$(45/\text{-}45)_3$	7.8031	15.409	20.327	26.222	26.222	34.414	36.530	37.351
$(45/\text{-}45)_4$	7.8081	15.601	20.588	26.428	26.428	34.414	36.687	37.666
$(45/\text{-}45)_5$	7.8104	15.688	20.707	26.520	26.520	34.414	36.757	37.808
(30/-30)	6.9490	9.1825	15.756	18.348	18.445	23.455	28.765	31.448
$(30/\text{-}30)_2$	7.1845	13.391	21.041	21.928	24.244	31.103	31.540	33.396
$(30/\text{-}30)_3$	7.2099	13.940	21.384	22.688	24.934	31.511	32.028	33.397
$(30/\text{-}30)_4$	7.2180	14.122	21.499	22.938	25.158	31.519	32.312	33.398
$(30/\text{-}30)_5$	7.2217	14.205	21.551	23.050	25.259	31.521	32.440	33.398

Table 3.17: Non-dimensional natural frequencies of the Mindlin circular plate, $\lambda = \omega R^2 \sqrt{\frac{\rho h}{D}}$

BCs	Author	h/a	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
C	Present	0.05	10.144	38.855	84.995	146.40	220.72	305.71	399.31
	[99]		10.145	38.855	84.995	146.40	220.73	305.71	399.32
	Present	0.1	9.9408	36.478	75.664	123.31	176.41	232.96	291.70
	[99]		9.9408	36.479	75.664	123.32	176.42	232.97	291.71
	Present	0.15	9.6286	33.393	65.550	102.08	140.93	180.98	221.62
	[99]		9.6286	33.393	65.551	102.09	140.93	180.99	221.62
	Present	0.2	9.2400	30.210	56.682	85.571	115.55	145.94	174.97
	[99]		9.2400	30.211	56.682	85.571	115.56	145.94	174.97
	Present	0.25	8.8068	27.252	49.420	73.054	97.197	117.90	122.42
	[99]		8.8068	27.253	49.420	73.054	97.198	117.90	122.43
S	Present	0.05	4.9247	29.323	71.756	130.34	202.80	286.78	380.12
	[99]		4.9247	29.323	71.756	130.35	202.81	286.79	380.13
	Present	0.1	4.8938	28.240	65.942	113.57	167.53	225.34	285.43
	[99]		4.8938	28.240	65.942	113.57	167.53	225.34	285.44
	Present	0.15	4.8440	26.714	59.062	96.774	136.97	178.23	219.85
	[99]		4.8440	26.715	59.062	96.775	136.98	178.23	219.86
	Present	0.2	4.7773	24.994	52.513	82.765	113.87	145.12	166.29
	[99]		4.7773	24.994	52.514	82.766	113.87	145.13	166.29
	Present	0.25	4.6963	23.254	46.774	71.603	96.609	108.27	121.49
	[99]		4.6963	23.254	46.775	71.603	96.609	108.27	121.50
F	Present	0.05	8.9686	37.787	84.442	146.75	222.37	308.97	404.44
	[99]		8.9686	37.787	84.443	146.76	222.38	308.98	404.44
	Present	0.1	8.8679	36.040	76.675	126.27	181.46	239.98	300.38
	[99]		8.8697	36.041	76.676	126.27	181.46	239.98	300.38
	Present	0.15	8.7095	33.674	67.827	106.39	146.83	187.79	228.38
	[99]		8.7095	33.674	67.827	106.40	146.83	187.79	228.39
	Present	0.2	8.5051	31.110	59.645	90.059	120.56	149.62	171.18
	[99]		8.5051	31.111	59.645	90.059	120.57	149.63	171.18
	Present	0.25	8.2674	28.605	52.584	76.935	99.545	114.52	126.34
	[99]		8.2674	28.605	52.584	76.936	99.545	114.53	126.34

Table 3.18: Non-dimensional natural frequencies of the Mindlin annular plate, $\beta = 0.5, \lambda = \omega R^2\sqrt{\frac{\rho h}{D}}$

BCs	Author	h/a	ω_1	ω_2	ω_3	ω_4	ω_5
CS	Present	0.1	51.219	142.71	252.22	369.85	490.92
	[95]		51.219	142.71	252.22	369.86	490.92
	Present	0.2	38.363	93.780	152.96	173.62	213.65
	[95]		38.363	93.781	152.96	173.63	213.66
CC	Present	0.1	70.277	159.78	265.44	378.42	496.03
	[95]		70.277	159.78	265.44	378.42	496.03
	Present	0.2	48.310	97.388	155.46	196.79	220.19
	[95]		48.310	97.389	155.47	196.79	220.20
SS	Present	0.1	37.326	127.17	240.54	362.48	487.00
	[95]		37.326	127.17	240.54	362.48	487.01
	Present	0.2	31.871	90.636	152.76	162.98	201.89
	[95]		31.871	90.637	152.76	162.99	201.89
SC	Present	0.1	55.090	145.82	254.73	371.67	492.22
	[95]		55.090	145.82	254.73	371.67	492.23
	Present	0.2	41.623	95.269	154.24	174.52	214.77
	[95]		41.624	95.269	154.24	174.52	214.78
SF	Present	0.1	4.0915	55.120	153.61	270.05	391.95
	[95]		4.0915	55.120	153.62	270.05	391.95
	Present	0.2	4.0077	44.644	104.94	156.56	178.69
	[95]		4.0077	44.644	104.95	156.57	178.70
FS	Present	0.1	5.0321	59.530	157.06	273.13	394.78
	[95]		5.0321	59.531	157.06	273.13	394.78
	Present	0.2	4.9063	48.561	107.48	159.81	176.41
	[95]		4.9063	48.562	107.48	159.81	176.41
CF	Present	0.1	12.567	69.583	168.72	280.47	398.43
	[95]		12.568	69.583	168.73	280.47	398.44
	Present	0.2	11.461	49.270	108.36	157.40	196.40
	[95]		11.461	49.270	108.37	157.40	196.40
FC	Present	0.1	17.023	77.238	174.89	285.68	402.83
	[95]		17.024	77.239	174.90	285.69	402.84
	Present	0.2	15.397	55.273	112.29	160.61	197.53
	[95]		15.397	55.274	112.29	160.61	197.53
FF	Present	0.1	9.1019	81.031	184.33	302.86	422.94
	[95]		9.1019	81.031	184.33	302.87	422.94
	Present	0.2	8.5531	64.011	117.99	174.18	180.85
	[95]		8.5531	64.011	118.00	174.18	180.86

Table 3.19: Non-dimensional natural frequencies of the sector annular mindlin plate, $\beta = 0.25, \alpha = 120, \delta = 0.2, \lambda = \omega R^2\sqrt{\frac{\rho h}{D}}$

BCs	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
CCCC	Present	31.056	41.813	55.950	62.419	71.127	72.862
	[105]	31.056	41.814	55.951	62.420	71.127	72.862
SSSS	Present	20.611	32.633	48.185	54.668	64.697	66.211
	[105]	20.612	32.633	48.186	54.668	64.697	66.211
CSCS	Present	29.161	38.136	52.247	61.628	67.946	71.004
	[105]	29.161	38.137	52.248	61.629	67.946	71.004
CFCF	Present	26.782	28.096	34.203	46.051	58.043	60.249
	[105]	26.783	28.096	34.204	46.051	58.044	60.249
FCSC	Present	20.779	35.808	42.009	51.555	64.148	67.563
	[105]	20.780	35.808	42.010	51.556	64.148	67.564
SCFC	Present	7.4836	16.412	27.889	29.702	40.878	43.775
	[105]	7.484	16.412	27.889	29.703	40.878	43.776
SSCC	Present	27.7041	39.385	54.047	59.945	69.559	70.877
	[105]	27.704	39.385	54.947	59.945	69.560	70.878
FFFF	Present	9.5465	14.512	20.773	29.123	33.611	36.328
FCFC	Present	7.3338	16.372	25.439	28.105	40.847	42.954
SCSC	Present	29.161	38.136	52.247	61.628	67.946	71.004

Table 3.20: Non-dimensional natural frequencies of the in-plane vibration of the isotropic classical plate theory, $\lambda = \omega a\sqrt{\frac{\rho(1-\nu^2)}{E}}$

BCs	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
SSSS1	Present	0.9292	0.9292	1.3142	1.8585	1.8585	2.0779	2.0779	2.2214
SSSS2	Present	1.3142	1.5707	1.5707	2.0779	2.0779	2.2214	2.6284	2.9386
	[69]	1.3142	1.5708	1.5708	2.0780	2.0780	2.2214	2.6285	2.9387
CCCC	Present	3.7268	3.7268	4.4394	5.4360	6.1415	6.1790		
	[69]	3.7269	3.7269	4.4395	5.4361	6.1415	6.1790		
CFCF	Present	1.7748	3.1635	3.2710	3.5125	3.9239	4.0908		
	[69]	1.7751	3.1640	3.2711	3.5127	3.9248	4.0909		

Table 3.21: Non-dimensional natural frequencies of the in-plane vibration of the isotropic plate, $\lambda = \omega a\sqrt{\frac{\rho(1-\nu^2)}{E}}$

BCs	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
FFFF	Present	2.3191	2.4707	2.4707	2.6299	2.9862	3.4280	3.6817	3.7187
	[68]	2.3206	2.4716	2.4716	2.6284	2.9874	3.4522	3.7232	3.7232
CCCC	Present	3.5552	3.5552	4.2350	5.1857	5.8586	5.8944	5.8944	6.7077
	[68]	3.5552	3.5552	4.2350	5.1859	5.8587	5.8946	5.8946	6.7080
S2FS2F	Present	1.4075	2.6284	3.1415	3.2487	3.3639	3.4987	3.7326	5.0348
	[68]	1.407	2.628	3.142	3.248	3.363	3.498		
S1CS1C	Present	1.8586	3.2752	3.4946	3.7172	4.4105	4.9572	5.5758	5.6212
	[68]	1.858	3.275	3.494	3.718	4.411	4.957		
S1FS1F	Present	1.4075	1.8586	2.6285	3.2486	3.3642	3.4988	3.7146	3.7329
	[68]	1.408	1.858	2.629	3.249	3.364	3.499		
S2CS2C	Present	3.1416	3.2752	3.4946	4.4105	4.9572	5.6212	5.6289	5.9978
	[68]	3.140	3.275	3.494	4.411	4.957	5.622		
SSSS1	Present	1.8586	1.8586	2.6284	3.7172	3.7172	4.1559	4.1559	4.4429
SSSS2		2.6284	3.1416	3.1416	4.1559	4.1559	4.4429	5.2569	5.8774
CFCF		1.6930	3.0178	3.1204	3.3507	3.7432	3.9024	5.0906	5.0937
CCCF		2.2698	3.1630	3.4089	4.3058	4.7181	4.9576	5.2057	5.9622
CCFF		1.4698	1.8839	2.2545	3.4024	3.8428	4.2320	4.3286	4.9546
CFFF		0.6278	1.5068	1.6904	2.6862	2.8967	3.0744	3.8751	4.0796
S1CS1F		0.9293	1.9775	2.7879	3.0615	3.4391	4.1624	4.4610	4.6465
S1CS2F		1.5984	1.7596	2.6500	3.2377	4.1247	4.2687	4.3881	5.1143

Table 3.22: Non-dimensional natural frequencies of the isotropic skew Mindlin plate, $h/a = 0.2, \lambda = \omega a^2\sqrt{\frac{\rho h}{D}}$

BCs	angle	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
CCCC	15	Present	2.8057	4.6296	5.0962	6.3068	7.4051	7.7177
		[87]	2.8058	4.6298	5.0963	6.3070	7.4052	7.7179
	45	Present	4.1588	5.9019	7.5420	7.7903	9.2156	10.091
		[87]	4.1590	5.9021	7.5422	7.7907	9.2159	10.0921
SSSS	15	Present	1.8610	3.7920	4.2803	5.5878	6.8437	7.0728
		[87]	1.8560	3.7856	4.2763	5.5784	6.8385	7.0702
	45	Present	2.9606	4.9670	6.7741	7.0336	8.6006	9.6533
		[87]	2.9129	4.8736	6.6622	7.0148	8.4831	9.5878

Table 3.23: Non-dimensional natural frequencies for the in-plane vibration of skew plate with $\nu = 0.3, \lambda = \omega a^2/\pi\sqrt{\rho/G}$

Author	α	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
				CCCC			
Present	0	3.7268	3.7268	4.4394	5.4360	6.1415	6.1790
[67]		3.7272	3.7270	4.4395	5.4363	6.1601	6.1741
Present	15	3.6813	3.9741	4.5708	5.4860	6.1667	6.2887
[67]		3.6814	3.9743	4.5708	5.4862	6.1674	6.2888
Present	30	3.8497	4.4855	5.0125	5.6949	6.5249	6.7837
[67]		3.8498	4.4861	5.0125	5.6953	6.5262	6.7847
Present	45	4.3406	5.4574	5.9590	6.2328	7.4378	7.8430
[67]		4.3407	5.4588	5.9608	6.2334	7.4494	7.9611
Present	60	5.5368	7.5134	7.5419	8.0486	9.3356	9.5452
[67]		5.5372	7.5428	7.5231	8.0737	9.3394	9.7038
				CCCF			
Present	0	2.3794	3.3156	3.5735	4.5137	4.9459	5.1969
[67]		2.3801	3.3165	3.5742	4.5148	4.9459	5.1970
Present	15	2.4396	3.3871	3.6721	4.5570	5.0908	5.2670
[67]		2.4390	3.3865	3.6678	4.5572	5.0827	5.2633
Present	30	2.6430	3.6250	4.0024	4.7483	5.4922	5.6177
[67]		2.6360	3.6174	3.9792	4.7449	5.1358	5.6158
Present	45	3.0850	4.1273	4.6995	5.2410	6.2092	6.4920
[67]		3.0521	4.1027	4.6204	5.2293	6.1184	6.4821
Present	60	4.0907	5.2175	6.1764	6.4591	7.9026	7.9026
[67]		3.9428	5.1716	5.9077	6.4405	7.6716	7.9042
				CFCF			
Present	0	1.7748	3.1635	3.2710	3.5125	3.9239	4.0908
[67]		1.7758	3.1653	3.2713	3.5130	3.9266	4.0912
Present	15	1.8259	3.2316	3.3224	3.6053	4.0046	4.1488
[67]		1.8283	3.2360	3.3225	3.6059	4.0097	4.1498
Present	30	1.9966	3.4587	3.5191	3.8485	4.3172	4.3553
[67]		2.0040	3.4737	3.5010	3.8486	4.3282	4.3569
Present	45	2.3576	3.9373	3.9726	4.2483	4.8266	5.0414
[67]		2.3776	3.7975	3.9587	4.2501	4.8358	5.0693
Present	60	3.1366	4.9109	4.9656	5.1440	5.9045	6.4776
[67]		3.1955	4.9137	5.0550	5.1719	5.9619	6.5640

Table 3.24: Non-dimensional natural frequencies for the in-plane vibration of skew plate with $\nu = 0.3, \lambda = \omega a^2/\pi\sqrt{\rho/G}$

α	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
				SSSS1				
0	1.9483	1.9483	2.7553	3.8966	3.8966	4.3566	4.3566	4.6574
15	1.7765	2.2028	2.7021	3.9848	4.0407	4.0805	4.7341	4.7887
30	1.6577	2.5977	2.5977	3.7966	4.2787	4.4994	5.1955	5.1955
45	1.5656	2.5015	3.2660	3.6161	4.5576	4.8953	5.6644	5.9112
60	1.4493	2.4314	3.4742	4.4500	4.6022	5.4752	6.2274	6.4465
				SSSS2				
0	2.7553	3.2932	3.2932	4.3566	4.3566	4.6574	5.5107	6.1611
15	2.8436	3.3269	3.4238	4.3344	4.6276	4.6493	5.5728	6.2732
30	3.1405	3.4723	3.9242	4.4413	4.6888	5.2540	5.8974	6.5439
45	3.7678	3.8789	4.8353	4.8353	4.9535	6.4323	6.5963	7.2400
60	4.8649	5.4973	5.7245	6.6567	6.6567	6.7986	8.0784	8.0784
				FFFF				
0	2.4352	2.5896	2.5896	2.7537	3.1378	3.5965	3.9040	3.9277
15	2.1468	2.4157	2.8105	2.9878	3.1263	3.7394	3.8519	3.8692
30	2.0448	2.1913	3.2423	3.2423	3.2464	3.2464	3.7899	3.8225
45	1.3875	2.0910	2.4129	2.9901	3.7649	3.8327	3.8747	4.7259
60	0.9519	1.4901	2.2682	2.4089	2.9856	3.8195	3.8525	4.6185

Table 3.25: Non-dimensional natural frequencies for the in-plane vibration of skew plate with $\nu = 0.3, \beta = 0.8, \lambda = \omega a^2/\pi\sqrt{\rho/G}$

α	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
				S1CS1C				
0	0.8333	1.5406	1.6667	1.7846	2.2493	2.3915	2.5000	2.5735
15	0.8627	1.5747	1.6831	1.8534	2.3007	2.3830	2.5938	2.6352
30	0.9593	1.6890	1.7701	2.0379	2.4494	2.4897	2.7643	2.8328
45	1.1577	1.9312	1.9915	2.3484	2.6697	2.9271	3.0215	3.2315
60	1.5603	2.4587	2.5030	2.9391	3.2167	3.5651	3.7280	4.0222
				S2FS2F				
0	0.7974	1.2751	1.4086	1.7284	1.7601	1.7804	1.9249	2.4207
15	0.7830	1.2828	1.4608	1.7856	1.7856	1.8590	1.8590	2.4636
30	0.7381	1.3036	1.6432	1.8894	1.8894	1.9086	1.9086	2.5908
45	0.1294	2.2771	2.2771	2.5649	2.9993	2.9993	3.0479	3.0479
60	0.2436	1.4130	2.2030	2.2030	2.5996	2.5996	3.0545	3.3464
				S1CS1F				
0	0.4167	0.9970	1.2500	1.5900	1.8388	2.0492	2.0783	2.0833
15	0.4242	1.0087	1.2458	1.6186	1.8930	1.9766	2.1152	2.2171
30	0.4453	1.0242	1.2674	1.6990	1.9304	2.0632	2.2650	2.4019
45	0.4719	0.9903	1.4057	1.7668	2.0260	2.2597	2.5267	2.7489
60	0.4719	0.9018	1.5691	1.8828	2.3400	2.4711	2.8653	3.0948
				S1CS2F				
0	0.7359	0.9029	1.3971	1.5252	1.9594	2.0398	2.2923	2.4121
15	0.7246	0.9740	1.3979	1.5857	2.0082	2.0590	2.3224	2.4456
30	0.7434	1.0870	1.4448	1.7314	2.0335	2.2563	2.3575	2.7027
45	0.7824	1.2939	1.6250	2.0323	2.0323	2.5534	2.5534	3.0001
60	0.8241	1.7592	2.1774	2.1774	2.4850	3.0661	3.0661	3.5670

Table 3.26: Non-dimensional natural frequencies for the in-plane vibration of sector annular plate with $\nu = 0.33, \beta = 0.5, \lambda = \omega a^2\sqrt{\rho/E}$

Author	α	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
				CCCC			
Present	30	8.1585	8.8369	10.1084	12.7629	13.2113	13.3264
[67]		8.1592	8.8379	10.1089	12.7648	13.2623	13.2555
Present	45	6.6082	7.2710	8.1844	10.3453	10.5967	10.8608
[67]		6.6113	7.2730	8.1863	10.3517	10.6104	10.8649
Present	60	5.5885	6.9190	7.4246	8.6194	9.0236	9.2407
[67]		5.5963	6.6935	7.4305	8.6325	9.0380	9.2545
Present	75	5.0466	6.6967	7.0530	7.4496	8.0645	8.5850
[67]		5.0629	6.7002	7.0684	7.4864	8.0930	8.6033
Present	90	4.7277	6.3392	6.8381	6.9152	7.5183	8.0357
[67]		4.7576	6.3458	6.9512	6.9793	7.5749	8.0831
				CCCF			
Present	30	5.3142	6.8471	7.4536	10.2267	10.7836	11.5196
[67]		5.3158	6.8500	7.4560	10.2296	10.7879	11.5210
Present	45	4.4903	6.5498	6.9076	8.5761	8.5973	9.3191
[67]		4.4936	6.5548	6.9123	8.5881	8.5840	9.3230
Present	60	4.1863	6.3407	6.7048	7.1367	7.7893	8.1870
[67]		4.1930	6.3518	6.7162	7.1459	7.8030	8.1970
Present	75	4.0493	5.8818	6.5864	6.6484	7.2869	7.6837
[67]		4.0631	5.9032	6.6534	6.6604	7.3121	7.7075
Present	90	3.9788	5.3974	6.4451	6.5854	6.9786	7.3158
[67]		4.0055	5.4345	6.5046	6.6321	7.0237	7.3563
				CFCF			
Present	30	3.4462	6.2267	6.4023	8.3276	8.3819	9.4086
[67]		3.4489	6.2700	6.4037	8.3676	8.3358	9.4104
Present	45	3.6081	5.8848	6.4608	6.6792	7.5550	7.6660
[67]		3.6116	5.8880	6.4635	6.6862	7.5640	7.6711
Present	60	3.6847	5.1754	6.3967	6.4777	7.0244	7.2447
[67]		3.6912	5.1823	6.4017	6.4919	7.0363	7.2645
Present	75	3.7292	4.7201	6.2648	6.4724	6.7256	7.0426
[67]		3.7422	4.7354	6.3011	6.5000	6.7543	7.0761
Present	90	3.7583	4.4493	6.0012	6.4907	6.5517	6.7654
[67]		3.7834	4.4797	6.0711	6.5393	6.5952	6.7786

Chapter 4

Shells

Shells are three-dimensional bodies bounded by two surfaces or bounded by four surfaces if it is shell panel. The 3D equations of elasticity are complicated when written in curvilinear or shell coordinates. All shell theories such as thin, thick, deep and shallow reduce 3D elasticity problem into the 2D theory. This is done by eliminating the coordinates normal to the shell surface in the shell equations. The accuracy of thin and thick theories are assessed in compared with the 3D theory. In this chapter, the equations of thin, thick shell both deep and shallow are discussed.

A shell is confined by two parallel (unless the thickness is varying) surfaces. Generally, the distance between top and bottom surfaces of shell which is thickness is small compared with other shell parameters such as length, width and radii of curvature.

4.1 Thin shells

If the shell thickness is less than 1/20 of the wavelength of the deformation and/or radii of curvature and shear deformation and rotary inertia are negligible, a thin shell theory is acceptable. Depending on various assumptions considered for derivation of the strain-displacements relations, stress-strain relations and equilibrium equations, many thin shell theories can be derived. Among the most common of these Love, Reissner, Naghdi, Sanders and Flugge are well known shell theories. For thin shells, some additional assumptions can simplify the equations: first one is neglecting the term including z/R because it is negligible in compared with unity and the second one is neglecting the shear deformation in the equations. It allows the in-plane displacement to be linearly varying through the thickness.

4.1.1 Deep thin shells

Love's theory is fundamental theory of deep shells. Other theories related to this case are Sanders, Flugge, etc. Love's approximation defines in the follows

1- Thickness of the shell is small compared to other dimensions, such as length, width and radii of curvature of the middle surface of the shell. This definition defines a thin shell theory. The ratio of h/R in engineering application is assumed less than 1/50 and the ratio of $z/R_{\alpha,\beta}$ is less than 1/100 and so this ratio is negligible.

2- Normal stress in z direction is zero $\sigma_{zz} = 0$.

3- Transverse normal strains are zeros $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$.

Equations of motion according to Love' approximation are

$$
\begin{aligned}
&\frac{\partial}{\partial \alpha}(BN_\alpha) + \frac{\partial}{\partial \beta}(AN_{\beta\alpha}) + \frac{\partial A}{\partial \beta}N_{\alpha\beta} - \frac{\partial B}{\partial \alpha}N_\beta \\
&\qquad + \frac{AB}{R_\alpha}Q_{\alpha z} + \frac{AB}{R_{\alpha\beta}}Q_{\beta z} + ABq_\alpha = AB(\bar{I}_1 \ddot{u}_0^2) \\
&\frac{\partial}{\partial \beta}(AN_\beta) + \frac{\partial}{\partial \alpha}(BN_{\alpha\beta}) + \frac{\partial B}{\partial \alpha}N_{\beta\alpha} - \frac{\partial A}{\partial \beta}N_\alpha \\
&\qquad + \frac{AB}{R_\beta}Q_{\beta z} + \frac{AB}{R_{\alpha\beta}}Q_{\alpha z} + ABq_\beta = AB(\bar{I}_1 \ddot{v}_0^2) \\
&-AB\Big(\frac{N_\alpha}{R_\alpha} + \frac{N_\beta}{R_\beta} + \frac{N_{\alpha\beta} + N_{\beta\alpha}}{R_{\alpha\beta}}\Big) + \frac{\partial}{\partial \alpha}(BQ_{\alpha z}) + \frac{\partial}{\partial \beta}(AQ_{\beta z}) \\
&\qquad + ABq_n = AB(\bar{I}_1 \ddot{w}_0^2)
\end{aligned} \tag{4.1}
$$

where force resultants are

$$
\begin{aligned}
Q_\alpha &= \frac{1}{AB}\Big(\frac{\partial}{\partial \alpha}(BM_\alpha) + \frac{\partial}{\partial \beta}(AM_{\beta\alpha}) + \frac{\partial A}{\partial \beta}M_{\alpha\beta} - \frac{\partial B}{\partial \alpha}M_\beta\Big) \\
Q_\beta &= \frac{1}{AB}\Big(\frac{\partial}{\partial \alpha}(BM_{\alpha\beta}) + \frac{\partial}{\partial \beta}(AM_\beta) + \frac{\partial B}{\partial \alpha}M_{\beta\alpha} - \frac{\partial A}{\partial \beta}M_\alpha\Big)
\end{aligned} \tag{4.2}
$$

and nonzero components of strains are as

$$
\begin{aligned}
\epsilon_{\alpha\alpha} &= \epsilon^0_{\alpha\alpha} + z\mathcal{K}_{\alpha\alpha} \\
\epsilon_{\beta\beta} &= \epsilon^0_{\beta\beta} + z\mathcal{K}_{\beta\beta} \\
\gamma_{\alpha\beta} &= \gamma^0_{\alpha\beta} + z\mathcal{K}_{\alpha\beta}
\end{aligned} \tag{4.3}
$$

where the membrane and the bending strains are

$$
\begin{aligned}
\epsilon^0_{\alpha\alpha} &= \frac{1}{A}\frac{\partial u_0}{\partial \alpha} + \frac{v_0}{AB}\frac{\partial A}{\partial \beta} + \frac{w_0}{R_\alpha} \\
\epsilon^0_{\beta\beta} &= \frac{1}{B}\frac{\partial v_0}{\partial \beta} + \frac{u_0}{AB}\frac{\partial B}{\partial \alpha} + \frac{w_0}{R_\beta} \\
\gamma^0_{\alpha\beta} &= \frac{1}{A}\frac{\partial v_0}{\partial \alpha} - \frac{u_0}{AB}\frac{\partial A}{\partial \beta} + \frac{1}{B}\frac{\partial u_0}{\partial \beta} - \frac{v_0}{AB}\frac{\partial B}{\partial \alpha} + 2\frac{w_0}{R_{\alpha\beta}} \\
\mathcal{K}_{\alpha\alpha} &= \frac{1}{A}\frac{\partial \psi_\alpha}{\partial \alpha} + \frac{\psi_\beta}{AB}\frac{\partial A}{\partial \beta} \\
\mathcal{K}_{\beta\beta} &= \frac{1}{B}\frac{\partial \psi_\beta}{\partial \beta} + \frac{\psi_\alpha}{AB}\frac{\partial B}{\partial \alpha} \\
\mathcal{K}_{\alpha\beta} &= \frac{1}{A}\frac{\partial \psi_\beta}{\partial \alpha} - \frac{\psi_\alpha}{AB}\frac{\partial A}{\partial \beta} + \frac{1}{B}\frac{\partial \psi_\alpha}{\partial \beta} - \frac{\psi_\beta}{AB}\frac{\partial B}{\partial \alpha}
\end{aligned} \tag{4.4}
$$

and also

$$\psi_\alpha = \frac{u}{R_\alpha} + \frac{v_0}{R_{\alpha\beta}} - \frac{1}{A}\frac{\partial w}{\partial \alpha} \qquad \psi_\beta = \frac{v}{R_\beta} + \frac{u_0}{R_{\alpha\beta}} - \frac{1}{A}\frac{\partial w}{\partial \beta} \tag{4.5}$$

where A and B are coefficients of the first fundamental form of reference surface which for cylindrical, spherical and toroidal shells are constant and unity.

Applying Kirchhoff theory and neglecting shear deformation and also $\sigma_{\alpha z}, \sigma_{\beta z}$ and σ_{zz}, the stress-strain equations for composite laminated thin shell can be written as

$$\left\{ \begin{array}{c} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\alpha\beta} \end{array} \right\}_k = \left[\begin{array}{ccc} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{16} \\ \tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{26} \\ \tilde{C}_{16} & \tilde{C}_{26} & \tilde{C}_{66} \end{array} \right] \left\{ \begin{array}{c} \epsilon_{\alpha\alpha} \\ \epsilon_{\beta\beta} \\ \gamma_{\alpha\beta} \end{array} \right\}_k \tag{4.6}$$

where $\sigma_{\alpha\alpha}$ and $\sigma_{\beta\beta}$ are normal stresses, $\sigma_{\alpha\beta}$ is the in-plane shear stress, $\epsilon_{\alpha\alpha}$ and $\epsilon_{\beta\beta}$ are normal strains and $\gamma_{\alpha\beta}$ is the in-plane shear strain. Evaluation of $\tilde{C}_{ij}$ which is the elastic stiffness coefficients and values of C_{ij} are expressed in section (3.2.1). Stresses over the shell thickness are integrated to obtain the force and moment resultants. For thin shell z_k/R_α and z_k/R_β are much less than unity and they are neglected. Therefore, the in-plane force resultants are equal $N_{\alpha\beta} = N_{\beta\alpha}$ and also the twisting moments are equal $M_{\alpha\beta} = M_{\beta\alpha}$.

Stress resultants are as

$$\begin{aligned} \left\{ \begin{array}{c} N_{\alpha\alpha} \\ N_{\beta\beta} \\ N_{\alpha\beta} \end{array} \right\} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{ \begin{array}{c} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\alpha\beta} \end{array} \right\} dz \\ \left\{ \begin{array}{c} M_{\alpha\alpha} \\ M_{\beta\beta} \\ M_{\alpha\beta} \end{array} \right\} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{ \begin{array}{c} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\alpha\beta} \end{array} \right\} zdz \end{aligned} \tag{4.7}$$

Force and moments are related to the strains by the relations

$$\left\{ \begin{array}{c} N_\alpha \\ N_\beta \\ N_{\alpha\beta} \\ M_\alpha \\ M_\beta \\ M_{\alpha\beta} \end{array} \right\} = \left[\begin{array}{cccccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array} \right] \left\{ \begin{array}{c} \epsilon_{0\alpha\alpha} \\ \epsilon_{0\beta\beta} \\ \epsilon_{0\alpha\beta} \\ \mathcal{K}_\alpha \\ \mathcal{K}_\beta \\ \mathcal{K}_{\alpha\beta} \end{array} \right\} \tag{4.8}$$

where A_{ij}, B_{ij}, D_{ij} are defined as follow

$$A_{ij} = \sum_{k=1}^{N} \tilde{C}_{ij}^{(k)} (h_k - h_{k-1})$$

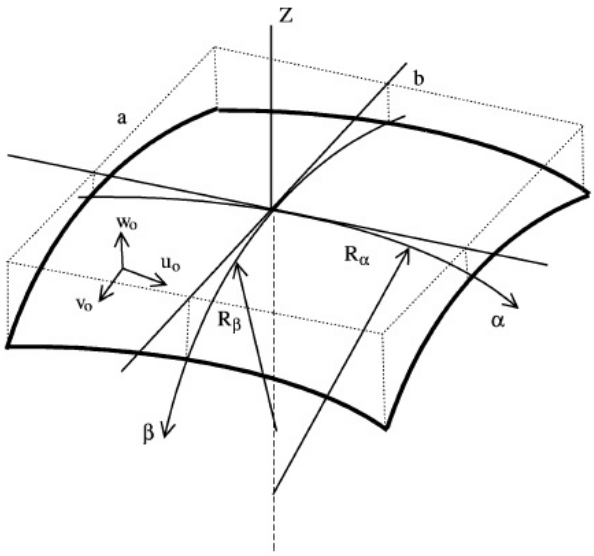

Figure 4.1: Shallow shell with plate geometry

$$B_{ij} = \sum_{k=1}^{N} \tilde{C}_{ij}^{(k)} (h_k^2 - h_{k-1}^2)$$
$$D_{ij} = \sum_{k=1}^{N} \tilde{C}_{ij}^{(k)} (h_k^3 - h_{k-1}^3) \quad (4.9)$$

where z_k is distance from the mid surface to the surface of the kth layer. For shell laminated symmetrically with respect to mid surface all the B_{ij} are zero.

$$F_{12} = N_{12} + \frac{M_{12}}{R_\beta}$$
$$F_{21} = N_{21} + \frac{M_{21}}{R_\alpha}$$
$$V_{13} = \frac{1}{AB}\Big(\frac{\partial(M_{11}B)}{\partial\alpha} + \frac{\partial(M_{21}A)}{\partial\beta} + M_{12}\frac{\partial A}{\partial\beta} - M_{22}\frac{\partial B}{\partial\alpha}\Big) + \frac{1}{B}\frac{\partial M_{12}}{\partial\beta}$$
$$V_{23} = \frac{1}{AB}\Big(\frac{\partial(M_{12}B)}{\partial\alpha} + \frac{\partial(M_{22}A)}{\partial\beta} + M_{12}\frac{\partial B}{\partial\alpha} - M_{11}\frac{\partial A}{\partial\beta}\Big) + \frac{1}{A}\frac{\partial M_{21}}{\partial\alpha} \quad (4.10)$$

where M_{11}, M_{22}, M_{12} are moment resultants, N_{11}, N_{22}, N_{12} are force resultants F_{12}, F_{21} are in-plane shear force resultants, V_{13}, V_{23} are transverse shear force resultants.

All of these relations are corresponded to deep thin shell theories, deep thick shell theories will describe in next section.

4.1.2 Shallow thin shells

The relations for deep shell theory can be simplified by Donnel-Mushtari-Vlasov theory which is well-known theory for shallow shells (see Figure 4.1). The following assumptions lead to the DMV theory:

1- The contribution of in-plane displacements for bending strains are negligible.

Bending strains are as follow

$$
\begin{aligned}
\mathcal{K}_{11} &= \frac{1}{A}\frac{\partial}{\partial\alpha}(-\frac{1}{A}\frac{\partial w}{\partial\alpha}) - \frac{1}{AB^2}\frac{\partial w}{\partial\beta}\frac{\partial A}{\partial\beta} \\
\mathcal{K}_{22} &= \frac{1}{B}\frac{\partial}{\partial\beta}(-\frac{1}{B}\frac{\partial w}{\partial\beta}) - \frac{1}{A^2B}\frac{\partial w}{\partial\alpha}\frac{\partial B}{\partial\alpha} \\
\mathcal{K}_{12} &= \frac{A}{B}\frac{\partial}{\partial\beta}(-\frac{1}{A^2}\frac{\partial w}{\partial\alpha}) + \frac{B}{A}\frac{\partial}{\partial\alpha}(-\frac{1}{B^2}\frac{\partial w}{\partial\beta})
\end{aligned}
\tag{4.11}
$$

2- The effects of shear terms $Q_{\alpha z}/R_\alpha$, $Q_{\beta z}/R_\beta$, $Q_{\beta z}/R_{\alpha\beta}$, and $Q_{\alpha z}/R_{\alpha\beta}$ in the equations of motion are negligible.

Therefore, shallow cylindrical shell's bending since $\alpha = x, \beta = \theta, R_\alpha = \infty, R_\beta = R$ are as follow

$$
\begin{aligned}
\mathcal{K}_{11} &= -\frac{\partial^2 w}{\partial\alpha^2} \\
\mathcal{K}_{22} &= -\frac{\partial^2 w}{\partial\beta^2} \\
\mathcal{K}_{12} &= -2\frac{\partial^2 w}{\partial\alpha\partial\beta}
\end{aligned}
\tag{4.12}
$$

and also strain relations are

$$
\begin{aligned}
\epsilon_{11}^0 &= \frac{\partial u}{\partial\alpha} + \frac{w}{R_\alpha} \\
\epsilon_{22}^0 &= \frac{\partial v}{\partial\beta} + \frac{w}{R_\beta} \\
\epsilon_{12}^0 &= \frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha}
\end{aligned}
\tag{4.13}
$$

Therefore, equations of motion for isotropic shallow cylindrical shells are

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2R^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{v}{R}\frac{\partial w}{\partial x} + \frac{1+\nu}{2R}\frac{\partial^2 v}{\partial x\partial\theta} = \frac{(1-\nu^2)\rho}{E}\frac{\partial^2 u}{\partial t^2} \\
&\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{1}{R^2}\frac{\partial w}{\partial\theta} + \frac{1+\nu}{2R}\frac{\partial^2 u}{\partial x\partial\theta} = \frac{(1-\nu^2)\rho}{E}\frac{\partial^2 v}{\partial t^2} \\
&-(\frac{v}{R}\frac{\partial u}{\partial x} + \frac{1}{R^2}\frac{\partial v}{\partial\theta} + \frac{w}{R^2}) \\
&-\frac{h^2}{12}(\frac{\partial^4 w}{\partial x^4} + \frac{2}{R^2}\frac{\partial^4 w}{\partial x^2\partial\theta^2} + \frac{1}{R^4}\frac{\partial^4 w}{\partial\theta^4}) = \frac{(1-\nu^2)\rho}{E}\frac{\partial^2 w}{\partial t^2}
\end{aligned}
\tag{4.14}
$$

Using $\zeta = x/a, \eta = y/b, \delta = a/h, \phi = a/b, \beta = a/R_x, \gamma = R_x/R_y$ for doing dimensionless the equations of motion

$$12\delta^2\frac{\partial^2 u}{\partial\zeta^2}+12\frac{1-\nu}{2}\delta^2\lambda^2\frac{\partial^2 u}{\partial\eta^2}+12(\frac{1+\nu}{2})\delta^2\lambda\frac{\partial^2 v}{\partial\zeta\partial\eta}+12\delta^2\beta(1+\nu\gamma)\frac{\partial w}{\partial\zeta}+\Omega^2 U=0$$
$$12(1+\nu 2)\delta^2\lambda\frac{\partial^2 u}{\partial\zeta\partial\eta}+12(\frac{1-\nu}{2})\delta^2\frac{\partial^2 v}{\partial\zeta^2}+12\delta^2\lambda^2\frac{\partial^2 v}{\partial\eta^2}+12\delta^2\beta\lambda(\nu+\gamma)\frac{\partial^2 w}{\partial\eta}+\Omega^2 v=0$$
$$12\delta^2\beta(1+\nu\gamma)\frac{\partial u}{\partial\zeta}+12\delta^2\beta\lambda(1+\gamma)\frac{\partial v}{\partial\eta}+(\frac{\partial^4 w}{\partial\zeta^4}+2\lambda^2\frac{\partial^4 w}{\partial\zeta^2\partial\eta^2}+\lambda^4\frac{\partial^4 w}{\partial\eta^4})$$
$$+12\delta^2\beta^2(1+2\nu\gamma+\gamma^2)w-\Omega^2 w=0 \tag{4.15}$$

Boundary conditions:

C: $\begin{cases} x: u=v=w=\frac{\partial w}{\partial x}=0 \\ y: u=v=w=\frac{\partial w}{\partial y}=0 \end{cases}$

S1(hard): $\begin{cases} x: u=v=w=M_{11}=0 \\ y: u=v=w=M_{22}=0 \end{cases}$ S2(soft): $\begin{cases} x: v=w=N_{11}=M_{11}=0 \\ y: u=w=N_{22}=M_{22}=0 \end{cases}$

Table 4.1: Non-dimensional natural frequencies of shallow shell, $\lambda=\omega a^2\sqrt{\frac{\rho h}{D}}, \phi=1, \beta=0.1$

γ	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
					CCCC				
0	Present	46.280	74.656	79.289	110.34	132.54	135.55	165.81	166.96
	[134]	46.28	74.65	79.28	110.3	132.5	135.5	165.7	166.9
1	Present	58.296	81.754	81.754	114.15	136.00	137.72	168.77	168.77
	[134]	58.30	81.75	81.75	114.1	136.0	137.7	168.7	168.7
−1	Present	50.750	79.151	79.151	110.69	135.25	135.73	166.75	166.75
	[134]	50.75	79.14	79.14	110.7	135.2	135.7	166.7	166.7
					SCSS				
0	Present	29.305	52.282	64.346	87.798	100.36	117.05	134.22	142.70
	[134]	29.30	52.28	64.34	87.79	100.4	117.1	134.2	143.7
1	Present	41.803	61.543	67.941	92.509	105.60	118.19	137.88	144.83
	[134]	41.80	61.54	67.94	92.51	105.6	118.2	137.9	144.8
−1	Present	29.379	56.632	62.888	87.178	103.95	116.47	134.82	141.81
	[134]	29.38	56.63	62.89	87.17	104.0	116.5	134.8	141.8

$\gamma=0,1,-1$ is correspond to cylindrical, spherical and hyperbolic paraboloidal shells. Procedure for solving thin shell such as shallow and deep are similar to non-symmetric classical composite laminated plate in section (3.2.2). Table (4.1) is shown results for isotropic shallow shell for cylindrical, spherical and hyperbolic paraboloidal. Results are compares with super-position Galerkin method and as seen two methods are very close to each other. Table (4.2) also is shown analysis of shallow shell panel for fully clamped and fully simply supported in terms of a/R_x which is the shallowness of the thin shell

Table 4.2: Non-dimensional natural frequencies of the cylindrical shallow shell with $\phi = 1, a/h = 100, \lambda = \omega a^2\sqrt{\frac{\rho h}{D}}$

BCs	β	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
CCCC	0	35.985	73.393	73.393	108.21	131.58	132.20	165.00	165.00
	0.1	46.280	74.656	79.290	110.34	132.54	135.55	165.81	166.96
	0.2	67.681	78.294	94.610	116.46	134.99	145.76	168.32	172.71
	0.3	83.923	90.396	115.047	125.86	140.52	161.25	172.80	181.82
	0.4	91.065	108.46	136.888	137.70	152.42	180.12	180.40	193.74
	0.5	99.262	118.99	151.126	156.34	172.52	192.43	201.67	207.80
SSSS	0	19.739	49.348	49.348	78.956	98.696	98.696	128.30	128.30
	0.1	36.840	51.575	58.382	82.301	99.526	103.65	129.41	131.10
	0.2	57.708	63.833	79.216	91.541	102.83	117.23	132.85	139.14
	0.3	66.573	85.624	103.770	104.95	113.69	136.67	139.33	151.49
	0.4	77.079	95.126	120.832	125.75	137.70	151.89	159.39	167.01
	0.5	88.431	99.889	137.823	140.07	166.63	171.87	174.15	182.94

from 0 (plate), 0.1 to 0.5 which is not so shallow. $\beta = 2$ is so deep and $\beta = 1$ is moderately deep. Table (4.3) also is shown composite laminated simply-supported shallow shell for three type, cylindrical, spherical and hyperbolic paraboloidal in terms of varying the shallowness. Results are compared with GDQ method and they are so close.

4.2 Thick shells

Thick shells theory are defined using the value of thickness of shell over length or radii of curvature which is between 1/10 to 1/20. It is assumed that normals to the mid surface remains straight during deformation but not normal to the surface. In this theory, shell displacements are expanded in terms of shell thickness which can be of a first-order. Difference of this approach with 3D is in assumption that the normal strain acting on the plane parallel to the mid surface is negligible compared with other strain components. This assumption is valid except within the vicinity of a highly concentrated force. In other words, $\epsilon_{zz} = 0$. The displacements can be written as

$$\begin{aligned} u(\alpha, \beta, z) &= u_0(\alpha, \beta) + z\psi_\alpha(\alpha, \beta) \\ v(\alpha, \beta, z) &= v_0(\alpha, \beta) + z\psi_\beta(\alpha, \beta) \\ w(\alpha, \beta, z) &= w_0(\alpha, \beta) \end{aligned} \tag{4.16}$$

where u_0, v_0 and w_0 are mid surface displacements of the shell and ψ_x and ψ_y are mid surface rotations. Third part of the above equation leads to $\epsilon_{zz} = 0$. This definition is called first-order shear deformation theory which is applied on thick shell whether deep or shallow.

The strains at any points in the shells can be written in terms of mid surface strains

Table 4.3: Non-dimensional natural frequencies of the simply-supported cross-ply (0/90/90/0) shallow composite shell, $\lambda = \omega a^2\sqrt{\frac{\rho}{E_2 h^2}}, h/a = 0.01\ a/b = 1, E_1/E_2 = 15, G_{12}/E_2 = 0.5, G_{13}/E_2 = 0.5, G_{23}/E_2 = 0.5, \nu_{12} = 0.25$

γ	β	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
-1	0.1	Present	12.264	24.999	43.859	47.786	49.096	65.582	80.149
		[155]	12.264						
	0.5	Present	11.977	48.800	49.354	63.805	70.162	80.545	102.49
		[155]	11.977						
0	0.1	Present	13.970	26.571	42.909	48.921	49.555	66.111	81.058
		[155]	13.970						
	0.5	Present	35.183	45.460	59.220	66.635	81.599	94.957	95.936
		[155]	35.183						
1	0.1	Present	18.129	28.197	45.584	49.604	50.883	66.898	81.375
		[155]	18.129						
	0.5	Present	66.577	81.304	82.489	88.705	96.195	104.20	121.57
		[155]	66.577						

and curvature changes as

$$
\begin{aligned}
\epsilon_\alpha &= \frac{1}{1+\frac{z}{R_\alpha}}(\epsilon_{0\alpha} + z\mathcal{K}_\alpha) \\
\epsilon_\beta &= \frac{1}{1+\frac{z}{R_\beta}}(\epsilon_{0\beta} + z\mathcal{K}_\beta) \\
\epsilon_{\alpha\beta} &= \frac{1}{1+\frac{z}{R_\alpha}}(\epsilon_{0\alpha\beta} + z\mathcal{K}_{\alpha\beta}) \\
\epsilon_{\beta\alpha} &= \frac{1}{1+\frac{z}{R_\beta}}\Big(\epsilon_{0\beta\alpha} + z\mathcal{K}_{\beta\alpha}\Big) \\
\gamma_{\alpha z} &= \frac{1}{1+\frac{z}{R_\alpha}}\Big(\gamma_{0\alpha z} + z(\psi_\alpha/R_\alpha)\Big) \\
\gamma_{\beta z} &= \frac{1}{1+\frac{z}{R_\beta}}\Big(\gamma_{0\beta z} + z(\psi_\beta/R_\beta)\Big)
\end{aligned}
\tag{4.17}
$$

where the midsurface strains are as

$$
\begin{aligned}
\epsilon_{0\alpha} &= \frac{1}{A}\frac{\partial u_0}{\partial \alpha} + \frac{v_0}{AB}\frac{\partial A}{\partial \beta} + \frac{w_0}{R_\alpha} \\
\epsilon_{0\beta} &= \frac{1}{B}\frac{\partial v_0}{\partial \beta} + \frac{u_0}{AB}\frac{\partial B}{\partial \alpha} + \frac{w_0}{R_\beta} \\
\epsilon_{0\alpha\beta} &= \frac{1}{A}\frac{\partial v_0}{\partial \alpha} + \frac{u_0}{AB}\frac{\partial A}{\partial \beta} + \frac{w_0}{R_{\alpha\beta}} \\
\epsilon_{0\beta\alpha} &= \frac{1}{B}\frac{\partial u_0}{\partial \beta} + \frac{v_0}{AB}\frac{\partial B}{\partial \alpha} + \frac{w_0}{R_{\alpha\beta}} \\
\gamma_{0\alpha z} &= \frac{1}{A}\frac{\partial w_0}{\partial \alpha} - \frac{u_0}{R_\alpha} - \frac{v_0}{R_{\alpha\beta}} + \psi_\alpha \\
\gamma_{0\beta z} &= \frac{1}{B}\frac{\partial w_0}{\partial \beta} - \frac{v_0}{R_\beta} - \frac{u_0}{R_{\alpha\beta}} + \psi_\beta
\end{aligned}
\tag{4.18}
$$

and the curvature and twist changes are as

$$\mathcal{K}_\alpha = \frac{1}{A}\frac{\partial \psi_\alpha}{\partial \alpha} + \frac{\psi_\beta}{AB}\frac{\partial A}{\partial \beta}, \qquad \mathcal{K}_\beta = \frac{1}{B}\frac{\partial \psi_\beta}{\partial \beta} + \frac{\psi_\alpha}{AB}\frac{\partial B}{\partial \alpha}$$
$$\mathcal{K}_{\alpha\beta} = \frac{1}{A}\frac{\partial \psi_\beta}{\partial \alpha} - \frac{\psi_\alpha}{AB}\frac{\partial A}{\partial \beta}, \qquad \mathcal{K}_{\beta\alpha} = \frac{1}{B}\frac{\partial \psi_\alpha}{\partial \beta} - \frac{\psi_\beta}{AB}\frac{\partial B}{\partial \alpha} \tag{4.19}$$

The above equations include the term $(1 + z/R_n)$ where n is either α or β for thick shallow shells are neglected in all strain components. Carrying out the integration of this term creates difficulties which in the case of plate don't exist. This term ignore in most thin shell theories, but in some theories like Vlasov this term is expand in a geometric series form. Numerical investigations is shown that this expansion does not lead to better results because the value if this term in thin shells are between 0.98 and 1.02 depending on value of z. Applying Mindlin theory, stress-strain equations for composite laminated thin shell is written as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{23} \end{Bmatrix} \tag{4.20}$$

For thick shell, shear deformation should be considered. The normal and shear forces are

$$\begin{Bmatrix} N_\alpha \\ N_{\alpha\beta} \\ Q_\alpha \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{Bmatrix} (1 + \frac{z}{R_\beta}) dz$$
$$\begin{Bmatrix} N_\beta \\ N_{\alpha\beta} \\ Q_\beta \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{\beta\beta} \\ \sigma_{\alpha\beta} \\ \sigma_{\beta z} \end{Bmatrix} (1 + \frac{z}{R_\alpha}) dz \tag{4.21}$$

The bending and twisting moments and higher-order shear force terms are as

$$\begin{Bmatrix} M_\alpha \\ M_{\alpha\beta} \\ P_\alpha \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{Bmatrix} (1 + \frac{z}{R_\beta}) dz$$
$$\begin{Bmatrix} M_\beta \\ M_{\alpha\beta} \\ P_\beta \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{\beta\beta} \\ \sigma_{\alpha\beta} \\ \sigma_{\beta z} \end{Bmatrix} (1 + \frac{z}{R_\alpha}) dz \tag{4.22}$$

P_α and P_β are higher-order shear force terms, needed only if the radius of twist curvature exists ($R_{\alpha\beta} \neq 0$). Although $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$, stress resultants $N_{\alpha\beta} \neq N_{\beta\alpha}$ and $M_{\alpha\beta} \neq M_{\beta\alpha}$ even $R_\alpha = R_\beta$ which is happened in plates.

Relation between stress resultants and strains are as

$$\left\{\begin{array}{c} N_{\alpha\alpha} \\ N_{\alpha\beta} \\ N_{\beta\alpha} \\ N_{\beta\beta} \\ M_{\alpha\alpha} \\ M_{\alpha\beta} \\ M_{\beta\alpha} \\ M_{\beta\beta} \end{array}\right\} = \left[\begin{array}{cccccccc} A_{11} & A_{12} & A_{16} & A_{16} & B_{11} & B_{12} & B_{16} & B_{16} \\ A_{12} & A_{22} & A_{26} & A_{26} & B_{12} & B_{22} & B_{26} & B_{26} \\ A_{16} & A_{26} & A_{66} & A_{66} & B_{16} & B_{26} & B_{66} & B_{66} \\ A_{16} & A_{26} & A_{66} & A_{66} & B_{16} & B_{26} & B_{66} & B_{66} \\ B_{11} & B_{12} & B_{16} & B_{16} & D_{11} & D_{12} & D_{16} & D_{16} \\ B_{12} & B_{22} & B_{26} & B_{26} & D_{12} & D_{22} & D_{26} & D_{26} \\ B_{16} & B_{26} & B_{66} & B_{66} & D_{16} & D_{26} & D_{66} & D_{66} \\ B_{16} & B_{26} & B_{66} & B_{66} & D_{16} & D_{26} & D_{66} & D_{66} \end{array}\right] \left\{\begin{array}{c} \epsilon_{0\alpha\alpha} \\ \epsilon_{0\beta\beta} \\ \epsilon_{0\alpha\beta} \\ \epsilon_{0\beta\alpha} \\ \mathcal{K}_{\alpha} \\ \mathcal{K}_{\beta} \\ \mathcal{K}_{\alpha\beta} \\ \mathcal{K}_{\beta\alpha} \end{array}\right\}$$

$$\left\{\begin{array}{c} Q_{\alpha} \\ Q_{\beta} \\ P_{\alpha} \\ P_{\beta} \end{array}\right\} = \left[\begin{array}{cccc} A_{55} & A_{45} & B_{55} & B_{45} \\ A_{45} & A_{44} & B_{45} & B_{44} \\ B_{55} & B_{45} & D_{55} & D_{45} \\ B_{45} & B_{44} & D_{45} & D_{44} \end{array}\right] \left\{\begin{array}{c} \gamma_{0\alpha z} \\ \gamma_{0\beta z} \\ \psi_{\alpha}/R_{\alpha} \\ \psi_{\beta}/R_{\beta} \end{array}\right\} \tag{4.23}$$

Combination of eqs. (4.23) yields in a well known matrix 12×12 as following:

$$\left\{\begin{array}{c} N_{\alpha\alpha} \\ N_{\alpha\beta} \\ Q_{\alpha} \\ N_{\beta\alpha} \\ N_{\beta\beta} \\ Q_{\beta} \\ M_{\alpha\beta} \\ M_{\alpha\alpha} \\ P_{\alpha} \\ M_{\beta\beta} \\ M_{\beta\alpha} \\ P_{\beta} \end{array}\right\} = \left[\begin{array}{cccccccccccc} A_{11} & A_{16} & 0 & A_{16} & A_{12} & 0 & B_{16} & B_{11} & 0 & B_{12} & B_{16} & 0 \\ A_{16} & 2A_{66} & 0 & 0 & A_{26} & 0 & B_{66} & B_{16} & 0 & -B_{26} & B_{66} & 0 \\ 0 & 0 & A_{55} & 0 & 0 & A_{45} & 0 & 0 & B_{55} & 0 & 0 & B_{45} \\ A_{16} & 0 & 0 & 2A_{66} & A_{26} & 0 & B_{66} & -B_{16} & 0 & B_{26} & B_{66} & 0 \\ A_{12} & A_{26} & 0 & A_{26} & A_{22} & 0 & B_{26} & B_{12} & 0 & B_{22} & B_{26} & 0 \\ 0 & 0 & A_{45} & 0 & 0 & A_{44} & 0 & 0 & B_{45} & 0 & 0 & B_{44} \\ B_{16} & B_{66} & 0 & B_{66} & B_{26} & 0 & 2D_{66} & D_{16} & 0 & D_{26} & 0 & 0 \\ B_{11} & B_{16} & 0 & -B_{16} & B_{12} & 0 & D_{16} & D_{11} & 0 & -D_{12} & D_{16} & 0 \\ 0 & 0 & B_{55} & 0 & 0 & B_{45} & 0 & 0 & D_{55} & 0 & 0 & D_{45} \\ B_{12} & -B_{26} & 0 & B_{26} & B_{22} & 0 & D_{26} & -D_{12} & 0 & D_{22} & D_{26} & 0 \\ B_{16} & B_{66} & 0 & B_{66} & B_{26} & 0 & 0 & D_{16} & 0 & D_{26} & 2D_{66} & 0 \\ 0 & 0 & B_{45} & 0 & 0 & B_{44} & 0 & 0 & D_{45} & 0 & 0 & D_{44} \end{array}\right] \left\{\begin{array}{c} \epsilon_{0\alpha\alpha} \\ \epsilon_{0\alpha\beta} \\ \epsilon_{0\alpha z} \\ \epsilon_{0\beta\alpha} \\ \epsilon_{0\beta\beta} \\ \epsilon_{0\beta z} \\ \mathcal{K}_{\alpha\beta} \\ \mathcal{K}_{\alpha\alpha} \\ \mathcal{K}_{\alpha z} \\ \mathcal{K}_{\beta\beta} \\ \mathcal{K}_{\beta\alpha} \\ \mathcal{K}_{\beta z} \end{array}\right\}$$

This matrix is symmetric and as seen for equilibrium, the sign of coefficients which are multiplied by $\mathcal{K}_{\alpha\alpha}, \mathcal{K}_{\beta\beta}$ and $\epsilon_{0\alpha\beta}, \epsilon_{0\beta\alpha}$ in equations of $N_{\alpha\beta}, N_{\beta\alpha}, M_{\alpha\alpha}, M_{\beta\beta}$ are negative. A_{ij}, B_{ij}, D_{ij} are defined as follow

$$A_{ij} = \sum_{k=1}^{N} \tilde{C}_{ij}^{(k)}(h_k - h_{k-1}), \quad B_{ij} = \sum_{k=1}^{N} \tilde{C}_{ij}^{(k)}(h_k^2 - h_{k-1}^2), \quad D_{ij} = \sum_{k=1}^{N} \tilde{C}_{ij}^{(k)}(h_k^3 - h_{k-1}^3) \tag{4.24}$$

for $i, j = 1, 2, 6$ and

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{N} \kappa_i \kappa_j \tilde{C}_{ij}^{(k)}(h_k - h_{k-1}) \\ B_{ij} &= \sum_{k=1}^{N} \kappa_i \kappa_j \tilde{C}_{ij}^{(k)}(h_k^2 - h_{k-1}^2) \\ D_{ij} &= \sum_{k=1}^{N} \kappa_i \kappa_j \tilde{C}_{ij}^{(k)}(h_k^3 - h_{k-1}^3) \end{aligned} \tag{4.25}$$

for $i, j = 4, 5$ and also $\kappa_i \kappa_j = 5/6$ which both of them are shear correction factor in i and j directions.

The equations of motion are

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}(BN_\alpha)+\frac{\partial}{\partial\beta}(AN_{\beta\alpha})+\frac{\partial A}{\partial\beta}N_{\alpha\beta}-\frac{\partial B}{\partial\alpha}N_\beta\\
&\qquad\qquad+\frac{AB}{R_\alpha}Q_\alpha+\frac{AB}{R_{\alpha\beta}}Q_\beta+ABq_\alpha=AB(\bar{I}_1\ddot{u}_0^2+\bar{I}_2\ddot{\psi}_\alpha^2)\\
&\frac{\partial}{\partial\beta}(AN_\beta)+\frac{\partial}{\partial\alpha}(BN_{\alpha\beta})+\frac{\partial B}{\partial\alpha}N_{\beta\alpha}-\frac{\partial A}{\partial\beta}N_\alpha\\
&\qquad\qquad+\frac{AB}{R_\beta}Q_\beta+\frac{AB}{R_{\alpha\beta}}Q_\alpha+ABq_\beta=AB(\bar{I}_1\ddot{v}_0^2+\bar{I}_2\ddot{\psi}_\beta^2)\\
&-AB\Big(\frac{N_\alpha}{R_\alpha}+\frac{N_\beta}{R_\beta}+\frac{N_{\alpha\beta}+N_{\beta\alpha}}{R_{\alpha\beta}}\Big)+\frac{\partial}{\partial\alpha}(BQ_\alpha)+\frac{\partial}{\partial\beta}(AQ_\beta)+ABq_n=AB(\bar{I}_1\ddot{w}_0^2)\\
&\frac{\partial}{\partial\alpha}(BM_\alpha)+\frac{\partial}{\partial\beta}(AM_{\beta\alpha})+\frac{\partial A}{\partial\beta}M_{\alpha\beta}-\frac{\partial B}{\partial\alpha}M_\beta\\
&\qquad\qquad-ABQ_\alpha+\frac{AB}{R_\alpha}P_\alpha+ABm_\alpha=AB(\bar{I}_2\ddot{u}_0^2+\bar{I}_3\ddot{\psi}_\alpha^2)\\
&\frac{\partial}{\partial\beta}(AM_\beta)+\frac{\partial}{\partial\alpha}(BM_{\alpha\beta})+\frac{\partial B}{\partial\alpha}M_{\beta\alpha}-\frac{\partial A}{\partial\beta}M_\alpha\\
&\qquad\qquad-ABQ_\beta+\frac{AB}{R_\beta}P_\beta+ABm_\beta=AB(\bar{I}_2\ddot{v}_0^2+\bar{I}_3\ddot{\psi}_\beta^2)
\end{aligned}\tag{4.26}$$

where $N_\alpha, N_\beta, N_{\alpha\beta}, N_{\beta\alpha}$ are force resultants tangent to the midsurface, Q_α, Q_β are the transverse shear force resultants, m_α, m_β are body couples (moments per unit length), q_α, q_β are external forces (per unit area), and

$$\bar{I}_i=\Big(I_i+I_{i+1}(\frac{1}{R_\alpha}+\frac{1}{R_\beta})+\frac{I_{i+2}}{R_{\alpha\beta}}\Big)\qquad i=1,2,3\tag{4.27}$$

and also

$$(I_1,I_2,I_3,I_4,I_5)=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\rho^{(k)}(1,z,z^2,z^3,z^4)dz\tag{4.28}$$

Substituting eq. (4.23) into eq. (4.26), yields equations of motion in matrix form as follow

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15}\\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25}\\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35}\\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45}\\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55}\end{bmatrix}\begin{Bmatrix} U\\ V\\ W\\ \psi_x\\ \psi_y\end{Bmatrix}=-\lambda^2\begin{bmatrix} M_{11} & 0 & 0 & M_{14} & 0\\ 0 & M_{22} & 0 & 0 & M_{25}\\ 0 & 0 & M_{33} & 0 & 0\\ M_{14} & 0 & 0 & M_{44} & 0\\ 0 & M_{25} & 0 & 0 & M_{55}\end{bmatrix}\begin{Bmatrix} U\\ V\\ W\\ \psi_x\\ \psi_y\end{Bmatrix}\tag{4.29}$$

where

$$
\begin{aligned}
K_{11} &= A_{11}\frac{\partial^2}{\partial\alpha^2} + 2\varrho A_{16}\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 A_{66}\frac{\partial^2}{\partial\beta^2} - \beta^2\gamma^2 A_{55} \\
K_{12} &= A_{16}\frac{\partial^2}{\partial\alpha^2} + \varrho(A_{12}+A_{66})\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 A_{26}\frac{\partial^2}{\partial\beta^2} - \beta^2\gamma A_{45} \\
K_{13} &= \beta h A_{12}\frac{\partial}{\partial\alpha} + \varrho\beta h A_{26}\frac{\partial}{\partial\beta} + \beta\gamma h A_{11}\frac{\partial}{\partial\alpha} + \beta\gamma\varrho h A_{16}\frac{\partial}{\partial\beta} + \beta\gamma h A_{55}\frac{\partial}{\partial\alpha} + \beta\gamma\varrho h A_{45}\frac{\partial}{\partial\beta} \\
K_{14} &= B_{11}\frac{\partial^2}{\partial\alpha^2} + 2\varrho B_{16}\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 B_{66}\frac{\partial^2}{\partial\beta^2} + \beta\gamma A_{55} \\
K_{15} &= B_{16}\frac{\partial^2}{\partial\alpha^2} + \varrho(B_{12}+B_{66})\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 B_{26}\frac{\partial^2}{\partial\beta^2} + \beta\gamma A_{45} \\
K_{21} &= A_{16}\frac{\partial^2}{\partial\alpha^2} + \varrho(A_{12}+A_{66})\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 A_{26}\frac{\partial^2}{\partial\beta^2} - \beta^2\gamma A_{45} \\
K_{22} &= A_{66}\frac{\partial^2}{\partial\alpha^2} + 2\varrho A_{26}\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 A_{22}\frac{\partial^2}{\partial\beta^2} - \beta^2 A_{44} \\
K_{23} &= \beta h(A_{26}+A_{45})\frac{\partial}{\partial\alpha} + \varrho\beta h(A_{22}+A_{44})\frac{\partial}{\partial\beta} + \beta\gamma\varrho h A_{12}\frac{\partial}{\partial\beta} + \beta\gamma h A_{16}\frac{\partial}{\partial\alpha} \\
K_{24} &= B_{16}\frac{\partial^2}{\partial\alpha^2} + \varrho(B_{12}+B_{66})\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 B_{26}\frac{\partial^2}{\partial\beta^2} + \beta A_{45} \\
K_{25} &= B_{66}\frac{\partial^2}{\partial\alpha^2} + 2\varrho B_{26}\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 B_{22}\frac{\partial^2}{\partial\beta^2} + \beta A_{44} \\
K_{31} &= -\beta A_{12}\frac{\partial}{\partial\alpha} - \varrho\beta A_{26}\frac{\partial}{\partial\beta} - \beta\gamma A_{11}\frac{\partial}{\partial\alpha} - \beta\gamma\varrho A_{16}\frac{\partial}{\partial\beta} - \beta\gamma A_{55}\frac{\partial}{\partial\alpha} - \beta\gamma\varrho A_{45}\frac{\partial}{\partial\beta} \\
K_{32} &= -\beta(A_{26}+A_{45})\frac{\partial}{\partial\alpha} - \varrho\beta(A_{22}+A_{44})\frac{\partial}{\partial\beta} - \beta\gamma\varrho A_{12}\frac{\partial}{\partial\beta} - \beta\gamma A_{16}\frac{\partial}{\partial\alpha} \\
K_{33} &= hA_{55}\frac{\partial^2}{\partial\alpha^2} + 2\varrho h A_{45}\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 h A_{44}\frac{\partial^2}{\partial\beta^2} - \beta^2 h A_{22} - \beta\gamma h A_{11} - 2\beta^2\gamma h A_{12} \\
K_{34} &= A_{55}\frac{\partial}{\partial\alpha} + \varrho A_{45}\frac{\partial}{\partial\beta} - \beta B_{12}\frac{\partial}{\partial\alpha} - \varrho\beta B_{26}\frac{\partial}{\partial\beta} - \beta\gamma B_{11}\frac{\partial}{\partial\alpha} - \beta\gamma\varrho B_{16}\frac{\partial}{\partial\beta} \\
K_{35} &= A_{45}\frac{\partial}{\partial\alpha} + \varrho A_{44}\frac{\partial}{\partial\beta} - \beta B_{26}\frac{\partial}{\partial\alpha} - \beta\varrho B_{22}\frac{\partial}{\partial\beta} - \beta\gamma\varrho B_{12}\frac{\partial}{\partial\beta} - \beta\gamma B_{16}\frac{\partial}{\partial\alpha} \\
K_{41} &= B_{11}\frac{\partial^2}{\partial\alpha^2} + 2\varrho B_{16}\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 B_{66}\frac{\partial^2}{\partial\beta^2} + \beta\gamma A_{55} \\
K_{42} &= B_{16}\frac{\partial^2}{\partial\alpha^2} + \varrho(B_{12}+B_{66})\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 B_{26}\frac{\partial^2}{\partial\beta^2} + \beta A_{45} \\
K_{43} &= \beta h B_{12}\frac{\partial}{\partial\alpha} + \varrho\beta h B_{26}\frac{\partial}{\partial\beta} - hA_{55}\frac{\partial}{\partial\alpha} - \varrho h A_{45}\frac{\partial}{\partial\beta} + \beta\gamma h B_{11}\frac{\partial}{\partial\alpha} + \beta\gamma\varrho h B_{16}\frac{\partial}{\partial\beta} \\
K_{44} &= D_{11}\frac{\partial^2}{\partial\alpha^2} + 2\varrho D_{16}\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 D_{66}\frac{\partial^2}{\partial\beta^2} - A_{55} \\
K_{45} &= D_{16}\frac{\partial^2}{\partial\alpha^2} + \varrho(D_{12}+D_{66})\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 D_{26}\frac{\partial^2}{\partial\beta^2} - A_{45} \\
K_{51} &= B_{16}\frac{\partial^2}{\partial\alpha^2} + \varrho(B_{12}+B_{66})\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 B_{26}\frac{\partial^2}{\partial\beta^2} + \beta\gamma A_{45}
\end{aligned}
$$

$$K_{52} = B_{66}\frac{\partial^2}{\partial\alpha^2} + 2\varrho B_{26}\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 B_{22}\frac{\partial^2}{\partial\beta^2} + \beta A_{44}$$
$$K_{53} = \beta h B_{26}\frac{\partial}{\partial\alpha} + \beta\varrho h B_{22}\frac{\partial}{\partial\beta} - hA_{45}\frac{\partial}{\partial\alpha} - \varrho h A_{44}\frac{\partial}{\partial\beta} + \beta\gamma\varrho h B_{12}\frac{\partial}{\partial\beta} + \beta\gamma h B_{16}\frac{\partial}{\partial\alpha}$$
$$K_{54} = D_{16}\frac{\partial^2}{\partial\alpha^2} + \varrho(D_{12} + D_{66})\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 D_{26}\frac{\partial^2}{\partial\beta^2} - A_{45}$$
$$K_{55} = D_{66}\frac{\partial^2}{\partial\alpha^2} + 2\varrho D_{26}\frac{\partial^2}{\partial\alpha\partial\beta} + \varrho^2 D_{22}\frac{\partial^2}{\partial\beta^2} - A_{44}$$

and

$$\begin{array}{lllll} M_{11} = \bar{I}_1 & M_{22} = \bar{I}_1 & M_{33} = \bar{I}_1 & M_{44} = \bar{I}_3 & M_{55} = \bar{I}_3 \\ M_{14} = \bar{I}_2 & M_{25} = \bar{I}_2 & & & \end{array} \tag{4.30}$$

Using $\varrho = a/b, a/R_\beta = \beta, R_\beta/R_\alpha = \gamma$ for doing dimensionless equations of motion.

Boundary conditions at $\zeta = 0$

Clamped ($U = V = W = \psi_x = \psi_y = 0$)

$$\begin{array}{lll} \mathcal{B}_{11} = (e_1^T \otimes I) & \mathcal{B}_{22} = (e_1^T \otimes I) & \mathcal{B}_{33} = (e_1^T \otimes I) \\ \mathcal{B}_{44} = (e_1^T \otimes I) & \mathcal{B}_{55} = (e_1^T \otimes I) & \end{array} \tag{4.31}$$

Simply-supported ($M_{xx} = V = W = N_{xx} = \psi_y = 0$)

$$\begin{aligned}
\mathcal{B}_{11} &= B_{11}(e_1^T D^{(1)} \otimes I) + B_{16}(e_1^T \otimes D^{(1)}) \\
\mathcal{B}_{12} &= B_{12}(e_1^T \otimes D^{(1)}) + B_{16}(e_1^T D^{(1)} \otimes I) \\
\mathcal{B}_{13} &= B_{11}\beta(e_1^T \otimes I) + B_{12}(e_1^T \otimes I) \\
\mathcal{B}_{14} &= D_{11}(e_1^T D^{(1)} \otimes I) + D_{16}(e_1^T \otimes D^{(1)}) \\
\mathcal{B}_{15} &= D_{12}(e_1^T \otimes D^{(1)}) + D_{16}(e_1^T D^{(1)} \otimes I) \\
\mathcal{B}_{22} &= (e_1^T \otimes I) \qquad \mathcal{B}_{33} = (e_1^T \otimes I) \\
\mathcal{B}_{41} &= A_{11}(e_1^T D^{(1)} \otimes I) + A_{16}(e_1^T \otimes D^{(1)}) \\
\mathcal{B}_{42} &= A_{12}(e_1^T \otimes D^{(1)}) + A_{16}(e_1^T D^{(1)} \otimes I) \\
\mathcal{B}_{43} &= A_{11}\beta(e_1^T \otimes I) + A_{12}(e_1^T \otimes I) \\
\mathcal{B}_{44} &= B_{11}(e_1^T D^{(1)} \otimes I) + B_{16}(e_1^T \otimes D^{(1)}) \\
\mathcal{B}_{45} &= B_{12}(e_1^T \otimes D^{(1)}) + B_{16}(e_1^T D^{(1)} \otimes I) \\
\mathcal{B}_{55} &= (e_1^T \otimes I)
\end{aligned} \tag{4.32}$$

Boundary conditions at $\eta = 0$

Clamped ($U = V = W = \psi_x = \psi_y = 0$)

$$\begin{aligned} &\mathcal{B}_{11} = (I \otimes e_1^T) \quad \mathcal{B}_{22} = (I \otimes e_1^T) \quad \mathcal{B}_{33} = (I \otimes e_1^T) \\ &\mathcal{B}_{44} = (I \otimes e_1^T) \quad \mathcal{B}_{55} = (I \otimes e_1^T) \end{aligned} \tag{4.33}$$

Simply-supported ($U = M_{yy} = W = \psi_x = N_{yy} = 0$)

$$\begin{aligned} &\mathcal{B}_{11} = (e_1^T \otimes I) \\ &\mathcal{B}_{21} = B_{12}(D^{(1)} \otimes e_1^T) + B_{26}(I \otimes e_1^T D^{(1)}) \\ &\mathcal{B}_{22} = B_{22}(I \otimes e_1^T D^{(1)}) + B_{26}(D^{(1)} \otimes e_1^T) \\ &\mathcal{B}_{23} = B_{12}\beta(I \otimes e_1^T) + B_{22}(I \otimes e_1^T) \\ &\mathcal{B}_{24} = D_{12}(D^{(1)} \otimes e_1^T) + D_{26}(I \otimes e_1^T D^{(1)}) \\ &\mathcal{B}_{25} = D_{22}(I \otimes e_1^T D^{(1)}) + D_{26}(D^{(1)} \otimes e_1^T) \\ &\mathcal{B}_{33} = (e_1^T \otimes I) \qquad \mathcal{B}_{44} = (e_1^T \otimes I) \\ &\mathcal{B}_{51} = A_{12}(D^{(1)} \otimes e_1^T) + A_{26}(I \otimes e_1^T D^{(1)}) \\ &\mathcal{B}_{52} = A_{22}(I \otimes e_1^T D^{(1)}) + A_{26}(D^{(1)} \otimes e_1^T) \\ &\mathcal{B}_{53} = A_{12}\beta(I \otimes e_1^T) + A_{22}(I \otimes e_1^T) \\ &\mathcal{B}_{54} = B_{12}(D^{(1)} \otimes e_1^T) + B_{26}(I \otimes e_1^T D^{(1)}) \\ &\mathcal{B}_{55} = B_{22}(I \otimes e_1^T D^{(1)}) + B_{26}(D^{(1)} \otimes e_1^T) \end{aligned} \tag{4.34}$$

This problem is solved like problem section (3.3.2). Table (4.4) is shown the results for cylindrical panel shell for non-symmetric composite laminated both cross-ply and angle-ply for many boundary conditions. Results are compared with differential quadrature method based on the first-order shear deformation theory. As seen results are close.

Table 4.4: Non-dimensional natural frequencies of composite laminated cylindrical shell, $\lambda = \omega a^2\sqrt{\frac{\rho h}{D}}, a/h = 10, a/R = 2$

Lay-up	BCs	ω_1	ω_2	ω_3	ω_4	ω_5
$(0, 90)_3$	CCCC	28.787	40.716	40.740	48.720	56.302
	[156]	28.319	40.005	40.196	47.953	55.815
	SSSS	14.715	25.041	32.495	36.373	42.928
	[156]	14.286	24.242	32.069	35.575	42.031
	CSCS	17.243	26.841	34.046	37.979	43.989
	[156]	17.039	26.220	33.748	37.342	43.191
	CFCF	14.258	15.780	21.582	22.329	29.582
	[156]	14.170	15.680	21.787	29.425	31.522
	CFSF	12.020	13.755	20.987	28.448	30.647
	[156]	11.923	13.639	20.526	28.278	30.478
	FCFS	6.4404	12.066	22.289	23.700	24.264
	[156]	6.3302	11.691	21.817	23.266	24.229
$(-45, 45)_3$	CCCC	39.578	41.011	47.437	50.069	56.787
	[156]	39.462	40.872	46.916	49.538	56.500
	SSSS	24.859	26.993	41.180	41.518	43.026
	[156]	24.711	27.023	40.721	41.241	42.692
	CSCS	29.705	36.948	41.861	47.041	52.652
	[156]	29.541	36.743	41.874	46.489	51.782
	CFCF	15.953	18.418	27.966	30.240	32.873
	[156]	16.158	18.741	27.508	30.034	33.255
	CFSF	13.908	14.336	23.002	23.998	29.321
	[156]	14.048	14.661	22.691	23.639	29.046
	FCFS	4.9677	17.031	18.183	29.243	29.477
	[156]	4.9697	16.198	18.107	28.769	29.241

Chapter 5

Carrera's Unified Formulation (CUF)

5.1 Doubly curved shells and rectangular plates

Carrera's unified formulation is a technique for implementing so higher-order theory of structures such as beam, plate and shell. This analysis is two-dimensional or quasi-3D. In this chapter, spectral collocation method is applied on the (CUF) model for free vibration analysis of composite plates and shells. This technique can cover all the thickness ratio (thin and thick), lay-up of composite laminated such as symmetric and non-symmetric structures and so higher-order expansion of variables. In this technique, all variables (u, v, w) are assumed in higher-order in terms of thickness direction not only (u, v) similar to FSDT or TSDT but also w. Implementing of formulation is based on principle of virtual displacement (PVD). Variation of internal work is equal to strain energy. Stresses are obtained from Hook's law and the strains from the geometrical relations, which for each layer k states as

$$\int_{\Omega^k}\int_{z_k}^{z_{k+1}} \delta\epsilon^{k^T}\sigma^k dz_k d\Omega^k = -\int_{\Omega^k}\int_{z_k}^{z_{k+1}} \delta\mathbf{u}^{k^T}\rho\ddot{\mathbf{u}} dz_k d\Omega^k \tag{5.1}$$

σ^k and ϵ^k are stresses and strains for each lamina, respectively which are divided into in-plane and out-of-plane (normal) components as

$$\sigma_p^k = \left\{ \begin{array}{c} \sigma_{\alpha\alpha}^k \\ \sigma_{\beta\beta}^k \\ \sigma_{\alpha\beta}^k \end{array} \right\} \qquad \sigma_n^k = \left\{ \begin{array}{c} \sigma_{\alpha z}^k \\ \sigma_{\beta z}^k \\ \sigma_{zz}^k \end{array} \right\} \qquad \epsilon_p^k = \left\{ \begin{array}{c} \epsilon_{\alpha\alpha}^k \\ \epsilon_{\beta\beta}^k \\ \epsilon_{\alpha\beta}^k \end{array} \right\} \qquad \epsilon_n^k = \left\{ \begin{array}{c} \epsilon_{\alpha z}^k \\ \epsilon_{\beta z}^k \\ \epsilon_{zz}^k \end{array} \right\} \tag{5.2}$$

where α, β are curvilinear orthogonal coordinates and are in-plane coordinates, and z is the normal coordinate or out-of-plane coordinate.

After these definitions eq. (5.1) could be rewritten in matrix form as

$$\begin{aligned} &\int_{\Omega^k}\int_{z_k}^{z_{k+1}} \left[\begin{array}{cc} \delta\epsilon_{pG}^{k^T} & \delta\epsilon_{nG}^{k^T} \end{array} \right] \left\{ \begin{array}{c} \sigma_{pH}^k \\ \sigma_{nH}^k \end{array} \right\} dz_k d\Omega^k \\ &\quad = \int_{\Omega^k}\int_{z_k}^{z_{k+1}} (\delta\epsilon_{pG}^{k^T}\sigma_{pH}^k + \delta\epsilon_{nG}^{k^T}\sigma_{nH}^k) dz_k d\Omega^k = -\int_{\Omega^k}\int_{z_k}^{z_{k+1}} \delta\mathbf{u}^{k^T}\rho\ddot{\mathbf{u}} dz_k d\Omega^k \end{aligned} \tag{5.3}$$

where subscript G for strains means that strains which are obtained from geometrical relations, subscript H for stresses means stresses which are evaluated using Hooke's law, subscript p, n are related to in-plane and out of plane strains and stresses and superscript T means transpose of array. The following Nth-order expansion have been chosen in the thickness direction of each layer ($k = 1, 2, ...N_l$).

The constitutive equation for each lamina can be written as

$$\begin{aligned} \sigma^k_{pH} &= \tilde{C}^k_{pp}\epsilon^k_{pG} + \tilde{C}^k_{pn}\epsilon^k_{nG} \\ \sigma^k_{nH} &= \tilde{C}^k_{np}\epsilon^k_{pG} + \tilde{C}^k_{nn}\epsilon^k_{nG} \end{aligned} \tag{5.4}$$

and the relation among displacements and strains are as follow

$$\begin{aligned} \epsilon^k_{pG} &= D_p\mathbf{u}^k + A_p\mathbf{u}^k \\ \epsilon^k_{nG} &= D_n\mathbf{u}^k + \lambda_D A_n\mathbf{u}^k \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} D_p &= \begin{bmatrix} \frac{\partial_\alpha}{H^k_\alpha} & 0 & 0 \\ 0 & \frac{\partial_\beta}{H^k_\beta} & 0 \\ \frac{\partial_\beta}{H^k_\beta} & \frac{\partial_\alpha}{H^k_\alpha} & 0 \end{bmatrix}, \quad D_n = \begin{bmatrix} \partial_z & 0 & \frac{\partial_\alpha}{H^k_\alpha} \\ 0 & \partial_z & \frac{\partial_\beta}{H^k_\beta} \\ 0 & 0 & \partial_z \end{bmatrix} \\ A_p &= \begin{bmatrix} 0 & 0 & \frac{1}{H^k_\alpha R^k_\alpha} \\ 0 & 0 & \frac{1}{H^k_\beta R^k_\beta} \\ 0 & 0 & 0 \end{bmatrix}, \quad A_n = -\begin{bmatrix} \frac{1}{H^k_\alpha R^k_\alpha} & 0 & 0 \\ 0 & \frac{1}{H^k_\beta R^k_\beta} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{5.6}$$

and also

$$H^k_\alpha = A(1 + \frac{z_k}{R^k_\alpha}), \quad H^k_\beta = B(1 + \frac{z_k}{R^k_\beta}), \quad H^k_z = 1 \tag{5.7}$$

R^k_α, R^k_β are radius of curvature in the associated coordinates. Eq. (5.7) is associated to the thick shells because for plates and thin shells $(1 + \frac{z_k}{R^k_{\alpha,\beta}})$ are one. A_p, A_n are just used for shells such as thin or thick. λ_D is a parameter which is one for general shell and zero for Donnell-Mushtary type shallow shell. D_p and D_n are differential operator matrix apply on displacements but A_p and A_n are matrices which are multiplied by displacements. A, B are coefficients of the first fundamental form of Ω_k. For simplicity in this section, it is assumed that $A = B = 1$. Curvature of cylindrical, spherical and toroidal shell are constant, if curvature or curvatures are not constant A or B, or A and B are not constant. For plates also these mentioned coefficients are always identity and the curvatures associated to α, β coordinates are infinity.

In-plane and out-of-plane devision is also done for $\tilde{C}^k_{ij}$, thus the stiffness coefficients are as

$$\tilde{C}^k_{pp} = \begin{bmatrix} \tilde{C}^k_{11} & \tilde{C}^k_{12} & \tilde{C}^k_{16} \\ \tilde{C}^k_{12} & \tilde{C}^k_{22} & \tilde{C}^k_{26} \\ \tilde{C}^k_{16} & \tilde{C}^k_{26} & \tilde{C}^k_{66} \end{bmatrix}, \qquad \tilde{C}^k_{pn} = \tilde{C}^{k^T}_{pn} = \begin{bmatrix} 0 & 0 & \tilde{C}^k_{13} \\ 0 & 0 & \tilde{C}^k_{23} \\ 0 & 0 & \tilde{C}^k_{36} \end{bmatrix}$$
$$\tilde{C}^k_{nn} = \begin{bmatrix} \tilde{C}^k_{55} & \tilde{C}^k_{45} & 0 \\ \tilde{C}^k_{45} & \tilde{C}^k_{44} & 0 \\ 0 & 0 & \tilde{C}^k_{33} \end{bmatrix} \tag{5.8}$$

The $\tilde{C}^k_{ij}$ coefficients are evaluated from the below formula as

Hook's law in three-dimensional analysis is

$$\sigma = C\epsilon \rightarrow \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & C_{13} \\ C_{12} & C_{22} & 0 & 0 & 0 & C_{23} \\ 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ C_{13} & C_{23} & 0 & 0 & 0 & C_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{23} \\ \epsilon_{33} \end{Bmatrix} \tag{5.9}$$

where stiffness coefficients and poison' ratio for an orthotropic material are as:

$$\begin{array}{lll} C_{11} = E_1 \dfrac{1-\nu_{23}\nu_{32}}{\Delta} & C_{22} = E_2 \dfrac{1-\nu_{13}\nu_{31}}{\Delta} & C_{33} = E_3 \dfrac{1-\nu_{12}\nu_{21}}{\Delta} \\ C_{12} = E_1 \dfrac{\nu_{21}+\nu_{23}\nu_{31}}{\Delta} & C_{13} = E_1 \dfrac{\nu_{31}+\nu_{21}\nu_{32}}{\Delta} & C_{23} = E_2 \dfrac{\nu_{32}+\nu_{12}\nu_{31}}{\Delta} \\ C_{44} = G_{23} & C_{55} = G_{13} & C_{66} = G_{12} \\ \nu_{21} = \dfrac{E_2}{E_1}\nu_{12} & \nu_{31} = \dfrac{E_3}{E_1}\nu_{13} & \nu_{32} = \dfrac{E_3}{E_2}\nu_{23} \\ \Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{12}\nu_{32}\nu_{13} & & \end{array} \tag{5.10}$$

For composite laminated, stiffness matrix can be obtained as following

$$\tilde{C} = TCT' = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{16} & 0 & 0 & \tilde{C}_{13} \\ \tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{26} & 0 & 0 & \tilde{C}_{23} \\ \tilde{C}_{16} & \tilde{C}_{26} & \tilde{C}_{66} & 0 & 0 & \tilde{C}_{36} \\ 0 & 0 & 0 & \tilde{C}_{55} & \tilde{C}_{45} & 0 \\ 0 & 0 & 0 & \tilde{C}_{45} & \tilde{C}_{44} & 0 \\ \tilde{C}_{13} & \tilde{C}_{23} & \tilde{C}_{36} & 0 & 0 & \tilde{C}_{33} \end{bmatrix} \tag{5.11}$$

where transformation matrix is as

$$T = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta & 0 & 0 & 0 \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta & 0 & 0 & 0 \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.12}$$

and θ is orientation angle of each lamina.

According to the Carrera's unified formulation (CUF) and assuming harmonic motion with circular frequency ω, the displacements vector for each layer is expressed as

$$\mathbf{u}^k(\zeta,\eta,\xi_k,t) = \left\{ \begin{array}{c} u^k(\zeta,\eta,\xi_k,t) \\ v^k(\zeta,\eta,\xi_k,t) \\ w^k(\zeta,\eta,\xi_k,t) \end{array} \right\} = \left\{ \begin{array}{c} F_\tau(\xi_k)u^k_\tau(\zeta,\eta) \\ F_\tau(\xi_k)v^k_\tau(\zeta,\eta) \\ F_\tau(\xi_k)w^k_\tau(\zeta,\eta) \end{array} \right\} e^{i\omega t} = F_\tau(\xi_k)\mathbf{u}^k_\tau(\zeta,\eta)e^{i\omega t} \quad (5.13)$$

$\tau = t, b, r(r = 2, ..., N_n)$ is an index related on the chosed theory, $F_\tau(\xi_k)$ are assumed thickness function and N_n is order of theory. ξ_k is a local dimensionless layer coordinate $(-1 \leq \xi_k \leq 1)$ related to the global thickness coordinate z and is defined as

$$\xi_k = \frac{2}{z_{topk} - z_{botk}} z - \frac{z_{topk} + z_{botk}}{z_{topk} - z_{botk}} \quad (5.14)$$

where z_{topk} and z_{botk} are correspond to top and bottom surface of each layer k. It is noted that in (CUF) technique in spite of other theories of plates and shells does not need to do dimensionless over displacements. Consider a plate or shell with a length and b width in (α, β) coordinates, only parameter $\beta = a/b$ is used as dimensionless parameter. In eq. (5.13) displacements vector is assumed as

$$\mathbf{u}^k = (F_t\mathbf{u}^k_t + F_b\mathbf{u}^k_b + F_r\mathbf{u}^k_r)e^{i\omega t} = F_\tau\mathbf{u}^k_\tau e^{i\omega t} \qquad \tau = 0, 1, ..., N_n \quad (5.15)$$

and also the summation convention over indices is as

$$\mathbf{u}^k = (F_t\mathbf{u}^k_t + F_2\mathbf{u}^k_2 + ... + F_N\mathbf{u}^k_N + F_b\mathbf{u}^k_b)e^{i\omega t} \quad (5.16)$$

Subscript t is correspond to top layer and subscript b is correspond to bottom layer and subscript r is defined for the layers in between. F_0 and F_1 define functions related to top and bottom layer or vice versa and other functions are correspond to other layers. Thus the above relation can be written in the form below

$$\mathbf{u}^k = F_0\mathbf{u}^k_0 + F_1\mathbf{u}^k_1 + F_r\mathbf{u}^k_r \qquad r = 2, ..., N_n \quad (5.17)$$

where $F_0 = 1$, $F_1 = z$, $F_{N_n} = Z^{N_n}$.

Thickness functions $F_\tau(\xi_k)$ can be arbitrarily chosed for better description of deformed plates and shells in each lamina which can be in power series form $(1, z, z^2, ..., z^N_n)$ or other functions like zig-zag function. Basically with this assumption, displacements assumed in multiplication of two variables F_τ and u_τ which are acting on thickness direction and surface, respectively. Thus, F_τ just has differentiation with respect to z and u_τ has differentiation with respect to surface coordinates. Expansion of eq. (5.17) for all three displacements are (see Figure 5.4)

$$\begin{aligned} u(\zeta,\eta,\xi_k) &= u^k_0 + zu^k_1 + z^2u^k_2 + ... + z^{N_n}u^k_{N_n} \\ v(\zeta,\eta,\xi_k) &= v^k_0 + zv^k_1 + z^2v^k_2 + ... + z^{N_n}v^k_{N_n} \\ w(\zeta,\eta,\xi_k) &= w^k_0 + zw^k_1 + z^2w^k_2 + ... + z^{N_n}w^k_{N_n} \end{aligned} \quad (5.18)$$

This type of definition for the thickness function leads to a theory which is called Equivalant single layer (ESL) naming EDN_n. i.e., expansion of $ED2$ based on this theory is

$$\begin{aligned} u^k &= u_t^k + zu_b^k + z^2u_2^k \\ v^k &= u_t^k + zu_b^k + z^2u_2^k \\ w^k &= u_t^k + zu_b^k + z^2u_2^k \end{aligned} \tag{5.19}$$

In this theory entire thickness assumes as a layer without changing in displacement and behavior with changing lamina, and the displacement degrees of freedom for a given EDN_n is $3(N_n + 1)(N + 1)^2$. But an effective way is to not considering whole thickness as one layer and to satisfy the inter-laminar continuity of the displacements is to choose

$$\begin{aligned} F_t(_k) &= \frac{1+\xi_k}{2} \\ F_t(_k) &= \frac{1-\xi_k}{2} \\ F_r(_k) &= P_r(\xi_k) - P_{r-2}(\xi_k) \qquad r = 2, 3, ..., N_n \end{aligned} \tag{5.20}$$

where $P_i(\xi_k)$ is the Legendre polynomial of ith order, and the inter-laminar continuity among layers is

$$\mathbf{u}_t^k = \mathbf{u}_b^{k+1} \qquad k = 1, ..., N_l - 1 \tag{5.21}$$

It means that the displacement for the top surface of kth lamina should be equal to displacement for the bottom surface of the $(k + 1)$th lamina. This type of formulation for the thickness function proceeds to layer-wise theory and naming LDN_n (see Figure 5.5). for example based on this theory formulation of $LD2$ is

$$\begin{aligned} u^k &= \frac{1+\xi_k}{2}u_t^k + \frac{1-\xi_k}{2}u_b^k + \frac{3\xi_k^2-3}{2}u_2^k \\ v^k &= \frac{1+\xi_k}{2}v_t^k + \frac{1-\xi_k}{2}v_b^k + \frac{3\xi_k^2-3}{2}v_2^k \\ w^k &= \frac{1+\xi_k}{2}w_t^k + \frac{1-\xi_k}{2}w_b^k + \frac{3\xi_k^2-3}{2}w_2^k \end{aligned} \tag{5.22}$$

For layer-wise description, displacements of each layer considered independently. The properties of the chosen functions, compatibility of the displacement and equilibrium of the transverse stress components and the additional C_z^0 requirements are as

$$\zeta_k = \begin{cases} 1 : F_t = 1, & F_b = 0, \quad F_r = 0 \\ -1 : F_t = 0, & F_b = 1, \quad F_r = 0 \end{cases} \qquad \begin{cases} u_t^k = u_b^{(k+1)} & k = 1, N_l - 1 \\ \sigma_{nt}^k = \sigma_{nb}^{(k+1)} & k = 1, N_l - 1 \end{cases} \tag{5.23}$$

Displacement degrees of freedom for a given LDN_n is $\Big(3(N_n+1)N_l - 3(N_l-1)\Big)(N+1)^2$.

Substituting eqs. (5.4), (5.5) and (5.15) into eq. (5.3), yields

$$\int_{\Omega^k}\int_{z_k}^{z_{k+1}}\Big(\delta(D_p\mathbf{u_\tau}^k + A_p\mathbf{u_\tau}^k)^T[\tilde{C}_{pp}(D_p\mathbf{u_s}^k + A_p\mathbf{u_s}^k) + \tilde{C}_{pn}(D_n\mathbf{u_s}^k + A_n\mathbf{u_s}^k)]\Big)dz_k d\Omega^k + ...$$
$$= \int_{\Omega^k}\int_{z_k}^{z_{k+1}}\Big(\delta\mathbf{u_\tau}^{k^T}(D_p^T + A_p^T)[\tilde{C}_{pp}(D_p\mathbf{u_s}^k + A_p\mathbf{u_s}^k) + \tilde{C}_{pn}(D_n\mathbf{u_s}^k + A_n\mathbf{u_s}^k)]\Big)dz_k d\Omega^k + ...$$
$$= -\int_{\Omega^k}\int_{z_k}^{z_{k+1}}\Big(\delta\mathbf{u_\tau}^{k^T}D_p^T[\tilde{C}_{pp}(D_p\mathbf{u_s}^k + A_p\mathbf{u_s}^k) + \tilde{C}_{pn}(D_n\mathbf{u_s}^k + A_n\mathbf{u_s}^k)]\Big)dz_k d\Omega^k$$
$$+ \int_{\Gamma^k}\int_{z_k}^{z_{k+1}}\Big(\delta\mathbf{u_\tau}^{k^T}[\tilde{C}_{pp}(D_p\mathbf{u_s}^k + A_p\mathbf{u_s}^k) + \tilde{C}_{pn}(D_n\mathbf{u_s}^k + A_n\mathbf{u_s}^k)]\Big)dz_k d\Gamma^k$$
$$+ \int_{\Omega^k}\int_{z_k}^{z_{k+1}}\Big(\delta\mathbf{u_\tau}^{k^T}A_p^T[\tilde{C}_{pp}(D_p\mathbf{u_s}^k + A_p\mathbf{u_s}^k) + \tilde{C}_{pn}(D_n\mathbf{u_s}^k + A_n\mathbf{u_s}^k)]\Big)dz_k d\Omega^k + ...$$
$$= -\int_{\Omega^k}\int_{z_k}^{z_{k+1}}\delta\mathbf{u_\tau}^{k^T}\rho\ddot{\mathbf{u}}_\mathbf{s} dz_k d\Omega^k \tag{5.24}$$

It is to be noted that, multiplication of D_p and D_n which are 3×3 matrices by $\mathbf{u}$ which is 3×1 vector yields a 3×1 vector. σ_p and σ_n also are obtained from multiplication of strains which are (3×1) matrices by determined coefficients which yield 3×1 matrices. Regarding eq. (5.3) transpose of strains which are 1×3 multiply by stresses which are 3×1 yields (1×1) matrix including many terms such as equations of motion and natural boundary conditions after doing integration by parts.

After doing integration by parts for all terms, Eq. (5.24) is simplified in the form below as

$$-\int_{\Omega^k}\int_{z_k}^{z_{k+1}}\delta\mathbf{u}_\tau^{k^T}\mathcal{L}^{k\tau s}\mathbf{u}_s^k dz_k d\Omega^k + \int_{\Gamma^k}\int_{z_k}^{z_{k+1}}\delta\mathbf{u}_\tau^{k^T}\mathcal{B}^{k\tau s}\mathbf{u}_s^k dz d\Gamma^k$$
$$= -\int_{\Omega^k}\int_{z_k}^{z_{k+1}}\delta\mathbf{u}_\tau^{k^T}\rho\ddot{\mathbf{u}}_s^k dz_k d\Omega^k \tag{5.25}$$

where $\mathcal{L}^{k\tau s}$ and $\mathcal{B}^{k\tau s}$ are the nuclei of stiffness and boundary condition matrices, respectively. Apply integral in the thickness direction using the formula

$$\Big(J^{k\tau s}, J_\alpha^{k\tau s}, J_\beta^{k\tau s}, J_{\frac{\alpha}{\beta}}^{k\tau s}, J_{\frac{\beta}{\alpha}}^{k\tau s}, J_{\alpha\beta}^{k\tau s}\Big) = \int_{z_k}^{z_{k+1}} F_\tau F_s\Big(1, H_\alpha^k, H_\beta^k, \frac{H_\alpha^k}{H_\beta^k}, \frac{H_\beta^k}{H_\alpha^k}, H_\alpha^k H_\beta^k\Big)dz$$
$$\Big(J^{k\tau_z s}, J_\alpha^{k\tau_z s}, J_\beta^{k\tau_z s}, J_{\frac{\alpha}{\beta}}^{k\tau_z s}, J_{\frac{\beta}{\alpha}}^{k\tau_z s}, J_{\alpha\beta}^{k\tau_z s}\Big) = \int_{z_k}^{z_{k+1}} F_{\tau_z} F_s\Big(1, H_\alpha^k, H_\beta^k, \frac{H_\alpha^k}{H_\beta^k}, \frac{H_\beta^k}{H_\alpha^k}, H_\alpha^k H_\beta^k\Big)dz$$
$$\Big(J^{k\tau s_z}, J_\alpha^{k\tau s_z}, J_\beta^{k\tau s_z}, J_{\frac{\alpha}{\beta}}^{k\tau s_z}, J_{\frac{\beta}{\alpha}}^{k\tau s_z}, J_{\alpha\beta}^{k\tau s_z}\Big) = \int_{z_k}^{z_{k+1}} F_\tau F_{sz}\Big(1, H_\alpha^k, H_\beta^k, \frac{H_\alpha^k}{H_\beta^k}, \frac{H_\beta^k}{H_\alpha^k}, H_\alpha^k H_\beta^k\Big)dz$$
$$\Big(J^{k\tau_z s_z}, J_\alpha^{k\tau_z s_z}, J_\beta^{k\tau_z s_z}, J_{\frac{\alpha}{\beta}}^{k\tau_z s_z}, J_{\frac{\beta}{\alpha}}^{k\tau_z s_z}, J_{\alpha\beta}^{k\tau_z s_z}\Big) = \int_{z_k}^{z_{k+1}} F_{\tau_z} F_{sz}\Big(1, H_\alpha^k, H_\beta^k, \frac{H_\alpha^k}{H_\beta^k}, \frac{H_\beta^k}{H_\alpha^k}, H_\alpha^k H_\beta^k\Big)dz \tag{5.26}$$

and simplified form for plates are as

$$J^{k\tau s} = J_\alpha^{k\tau s} = J_\beta^{k\tau s} = J_{\alpha\beta}^{k\tau s} = J_{\alpha/\beta}^{k\tau s} = J_{\beta/\alpha}^{k\tau s} = E^{k\tau s}$$
$$J^{k\tau s_z} = J_\alpha^{k\tau s_z} = J_\beta^{k\tau s_z} = J_{\alpha\beta}^{k\tau s_z} = J_{\alpha/\beta}^{k\tau s_z} = J_{\beta/\alpha}^{k\tau s_z} = E^{k\tau s_z}$$
$$J^{k\tau_z s} = J_\alpha^{k\tau_z s} = J_\beta^{k\tau_z s} = J_{\alpha\beta}^{k\tau_z s} = J_{\alpha/\beta}^{k\tau_z s} = J_{\beta/\alpha}^{k\tau_z s} = E^{k\tau_z s}$$
$$J^{k\tau_z s_z} = J_\alpha^{k\tau_z s_z} = J_\beta^{k\tau_z s_z} = J_{\alpha\beta}^{k\tau_z s_z} = J_{\alpha/\beta}^{k\tau_z s_z} = J_{\beta/\alpha}^{k\tau_z s_z} = E^{k\tau_z s_z} \tag{5.27}$$

Eq. (5.25) is simplified as

$$\begin{aligned}\mathcal{L}^{k\tau s}\mathbf{u}_s^k &= \rho^k J_{\alpha\beta}^{\tau s\,k}\ddot{\mathbf{u}}_s^k\\ \mathcal{B}^{k\tau s}\mathbf{u}_s^k &= 0\end{aligned} \tag{5.28}$$

$\mathcal{L}^{k\tau s}$ and $\mathcal{B}^{k\tau s}$ are the nuclei with $3(N+1)^2 \times 3(N+1)^2$ size. In Ritz or finite element methods each of these elements are scalar and so nuclei is (3×3). After expansion through the τ and s, assemblage based on ESL or LW description, new matrix repeats through rows and columns in a big matrix. The number of these rows and columns are equal and also equal to order of Ritz expansion or number of nodes of the considered 2D structural elements, respectively.

But in the presented method each of the elements of nuclei are a matrix by order $(N+1)^2\times(N+1)^2$ (see Figure 5.1) and so nuclei itself is $3(N+1)^2\times3(N+1)^2$. Nuclei is expanded via τ and s which comes from the chosed theory, i.e., for $ED1$ or $LD1$, $N_n = 1$ and $(\tau, s = 0, 1)$ or in $ED2$ or $LD2$, $N_n = 2$ and $(\tau, s = 0, 1, 2)$. Finally, according to equivalent single layer theory or layer-wise theory matrix is assembled. Therefore, finite element discretization is applied at the end of procedure for analysis of (CUF) based on finite element but discretization based on the chebyshev polynimial is applied at first during (CUF) analysis based on chebyshev spectral method.

Generally, for all structures after multiplication of matrices and vectors inside of integrals and doing integration by parts and factorization with respect to $\delta u^{k^T}, \delta v^{k^T}$ and δw^{k^T}, it is summarized as

$$\begin{aligned}
&-\int_{\Omega^k}\int_{z_k}^{z_{k+1}} \delta u^{k^T}(\mathcal{L}_{11}^{\tau s^k}u^k + \mathcal{L}_{12}^{\tau s^k}v^k + \mathcal{L}_{13}^{\tau s^k}w^k)dz_k d\Omega_k + \int_{\Gamma^k}\int_{z_k}^{z_{k+1}} \delta u^{k^T}(\mathcal{B}_{11}^{\tau s^k}u^k + \mathcal{B}_{12}^{\tau s^k}v^k + \mathcal{B}_{13}^{\tau s^k}w^k)dz_k d\Gamma_k\\
&-\int_{\Omega^k}\int_{z_k}^{z_{k+1}} \delta v^{k^T}(\mathcal{L}_{21}^{\tau s^k}u^k + \mathcal{L}_{22}^{\tau s^k}v^k + \mathcal{L}_{23}^{\tau s^k}w^k)dz_k d\Omega_k + \int_{\Gamma^k}\int_{z_k}^{z_{k+1}} \delta v^{k^T}(\mathcal{B}_{21}^{\tau s^k}u^k + \mathcal{B}_{22}^{\tau s^k}v^k + \mathcal{B}_{23}^{\tau s^k}w^k)dz_k d\Gamma_k\\
&-\int_{\Omega^k}\int_{z_k}^{z_{k+1}} \delta w^{k^T}(\mathcal{L}_{31}^{\tau s^k}u^k + \mathcal{L}_{32}^{\tau s^k}v^k + \mathcal{L}_{33}^{\tau s^k}w^k)dz_k d\Omega_k + \int_{\Gamma^k}\int_{z_k}^{z_{k+1}} \delta w^{k^T}(\mathcal{B}_{31}^{\tau s^k}u^k + \mathcal{B}_{32}^{\tau s^k}v^k + \mathcal{B}_{33}^{\tau s^k}w^k)dz_k d\Gamma_k\\
&= -\int_{\Omega^k}\int_{z_k}^{z_{k+1}} (\delta u^{k^T}\rho^k\ddot{u}^k + \delta v^{k^T}\rho^k\ddot{v}^k + \delta w^{k^T}\rho^k\ddot{w}^k)dz_k d\Omega_k
\end{aligned} \tag{5.29}$$

Therefore, the equations result into nuclei of stiffness and nuclei of natural boundary conditions, respectively as

$$\begin{aligned}
\delta u^{k^T} &: \mathcal{L}_{11}^{\tau s^k}u^k + \mathcal{L}_{12}^{\tau s^k}v^k + \mathcal{L}_{13}^{\tau s^k}w^k = \rho^k\ddot{u}^k\\
\delta v^{k^T} &: \mathcal{L}_{21}^{\tau s^k}u^k + \mathcal{L}_{22}^{\tau s^k}v^k + \mathcal{L}_{23}^{\tau s^k}w^k = \rho^k\ddot{v}^k\\
\delta w^{k^T} &: \mathcal{L}_{31}^{\tau s^k}u^k + \mathcal{L}_{32}^{\tau s^k}v^k + \mathcal{L}_{33}^{\tau s^k}w^k = \rho^k\ddot{w}^k
\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}
\delta u^{k^T} &: \mathcal{B}_{11}^{\tau s^k}u^k + \mathcal{B}_{12}^{\tau s^k}v^k + \mathcal{B}_{13}^{\tau s^k}w^k = 0\\
\delta v^{k^T} &: \mathcal{B}_{21}^{\tau s^k}u^k + \mathcal{B}_{22}^{\tau s^k}v^k + \mathcal{B}_{23}^{\tau s^k}w^k = 0\\
\delta w^{k^T} &: \mathcal{B}_{31}^{\tau s^k}u^k + \mathcal{B}_{32}^{\tau s^k}v^k + \mathcal{B}_{33}^{\tau s^k}w^k = 0
\end{aligned} \tag{5.31}$$

Nuclei of shell' stiffness

$$
\begin{aligned}
\mathcal{L}_{11}^{\tau s^k} &= -\Big(\tilde{C}^k_{11} J^{\tau s}_{\frac{\beta}{\alpha}} \partial_{\alpha\alpha} + 2\tilde{C}^k_{16} J^{\tau s}\partial_{\alpha\beta} + \tilde{C}^k_{66} J^{\tau s}_{\frac{\alpha}{\beta}}\partial_{\beta\beta}\Big) + \tilde{C}^k_{55} J^{\tau_z s_z}_{\alpha\beta} \\
&\quad +\lambda \tilde{C}^k_{55}\Big(\frac{1}{R_\alpha^2} J^{\tau s}_{\frac{\beta}{\alpha}} - \frac{1}{R_\alpha} J^{\tau_z s}_{\beta} - \frac{1}{R_\alpha} J^{\tau s_z}_\beta\Big) \\
\mathcal{L}_{12}^{\tau s^k} &= -\Big((\tilde{C}^k_{12}+\tilde{C}^k_{66})J^{\tau s}\partial_{\alpha\beta} + \tilde{C}^k_{16} J^{\tau s}_{\frac{\beta}{\alpha}}\partial_{\alpha\alpha} + \tilde{C}^k_{26} J^{\tau s}_{\frac{\alpha}{\beta}}\partial_{\beta\beta}\Big) + \tilde{C}^k_{45} J^{\tau_z s_z}_{\alpha\beta} \\
&\quad +\lambda\tilde{C}^k_{45}\Big(\frac{1}{R_\alpha R_\beta}J^{\tau s} - \frac{1}{R_\alpha}J^{\tau s_z}_\beta - \frac{1}{R_\beta}J^{\tau_z s}_\alpha\Big) \\
\mathcal{L}_{13}^{\tau s^k} &= -\Big(\frac{\tilde{C}^k_{11}}{R_\alpha}J^{\tau s}_{\frac{\beta}{\alpha}} + \frac{\tilde{C}^k_{12}}{R_\beta}J^{\tau s} + \tilde{C}_{13} J^{\tau s_z}_\beta - \tilde{C}^k_{55} J^{\tau_z s}_\beta\Big)\partial_\alpha \\
&\quad -\Big(\frac{\tilde{C}^k_{16}}{R_\alpha}J^{\tau s} + \frac{\tilde{C}^k_{26}}{R_\beta}J^{\tau s}_{\frac{\alpha}{\beta}} + \tilde{C}^k_{36}J^{\tau s_z}_\alpha - \tilde{C}^k_{45}J^{\tau_z s}_\alpha\Big)\partial_\beta \\
&\quad -\lambda\Big(\frac{\tilde{C}^k_{55}}{R_\alpha}J^{\tau s}_{\frac{\beta}{\alpha}}\partial_\alpha + \frac{\tilde{C}^k_{45}}{R_\alpha}J^{\tau s}\partial_\beta\Big) \\
\mathcal{L}_{21}^{\tau s^k} &= -\Big((\tilde{C}^k_{12}+\tilde{C}^k_{66})J^{\tau s}\partial_{\alpha\beta} + \tilde{C}^k_{16} J^{\tau s}_{\frac{\beta}{\alpha}}\partial_{\alpha\alpha} + \tilde{C}^k_{26} J^{\tau s}_{\frac{\alpha}{\beta}}\partial_{\beta\beta}\Big) + \tilde{C}^k_{45} J^{\tau_z s_z}_{\alpha\beta} \\
&\quad +\lambda\tilde{C}^k_{45}\Big(\frac{1}{R_\alpha R_\beta}J^{\tau s} - \frac{1}{R_\alpha}J^{\tau_z s}_\beta - \frac{1}{R_\beta}J^{\tau s_z}_\alpha\Big) \\
\mathcal{L}_{22}^{\tau s^k} &= -\Big(\tilde{C}^k_{22}J^{\tau s}_{\frac{\alpha}{\beta}}\partial_{\beta\beta} + 2\tilde{C}^k_{26}J^{\tau s}\partial_{\alpha\beta} + \tilde{C}^k_{66}J^{\tau s}_{\frac{\beta}{\alpha}}\partial_{\alpha\alpha} - \tilde{C}^k_{44}J^{\tau_z s_z}_{\alpha\beta}\Big) \\
&\quad +\lambda\tilde{C}^k_{44}\Big(\frac{1}{R_\beta^2}J^{\tau s}_{\frac{\alpha}{\beta}} - \frac{1}{R_\beta}J^{\tau s_z}_\alpha - \frac{1}{R_\beta}J^{\tau_z s}_\alpha\Big) \\
\mathcal{L}_{23}^{\tau s^k} &= -\Big(\frac{\tilde{C}^k_{16}}{R_\alpha}J^{\tau s}_{\frac{\beta}{\alpha}} + \frac{\tilde{C}^k_{26}}{R_\beta}J^{\tau s} + \tilde{C}_{36}J^{\tau s_z}_\beta - \tilde{C}^k_{45}J^{\tau_z s}_\beta\Big)\partial_\alpha \\
&\quad -\Big(\frac{\tilde{C}^k_{12}}{R_\alpha}J^{\tau s} + \frac{\tilde{C}^k_{22}}{R_\beta}J^{\tau s}_{\frac{\alpha}{\beta}} + \tilde{C}^k_{23}J^{\tau s_z}_\alpha - \tilde{C}^k_{44}J^{\tau_z s}_\alpha\Big)\partial_\beta \\
&\quad -\lambda\Big(\frac{\tilde{C}^k_{45}}{R_\beta}J^{\tau s}\partial_\alpha + \frac{\tilde{C}^k_{44}}{R_\beta}J^{\tau s}_{\frac{\alpha}{\beta}}\partial_\beta\Big) \\
\mathcal{L}_{31}^{\tau s^k} &= \Big(\frac{\tilde{C}^k_{11}}{R_\alpha}J^{\tau s}_{\frac{\beta}{\alpha}} + \frac{\tilde{C}^k_{12}}{R_\beta}J^{\tau s} + \tilde{C}_{13}J^{\tau_z s}_\beta - \tilde{C}^k_{55}J^{\tau s_z}_\beta\Big)\partial_\alpha \\
&\quad +\Big(\frac{\tilde{C}^k_{16}}{R_\alpha}J^{\tau s} + \frac{\tilde{C}^k_{26}}{R_\beta}J^{\tau s}_{\frac{\alpha}{\beta}} + \tilde{C}^k_{36}J^{\tau_z s}_\alpha - \tilde{C}^k_{45}J^{\tau s_z}_\alpha\Big)\partial_\beta \\
&\quad +\lambda\Big(\frac{\tilde{C}^k_{55}}{R_\alpha}J^{\tau s}_{\frac{\beta}{\alpha}}\partial_\alpha + \frac{\tilde{C}^k_{45}}{R_\alpha}J^{\tau s}\partial_\beta\Big) \\
\mathcal{L}_{32}^{\tau s^k} &= \Big(\frac{\tilde{C}^k_{16}}{R_\alpha}J^{\tau s}_{\frac{\beta}{\alpha}} + \frac{\tilde{C}^k_{26}}{R_\beta}J^{\tau s} + \tilde{C}_{36}J^{\tau_z s}_\beta - \tilde{C}^k_{45}J^{\tau s_z}_\beta\Big)\partial_\alpha \\
&\quad +\Big(\frac{\tilde{C}^k_{12}}{R_\alpha}J^{\tau s} + \frac{\tilde{C}^k_{22}}{R_\beta}J^{\tau s}_{\frac{\alpha}{\beta}} + \tilde{C}^k_{23}J^{\tau_z s}_\alpha - \tilde{C}^k_{44}J^{\tau s_z}_\alpha\Big)\partial_\beta \\
&\quad +\lambda\Big(\frac{\tilde{C}^k_{45}}{R_\beta}J^{\tau s}\partial_\alpha + \frac{\tilde{C}^k_{44}}{R_\beta}J^{\tau s}_{\frac{\alpha}{\beta}}\partial_\beta\Big)
\end{aligned}
$$

$$\begin{aligned}
\mathcal{L}_{33}^{\tau s^k} &= -\Big(\tilde{C}_{55}^k J_{\frac{\beta}{\alpha}}^{\tau s}\partial_{\alpha\alpha} + 2\tilde{C}_{45}^k J^{\tau s}\partial_{\alpha\beta} + \tilde{C}_{44}^k J_{\frac{\alpha}{\beta}}^{\tau s}\partial_{\beta\beta}\Big) + \tilde{C}_{33}^k J_{\alpha\beta}^{\tau_z s_z} \\
&+\frac{1}{R_\alpha}\Big(\frac{\tilde{C}_{11}^k}{R_\alpha}J_{\frac{\beta}{\alpha}}^{\tau s} + 2\frac{\tilde{C}_{12}^k}{R_\beta}J^{\tau s} + \tilde{C}_{13}^k J_\beta^{\tau s_z} + \tilde{C}_{13}^k J_\beta^{\tau_z s}\Big) \\
&+\frac{1}{R_\beta}\Big(\frac{\tilde{C}_{22}^k}{R_\beta}J_{\frac{\alpha}{\beta}}^{\tau s} + \tilde{C}_{23}^k J_\alpha^{\tau s_z} + \tilde{C}_{23}^k J_\alpha^{\tau_z s}\Big)
\end{aligned}$$

$$\mathcal{M}_{11}^{\tau s^k} = \rho^k J_{\alpha\beta}^{\tau s k}, \quad \mathcal{M}_{11}^{\tau s^k} = \mathcal{M}_{22}^{\tau s^k} = \mathcal{M}_{33}^{\tau s^k} \tag{5.32}$$

Nuclei of shell's natural boundary conditions

$$\begin{aligned}
\mathcal{B}_{11}^{\tau s^k} &= n_\alpha\Big(\tilde{C}_{11}^k J_{\frac{\beta}{\alpha}}^{k\tau s}\partial_\alpha + \tilde{C}_{16}^k J^{k\tau s}\partial_\beta\Big) + n_\beta\Big(\tilde{C}_{16}^k J^{k\tau s}\partial_\alpha + \tilde{C}_{66}^k J_{\frac{\alpha}{\beta}}^{k\tau s}\partial_\beta\Big) \\
\mathcal{B}_{12}^{\tau s^k} &= n_\alpha\Big(\tilde{C}_{16}^k J_{\frac{\beta}{\alpha}}^{k\tau s}\partial_\alpha + \tilde{C}_{12}^k J^{k\tau s}\partial_\beta\Big) + n_\beta\Big(\tilde{C}_{66}^k J^{k\tau s}\partial_\alpha + \tilde{C}_{26}^k J_{\frac{\alpha}{\beta}}^{k\tau s}\partial_\beta\Big) \\
\mathcal{B}_{13}^{\tau s^k} &= n_\alpha\Big(\frac{1}{R_\alpha^k}\tilde{C}_{11}^k J_{\frac{\beta}{\alpha}}^{k\tau s} + \frac{1}{R_\beta^k}\tilde{C}_{12}^k J^{k\tau s} + \tilde{C}_{13}^k J_\beta^{k\tau s_z}\Big) \\
&+n_\beta\Big(\frac{1}{R_\alpha^k}\tilde{C}_{16}^k J^{k\tau s} + \frac{1}{R_\beta^k}\tilde{C}_{26}^k J_{\frac{\alpha}{\beta}}^{k\tau s} + \tilde{C}_{36}^k J_\alpha^{k\tau s_z}\Big) \\
\mathcal{B}_{21}^{\tau s^k} &= n_\alpha\Big(\tilde{C}_{16}^k J_{\frac{\beta}{\alpha}}^{k\tau s}\partial_\alpha + \tilde{C}_{66}^k J^{k\tau s}\partial_\beta\Big) + n_\beta\Big(\tilde{C}_{12}^k J^{k\tau s}\partial_\alpha + \tilde{C}_{26}^k J_{\frac{\alpha}{\beta}}^{k\tau s}\partial_\beta\Big) \\
\mathcal{B}_{22}^{\tau s^k} &= n_\alpha\Big(\tilde{C}_{66}^k J_{\frac{\beta}{\alpha}}^{k\tau s}\partial_\alpha + \tilde{C}_{26}^k J^{k\tau s}\partial_\beta\Big) + n_\beta\Big(\tilde{C}_{22}^k J_{\frac{\alpha}{\beta}}^{k\tau s}\partial_\beta + \tilde{C}_{26}^k J^{k\tau s}\partial_\beta\Big) \\
\mathcal{B}_{23}^{\tau s^k} &= n_\alpha\Big(\tilde{C}_{36}^k J_\beta^{k\tau s_z} + \frac{1}{R_\alpha^k}\tilde{C}_{16}^k J_{\frac{\beta}{\alpha}}^{k\tau s} + \tilde{C}_{26}^k J^{k\tau s}\Big) \\
&+n_\beta\Big(\tilde{C}_{23}^k J_\alpha^{k\tau s_z} + \frac{1}{R_\alpha^k}\tilde{C}_{12}^k J^{k\tau s} + \frac{1}{R_\beta^k}\tilde{C}_{22}^k J_{\frac{\alpha}{\beta}}^{k\tau s}\Big) \\
\mathcal{B}_{31}^{\tau s^k} &= n_\alpha\Big(\tilde{C}_{55}^k J_\beta^{k\tau s_z} - (\frac{\lambda}{R_\alpha})\tilde{C}_{55}^k J_{\frac{\beta}{\alpha}}^{k\tau s}\Big) + n_\beta\Big(\tilde{C}_{45}^k J^{k\tau s} - (\frac{\lambda}{R_\alpha})\tilde{C}_{45}^k J_\alpha^{k\tau s_z}\Big) \\
\mathcal{B}_{32}^{\tau s^k} &= n_\alpha\Big(\tilde{C}_{45}^k J_\beta^{k\tau s_z} - (\frac{\lambda}{R_\beta})\tilde{C}_{45}^k J^{k\tau s}\Big) + n_\beta\Big(\tilde{C}_{44}^k J_\alpha^{k\tau s_z} - (\frac{\lambda}{R_\beta})\tilde{C}_{44}^k J_{\frac{\alpha}{\beta}}^{k\tau s}\Big) \\
\mathcal{B}_{33}^{\tau s^k} &= n_\alpha\Big(\tilde{C}_{55}^k J_{\frac{\beta}{\alpha}}^{k\tau s}\partial_\alpha + \tilde{C}_{45}^k J^{k\tau s}\partial_\beta\Big) + n_\beta\Big(\tilde{C}_{44}^k J_{\frac{\alpha}{\beta}}^{k\tau s}\partial_\beta + \tilde{C}_{45}^k J^{k\tau s}\partial_\alpha\Big)
\end{aligned} \tag{5.33}$$

For composite structures in ESL (Equivalent Single Layer) theory for layer expansion, all the layers accumulate (see Figure 5.2) but in layer-wise approach all the layers expand like finite element assemblage with considering the continuity condition between the layers (see Figure 5.3).

Note that the first-order shear deformation theory (FSDT) can be easily recovered from $ED1$ theory after imposing the condition of null transverse normal stresses σ_{zz}^k and introducing a shear zz correction factor κ. The corresponding modified elastic coefficients are given by:

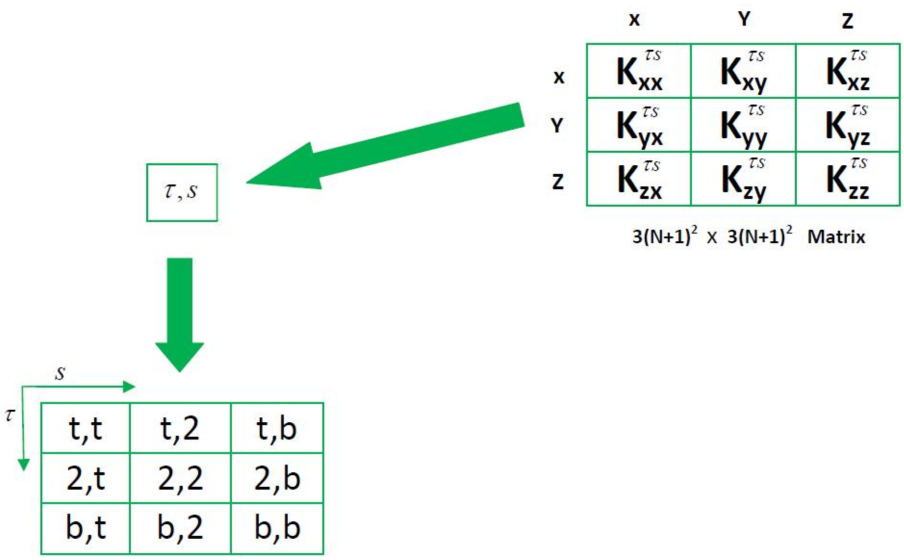

Figure 5.1: Assemblage of nuclei via τ and s

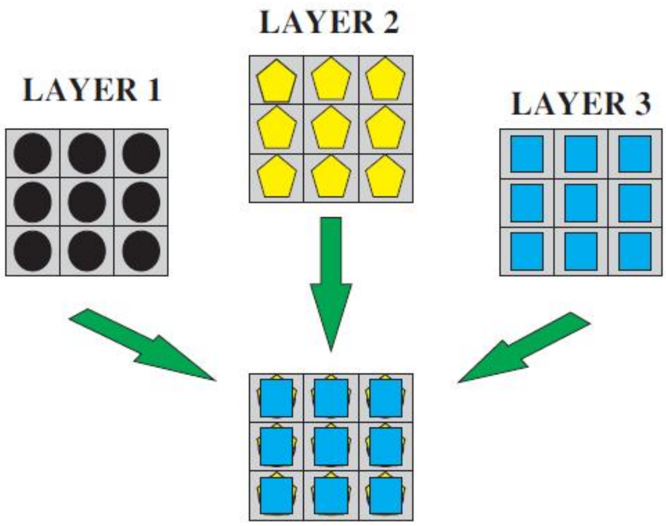

Figure 5.2: Assemblage from layer to multi-layered level in ESL description

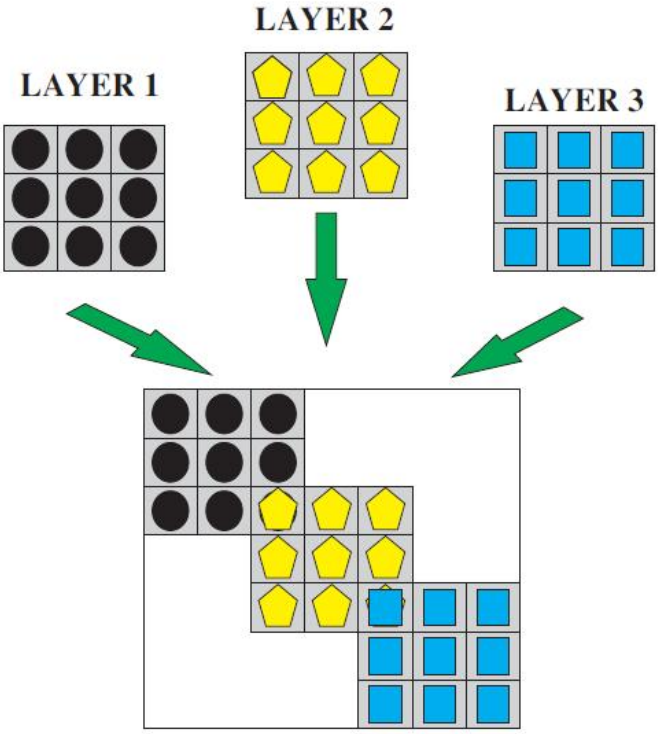

Figure 5.3: Assemblage from layer to multi-layered level in layer-wise description

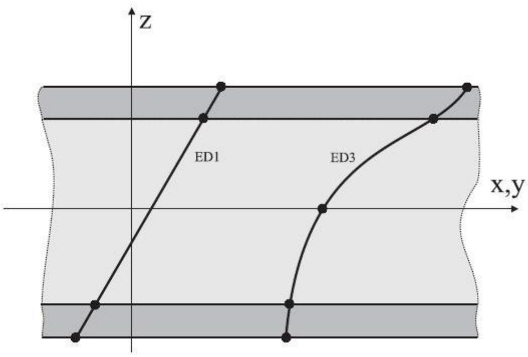

Figure 5.4: ESLM assumption. Linear and cubic cases

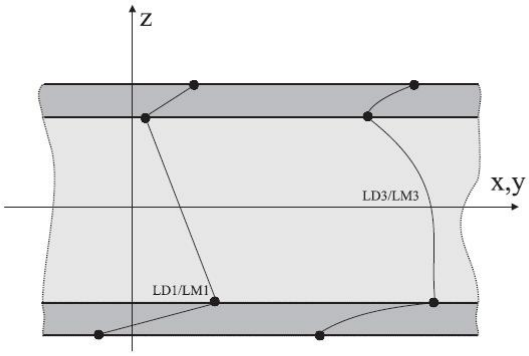

Figure 5.5: LWM assumption. Linear and cubic cases

$$\tilde{C}_{ij}^k = C_{ij}^k - \frac{C_{i3}^k C_{j3}^k}{C_{33}^k} \qquad (i,j) = 1,2$$
$$\tilde{C}_{ii}^k = C_{ii}^k \qquad i = 4,5 \tag{5.34}$$

In the case of isotropic thin rectangular plates for evaluation of natural frequencies of fully clamped, fully simply supported and CSCS boundary condition theory of ED3 is enough for CFCF and SFSF boundaries theory of ED4 is enough and for SFFF it is worth to use theory of ED5. For cantilever plate, in the case of SSFF and CCFF boundaries it is better using eq. (5.34) to recover them from ED1 because in thin plates results don't converge to good results.

5.2 Sector annular plates

Assume a sector annular plate with inner radius R_i, outer radius R_0 and thickness h. Radial direction is defined by $R_i \leq r \leq R_0$ which is after normalization by $\gamma = R_0, \zeta = r/\gamma$ and using $\beta = R_i/R_o$ becomes $\beta \leq \zeta \leq 1$ and in circumferential direction which is defined by $0 \leq \theta \leq \phi$ and $0 < \phi < 2\pi$ which is after normalization by $\eta = \frac{\theta}{\phi}$ becomes $0 \leq \eta \leq 1$, and thickness direction is $-\frac{h}{2} \leq z \leq \frac{h}{2}$ which is after normalization becomes $-1 \leq \xi_k \leq 1$.

The displacements vector $\mathbf{u} = \mathbf{u}(\zeta, \eta, \xi_k, t)$ of a generic point of the plate is given by:

$$\mathbf{u}(\zeta, \eta, \xi_k, t) = \begin{Bmatrix} u_\zeta(\zeta, \eta, \xi_k, t) \\ u_\eta(\zeta, \eta, \xi_k, t) \\ u_z(\zeta, \eta, \xi_k, t) \end{Bmatrix} \tag{5.35}$$

Strain components are grouped into an in-plane strain vector ϵ_p and out-of-plane strain vector ϵ_n as follow

$$\epsilon_p = \begin{Bmatrix} \epsilon_{\zeta\zeta} \\ \epsilon_{\eta\eta} \\ \epsilon_{\zeta\eta} \end{Bmatrix} \qquad \epsilon_n = \begin{Bmatrix} \epsilon_{\zeta z} \\ \epsilon_{\eta z} \\ \epsilon_{zz} \end{Bmatrix} \tag{5.36}$$

In linear framework, strains vector are related to displacements through the following relations

$$\epsilon_p = D_p u^k$$
$$\epsilon_n = D_n u^k \tag{5.37}$$

where

$$D_p = \frac{1}{\gamma}\begin{bmatrix} \frac{\partial}{\partial_\zeta} & 0 & 0 \\ \frac{1}{\zeta} & \frac{1}{\zeta}\frac{\partial}{\partial_\eta} & 0 \\ \frac{1}{\zeta}\frac{\partial}{\partial_\eta} & \frac{\partial}{\partial_\zeta} - \frac{1}{\zeta} & 0 \end{bmatrix}, \quad D_n = \frac{1}{\gamma}\begin{bmatrix} \partial_z & 0 & \frac{\partial}{\partial_\zeta} \\ 0 & \partial_z & \frac{1}{\zeta}\frac{\partial}{\partial_\eta} \\ 0 & 0 & \partial_z \end{bmatrix} \tag{5.38}$$

Three-dimensional Hook's law and stiffness coefficients are similar to previous section. According to the approach developed by Carrera, an entire class of two-dimensional higher-order plate theories can be described through the following indicial notation as

$$\mathbf{u}(\zeta,\eta,\xi_k,t) = F_\tau(z)\mathbf{u}_\tau^k(\zeta,\eta,t) \qquad (\tau = 0,1,...,N_n) \tag{5.39}$$

where $u_\tau(\zeta,\eta,t)$ is the displacements vector containing the unknown kinematic variables related to the specific plate theory, τ is an integer index related to the order N_n of the theory and $F_\tau(z)$ are selected functions in the thickness directions. Therefore, displacements vector can be expressed as

$$\mathbf{u}(\zeta,\eta,\xi_k,t) = F_\tau(\xi_k)\left\{\begin{array}{c} u^k(\zeta,\eta) \\ v^k(\zeta,\eta) \\ w^k(\zeta,\eta) \end{array}\right\} e^{i\omega t} \tag{5.40}$$

Principle of virtual displacement describes as:

$$\begin{aligned}
&\int_0^1\int_\beta^1\int_{-\frac{h}{2}}^{\frac{h}{2}} \Big(\delta(\epsilon_p)^T(\tilde{C}^k_{pp}\epsilon_p + \tilde{C}^k_{pn}\epsilon_n) + \delta(\epsilon_n)^T(\tilde{C}^k_{np}\epsilon_p + \tilde{C}^k_{nn}\epsilon_n)\Big)dz_k\zeta d\zeta d\eta \\
&= -\int_0^1\int_\beta^1\int_{-\frac{h}{2}}^{\frac{h}{2}} \Big(\delta(u^k)\ddot{u}^k + \delta(v^k)\ddot{v}^k + \delta(w^k)\ddot{w}^k\Big)dz_k\zeta d\zeta d\eta
\end{aligned} \tag{5.41}$$

After some substitution and integrations by part, nuclei of stiffness matrix and natural boundary condition, respectively are as follow

$$\begin{aligned}
\mathcal{L}_{11}^{\tau s^k} &= -\Big(C_{11}(\frac{1}{\zeta}\frac{\partial}{\partial\zeta} + \frac{\partial^2}{\partial\zeta^2}) - \frac{C_{22}}{\zeta^2} + \frac{C_{66}}{\zeta^2\phi^2}\frac{\partial^2}{\partial\eta^2} + \frac{2C_{16}}{\zeta\phi}\frac{\partial^2}{\partial\zeta\partial\eta}\Big)E^k_{\tau s} + C_{55}E^k_{\tau_z s_z} \\
\mathcal{L}_{12}^{\tau s^k} &= \Big(-\frac{(C_{12}+C_{66})}{\zeta\phi}\frac{\partial}{\partial\zeta\partial\eta} + \frac{(C_{22}+C_{66})}{\zeta^2\phi}\frac{\partial}{\partial\eta} - C_{16}\frac{\partial^2}{\partial\zeta^2} \\
&\qquad + C_{26}(\frac{1}{\zeta}\frac{\partial}{\partial\zeta} - \frac{1}{\zeta^2} - \frac{1}{\zeta^2\phi^2}\frac{\partial}{\partial\eta^2})\Big)E^k_{\tau s} + C_{45}E^k_{\tau_z s_z} \\
\mathcal{L}_{13}^{\tau s^k} &= -\Big(C_{13}(\frac{\partial}{\partial\zeta} + \frac{1}{\zeta}) - \frac{C_{23}}{\zeta} + \frac{C_{36}}{\zeta\phi}\frac{\partial}{\partial\eta}\Big)E^k_{\tau s_z} + \Big(C_{55}\frac{\partial}{\partial\zeta} + \frac{C_{45}}{\zeta}\frac{\partial}{\partial\zeta}\Big)E^k_{\tau_z s} \\
\mathcal{L}_{21}^{\tau s^k} &= -\Big(\frac{(C_{12}+C_{66})}{\zeta\phi}\frac{\partial}{\partial\zeta\partial\eta} + \frac{(C_{22}+C_{66})}{\zeta^2\phi}\frac{\partial}{\partial\eta} + C_{16}(\frac{\partial^2}{\partial\zeta^2} + \frac{2}{\zeta}\frac{\partial}{\partial\zeta}) \\
&\qquad + C_{26}(\frac{1}{\zeta^2\phi^2}\frac{\partial^2}{\partial\eta^2} + \frac{1}{\zeta}\frac{\partial}{\partial\zeta} + \frac{1}{\zeta^2})\Big)E^k_{\tau s} + C_{45}E^k_{\tau_z s_z} \\
\mathcal{L}_{22}^{\tau s^k} &= -\Big(\frac{C_{22}}{\zeta^2\phi^2}\frac{\partial^2}{\partial\eta^2} + C_{66}(\frac{\partial^2}{\partial\zeta^2} + \frac{1}{\zeta}\frac{\partial}{\partial\zeta} - \frac{1}{\zeta^2}) + 2C_{26}\frac{1}{\zeta\phi}\frac{\partial}{\partial\zeta\partial\eta}\Big)E^k_{\tau s} + C_{44}E^k_{\tau_z s_z} \\
\mathcal{L}_{23}^{\tau s^k} &= -\Big(\frac{C_{23}}{\zeta\phi}\frac{\partial}{\partial\eta} + C_{36}(\frac{\partial}{\partial\zeta} + \frac{2}{\zeta})\Big)E^k_{\tau s_z} + \Big(\frac{C_{44}}{\zeta\phi}\frac{\partial}{\partial\eta} + C_{45}\frac{\partial}{\partial\zeta}\Big)E^k_{\tau_z s} \\
\mathcal{L}_{31}^{\tau s^k} &= \Big(C_{13}\frac{\partial}{\partial\zeta} + \frac{C_{23}}{\zeta} + \frac{C_{36}}{\zeta\phi}\frac{\partial}{\partial\eta}\Big)E^k_{\tau_z s} - \Big(C_{55}(\frac{1}{\zeta} + \frac{\partial}{\partial\zeta}) + \frac{C_{45}}{\zeta\phi}\frac{\partial}{\partial\eta}\Big)E^k_{\tau s_z} \\
\mathcal{L}_{32}^{\tau s^k} &= \Big(\frac{C_{23}}{\zeta\phi}\frac{\partial}{\partial\eta} + C_{36}(\frac{\partial}{\partial\zeta} - \frac{1}{\zeta})\Big)E^k_{\tau_z s} - \Big(\frac{C_{44}}{\zeta\phi}\frac{\partial}{\partial\eta} + C_{45}(\frac{\partial}{\partial\zeta} + \frac{1}{\zeta})\Big)E^k_{\tau s_z} \\
\mathcal{L}_{33}^{\tau s^k} &= -\Big(\frac{C_{44}}{\zeta^2\phi^2}\frac{\partial^2}{\partial\eta^2} + C_{55}(\frac{1}{\zeta}\frac{\partial}{\partial\zeta} + \frac{\partial^2}{\partial\zeta^2}) - C_{45}(\frac{1}{\zeta}\frac{\partial^2}{\partial\zeta^2} + \frac{1}{\zeta\phi}\frac{\partial^2}{\partial\zeta\partial\eta})\Big)E^k_{\tau s} + C_{33}E^k_{\tau_z s_z}
\end{aligned}$$

$$\mathcal{M}_{11}^{\tau s^k} = \rho^k E_{\tau s}^k, \quad \mathcal{M}_{11}^{\tau s^k} = \mathcal{M}_{22}^{\tau s^k} = \mathcal{M}_{33}^{\tau s^k} \tag{5.42}$$

Natural boundary condition on straight sides

$$\begin{aligned}
\mathcal{B}_{11}^{\tau s^k} &= \Big(C_{11}\frac{\partial}{\partial\zeta} + \frac{C_{12}}{\zeta} + \frac{C_{16}}{\zeta\phi}\frac{\partial}{\partial\eta}\Big)E_{\tau s}^k \\
\mathcal{B}_{12}^{\tau s^k} &= \Big(\frac{C_{12}}{\zeta\phi}\frac{\partial}{\partial\eta} + C_{16}(\frac{\partial}{\partial\zeta} - \frac{1}{\zeta})\Big)E_{\tau s}^k \\
\mathcal{B}_{13}^{\tau s^k} &= C_{13}E_{\tau s_z}^k \\
\mathcal{B}_{21}^{\tau s^k} &= \Big(\frac{C_{66}}{\zeta\phi}\frac{\partial}{\partial\eta} + C_{16}\frac{\partial}{\partial\zeta} + \frac{C_{26}}{\zeta}\Big)E_{\tau s}^k \\
\mathcal{B}_{22}^{\tau s^k} &= \Big(C_{66}(\frac{\partial}{\partial\zeta} - \frac{1}{\zeta}) + \frac{C_{26}}{\zeta\phi}\frac{\partial}{\partial\eta}\Big)E_{\tau s}^k \\
\mathcal{B}_{23}^{\tau s^k} &= C_{36}E_{\tau s_z}^k \\
\mathcal{B}_{31}^{\tau s^k} &= C_{55}E_{\tau s_z}^k \\
\mathcal{B}_{32}^{\tau s^k} &= C_{45}E_{\tau s_z}^k \\
\mathcal{B}_{33}^{\tau s^k} &= \Big(C_{55}\frac{\partial}{\partial\zeta} + \frac{C_{45}}{\zeta}\frac{\partial}{\partial\zeta}\Big)E_{\tau s}^k
\end{aligned} \tag{5.43}$$

And also natural boundary condition on curved sides

$$\begin{aligned}
\mathcal{B}_{11}^{\tau s^k} &= \Big(\frac{C_{66}}{\zeta^2\phi}\frac{\partial}{\partial\eta} + \frac{C_{16}}{\zeta}\frac{\partial}{\partial\zeta} + \frac{C_{26}}{\zeta^2}\Big)E_{\tau s}^k \\
\mathcal{B}_{12}^{\tau s^k} &= \Big(\frac{C_{66}}{\zeta}(\frac{\partial}{\partial\zeta} - \frac{1}{\zeta}) + \frac{C_{26}}{\zeta^2\phi}\frac{\partial}{\partial\eta}\Big)E_{\tau s}^k \\
\mathcal{B}_{13}^{\tau s^k} &= \frac{C_{36}}{\zeta}E_{\tau s_z}^k \\
\mathcal{B}_{21}^{\tau s^k} &= \Big(\frac{C_{12}}{\zeta}\frac{\partial}{\partial\zeta} + \frac{C_{22}}{\zeta^2} + \frac{C_{26}}{\zeta^2\phi}\frac{\partial}{\partial\eta}\Big)E_{\tau s}^k \\
\mathcal{B}_{22}^{\tau s^k} &= \Big(\frac{C_{22}}{\zeta^2\phi}\frac{\partial}{\partial\eta} + \frac{C_{26}}{\zeta}(\frac{\partial}{\partial\zeta} - \frac{1}{\zeta})\Big)E_{\tau s}^k \\
\mathcal{B}_{23}^{\tau s^k} &= \frac{C_{23}}{\zeta}E_{\tau s_z}^k \\
\mathcal{B}_{31}^{\tau s^k} &= \frac{C_{45}}{\zeta}E_{\tau s_z}^k \\
\mathcal{B}_{32}^{\tau s^k} &= \frac{C_{44}}{\zeta}E_{\tau s_z}^k \\
\mathcal{B}_{33}^{\tau s^k} &= \Big(\frac{C_{44}}{\zeta^2\phi}\frac{\partial}{\partial\eta} + \frac{C_{45}}{\zeta}\frac{\partial}{\partial\zeta}\Big)E_{\tau s}^k
\end{aligned} \tag{5.44}$$

For $\phi = 2\pi$ (complete circular annular plate) continuity condition should be apply on ($\phi = 0$ & $\phi = 2\pi$). It means that natural boundary conditions at $\phi = 0$ minus natural boundary conditions at $\phi = 2\pi$ in the straight sides direction should be zero as follow

$$\left(\begin{bmatrix} \mathcal{B}_{11}^{\tau s^k} & \mathcal{B}_{12}^{\tau s^k} & \mathcal{B}_{13}^{\tau s^k} \\ \mathcal{B}_{21}^{\tau s^k} & \mathcal{B}_{22}^{\tau s^k} & \mathcal{B}_{23}^{\tau s^k} \\ \mathcal{B}_{31}^{\tau s^k} & \mathcal{B}_{32}^{\tau s^k} & \mathcal{B}_{33}^{\tau s^k} \end{bmatrix}_{\phi=0} - \begin{bmatrix} \mathcal{B}_{11}^{\tau s^k} & \mathcal{B}_{12}^{\tau s^k} & \mathcal{B}_{13}^{\tau s^k} \\ \mathcal{B}_{21}^{\tau s^k} & \mathcal{B}_{22}^{\tau s^k} & \mathcal{B}_{23}^{\tau s^k} \\ \mathcal{B}_{31}^{\tau s^k} & \mathcal{B}_{32}^{\tau s^k} & \mathcal{B}_{33}^{\tau s^k} \end{bmatrix}_{\phi=2\pi}\right)\begin{Bmatrix} u_s^k \\ v_s^k \\ w_s^k \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (5.45)$$

It means that instead of sector annular plate which there are four set of equations for four sides, for complete annular plate there are just three set of equations, two set equations for two curved sides and one set equation for applying continuity conditions between two straight sides.

Thickness integrals are similar to rectangular plates as

$$\begin{aligned} E_{\tau s}^k &= \int_{-\frac{h}{2}}^{\frac{h}{2}} F_\tau F_s dz \qquad & E_{\tau s_z}^k &= \int_{-\frac{h}{2}}^{\frac{h}{2}} F_\tau F_{sz} dz \\ E_{\tau_z s}^k &= \int_{-\frac{h}{2}}^{\frac{h}{2}} F_{\tau_z} F_s dz \qquad & E_{\tau_z s_z}^k &= \int_{-\frac{h}{2}}^{\frac{h}{2}} F_{\tau_z} F_{sz} dz \end{aligned} \quad (5.46)$$

5.3 Conical shells

Assume a conical shell with length a which is $a = s_o - s_i$ and thickness h (see Figure 5.6). Length direction is defined by $s_i \leq s \leq s_o$ which is after normalization by $\zeta = s/\delta, \delta = s_o$ and using $\beta = s_i/s_o$ becomes $\beta \leq \zeta \leq 1$ and the angle between the length direction and axes of conical shell is defined by $(0 < \alpha < \pi/2)$ and in circumferential direction which is defined by $0 \leq \theta \leq \phi$ and $0 < \phi < 2\pi$ which is after normalization by $\eta = \frac{\theta}{\phi}$ becomes $0 \leq \eta \leq 1$ and in thickness direction be $-\frac{h}{2} \leq z \leq \frac{h}{2}$ which is after normalization becomes $-1 \leq \xi_k \leq 1$. Half of distance between two opposite points inside of conical shell in each point is evaluated using $R(x) = R_o + x\sin\alpha$ or $R_A = R_o + a\sin\alpha$ and also the coefficients of first fundamental form of Ω_k are $A = 1, B = \zeta\sin\alpha$. As seen in conical shell parameters A, B are not constant.

The displacements vector $\mathbf{u} = \mathbf{u}(\zeta, \eta, z, t)$ of a generic point of the shell is given by

$$\mathbf{u}(\zeta, \eta, z, t) = F_\tau(\xi_k)\begin{Bmatrix} u(\zeta,\eta) \\ v(\zeta,\eta) \\ w(\zeta,\eta) \end{Bmatrix} e^{i\omega t} \quad (5.47)$$

Strain components are grouped into an in-plane strain vector ϵ_p and out-of-plane strain vector ϵ_n are as follow

$$\epsilon_p = \begin{Bmatrix} \epsilon_{\zeta\zeta} \\ \epsilon_{\eta\eta} \\ \epsilon_{\zeta\eta} \end{Bmatrix} \qquad \epsilon_n = \begin{Bmatrix} \epsilon_{\zeta z} \\ \epsilon_{\eta z} \\ \epsilon_{zz} \end{Bmatrix} \quad (5.48)$$

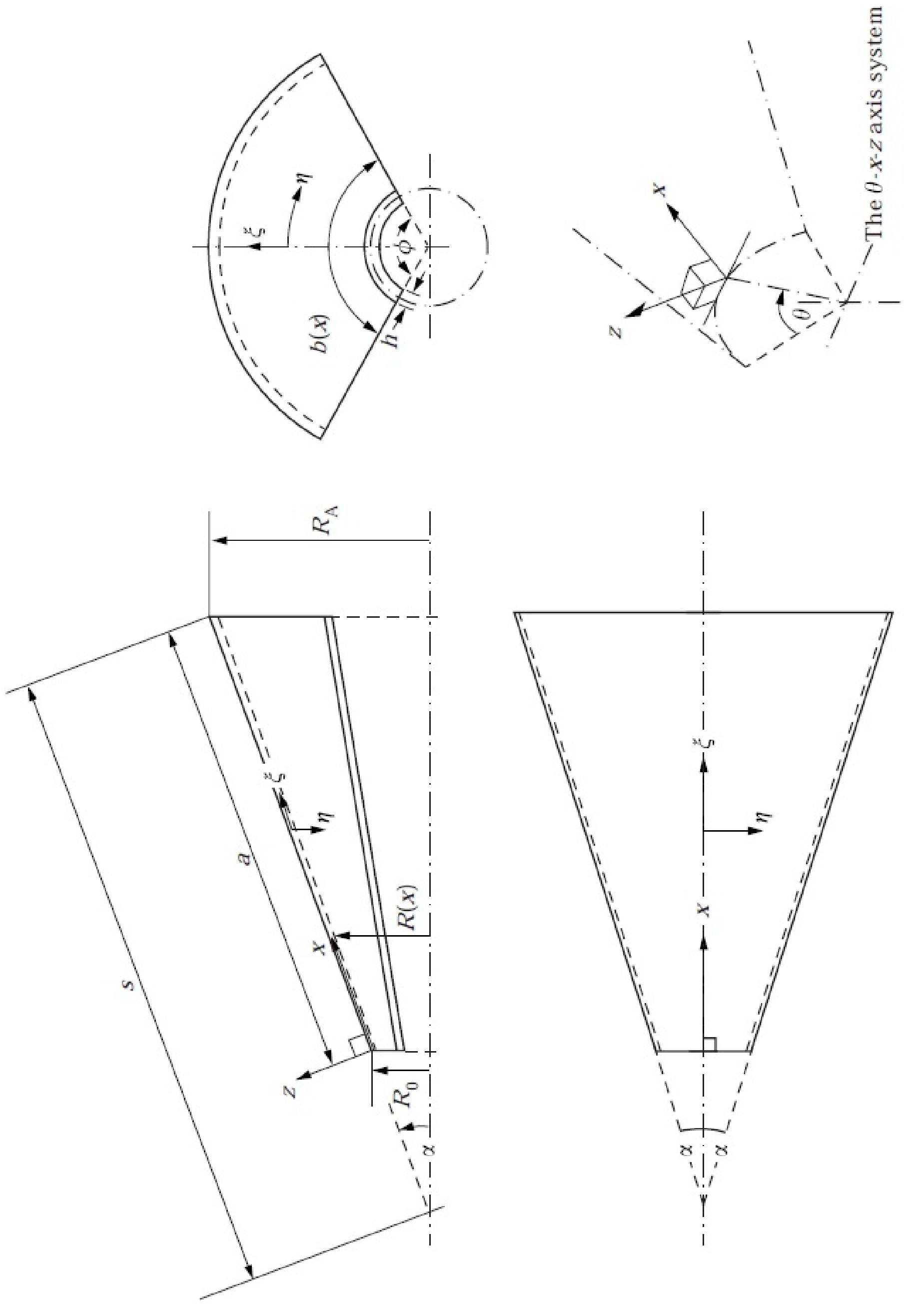

Figure 5.6: Conical shell geometry

In linear framework, strains vector are related to displacements through the following relations

$$\begin{aligned} \epsilon_p &= D_p u^k + A_p u^k \\ \epsilon_n &= D_n u^k + A_n u^k \end{aligned} \tag{5.49}$$

where

$$D_p = \frac{1}{\delta}\begin{bmatrix} \frac{\partial}{\partial_\zeta} & 0 & 0 \\ \frac{1}{\zeta} & \frac{1}{\zeta \sin\alpha}\frac{\partial}{\partial_\eta} & 0 \\ \frac{1}{\zeta \sin\alpha}\frac{\partial}{\partial_\eta} & \frac{\partial}{\partial_\zeta} - \frac{1}{\zeta} & 0 \end{bmatrix}, \quad D_n = \frac{1}{\delta}\begin{bmatrix} \partial_z & 0 & \frac{\partial}{\partial_\zeta} \\ 0 & \partial_z & \frac{1}{\zeta \sin\alpha}\frac{\partial}{\partial_\eta} \\ 0 & 0 & \partial_z \end{bmatrix}$$

$$A_p = \frac{1}{\delta}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\zeta \tan\alpha} \\ 0 & 0 & 0 \end{bmatrix}, \quad A_n = -\frac{1}{\delta}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\zeta \tan\alpha} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.50}$$

Three-dimensional Hook's law and stiffness coefficients are similar to previous sections. According to the approach developed by Carrera, an entire class of two-dimensional higher-order plate theories can be described through the following indicial notation

$$\mathbf{u}(\zeta, \eta, \xi_k, t) = F_\tau(\xi_k) u_\tau^k(\zeta, \eta, t) \qquad (\tau = 0, 1, ..., N_n) \tag{5.51}$$

where $u_\tau(\zeta, \eta, t)$ is the displacements vector containing the unknown kinematic variables related to the specific plate theory, τ is an integer index related to the order N_n of the theory anf $F_\tau(z)$ are selected functions in the thickness directions. Therefore, displacements vector can be expressed as

$$\mathbf{u}(\zeta, \eta, \xi_k, t) = F_\tau(\xi_k)\begin{Bmatrix} u(\zeta, \eta) \\ v(\zeta, \eta) \\ w(\zeta, \eta) \end{Bmatrix} e^{i\omega t} \tag{5.52}$$

Principle of virtual displacement is described as

$$\begin{aligned} &\int_0^1 \int_\beta^1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \Big(\delta(\epsilon_p)^T(\tilde{C}_{pp}^k \epsilon_p + \tilde{C}_{pn}^k \epsilon_n) + \delta(\epsilon_n)^T(\tilde{C}_{np}^k \epsilon_p + \tilde{C}_{nn}^k \epsilon_n)\Big) dz_k(\zeta \sin\alpha + z_k \cos\alpha) d\zeta d\eta \\ &= -\int_0^1 \int_\beta^1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \Big(\delta(u^k)\ddot{u}^k + \delta(v^k)\ddot{v}^k + \delta(w^k)\ddot{w}^k\Big) dz_k(\zeta \sin\alpha + z_k \cos\alpha) d\zeta d\eta \end{aligned} \tag{5.53}$$

After some substitution and integrations by part, nuclei of stiffness matrix and natural boundary condition, respectively are as follow

$$
\begin{aligned}
\mathcal{L}_{11}^{\tau s^k} &= -\Big(C_{11}\sin\alpha(\frac{1}{\zeta}\frac{\partial}{\partial\zeta}+\frac{\partial^2}{\partial\zeta^2})E_{\tau s}^k + C_{11}\cos\alpha\frac{1}{\zeta}\frac{\partial^2}{\partial\zeta^2}Ez_{\tau s}^k - \frac{C_{22}\sin\alpha}{\zeta^2}E_{\tau s}^{con^k}\\
&\quad+\frac{C_{66}}{\zeta^2\phi^2\sin\alpha}\frac{\partial^2}{\partial\eta^2}E_{\tau s}^{con^k} + C_{55}\sin\alpha E_{\tau_z s_z}^k + C_{55}\frac{1}{\zeta}\cos\alpha Ez_{\tau_z s_z}^k\\
\mathcal{L}_{12}^{\tau s^k} &= -\frac{(C_{12}+C_{66})}{\zeta\phi}\frac{\partial}{\partial\zeta\partial\eta}E_{\tau s}^k + \frac{(C_{22}+C_{66})}{\zeta^2\phi}\frac{\partial}{\partial\eta}E_{\tau s}^{con^k}\\
\mathcal{L}_{13}^{\tau s^k} &= -\Big(C_{12}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\cos\alpha E_{\tau s}^k + C_{13}\sin\alpha(\frac{\partial}{\partial\zeta}+\frac{1}{\zeta})E_{\tau s_z}^k + C_{13}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\cos\alpha Ez_{\tau s_z}^k\\
&\quad-\frac{C_{22}\cos\alpha}{\zeta^2}E_{\tau s}^{con^k} - \frac{C_{23}}{\zeta}\sin\alpha E_{\tau s_z}^k - C_{55}\frac{\partial}{\partial\zeta}\sin\alpha E_{\tau_z s}^k + C_{55}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\cos\alpha Ez_{\tau_z s}^k\Big)\\
\mathcal{L}_{21}^{\tau s^k} &= -\frac{(C_{12}+C_{66})}{\zeta\phi}\frac{\partial}{\partial\zeta\partial\eta}E_{\tau s}^k - \frac{(C_{22}+C_{66})}{\zeta^2\phi}\frac{\partial}{\partial\eta}E_{\tau s}^{con^k}\\
\mathcal{L}_{22}^{\tau s^k} &= -\Big(\frac{C_{22}}{\zeta^2\phi^2\sin\alpha}\frac{\partial^2}{\partial\eta^2}E_{\tau s}^{con^k} + C_{66}\sin\alpha(\frac{\partial^2}{\partial\zeta^2}+\frac{1}{\zeta}\frac{\partial}{\partial\zeta})E_{\tau s}^k - C_{66}\frac{1}{\zeta^2}\sin\alpha E_{\tau s}^{con^k}\\
&\quad+C_{66}\frac{1}{\zeta}\frac{2}{\partial\zeta^2}\cos\alpha Ez_{\tau s}^k - C_{44}\sin\alpha E_{\tau_z s_z}^k - C_{44}\frac{1}{\zeta}\cos\alpha Ez_{\tau_z s_z}^k + C_{44}\frac{1}{\zeta}\cos\alpha E_{\tau_z s}^k\\
&\quad+C_{44}\frac{1}{\zeta}\cos\alpha E_{\tau s_z}^k - C_{44}\frac{1}{\zeta^2}\frac{\cos^2\alpha}{\sin\alpha}E_{\tau s}^{con^k}\\
\mathcal{L}_{23}^{\tau s^k} &= -\Big(\frac{C_{23}}{\zeta\phi}\frac{\partial}{\partial\eta}E_{\tau s_z}^k - \frac{C_{44}}{\zeta\phi}\frac{\partial}{\partial\eta}E_{\tau_z s}^k + C_{44}\frac{1}{\zeta^2\tan\alpha}\frac{\partial}{\partial\eta}E_{\tau s}^{con^k} + C_{22}\frac{1}{\zeta^2\tan\alpha}\frac{\partial}{\partial\eta}E_{\tau s}^{con^k}\Big)\\
\mathcal{L}_{31}^{\tau s^k} &= \Big(C_{12}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\cos\alpha E_{\tau s}^k - C_{55}\sin\alpha(\frac{\partial}{\partial\zeta}+\frac{1}{\zeta})E_{\tau s_z}^k - C_{55}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\cos\alpha Ez_{\tau s_z}^k\\
&\quad+\frac{C_{22}\cos\alpha}{\zeta^2}E_{\tau s}^{con^k} + \frac{C_{23}}{\zeta}\sin\alpha E_{\tau_z s}^k + C_{13}\frac{\partial}{\partial\zeta}\sin\alpha E_{\tau_z s}^k + C_{13}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\cos\alpha Ez_{\tau_z s}^k\Big)\\
\mathcal{L}_{32}^{\tau s^k} &= \Big(\frac{C_{23}}{\zeta\phi}\frac{\partial}{\partial\eta}E_{\tau_z s}^k - \frac{C_{44}}{\zeta\phi}\frac{\partial}{\partial\eta}E_{\tau s_z}^k + C_{44}\frac{1}{\zeta^2\tan\alpha}\frac{\partial}{\partial\eta}E_{\tau s}^{con^k} + C_{22}\frac{1}{\zeta^2\tan\alpha}\frac{\partial}{\partial\eta}E_{\tau s}^{con^k}\Big)\\
\mathcal{L}_{33}^{\tau s^k} &= \Big(\frac{C_{22}\cos^2\alpha}{\zeta^2\sin\alpha} - \frac{C_{44}}{\zeta^2\phi^2\sin\alpha}\frac{\partial^2}{\partial\eta^2}\Big)E_{\tau s}^{con^k} - C_{55}\sin\alpha(\frac{1}{\zeta}\frac{\partial}{\partial\zeta}+\frac{\partial^2}{\partial\zeta^2})E_{\tau s}^k\\
&\quad-C_{55}\frac{1}{\zeta}\frac{\partial^2}{\partial\zeta^2}\cos\alpha Ez_{\tau s}^k + C_{23}\frac{1}{\zeta}\cos\alpha E_{\tau s_z}^k + C_{33}\sin\alpha E_{\tau_z s_z}^k\\
&\quad+C_{23}\frac{1}{\zeta}\cos\alpha E_{\tau_z s}^k + C_{33}\frac{1}{\zeta}\cos\alpha Ez_{\tau_z s_z}^k
\end{aligned}
$$

$$
\mathcal{M}_{11}^{\tau s^k} = \rho^k(\sin\alpha E_{\tau s}^k + \frac{1}{\zeta}\cos\alpha Ez_{\tau s}^k) \quad \mathcal{M}_{11}^{\tau s^k} = \mathcal{M}_{22}^{\tau s^k} = \mathcal{M}_{33}^{\tau s^k} \tag{5.54}
$$

Natural boundary condition on straight sides

$$
\begin{aligned}
\mathcal{B}_{11}^{\tau s^k} &= \Big(C_{11}\frac{\partial}{\partial\zeta}\sin\alpha + \frac{C_{12}}{\zeta}\sin\alpha\Big)E_{\tau s}^k + \frac{C_{11}}{\zeta}\frac{\partial}{\partial\zeta}\cos\alpha Ez_{\tau s}\\
\mathcal{B}_{12}^{\tau s^k} &= \frac{C_{12}}{\zeta\phi}\frac{\partial}{\partial\eta}E_{\tau s}^k
\end{aligned}
$$

$$\begin{aligned}
\mathcal{B}_{13}^{\tau s^k} &= C_{13}\sin\alpha E_{\tau s_z}^k + \frac{C_{13}}{\zeta}\cos\alpha Ez_{\tau s_z} + \frac{C_{12}}{\zeta}\cos\alpha E_{\tau s} \\
\mathcal{B}_{21}^{\tau s^k} &= \frac{C_{66}}{\zeta\phi}\frac{\partial}{\partial\eta}E_{\tau s}^k \\
\mathcal{B}_{22}^{\tau s^k} &= C_{66}(\frac{\partial}{\partial\zeta} - \frac{1}{\zeta})\sin\alpha E_{\tau s}^k + \frac{C_{66}}{\zeta}\frac{\partial}{\partial\zeta}\cos\alpha Ez_{\tau s} \\
\mathcal{B}_{23}^{\tau s^k} &= 0 \\
\mathcal{B}_{31}^{\tau s^k} &= C_{55}\sin\alpha E_{\tau s_z}^k + \frac{C_{55}}{\zeta}\cos\alpha Ez_{\tau s_z} \\
\mathcal{B}_{32}^{\tau s^k} &= 0 \\
\mathcal{B}_{33}^{\tau s^k} &= C_{55}\frac{\partial}{\partial\zeta}\sin\alpha E_{\tau s}^k + \frac{C_{55}}{\zeta}\frac{\partial}{\partial\zeta}\cos\alpha Ez_{\tau s}^k
\end{aligned} \tag{5.55}$$

Natural boundary condition on curved sides

$$\begin{aligned}
\mathcal{B}_{11}^{\tau s^k} &= \frac{C_{66}}{\zeta^2\phi\sin\alpha}\frac{\partial}{\partial\eta}E_{\tau s}^{con^k} \\
\mathcal{B}_{12}^{\tau s^k} &= \frac{C_{66}}{\zeta}\frac{\partial}{\partial\zeta}E_{\tau s}^k - \frac{C_{66}}{\zeta^2}E_{\tau s}^{con^k} \\
\mathcal{B}_{13}^{\tau s^k} &= 0 \\
\mathcal{B}_{21}^{\tau s^k} &= \frac{C_{12}}{\zeta}\frac{\partial}{\partial\zeta}E_{\tau s}^k + \frac{C_{22}}{\zeta^2}E_{\tau s}^{con^k} \\
\mathcal{B}_{22}^{\tau s^k} &= \frac{C_{22}}{\zeta^2\phi\sin\alpha}\frac{\partial}{\partial\eta}E_{\tau s}^{con^k} \\
\mathcal{B}_{23}^{\tau s^k} &= \frac{C_{22}}{\zeta^2\tan\alpha}E_{\tau s_z}^{con^k} + \frac{C_{23}}{\zeta}E_{\tau s_z}^k \\
\mathcal{B}_{31}^{\tau s^k} &= 0 \\
\mathcal{B}_{32}^{\tau s^k} &= \frac{C_{44}}{\zeta}E_{\tau s_z}^k - \frac{C_{44}}{\zeta^2\tan\alpha}E_{\tau s}^{con^k} \\
\mathcal{B}_{33}^{\tau s^k} &= \frac{C_{44}}{\zeta^2\phi\sin\alpha}\frac{\partial}{\partial\eta}E_{\tau s}^{con^k}
\end{aligned} \tag{5.56}$$

where thickness integrals of conical shell are as

$$\begin{aligned}
Ez_{\tau s}^k &= \int_{-\frac{h}{2}}^{\frac{h}{2}} F_\tau F_s z dz \qquad Ez_{\tau s_z}^k = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_\tau F_{sz} z dz \\
Ez_{\tau_z s}^k &= \int_{-\frac{h}{2}}^{\frac{h}{2}} F_{\tau_z} F_s z dz \qquad Ez_{\tau_z s_z}^k = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_{\tau_z} F_{sz} z dz \\
E_{\tau s}^{con^k} &= E_{\tau s}^k - \frac{cot\alpha}{\zeta}Ez_{\tau s}^k
\end{aligned} \tag{5.57}$$

According to above equations some thickness integrals for conical shells in comparison with sector annular plates has additional z multiplied by the integrand and some of them like $E_{\tau s}^{con^k}$ are vary with ζ in the length direction. Based on the Taylor series,

$$\frac{1}{1+(z/\zeta)} = 1 - (z/\zeta) + (z/\zeta)^2 + ...$$

Thickness integrals are assumed in an expansion but just with two first terms. Conical shell panel with $\alpha = \pi/2$ becomes an annular sector plate, and with $\alpha = 0$ becomes a cylindrical shell panel with $\zeta \tan\alpha = \zeta \sin\alpha = R_\beta$. For solving complete conical shell problem we can also use eq. (5.45).

Generally, with increasing N_n more accurate results can be obtained in both approaches and results are near to the three-dimensional analysis. Because this technique is quasi-3D and with increasing the order of theory they reach to results of 3D analysis. But in some cases it does n't need to evaluate higher-order theories. However, it depends on how many digits accuracy we want to have. In this thesis, five digits accuracy is assumed. Most of the uniform boundary conditions such as fully clamped, simply-supported and free has a good convergence without very high-order theory and sometimes they just need increasing points. But non-uniform boundaries and boundaries including free condition or conditions need higher-order theory rather than the uniform ones.

Solving problems using CUF technique in the Chebyshev spectral collocation framework is similar to first-order shear deformation theory just with more variables, and the number of variables depends on the theories which are expressed before. For single layer theory number of variables are $3(N_n + 1)$ and for layer-wise theory number of variables are $3(N_n N_l + 1)$.

For conical isotropic shell in the case of fully clamped as seen in the table (5.1) ED3 theory is enough and the presented results are very close to hp-finite element analysis based on the three-dimensional theory rather than the other methods. If β, ϕ are constant, with increasing α frequencies decreased. If β, α are constant, with increasing ϕ frequencies decreased. If α, ϕ are constant, with increasing β frequencies increased.

In table (5.2) CUF technique analysis based on spectral collocation method is compared with first-order shear deformation theory based on the GDQ method and three-dimensional theory for evaluation of natural frequencies of isotropic spherical shell panel. As seen, results for spectral method are very close to 3D analysis. For thin isotropic spherical panel shell results are more close to 3D results rather than thick spherical shell panel and for thin panel ED5 theory is enough but for thick and moderately thick panel ED6 theory with more collocation points is needed. Table (5.8) is also shown results for isotropic moderately thick spherical shell panel for some boundary conditions.

For the results of isotropic sector plate which is tabulated in table (5.3), as seen using ED6 is essential for calculating good results. For isotropic sector plate for all considered boundary conditions, fully clamped, fully simply supported, CFCF and FCFC for ϕ is constant with increasing β frequency decreased. If β is constant with increasing of ϕ in all boundary condition except CFCF frequency decreased. In table (5.4) ED6 theory with different number of collocation points are presented. one can see convergence of presented is very fast and with fifteen collocation points four digits accuracy is available for all boundary conditions which are surveyed in this table.

Laminated composite sector annular plate is considered in table (5.5) and ED6 and LD4 theories are shown and as seen for $\beta = 0.1$ changing the frequencies from ED1 to ED6 is not so much. Results are compared with analytical solution using Chebyshev polynomials based on the FDST.

In the case of fully simply-supported isotropic cylindrical shells panel which is shown in table (5.9) for all a/R and a/h for evaluating good results theory of ED3 is enough. For fully clamped composite cylindrical shell panel as seen in table (5.10) even the higher-order equivalent single layer theory ED6 can not given good results and for accurate results using higher-order layer-wise theory like ED4 is advised. (CUF) results were compared with 3D analysis, FSDT and FSDT with some modification by Qatu. As seen, presented results are so close to the three-dimensional analysis rather than 2D analysis based on the first-order and first-order-like shear deformation theory.

Table (5.11) is shown results for many boundaries of isotropic moderately thick plates. In the case of fully simply supported, fully free and CSCS ED3 theory is enough for good results but increasing the number of points is recommended. Evaluated results in the case of SSSS, FFFF and CFFF are compared with 3D analysis based on B-Spline Ritz method and results which are obtained for CFCF and CSCS are compared with (CUF) technique based on trigonometric Ritz method. For CFFF, CFCF and SFFF using higher-order rather than ED3 is better. For other boundaries, there are no results for 3D and they are compared with first-order shear deformation theory and as seen FSDT theory can not cover all natural frequencies such as axisymmetric modes like CUF technique. Fully clamped case is shown in table (5.6) and as seen by ED6 theory good results can be obtained. The purpose of author by writing FSDT in this table is FSDT results obtained by the author.

Table (5.12) is shown results for many boundaries of isotropic thin plates. In the case of fully simply supported ED3 theory is really enough but for fully clamped and fully free higher-order theory rather than ED3 should be used. Obtained results for CCCC and SSSS are compared with 3D analysis based on B-Spline Ritz method. For CSCS, CFCF, SFSF, FFFF and SFFF results are compared with classical plate theory. For CFFF, CCFF and SSFF higher-order theories don't show us good converge and good results and it is recommended to use modified elastic coefficients (see eq. (5.34)) and recover them to ED1 theory. The purpose of author by writing CLPT in this table is CLPT results obtained by the author.

In tables (5.7) and (5.13) composite laminated thick rectangular plate is considered and a comparison between applying CUF technique on spectral collocation method and Ritz method using trigonometric functions were presented. As seen results are more close to each other for both approaches such as ESL and LW description.

In table (5.14) results for cylindrical shells panel non-symmetric composite laminated both cross-ply and angle-ply considering many boundary conditions are tabulated and compared with GDQ method.

In table (5.15) results for isotropic moderately thick and shallow spherical shells panel of fully simply-supported, CSCS and SFSF are tabulated and they are compared with FSDT based on analytical solution and 3D based on finite element method.

Three following tables were presented results for isotropic thin plates which are compared with analytical solution and differential quadrature methods, isotropic in-plane vibration of sector annular plates for various boundary conditions from angle 30^o to 90^o were investigated in the next table and as seen with increasing sector angle, natural frequencies get decreased, and also results based on the third-order shear deformation theory were tabulated in the last tables for symmetric composite laminated and results just for fundamental frequency are compared with Reddy results.

5.4 CUF results

Table 5.1: Non-dimensional natural frequencies of fully clamped isotropic conical shell, $\lambda = \omega a^2\sqrt{\frac{\rho h}{D}}, a/s = 0.6, \alpha = 60, \phi = 30, h/a = 0.01$

Theory	ω_1	ω_2	ω_3	ω_4
ED1	221.8769	275.2954	323.8748	371.8919
ED2	209.1731	255.4765	306.6505	349.5340
ED3	209.1086	255.3170	306.5358	349.3015
ED4	209.1082	255.3165	306.5352	349.3005
ED5	209.1081	255.3163	306.5350	349.3002
ED6	209.1081	255.3163	306.5350	349.3002
[170]	209.84	257.11	307.90	351.90
[172]	207.53	266.96	318.53	367.95
[171]	213.4	262.5	314.7	358.6
Present FSDT	205.3911	249.8287	295.6223	337.1081

Table 5.2: Non-dimensional natural frequencies of isotropic fully clamped spherical panel, $\lambda = \omega a\sqrt{\frac{\rho}{E}}, R_/a = 2$

h/a	Theory	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.01	ED4	0.5803	0.5803	0.5950	0.6343	0.6529	0.7310
	[169]	0.5809	0.5809	0.5959	0.6353	0.6542	0.7329
	[161]	0.5763	0.5763	0.5913	0.6303	0.6476	0.7260
0.1	ED4	1.2047	1.9451	1.9451	2.6956	3.1561	3.2018
	[169]	1.1886	1.9122	1.9122	2.6625	3.1059	3.1579
	[161]	1.1988	1.9301	1.9340	2.6828	3.1288	3.1774
0.2	ED4	1.7625	2.8499	2.8499	3.7794	3.7794	3.8449
	[169]	1.7360	2.8062	2.8062	3.7322	3.7322	3.8044
	[161]	1.7405	2.8036	2.8091	3.7504	3.7572	3.7904

Table 5.3: Non-dimensional natural frequencies of isotropic annular sector plate, $\lambda = \omega R_0^2 \sqrt{\frac{\rho h}{D}}, \beta = 0.25, \alpha = 120, h/(R_o - R_i) = 0.25, N = 10$

BCs	Theory	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
CCCC	ED1	35.3986	47.7797	62.7673	64.1837	71.9614	80.6985
	ED2	33.6573	45.5026	61.3768	62.7186	69.2809	78.6371
	ED3	32.7619	44.2491	59.5719	62.6967	67.0613	76.2618
	ED4	32.7020	44.1729	59.4590	62.6873	66.8765	76.0791
	ED5	32.6732	44.1387	59.4125	62.6866	66.8005	76.0269
	ED6	32.6717	44.1374	59.4109	62.6849	66.7975	76.0251
	[132]	32.649	44.109	59.368	62.677	66.765	75.8690
SSSS	ED1	23.1489	26.2510	36.9344	39.8405	48.6116	54.8371
	ED2	21.3039	26.2147	34.1419	39.8253	48.6116	51.1613
	ED3	21.0572	26.2396	33.6369	39.8379	48.6116	50.1696
	ED4	21.0532	26.2314	33.6268	39.8342	48.6116	50.1382
	ED5	21.0524	26.2304	33.6264	39.8336	48.6116	50.1377
	ED6	21.0523	26.2302	33.6263	39.8336	48.6116	50.1377
	[132]	21.069	26.194	33.633	39.806	48.611	50.1450
CFCF	ED1	30.3367	31.4445	38.2895	45.5080	52.0806	56.5740
	ED2	29.1072	30.4584	36.8175	45.4947	49.5717	56.5183
	ED3	28.2890	29.6087	35.8923	45.4913	48.3638	56.4947
	ED4	28.2344	29.5671	35.8541	45.4900	48.3203	56.4851
	ED5	28.2067	29.5438	35.8321	45.4898	48.3115	56.4839
	ED6	28.2059	29.5480	35.8357	45.4900	48.3230	56.4873
	[132]	28.181	29.502	35.816	45.545	48.301	56.486
FCFC	ED1	7.9138	17.5928	28.8405	29.5850	30.5367	34.1218
	ED2	7.5333	17.0891	27.0622	29.5739	29.7722	34.0975
	ED3	7.4718	16.8350	26.5176	29.1703	29.5703	34.0822
	ED4	7.4694	16.8302	26.4922	29.1560	29.5699	34.0772
	ED5	7.4677	16.8269	26.4802	29.1506	29.5694	34.0761
	ED6	7.4679	16.8273	26.4806	29.1507	29.5696	34.0752
	[132]	7.4547	16.806	26.472	29.111	29.591	34.0820

Table 5.4: Non-dimensional natural frequencies of isotropic annular sector plate, $ED6$, $\lambda = \omega R_0^2/\pi^2\sqrt{\frac{\rho h}{D}}, \beta = 0.4, \alpha = 90, h/(R_o - R_i) = 1/6$

BCs	Theory	points	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
CCCC	ED6	5	6.1022	12.4784	14.5416	17.0971	19.2707	20.4228
		10	5.9344	7.9000	10.9296	13.1516	14.4776	14.5668
		15	5.9294	7.8911	10.9136	13.1393	14.4850	14.5641
		20	5.9282	7.8896	10.9112	13.1376	14.4822	14.5634
	[132]		5.9274	7.8885	10.910	13.135	14.480	14.563
SSSS	ED6	5	3.3962	6.0694	9.0964	10.9577	11.1684	14.3688
		10	3.4487	5.5706	6.0716	8.6915	9.9610	10.2450
		15	3.4469	5.5697	6.0829	8.6811	9.9634	10.2479
		20	3.4477	5.5700	6.0802	8.6811	9.9621	10.2473
	[132]		3.4476	5.5699	6.0807	8.6811	9.9620	10.248
CSCS	ED6	5	5.8255	10.2205	11.1684	14.0378	17.6629	18.6434
		10	5.6682	7.1045	9.8119	11.1946	13.0175	13.3632
		15	5.6643	7.1014	9.7995	11.1946	13.0060	13.3056
		20	5.6633	7.1005	9.7988	11.1946	13.0044	13.3048
	[132]		5.6625	7.0999	9.7982	11.194	13.002	13.304
CFCF	ED6	5	5.4121	6.1030	6.3060	10.2239	10.2239	11.0141
		10	5.2794	5.4918	6.5669	8.6003	10.5850	12.0155
		15	5.2633	5.5012	6.5191	8.6298	10.5802	11.7286
		20	5.2621	5.4988	6.5180	8.6268	10.5834	11.7292
	[132]		5.2615	5.4983	6.5172	8.6262	10.584	11.728
SFSF	ED6	5	2.4166	2.6017	3.3787	4.4025	5.7078	5.7078
		10	2.6832	3.0558	3.2128	4.9372	5.7993	7.4574
		15	2.6593	3.1078	3.2124	4.8665	5.8230	7.4744
		20	2.6605	3.1024	3.2109	4.8670	5.8187	7.4714
	[132]		2.6606	3.1058	3.2107	4.8665	5.8203	7.4713

Table 5.5: Non-dimensional natural frequencies of composite laminated annular sector plate (0,90), $\lambda = \omega R_0^2/h\sqrt{\frac{\rho}{E_2}}, \beta = 0.1, \alpha = 60, h/R_o = 0.2$

BCs	Theory	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
CCCC	ED1	16.1094	24.7174	24.8425	33.9087	34.2128	34.3279
	[133]	16.113	24.720	24.846	33.913	34.216	34.331
	ED6	15.8618	24.3087	24.8344	33.5711	33.7167	33.9794
	LD4	16.0183	24.5666	25.0621	33.8683	34.1555	34.3579
SSSS	ED1	12.3632	12.6608	16.7998	22.9968	23.0005	23.1540
	[133]	12.363	12.695	16.962	22.997	23.021	23.176
	ED6	11.8404	12.3632	16.3722	21.7090	21.8228	22.9969
	LD4	11.9518	12.3632	16.4511	21.9456	22.0521	22.9968
CSCS	ED1	12.3632	13.9785	22.9968	23.1404	24.1258	32.7192
	[133]	12.363	13.983	22.997	23.141	24.130	32.722
	ED6	12.3632	13.3205	22.1404	22.9968	23.0517	31.5083
	LD4	12.3632	13.4133	22.3318	22.9968	23.3013	31.8956
FCFC	ED1	8.9157	16.3227	17.8928	25.3442	25.6553	27.7412
	[133]	8.918	16.323	17.906	25.352	25.688	27.767
	ED6	8.5433	15.8179	17.6881	24.5587	24.6480	27.3918
	LD4	8.6238	15.9855	17.8008	24.6746	24.8778	27.4883
CSFS	ED1	0.8991	5.4611	15.0103	15.1506	16.7246	22.3287
	[133]	0.899	5.460	15.022	15.151	16.725	22.375
	ED6	0.8995	5.1803	13.9355	14.1580	16.7246	22.0869
	LD4	0.9002	5.1735	13.9467	14.2799	16.7244	22.1309
SSFS	ED1	0.8991	5.3782	14.3496	15.1332	16.7246	17.6225
	[133]	0.899	5.377	14.388	15.133	16.725	17.681
	ED6	0.8993	5.0778	13.4483	14.1407	16.7244	17.1805
	LD4	0.9002	5.0740	13.4296	14.2632	16.7244	17.2194

Table 5.6: Non-dimensional natural frequencies of isotropic fully clamped square plate, $\lambda = \omega a^2 \sqrt{\frac{\rho h}{D}}, h/a = 0.1$

Theory	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
ED1	35.926	68.464	68.464	95.873	112.93	114.04
[180]	35.927	68.465	68.465	95.874	112.94	114.04
ED2	33.122	63.611	63.611	89.503	105.78	106.76
[180]	33.127	63.619	63.619	89.519	105.80	106.78
ED3	32.782	62.651	62.651	87.918	103.68	104.67
[180]	32.785	62.655	62.655	87.924	103.69	104.67
ED4	32.754	62.589	62.589	87.824	103.56	104.54
[180]	32.768	62.610	62.610	87.862	103.60	104.58
ED5	32.745	62.569	62.569	87.793	103.52	104.50
[180]	32.759	62.592	62.592	87.831	103.56	104.54
[191]	32.743	62.562	62.562	87.783	103.51	104.49
[131]	32.749	62.577	62.577	87.801	103.60	104.59

Table 5.7: Non-dimensional natural frequencies of composite fully clamped square plate, $\lambda = \omega a^2/h\sqrt{\frac{\rho}{E}}$, $h/a = 0.25, E_1/E_2 = 25, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, \nu = 0.25$

Lay-up	Theory	Author	ω_1	ω_2	ω_3	ω_4
(-30/45)	LD1	Present	9.1092	14.0432	15.2005	19.2441
		[181]	9.103	14.043	15.191	19.246
	LD2	Present	8.9607	13.7798	14.9869	18.8701
		[181]	8.954	13.777	14.979	18.865
	LD3	Present	8.7491	13.4836	14.6418	18.4747
		[181]	8.740	13.480	14.630	18.467
	LD4	Present	8.7445	13.4740	14.6307	18.4588
		[181]	8.736	13.471	14.620	18.452

Table 5.8: Non-dimensional natural frequencies of isotropic spherical panel shell, ED3, $\lambda = \omega a\sqrt{\frac{\rho}{E}}$, $a/h = 10, a/R = 0.2$

BCs	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
SSSS	6.06	13.88	13.88	19.25	19.25	21.22	25.84	25.84
CSCS	8.38	15.01	18.10	19.25	24.03	26.40	30.94	34.00
SFSF	2.87	4.64	10.39	11.03	12.97	14.65	18.86	19.25

Table 5.9: Non-dimensional natural frequencies of isotropic cylindrical panel shell, $\lambda = \omega a\sqrt{\frac{\rho}{E}}$

a/h	a/R	Theory	ω_1	ω_2	ω_3	ω_4	ω_5
20	1	ED1	11.3328	16.0687	22.2990	26.8150	31.1816
		ED2	11.0456	14.6575	21.3004	24.7001	28.3594
		ED3	11.0168	14.4443	21.1659	24.3685	27.8845
		[155]	11.017	14.444	21.166	24.368	27.884
20	2	ED1	15.8645	18.0008	29.8369	30.1451	33.9856
		ED2	14.7246	17.9045	27.1787	28.5443	33.4681
		ED3	14.5617	17.9453	26.7273	28.3241	33.4551
		[155]	14-.562	17.945	26.727	28.324	33.455
20	0.5	ED1	8.0896	16.1370	17.9155	25.8598	31.5286
		ED2	7.6166	14.6503	16.5872	23.5739	28.6643
		ED3	7.5523	14.4231	16.3896	23.2063	28.1834
		[155]	7.5523	14.423	16.390	23.206	28.183
10	1	ED1	7.5363	14.8044	16.7495	19.4833	19.4912
		ED2	7.1361	13.5547	15.6516	19.4833	19.4912
		ED3	7.0810	13.3097	15.4528	19.8692	19.8771
		[155]	7.0810	13.309	15.452	19.869	19.877
10	2	ED1	9.7346	13.6845	19.4833	23.2538	26.7183
		ED2	9.5723	12.6115	19.4833	21.7269	24.6695
		ED3	9.5791	12.3982	19.6570	21.4038	24.1363
		[155]	9.5791	12.397	19.657	21.401	24.131
10	0.5	ED1	6.7037	15.1106	15.6176	19.4833	19.4853
		ED2	6.1750	13.8133	14.3646	19.4833	19.4853
		ED3	6.0921	13.5601	14.1245	19.8692	19.8712
		[155]	6.0921	13.560	14.124	19.869	19.871

Table 5.10: Non-dimensional natural frequencies of composite fully clamped cylindrical panel, $\lambda = \omega a^2/h\sqrt{\frac{\rho}{E_2}}$, $h/a = 0.1, R/a = 0.5, E_1/E_2 = 25, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, \nu = 0.25$

Lay-up	Theory	ω_1	ω_2	ω_3	ω_4	ω_5
$(0/90)_3$	ED1	30.0179	42.3381	42.4674	50.9338	58.8817
	ED2	29.8503	42.1047	42.2398	50.6407	58.5555
	ED3	27.9026	39.4615	39.7030	47.3786	55.2173
	ED4	27.8579	39.4010	39.6305	47.2839	55.1055
	ED5	27.8361	39.3607	39.6054	47.2393	55.0671
	ED6	27.6163	39.0386	39.3285	46.8647	54.6645
	LD1	26.5404	37.4972	38.0157	45.0429	52.9214
	LD2	26.3778	37.2541	37.8167	44.7640	52.6438
	LD3	26.3446	37.1903	37.7761	44.6925	52.5785
	LD4	26.3444	37.1899	37.7759	44.6922	52.5781
	[156]	26.249	37.072	37.624	44.511	52.327
	[195]	28.319	40.005	40.196	47.953	55.815
	[195]	28.417	40.157	40.346	48.160	56.039
$(45/\text{-}45)_3$	ED1	40.8623	42.4399	48.2068	51.7877	58.7900
	ED2	40.7293	42.3110	48.0840	51.6059	58.5805
	ED3	39.1107	40.5967	46.6800	49.2976	55.9288
	ED4	39.0344	40.5094	46.6179	49.1983	55.8203
	ED5	39.0198	40.4848	46.6055	49.1703	55.7955
	ED6	38.8423	40.2739	46.4503	48.9275	55.4854
	LD1	37.7460	38.9171	45.7545	47.8411	53.9360
	LD2	37.5960	38.7264	45.6185	47.6556	53.6998
	LD3	37.5828	38.6984	45.6079	47.6308	53.6697
	LD4	37.5828	38.6982	45.6078	47.6306	53.6696
	[156]	37.562	38.711	45.369	47.324	53.543
	[195]	39.462	40.872	46.916	49.538	56.500
	[195]	39.578	41.011	47.437	50.070	56.788
(30/60/45)	ED1	31.0278	38.2398	47.0649	48.4109	52.2447
	ED2	30.8300	37.9586	46.7812	48.0276	51.9510
	ED3	29.9638	36.8070	45.6621	46.8861	51.0476
	ED4	29.8869	36.7278	45.5507	46.7917	50.8401
	ED5	29.8520	36.6726	45.4993	46.7301	50.7885
	ED6	29.8348	36.6550	45.4704	46.7075	50.7497
	LD1	30.3188	37.4267	46.1967	47.5967	51.7574
	LD2	29.8627	36.7875	45.5764	46.8988	51.1529
	LD3	29.8021	36.6964	45.4793	46.7904	51.1068
	LD4	29.7997	36.6927	45.4757	46.7856	51.1048
	[156]	29.778	36.579	45.383	46.621	50.667
	[195]	29.832	36.505	45.445	46.441	51.166
	[195]	29.908	36.690	45.599	46.551	51.587

Table 5.11: Non-dimensional natural frequencies of isotropic plate $\lambda = \omega a^2 \sqrt{\frac{\rho h}{D}}, h/a = 0.1$

BCs	Theory	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
SSSS	ED1	21.089	50.279	50.279	64.383	64.383	77.112	91.052	93.925
	ED2	19.149	45.938	45.938	64.383	64.383	70.817	86.513	86.513
	ED3	19.090	45.621	45.621	64.383	64.383	70.112	85.501	85.501
	[131]	19.090	45.619	45.619	64.383	64.383	70.104	85.487	85.487
CFFF	ED1	3.7242	8.2872	21.248	21.828	27.675	29.546	50.115	52.365
	ED2	3.3569	8.0628	20.326	21.817	25.647	28.637	48.236	52.329
	ED3	3.4271	8.0753	20.146	21.820	25.541	28.337	47.689	52.356
	[131]	3.4387	8.0746	20.152	21.796	25.541	28.325	47.677	52.302
FFFF	ED1	12.839	19.325	26.293	32.974	32.978	58.107	60.377	60.377
	ED2	12.796	18.983	23.398	32.167	32.167	55.684	56.004	56.374
	ED3	12.739	18.949	23.343	31.959	31.959	55.480	55.590	55.898
	[131]	12.723	18.954	23.345	31.955	31.955	55.490	55.490	55.821
CFCF	ED1	22.477	25.590	41.576	57.406	58.715	61.293	74.948	77.680
	ED2	20.938	24.405	39.117	53.935	58.406	58.704	69.507	73.808
	ED3	20.768	24.167	38.719	53.159	57.523	58.700	68.765	72.634
	[180]	20.779	24.179	38.730	53.182	57.540	58.722	68.785	
CSCS	ED1	29.469	54.255	64.383	65.359	86.962	95.884	111.795	113.713
	ED2	27.060	49.830	60.616	64.383	80.688	88.516	104.594	113.680
	ED3	26.831	49.363	59.745	64.383	79.487	87.392	102.560	113.067
	[180]	26.840	49.368	59.764	64.383	79.502	87.395	102.59	
SFSF	ED1	10.097	15.898	36.372	39.337	45.356	48.764	64.382	66.548
	ED2	9.4635	15.468	34.112	36.658	43.226	48.755	62.973	64.383
	ED3	9.4461	15.407	33.909	36.438	42.899	48.755	62.337	64.383
	FSDT	9.4406	15.389	33.859	36.357	42.792		62.146	
SSFF	ED1	3.3102	17.066	20.182	37.225	40.194	49.648	53.507	68.701
	ED2	3.2948	16.628	18.585	35.429	40.191	47.029	49.312	64.781
	ED3	3.2885	16.572	18.523	35.190	40.193	46.708	48.968	64.138
	FSDT	3.2859	16.560	18.505	35.127		46.601	48.842	63.936
CCFF	ED1	7.0581	23.168	27.030	44.772	51.041	58.728	63.018	65.403
	ED2	6.7444	22.370	24.987	42.641	51.028	55.839	58.424	65.388
	ED3	6.6775	22.210	24.819	42.191	51.072	55.248	57.779	65.411
	FSDT	6.6731	22.143	24.721	41.998		54.903	57.425	
SFFF	ED1	6.4142	15.317	24.519	26.405	46.140	50.071	55.102	56.585
	ED2	6.3925	14.518	23.868	24.702	44.341	46.547	53.121	56.583
	ED3	6.3719	14.487	23.738	24.629	44.122	45.940	52.846	56.590

Table 5.12: Non-dimensional natural frequencies of isotropic plate $\lambda = \omega a^2 \sqrt{\frac{\rho h}{D}}, h/a = 0.01$

BCs	Theory	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
CCCC	ED1	39.780	81.057	81.057	119.40	145.12	145.82	181.83	181.83
	ED2	35.993	73.349	73.349	108.06	131.35	131.98	164.60	164.60
	ED3	35.987	73.329	73.329	108.02	131.30	131.93	164.52	164.52
	[131]	35.977	73.315	73.315	108.00	131.39	132.02	164.56	164.56
SSSS	ED1	21.839	54.567	54.567	87.260	109.03	109.03	141.66	141.66
	ED2	19.733	49.309	49.309	78.857	98.540	98.540	128.04	128.04
	ED3	19.732	49.304	49.304	78.846	98.523	98.523	128.01	128.01
	[131]	19.732	49.305	49.305	78.846	98.525	98.525	128.01	128.01
FFFF	ED1	13.454	21.720	27.584	35.786	35.873	61.248	66.380	67.078
	ED2	13.390	19.609	24.301	34.744	34.755	61.081	61.081	63.738
	ED3	13.783	19.581	24.260	34.784	34.809	61.000	61.000	62.934
	[31]	13.419	19.589	24.258	34.669	34.669	61.016	61.016	63.355
CSCS	ED1	32.012	60.509	76.581	104.43	112.89	142.41	154.68	170.65
	ED2	28.954	54.704	69.285	94.463	102.05	128.88	139.89	154.42
	ED3	28.970	54.713	69.341	94.509	102.04	128.95	139.91	154.47
	[31]	28.950	54.743	69.327	94.585	102.21	129.09	140.20	154.77
CFCF	ED1	24.227	28.087	46.810	66.812	72.084	86.742	94.171	131.14
	ED2	22.195	26.358	43.578	61.187	67.028	79.505	87.505	119.97
	ED3	22.186	26.654	43.367	61.172	67.567	79.681	87.159	119.91
	CLPT	22.165	26.402	43.591	61.170	67.166	79.813	87.587	120.09
	[31]	22.272	26.529	43.664	61.466	67.549	79.904		
SFSF	ED1	10.357	16.607	39.284	42.308	49.483	75.559	81.888	95.897
	ED2	9.6529	16.082	36.716	38.966	46.615	70.692	75.003	87.926
	ED3	9.6236	16.404	36.489	38.887	47.122	70.259	75.138	87.773
	CLPT	9.6314	16.134	36.725	38.945	46.738	70.740	75.283	87.986
	[31]	9.6314	16.134	36.725	38.945	46.738	70.740	75.283	87.986
SFFF	ED1	6.6618	15.837	26.124	27.901	50.984	54.817	61.319	71.146
	ED2	6.6290	14.925	25.301	26.007	48.411	50.545	58.541	65.064
	ED3	6.8310	14.880	25.696	25.962	48.211	50.410	58.974	65.106
	CLPT	6.6797	14.904	25.453	26.047	48.457	50.699	58.767	65.275
	[31]	6.6480	15.023	25.492	26.126	48.711	50.849		
CFFF	ED1	3.4696	8.5079	21.261	27.144	30.907	54.924	61.145	63.980
	[131]	3.4712	8.4826	21.273	27.150	30.861	53.951	61.278	64.078
CCFF	ED1	6.9325	23.865	26.564	47.497	62.582	65.424	85.324	88.018
	[31]	6.4921	24.034	26.681	47.785	63.039	65.833		
SSFF	ED1	3.3792	17.303	19.290	38.134	50.965	53.425	72.714	74.414
	[31]	3.3687	17.407	19.367	38.291	51.324	53.738		

Table 5.13: Non-dimensional natural frequencies of composite laminated plate, $\lambda = \omega a^2/h\sqrt{\frac{\rho}{E_2}}$

Case	Theory	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
CCCC	ED1	9.1684	14.154	15.337	19.431	20.333	22.577
(-30,45), $a/h = 4$	ED2	9.1094	14.046	15.213	19.257	20.164	22.361
	ED3	8.8940	13.701	14.895	18.765	19.684	21.948
	ED4	8.8692	13.664	14.849	18.723	19.637	21.865
	[181](ED4)	8.8590	13.660	14.837	18.715		
	LD1	9.1092	14.043	15.200	19.244	20.162	22.333
	LD2	8.9606	13.779	14.986	18.869	19.775	22.048
	LD3	8.7490	13.483	14.642	18.474	19.381	21.562
	[181](LD3)	8.740	13.480	14.630	18.467		
FCCF	ED1	2.8606	7.766	8.0184	11.373	15.620	15.653
$(0, 90)$, $a/h = 4$	ED2	2.8425	7.701	7.9465	11.305	15.310	15.430
	ED3	2.8113	7.566	7.8014	11.132	14.949	15.084
	ED4	2.8040	7.523	7.7576	11.093	14.829	14.910
	[181](ED4)	2.804	7.510	7.748	11.106		
	LD1	2.8454	7.673	7.9282	11.313	15.256	15.280
	LD2	2.8210	7.587	7.8266	11.184	14.980	14.988
	LD3	2.7875	7.434	7.6663	10.983	14.653	14.663
	[181](LD3)	2.788	7.435	7.667	10.986		
FCFC	ED1	5.2478	6.366	10.949	11.351	12.619	12.947
$(0, 45)$, $a/h = 4$	ED2	5.2039	6.111	10.822	11.338	12.103	12.875
	ED3	5.0380	5.869	10.497	11.027	12.045	12.471
	ED4	5.0270	5.917	10.505	10.928	12.272	12.272
	[181](ED4)	4.946	5.956	10.018	10.591		
	LD1	5.1096	6.148	10.168	10.810	11.556	13.143
	LD2	4.9629	5.986	10.030	10.604	11.235	12.955
	LD3	4.8995	5.910	9.9491	10.533	11.085	12.729
	[181](LD3)	4.903	5.913	9.954	10.541		
SSSS	ED1	10.665	24.334	24.334	26.415	26.415	36.576
$(0, 90)$, $a/h = 4$	ED2	10.463	24.334	24.334	25.395	25.395	35.199
	ED3	10.431	24.334	24.334	25.214	25.214	34.908
	ED4	10.364	24.334	24.334	24.757	24.757	34.332
	[181](ED4)	10.364					
	LD1	10.453	24.334	24.334	25.177	25.177	34.937
	LD2	10.414	24.334	24.334	25.065	25.065	34.754
	LD3	10.336	24.334	24.334	24.590	24.590	34.109
	[181](LD4)	10.336					

Table 5.14: Non-dimensional natural frequencies of composite laminated shell, $\lambda = \frac{\omega a^2}{h}\sqrt{\frac{\rho}{E_2}}, a/h = 10, a/R = 2$

Lay-up	BCs	Theory	ω_1	ω_2	ω_3	ω_4	ω_5
$(-45,45)_3$	CCCC	LD2	26.719	37.853	38.238	45.391	52.993
		[156]	26.249	37.072	37.624	44.511	52.327
	SSSS	LD2	22.510	26.135	37.383	39.601	41.207
		[156]	22.842	26.432	37.744	38.975	41.111
	CSCS	LD2	27.515	35.046	38.139	44.516	49.094
		[156]	27.675	35.182	38.499	44.261	48.880
	CFCF	LD2	15.156	17.636	26.840	28.633	30.885
		[156]	15.344	17.798	26.640	28.518	31.178
	CFSF	LD2	11.358	13.268	19.794	26.367	28.726
		[156]	11.251	13.155	19.310	26.175	28.562
	FCFS	LD2	5.870	11.777	20.315	22.092	23.994
		[156]	5.846	11.398	19.907	21.586	24.115
$(0,90)_3$	CCCC	LD2	37.487	38.597	45.676	47.692	53.624
		[156]	37.562	38.711	45.369	47.324	53.543
	SSSS	LD2	14.445	22.983	31.169	33.616	38.710
		[156]	14.018	22.265	30.663	32.825	37.790
	CSCS	LD2	16.581	24.562	32.640	35.181	39.624
		[156]	16.338	24.020	32.222	34.513	38.824
	CFCF	LD2	13.287	15.061	20.798	27.454	29.723
		[156]	13.185	14.946	20.394	27.278	29.563
	CFSF	LD2	12.950	13.931	21.850	23.089	27.703
		[156]	13.127	14.149	21.550	23.253	27.606
	FCFS	LD2	4.6021	15.610	16.849	27.363	28.443
		[156]	4.733	15.322	17.011	27.282	28.683

Table 5.15: Non-dimensional natural frequencies of isotropic spherical panel shell, $h/a = 0.1, R/a = 5, \lambda = \omega a^2/h\sqrt{\frac{\rho}{E}}$

Theory	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}	ω_{15}	ω_{16}	ω_{17}
									SSSS								
ED1	6.31	14.56	14.56	19.63	19.63	22.31	27.19	27.19	27.76	34.16	34.16	39.27	39.27	42.87	42.87	43.90	43.90
ED2	6.00	13.82	13.82	19.63	19.63	21.23	25.92	25.92	27.76	32.62	32.62	39.27	39.27	41.04	41.04	43.06	43.90
ED3	5.99	13.73	13.73	19.63	19.63	21.01	25.61	25.61	27.76	32.16	32.16	39.27	39.27	40.35	40.35	42.31	43.90
[157, 158]	6.07	13.88		19.48			25.78		27.55					40.46			43.56
FEM (3D)	6.08	13.93	13.93	19.49	19.49	21.32											
									CSCS								
ED1	8.74	15.75	19.02	19.63	25.29	27.80	32.61	34.33	35.97	36.25	38.28	39.27	43.20	46.52	47.78	48.42	50.12
ED2	8.34	14.99	18.20	19.63	24.20	26.52	31.36	34.32	34.46	36.23	36.81	39.27	41.38	45.97	46.52	46.75	48.14
ED3	8.28	14.85	17.94	19.63	23.84	26.18	30.74	33.88	34.31	36.07	36.23	39.27	40.66	45.00	45.63	46.51	47.18
[157, 158]	8.35		18.00	19.49			30.70		34.34		36.70			45.40			
FEM (3D)	8.41	15.06	18.18	19.49	24.15	26.51											
									SFSF								
ED1	2.97	4.76	10.81	11.52	13.46	14.79	19.63	19.68	21.07	24.38	26.13	27.77	29.46	32.12	34.13	34.23	35.12
ED2	2.86	4.68	10.43	11.03	13.05	14.79	19.07	19.63	20.14	23.38	25.26	27.76	28.40	31.09	33.56	34.12	34.21
ED3	2.85	4.66	10.37	10.96	12.95	14.79	18.87	19.63	19.97	23.11	24.95	27.76	28.03	30.64	33.13	34.12	34.21
[157, 158]	2.87	4.62	10.40	11.03	12.93	14.82	18.86	19.51									
FEM (3D)	2.88	4.64	10.43	11.09	13.03	14.80											

Non-dimensional natural frequencies of the Square Kirchhoff plate theory, $\lambda = \omega a^2 \sqrt{\frac{\rho h}{D}}$

BCs	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
CCCC	Present	35.985	73.393	73.393	108.21	131.58	132.20	165.00	165.00
	[31]	35.992	73.413	73.413	108.27	131.64	132.24		
	[37]	35.986	73.399	73.399	108.23	131.41			
SSSS	Present	19.739	49.348	49.348	78.956	98.696	98.696	128.30	128.30
	[31]	19.739	49.348	49.348	78.956	98.696	98.696	128.30	128.30
	[37]	19.739	49.349	49.349	78.958	98.415			
CSCS	Present	28.950	54.743	69.327	94.585	102.21	129.09	140.20	154.77
	[31]	28.950	54.743	69.327	94.585	102.21	129.09	140.20	154.77
	[37]	28.951	54.745	69.329	94.589	101.95			
CCSS	Present	27.054	60.538	60.786	92.836	114.55	114.70	145.78	146.08
	[31]	27.056	60.544	60.791	92.865	114.57	114.72		
	[37]	27.054	60.540	60.788	92.834	114.20			
CFCF	Present	22.165	26.402	43.591	61.170	67.166	79.813	87.588	120.09
	[31]	22.272	26.529	43.664	61.466	67.549	79.904		
	[37]	22.237	26.594	43.871	61.407	67.659			
SFSF	Present	9.6314	16.134	36.725	38.945	46.738	70.740	75.283	87.986
	[31]	9.6314	16.134	36.725	38.945	46.738	70.740		
	[37]	9.631	16.135	36.726	38.945	46.739			
CFFF	Present	3.4349	8.6160	21.272	27.352	30.999	54.224	61.317	64.186
	[31]	3.4917	8.5246	21.429	27.331	31.111	54.443		
	[37]	3.485	8.604	21.586	27.230	31.358			
SFFF	Present	6.6917	14.905	25.478	26.062	48.459	50.736	58.774	65.307
	[31]	6.6480	15.023	25.492	26.126	48.711	50.849		
	[37]	6.636	14.901	25.388	26.003	48.469			
FFFF	Present	13.620	19.596	24.325	34.977	34.990	61.140	61.160	64.001
	[31]	13.468	19.596	24.271	34.801	34.801	61.111	61.111	69.279
	[37]	13.454	19.597	24.271	34.815	34.817			

Non-dimensional natural frequencies for the in-plane vibration of sector annular plate, $\lambda = \omega a^2 \sqrt{\frac{\rho}{E}}, \nu = 0.33, \beta = 0.5$

α	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
				SSSS1				
30	3.9198	4.2857	7.1534	7.7406	8.2282	9.6151	10.774	11.440
45	2.9825	3.9198	5.6094	5.7644	7.7406	8.2282	8.5126	8.5878
60	2.3779	3.9198	4.2857	5.1111	6.2682	7.1534	7.7406	7.8930
75	2.0534	3.4962	3.9198	4.7476	5.0805	6.3109	6.6621	7.5359
90	1.8624	2.9825	3.9198	4.2857	4.5254	5.6094	5.7644	6.9241
				SSSS2				
30	6.1118	6.7725	7.7608	9.2674	10.158	12.039	12.733	12.797
45	4.7603	5.6021	6.7725	7.4362	8.4225	9.6103	9.9957	10.158
60	3.8027	4.8741	6.1118	6.7725	7.7608	8.0269	8.1133	8.6481
75	3.1037	4.5984	5.3324	6.4181	6.7725	6.9003	7.7122	8.2796
90	2.6101	4.4609	4.7603	5.6021	6.1118	6.7725	7.4362	7.4704
				FFFF				
30	4.8464	5.6189	5.6785	6.0716	7.7172	8.5038	9.2103	9.6426
45	3.9516	4.5785	4.7464	5.3662	6.3297	6.8614	6.9257	7.6830
60	2.8185	3.7550	4.0170	5.3325	5.6191	5.8646	6.2471	6.5695
75	2.0687	3.1003	3.4998	4.8642	5.1433	5.1969	5.6156	5.9452
90	1.5608	2.6721	2.9624	4.4167	4.4376	4.6210	5.5091	5.5917
				S1CS1C				
30	4.9617	7.5080	8.5318	9.9073	10.6550	11.8665	12.9769	13.2176
45	3.4926	5.2923	7.2431	7.2887	8.3676	8.8092	9.9373	10.5923
60	2.7576	4.2218	5.9242	6.0710	7.1480	7.8751	8.3615	8.8992
75	2.3338	3.5739	5.0402	5.1742	6.5946	6.6237	7.7871	7.8661
90	2.0690	3.1224	4.3185	4.7382	5.6409	6.0900	6.9810	7.1982
				S2FS2F				
30	2.8739	5.8308	6.5960	7.0846	8.0896	8.5094	9.7528	10.3170
45	3.2971	4.6365	5.5711	6.6478	7.1224	7.1636	7.7245	9.0620
60	3.5138	3.6876	4.7016	5.9752	6.6815	7.1667	7.3023	7.3557
75	3.0480	3.6325	4.2456	5.3073	6.2937	6.3970	7.1081	7.1929
90	2.5830	3.6967	4.0297	4.7260	5.5864	5.9957	6.8404	6.8424
				S1CS2F				
30	3.6889	4.6148	5.9073	8.7098	9.3844	9.4403	10.5557	12.0189
45	2.5659	4.1016	5.3536	6.7821	6.9963	8.3148	8.8595	9.7370
60	2.0287	3.9026	5.1693	5.4703	5.6916	7.0764	8.0003	8.2896
75	1.7338	3.7748	4.4813	4.9488	5.2883	6.3398	6.9092	7.0416
90	1.5579	3.5473	4.0040	4.5907	5.2217	5.7919	5.9877	6.4493

Non-dimensional natural frequencies of the fully simply-supported TSDT composite laminated plate, $(0/90/90/0), h/a = 0.1, \lambda = \omega a^2/h\sqrt{\frac{\rho}{E_2}}$

$E1/E2$	Author	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
40	Present	15.1438	18.7674	26.5480	29.3436	29.3436	32.2886	37.6659	39.2598
	[194]	15.143							
10	Present	9.85339	18.8813	27.7476	33.2736	33.9522	44.6962	49.8431	49.9142
	[194]	9.853							
3	Present	7.25211	15.6289	19.4222	26.3661	28.7860	36.9583	37.9758	42.7148
	[194]	7.247							

Bibliography

[1] D. Gottlieb and S.A. Orszag, Numerical analysis of spectral methods: Theory and Applications, SIAM, (1977)

[2] B. Yagci, S. Filiz, L.L. Romero and O.B. Ozdoganlar, A spectral- Tchebychev technique for solving linear and nonlinear beam equations, Journal of Sound and Vibration, 321 (2009) 375–404.

[3] C.H. Lin, and M.R. Jen, Analysis of non-rectangular laminated anisotropic plates by Chebyshev collocation method, JSME International Journal, 47(2) (2004) 146–156.

[4] C.H. Lin, and M.R. Jen, Analysis of a laminated anisotropic plates By Chebyshev collocation method, Composites Part B: Engineering, 36 (2005) 155–169.

[5] V. Deshmukh, Spectral collocation-based optimization in parameter estimation for non-linear time-varying dynamical systems, Journal of Computational and Nonlinear Dynamics, 3 (2008) 1–7.

[6] L.N. Trefethen, Spectral methods in MATLAB, Software, Enviroments, and Tools. SIAM, Philadelphia, (2000)

[7] Y. Luo, Polynomial time-marching for three-dimensional wave equations, Journal of Scientific Computing, 12(4) (1997) 465–477.

[8] U. Ehrenstein and R. Peyret, A Chebyshev collocation method for the Navier-Stokes equations with application to double diffusive convection, International Journal for Numerical Methods in Fluids, 9(4) (2005) 427–452.

[9] V. Deshmukh, E.A. Butcher and E. Bueler, Dimensional reduction of nonlinear delay differential equations with periodic coefficients using Chebyshev spectral collocation, Nonlinear Dynamics, 53 (2008) 137–149.

[10] D. Zhou, Y.K. Cheung, S.H. Lo and F.T.K Au, Three dimensional vibration analysis of rectangular thick plates on Pasternak foundation, International Journal for numerical methods in Engineering, 59 (2004) 1313–1324.

[11] S.C. Sinha and E.A. Butcher, Solution and stability of a set of Pth order linear differential equations with periodic coefficients via Chebyshev polynomials, Mathematical Problems in Engineering, 2 (1996) 165–190.

[12] Eric Gourgoulhon, Introduction to spectral methods, 4th EU Network Meeting, Palma de Mallorca, Sept. 2002

[13] A.J.M Ferreira and G.E. Fasshauer, Analysis of natural frequencies of composite plates by an RBF-pseudospectral method, Composite Structures 79 (2007) 202–210.

[14] A. Karageorghis, A note on the satisfaction of the boundary conditions for Chebyshev collocation methods in rectangular domains, J. scientific computing, 6(1) (1991) 21–25.

[15] D. Gottlieb, M.Y. Hussaini and S.A. Orszag, Theory and applications of spectral methods, for partial differential equations, Ed. by R.G. Voigt, D. Gottlieb, M.Y. Hussaini, SIAM-CBMS, (1984) 1–54.

[16] C. Canuto, M.Y. Hussaini, A. Quarteroni and Th.A. Zang, Spectral Methods: Fundamentals in single domain (2006)

[17] J.P. Boyd, Chebyshev and Fourier spectral methods, Second edition, Dover, New York (2001)

[18] J. Thomas and B.A.H. Abbas, Finite element model for dynamic analysis of Timoshenko beam, Journal of Sound and Vibration, 41 (1975) 291–299.

[19] Z. Friedman and J.B. Kosmatka, An improved two-node Timoshenko beam finite element, Computers Structures, 47(3) (1993) 473–481.

[20] P.A.A. Laura and R.H. Gutierrez, Analysis of vibrating Timoshenko beams using the method of differential quadrature, Shock and Vibration, 1(1) (1993) 89–93.

[21] B.A.H. Abbas, Vibrations of Timoshenko beams with elastically restrained ends, Journal of Sound and Vibration, 97(4) (1984) 541–548.

[22] R.0. tafford and V. Giurgiutiu, Semi-analytic method for rotating Timoshenko beams, International Journal of Mechanical Sciences, 17 (1975) 719–727.

[23] T. Yokoyama, Free vibration characteristics of rotating Timoshenko beams, International Journal of Mechanical Sciences, 30 (1988) 743–755.

[24] S.Y. Lee and Y.H. Kuo, Bending frequency of a rotating Timoshenko beam with general elastically restrained root, Journal of Sound and Vibration, 162 (1993) 243–250.

[25] J.R. Banerjee, Dynamic stiffness formulation and free vibration analysis of centrifugally stiffened Timoshenko beams, Journal of Sound and Vibration, 247(1) (2001) 97–115.

[26] H. Du, M.K. Lim and K.M. Liew, A power series solution for vibrations of a rotating Timoshenko beam, Journal of Sound and Vibration, 175(4) (1994) 505–523.

[27] M.O. Kaya, Free vibration analysis of a rotating Timoshenko beam by differential transform method, Aircraft Engineering and Aerospace Technology, 78(3) (2006) 194–203.

[28] J. Lee and W.W. Schultz, Eigenvalue analysis of Timoshenko beams and axisymmetric Mindlin plates by the pseudospectral method, Journal of Sound and Vibration, 269 (2004) 609–621.

[29] M.A. Sprague and T.L. Geers, Legendre spectral finite elements for structural dynamics analysis, Communications in numerical methods in engineering, (2007) 1–13.

[30] S. B. Cokun, M. T. Atay and B. ztrk, Transverse vibration analysis of Euler-Bernoulli beams using analytical approximate techniques, Advances in Vibration Analysis Research, (2011).

[31] A.W. Leissa, The free vibration of rectangular plates, Journal of Sound and Vibration, 31(3) (1973) 257–293.

[32] S.M. Dickinson and E.K.H. Li, On the use of simply supported plate functions in the Rayleigh-Ritz method applied to the vibration of rectangular plates, Journal of Sound and Vibration, 80 (1982) 292–297.

[33] P. Cupial, Calculation of the natural frequencies of composite plates by the Rayleigh-Ritz method with orthogonal polynomials, Journal of Sound and Vibration, 201 (1997) 385–387.

[34] R.B. Bhat, Natural frequencies of rectangular plates using characteristic orthogonal polynomials in the Rayleigh-Ritz method, Journal of Applied Mechanics, 102 (1985) 493–499.

[35] L. Dozio, On the use of the trigonometric Ritz method for general vibration analysis of rectangular Kirchhoff plates, Thin-Walled Structures 49 (2011) 129–144.

[36] W.L. Li, X. Zhang, J. Du and Z. Liu, An exact series solution for the transverse vibration of rectangular plates with general elastic boundary supports, Journal of Sound and Vibration, 321 (2009) 254–269.

[37] C. Shu and H. Du, A generalized approach for implementing general boundary conditions in the GDQ free vibration analysis of plates, International Journal of Solids Structures, 34(7) (1997) 837–846.

[38] C.W. Bert, X. Wang, A.G. Striz, Differential quadrature for static and free vibrational analysis of anisotropic plates, International Journal of Solids Structures, 30 (1993) 1737–1744.

[39] R.F.S. Hearmon, The frequency of flexural vibration of rectangular orthotropic plates with clamped or supported edges, Journal of Applied Mechanics, 26 (1959) 537–540.

[40] S.M. Dickinson and A. Di Blasio, On the use of orthogonal polynomials in the Reyleigh-Ritz method for the study of the flexural vibration and buckling of isotropic and orthotropic rectangular plates, Journal of Sound and Vibration, 108(1) (1986) 51–62.

[41] R.E. Rossi, D.V. Bambill and P.A.A. Laura, Vibrations of a rectangular orthotropic plate with a free edge a comparison of analytical and numerical results, Ocean Eng, 25(7) (1998) 521–527.

[42] D.J. Gorman, Accurate free vibration analysis of clamped orthotropic plates by the method of superposition, Journal of Sound and Vibration, 140(3) (1990) 391–411.

[43] D.J. Gorman, Accurate free vibration analysis of the completely free orthotropic rectangular plate by the method of superposition, Journal of Sound and Vibration, 165(3) (1993) 409–420.

[44] D.J. Gorman and W. Ding, Accurate free vibration analysis of completely free symmetric cross-ply rectangular laminated plates, Composite Structures, 60 (2003) 359–365.

[45] S. Yu and W. Cleghorn, Generic free vibration of orthotropic rectangular plates with clamped and simply supported edges, Journal of Sound and Vibration, 163(3) (1993) 439–50.

[46] M. Dalaei and A.D. Kerr, Natural vibration analysis of clamped rectangular orthotropic plates, Journal of Sound and Vibration, 189(3) (1996) 399–406.

[47] L.V. Kantorovich and V.L. Krylov, Approx Methods Higher Anal. The Netherlands: Groningen, Noordhoff, (1964)

[48] T. Sakata, K. Takahashi and R.B. Bhat, Natural frequencies of orthotropic rectangular plates obtained by iterative reduction of the partial differential equation, Journal of Sound and Vibration, 189(1) (1996) 89–101.

[49] R.L. Ramkumar, P.C. Chen and W.J. Sanders, Free vibration solution for clamped orthotropic plates using Lagrangian multiplier technique, Am Inst Aeronaut Astronaut J, 25(1) (1987) 146–151.

[50] M. Huang, X.Q. Ma, T. Sakiyama, H. Matuda and C. Morita, Free vibration analysis of orthotropic rectangular plates with variable thickness and general boundary conditions, Journal of Sound and Vibration, 288 (2005) 931–955.

[51] Y.F. Xing and B. Liu, New exact solutions for free vibrations of thin orthotropic plates, Composite Structures 89(4) (2009) 567-574.

[52] C.W. Bert, and M. Malik, Free vibration analysis of tapered rectangular plates by differential quadrature method: A semi-analytical approach, Journal of Sound and Vibration, 190(1) (1996) 41–63.

[53] M. Sari, M. Nazari and E.A. Butcher, Effects of damaged boundaries on the free vibration of Kirchoff plates: Comparison of perturbation and spectral collocation solutions, Journal of Computational and Nonlinear Dynamics, 7 (2012) 011011-1, DOI: 10.1115/1.4004808

[54] Y.B. Zhao, G.W. Wei, Y. Xiang, Discrete singular convolution for the prediction of high frequency vibration of plates, International Journal of Solids and Structures, 39 (2002) 65–88.

[55] C. Shu and C.M. Wang, Treatment of mixed and nonuniform boundary conditions in GDQ vibration analysis of rectangular plates, Engineering Structures 21 (1999) 125–134.

[56] A.S. Shour, Vibration of angle-ply symmetric laminated composite plates with edges elastically restrained, J. Composite Structures, 74 (2006) 294–302.

[57] S.T. Chow, K.M. Liew and K.Y. Lam, Transverse vibration of symmetrically laminated rectangular composite plates, J. Composite Structures, 20 (1992) 213–226.

[58] A.W. Leissa and Y. Narita, Vibration studies for simply supported symmetrically laminated rectangular plates, J. Composite Structures, 12 (1989) 113–132.

[59] C.H.W. Ng, Y.B. Zhao and G.W. Wei, Comparison of discrete singular convolution and generalized differential quadrature for the vibration analysis of rectangular plates, J. Comput. Methods Appl. Mech. Eng. 193 (2004) 2483–2506.

[60] G.W Wei, Y.B Zhao and Y Xiang, The determination of natural frequencies of rectangular plates with mixed boundary conditions by discrete singular convolution, 43 (2001) 1731–1746.

[61] A.W. Leissa, P.A.A. Laura and R.H. Gutierrez, Vibrations of rectangular plates with nonuniform elastic edges supportes, 47 (1980) 891–895.

[62] G. Karami, S. A. Shahpari and P. Malekzadeh, DQM analysis of skewed and trapezoidal laminated plates, Composite Structures 59 (2003) 393–402.

[63] P. Yousefi, M.H. Kargarnovin and S.H. Hosseini-Hashemi, Free vibration of generally laminated plates with various shapes, Journal of polymer composites, (2011)

[64] G. Catania1 and S. Sorrentino, Spectral modeling of vibrating plates with general shape and general boundary conditions, Journal of vibration and control, Journal of vibration and control, (2011) doi: 10.1177/1077546311426593

[65] A. Secgin and A.S. Sarigul, Free vibration analysis of symmetrically laminated thin composite plates by using discrete singular convolution (DSC) approach: Algorithm and verification, Journal of Sound and Vibration 315 (2008) 197–211.

[66] C.B. Kim, H. S. Cho and H.G. Beom, Exact solutions of in-plane natural vibration of a circular plate with outer edge restrained elastically, Journal of Sound & Vibration 331 (2012) 2173–2189.

[67] A.V. Singh and T. Muhammad, Free in-plane vibration of isotropic non-rectangular plates, Journal of Sound and Vibration 273 (2004) 219–231.

[68] J. Dua, W.L. Lib, G. Jina, T. Yanga and Z. Liua, An analytical method for the in-plane vibration analysis of rectangular plates with elastically restrained edges, Journal of Sound and Vibration 306 (2007) 908927.

[69] L. Dozio, Free in-plane vibration analysis of rectangular plates with arbitrary elastic boundaries, Mechanics Research Communications 37 (2010) 627–635.

[70] T. Irie, G. Yamada and Y. Muramoto, Natural frequencies of in-plane vibration of annular plates, Journal of Sound and Vibration 97 (1984) 171–175.

[71] D.J. Gorman, Free in-plane vibration analysis of rectangular plates by the method of superposition, Journal of Sound and Vibration 272 (2004) 831–851.

[72] M.R. Karamooz Ravari and M.R. Forouzan, Frequency equations for the in-plane vibration of orthotropic circular annular plate, Arch Appl Mech 81 (2011) 1307–1322.

[73] Chan Il Park, Frequency equation for the in-plane vibration of a clamped circular plate, Journal of Sound and Vibration 313 (2008) 325–333.

[74] B. Liu and Y. Xing, Exact solutions for free in-plane vibrations of rectangular plates, 24(6) (2011) 556–567.

[75] S. Bashmal, R. Bhat and S. Rakheja, In-plane free vibration of circular annular disks, Journal of Sound and Vibration 322 (2009) 216226.

[76] N. Fantuzzi, F. Tornabene, E. Viola, Generalized differential quadrature finite element method for vibration analysis of arbitrarily shaped membranes, International Journal of Mechanical Sciences 79(1) (2014) 216–251.

[77] E. Viola, F. Tornabene, N. Fantuzzi, Generalized differential quadrature finite element method for cracked composite structures of arbitrary shape, Composite Structures 106(1) (2013) 815–834.

[78] N. Fantuzzi, F. Tornabene, Strong formulation finite element method for arbitrarily shaped laminated plates-I, Theoretical Analysis, Advances in Aircraft and Spacecraft Science 1(2) (2014) 125–143.

[79] R.D. Mindlin, Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates, Journal of Applied Mechanics, 18 (1951) 31–38.

[80] M. Levinson, Free vibrations of a simply supported rectangular plate, an exact elasticity solution, Journal of Sound and Vibration, 98 (1985) 289–298.

[81] S. Srinivas and A.K. Rao, Bending and buckling of simply supported thick orthotropic rectangular plates and laminates, International Journal of Solids Structures, 6 (1970) 1463–1481.

[82] L.W. Chen and J.L. Doong, Large amplitude vibration of initially stressed moderately thick plate, Journal of Sound and Vibration, 78 (1983) 388–497.

[83] D.J. Dawe and O.L. Roufaeil, Rayleigh-Ritz vibration analysis of Mindlin plates, Journal of Sound and Vibration, 69(3) (1980) 345–359.

[84] L.F. Greimann and P.P. Lynn, Finite element analysis of plate bending with transverse shear deformation, Nucl. Engng Des, 14 (1970) 223–230.

[85] T.A. Rock and H. Hinton, Free vibration and transient response of thick and thin plates using the finite element method," Earthquake Engng. Struct. Dyn., 3 (1974) 57–63.

[86] K.M. Liew, K.C. Hung and M.K. Lim, Vibration of Mindlin plates using boundary characteristic orthogonal polynomials, Journal of Sound and Vibration, 182(1) (1995) 77–90.

[87] K.M. Liew, Y. Xiang, S. Kitipornchai and C.M. Wang, Vibration of thick skew plates based on Mindlin shear deformation plate theory, Journal of Sound and Vibration 168(1) (1993) 39–69.

[88] J.H. Chung, T.Y. Chung and K.C. Kim, Vibration analysis of orthotropic Mindlin plates with edges elastically restrained against rotation, Journal of Sound and Vibration, 163 (1993) 151–163.

[89] K.N. Saha, R.C. Kar and P.K. Datta, Free vibration analysis of rectangular plates with elastic restraints uniformly distributed along the edges," Journal of Sound and Vibration, 192(4) (1996) 885–904.

[90] S.S. Rao and A.S. Prasad, Vibrations of annular plates including the effects of rotatory inertia and transverse shear deformation, Journal of Sound and Vibration, 42(3) (1975) 305–324.

[91] T. Irie, G. Yamada and S. Aomura, Free vibration of a mindlin annular plate of varying thickness, Journal of Sound and Vibration, 66(2) (1979) 187–197.

[92] T. Irie, G. Yamada and K. Takagi, Natural frequencies of thick annular plates, ASME Journal of Applied Mechanics, 49(3) (1982) 633–638.

[93] S.K. Sinha, Determination of natural frequencies of a thick spinning annular disk using a numerical Rayleigh-Ritz's trial function, Journal of Acoustics Society of America, 81(2) (1987) 357–369.

[94] S.L. Nayar, K.K. Raju and G.V. Rao, Axisymmetric free vibrations of internally compressed moderately thick annular plates, Computers and Structures, 53(3) (1994) 759–765.

[95] J.B. Han and K.M. Liew, Axisymmetric free vibration of thick annular plates, International Journal of Mechanical Sciences, 41 (1999) 1089–1109.

[96] O. Civalek and M. Giirses, Free vibration of annular Mindlin plates with free inner edge via discrete singular convolution method, The Arabian Journal of Science and Engineering, 34(IB) (2008) 81–90.

[97] G.C. Pardoen, Static vibration and buckling analysis of axisymmetric circular plates using finite element, Computers and Structures, 3 (1973) 355–375.

[98] K.M. Liew, J.B. Han and Z.M. Xiao, Vibration analysis of circular Mindlin plates using the differential quadrature method, Journal of Sound and Vibration, 205(5) (1997) 617–630.

[99] J. Lee and W.W. Schultz, Eigenvalue analysis of Timoshenko beams and axisymmetric Mindlin plates by the pseudospectral method, Journal of Sound and Vibration, 269 (2004) 609–621.

[100] P. Guruswamy and T.Y. Yang, A sector element for dynamic analysis of thick plates, Journal of Sound and vibration 62 (1979) 505–516.

[101] M.S. Cheung and M.Y.T. Chan, Static and dynamic analysis of thin and thick sectoral plates by the finite strip method, Computers and Structures, 14 (1981) 79–88.

[102] R.S. Srinivasan and V. Thiruvenkatachari, Free vibration of transverse isotropic annular sector Mindlin plates," Journal of Sound and Vibration, 101 (1985) 193–210.

[103] T. Mizusawa, Vibration of thick annular sector plates using semi- analytical methods, Journal of Sound and Vibration, 150 (1991) 245–259.

[104] Y. Xiang, K.M. Liew and S. Kitipornchai, Transverse vibration of thick annular sector plates," Proceedings of the American Society of Civil Engineers, Journal of Engineering Mechanics, 119 (1993) 1579–1599.

[105] K.M. Liew and F.L. Liu, Differential quadrature method for vibration analysis of shear deformable annular sector plates, Journal of Sound and Vibration, 230(2) (2000) 335–356.

[106] W. Lanhe, L. Hua and W. Daobin, Vibration analysis of generally laminated composite plates by the moving least squares differential quadrature method, Composite Structures 68 (2005) 319–330.

[107] Song Xiang, Shao-xi Jiang, Ze-yang Bi, Yao-xing Jin, Ming-sui Yang, A nth-order mesh less generalization of Reddys third-order shear deformation theory for the free vibration on laminated composite plates, Composite Structures 93 (2011) 299–307.

[108] K. M. Liew and F.L. Liu, Differential quadrature method for vibration analysis of shear deformable annular sector plates, Journal of Sound and vibration 230(2) (2000) 335–356.

[109] K. M. Liew, J. B. Han and Z. M. Xiao, Vibration analysis of circular Mindlin plates using the differential quadrature method, Journal of Sound and Vibration 205(5) (1997) 617–630.

[110] K.K. Viswanathan and Kyung Su Kim, Free vibration of antisymmetric angle-ply-laminated plates including transverse shear deformation: Spline method, International Journal of Mechanical Sciences 50 (2008) 1476–1485.

[111] Huu-Tai Thai, Seung-Eock Kim, Free vibration of laminated composite plates using two variable refined plate theory, International Journal of Mechanical Sciences, 52 (2010) 626–633.

[112] M. Gurses, O. Civalek, H. Ersoy and Okyay Kiracioglu, Analysis of shear deformable laminated composite trapezoidal plates, Materials and Design 30 (2009) 3030–3035.

[113] T. Kant and K. Swaminathan, Free vibration of isotropic, orthotropic, and multilayer plates based on higher-order refined theories, Journal of Sound and vibration 241(2) (2001) 319–327.

[114] T. Kant and K. Swaminathan, Analytical solution for free vibration of laminated composite and sandwich plates based on a higher-order refined theory, Journal of composite structures 53 (2001) 73–85.

[115] G. Karami, P. Malekzadeh, S.R. Mohebpour, DQM free vibration analysis of moderately thick symmetric laminated plates with elastically restrained edges, Journal of composite structures 74 (2006) 115–125.

[116] S. Wang, Free vibration analysis of skew fiber-reinforced composite laminated based on first-order shear deformation plate theory, Journal of composite and structures, 63(3), 525–538.

[117] K.Y. Daia, G.R. Liua, K.M. Lima and X.L. Chen, A mesh-free method for static and free vibration analysis of shear deformable laminated composite plates, Journal of Sound and Vibration 269 (2004) 633–652.

[118] O. Civalek, Free vibration analysis of symmetrically laminated composite plates with first-order shear deformation theory (FSDT) by discrete singular convolution method, Finite Elements in Analysis and Design 44 (2008) 725–731.

[119] T. Kant and Mallikarjuna, Higher-order theory for free vibration of unsymmetrically composite laminated and sandwich plate finite element evaluations, Journal of composite and structures, 32(5) (1989) 1125–1132.

[120] D. Ngo-Conga, N. Mai-Duya, W. Karunasenab and T. Tran-Conga, Free vibration analysis of laminated composite plates based on FSDT using one-dimensional IRBFN method, Journal of Computers and Structures, 89(1-2), 1–13. ISSN 0045-7949.

[121] H. Nguyen-Van, N. Mai-Duy and T. Tran-Cong, Free vibration analysis of laminated plate/shell structures based on FSDT with a stabilized nodal-integrated quadrilateral element, Journal of Sound & Vibration, 313(12) (2008) 205–223.

[122] K.M. Liew, Y.Q. Huang and J.N. Reddy, Vibration analysis of symmetrically laminated plates based on FSDT using the moving least squares differential quadrature method, Comput. Methods Appl. Mech. Eng. 192 (2003) 2203–2222.

[123] E. Asadi, M.S. Qatu, Static analysis of thick laminated shells with different boundary conditions using GDQ. Thin-walled Structures 51 (2012) 76–81.

[124] K. M. Liew, Solving the vibration of thick symmetric laminates by Reissner/Mindlin plate theory and the P-RITZ method, Journal of Sound and Vibration, 198(3), 1996, 343–360.

[125] K. M. Liew, K. C. Hung and M. K. Lim, Vibration of Mindlin plates using boundary characteristic orthogonal polynomials, Journal of Sound and Vibration, 182(1) (1995) 77–90.

[126] J.B. Han and K.M. Liew, Axisymmetric free vibration of thick annular plates, International Journal of Mechanical Sciences 41 (1999) 1089–1109.

[127] M. Sari and E.A. Butcher, Free vibration analysis of rectangular and annular Mindlin plates with undamaged and damaged boundaries by the spectral collocation method, Journal of Vibration and Control, (2012) doi:10.1177/1077546311422242

[128] A.A. Khdeira and J.N. Reddy, Free vibrations of laminated composite plates using second-order shear deformation theory, Computers and Structures 71 (1999) 617–626.

[129] Sh. Hosseini-Hashemi, K. Khorshidi1, and H. Payandeh, Vibration analysis of moderately thick rectangular plates with internal line support using the Rayleigh-Ritz approach, B: Mechanical Engineering, 16(1) (2009) 22–39.

[130] M. Zamani, A. Fallah and M.M. Aghdam, Free vibration analysis of moderately thick trapezoidal symmetrically laminated plates with various combinations of boundary conditions, European Journal of Mechanics A/Solids, 36 (2012) 204–212.

[131] H. Naginoa, T. Mikamia and T. Mizusawab, Three-dimensional free vibration analysis of isotropic rectangular plates using the B-spline Ritz method, Journal of Sound and Vibration 317 (2008) 329–353.

[132] D. Zhoua, S.H. Lob and Y.K. Cheungb, 3-D vibration analysis of annular sector plates using the ChebyshevRitz method, Journal of sound and vibration, 320(1-2) (2009) 421–437.

[133] Ashish Sharmaa, H.B. Shardaa and Y. Nathb, Stability and vibration of thick laminated composite sector plates, Journal of Sound and Vibration 287 (2005) 1–23.

[134] Y. Mochida, S. Ilanko, M. Duke and Y. Narita, Free vibration analysis of doubly curved shallow shells using the Super position-Galerkin method, Journal of Sound and Vibration 331(6) (2012) 1413–1425.

[135] L. E. Monterrubio, Free vibration of shallow shells using the Rayleigh-Ritz method and penalty parameters, Proc. IMechE Vol. 223 Part C: J. Mechanical Engineering Science.

[136] K.M. Liew and C.W. Lim, Vibration of doubly-curved shallow shells, Acta Mechanica, 114 (1996) 95-119.

[137] R.A. Ruotolo, comparison of some thin shell theories used for the dynamic analysis of stiffened cylinders. J Sound Vibration (2001) 243:847.

[138] O. Civalek, Free vibration analysis of composite conical shells using the discrete singular convolution algorithm. Steel Comp Struct (2006) 6:353.

[139] O. Civalek, Numerical analysis of free vibrations of laminated composite conical and cylindrical shells: discrete singular convolution (DSC) approach. J Comp Appl Math (2007) 205:251.

[140] C.W. Lim, K.M. Liew and S. Kitipornchai, Free vibration of pretwisted, cantilevered composite shallow conical shells, AIAA Journal 35(2) (1997).

[141] H.R.H. Kabir, Free vibration response of shear deformable antisymmetric cross-ply cylindrical panel, Journal of Sound and Vibration, 217(4) (1998) 601–618.

[142] H.R.H. Kabir and R.A. Chaudhuri, Free vibration of shear flexible anti-symmetric angle-ply doubly curved panels, International Journal of Solids and Structures, 28(1) (1991) 17–32.

[143] H.R.H. Kabir and R.A. Chaudhuri, Gibs-phenomenon-free Fourier solution for finite shear-flexible laminated clamped curved panels, International Journal of Engineering Science, 32(3) (1994) 501–520.

[144] A. Noseir and J.N. Reddy, Vibration and stability analysis of cross-ply laminated circular shells, Journal of Sound and Vibration, 157(1) (1992) 139–159.

[145] K.P. Soldatos, A comparison of some shell theories used for the dynamic analysis of cross-ply laminated circular cylindrical panels, Journal of Sound and Vibration, 97(2) (1984) 305–319.

[146] K.P. Soldatos, Refined laminated plate and shell theory with applications, Journal of Sound and Vibration, 144(1) (1991) 109–129.

[147] C. Shu, An efficient approach for free vibration analysis of conical shell, International Journal of Mechanical Sciences, 38 (1996) 935–949.

[148] C. Shu, Free vibration analysis of composite laminated conical shells by generalized differential quadrature, Journal of Sound and Vibration, 194(4) (1996) 587–604.

[149] C. Shu and H. Du, Free vibration analysis of laminated composite cylindrical shells by DQM, Journal of composite part B: Engineering, 28B (1997) 267–274.

[150] M. Ganapathi and M. Haboussi, Free vibrations of thick laminated anisotropic noncircular cylindrical shells. Comp Struct (2003) 60:125–33.

[151] M. Ganapathi, B.P. Patel and D.S. Pawargi, Dynamic analysis of laminated cross-ply composite non-circular thick cylindrical shells using higher-order theory. Int J Solids Struct 39 (2002) 5945–5962.

[152] A.W. Leissa, J.K. Lee and A.J. Wang, Vibration of cantilevered shallow cylindrical shells of rectangular platform, Journal of Sounfd and Vibration, 78(3) (1981) 311-328.

[153] A.W. Leissa and A. S. Kadi, Curvature effects on shallow shell vibrations, journal of sound and vibration, 16(2) (1971) 173–187.

[154] K.P. Soldatos and V.P. Hadjigeorgiou, Three-dimensional solution of the free vibration problem of homogeneous isotropic cylindrical shells and panels, Journal of sound and vibration, 137(3) (1990) 369–384.

[155] E. Asadi, W. Wanga and M.S. Qatu, Static and vibration analyses of thick deep laminated cylindrical shells using 3D and various shear deformation theories, Composite Structures 94 (2012) 494–500.

[156] E. Asadi and M. S. Qatu, Free vibration of thick laminated cylindrical shells with different boundary conditions using general differential quadrature, Journal of vibration and control, (2012)

[157] Sh. Hosseini-Hashemi and M. Fadaee, On the free vibration of moderately thick spherical shell panel-A new exact closed-form procedure, Journal of Sound and vibration, 330 (2011) 4352–4367.

[158] Sh. Hosseini-Hashemi, S. R. Atashipour, M. Fadaee and U. A. Girhammar, An exact closed-form procedure for free vibration analysis of laminated spherical shell panels based on Sanders theory, Arch Appl Mech 82 (2012) 985–1002.

[159] M.S. Qatu, Recent research advances in the dynamic behavior of shells: 1989-2000, Part 1: Laminated composite shells. Journal of Applied Mechanics Review, 55(4) (2002) 325–349.

[160] M.S. Qatu, R.W. Sullivan and W. Wanga, Recent research advances on the dynamic analysis of composite shells: 2000–2009, Composite Structures 93 (2010) 14–31.

[161] K.M. Liew, L.X. Pengb and T.Y. Ngc, Three-dimensional vibration analysis of spherical shell panels subjected to different boundary conditions, International Journal of Mechanical Sciences 44 (2002) 2103–2117.

[162] W.Q. Chen and H.J. Ding, Free vibration of multi-layered spherically isotropic hollow spheres, International Journal of Mechanical Sciences 43 (2001) 667–680.

[163] K.M. Liew and C.W. Lim, A Ritz vibration analysis of doubly-curved rectangular shallow shells using a refined first-order theory, Comput. Methods Appl. Mech. Eng. 127 (1995) 145–162.

[164] Yeong-Chyuan Chern and C.C. Chao, Comparison of natural frequencies of laminates by 3-D theory, part II: Curved panels, Journal of Sound and vibration 230(5) (2000) 1009–1030.

[165] S.D. Yu, W.L. Cleghorn and R.G. Fenton, On the accurate analysis of free vibration of open circular cylindrical shells, Journal of Sound and Vibration, 188(3) (1995) 315–336.

[166] G.R. Buchanan and B.S. Rich, Effect of boundary conditions on free vibration of thick isotropic spherical shells, Journal of vibration and control, (2008)

[167] E. Viola, F. Tornabene and N. Fantuzzi, Static analysis of completely doubly-curved laminated shells and panels using general higher-order shear deformation theories, Composite Structures 101 (2013) 59–93.

[168] E. Viola, F. Tornabene and N. Fantuzzi, General higher-order shear deformation theories for the free vibration analysis of completely doubly-curved laminated shells and panels, Composite Structures 95 (2013) 639–666.

[169] F. Tornabene, Free vibrations of laminated composite doubly-curved shells and panels of revolution via the GDQ method, Comput. Methods Appl. Mech. Eng. 200 (2011) 931–952.

[170] N.S. Bardell, J. M. Dunsdon and R. S. Langley, Free vibration of thin, isotropic, open, conical panels, Journal of Sound and Vibration 217(2) (1998) 297–320.

[171] Y.K. Cheung, W.Y. Li and L.G. Tham, Free vibration analysis of singly curved shell by spline finite strip method, Journal of Sound and Vibration 128 (1989) 411–422.

[172] X. Zhao, Q. Li, K.M. Liew and T.Y. Ng, The element-free hp-Ritz method for free vibration analysis of conical shell panels, Journal of Sound and Vibration 295 (2006) 906–922.

[173] E. Carrera, Theories and finite elements for multilayered, Anisotropic, Composite Plates and Shells, Arch. Comput. Meth. Engng. 9(2) (2002) 87–140.

[174] E. Carrera, Theories and finite elements for multilayered plates and shells: A unified compact formulation with numerical assessment and benchmarking, Arch. Comput. Meth. Engng. 10(3) (2003) 215–296.

[175] E. Carrera, Multi-layered shell theories accounting for layer-wise mixed description, Part I: Governing equations, AIAA Journal, 37(9) (1999)

[176] E. Carrera, Multi-layered shell theories accounting for layer-wise mixed description, Part II: Numerical evaluation, AIAA Journal, 37(9) (1999)

[177] E. Carrera, Mixed layer-wise models for multi-layered plates analysis, Composite Structures 43 (1998) 57–70.

[178] E. Carrera, Evaluation of layer-wise mixed theories for laminated plates analysis, AIAA Journal, 36(5) (1998)

[179] L. Demasi, Quasi-3D analysis of free vibration of anisotropic plates, Composite Structures 74 (2006) 449–457.

[180] L. Dozio and E. Carrera, A variable kinematic Ritz formulation for vibration study of quadrilateral plates with arbitrary thickness, Journal of Sound and Vibration 330 (2011) 4611–4632.

[181] L. Dozio and E. Carrera, Ritz analysis of vibrating rectangular and skew multilayered plates based on advanced variable-kinematic models, Composite Structures 94 (2012) 2118–2128.

[182] L. Dozio, Computation of eigenvalues for thick and thin circular and annular plates using a unified Ritz-based formulation, proceedings of the eleventh international conference on computational structures technology, Stirlingshire, United Kingdom, (2012).

[183] F. Tornabene, E. Viola and N. Fantuzzi, General higher-order equivalent single layer theory for free vibrations of doubly-curved laminated composite shells and panels, Journal of Composite Structures 104 (2013) 94–117.

[184] A.J.M. Ferreira, E. Carrera, M. Cinefra, C.M.C. Rouque and O. Polit, Analysis of laminated shells by a sinusoidal shear deformation theory and radial basis functions collocation, accounting for through-the-thickness deformations, Journal of composite: Part B, 42 (2011) 1276–1284.

[185] A.J.M. Ferreira, C.M.C. Roque, E. Carrera, M. Cinefra and O. Polit, Analysis of laminated plates by trigonometric theory, radial basis, and unified formulation. AIAA J 49 (2011) 1559–1562.

[186] A.J.M. Ferreira, C.M.C. Roque, E. Carrera and M. Cinefra, Analysis of thick isotropic and cross-ply laminated plates by radial basis functions and a unified formulation, Composites Part B: Engineering. 58 (2014) 544–552.

[187] E. Carrera and S. Brischetto, Analysis of thickness locking in classical, refined and mixed multi-layered plate theories, Journal of Composite Structures 82 (2008) 549–562.

[188] E. Carrera and S. Brischetto, Analysis of thickness locking in classical, refined and mixed multi-layered plate theories, Composite Structures 82 (2008) 549–562.

[189] E. Carrera, A study of transverse normal stress effect on vibration of multi-layered plates and shells, Journal of Sound and Vibration 225 (1999) 803–829.

[190] F.A. Fazzolari and E. Carrera, Advanced variable kinematics Ritz and Galerkin formulations for accurate buckling and vibration analysis of anisotropic laminated composite plates. Comp. Struct. 94 (2011) 50–67.

[191] D. Zhou, Y.K. Cheung, F.T.K. Au, S.H. Lo, Three-dimensional vibration analysis of thick rectangular plates using Chebyshev polynomial and Ritz method, International Journal of Solids and Structures 39 (2002) 6339–6353.

[192] S.T. Chow, K.M. Liew and K.Y. Lam, Transverse vibration of symmetrically laminated rectangular composite plates, Composite Structures 20 (4) (1992) 213226.

[193] A.W. Leissa, Y. Narita, Vibration studies for simply supported symmetrically laminated rectangular plates, Composite Structures 12 (1989) 113–132.

[194] J.N. Reddy, Mechanics of laminated composite plates and shells, Theory and analysis. Second edition.

[195] M.S. Qatu, Vibration of laminated shells and plates, (2004)

[196] W. Soedel, Vibrations of Shells and Plates. Third edition.

[197] A. W. Leissa & Mohammad S. Qatu, Vibrations of continuous systems.

[198] S.S. Rao, Vibrations of continuous systems.

[199] M.S. Qatu, Review of shallow shell vibration research, J. Shock. vib. Dig. 24(9) 3–15.

[200] K.M. Liew, C.W. Lim, S. Kitipornchai, Vibration of shallow shells: A review with bibliography, Appl. Mech. Rev. 50(8) 431–444.

[201] A.W. Leissa, Vibration of shells, NASA SP288, US government printing office, Washington DC, republished 1993, Acoustical Society of America.

Conclusion and future development

An efficient numerical technique for studying natural frequencies of beams, plates and shells has been introduced by applying Chebyshev polynomial on collocation method using Kronecker product operator which is easy for implementation of the differential equations. Plates with constrained boundaries was studied in which the constrained boundaries were modeled using torsional and translational springs. The governing equations and boundary conditions were expressed in terms of Chebyshev polynomials and Kronecker product. It was concluded that Chebyshev polynomials can be used to solve 1-D, 2-D and quasi 3-D linear vibration problems, since they show fast convergence and accurate results compared to other techniques available in the literature. This property comes from using non-uniform points instead of uniform points. For increasing the accuracy, increasing the order of polynomial N is essential but this leads to increasing the error. But with this type of polynomial without increasing the error can reach to more accurate results.

This proposed method was applied to rod problem, Euler-Bernoulli and Timoshenko beams, in-plane vibration, Classical, first-order and third-order shear deformation theories of plates both isotropic and composite symmetric and non-symmetric, rectangular, annular, circular, sector annular and skew plates. Effect of constrained boundaries is considered on both thin and thick plates, isotropic and composite. The natural frequencies of several cases were calculated and compared with the well-known results of Leissa [31]. Thin shallow shell was considered in cylindrical, spherical and hyperbolic-paraboloidal geometry also thin shell is studied from shallow to deep. Thick shell such as shallow or deep was considered and for thick deep non-symmetric composite laminated shell, many boundary conditions were studied.

Finally, the quasi 3-D technique so called (CUF) was studied for composite laminated doubly curved shells and plates, sector annular plates and conical shells. Results for thin, thick and moderately thick plates both isotropic and laminated composites are evaluated for higher-order theories and comparison according to layer-level theories between equivalent single layer and layer-wise theory were studied. Results for plates are very close to 3D-Ritz method results. For shells most of the results are correspond to the isotropic results for the lack of information and the obtained results are close to 3D results. For isotropic sector annular plates results are so close to the 3D-Chebyshev Ritz method and in the case of isotropic conical shells results are close to 3D hp-fem. In the case of composite laminated sector annular plates results are so close to the analytical method using Chebyshev polynomials based on the first-order shear deformation theory. It is to be noted that ESL theory is useful for isotropic plates and LW is also useful for composite laminated plates.

Ideas for Future Work

It is recommended to apply the Chebyshev collocation method to continuous systems with viscoelastic and piezoelectric properties, since in many engineering applications these properties could exist. Moreover, studying the free vibration of continuous systems with constrained boundaries and internal cracks is highly recommended, since internal crack is possible to exist. Furthermore, it is specified that the collocation method could also be applied to nonlinear problems, as in [9], where this method was applied to reduce the dimensions of nonlinear delay differential equations with periodic coefficients and also it is advised to applying the proposed method to study the inverse problems in which the frequencies and mode shapes are known. Finally, it is so useful to apply this method on the beam vibration analysis using (CUF) technique under presented method and solving complicated cross section of beam. Aeroelastic analysis of beam with airfoil cross section using presented method can be of interest.

Appendix

Derivation of CUF nuclei from principal of virtual work for doubly curved shells and rectangular plates

$$\sum_{k=1}^{N_l}\int_{\Omega}\int_{z_k}^{z_{k+1}}(\delta\epsilon_{pG}^{k^T}\sigma_{pH}^k+\delta\epsilon_{nG}^{k^T}\sigma_{nH}^k)dzd\Omega=-\sum_{k=1}^{N_l}\int_{\Omega}\int_{z_k}^{z_{k+1}}\delta u^{k^T}\rho^k\ddot{u}^k dzd\Omega$$

$$\begin{aligned}
&u^k=F_\tau u_\tau^k\\
&\epsilon_{pG}^k=D_p u^k+A_p u_k=D_pF_\tau u_\tau^k+A_pF_\tau u_\tau^k\\
&\epsilon_{nG}^k=D_n u^k+\lambda_D A_n u_k+\frac{\partial}{\partial z}u^k=D_nF_\tau u_\tau^k+\lambda_D A_nF_\tau u_\tau^k+F_{\tau,z}u_\tau^k\\
&\sigma_{pH}^k=\tilde{C}_{pp}^k\epsilon_p^k+\tilde{C}_{pn}^k\epsilon_n^k=\tilde{C}_{pp}^k(D_pF_\tau u_\tau^k+A_pF_\tau u_\tau^k)+\tilde{C}_{pn}^k[D_nF_\tau u_\tau^k+\lambda_D A_nF_\tau u_\tau^k+F_{\tau,z}u_\tau^k]\\
&\sigma_{nH}^k=\tilde{C}_{np}^k\epsilon_p^k+\tilde{C}_{nn}^k\epsilon_n^k=\tilde{C}_{np}^k(D_pF_\tau u_\tau^k+A_pF_\tau u_\tau^k)+\tilde{C}_{nn}^k[D_nF_\tau u_\tau^k+\lambda_D A_nF_\tau u_\tau^k+F_{\tau,z}u_\tau^k]
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega}\sum_{k=1}^{N_l}\int_{z_k}^{z_{k+1}}[\delta(D_pF_\tau u_\tau^k)^T\sigma_{pH}^k+\delta(A_pF_\tau u_\tau^k)^T\sigma_{pH}^k+\delta(D_nF_\tau u_\tau^k)^T\sigma_{nH}^k\\
&+\delta(\lambda A_nF_\tau u_\tau^k)^T\sigma_{nH}^k+\delta(F_{\tau,z}u_\tau^k)^T\sigma_{nH}^k]dzd\Omega=-\int_{\Omega}\sum_{k=1}^{N_l}\int_{z_k}^{z_{k+1}}F_\tau\delta u_\tau^{k^T}\rho^kF_s\ddot{u}_s^k dzd\Omega
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega}\sum_{k=1}^{N_l}\Big[(D_p\delta u_\tau^k)^T\int_{z_k}^{z_{k+1}}F_\tau\sigma_{pH}^k dz+(A_p\delta u_\tau^k)^T\int_{z_k}^{z_{k+1}}F_\tau\sigma_{pH}^k dz\\
&+(D_n\delta u_\tau^k)^T\int_{z_k}^{z_{k+1}}F_\tau\sigma_{nH}^k dz+(\lambda A_n\delta u_\tau^k)^T\int_{z_k}^{z_{k+1}}F_\tau\sigma_{nH}^k dz\\
&+(\delta u_\tau^k)^T\int_{z_k}^{z_{k+1}}F_{\tau,z}\sigma_{nH}^k dz\Big]d\Omega=-\int_{\Omega}\sum_{k=1}^{N_l}\delta u_\tau^k)^T\int_{z_k}^{z_{k+1}}F_\tau F_s\rho^k dz\ddot{u}_s^k d\Omega
\end{aligned}$$

$$\begin{aligned}
R_{p\tau}^k&=\int_{z_k}^{z_{k+1}}F_\tau\sigma_{pH}^k dz\\
R_{n\tau}^k&=\int_{z_k}^{z_{k+1}}F_\tau\sigma_{nH}^k dz\\
R_{n\tau,z}^k&=\int_{z_k}^{z_{k+1}}F_{\tau,z}\sigma_{nH}^k dz\\
J_{\alpha\beta}^k&=\int_{z_k}^{z_{k+1}}F_\tau F_sH_\alpha^kH_\beta^k dz
\end{aligned}$$

$$\int_\Omega \sum_{k=1}^{N_l} \Big[(D_p \delta u_\tau^k)^T R_{p\tau}^k + (A_p \delta u_\tau^k)^T R_{p\tau}^k + (D_n \delta u_\tau^k)^T R_{n\tau}^k + (\lambda A_n \delta u_\tau^k)^T R_{n\tau}^k + (\delta u_\tau^k)^T R_{n\tau,z}^k \Big] d\Omega$$
$$= - \int_\Omega \sum_{k=1}^{N_l} \delta u_\tau^k)^T J_{\alpha\beta}^k \rho^k \ddot{u}_s^k d\Omega$$

$$\int_\Omega (D_\zeta \delta u_\tau^k)^T v^k d\Omega = - \int_\Omega \delta u_\tau^{kT} (D_\zeta^T v^k) d\Omega + \int_{\Gamma^k} \delta u_\tau^{kT} I_\zeta^T v^k d\Gamma^k \qquad \zeta = p, n$$

where

$$I_p = \begin{bmatrix} \frac{n_\alpha}{H_\alpha^k} & 0 & 0 \\ 0 & \frac{n_\beta}{H_\beta^k} & 0 \\ \frac{n_\beta}{H_\beta^k} & \frac{n_\alpha}{H_\alpha^k} & 0 \end{bmatrix}, \quad I_n = \begin{bmatrix} 0 & 0 & \frac{n_\alpha}{H_\alpha^k} \\ 0 & 0 & \frac{n_\beta}{H_\beta^k} \\ 0 & 0 & 0 \end{bmatrix}$$

I_p and I_n are depend on the boundary geometry. For each side of the geometry normal vector to the edge is zero and shear vector is identity.

$$\downarrow$$

$$- \sum_{k=1}^{N_l} \int_\Omega \delta u_\tau^{kT} D_p^T R_{p\tau}^k d\Omega - \sum_{k=1}^{N_l} \int_\Omega \delta u_\tau^{kT} D_n^T R_{n\tau}^k d\Omega + \sum_{k=1}^{N_l} \int_\Omega \delta u_\tau^{kT} R_{n\tau,z}^k d\Omega$$
$$+ \sum_{k=1}^{N_l} \int_\Omega \delta u_\tau^{kT} A_p^T R_{p\tau}^k d\Omega + \sum_{k=1}^{N_l} \int_\Omega \delta u_\tau^{kT} A_n^T R_{n\tau}^k d\Omega + \sum_{k=1}^{N_l} \int_{\Gamma^k} \delta u_\tau^{kT} I_p^T R_{p\tau}^k d\Gamma^k$$
$$+ \sum_{k=1}^{N_l} \int_{\Gamma^k} \delta u_\tau^{kT} I_n^T R_{n\tau}^k d\Gamma^k = - \sum_{k=1}^{N_l} \int_\Omega \delta u_\tau^{kT} J_{\alpha\beta}^k \rho^k \ddot{u}_s^k d\Omega$$

$$\sum_{k=1}^{N_l} \int_\Omega \delta u_\tau^{kT} [-D_p^T R_{p\tau}^k - D_n^T R_{n\tau}^k + A_p^T R_{p\tau}^k + \lambda A_n^T R_{n\tau}^k + R_{n\tau,z}^k] d\Omega$$
$$+ \sum_{k=1}^{N_l} \int_{\Gamma^k} \delta u_\tau^{kT} [I_p^T R_{p\tau}^k + I_n^T R_{n\tau}^k] d\Gamma^k = - \sum_{k=1}^{N_l} \int_\Omega \delta u_\tau^{kT} J_{\alpha\beta}^k \rho^k \ddot{u}_s^k d\Omega$$

Equation of motion:

$$\delta u_\tau^k : D_p^T R_{p\tau}^k + D_n^T R_{n\tau}^k - A_p^T R_{p\tau}^k - \lambda A_n^T R_{n\tau}^k - R_{n\tau,z}^k = J_{\alpha\beta}^k \rho^k \ddot{u}_s^k \qquad k = 1, ..., N_l, \tau = 0, ..., N$$

Boundary condition on Γ:

Geometric boundary condition: $u_\tau^k = 0 \quad k = 1, ..., N_l, \tau = 0, ..., N$ (Clamped edge)
Natural boundary condition: $I_p^T R_{p\tau}^k + I_n^T R_{n\tau}^k = 0 \quad k = 1, ..., N_l, \tau = 0, ..., N$ (Free edged)

After substitution Nuclei of equation of motion and natural boundary condition can be derived. Operator matrix form of equation of motion is

$$\begin{bmatrix} \mathcal{L}_{11}^{\tau s^k} & \mathcal{L}_{12}^{\tau s^k} & \mathcal{L}_{13}^{\tau s^k} \\ \mathcal{L}_{21}^{\tau s^k} & \mathcal{L}_{22}^{\tau s^k} & \mathcal{L}_{23}^{\tau s^k} \\ \mathcal{L}_{31}^{\tau s^k} & \mathcal{L}_{32}^{\tau s^k} & \mathcal{L}_{33}^{\tau s^k} \end{bmatrix} \begin{Bmatrix} u_s^k \\ v_s^k \\ w_s^k \end{Bmatrix} = \rho^k J_{\alpha\beta}^k \begin{Bmatrix} \ddot{u}_s^k \\ \ddot{v}_s^k \\ \ddot{w}_s^k \end{Bmatrix}$$

and also, operator matrix form of natural boundary conditions (free edge) is:

$$\begin{bmatrix} \mathcal{B}_{11}^{\tau s^k} & \mathcal{B}_{12}^{\tau s^k} & \mathcal{B}_{13}^{\tau s^k} \\ \mathcal{B}_{21}^{\tau s^k} & \mathcal{B}_{22}^{\tau s^k} & \mathcal{B}_{23}^{\tau s^k} \\ \mathcal{B}_{31}^{\tau s^k} & \mathcal{B}_{32}^{\tau s^k} & \mathcal{B}_{33}^{\tau s^k} \end{bmatrix} \begin{Bmatrix} u_s^k \\ v_s^k \\ w_s^k \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For simply supported at α side: $(M_{\alpha\alpha} = v = w = 0)$

$$\begin{bmatrix} \mathcal{B}_{11}^{\tau s^k} & \mathcal{B}_{12}^{\tau s^k} & \mathcal{B}_{13}^{\tau s^k} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} u_s^k \\ v_s^k \\ w_s^k \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For simply supported at β side: $(u = M_{\beta\beta} = w = 0)$

$$\begin{bmatrix} I & 0 & 0 \\ \mathcal{B}_{21}^{\tau s^k} & \mathcal{B}_{22}^{\tau s^k} & \mathcal{B}_{23}^{\tau s^k} \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} u_s^k \\ v_s^k \\ w_s^k \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

I is unity matrix by order $(N_n + 1) \times (N + 1)^2$.