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STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY BROWNIAN MOTION AND COMPOUND POISSON PROCESS

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Abstract

The purpose of this work is to provide an introduction to stochastic differential equations (SDEs) driven by brownian motion and compound Poisson process for students who have already attended a rigorous course on SDEs driven by brownian motion.
Thus, we want to add to the diffusive noise the jumps of a compound Poisson process by preserving the accuracy of a master course but without facing the mathematical sophistication required by the stochastic integration with respect to semimartingales.
Our intent is justified by the fact that such approach is not frequent in literature.

Sunto

Lo scopo di questo lavoro è fornire un'introduzione allo studio di equazioni differenziali stocastiche (EDS) guidate da moto browniano e processo di Poisson composto per studenti che abbiano già seguito un corso rigoroso di EDS guidate da moto browniano.
Si vuole quindi aggiungere al rumore diffusivo i salti di un processo di Poisson composto, mantenendo il rigore tipico di un corso di laurea magistrale ma senza dover affrontare le difficoltà matematiche dell'integrazione stocastica rispetto alle semimartingale.
Il nostro intento è motivato dal fatto che un tale approccio non è frequente in letteratura.

Introduction

The purpose of this work is to provide a rigorous introduction to stochastic differential equations (SDEs) driven by brownian motion and compound Poisson process.
In particular, we want to do this by relying on the knowledge which can typically be acquired during a first master level course on SDEs driven only by brownian motion, as for instance the course Stochastic Differential Equations provided by the Master of Science in Mathematical Engineering of Politecnico di Milano.
As the most simple and important exponent of the family of noises, brownian motion allows to provide a mathematical representation for a fair range of phenomena in different scientific areas. However, it has continuous sample paths, so that it is no more sufficient to model quantities whose time evolution can have discontinuities. Think for instance about finance, where the prices of assets can jump as a consequence of random unpredictable events. If we want to deal with such situations, we have to consider a different kind of noise. For many applications it is sufficient to combine brownian motion with the compound Poisson process, a stochastic process which moves only by jumps of finite size.
Thus, our intent is to extend some of the main results given in the course Stochastic Differential Equations mentioned above to the case where the noise has a brownian component and a compound Poisson component. In particular, we aim at maintaining the language used during the course and the same level of mathematical accuracy and sophistication.
This requirement led us through the exploration of literature to find an appropriate text whence to deduce what we deserve to accomplish our task. As the subject is relevant and widespread, we found many texts: we can divide them in two groups.
The texts belonging to the first group deal significantly with applications: if this represents an advantage in that makes their language clear and accessible, on the other hand it prevents them from lingering much on theoretical details. Thus, they do not match the level of mathematical accuracy we

want to maintain, and since they present stochastic calculus but do not talk about SDEs, then they do not fit our need. In particular, these texts are S.E. Shreve (2004) [SH] and R. Cont, P. Tankov (2004) [CT]. Being focused on the practical perspective, most of the prooves in [SH] are only supposed to give the reader an intuitive idea of how jumps are managed, while [CT] does not provide prooves when they are too long. However, these texts have given us materials, especially for the first part of this work.
The second group contains texts whose level of mathematical sophistication is substantially higher than the one we want to preserve. In particular they are D. Applebaum (2009) [AP], whose approach to stochastic integration is based on *Lévy processes*, then R. Situ (2005) [SI], P.E. Protter (2005) [PR], M. Métivier (1982) [ME], S. He, J. Wan, J. Yan (1992) [HWY], all of which base the theory on *semimartingales*, while N. Ikeda, S. Watanabe (1989) [IW] deals only with the diffusive case.
The outcome of the search seems to suggest that our approach is not frequent in literature. Thus, we have selected a text among those in the second group and decided to simplify the theory according to our intent. In particular our choice has been [AP] because its setting, based on Lévy processes, provide the most simple generalisation of the one we need. We went through its pages and tried to adapt its arguments to our framework: for such operation it has been necessary to study its prooves and understand to what extent they could be simplified as a consequence of the stronger hypotheses we require in our context.
At the same time, the main results we are interested in, that is Itô's formula and the existence and uniqueness theorem for strong solutions to SDEs, are prooved in [AP] by means of a technique, called *interlacing*, which partially conceals some mathematical details of its arguments. Since we want to maintain a certain level of accuracy, then in this situation we could not omit these details but instead we had to make them emerge.
In particular, we had to extend in a clear way Itô's formula for Itô processes and the existence and uniqueness theorem for strong solutions to SDEs driven by brownian motion to the case where the extremes of integration are no more deterministic instants but instead stopping times: we found in principle two possible approaches to such task.
The first consists in three steps: first, approximating the stopping times which constitute the extremes of integration with decreasing sequences of discrete valued stopping times; second, extending the highlighted results to the case where the extremes of integration are discrete valued stopping times; third, deducing the thesis by a limit argument. This approach works for Itô's formula, but when applied to SDEs it requires to generalise the results about dependence on the initial data to the case where the extremes of integration

are stopping times. This inevitably complicates the argument, so that we decided to leave such direction: we only include the proof for Itô's formula in the appendix.
The alternative approach consists in resetting the time scale in concurrence with the realisation of a given stopping time. Then the prooves can be carried out by means of the strong Markov property for brownian motion and of another important result, that is Itô's integral invariance with respect to the above time change. Actually this last result is not part of the program of the course Stochastic Differential Equations but we found it in literature (more details will be given in chapter 1). This approach allows to accomplish our task without adding too more materials to the course.
Finally, an important part of this work is the placement of our framework within the more general setting of Lévy processes. The full exposition of this subject would require more than few words, but since this constitutes in fact a diversion from our purpose, then we try to make our account as concise and self-contained as possible.
We have thus briefly described the genesis and the main points of the present work. We hope it can be useful for those who face problems involving our type of SDEs and need to acquire the theoretical bases without diving deep into a much complex mathematical setting.

Introduzione

Lo scopo di questo elaborato è fornire un'introduzione rigorosa allo studio di equazioni differenziali stocastiche (EDS) guidate da moto browniano e processo di Poisson composto.
In particolare, vogliamo farlo sulla base delle conoscenze che sono tipicamente acquisite durante un primo corso di EDS guidate dal solo moto browniano, come per esempio il corso Stochastic Differential Equations erogato dalla Laurea Magistrale in Ingegneria Matematica del Politecnico di Milano.
Il moto browniano è il più semplice e importante esempio di rumore, e consente di fornire una rappresentazione matematica per un buon numero di fenomeni in diversi campi scientifici. Tuttavia ha traiettorie continue, e dunque non è più sufficiente per modellizzare quantità la cui evoluzione temporale presenti discontinuità. Si pensi per esempio alla finanza, dove i prezzi dei titoli possono avere dei salti in conseguenza di eventi imprevedibili.
Se si vuole trattare simili situazioni, occorre quindi considerare un differente tipo di rumore. Per molte applicazioni è sufficiente combinare il moto browniano con il processo di Poisson composto, un processo stocastico che varia nel tempo solo attraverso salti di grandezza finita.
Quindi, il nostro intento è estendere alcuni dei risultati principali visti durante il corso Stochastic Differential Equations di cui sopra al caso in cui il rumore ha una componente browniana e una di Poisson composto. In particolare, cerchiamo di mantenere un linguaggio e un livello di rigore e astrazione matematica simili a quelli usati nel corso.
Tale requisito ci ha guidati nell'esplorazione della letteratura alla ricerca di un testo appropriato dal quale dedurre ciò di cui abbiamo bisogno per svolgere il compito prefissato. Essendo la materia importante e diffusa, abbiamo trovato molti testi: essi possono essere divisi in due gruppi.
I testi appartenenti al primo gruppo trattano in maniera importante gli aspetti applicativi della materia: se da un lato ciò rappresenta un vantaggio in quanto rende il loro linguaggio esplicativo e accessibile, dall'altro impedisce loro di indugiare sui dettagli teorici. Di conseguenza, non raggiungono il livello di rigore desiderato, e poichè inoltre presentano il calcolo stocastico

senza arrivare a parlare di EDS, essi non sono adeguati per il nostro scopo. In particolare, tali testi sono S.E. Shreve (2004) [SH] e R. Cont, P. Tankov (2004) [CT]. Essendo concentrato sugli aspetti pratici, molte delle dimostrazioni in [SH] intendono solo fornire al lettore un'idea intuitiva del modo in cui i salti sono gestiti, mentre [CT] non fornisce le dimostrazioni quando esse richiederebbero troppo tempo. Tuttavia, da questi due testi abbiamo tratto materiale, in special modo per la prima parte dell'elaborato.
Il secondo gruppo contiene testi il cui livello di astrazione matematica è sostanzialmente maggiore di quello che vogliamo mantenere. In particolare essi sono D. Applebaum (2009) [AP], il cui approccio all'integrazione stocastica è basato sui *processi di Lévy*, poi R. Situ (2005) [SI], P.E. Protter (2005) [PR], M. Métivier (1982) [ME], S. He, J. Wan, J. Yan (1992) [HWY], ognuno dei quali basa la teoria sulle *semimartingale*, mentre N. Ikeda, S. Watanabe (1989) [IW] tratta solo il caso diffusivo.
L'esito della ricerca sembra suggerire che il nostro approccio non è frequente in letteratura. Quindi, abbiamo selezionato un testo del secondo gruppo e deciso di semplificare la teoria in accordo con i nostri intenti. In particolare la nostra scelta è stata [AP] poichè il suo contesto, basato sui processi di Lévy, costituisce la più semplice generalizzazione di quello che ci occorre. Ci siamo perciò addentrati nei suoi capitoli e cercato di adattarne gli argomenti al nostro contesto: per tale operazione è stato necessario studiare le sue dimostrazioni e capire in che misura potessero essere semplificate in virtù delle nostre ipotesi più forti.
Parallelamente, i risultati principali a cui siamo interessati, cioè la formula di Itô e il teorema di esistenza e unicità per soluzioni forti di EDS, sono dimostrate in [AP] tramite una tecnica, detta *interlacing*, che nasconde parzialmente alcuni dettagli dei suoi argomenti. Poichè vogliamo mantenere, come dichiarato, un certo livello di rigore, in tale situazione non abbiamo potuto omettere questi dettagli ma al contrario abbiamo cercato di farli emergere.
In particolare, abbiamo dovuto estendere in modo chiaro la formula di Itô per processi di Itô e il teorema di esistenza e unicità per soluzioni forti di EDS guidate da moto browniano al caso in cui gli estremi di integrazione non sono più istanti deterministici ma invece tempi d'arresto: abbiamo trovato due possibili modi per farlo.
Il primo consiste in tre fasi: primo, approssimare i tempi d'arresto che costituiscono gli estremi di integrazione con successioni decrescenti di tempi d'arresto a valori discreti; secondo, generalizzare i risultati evidenziati al caso in cui gli estremi di integrazione sono tempi d'arresto a valori discreti; terzo, ricavare la tesi con un argomento di limite. Tale approccio funziona per la formula di Itô, ma quando applicato alle EDS richiede di estendere i risultati sulla dipendenza dai dati iniziali al caso in cui gli estremi di inte-

grazione sono tempi d'arresto. Ciò complica inevitabilmente i ragionamenti necessari, e dunque abbiamo deciso di non procedere in questa direzione: ci limitiamo a includere la dimostrazione della formula di Itô svolta con questa tecnica in appendice.
L'approccio alternativo consiste nell'azzerare l'asse temporale in concomitanza con la realizzazione del tempo d'arresto che costituisce l'estremo di integrazione inferiore. In tal modo le dimostrazioni possono allora essere realizzate attraverso la proprietà di Markov forte per il moto browniano e attraverso un altro importante risultato, ossia l'invarianza dell'integrale di Itô rispetto al cambio di tempo appena descritto. Quest'ultimo risultato non fa parte del corso Stochastic Differential Equations, ma lo abbiamo trovato nella letteratura (diamo più dettagli in proposito nel capitolo 1). Questo secondo approccio consente di svolgere il compito prefissato senza aggiungere troppo materiale a quanto visto durante il corso.
Infine, una parte importante di questo elaborato è l'inquadramento del contesto in cui lavoriamo nella più ampia cornice dei processi di Lévy. Una discussione diffusa in proposito richiederebbe tempo, ma poichè ciò costitisce di fatto una diversione dal nostro scopo, cerchiamo di fornirne un resoconto più conciso e autosufficiente possibile.
Con queste righe abbiamo brevemente descritto l'origine e i punti principali del presente elaborato. La nostra speranza è che possa rivelarsi utile per coloro che affrontino problemi riguardanti la nostra categoria di EDS e volessero acquisire le basi teoriche senza esplorare in profondità un contesto matematico eccessivamente complesso.

Contents

Chapter 1

Preliminaries

In this chapter we introduce some notation and summarize known results that will be used throughout the following chapters.
The first section is mostly devoted to generalities concerning probability spaces and stochastic processes. The second recalls some of the main points of any master course on SDEs driven by brownian motion, and in particular our account is based on [BA]. The last section introduces a change of time by means of a stopping time and describes its effects on filtrations and brownian stochastic integrals.
We assume the reader is experienced in the above subjects to some extent, so that we will not go deep into details when it is not necessary.

1.1 On probability spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, that is $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is a complete probability measure on $\mathcal{F}$.
Given a class $\mathcal{A} \subseteq \mathcal{F}$ we call $\sigma(\mathcal{A})$ the σ-algebra generated on Ω by $\mathcal{A}$. We say that $\mathcal{A}$ is a π-system on Ω if it is stable with respect to finite intersections.
A finite collection of classes $\mathcal{A}_1, ..., \mathcal{A}_n \subseteq \mathcal{F}$, $n \in \mathbb{N}$, is said to be independent if for each choice of $A_i \in \mathcal{A}_i$, $1 \le i \le n$, we have

$$\mathbb{P}(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} \mathbb{P}(A_i).$$

A family of classes $\{\mathcal{A}_i\}_{i \in \mathbb{I}} \subseteq \mathcal{F}$ is independent if each one of its finite subcollection is independent.

Proposition 1.1. *If $\mathcal{A}_1, ..., \mathcal{A}_n \subseteq \mathcal{F}$, $n \in \mathbb{N}$, are independent and $\mathcal{A}_i$ is a π-system for all $1 \le i \le n$, then $\sigma(\mathcal{A}_1), ..., \sigma(\mathcal{A}_n)$ are independent.*

Given any measurable space $(E, \mathcal{E})$, a mapping $X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ is said to be an E-valued random variable if it is measurable from $\mathcal{F}$ to $\mathcal{E}$. Given $C \in \mathcal{E}$, then $X^{-1}(C) = \{X \in C\}$ is called the counterimage of C with respect to X; the mapping $p_X : \mathcal{E} \to [0,1]$ such that $p_X(C) = \mathbb{P}(X \in C)$ is called the *law* of X.
Given a random variable X, the σ-algebra generated on Ω by X is called $\sigma(X)$. A collection of random variables $\{X_i\}_{i\in\mathbb{I}}$ is said to be independent if $\{\sigma(X_i)\}_{i\in\mathbb{I}}$ is independent. Analogously, X is said to be independent from a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ if $\sigma(X)$ and $\mathcal{G}$ are independent.

Proposition 1.2. *Let $\mathcal{C}$ be a class of subset of E such that $\sigma(\mathcal{C}) = \mathcal{E}$. In order for an E-valued mapping X on $(\Omega, \mathcal{F}, \mathbb{P})$ to be a random variable it is necessary and sufficient that $X^{-1}(C) \in \mathcal{F}$ $\forall$ $C \in \mathcal{C}$.*

We will often have $E = \mathbb{R}^m$ and $\mathcal{E} = \mathcal{B}(\mathbb{R}^m)$ for some $m \in \mathbb{N}$; thus the random variables introduced from now on are supposed to take values in $\mathbb{R}^m$ if it is not specified. We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^m$.
Given a random variable X, $\mathbb{E}[X]$ denotes its *expectation* (if it exists). Given a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, $\mathbb{E}[X|\mathcal{G}]$ denotes the *conditional expectation* of X with respect to $\mathcal{G}$ (if it exists).
A certain proposition is said to hold almost surely (a.s.) if it holds on a subset of Ω which contains an event having probability equal to 1: recall that by the assumption of completeness such a subset belongs to $\mathcal{F}$ and thus it is in fact an event. Given two random variables X, Y such that $X = Y$ a.s., they are said to belong to the same equivalence class with respect to almost sure equivalence, or simply to the same equivalence class.
Given $p \geq 1$, $L^p(\Omega, \mathcal{F}, \mathbb{P})$ denotes the Banach space of the equivalence classes of all $\mathcal{F}$-measurable random variables such that $\mathbb{E}[|X|^p] < \infty$: for the sake of simplicity we still call random variables the elements of $L^p(\Omega, \mathcal{F}, \mathbb{P})$. We write L^p instead of $L^p(\Omega, \mathcal{F}, \mathbb{P})$ when the context is clear.
Given a random variable X we define as usual its characteristic function $\Phi_X : \mathbb{R}^m \to \mathbb{C}$ such that

$$\Phi_X(z) = \mathbb{E}[e^{i\langle z, X\rangle}].$$

It is well known that Φ_X characterizes the law of X.
We now recall some useful results.

Proposition 1.3. *Fix $n \in \mathbb{N}$ and let $X_1, ..., X_n$ be random variables defined on the same probability space. Then they are mutually independent if and only if*

$$\mathbb{E}[e^{i\sum_{i=1}^{n}\langle z_i, X_i\rangle}] = \prod_{i=1}^{n} \Phi_{X_i}(z_i) \quad \forall z_1, ..., z_n \in \mathbb{R}^m.$$

Proposition 1.4 (Freezing Lemma)**.** *Let $\mathcal{D}, \mathcal{G}$ be two independent sub σ-algebras of $\mathcal{F}$, let X be a $\mathcal{D}$-measurable random variable taking values in some measurable space $(E, \mathcal{E})$ and let ψ be an $\mathcal{E} \otimes \mathcal{G}$-measurable mapping on $E \times \Omega$ such that $\omega \mapsto \psi(X(\omega), \omega)$ is integrable and $\omega \mapsto \psi(x, \omega)$ is integrable for all $x \in E$. Then*

$$\mathbb{E}[\psi(X, \cdot) \mid \mathcal{D}] = \phi(X) \quad a.s.$$

where $\phi(x) = \mathbb{E}[\psi(x, \cdot)]$ for all $x \in E$.

The following result goes under the name of measurability theorem. Actually its versions are many: we will use the one given in [BR], appendix A1, theorem T4.

Proposition 1.5 (Measurability theorem)**.** *Let D be a set and $\mathcal{C}$ a π-system on D. Let $\mathcal{H}$ be a vector space of real valued mappings on D such that*

1. $1 \in \mathcal{H}$, $1_C \in \mathcal{H}$ $\forall C \in \mathcal{C}$
2. *if $\{f_n\}_{n \in \mathbb{N}}$ is an increasing sequence of nonnegative mappings of $\mathcal{H}$ such that $\lim_n f_n$ is bounded, then $\lim_n f_n \in \mathcal{H}$*

Then $\mathcal{H}$ contains all the real valued bounded mappings which are measurable with respect to $\sigma(\mathcal{C})$.

A *stochastic process* is a family of random variables $(X_t)_{t \geq 0}$ all defined on the same probability space. Two stochastic processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ defined on the same probability space are said to be *independent* if, for all $m, n \in \mathbb{N}$, all $0 \leq t_1 < ... < t_n < \infty$ and all $0 \leq s_1 < ... < s_m < \infty$, the σ-algebras $\sigma(X_{t_1}, ..., X_{t_n})$ and $\sigma(Y_{s_1}, ..., Y_{s_m})$ are independent. A similar definition is given for independence between a single stochastic process and a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$.
A process is said to have bounded variation, or to be BV, provided almost all its sample paths have bounded variation on the compact subsets of the time line $\mathbb{R}^+$.
A process $(X_t)_{t \geq 0}$ has *independent increments* if for all $n \in \mathbb{N}$ and all $0 \leq t_1 < ... < t_{n+1} < \infty$ the random variables $X_{t_{j+1}} - X_{t_j}$, $1 \leq j \leq n$, are independent. It has *stationary increments* if $X_{t_{j+1}} - X_{t_j}$ and $X_{t_{j+1} - t_j} - X_0$ have the same law for all $1 \leq j \leq n$.
A process $(X_t)_{t \geq 0}$ is *stochastically continuous* if

$$\lim_{s \to t} \mathbb{P}(|X_t - X_s| > a) = 0 \quad \forall a > 0 \quad \forall t \geq 0,$$

that is if it is continuous in probability with respect to time.
Given a process $(X_t)_{t\geq 0}$, we denote $\Phi_t(z), z \in \mathbb{R}^m$, the characteristic function of the random variable X_t.
Given $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space, a *filtration* is a family $(\mathcal{F}_t)_{t\geq 0}$ of sub σ-algebras of $\mathcal{F}$ such that for every fixed $0 \leq s \leq t < \infty$ then $\mathcal{F}_s \subseteq \mathcal{F}_t$; we set as usual $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$.
A filtration $(\mathcal{F}_t)_{t\geq 0}$ is said to satisfy the usual hypotheses if it is right continuous and such that

$$\mathcal{N} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0\} \subseteq \mathcal{F}_t \quad \forall t \geq 0.$$

Let now $(X_t)_{t\geq 0}$ be a process and $(\mathcal{F}_t)_{t\geq 0}$ a filtration defined on the same probability space.
The process $(X_t)_{t\geq 0}$ is said to be adapted to $(\mathcal{F}_t)_{t\geq 0}$, or simply $\mathcal{F}_t$-adapted, if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$; $(X_t)_{t\geq 0}$ is said to be $\mathcal{F}_t$-progressively measurable if for all $u \geq 0$ the mapping $(t, \omega) \mapsto X_t(\omega)$ restricted on $t \in [0, u]$ is measurable with respect to $\mathcal{B}([0, u]) \otimes \mathcal{F}_u$.
The process $(X_t)_{t\geq 0}$ is said to have independent increments with respect to $(\mathcal{F}_t)_{t\geq 0}$ if for all $0 \leq s \leq t < \infty$ the random variable $X_t - X_s$ is independent of $\mathcal{F}_s$.
The process $(X_t)_{t\geq 0}$ is said to be an $\mathcal{F}_t$-martingale if it is adapted to $(\mathcal{F}_t)_{t\geq 0}$, $\mathbb{E}[\,|X_t|\,] < \infty$ $\forall t \geq 0$ and $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ $\forall\ 0 \leq s \leq t$.
A nonnegative random variable τ is said to be a *stopping time* with respect to $(\mathcal{F}_t)_{t\geq 0}$ if $\{\tau \leq t\} \in \mathcal{F}_t$ $\forall t \geq 0$. Given a stopping time τ, the collection

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t \ \forall t \geq 0\}$$

is the σ-algebra containing the information acquired from $(\mathcal{F}_t)_{t\geq 0}$ up to and including τ.
The process $(X_t)_{t\geq 0}$ is said to be an $\mathcal{F}_t$-local martingale if there exists a sequence of $\mathcal{F}_t$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ diverging a.s. and such that for all $n \in \mathbb{N}$ the process $(X_{t\wedge\tau_n})_{t\geq 0}$ is an $\mathcal{F}_t$-martingale.
We now recall two result about processes and stopping times.

Proposition 1.6. *Let $(X_t)_{t\geq 0}$ be an $\mathcal{F}_t$-progressively measurable process and τ an a.s. finite $\mathcal{F}_t$-stopping time. Then the random variable X_τ is a.s. well defined and $\mathcal{F}_\tau$-measurable.*

Proposition 1.7. *Let $(X_t)_{t\geq 0}$ be a right-continuous $\mathcal{F}_t$-martingale and τ an $\mathcal{F}_t$-stopping time. Then the process $(X_{t\wedge\tau})_{t\geq 0}$ is a right-continuous $\mathcal{F}_t$-martingale.*

1.2 On SDEs driven by brownian motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses.
Let $(B_t)_{t\geq 0}$ be a continuous $\mathbb{R}^r$-valued $\mathcal{F}_t$-brownian motion and call $(B_t^j)_{t\geq 0}$ its components, $1 \leq j \leq r$.
When saying $\mathcal{F}_t$-brownian motion we mean that the process $(B_t)_{t\geq 0}$ is a brownian motion, so that $B_0 = 0$ a.s. and $B_t - B_s \sim N(0, (t-s)I)\ \forall\ 0 \leq s \leq t$, and that it is $\mathcal{F}_t$-adapted and has independent increments with respect to $(\mathcal{F}_t)_{t\geq 0}$.
We call $(B_t^\Sigma)_{t\geq 0}$ an $\mathbb{R}^m$-valued $\mathcal{F}_t$-brownian motion with covariance matrix $\Sigma \in \mathbb{R}^{m\times m}$ if $B_t^\Sigma = \sigma B_t\ \forall t \geq 0$ where σ is a $m \times r$ real valued matrix such that $\sigma\sigma^T = \Sigma$.
Given a time horizon $0 < T < \infty$ we recall the definition of the following vector spaces of equivalence classes of real valued stochastic processes with respect to the measure $l \otimes \mathbb{P}$, where l denotes the Lebesgue measure on $[0, T]$,

$$M^p([0,T]) = \left\{ (X_t)_{t\geq 0} \text{ progressively measurable} : \mathbb{E}\left[\int_0^T |X_t|^p dt \right] < \infty \right\}$$
$$M^p_{loc}([0,T]) = \left\{ (X_t)_{t\geq 0} \text{ progressively measurable} : \int_0^T |X_t|^p dt < \infty \ a.s. \right\},$$

for all $p \geq 1$.
As we do for random variables, we still call processes the members of these spaces for the sake of simplicity.
A real valued process $(X_t)_{t\geq 0}$ is said to belong to $M^p([0,\infty))$ if it belongs to $M^p([0,T])$ for all $T < \infty$; $M^p_{loc}([0,\infty))$ is defined analogously.
Fix $m \in \mathbb{N}$ and consider the real matrix valued process $(X_t)_{t\geq 0}$ such that $X_t = (X_t^{ij})_{ij}$ for all $t \geq 0$, where $1 \leq i \leq m$ and $1 \leq j \leq r$: we say that such process belongs to $M^2_{loc}([0,T])$ provided all of its components do.
Thus, if $(X_t)_{t\geq 0} \in M^2_{loc}([0,T])$, we denote by $I(X) = \int_0^T X_t dB_t$ the multi-dimensional brownian integral defined with the usual Itô's procedure, and $(I(X)_t)_{t\in[0,T]}$ the associated $\mathbb{R}^m$-valued continuous stochastic process, that is $I(X)_t = \int_0^t X_s dB_s$.
We recall some important results.

Proposition 1.8 (Localization lemma). *Let $0 \leq a < b \leq T$, $\Omega_0 \in \mathcal{F}$ and let $(X_t)_{t\geq 0}, (Y_t)_{t\geq 0} \in M^2_{loc}([0,T])$ be such that $X_t = Y_t$ on Ω_0 for all $t \in [a,b]$. Then*

$$\int_a^b X_t dB_t = \int_a^b Y_t dB_t \ a.s. \ on \ \Omega_0$$

Proposition 1.9. *Let* $(X_t^n)_{t\geq 0}, (X_t)_{t\geq 0} \in M^2_{loc}([0,T])$ *be such that*

$$\int_0^T |X_t^n - X_t|^2 dt \to 0 \quad \textit{in probability.}$$

Then

$$\sup_{t\in[0,T]} \left| \int_0^t X_s^n dB_s - \int_0^t X_s dB_s \right| \to 0 \quad \textit{in probability.}$$

An $\mathcal{F}_t$-adapted and a.s. continuous $\mathbb{R}^m$-valued process $(X_t)_{t\geq 0}$ is said to be an Itô process on $[0,T]$ provided there exist an $\mathcal{F}_0$-measurable random variable X_0, $F = (F^1, ..., F^m)$, $F \in M^1_{loc}([0,T])$, $G = (G^{ij})_{ij}, 1 \leq i \leq m, 1 \leq j \leq r$, $G \in M^2_{loc}([0,T])$ such that

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t G_s dB_s \quad \forall t \in [0,T] \quad a.s.$$

Equivalently we say that $(X_t)_{t\geq 0}$ has stochastic differential given by

$$dX_t = F_t dt + G_t dB_t \quad t \in [0,T].$$

Theorem 1.10 (Itô's formula)**.** *Let* $h : \mathbb{R}^m \to \mathbb{R}$ *such that* $h \in C^2(\mathbb{R}^m)$ *and* $(X_t)_{t\geq 0}$ *be an* $\mathbb{R}^m$*-valued process with stochastic differential given by*

$$dX_t = F_t dt + G_t dB_t \quad t \in [0,T].$$

Then $(h(X_t))_{t\geq 0}$ *is a real valued Itô process with stochastic differential given by*

$$\begin{aligned} dh(X_t) = &\sum_{i=1}^m h_{x_i}(X_t) F_t^i dt + \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \sum_{j=1}^r h_{x_i x_l}(X_t) G_t^{ij} G_t^{lj} dt \\ &+ \sum_{i=1}^m \sum_{j=1}^r h_{x_i}(X_t) G_t^{ij} dB_t^j \quad t \in [0,T]. \end{aligned}$$

Now let $b : \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times r}$ be lipschitz continuous mappings, so that there exists $L > 0$ such that for all $x, y \in \mathbb{R}^m$

$$\begin{aligned} |b(x) - b(y)| &\leq L|x-y| \\ |\sigma(x) - \sigma(y)| &\leq L|x-y|, \end{aligned}$$

where $|\sigma|$ is the matrix norm given by $|\sigma|^2 = tr(\sigma\sigma^T) = \sum_{ij} \sigma_{ij}^2$.
Now we recall the result that states existence and uniqueness for strong solutions to SDEs driven by brownian motion. In fact the final goal of this work is to extend it to the case where the driving noise is given by a brownian motion and a compound Poisson process.

Theorem 1.11. *Let $v \geq 0$ and let $\xi \in L^2(\Omega, \mathcal{F}_v, \mathbb{P})$ be an $\mathbb{R}^m$-valued random variable. Then there exists a unique solution to*

$$\begin{cases} X_t = X_v + \int_v^t b(X_s)ds + \int_v^t \sigma(X_s)dB_s \quad t \in [v, T] \\ X_v = \xi \end{cases}$$

The solution is an Itô process on $[v, T]$.

Remark. Since the brownian motion $(B_t)_{t\geq 0}$ is fixed and given as data, when saying solution we mean in fact what literature calls strong solution. Moreover, by uniqueness we mean that all the solutions to the above equation are indistinguishable on $[v, T]$.
We recall that the proof of the above result is typically carried out in a more general setting where the mappings b and σ depend also on time, that is $b : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m\times r}$. Thus, the requirements on b and σ are different from ours. In particular it is sufficient for them to be lipschitz continuous on $\mathbb{R}^m$ uniformly in $t \in [0, T]$ and to have sublinear growth on $\mathbb{R}^m$ uniformly in $t \in [0, T]$. But since

$$\begin{aligned} |b(x)| &\leq |b(x) - b(0)| + |b(0)| \\ &\leq L|x| + |b(0)| \\ &\leq (L \vee |b(0)|)(1 + |x|), \end{aligned}$$

then in our case the sublinear growth condition is an immediate consequence of the lipschitz continuity.

With differential notation it is equivalently said that there exists a unique solution to the SDE

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dB_t \quad t \in [v, T] \\ X_v = \xi \end{cases}$$

We will mostly employ the integral notation instead of the differential one because it allows to identify clearly the extremes of integration.

Proposition 1.12. *Let $\xi_i \in L^2(\Omega, \mathcal{F}_v, \mathbb{P})$, $i \in \{1, 2\}$, and let $(X_t^i)_{t\geq 0}$, $i \in \{1, 2\}$, be respectively the solutions to the SDEs*

$$\begin{cases} X_t = X_v + \int_v^t b(X_s)ds + \int_v^t \sigma(X_s)dB_s \quad t \in [v, T] \quad a.s. \\ X_v = \xi_i \qquad i \in \{1, 2\} \end{cases}$$

Then on $\{\omega \in \Omega : \xi_1(\omega) = \xi_2(\omega)\}$ the processes $(X_t^1)_{t\geq 0}$ and $(X_t^2)_{t\geq 0}$ are indistinguishable in $[v, T]$.

1.3 On τ-changed filtrations, stochastic processes and stochastic integrals

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses.
Let τ be an $\mathcal{F}_t$-stopping time: we define the σ-algebras $\mathcal{F}^\tau_u = \mathcal{F}_{\tau+u}$ $\forall u \geq 0$. Thus, given any $u \geq 0$, $\mathcal{F}^\tau_u$ contains the information gained by one who resets the time in τ without forgetting what has happened before τ and keeps on acquiring information for a further time interval of deterministic duration u.

Proposition 1.13. *The family $(\mathcal{F}^\tau_u)_{u\geq 0}$ is a filtration. Furthermore, if $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual hypotheses, then also $(\mathcal{F}^\tau_u)_{u\geq 0}$ does.*

Proof. We only verify the right-continuity: we show that given any $u \geq 0$ then $\mathcal{F}^\tau_u = \mathcal{F}^\tau_{u^+}$, that is $\mathcal{F}_{\tau+u} = \mathcal{F}_{(\tau+u)^+}$, where $\mathcal{F}_{(\tau+u)^+} = \bigcap_{\epsilon>0} \mathcal{F}_{\tau+u+\epsilon}$.
Fix $u \geq 0$. Clearly $\mathcal{F}_{\tau+u} \subseteq \mathcal{F}_{(\tau+u)^+}$.
In the following we take $C \in \mathcal{F}_{(\tau+u)^+}$ and show that $C \in \mathcal{F}_{\tau+u}$ to verify the opposite inclusion.
By hypothesis we have $C \in \mathcal{F}_{\tau+u+\frac{1}{n}}$ for all $n \in \mathbb{N}$: since $\tau + u + \frac{1}{n}$ is an $\mathcal{F}_t$-stopping time for all $n \in \mathbb{N}$, then we get $C \in \mathcal{F}_\infty$ and

$$\left\{C,\ \tau + u + \frac{1}{n} \leq s\right\} \in \mathcal{F}_s \quad \forall s \geq 0 \quad \forall n \in \mathbb{N}. \tag{1.1}$$

We need to show that $\{C,\ \tau + u \leq s\} \in \mathcal{F}_s$ $\forall s \geq 0$: to do this we fix $s \geq 0$ and write

$$\{C,\ \tau + u \leq s\} = \{C,\ \tau + u < s\} \cup \{C,\ \tau + u = s\}$$

and analyse separately the above sets.
Thanks to (1.1) we have

$$\begin{aligned}\{C,\ \tau + u < s\} &= \left\{C,\ \bigcup_{n\in\mathbb{N}} \left\{\tau + u + \frac{1}{n} \leq s\right\}\right\} \\ &= \bigcup_{n\in\mathbb{N}} \left\{C,\ \tau + u + \frac{1}{n} \leq s\right\} \in \mathcal{F}_s.\end{aligned}$$

Then, thanks to (1.1) and since $\tau + u$ is an $\mathcal{F}_t$-stopping time we get

$$\begin{aligned}\{C,\ \tau + u = s\} &= \left\{C,\ \tau + u + \frac{1}{n} = s + \frac{1}{n}\right\} \\ &= \left\{C,\ \tau + u + \frac{1}{n} \leq s + \frac{1}{n},\ \tau + u = s\right\} \in \mathcal{F}_{s+\frac{1}{n}} \quad \forall n \in \mathbb{N}.\end{aligned}$$

This implies $\{C, \tau + u = s\} \in \mathcal{F}_{s^+}$ and the required result follows by the right-continuity of $(\mathcal{F}_t)_{t\geq 0}$.

■

From now on, given the filtration $(\mathcal{F}_t)_{t\geq 0}$, we call $(\mathcal{F}^\tau_u)_{u\geq 0}$ the associate τ-changed filtration.
We now recall the strong Markov property for brownian motion.

Proposition 1.14. *Let τ be an a.s. finite $\mathcal{F}_t$-stopping time. Then the process $(B^\tau_u)_{u\geq 0}$ where $B^\tau_u = B_{\tau+u} - B_\tau$ $\forall u \geq 0$ is a brownian motion with respect to the filtration $(\mathcal{F}^\tau_u)_{u\geq 0}$.*

For a proof see [CA], teorema 3.28.
From now on, given an a.s. finite stopping time τ we call τ-changed brownian motion the process $(B^\tau_u)_{u\geq 0}$ introduced in the above proposition.
Now let τ be a finite $\mathcal{F}_t$-stopping time and consider the associate τ-changed filtration and the τ-changed brownian motion.
Let $(X_t)_{t\geq 0}$ be an $\mathcal{F}_t$-progressive measurable process and define the associated τ-changed stochastic process $X^\tau_u = X_{\tau+u}$ $\forall u \geq 0$. Notice that this operation is different from the one we did on $(B_t)_{t\geq 0}$ in that here we are not subtracting X_τ.

Proposition 1.15. *Let $(X_t)_{t\geq 0}$ be $\mathcal{F}_t$-progressively measurable. Then $(X^\tau_u)_{u\geq 0}$ is $\mathcal{F}^\tau_u$-progressively measurable.*

For the proof see [RY], proposition 1.4.
Given $X \in M^1_{loc}([0,\infty))$, then by the above proposition we get $(X^\tau_u)_{u\geq 0} \in M^1_{loc}([0,\infty); \mathcal{F}^\tau_u)$, where we specify the filtration with respect to which the process is progressively measurable. Furthermore, since the Lebesgue integral is done pathwise, then by a change of variable we get

$$\int_\tau^{\tau+u} X_s ds = \int_0^u X^\tau_v dv \quad \forall u \geq 0.$$

We also have the following crucial result.

Proposition 1.16. *Let $(X_t)_{t\geq 0} \in M^2_{loc}([0,\infty))$.*
Then $(X^\tau_u)_{u\geq 0} \in M^2_{loc}([0,\infty); \mathcal{F}^\tau_u)$ and

$$\int_\tau^{\tau+u} X_s dB_s = \int_0^u X^\tau_v dB^\tau_v \quad \forall u \geq 0 \quad a.s.$$

For the proof see [RY], proposition 1.5.

Remark. The result given in [RY] is slightly different from the one above. Actually it does not even mention the τ-changed brownian motion $(B_u^\tau)_{u\geq 0}$ but instead it uses the process $(B_{\tau+u})_{u\geq 0}$. Such process is not a brownian motion and using it as integrator gives an object we have never defined before. Nevertheless, notice that given an $\mathcal{F}_0$-measurable random variable X there is no difference between integrating with respect to $(B_t)_{t\geq 0}$ or $(B_t + X)_{t\geq 0}$ in that their increments coincide. This implies the effectiveness of the above proposition.

Chapter 2

The compound Poisson process

In this chapter we introduce the notion of *compound Poisson process* by following closely sections 2.5 and 3.2 in [CT].

2.1 The Poisson process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 2.1. *Let $\{T_i\}_{i\geq 1}$ be a sequence of i.i.d. exponential random variables with parameter λ and $\tau_n = \sum_{i=1}^n T_i$. The process $(N_t)_{t\geq 0}$ defined by*

$$N_t = \sum_{n\geq 1} 1_{[0,t]}(\tau_n)$$

is called Poisson process with intensity λ.
The random variables τ_n are called the jump times, or arrival times, of $(N_t)_{t\geq 0}$.

We set as a convention $\tau_0 \equiv 0$.
In words, a Poisson process evaluated at a given time t counts how many arrival times τ_n occur between 0 and t, where $\{\tau_n - \tau_{n-1}\}_{n\geq 1}$ is an i.i.d. sequence of exponential random variables.
From now on we denote $(N_t)_{t\geq 0}$ a Poisson process as defined above.

Remark. Of course, as mathematicians actually know, when giving a definition one should ensure that the object he is defining does really exist. Thus, we should build a probability space on which it is possible to define a Poisson process. While this could be done without diverting much from our direction, more complex objects will be introduced for which the building of a probability space requires more sophistications (think for instance about the Wiener process). We shall not deepen these discussions because they go beyond the purpose of this work: one should not worry anyway, since mathematicians would not have permitted the development of such a wide literature about non-existing objects.

Let's now analyse the main properties of the process $(N_t)_{t\geq 0}$.

Proposition 2.1. *For any $t \geq 0$ the random variable N_t is a.s. finite.*

Proof. Let $\Omega_1 = \{\omega \in \Omega : \frac{\tau_n}{n}(\omega) \to \frac{1}{\lambda}\}$ so that by the law of large numbers $\mathbb{P}(\Omega_1) = 1$. Since $\tau_n(\omega) \to \infty$ for all $\omega \in \Omega_1$, then for any fixed $t \geq 0$ we have that

$$\forall \omega \in \Omega_1 \;\; \exists\, n_0 = n_0(\omega) : \tau_n(\omega) > t \;\; \forall n \geq n_0$$

so that $\mathbb{P}(N_t < \infty) \geq \mathbb{P}(\Omega_1) = 1$.

■

We then see immediately from the definition that the sample paths of $(N_t)_{t\geq 0}$ are a.s. piecewise constant and increase by jumps of size 1; it is also clear that they are a.s. càdlàg and BV.
We now give the following important results.

Proposition 2.2. *For any $t > 0$, the random variable N_t follows a Poisson distribution with parameter λt:*

$$\mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \quad \forall n \in \mathbb{N}$$

Proof. Consider the sequence of i.i.d. exponential random variables $\{T_n\}_{n\in\mathbb{N}}$ which defines $(N_t)_{t\geq 0}$ and the arrival times $\{\tau_n\}_{n\in\mathbb{N}}$. Call $X = (T_1, ..., T_d)$, $Y = (\tau_1, ..., \tau_d)$ and take g to be the linear transormation given by the $d \times d$ lower triangular matrix with entries equal to 1: its inverse is

$$\begin{bmatrix} 1 & 0 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & \\ \vdots & & \ddots & \ddots \end{bmatrix}$$

so that by a change of variable we get

$$f_Y(y) = \lambda^d \; e^{-\lambda t_d} \; 1_{0<t_1<...<t_d}(t_1, ..., t_d)$$

Now fix $n \in \mathbb{N}$ and write

$$\begin{aligned}\mathbb{P}(N_t = n) &= \mathbb{P}(\tau_n \le t < \tau_{n+1}) \\ &= \int_{0<t_1<...<t_n<t<t_{n+1}} \lambda^{n+1} e^{-\lambda t_{n+1}} dt_1 ... dt_{n+1} \\ &= e^{-\lambda t} \frac{(\lambda t)^n}{n!}\end{aligned}$$

which is the required result.

∎

Proposition 2.3. *The process $(N_t)_{t\ge 0}$ is such that $N_0 = 0$ a.s., it has independent and stationary increments and it is stochastically continuous.*

Proof. We start from the stationarity by arbitrarily fixing $n \in \mathbb{N}$ and verifying the equality below

$$\mathbb{P}(N_t - N_s = n) = \mathbb{P}(N_{t-s} = n).$$

We consider the left hand side and write

$$\begin{aligned}\mathbb{P}(N_t - N_s = n) &= \sum_{k\in\mathbb{N}} \mathbb{P}(N_t - N_s = n, N_s = k) \\ &= \sum_{k\in\mathbb{N}} \mathbb{P}(N_t = n + k, N_s = k) \\ &= \sum_{k\in\mathbb{N}} \mathbb{P}(\tau_{n+k} \le t, \tau_{n+k+1} > t, \tau_k \le s, \tau_{k+1} > s) \qquad (2.1)\end{aligned}$$

Recall that the random vector $(\tau_1, ..., \tau_{n+k+1})$ has joint law given by

$$\lambda^{n+k+1}\ e^{-\lambda t_{n+k+1}}\ 1_{0<t_1<...<t_{n+k+1}}(t_1, ..., t_{n+k+1}),$$

thus by a calculation similar to the one in proposition 2.2 we obtain

$$\mathbb{P}(\tau_{n+k} \le t, \tau_{n+k+1} > t, \tau_k \le s, \tau_{k+1} > s) = e^{-\lambda t} \frac{\lambda^{n+k}}{n!k!} (t - s)^n s^k$$

and substituting in (2.1) yields

$$\mathbb{P}(N_t - N_s = n) = e^{-\lambda(t-s)} \frac{(\lambda(t - s))^n}{n!}$$

which is the required result.
We now verify the independence of increments: we arbitrarily fix $n \in \mathbb{N}$, $0 \le t_1 < t_2 < ... < t_n < \infty$, $k_1, ..., k_n \in \mathbb{N}$ and compute the probability

$$\mathbb{P}(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, ..., N_{t_n} - N_{t_{n-1}} = k_n).$$

Define $j_i = \sum_{h=1}^{i} k_h, i \in 1, ..., n$, then the above probability can be written as

$$\mathbb{P}(\tau_{j_1} \leq t_1 < \tau_{j_1+1}, ..., \tau_{j_n} \leq t_n < \tau_{j_n+1})$$

and thus computed by integrating the joint law of $(\tau_1, ..., \tau_{j_n+1})$. This calculation, together with the stationarity of increments, allows to write the above probability as follows

$$\mathbb{P}(N_{t_1} = k_1) \prod_{i=2}^{n} \mathbb{P}(N_{t_i} - N_{t_{i-1}} = k_i),$$

which is the required result.
Now we must only prove the stochastic continuity. Since for any fixed $t \geq 0$ and $n \in \mathbb{N}$ we clearly have $\mathbb{P}(\tau_n = t) = 0$, then the set $\cup_{n \in \mathbb{N}} \{\tau_n = t\}$ has null probability: this implies that for any fixed $t \geq 0$ the process $(N_t)_{t \geq 0}$ is a.s. continuous at t. Since almost sure covergence entails convergence in probability, the process $(N_t)_{t \geq 0}$ enjoys the required property.

■

Proposition 2.2 implies that the characteristic function of the process $(N_t)_{t \geq 0}$ is given by

$$\Phi_t(z) = exp\{\lambda t(e^{iz} - 1)\} \quad \forall z \in \mathbb{R}.$$

Finally, thanks to proposition 2.3 we immediately obtain that the so-called compensated Poisson process $(N_t - \lambda t)_{t \geq 0}$ is a martingale with respect to the natural filtration of $(N_t)_{t \geq 0}$.

2.2 The compound Poisson process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 2.2. *Let $(N_t)_{t \geq 0}$ be a Poisson process with intensity λ and let $\{Y_n\}_{n \geq 1}$ be a sequence of $\mathbb{R}^d$-valued i.i.d. random variables independent of the process $(N_t)_{t \geq 0}$. We call compound Poisson process the process $(Q_t)_{t \geq 0}$ defined by*

$$Q_t = \sum_{n=1}^{N_t} Y_n \quad \forall t \geq 0.$$

From now on we denote $(Q_t)_{t\geq 0}$ a compound Poisson process as defined above, and we call f the law of every element of the sequence $\{Y_n\}_{n\geq 1}$.
Since for each $t \geq 0$ the random variable N_t is finite a.s., then also Q_t is, and thus the sample paths of $(Q_t)_{t\geq 0}$ are càdlàg piecewise constant functions.
By definition, $(Q_t)_{t\geq 0}$ can jump only in the jump times of the underlying Poisson process $(N_t)_{t\geq 0}$; nevertheless, if f gives non-zero probability to $0 \in \mathbb{R}^d$, then $(Q_t)_{t\geq 0}$ does not necessarily jump every time $(N_t)_{t\geq 0}$ does.
Notice that $(Q_t)_{t\geq 0}$ is BV, since the sequence of the jump times of the underlying Poisson process diverges a.s.
The difference between a Poisson process and a compound Poisson process lies in the sequence $\{Y_n\}_{n\geq 1}$: while the former takes values in $\mathbb{R}$ and has jumps of unitary size, the latter takes values in $\mathbb{R}^d$ and has the n-th jump size given by the random variable Y_n.
The proposition below gives the distribution of Q_t for all $t \geq 0$.

Proposition 2.4. *The characteristic function of the process $(Q_t)_{t\geq 0}$ is given by*

$$\Phi_t(z) = exp\left\{ \lambda t \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1) f(dx) \right\} \quad \forall z \in \mathbb{R}^d.$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^d$.

Proof. Using the definition $Q_t = \sum_{n=1}^{N_t} Y_n$ and recalling that the sequence $\{Y_n\}_{n\geq 1}$ and the process $(N_t)_{t\geq 0}$ are independent, the characteristic function can be easily calculated by means of proposition 1.4 conditioning on the random variable N_t.

■

We now give a result which is crucial for the following chapters.

Proposition 2.5. *The process $(Q_t)_{t\geq 0}$ is such that $Q_0 = 0$ a.s., it has independent and stationary increments and it is stochastically continuous.*

Proof. We start from the independence of increments: we fix $0 < s < t$ and prove that the random variables $Q_t - Q_s$ and Q_s are independent, the same argument applying to any finite number of increments. Define

$$\mathcal{G}_s = \sigma(N_u, 0 \leq u \leq s;\ Y_n 1_{\{n\leq N_s\}}, n \geq 1).$$

We can write $Q_s = \sum_{n=1}^{\infty} Y_n 1_{\{n \le N_s\}}$, so that Q_s is $\mathcal{G}_s$-measurable, and

$$\begin{aligned} Q_t - Q_s &= \sum_{n=N_s+1}^{N_t} Y_n = \sum_{n=1}^{N_t - N_s} Y_{N_s+n} 1_{\{N_s+n \le N_t\}} \\ &= \sum_{n=1}^{N_t-N_s} Y_{N_s+n} 1_{\{n \le N_t - N_s\}} = \sum_{n=1}^{N_t-N_s} Y_{N_s+n}. \end{aligned}$$

Now we show by following [BAn], proposition 5.3, that the random variable $N_t - N_s$ and the sequence $\{Y_{N_s+n}\}_{n\ge1}$ are jointly independent of $\mathcal{G}_s$.
Fix $m, k \in \mathbb{N}$, $A \in \mathcal{B}(\mathbb{R}^{md})$, $B \in \sigma(N_t - Ns)$, $C \in \mathcal{B}(\mathbb{R}^{kd})$ and $D \in \sigma(N_u, 0 \le u \le s)$. Since the jump sizes are i.i.d. and independent of $(N_t)_{t\ge0}$, and since $(N_t)_{t\ge0}$ has independent increments we get

$$\begin{aligned} &\mathbb{P}((Y_{N_s+1}, ..., Y_{N_s+m}) \in A,\ B,\ (Y_1, ..., Y_k) \in C,\ D,\ N_s \ge k) \\ &= \sum_{l \ge k} \mathbb{P}((Y_{l+1}, ..., Y_{l+m}) \in A,\ B,\ (Y_1, ..., Y_k) \in C,\ D,\ N_s = l) \\ &= \sum_{l \ge k} \mathbb{P}((Y_{l+1}, ..., Y_{l+m}) \in A) \mathbb{P}(B) \mathbb{P}((Y_1, ..., Y_k) \in C) \mathbb{P}(D,\ N_s = l) \\ &= \sum_{l \ge k} \mathbb{P}((Y_1, ..., Y_m) \in A) \mathbb{P}(B) \mathbb{P}((Y_1, ..., Y_k) \in C) \mathbb{P}(D,\ N_s = l) \\ &= \mathbb{P}((Y_1, ..., Y_m) \in A) \mathbb{P}(B) \mathbb{P}((Y_1, ..., Y_k) \in C) \sum_{l \ge k} \mathbb{P}(D,\ N_s = l) \\ &= \mathbb{P}((Y_1, ..., Y_m) \in A) \mathbb{P}(B) \mathbb{P}((Y_1, ..., Y_k) \in C) \mathbb{P}(D,\ N_s \ge k) \\ &= \mathbb{P}((Y_1, ..., Y_m) \in A,\ B) \mathbb{P}((Y_1, ..., Y_k) \in C,\ D,\ N_s \ge k) \end{aligned}$$

Thus, choosing $D = \Omega$ and $k = 0$ in the above equation we get

$$\mathbb{P}((Y_{N_s+1}, ..., Y_{N_s+m}) \in A,\ B) = \mathbb{P}((Y_1, ..., Y_m) \in A,\ B). \tag{2.2}$$

Thus, combining the above equalities yields

$$\begin{aligned} &\mathbb{P}((Y_{N_s+1}, ..., Y_{N_s+m}) \in A,\ B,\ (Y_1, ..., Y_k) \in C,\ D,\ N_s \ge k) \\ &= \mathbb{P}((Y_{N_s+1}, ..., Y_{N_s+m}) \in A,\ B) \mathbb{P}((Y_1, ..., Y_k) \in C,\ D,\ N_s \ge k) \end{aligned}$$

and the independence result follows via proposition 1.1.
By the above argument $Q_t - Q_s$ and Q_s are seen to be independent.
Let's now show the stationarity of increments: as we did for independence we fix $0 \le s \le t$ and show that $Q_t - Q_s$ and Q_{t-s} have the same distribution,

the same argument applying to any finite number of increments. By the independence of increments and proposition 1.3 we can write

$$\Phi_{Q_t}(z) = \mathbb{E}[e^{i\langle z, Q_t\rangle}] = \mathbb{E}[e^{i\langle z, Q_t - Q_s + Q_s\rangle}] = \Phi_{Q_t - Q_s}(z)\Phi_{Q_s}(z)$$

so that by proposition 2.4 we get

$$\Phi_{Q_t - Q_s}(z) = \frac{\Phi_{Q_t}(z)}{\Phi_{Q_s}(z)} = \Phi_{Q_{t-s}}(z)$$

which is the required result.
Finally, stochastic continuity follows from the fact that the jump times of $(Q_t)_{t\geq 0}$ coincide with the jump times of $(N_t)_{t\geq 0}$ and that the latter enjoys stochastic continuity.

■

Remark. By choosing $B = \Omega$ in (2.2) we get that $\{Y_{N_s+n}\}_{n\geq 1}$ is a sequence of i.i.d. random variables with law f as Y_1 for any $s \geq 0$.

Definition 2.3. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration.
We say that $(Q_t)_{t\geq 0}$ is a compound Poisson process with respect to $(\mathcal{F}_t)_{t\geq 0}$ if it is an $\mathcal{F}_t$-adapted compound Poisson process whose increments are independent with respect to $(\mathcal{F}_t)_{t\geq 0}$.*

Now let $(Q_t)_{t\geq 0}$ be an $\mathcal{F}_t$-compound Poisson process and let $\beta = \mathbb{E}[Y_n]\ \forall n \in \mathbb{N}$: it is not difficult to show that the so-called compensated compound Poisson process $(Q_t - \lambda\beta t)_{t\geq 0}$ is an $\mathcal{F}_t$-martingale.

Proposition 2.6. *Let $(Q_t)_{t\geq 0}$ be an $\mathcal{F}_t$-compound Poisson process. Then the random variable Y_n is measurable with respect to $\mathcal{F}_{\tau_n}$ for all $n \geq 1$.*

Proof. Since $(Q_t)_{t\geq 0}$ is $\mathcal{F}_t$-adapted and right-continuous, it is $\mathcal{F}_t$-progressively measurable. Thus the random variable Q_{τ_n} is $\mathcal{F}_{\tau_n}$-measurable for all $n \geq 1$ thanks to proposition 1.6. The required result follows by noticing that $\tau_{n-1} \leq \tau_n$ and $Y_n = Q_{\tau_n} - Q_{\tau_{n-1}}$ for all $n \geq 1$.

■

Chapter 3

Finite-activity Lévy processes

This chapter is divided in two sections: in the first we define finite-activity Lévy processes by combining brownian motion and compound Poisson process, in the second we highlight some of their properties and briefly investigate their connection with Lévy processes and semimartingales.
To do this we follow [AP], section 2.3 and 2.4: according to its account we do not start our discussion by fixing any filtration on our probability space. Filtrations will only come back at the end of the chapter.

3.1 Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 3.1. *Let $(B_t^\Sigma)_{t\geq 0}$ be an $\mathbb{R}^m$-valued brownian motion with covariance matrix Σ and let $(Q_t)_{t\geq 0}$ be an $\mathbb{R}^m$-valued compound Poisson process such that they are independent. Let $\gamma \in \mathbb{R}^m$.*
We define a finite-activity Lévy process $(X_t)_{t\geq 0}$ to be of the form

$$X_t = \gamma t \; + \; B_t^\Sigma + \; Q_t \quad \forall t \geq 0$$

From now on we denote $(B_t^\Sigma)_{t\geq 0}$, $(Q_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ the processes introduced above.
Finite-activity Lévy processes are important since their features make them suitable to serve as noises and since they provide an adequate mathematical representation for many applications.

3.2 Properties

In this section we briefly discuss the connection between the notion of finite-activity Lévy process and those of Lévy process and semimartingale.

We do not go deep into the theory concerning Lévy processes and semimartingales, but we concisely highlight to what extent they are more general than a finite-activity Lévy process.
We also give a result about usual hypotheses which assures that our approach is consistent and introduce the notion of counting random measure associated with a process, which will be important in the following chapters.

3.2.1 On the connection with Lévy processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 3.2. *A process $(W_t)_{t\geq 0}$ is called Lévy process if*

- $W_0 = 0$ *a.s.*
- $(W_t)_{t\geq 0}$ *has independent and stationary increments*
- $(W_t)_{t\geq 0}$ *is stochastically continuous*

Proposition 3.1. *A finite-activity Lévy process $(X_t)_{t\geq 0}$ is such that $X_0 = 0$ a.s, it has independent and stationary increments and it is stochastically continuous.*

Proof. First we recall that the process $(B_t^\Sigma)_{t\geq 0}$ enjoys the required properties, and so does $(Q_t)_{t\geq 0}$ thanks to proposition 2.5. We then note that it suffices to prove the thesis for the process $(B_t^\Sigma + Q_t)_{t\geq 0}$.
We start by establishing the independence of increments: we fix $0 < s < t$ and show that the random variables $B_t^\Sigma + Q_t - B_s^\Sigma - Q_s$ and $B_s^\Sigma + Q_s$ are independent, the same argument applying to any finite number of increments. We use proposition 1.3 and thanks to the above statement and the independence of the processes $(B_t^\Sigma)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ we can write for each $z_1, z_2, z_3, z_4 \in \mathbb{R}^m$

$$\begin{aligned}
\Phi_{B_t^\Sigma - B_s^\Sigma, Q_t - Q_s, B_s^\Sigma, Q_s}(z_1, z_2, z_3, z_4) &= \Phi_{B_t^\Sigma - B_s^\Sigma, B_s^\Sigma}(z_1, z_3)\Phi_{Q_t - Q_s, Q_s}(z_2, z_4) \\
&= \Phi_{B_t^\Sigma - B_s^\Sigma}(z_1)\Phi_{B_s^\Sigma}(z_3)\Phi_{Q_t - Q_s}(z_2)\Phi_{Q_s}(z_4) \\
&= \Phi_{B_t^\Sigma - B_s^\Sigma, Q_t - Q_s}(z_1, z_2)\Phi_{B_s^\Sigma, Q_s}(z_3, z_4)
\end{aligned}$$

The stationarity of the increments follows directly from the stationarity of the increments of both processes $(B_t^\Sigma)_{t\geq 0}$, $(Q_t)_{t\geq 0}$ and from their independence.
The stochastic continuity follows from the stochastic continuity of both processes $(B_t^\Sigma)_{t\geq 0}$, $(Q_t)_{t\geq 0}$.

■

By proposition 3.1 we immediately see that every finite-activity Lévy process is a Lévy process. Now we are interested in evaluating the size of the loss of generality between the two notions.
Let $(W_t)_{t\geq 0}$ be an $\mathbb{R}^m$-valued Lévy process and define its jump at time $t \geq 0$ to be the random variable

$$\Delta W_t = W_t - W_{t^-} = W_t - \lim_{s \nearrow t} W_s,$$

which is well defined and a.s. finite since $(W_t)_{t\geq 0}$ is càdlàg.
Call $E = \mathbb{R}^m \smallsetminus \{0\}$ and $\mathcal{E} = \mathcal{B}(E)$: given $A \in \mathcal{E}$ and $t \geq 0$ we introduce the random variable

$$N(t, A) = \#\{0 \leq s \leq t : \Delta W_s \in A\}.$$

For all $\omega \in \Omega$, $t \geq 0$, the set function $A \mapsto N(t, A)(\omega)$ is a counting measure on $\mathcal{E}$: it is also said that, for all $t \geq 0$, $A \mapsto N(t, A)$ is a random counting measure on $\mathcal{E}$. Furthermore, it is possible to show (see [AP], paragraph 2.3.1) that the mapping $([0, t], A) \mapsto N(t, A)$ constitutes a random counting measure on $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{E}$: we will denote it by $N(dt, dx)$ and call it the *random counting measure* associated with the process $(W_t)_{t\geq 0}$.
Define

$$\nu(A) = \mathbb{E}[N(1, A)] \quad \forall A \in \mathcal{E}.$$

Then ν is a positive Borel measure on $\mathcal{E}$: we call it the *intensity measure* associated with the process $(W_t)_{t\geq 0}$.
Given $A \in \mathcal{E}$, then the random variable $N(t, A)$ can fail to be a.s. finite for all $t \geq 0$. Indeed Lévy processes can have the following particular behaviour: for any fixed $[a, b] \subseteq \mathbb{R}^+$ and any threshold $M > 0$, the number of jumps in $[a, b]$ whose size's norm is greater than M is a.s. finite, while the number of jumps in $[a, b]$ whose size's norm is smaller than M can fail to be a.s. finite. In regards to this behaviour we say that Lévy process can have infinitely many jumps of small size.
A way to ensure that $N(t, A)$ is a.s. finite for all $t \geq 0$ is to take A *bounded below*, that is $0 \notin \bar{A}$, where $\bar{A}$ is the closure of A (for the proof see [AP], lemma 2.3.4): this allows in fact to exclude the potentially infinite small jumps.
It is possible to show that if $N(t, A)$ is a.s. finite for all $t \geq 0$, then the process $(N(t, A))_{t\geq 0}$ is a Poisson process with intensity given by $\nu(A)$. For the proof see theorem 2.2.13 and 2.3.5: [AP] requires A bounded below, but indeed its arguments hold whenever $N(t, A)$ is a.s. finite for all $t \geq 0$.
In particular, if $N(t, E)$ is a.s. finite for all $t \geq 0$, then $\nu(E)$ is the intensity of a Poisson process, so that $\nu(E) < \infty$ and thus the measure ν is finite.

Remark. Since for a general Lévy process taking $A \in \mathcal{E}$ bounded below yields $\nu(A) < \infty$, then the measure ν must be σ-finite: indeed the sequence $\{C_n\}_{n\geq 1}$, where

$$C_1 = \{x \in E : |x| > 1\}$$
$$C_n = \left\{x \in E : \frac{1}{n} < |x| \leq \frac{1}{n-1}\right\} \ \forall n \geq 2,$$

is such that C_n is bounded below for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} C_n = E$.

Theorem 3.2. *Let $(W_t)_{t\geq 0}$ be an $\mathbb{R}^m$-valued Lévy process. Then the following conditions are equivalent*

- *$N(t, E) < \infty$ a.s. $\forall t \geq 0$*
- *the associated intensity measure ν is finite*
- *there exist $\gamma \in \mathbb{R}^m$, an $\mathbb{R}^m$-valued brownian motion $(B_t^\Sigma)_{t\geq 0}$ and an $\mathbb{R}^m$-valued compound Poisson process $(Q_t)_{t\geq 0}$ independent of $(B_t^\Sigma)_{t\geq 0}$ such that*

$$W_t = \gamma t + B_t^\Sigma + Q_t \quad \forall t \geq 0$$

The last condition matches exactly our definition of finite-activity Lévy process, and since the first one means that the Lévy process $(W_t)_{t\geq 0}$ jumps a.s. finitely many times on the bounded subsets of the time line, then such a process is indeed a finite-activity Lévy process. Since a finite-activity Lévy process is a Lévy process jumping a.s. finitely many times on the bounded subsets of the time line, then the two notions are equivalent. This indeed justifies our choice for the name of finite-activity Lévy processes.
Since for our purpose we restrict the class of the processes which are suitable to model noises to Lévy processes, we can conclude that the most general noise jumping a.s. finitely many times on the bounded subsets of the time line can be represented by a finite-activity Lévy process.
Finally, let $(X_t)_{t\geq 0}$ be a finite-activity Lévy process whose jump part is given by the compound Poisson process $(Q_t)_{t\geq 0}$. Let $\{Y_n\}_{n\geq 1}$ be its jump sizes, with law f, and let $(N_t)_{t\geq 0}$ be the underlying Poisson process with intensity λ. Choose $A \in \mathcal{E}$ and consider the process $(N(t, A))_{t\geq 0}$ associated with $(X_t)_{t\geq 0}$ and its intensity measure ν. Then we get

$$N(t, A) = \sum_{n=1}^{N_t} 1_A(Y_n)$$
$$\nu(A) = \lambda \mathbb{P}(Y_n \in A) = \lambda f(A).$$

Notice that the random counting measure associated with $(X_t)_{t\geq 0}$ depends only on $(Q_t)_{t\geq 0}$, so that in fact it coincides with the random counting measure associated with $(Q_t)_{t\geq 0}$.
Notice also that the processes $(N(t,E))_{t\geq 0}$, $(N_t)_{t\geq 0}$ are indistinguishable if and only if $f(\{0\}) = 0$, since in fact

$$\mathbb{P}(N(t,E) = N_t \ \forall t \geq 0) = \mathbb{P}(Y_n \neq 0 \ \forall n \geq 1) = 1 \Leftrightarrow \mathbb{P}(Y_n \neq 0) = 1 \ \forall n \geq 1.$$

3.2.2 On the usual hypotheses

We now consider Lévy processes in full generality and use them to ensure that the usual hypotheses, which constitute the basis of the whole theory concerning stochastic calculus, do not entail any loss of generality for the purpose of this work.
In particular, from chapter 5 on, our setting will be given by fixed brownian motion and compound Poisson process such that they are independent. Thus, our goal is to show that the filtration they generate together satisfies the usual hypotheses.

Proposition 3.3. *Given a Lévy process $(W_t)_{t\geq 0}$ and any modification $(W'_t)_{t\geq 0}$, then $(W'_t)_{t\geq 0}$ is a Lévy process.*

Proof. Clearly $W'_0 = 0$ a.s.
We now show the stationarity of increments: fix $0 \leq s \leq t < \infty$ and define

$$\Omega_0 = \{\omega \in \Omega : W_s(\omega) = W'_s(\omega), W_t(\omega) = W'_t(\omega)\}.$$

Clearly $\mathbb{P}(\Omega_0) = 1$, and thus we have

$$\mathbb{P}(F, \Omega_0) = \mathbb{P}(F) \quad \forall F \in \mathcal{F}.$$

Now fix arbitrarily $A \in \mathcal{B}(\mathbb{R}^m)$ and write

$$\begin{aligned}\mathbb{P}(W'_t - W'_s \in A) &= \mathbb{P}(W_t - W_s \in A) \\ &= \mathbb{P}(W_{t-s} \in A) \\ &= \mathbb{P}(W'_{t-s} \in A)\end{aligned}$$

Then we get the required result since this argument can be applied to any finite number of increments.
Now we show that $(W'_t)_{t\geq 0}$ has independent increments: we fix arbitrarily $A, B \in \mathcal{B}(\mathbb{R}^m)$ and by the above argument we write

$$\begin{aligned}\mathbb{P}(W'_t - W'_s \in A, W'_s \in B) &= \mathbb{P}(W_t - W_s \in A, W_s \in B) \\ &= \mathbb{P}(W_t - W_s \in A)\mathbb{P}(W_s \in B) \\ &= \mathbb{P}(W'_t - W'_s \in A)\mathbb{P}(W'_s \in B)\end{aligned}$$

The required result follows since this argument can be applied to any finite number of increments.
We now establish the stochastic continuity by using again Ω_0 and writing for each $a > 0$

$$\mathbb{P}(|W'_t - W'_s| > a) = \mathbb{P}(|W_t - W_s| > a)$$

and noticing that the right hand side vanishes as s nears t.

■

It is possible to show that every Lévy process has a càdlàg modification: for a proof see [AP], theorems 2.1.7 and 2.1.8. Then by the above proposition the càdlàg modification is a Lévy process (thus, given any Lévy process, we shall henceforth consider its càdlàg modification by tacit agreement). Furthermore, the augmented natural filtration of every càdlàg Lévy process is found to be right continuous: for a proof see [AP], theorem 2.1.10.
These results assure that the usual hypotheses are not restrictive if we work with one Lévy process.
Finally, we give the following result.

Proposition 3.4. *Let $(W'_t)_{t\geq 0}$, $(W''_t)_{t\geq 0}$ be independent Lévy processes. Then $(W_t)_{t\geq 0}$ such that $W_t = (W'_t, W''_t)$ $\forall t \geq 0$ is a Lévy process.*

Proof. Suppose $(W'_t)_{t\geq 0}$, $(W''_t)_{t\geq 0}$ take values respectively in $\mathbb{R}^n$, $\mathbb{R}^k$ for some $n, k \in \mathbb{N}$.
Of course $W_0 = (0, 0)$.
Since a random vector converges in probability if and only if its components do, we immediately get the stochastic continuity.
Let's now show the stationarity of increments: we fix $0 \leq s < t,\ u \in \mathbb{R}^{n+k}$ such that $u = (u_1, u_2),\ u_1 \in \mathbb{R}^n,\ u_2 \in \mathbb{R}^k$ and using the independence and the stationarity of increments of the processes $(W'_t)_{t\geq 0},\ (W''_t)_{t\geq 0}$ we write

$$\begin{aligned}
\mathbb{E}[e^{i\langle u, W_t - W_s\rangle}] &= \mathbb{E}[e^{i(\langle u_1, W'_t - W'_s\rangle + \langle u_2, W''_t - W''_s\rangle)}] \\
&= \mathbb{E}[e^{i\langle u_1, W'_t - W'_s\rangle}]\ \mathbb{E}[e^{i\langle u_2, W''_t - W''_s\rangle}] \\
&= \mathbb{E}[e^{i\langle u_1, W'_{t-s}\rangle}]\ \mathbb{E}[e^{i\langle u_2, W''_{t-s}\rangle}] \\
&= \mathbb{E}[e^{i(\langle u_1, W'_{t-s}\rangle + \langle u_2, W''_{t-s}\rangle)}] \\
&= \mathbb{E}[e^{i\langle u, W_{t-s}\rangle}]
\end{aligned}$$

To show the independence of increments we fix $0 \leq s < t,\ u, v \in \mathbb{R}^{n+k}$ such that $u = (u_1, u_2),\ u_1 \in \mathbb{R}^n,\ u_2 \in \mathbb{R}^k, v = (v_1, v_2),\ v_1 \in \mathbb{R}^n,\ v_2 \in \mathbb{R}^k$ and using the independence of the processes $(W'_t)_{t\geq 0}$, $(W''_t)_{t\geq 0}$, we write

$$\begin{aligned}\mathbb{E}[e^{i(\langle u,W_t-W_s\rangle+\langle v,W_s\rangle)}] &= \mathbb{E}[e^{i(\langle u_1,W'_t-W'_s\rangle+\langle u_2,W''_t-W''_s\rangle+\langle v_1,W'_s\rangle+\langle v_2,W''_s\rangle)}] \\ &= \mathbb{E}[e^{i\langle u_1,W'_t-W'_s\rangle}]\ \mathbb{E}[e^{i\langle u_2,W''_t-W''_s\rangle}]\ \mathbb{E}[e^{i\langle v_1,W'_s\rangle}]\ \mathbb{E}[e^{i\langle v_2,W''_s\rangle}] \\ &= \mathbb{E}[e^{i\langle u,W_t-W_s\rangle}]\ \mathbb{E}[e^{i\langle v,W_s\rangle}]\end{aligned}$$

so that the required result follows by proposition 1.3.

■

Thus the augmented filtration generated by a brownian motion and an independent compound Poisson process satisfies the usual hypotheses.

Finally, suppose to be given a filtration $(\mathcal{F}_t)_{t\geq 0}$.

Definition 3.3. *The process $(W_t)_{t\geq 0}$ is a Lévy process with respect to $(\mathcal{F}_t)_{t\geq 0}$ if it is a Lévy process, if it is $\mathcal{F}_t$-adapted and if it has independent increments with respect to $(\mathcal{F}_t)_{t\geq 0}$.*

Given an $\mathcal{F}_t$-Lévy process $(W_t)_{t\geq 0}$, we are interested in the connection between $(\mathcal{F}_t)_{t\geq 0}$ and the random counting measure associated with $(W_t)_{t\geq 0}$.

Proposition 3.5. *Let $(W_t)_{t\geq 0}$ be a Lévy process with respect to $(\mathcal{F}_t)_{t\geq 0}$ and let $A \in \mathcal{E}$. Then the process $(N(t,A))_{t\geq 0}$ is $\mathcal{F}_t$-adapted and has independent increments with respect to $(\mathcal{F}_t)_{t\geq 0}$.*

Proof. Since $(W_t)_{t\geq 0}$ is $\mathcal{F}_t$-adapted, then by definition we see that also $(N(t,A))_{t\geq 0}$ is.
For independence fix $0 \leq s < t$ and write

$$N(t,A) - N(s,A) = \#\{s < u \leq t : \Delta W_u \in A\}.$$

Fix $u > s$, $B \in \mathcal{F}_s$, a bounded continuous mapping $g : \mathbb{R}^m \to \mathbb{R}$ and since $(W_t)_{t\geq 0}$ has independent increments with respect to $(\mathcal{F}_t)_{t\geq 0}$ and thanks to the dominated convergence theorem we can write

$$\begin{aligned}\mathbb{E}[\ g(\Delta W_u)1_B\ ] &= \mathbb{E}[\ g(W_u - \lim_{v\nearrow u} W_v)1_B\ ] = \mathbb{E}[\ \lim_{v\nearrow u} g(W_u - W_v)1_B\ ] \\ &= \lim_{v\nearrow u} \mathbb{E}[\ g(W_u - W_v)1_B\ ] = \lim_{v\nearrow u} \mathbb{E}[\ g(W_u - W_v)\ ]\mathbb{P}(B) \\ &= \mathbb{E}[\ g(\Delta W_u)\ ]\mathbb{P}(B).\end{aligned}$$

so that ΔW_u is independent of $\mathcal{F}_s$ for all $u > s$. Then for all $s < u_1 < ... < u_n < t$, $n \in \mathbb{N}$, by an induction argument we get that $(\Delta W_{u_1}, ..., \Delta W_{u_n})$ is independent of $\mathcal{F}_s$, and the required result follows via proposition 1.1.

■

3.2.3 On the connection with semimartingales

Definition 3.4. *Let* $(\mathcal{F}_t)_{t\geq 0}$ *be a filtration on* $(\Omega, \mathcal{F}, \mathbb{P})$*.*
We say that the process $(Z_t)_{t\geq 0}$ *is an* $\mathcal{F}_t$*-semimartingale if*

$$Z_t = Z_0 + M_t + C_t \quad \forall t \geq 0$$

where Z_0 *is an* $\mathcal{F}_0$*-measurable random variable,* $(M_t)_{t\geq 0}$ *is an* $\mathcal{F}_t$*-local martingale and* $(C_t)_{t\geq 0}$ *is a* BV $\mathcal{F}_t$*-adapted process.*

In the introduction we said that the approach to stochastic integration based on semimartingales is more general than the one based on Lévy process. Actually, the reason is given in the proposition below.

Proposition 3.6. *Every Lévy process is a semimartingale.*

For a proof see [AP], Proposition 2.7.1.
In the case of a finite-activity Lévy process the above decomposition is given by

$$Z_0 = 0, \; M_t = B_t^{\Sigma}, \; C_t = \gamma t + Q_t \quad \forall t \geq 0.$$

Chapter 4

Predictable processes and mappings

This chapter is divided in two sections: in the first we recall the definition of predictability for stochastic processes and provide, let us say, a concise phenomenology based on the account given in [ME], while in the second we extend the notion of predictability to a different type of mappings and give a measurability result.
In what follows, if not specified, all stochastic processes take values in $\mathbb{R}^m$ for any $m \in \mathbb{N}$ and their time indexes vary up to the finite horizon T, but the definitions and the results given below can be smoothly generalised to the case of infinite temporal horizon.

4.1 Predictable processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses.

Definition 4.1. *A subset A of $[0,T] \times \Omega$ is called adapted (respectively measurable, progressively measurable) if the function 1_A is an adapted (respectively measurable, progressively measurable) process.*
We call $\mathcal{M}_0$ (respectively $\mathcal{M}, \mathcal{M}_1$) the set of all the adapted (respectively measurable, progressively measurable) sets.

We have the following elementary result.

Proposition 4.1. *$\mathcal{M}_0, \mathcal{M}$ and $\mathcal{M}_1$ are σ-algebras.*
Moreover, a stochastic process is adapted (respectively measurable, progressively measurable) if and only if it is $\mathcal{M}_0$-measurable (respectively $\mathcal{M}$-measurable, $\mathcal{M}_1$-measurable).

Proof. We prove the result only for $\mathcal{M}_0$ and adapted processes, the others being proved in similar fashion.
Notice that by definition a set A belongs to $\mathcal{M}_0$ if and only if $\Omega_t^A \in \mathcal{F}_t \ \forall t \geq 0$, where for all $t \geq 0$

$$\Omega_t^A = \{\omega \in \Omega : (t, \omega) \in A\}.$$

Now we show that $\mathcal{M}_0$ fulfills the properties of a σ-algebra.

- $[0, T] \times \Omega \in \mathcal{M}_0$
  It is trivial since $\Omega_t^{[0,T]\times\Omega} = \Omega \ \forall t \geq 0$.

- $A \in \mathcal{M}_0 \Rightarrow A^c \in \mathcal{M}_0$
  It is an immediate since $\mathcal{F}_t$ is a σ-algebra for all $t \geq 0$.

- $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}_0 \Rightarrow A = \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}_0$
  It is an immediate consequence of the fact that for all $t \geq 0$ $\mathcal{F}_t$ is a σ-algebra and that

$$\bigcup_{n\in\mathbb{N}} \Omega_t^{A_n} = \Omega_t^A$$

We now prove that a process is adapted if and only if it is $\mathcal{M}_0$-measurable. Take $(X_t)_{t\geq 0}$ to be an adapted process and consider any $B \in \mathcal{B}(\mathbb{R}^m)$: then we have to verify that the counterimage of B with respect to the mapping $(t, \omega) \mapsto X(t, \omega)$, call it $X^{-1}(B)$, is an adapted set, i.e. that $1_{X^{-1}(B)}$ is an adapted process. Thus we have to show that, fixing $t \in [0, T]$, the mapping $\omega \mapsto 1_{X^{-1}(B)}(t, \omega)$ is $\mathcal{F}_t$-measurable. To do this, notice that

$$1_{X^{-1}(B)}(t, \omega) = 1_B(X_t(\omega))$$

and since $(X_t)_{t\geq 0}$ is adapted by hypothesis, then we get that every adapted process is $\mathcal{M}_0$-measurable.
The above equality also allows to prove the inverse implication, so that the proof is complete.

■

In particular we have $\mathcal{M} = \mathcal{B}([0, T]) \otimes \mathcal{F}$; since a progressively measurable process is also adapted and measurable, we have that $\mathcal{M}_1 \subseteq \mathcal{M}_0$ and $\mathcal{M}_1 \subseteq \mathcal{M}$.
We now recall a well known result.

Proposition 4.2. *Every adapted right-continuous process and every adapted left-continuous process are progressively measurable.*

We can now give the definition of predictable process.

Definition 4.2. *The σ-algebra of subsets of $[0,T]\times\Omega$ which is generated by the adapted continuous real valued processes is called the σ-algebra of predictable sets and is denoted by $\mathcal{P}$.*
A $\mathcal{P}$-measurable process is called predictable.

Proposition 4.2 implies that $\mathcal{P}\subseteq\mathcal{M}_1$, so that a predictable process is also progressively measurable.
We now state and prove the following fundamental result.

Proposition 4.3. *The σ-algebra $\mathcal{P}$ is generated by*

- *the adapted left-continuous real valued processes*
- *the family of sets*

$$\mathcal{R}=\{(s,t]\times F: 0\le s\le t\le T, F\in\mathcal{F}_s\}\cup\{\{0\}\times F, F\in\mathcal{F}_0\}.$$

Proof. Call $\mathcal{P}'$ the σ-algebra generated by the adapted left-continuous real valued processes: the inclusion $\mathcal{P}\subseteq\mathcal{P}'$ is trivial since every continuous process is also left-continuous.
Now we verify that $\mathcal{P}'\subseteq\sigma(\mathcal{R})$. To do this, we first show that any bounded adapted left-continuous real valued process can be obtained as a pointwise limit of $\sigma(\mathcal{R})$-measurable mappings; then we show that any adapted left-continuous real valued process can be obtained as a pointwise limit of bounded adapted left-continuous real valued processes, whence the required result.
Let $(X_t)_{t\in[0,T]}$ be a bounded adapted left-continuous real valued process, fix $n\in\mathbb{N}$, $k\in\{0,1,...,2^n\}$ and take $M=\sup_{(t,\omega)\in[0,T]\times\Omega}|X_t(\omega)|$.
Choose I_n to be the smallest integer such that $\frac{M}{I_n}<\frac{1}{2^n}$.
Define $\lambda_n^i=i\frac{M}{I_n}$, $i\in\{-I_n,...,I_n\}$,

$$F_{n,k}^i=\left\{\omega\in\Omega: X_{\frac{kT}{2^n}}(\omega)\in[\lambda_n^i,\lambda_n^{i+1})\right\},\quad i\in\{-I_n,-I_n+1,...,I_n-1\},$$

and $F_{n,k}^{I_n}=\{\omega\in\Omega: X_{\frac{kT}{2^n}}(\omega)=M\}$. Notice that $\{F_{n,k}^i\}_{i\in\{-I_n,...,I_n\}}$ is a partition of Ω.
Since the random variable $X_{\frac{kT}{2^n}}$ is $\mathcal{F}_{\frac{kT}{2^n}}$-measurable, we have that $F_{n,k}^i\in\mathcal{F}_{\frac{kT}{2^n}}$ $\forall i\in\{-I_n,...,I_n\}$. Thus the map

$$X_n(t,\omega)=\sum_{k=0}^{2^n-1}\sum_{i=-I_n}^{I_n}\lambda_n^i 1_{(\frac{kT}{2^n},\frac{(k+1)T}{2^n}]}(t)1_{F_{n,k}^i}(\omega)$$

is $\sigma(\mathcal{R})$-measurable for all $n \in \mathbb{N}$.
Futhermore, by construction for all n, k we have

$$\sup_{\omega\in\Omega} \left| X_{\frac{kT}{2^n}}(\omega) - \sum_{i=-I_n}^{I_n} \lambda_n^i 1_{F_{n,k}^i}(\omega) \right| \leq \frac{1}{2^n} \tag{4.1}$$

Now we show that (4.1) together with the left-continuity of $(X_t)_{t\in[0,T]}$ yields $X_n \to X$ pointwise on $[0,T]\times\Omega$. Fix $t \in [0,T]$, $\omega \in \Omega$: since $(X_t)_{t\in[0,T]}$ is left-continuous we have

$$\forall \epsilon > 0 \; \exists \; \delta > 0 : |X_t(\omega) - X_s(\omega)| < \epsilon \;\; \forall t - \delta < s < t.$$

Since

$$\forall \delta > 0 \; \exists \; n_0 \in \mathbb{N} : \forall n \geq n_0 \; \exists \; k : t - \delta < \frac{kT}{2^n} < t \leq \frac{(k+1)T}{2^n},$$

then

$$\forall \epsilon > 0 \; \exists \; n_0 \in \mathbb{N} : \forall n \geq n_0 \; \exists \; k : |X_t(\omega) - X_{\frac{kT}{2^n}}(\omega)| < \epsilon.$$

Thus by the definition of X_n and by (4.1) we get that $\forall \epsilon > 0 \; \exists \; n_1 = n_0 \vee \log_2 \frac{1}{\epsilon}$ such that $\forall n \geq n_1$

$$\begin{aligned} |X_t(\omega) - X_n(t,\omega)| &\leq |X_t(\omega) - X_{\frac{kT}{2^n}}(\omega)| + |X_{\frac{kT}{2^n}}(\omega) - X_n(t,\omega)| \\ &< \epsilon + \frac{1}{2^n} < 2\epsilon, \end{aligned}$$

which proves the required convergence.
Let now $(X_t)_{t\in[0,T]}$ be an adapted left-continuous real valued process, fix $n \in \mathbb{N}$ and define

$$X_n(t,\omega) = \begin{cases} -n & X(t,\omega) < -n \\ X(t,\omega) & |X(t,\omega)| \leq n \\ n & X(t,\omega) > n. \end{cases}$$

Then for each $n \in \mathbb{N}$ this mapping defines a bounded adapted left-continuous process and thanks to the above argument it is $\sigma(\mathcal{R})$-measurable. Furthermore, $X_n \to X$ pointwise on $[0,T]\times\Omega$.
We have thus proved the inclusion $\mathcal{P}' \subseteq \sigma(\mathcal{R})$, so that $\mathcal{P} \subseteq \sigma(\mathcal{R})$.
It remains to show that $\sigma(\mathcal{R}) \subseteq \mathcal{P}$: to do this we show that every member of $\mathcal{R}$ belongs to $\mathcal{P}$. Let $0 \leq s \leq t \leq T$, $F \in \mathcal{F}_s$ and take $R = (s,t] \times F \in \mathcal{R}$. Fix $n \in \mathbb{N}$ and define

$$\phi_n(u) = \begin{cases} 0 & 0 \leq u \leq s \\ 1 - e^{-n(u-s)} & s < u \leq t \\ \phi_n(t)e^{-n(u-t)} & t < u \leq T. \end{cases}$$

Then $\phi_n \to 1_{(s,t]}$ pointwise on $[0,T]$ as $n \to \infty$. Furthermore, since for each $n \in \mathbb{N}$ the map ϕ_n is continuous and null up to time s, then for each n the process defined by $\phi_n 1_F$ is continuous and adapted. The required result follows by noticing that $\phi_n 1_F \to 1_R$ pointwise as $n \to \infty$.

■

The notion of predictability is not fundamental within the theory of stochastic integration with respect to brownian motion (in fact it does not typically appear in the program of such courses). It comes to be fundamental when the integrator can have a jump component. To give an idea of the reason, we note that by the above proposition $\mathcal{P}$ is generated even by the adapted left-continuous processes and consider a sample path of such a process: if there is a jump at a given time $t \geq 0$, then the value at t can be "predicted" by the values at the preceeding instants. This is not the case of a right-continuous process, where it is not possible to evaluate the paths in their discontinuities before they happen.
In most modelling contexts, jumps represent sudden events, so that if the noise has a jump component then it must be right-continuous. On the other hand, the integrand is supposed to model quantities which must be forseeable in the above sense, so that the choice of left-continuity is natural instead of right-continuity. Then in the general theory the processes which are suitable as integrands turn out to be those generated by the adapted left-continuous processes, that is the predictable ones: they are a subclass of the progressive measurable processes and they do not contain in general the right-continuous ones.
The difference between progressive measurability and predictability lies in fact in the dependence on Ω, in absence of which the two notions would coincide, since the σ-algebra generated on $[0,T]$ by the continuous functions is $\mathcal{B}([0,T])$.
We note that, while predictability is important in the theory of stochastic integration, on the other hand for many applications to finance it is sufficient to consider left-continuity: for instance, [SH] cites predictability *a latere*, and [CT] emphasizes that all its explicit examples of predictable processes are actually left-continuous.

4.2 Predictable mappings

We now extend the notion of predictability for stochastic processes to a different kind of mappings and give a related measurability result.
Let $d \in \mathbb{N}$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

Definition 4.3. *We denote by $\mathcal{P}_A$ the σ-algebra generated on $[0,T] \times A \times \Omega$ by the real valued mappings F such that*

- *for each $0 \leq t \leq T$ the mapping $(x, \omega) \mapsto F(t, x, \omega)$ is $\mathcal{B}(A) \otimes \mathcal{F}_t$-measurable*
- *for each $x \in A, \omega \in \Omega$ the mapping $t \mapsto F(t, x, \omega)$ is left-continuous.*

A $\mathcal{P}_A$-measurable mapping F is said to be predictable.

Proposition 4.4. *Every predictable mapping is measurable with respect to $\mathcal{B}[0,T] \otimes \mathcal{B}(A) \otimes \mathcal{F}$.*

Proof. Define a new measurable space $(\Omega', \mathcal{F}')$ and endow it with the filtration $(\mathcal{F}'_t)_{t \geq 0}$, where $\Omega' = A \times \Omega$, $\mathcal{F}' = \mathcal{B}(A) \otimes \mathcal{F}$, $\mathcal{F}'_t = \mathcal{B}(A) \otimes \mathcal{F}_t$ for all $t \geq 0$.
In this space a mapping $F : [0,T] \times A \times \Omega \to \mathbb{R}$ is in fact an $\mathcal{F}'_t$-adapted stochastic process. Furthermore, we easily see that the predictability of F as a stochastic mapping is equivalent to the predictability of F as a process on $(\Omega', \mathcal{F}')$.
Thus, the process F is progressively measurable by the discussion in chapter 4, so that it is also measurable. By the definition of $(\Omega', \mathcal{F}')$, this means that the mapping $(t, x, \omega) \mapsto F(t, x, \omega)$ is measurable with respect to $\mathcal{B}[0,T] \otimes \mathcal{B}(A) \otimes \mathcal{F}_T \subseteq \mathcal{B}[0,T] \otimes \mathcal{B}(A) \otimes \mathcal{F}$.

■

Chapter 5

The Poisson stochastic integral

In this chapter we define the Poisson stochastic integral, that is a stochastic integral with respect to a compound Poisson process.
As known, Itô's brownian integral is not an ordinary Lebesgue-Stieltjes integral because brownian motion is not a BV process.
In section 2.2 we saw that compound Poisson process is BV: this is actually a good reason to suppose that we could in principle define an integral pathwise. Indeed it turns out that this is possible, as we will see as the following discussion proceeds. Nevertheless we will initially introduce the Poisson stochastic integral by means of a procedure similar to the Itô's one for brownian motion, the reason why being the proof of a number of desired properties fulfilled by the resulting object. To do this we will refer constantly to [AP], chapter 4, and include only the prooves it does not carry out. As a second step we give a pathwise definition of Poisson stochastic integral and show, by following closely [BR] (chapter III, theorem T13), that it is consistent with the first one.

5.1 Itô's way to Poisson integration

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses.
Take an $\mathbb{R}^d$-valued $\mathcal{F}_t$-compound Poisson process $(Q_t)_{t\geq 0}$, that is $Q_t = \sum_{n=1}^{N_t} Y_n$, where as usual $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ and arrival times $\{\tau_n\}_{n\in\mathbb{N}}$ and the jumps $\{Y_n\}_{n\geq 1}$ are i.i.d. with law f and are independent of $(N_t)_{t\geq 0}$.
Call $E = \mathbb{R}^d \setminus \{0\}$ and $\mathcal{E} = \mathcal{B}(\mathbb{R}^d \setminus \{0\})$: given $A \in \mathcal{E}$ and $t \geq 0$ we recall the random variable

$$N(t, A) = \#\{0 \leq n \leq N_t : Y_n \in A\}.$$

We already know from chapter 3 that for each $A \in \mathcal{E}$ the process $(N(t, A))_{t \geq 0}$ is a Poisson process with intensity given by $\nu(A) = \mathbb{E}[N(1, A)] = \lambda f(A)$. In particular, from proposition 3.5 we know that $(N(t, A))_{t \geq 0}$ is $\mathcal{F}_t$-adapted and has independent increments with respect to $(\mathcal{F}_t)_{t \geq 0}$, so that the compensated process $(M(t, A))_{t \geq 0}$ defined by $M(t, A) = N(t, A) - t\lambda f(A)$ for all $t \geq 0$ is an $\mathcal{F}_t$-martingale.
As in the case of $N(t, A)$, the set function $([0, t], A) \mapsto M(t, A)$ constitutes a random measure on $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{E}$, but it is neither positive nor a counting measure any more. We call it the *compensated Poisson measure* associated with $(Q_t)_{t \geq 0}$. We denote it by $M(dt, dx)$, so that we have

$$M(dt, dx) = N(dt, dx) - \lambda dt \otimes f(dx)$$
$$M((s, t], A) = M(t, A) - M(s, A) \; \forall \; 0 \leq s < t.$$

Now fix $A \in \mathcal{E}$, a finite time horizon T, $p \geq 1$ and recall the σ-algebra $\mathcal{P}_A$ together with the notion of predictability introduced at the end of chapter 4. Define $\mathcal{H}^p(T, A)$ to be the linear space of all equivalence classes of predictable mappings $F : [0, T] \times A \times \Omega \to \mathbb{R}$ which coincide almost everywhere with respect to the measure $l \otimes f \otimes \mathbb{P}$, where l is the Lebesgue measure on $[0, T]$, and which satisfy the condition

$$\int_0^T \int_A \mathbb{E}[|F(t, x)|^p] f(dx) dt < \infty.$$

The space $\mathcal{H}^p(T, A)$ is immediately seen to be a generalisation of $M^p([0, T])$. In particular, for $p = 2$, given the norm

$$\|F\|^2_{\mathcal{H}^2(T,A)} = \int_0^T \int_A \mathbb{E}[|F(t, x)|^2] \lambda f(dx) dt,$$

then $\mathcal{H}^2(T, A)$ is found to be an Hilbert space (see [AP], lemma 4.1.3).
Precisely as goes the procedure for brownian integrals we define the space $S(T, A)$ to be the linear space of all *simple* mappings in $\mathcal{H}^2(T, A)$, where F is simple if for some $m, n \in \mathbb{N}$ there exist $0 \leq t_1 \leq ... \leq t_{m+1} = T$ and a Borel partition $A_1, ..., A_n$ of A such that

$$F(t, x, \omega) = \sum_{j=1}^{m} \sum_{k=1}^{n} c_k 1_{(t_j, t_{j+1}]}(t) 1_{A_k}(x) F_j(\omega),$$

where each $c_k \in \mathbb{R}$ and each F_j is a bounded $\mathcal{F}_{t_j}$-measurable random variable.
Notice that such an F matches the requirements of the definition of predictable mapping. This is in fact the reason why simple mappings are defined

to be left-continuous in time and not right-continuous as we could take them for brownian integrals.
It is possible to show that $S(T,A)$ is dense in $\mathcal{H}^2(T,A)$ (see [AP], lemma 4.1.4).
Now we define the Poisson stochastic integral for $F \in S(T,A)$ to be the integral with respect to the compensated Poisson measure M, that is

$$\begin{aligned} I(F) &= \int_0^T \int_A F(t,x) M(dt,dx) \\ &= \sum_{j=1}^m \sum_{k=1}^n c_k F_j M((t_j,t_{j+1}],A_k). \end{aligned}$$

Notice that this is indeed a pathwise definition, as in the case of the Itô's procedure.
The application I is easily found to be linear; furthermore for all $F \in S(T,A)$ we have $\mathbb{E}[I(F)] = 0$ and

$$\mathbb{E}[I(F)^2] = \int_0^T \int_A \mathbb{E}[|F(t,x)|^2] \lambda f(dx) dt.$$

For the proof see [AP], lemma 4.2.2.
Thus, the application I is a linear isometry from $S(T,A)$ into $L^2(\Omega,\mathcal{F},\mathbb{P})$ and so by the density it extends uniquely to $\mathcal{H}^2(T,A)$. We will denote this extension by I and write for all $F \in \mathcal{H}^2(T,A)$

$$I(F) = \int_0^T \int_A F(t,x) M(dt,dx).$$

It follows by the above argument that for all $F \in \mathcal{H}^2(T,A)$ the random variable $I(F)$ has null expectation and its L^2 norm equals the $\mathcal{H}^2(T,A)$ norm of F.
We will denote $(I(F)_t)_{t\in[0,T]}$ the process given by the Poisson integral up to the time index $t \in [0,T]$: it is possible to show that it is a square-integrable $\mathcal{F}_t$-martingale (see [AP], theorem 4.2.3).
Let now Itô's procedure go on and define $\mathcal{P}^p(T,A)$ for all $p \geq 1$ to be the linear space of all equivalence classes of predictable mappings $F : [0,T] \times A \times \Omega \to \mathbb{R}$ which coincide almost everywhere with respect to the measure $l \otimes f \otimes \mathbb{P}$ and which satisfy the condition

$$\int_0^T \int_A |F(t,x)|^p f(dx) dt < \infty \quad a.s.$$

The space $\mathcal{P}^p(T,A)$ is immediately seen to be a generalisation of $M^p_{loc}([0,T])$: indeed $\mathcal{H}^p(T,A) \subseteq \mathcal{P}^p(T,A)$. We say that a sequence $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}^p(T,A)$ converges to $F \in \mathcal{P}^p(T,A)$ in $\mathcal{P}^p(T,A)$ if (see [AP], section 4.2.2)

$$\mathbb{P}\Big(\lim_n \int_0^T \int_A |F_n(t,x) - F(t,x)|^p f(dx)dt = 0 \Big) = 1$$

It is possible to show that $S(T,A)$ is dense in $\mathcal{P}^2(T,A)$ with respect to this notion of convergence (see [AP], exercise 4.2.7); furthermore, given $F \in S(T,A)$ then for all $C > 0, K \geq 0$ we have

$$\mathbb{P}\Big(\Big| \int_0^T \int_A F(t,x) M(dt,dx) \Big| > C \Big) \leq \frac{K}{C^2} + \mathbb{P}\Big(\int_0^T \int_A |F(t,x)|^2 \lambda f(dx)dt > K \Big).$$

For the proof see [AP], lemma 4.2.8.
Now take $F \in \mathcal{P}^2(T,A)$ and a sequence $\{F_n\}_{n\in\mathbb{N}} \subseteq S(T,A)$ such that $F_n \to F$ in $\mathcal{P}^2(T,A)$. This implies that the sequence given by

$$\Big\{ \int_0^T \int_A |F_n(t,x) - F(t,x)|^2 \lambda f(dx)dt \Big\}_{n\in\mathbb{N}}$$

converges to 0 in probability and thus is a Cauchy sequence in probability. By the above inequality, given any $m,n \in \mathbb{N}, K, \beta > 0$ we can write

$$\mathbb{P}\Big(\Big| \int_0^T \int_A (F_n(t,x) - F_m(t,x)) M(dt,dx) \Big| > \beta \Big) \leq$$
$$\leq \frac{K}{\beta^2} + \mathbb{P}\Big(\int_0^T \int_A |F_n(t,x) - F_m(t,x)|^2 \lambda f(dx)dt > K \Big). \quad (5.1)$$

Now define the mapping $X : \Omega \to L^2([0,T] \times A,\ \mathcal{B}[0,T] \otimes \mathcal{B}(A),\ l \otimes f)$ such that $X(\omega) = F(\cdot,\cdot,\omega)$, and the sequence $\{X_n\}_{n\in\mathbb{N}}$ where $X_n : \Omega \to L^2([0,T] \times A,\ \mathcal{B}[0,T] \otimes \mathcal{B}(A),\ l \otimes f)$ such that $X_n(\omega) = F_n(\cdot,\cdot,\omega)$ for all $n \in \mathbb{N}$.
Consider the metric on $L^2([0,T] \times A, \mathcal{B}[0,T] \otimes \mathcal{B}(A), l \otimes f)$ induced by its natural norm, that is

$$\|g\|_{L^2} = \int_0^T \int_A |g(t,x)|^2 \lambda f(dx)dt$$

for every $g \in L^2([0,T] \times A,\ \mathcal{B}[0,T] \otimes \mathcal{B}(A),\ l \otimes f)$. Then an open ball with center g and range $r > 0$ is

$$B^r_g = \{h \in L^2([0,T] \times A, \mathcal{B}[0,T] \otimes \mathcal{B}(A), l \otimes f) : \|h - g\|_{L^2} < r\}.$$

Take $\mathcal{L}$ to be the σ-algebra generated by the open balls and endow $L^2([0,T] \times A, \mathcal{B}[0,T] \otimes \mathcal{B}(A), l \otimes f)$ with $\mathcal{L}$.

Proposition 5.1. *The mappings X and X_n, $n \in \mathbb{N}$, are random variables. Moreover, $X_n \to X$ in probability.*

Proof. Fix n and any B_g^r and consider its counterimage with respect to X_n: we have

$$\begin{aligned} X_n^{-1}(B_g^r) &= \{\omega \in \Omega : \|X_n(\omega) - g\|_{L^2} < r\} \\ &= \{\omega \in \Omega : \|F_n(\omega) - g\|_{L^2} < r\} \\ &= \left\{\omega \in \Omega : \int_0^T \int_A |F_n(t,x,\omega) - g(t,x)|^2 \lambda f(dx)dt < r^2\right\} \end{aligned}$$

Since F_n is $\mathcal{B}[0,T] \otimes \mathcal{B}(A) \otimes \mathcal{F}$-measurable thanks to proposition 4.4, then by Fubini-Tonelli $X_n^{-1}(B_g^r) \in \mathcal{F}$. This being true for all B_g^r, then X_n is a random variable by proposition 1.2.
The same argument holds for X.
We now show the convergence: since $F_n \to F$ in $\mathcal{P}^2(T,A)$, then for all $\delta > 0$ we have

$$\mathbb{P}(\ \|X_n - X\|_{L^2} > \delta\ ) = \mathbb{P}(\ \|X_n - X\|_{L^2}^2 > \delta^2\ ) \to 0$$

which is the required result.
Notice that the event $\{\|X_n - X\|_{L^2} > \delta\}$ is measurable thanks to the choice of $\mathcal{L}$.

■

Thanks to the above proposition, for all $\beta, \gamma, \epsilon > 0$ there exists $n_0 = n_0(\beta, \gamma, \epsilon)$ such that

$$\mathbb{P}\left(\int_0^T \int_A |F_n(t,x) - F_m(t,x)|^2 \lambda f(dx)dt > \gamma\beta^2\right) < \epsilon \quad \forall n, m \geq n_0.$$

Now choose $K = \gamma\beta^2$ in (5.1) to deduce that the sequence

$$\left\{\int_0^T \int_A F_n(t,x)M(dt,dx)\right\}_{n\in\mathbb{N}}$$

is Cauchy in probability and thus has a limit in probability, which is unique up to almost sure agreement.
We denote this limit by I and write $I(F) = \int_0^T \int_A F(t,x)M(dt,dx)$ so that $I(F_n) \to I(F)$ in probability. Notice that we use the same notation as in the previous case because the integral on $\mathcal{P}^2(T,A)$ extends that on $\mathcal{H}^2(T,A)$: in fact for $\mathcal{H}^2(T,A)$ we have the L^2 convergence which entails the convergence in probability.

We again consider the process $(I(F)_t)_{t\in[0,T]}$ for $F \in \mathcal{P}^2(T, A)$: as in the brownian case it is not necessarily a martingale anymore, but it turns out to be an $\mathcal{F}_t$-local martingale. Furthermore it admits a càdlàg modification (see [AP], theorem 4.2.12), so that we will consider $(I(F)_t)_{t\in[0,T]}$ to be càdlàg by tacit agreement.
It is possible (see [AP], section 4.3.2, page 231) to extend the construction of the Poisson integral to the case where the integrand F no longer lies in $\mathcal{P}^2(T, A)$ but in $\mathcal{P}^1(T, A)$. In this case the resulting process is still a local martingale; in particular it is a martingale provided $F \in \mathcal{H}^1(T, A)$.

Now, in view of the result we show in the following section, we adjust the concepts and notations of the above discussion to a simplified setting.
Suppose $d = 1$, $Y_n \equiv 1$ for all $n \geq 1$ and $1 \in A$ so that $f = \delta_1$ and $Q_t = N(t, A) = N_t$ $\forall t \geq 0$. In this case in the definition of the Poisson integral there only remains the time integral, the integrands $F(t, x, \omega)$ become ordinary stochastic processes $F(t, 1, \omega)$ and the integrator $M(t, A)$ becomes the compensated Poisson process $(M_t)_{t\geq 0}$ such that $M_t = N_t - \lambda t$ for all $t \geq 0$.
Thus, we define for all $p \geq 1$ $\mathcal{H}^p(T)$ $(\mathcal{P}^p(T))$ to be the linear space of all equivalence classes of real valued processes $(F_t)_{t\geq 0}$ which coincide almost everywhere with respect to $l \otimes \mathbb{P}$, which are predictable and such that

$$\int_0^T \mathbb{E}[|F_t|^p]dt < \infty$$

$$\left(\int_0^T |F_t|^p dt < \infty \ a.s.\right)$$

Then we define $S(T)$ to be the linear space of simple processes $(F_t)_{t\geq 0}$, that is for some $n \in \mathbb{N}$ there exist $0 \leq t_1 \leq ... \leq t_{n+1} = T$ such that

$$F(t, \omega) = \sum_{j=1}^{n} F_j(\omega) 1_{(t_j, t_{j+1}]}(t),$$

where each F_j is a bounded $\mathcal{F}_{t_j}$-measurable random variable.
We denote by I the application such that, for $F \in S(T)$,

$$I(F) = \int_0^T F_t M(dt) = \sum_{j=1}^{n} F_j (M_{t_{j+1}} - M_{t_j}).$$

Clearly, all the results given in the above discussion hold true in this simplified case.

5.2 The pathwise definition

As in the previous section let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses.
Take an $\mathbb{R}^d$-valued $\mathcal{F}_t$-compound Poisson process $(Q_t)_{t\geq 0}$, that is $Q_t = \sum_{n=1}^{N_t} Y_n$, where as usual $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ and arrival times $\{\tau_n\}_{n\in\mathbb{N}}$ and the jumps $\{Y_n\}_{n\geq 1}$ are i.i.d. with law f and are independent of $(N_t)_{t\geq 0}$.
Call $E = \mathbb{R}^d \setminus \{0\}$ and $\mathcal{E} = \mathcal{B}(\mathbb{R}^d \setminus \{0\})$.
Fix $A \in \mathcal{E}$, a finite time horizon T and $F \in \mathcal{P}^1(T, A)$.
Let N be the random counting measure associated with $(Q_t)_{t\geq 0}$.
We can now define

$$\begin{aligned}\int_0^T \int_A F(t,x)N(dt,dx) &= \sum_{0\leq t\leq T} F(t, \Delta Q_t)1_A(\Delta Q_t)\\ &= \sum_{n=1}^{N_T} F(\tau_n, Y_n)1_A(Y_n).\end{aligned}$$

Now consider the associated compensated Poisson measure $M = N - \lambda l \otimes f$ and define the application

$$I'(F) = \int_0^T \int_A F(t,x)N(dt,dx) - \int_0^T \int_A F(t,x)\lambda f(dx)dt.$$

Notice that the requirement $F \in \mathcal{P}^1(T, A)$ is necessary for the definition of the last integral, while in principle it could be dropped for the integral in $N(dt, dx)$. Moreover, for the latter to be a well defined and $\mathcal{F}$-measurable random variable we do not even have to require the mapping F to be predictable, but only measurable with respect to $\mathcal{B}([0,T]) \otimes \mathcal{B}(A) \otimes \mathcal{F}$.
The application I' can be seen as a pathwise version of the Poisson stochastic integral: we are now interested in its connection with the object I as we have built it in the previous section.
In fact it can be shown that they coincide, that is

$$I(F) = I'(F) \quad a.s. \quad \forall F \in \mathcal{P}^1(T, A).$$

We are going to prove this result in the simplified case in which $d = 1$, $Y_n \equiv 1$ for all $n \geq 1$, $1 \in A$ and in which the integrands lie in $\mathcal{H}^2(T)$. To do this, we first have to adjust the definition of I' to this case: we simply take, for

$F \in \mathcal{H}^2(T)$,

$$\begin{aligned} I'(F) &= \int_0^T F_t N(dt) - \int_0^T F_t \lambda dt \\ &= \sum_{n=1}^{N_T} F_{\tau_n} - \int_0^T F_t \lambda dt. \end{aligned}$$

We now give a preliminary result.

Proposition 5.2. *The application*

$$I' : \mathcal{H}^2(T) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$$

is an isometry.

Proof. The proof is rather long: we carry it out in three steps.

Let's start by considering $F \in \mathcal{H}^2(T)$ such that $F(t, \omega) = 1_B(\omega) 1_{(u,v]}(t)$ for some $0 \le u \le v \le T$ and $B \in \mathcal{F}_u$. We want to show that $I'(F) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and

$$\mathbb{E}[\ |I'(F)|^2\ ] = \mathbb{E}\bigg[\int_0^T |F_t|^2 \lambda dt \bigg].$$

We only prove the isometry, since it entails the other result.
We have

$$\begin{aligned} \int_0^T F_t N(dt) &= 1_B (N_v - N_u) \\ \int_0^T F_t \lambda dt &= \lambda 1_B (v - u), \end{aligned}$$

so that

$$\begin{aligned} \mathbb{E}[\ |I'(F)|^2\ ] &= \mathbb{E}\bigg[\bigg| \int_0^T F_t N(dt) - \int_0^T F_t \lambda dt \bigg|^2 \bigg] \\ &= \mathbb{E}[\ 1_B |N_v - N_u - \lambda(v-u)|^2\ ]. \end{aligned}$$

Since $B \in \mathcal{F}_u$, by the independence of $N_v - N_u$ and $\mathcal{F}_u$ and by the properties of the conditional expectation we get

$$\mathbb{E}[\ |I'(F)|^2\ ] = \lambda(v-u)\mathbb{P}(B).$$

An easy calculation shows that

$$\mathbb{E}\left[\int_0^T |F_t|^2 \lambda dt\right] = \lambda(v-u)\mathbb{P}(B),$$

so that the first part of the proof is complete.

As a second step we are going to use proposition 1.5.
Define $\mathcal{H}$ to be the linear space of the processes F such that $I'(F) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and

$$\mathbb{E}[\ |I'(F)|^2\ ] = \mathbb{E}\left[\int_0^T |F_t|^2 \lambda dt\right].$$

Take $D = \Omega \times [0,T]$ and

$$\mathcal{C} = \{C \subseteq D : C = B \times (u,v], 0 \leq u \leq v \leq T,\ B \in \mathcal{F}_u\}.$$

Clearly $\mathcal{C}$ is a π-system; furthermore $\mathcal{C}$ it generates the predictable σ-algebra $\mathcal{P}$ on D. From the previous step of the proof we know that $\mathcal{H}$ contains the function 1 and the indicators of the elements of $\mathcal{C}$.
Now we show that, given an increasing sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ of nonnegative functions such that $\lim_n f_n \leq K < \infty$ pointwise on D, then $\lim_n f_n \in \mathcal{H}$. We define $g = \lim_n f_n$ and show that $I'(g) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and

$$\mathbb{E}[|I'(g)|^2] = \mathbb{E}\left[\int_0^T |g_t|^2 \lambda dt\right].$$

We only prove the isometry.
For all n we have

$$\mathbb{E}[\ |I'(f_n)|^2\ ] = \mathbb{E}\left[\int_0^T |f_n(t)|^2 \lambda dt\right]. \tag{5.2}$$

Now we take limits in the above equation.
The right hand side converges to $\mathbb{E}[\int_0^T |g_t|^2 \lambda dt]$ by the dominated convergence theorem.
The left hand side requires some more words: in the following we show that $I'(f_n) \to I'(g)$ in L^2.
To do that it suffices to prove that

1. $\sum_{k\in\mathbb{N}} f_n(\tau_k) 1_{[0,T]}(\tau_k) \to \sum_{k\in\mathbb{N}} g_{\tau_k} 1_{[0,T]}(\tau_k)$ in L^2
2. $\int_0^T f_n(t)dt \to \int_0^T g_t dt$ in L^2

We start by (2): we have to show that $\mathbb{E}[\ |\int_0^T (f_n(t) - g_t)dt|^2\ ]$ vanishes as $n \to \infty$. Since for all ω by the dominated convergence theorem the time integral is such that

$$\int_0^T (f_n(t) - g_t)dt \to 0$$

then applying again the dominated convergence theorem yields the required result.
Let's now consider (1): we have to show that $\mathbb{E}[\ |\sum_{k\in\mathbb{N}} f_n(\tau_k)1_{[0,T]}(\tau_k) - \sum_{k\in\mathbb{N}} g_{\tau_k}1_{[0,T]}(\tau_k)|^2\ ]$ vanishes as $n \to \infty$. By the pointwise convergence of $\{f_n\}_{n\in\mathbb{N}}$ to g and since for almost all ω the interval $[0,T]$ contains a finite number of arrival times, then we get

$$\sum_{k\in\mathbb{N}} f_n(\tau_k)1_{[0,T]}(\tau_k) - \sum_{k\in\mathbb{N}} g_{\tau_k}1_{[0,T]}(\tau_k) \to 0 \quad a.s.$$

Then the result follows thanks to the dominated convergence theorem by noticing that

$$\left|\sum_{k\in\mathbb{N}} f_n(\tau_k)1_{[0,T]}(\tau_k) - \sum_{k\in\mathbb{N}} g_{\tau_k}1_{[0,T]}(\tau_k)\right|^2 \leq 4\left|\sum_{k\in\mathbb{N}} g_{\tau_k}1_{[0,T]}(\tau_k)\right|^2$$

and that

$$\sum_{k\in\mathbb{N}} g_{\tau_k}1_{[0,T]}(\tau_k) = \sum_{k=1}^{N_T} g_{\tau_k} \leq KN_T \in L^2(\Omega, \mathcal{F}, \mathbb{P})$$

Thus we have shown that $I'(f_n) \to I'(g)$ in L^2, so that taking limits in (5.2) yields $g \in \mathcal{H}$.
Thus by the measurability theorem $\mathcal{H}$ contains all predictable bounded real valued processes.

Define $\mathcal{H}^2(T)_b$ to be the set containing the bounded elements of $\mathcal{H}^2(T)$. Thanks to the previous step we know that I' is an isometry from $\mathcal{H}^2(T)_b$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$. But $\mathcal{H}^2(T)_b$ is found to be dense in $\mathcal{H}^2(T)$: indeed for any $F \in \mathcal{H}^2(T)$ define for all $n \in \mathbb{N}$

$$F_n(t,\omega) = \begin{cases} -n & F(t,\omega) < -n \\ F(t,\omega) & |F(t,\omega)| \leq n \\ +n & F(t,\omega) > +n \end{cases}$$

so that we have

$$\begin{aligned}\mathbb{E}\Big[\int_0^T |F_n - F|^2 dt\Big] &= \int_{\Omega\times[0,T]} |F_n - F|^2 \mathbb{P}(d\omega)dt \\ &= \int_{\{(t,\omega):|F(t,\omega)|>n\}} (|F| - n)^2 \mathbb{P}(d\omega)dt \\ &\le \int_{\{(t,\omega):|F(t,\omega)|>n\}} |F|^2 \mathbb{P}(d\omega)dt \to 0\end{aligned}$$

where the last term vanishes thanks to the dominated convergence theorem. Since $\mathcal{H}^2(T)_b$ is dense in $\mathcal{H}^2(T)$, then the isometry I' can be uniquely extended from $\mathcal{H}^2(T)_b$ to $\mathcal{H}^2(T)$, call it $\tilde{I}'$. The proof is complete provided $\tilde{I}'(F) = I'(F)$ for all $F \in \mathcal{H}^2(T)$: we show it in the following.
Take any $F \in \mathcal{H}^2(T)$ and consider the sequence $\{F_n\}_{n\in\mathbb{N}}$ as defined above. Since $F_n \in \mathcal{H}^2(T)_b$ for all $n \in \mathbb{N}$ we have that

$$\tilde{I}'(F_n) = I'(F_n) \quad \forall n \in \mathbb{N}.$$

Now take limits in the above equation.
By definition we get $\tilde{I}'(F_n) \to \tilde{I}'(F)$ in L^2 so that it is possible to find a subsequence converging a.s.
It only remains to show that $I'(F_n) \to I'(F)$ a.s. To do that we recall that by the previous step we have

$$\sum_{k\in\mathbb{N}} f_n(\tau_k) 1_{[0,T]}(\tau_k) - \sum_{k\in\mathbb{N}} g_{\tau_k} 1_{[0,T]}(\tau_k) \to 0 \quad a.s.$$

Then, since $F_n \to F$ in $\mathcal{H}^2(T)$, then we get $F_n \to F$ in $L^2([0,T])$ a.s., so that

$$\int_0^T F_n(t)dt \to \int_0^T F_t dt \quad a.s.$$

Thus we find that $I'(F_n) \to I'(F)$ a.s., and the proof is complete.

■

We are now ready to prove the final result.

Proposition 5.3. *Given any $F \in \mathcal{H}^2(T)$, then $I(F) = I'(F)$ a.s.*

Proof. We already know from the previous section that the application

$$I : \mathcal{H}^2(T) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$$

is an isometry.
Since I and I' are both isometries and $S(T)$ is dense in $\mathcal{H}^2(T)$, it suffices to

show that they coincide on $S(T)$ to prove that they coincide on the whole of $\mathcal{H}^2(T)$: we do it in the following.
Take $F \in S(T)$ so that for some $n \in \mathbb{N}$ and $0 \leq t_1 \leq ... \leq t_{n+1} = T$ we have

$$F(t, \omega) = \sum_{j=1}^{n} F_j(\omega) 1_{(t_j, t_{j+1}]}(t).$$

By the definition of I on $S(T)$ we get

$$I(F) = \sum_{j=1}^{n} F_j \; [\; N_{t_{j+1}} - N_{t_j} - \lambda(t_{j+1} - t_j) \;].$$

Again by definition we have

$$\begin{aligned} I'(F) &= \sum_{i=1}^{N_T} F_{\tau_i} - \int_0^T F_t \lambda dt \\ &= \sum_{i=1}^{N_T} F_{k(i)}(N_{t_{k(i)+1}} - N_{t_{k(i)}}) - \sum_{j=1}^{n} F_j \lambda(t_{j+1} - t_j), \end{aligned}$$

where $k(i)$ is an integer valued random variable such that $t_{k(i)} < \tau_i \leq t_{k(i)+1}$ for all $i \leq N_T$. Since $N_{t_{j+1}} - N_{t_j} = 1$ if $j = k(i)$ for some i and 0 otherwise, the two expressions above are found to coincide.

■

Thus we have shown that the pathwise version and the Itô version of the Poisson stochastic integral coincide in the particular case where the integrator is a Poisson process and the integrands belong to $\mathcal{H}^2(T)$. The arguments above can be repeated for any $t \in [0, T]$, so that the integrals coincide up to any $t \in [0, T]$ and by right-continuity we get that the associated processes are indistinguishable: thus, they share the same properties.
The two definitions of Poisson stochastic integral are found to coincide in the general case where the integrator is a compound Poisson process and integrands belong to $\mathcal{P}^1(T, A)$ for all $A \in \mathcal{E}$ (see [AP], exercise 4.3.2). Thus the associated processes are indistinguishable even in the general case.
It is clear by the pathwise definition that

$$Q_t = \int_0^t \int_E x N(ds, dx) \quad \forall t \in [0, T],$$

while given $F \in \mathcal{P}^1(T, A)$ we have

$$\int_0^t \int_A F(s, x) N(ds, dx) = \sum_{n=1}^{N_t} F(\tau_n, Y_n) 1_A(Y_n) \quad \forall t \in [0, T],$$

so that for each A the process $(\int_0^t \int_A F(s,x)N(ds,dx))_{t\in[0,T]}$ has piecewise constant paths, its jump times are a subset of the jump times of the underlying Poisson process $(N_t)_{t\in[0,T]}$ and its jump sizes are given by the sequence $\{F(\tau_n, Y_n)1_A(Y_n)1_{[0,T]}(\tau_n)\}_{n\geq 1}$.

Remark. As known, a brownian integral process is not in general a brownian motion: integrating generic elements of $M^2_{loc}([0,T])$ does not guarantee in any way the independence of increments.
The same argument holds in our case: a Poisson integral process is not in general a compound Poisson process.

We can now discuss the role of predictability.
Within any theory of stochastic integration it is highly desirable that the stochastic integral of a process against a martingale as integrator should at least be a local martingale: think for instance about the typical case in finance where the integrand is a position in an asset, the integrator is the asset price and the integral is the final gain.
We now give an example concerning stochastic processes which shows that dropping the requirement of predictability entails the loss of the martingale property: to do this we follow [SH], example 11.4.6.
Consider a Poisson process $(N_t)_{t\geq 0}$ with intensity λ and arrival times $\{\tau_n\}_{n\in\mathbb{N}}$. Take the natural filtration of $(N_t)_{t\geq 0}$ and let $(M_t)_{t\geq 0}$ be the associated compensated Poisson process, so that it is a martingale. Let $(F_t)_{t\geq 0}$ be such that

$$F_t = 1_{[0,\tau_1]}(t) \ \forall t \geq 0.$$

Such process is adapted and left-continuous, so that it is predictable.
Now for all $t \geq 0$ we get

$$\int_0^t F_s dM_s = M_{t\wedge\tau_1} = N_{t\wedge\tau_1} - \lambda(t\wedge\tau_1) = 1_{\{\tau_1\leq t\}} - \lambda(t\wedge\tau_1).$$

Since $(M_t)_{t\geq 0}$ is a right-continuous martingale, then by proposition 1.7 the process $(M_{t\wedge\tau_1})_{t\geq 0}$ is a martingale too.
Now define $(F'_t)_{t\geq 0}$ such that

$$F'_t = 1_{[0,\tau_1)}(t) \ \forall t \geq 0.$$

This process is adapted and right-continuous, so that we do not know whether it is predictable or not.
Computing the Poisson integral yields

$$\int_0^t F_s dM_s = -\lambda(t\wedge\tau_1) \ \forall t \geq 0.$$

An easy calculation allows to obtain $\mathbb{E}[-\lambda(t \wedge \tau_1)] = e^{-\lambda t} - 1$, which is strictly decreasing in t so that the process given by the last integral fails to be a martingale. It is neither a local martingale, since for any stopping time $\tau > \tau_1$ we have $t \wedge \tau_1 \wedge \tau = t \wedge \tau_1$.

Chapter 6

Jump-diffusion stochastic calculus

In this chapter we define jump-diffusion processes, which are the generalisation of Itô processes in our framework, and accordingly extend Itô's formula.

6.1 Jump-diffusion processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses.
Let $(B_t)_{t\geq 0}$ be an $\mathbb{R}^r$-valued continuous $\mathcal{F}_t$-brownian motion, $B = (B^1, ..., B^r)$.
Let $(Q_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued $\mathcal{F}_t$-compound Poisson process independent of $(B_t)_{t\geq 0}$, with the usual representation $Q_t = \sum_{n=1}^{N_t} Y_n \ \forall t \geq 0$, where $(N_t)_{t\geq 0}$ is the underlying Poisson process, $\{\tau_n\}_{n\in\mathbb{N}}$ the sequence of its arrival times, $\{Y_n\}_{n\geq 1}$ the sequence of the jump sizes of $(Q_t)_{t\geq 0}$.

Remark. We stress that the requirement of independence between $(B_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ does not entail any loss of generality. Indeed given any brownian motion and any compound Poisson process both with respect to the same filtration, they are always found to be independent (see [IW], theorem 6.3).

Let $E = \mathbb{R}^d \setminus \{0\}$, $\mathcal{E} = \mathcal{B}(E)$: we associate to $(Q_t)_{t\geq 0}$ the random counting measure N as in the previous chapters.
Fix a finite time horizon T, $m \in \mathbb{N}$ and the indexes $i \in \{1, ..., m\}$, $j \in \{1, ..., r\}$.

Definition 6.1. *We say that an $\mathbb{R}^m$-valued $\mathcal{F}_t$-adapted càdlàg process $(X_t)_{t\geq 0}$ is a jump-diffusion process on $[0,T]$ provided there exist an $\mathcal{F}_0$-measurable random variable X_0, $F = (F^1, ..., F^m)$, $F \in M^1_{loc}([0,T])$, $G = (G^{ij})_{ij}$, $G \in M^2_{loc}([0,T])$, $1 \leq i \leq m$, $1 \leq j \leq r$, $K = (K^1, ..., K^m)$, K predictable, such* *that*

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t G_s dB_s + \int_0^t \int_E K(s,y)N(ds,dy) \quad \forall t \in [0,T] \quad a.s.$$

Equivalently we say that $(X_t)_{t\geq 0}$ has stochastic differential given by

$$dX_t = F_t dt + G_t dB_t + \int_E K(t,y)N(dt,dy) \quad t \in [0,T].$$

From now on we will consider $(X_t)_{t\geq 0}$ to be a jump-diffusion process as defined above.
Notice that $(X_t)_{t\geq 0}$ jumps at a given time t only if $(Q_t)_{t\geq 0}$ does, but it does not necessarily jump every time $(Q_t)_{t\geq 0}$ does because K can vanish.
Since $(X_t)_{t\geq 0}$ is $\mathcal{F}_t$-adapted and càdlàg, then it is $\mathcal{F}_t$-progressively measurable.
We can divide the continuous part of $(X_t)_{t\geq 0}$, call it $(X_t^c)_{t\geq 0}$, from its jump part, call it $(X_t^j)_{t\geq 0}$, so that for all $t \in [0,T]$

$$X_t^c = X_0 + \int_0^t F_s ds + \int_0^t G_s dB_s$$
$$X_t^j = \int_0^t \int_E K(s,y)N(ds,dy).$$

Of course both $(X_t^c)_{t\geq 0}$ and $(X_t^j)_{t\geq 0}$ are $\mathcal{F}_t$-progressively measurable.

6.2 Itô's formula for jump-diffusion processes

In the following we establish Itô's formula for jump-diffusion processes.

Remark. In literature, such a result is commonly proved by means of the so called interlacing technique, which consists in splitting a jump-diffusion process in its continuous and jump part, using Itô's formula for continuous processes (i.e. as stated in theorem 1.10) between consecutive arrival times τ_n and τ_{n+1} and then adding the contribution given by the jump part.
This is essentially what we do. But there is a problem: the second step of the above procedure cannot be managed by an immediate application of theorem 1.10, for two reasons.
First, the process $(X_t)_{t\geq 0}$ is not an Itô process over the whole of $[0,T]$. If it was, then we could apply Itô's formula so that the process $(h(X_t))_{t\geq 0}$ would also be an Itô process on $[0,T]$. Then the required result would follow by evaluating this process between τ_n and τ_{n+1}.

Anyway a jump-diffusion process is not an Itô process on $[0,T]$, but it is "Itô" only between τ_n and τ_{n+1}. Then one could hope to get Itô's formula between τ_n and τ_{n+1} by considering the partition $\{\Omega_t\}_{t\geq 0}$ of Ω, where $\Omega_t = \{\omega \in \Omega : \tau_n = t\}$, and to work separately on each of those sets. But since in fact this could be an uncountable partition and for all $t \geq 0$ we could have $\mathbb{P}(\Omega_t) = 0$, then it is not possible to use directly theorem 1.10 anymore. Moreover, we could neither use it if τ_n was discrete valued, because even in that case its hypotheses are not fulfilled.
As said in the introduction, in the appendix we provide a way to prove Itô's formula by means of discrete valued stopping times, but here we use another approach.

Since the argument is rather long, we split it in two steps: first we have to extend theorem 1.10 to the case in which we integrate from any arrival time to the next.
We start by setting F and G to be null on (T,∞), so that they are still progressively measurable and furthermore they satisfy the respective integrability conditions on every compact subset of the time line.
Let τ be an $\mathcal{F}_t$-stopping time such that $\tau \leq T$, introduce the τ-changed filtration $(\mathcal{F}_u^\tau)_{u\geq 0}$ and the τ-changed stochastic processes $F_u^\tau = F_{\tau+u}$, $G_u^\tau = G_{\tau+u}$ for all $u \geq 0$. Thus the processes $(F_u^\tau)_{u\geq 0}$, $(G_u^\tau)_{u\geq 0}$ are $\mathcal{F}_u^\tau$-progressively measurable thanks to proposition 1.15 and by the above argument we get

$$(F_u^\tau)_{u\geq 0} \in M^1_{loc}([0,\infty);\mathcal{F}_u^\tau),\ (G_u^\tau)_{u\geq 0} \in M^2_{loc}([0,\infty);\mathcal{F}_u^\tau). \tag{6.1}$$

Define $\tau_n' = \tau_n \wedge T$ for all $n \in \mathbb{N}$.

Proposition 6.1. *Let $(X_t)_{t\geq 0}$ be a jump-diffusion process in $[0,T]$, $h : \mathbb{R}^m \to \mathbb{R}$, $h \in C^2(\mathbb{R}^m)$. Then for all $n \in \mathbb{N}$*

$$\begin{aligned} h(X_{\tau_n'+u}) = h(X_{\tau_n'}) &+ \int_{\tau_n'}^{\tau_n'+u} \sum_{i=1}^m h_{x_i}(X_s) F_s^i ds \\ &+ \int_{\tau_n'}^{\tau_n'+u} \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \sum_{j=1}^r h_{x_i x_l}(X_s) G_s^{ij} G_s^{lj} ds \\ &+ \int_{\tau_n'}^{\tau_n'+u} \sum_{i=1}^m \sum_{j=1}^r h_{x_i}(X_s) G_s^{ij} dB_s^j \quad \forall u \in [0, \tau_{n+1}' - \tau_n') \quad a.s. \end{aligned}$$

Proof. Fix $n \in \mathbb{N}$. We have

$$X_t = \begin{cases} X_t & 0 \leq t \leq \tau_n' \\ X_t^c + X_{\tau_n'} - X_{\tau_n'}^c & \tau_n' < t < \tau_{n+1}'. \end{cases}$$

Define the process $(X_t^n)_{t\geq 0}$ such that

$$X_t^n = \begin{cases} 0 & 0 \leq t < \tau_n' \\ X_t^c + X_{\tau_n'} - X_{\tau_n'}^c & \tau_n' \leq t \leq T \\ X_T^n & t > T \end{cases}$$

Such process is $\mathcal{F}_t$-adapted, in fact given $t \geq 0$ and $C \in \mathbb{R}^m$ we can write

$$\begin{aligned}\{X_t^n \in C\} &= \{X_t^n \in C,\ 0 \leq t < \tau_n'\} \cup \{X_t^n \in C,\ \tau_n' \leq t \leq T\} \cup \{X_t^n \in C,\ t > T\} \\ &= \{0 \in C,\ 0 \leq t < \tau_n'\} \cup \{X_t^c + X_{\tau_n'} - X_{\tau_n'}^c \in C,\ \tau_n' \leq t \leq T\} \\ &\cup \{X_T^n \in C,\ t > T\}.\end{aligned}$$

The first event clearly belongs to $\mathcal{F}_t$, so does the second because of the progressive measurability of the processes $(X_t)_{t\geq 0}$, $(X_t^c)_{t\geq 0}$ and this entails that the third does too.
Since $(X_t^n)_{t\geq 0}$ is right-continuous by definition, it is also $\mathcal{F}_t$-progressively measurable.
Consider the process $(B_u^{\tau_n'})_{u\geq 0}$: thanks to proposition 1.14 it is a $\mathcal{F}_u^{\tau_n'}$-brownian motion.
Fix $u \geq 0$: by definition we get

$$X_{\tau_n'+u}^n = X_{\tau_n'}^n + \int_{\tau_n'}^{\tau_n'+u} F_s ds + \int_{\tau_n'}^{\tau_n'+u} G_s dB_s \quad a.s.$$

Define the process $(Z_v)_{v\geq 0}$ such that $Z_v = X_{\tau_n'+v}^n$ $\forall v \geq 0$. Then proposition 1.15 and 1.16 allow to rewrite the above equation as

$$Z_u = Z_0 + \int_0^u F_v^{\tau_n'} dv + \int_0^u G_v^{\tau_n'} dB_v^{\tau_n'} \quad a.s.$$

Since u was fixed arbitrarily, by the continuity of $(Z_v)_{v\geq 0}$ and by (6.1) we get that $(Z_u)_{u\geq 0}$ is an Itô process on every compact subset of the time line. Thus we can apply theorem 1.10 and obtain

$$\begin{aligned} h(Z_u) = h(Z_0) &+ \int_0^u \sum_{i=1}^m h_{x_i}(Z_v) F_v^{\tau_n'\ i} dv \\ &+ \int_0^u \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \sum_{j=1}^r h_{x_i x_l}(Z_v) G_v^{\tau_n'\ ij} G_v^{\tau_n'\ lj} dv \\ &+ \int_0^u \sum_{i=1}^m \sum_{j=1}^r h_{x_i}(Z_v) G_v^{\tau_n'\ ij} dB_v^{\tau_n'} \quad \forall u \geq 0 \quad a.s. \end{aligned}$$

Since the processes $(X_t^n)_{t\geq 0}$, $(F_t)_{t\geq 0}$, $(G_t)_{t\geq 0}$ are $\mathcal{F}_t$-progressively measurable and h is C^2-continuous, then the processes

$$(h_{x_i}(X_t^n)F_t^i)_{t\geq 0},\ (h_{x_i,x_l}(X_t^n)G_t^{ij}G_t^{lj})_{t\geq 0},\ (h_{x_i}(X_t^n)G_t^{ij})_{t\geq 0}$$

are $\mathcal{F}_t$-progressively measurable for all $i,\ l \in \{1,...,m\}$, $j \in \{1,...,r\}$.
Since the process $(X_t^n)_{t\geq 0}$ is right-continuous and h is C^2-continuous, then the processes

$$(h_{x_i}(X_t^n))_{t\geq 0},\ (h_{x_i,x_l}(X_t^n))_{t\geq 0}$$

are a.s. continuous except for τ_n', where there is a jump discontinuity. Thus they admit finite maximum and minimum on the compact subsets of the time line a.s., so that

$$(h_{x_i}(X_t^n)F_t^i)_{t\geq 0},\ (h_{x_i,x_l}(X_t^n)G_t^{ij}G_t^{lj})_{t\geq 0} \in M_{loc}^1([0,\infty))$$
$$(h_{x_i}(X_t^n)G_t^{ij})_{t\geq 0} \in M_{loc}^2([0,\infty))$$

for all $i,\ l \in \{1,...,m\}$, $j \in \{1,...,r\}$.
Thus, thanks to proposition 1.16 the last equation can be written as

$$\begin{aligned} h(X^n_{\tau_n'+u}) = h(X^n_{\tau_n'}) &+ \int_{\tau_n'}^{\tau_n'+u} \sum_{i=1}^m h_{x_i}(X_s^n)F_s^i ds \\ &+ \int_{\tau_n'}^{\tau_n'+u} \frac{1}{2}\sum_{i=1}^m\sum_{l=1}^m\sum_{j=1}^r h_{x_ix_l}(X_s^n)G_s^{ij}G_s^{lj} ds \\ &+ \int_{\tau_n'}^{\tau_n'+u} \sum_{i=1}^m\sum_{j=1}^r h_{x_i}(X_s^n)G_s^{ij}dB_s^j \quad \forall u \geq 0 \quad a.s. \end{aligned}$$

Since by definition $X_{\tau_n'+u} = X^n_{\tau_n'+u}$ $\forall u \in [0, \tau_{n+1}' - \tau_n')$, then by a localization argument we get the desired result.

■

We are now ready to establish Itô's formula for jump-diffusion processes.

Theorem 6.2 (Itô's formula for jump-diffusion processes)**.** *Let* $(X_t)_{t\geq 0}$ *be a jump-diffusion process on* $[0,T]$ *and* $h \in C^2(\mathbb{R}^m)$ *a real valued function.*

Then

$$h(X_t) = h(X_0) + \int_0^t \sum_{i=1}^m h_{x_i}(X_s) F_s^i ds$$
$$+ \int_0^t \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \sum_{j=1}^r h_{x_i x_l}(X_s) G_s^{ij} G_s^{lj} ds$$
$$+ \int_0^t \sum_{i=1}^m \sum_{j=1}^r h_{x_i}(X_s) G_s^{ij} dB_s^j$$
$$+ \int_0^t \int_E \Big[h(X_{s^-} + K(s,y)) - h(X_{s^-})\Big] N(ds, dy) \quad \forall t \in [0, T] \quad a.s.$$

Proof. Fix $t \in [0, T]$ and write

$$\begin{aligned} h(X_t) - h(X_0) &= \sum_{n\in\mathbb{N}} h(X_{t\wedge\tau_{n+1}}) - h(X_{t\wedge\tau_n}) \\ &= \sum_{n\in\mathbb{N}} h(X_{(t\wedge\tau_{n+1})^-}) - h(X_{t\wedge\tau_n}) \\ &+ \sum_{n\in\mathbb{N}} h(X_{t\wedge\tau_{n+1}}) - h(X_{(t\wedge\tau_{n+1})^-}). \end{aligned} \tag{6.2}$$

Define

$$K'(s, y, \omega) = h(X_{s^-}(\omega) + K(s, y, \omega)) - h(X_{s^-}(\omega)).$$

Then by the definition of Poisson integral we get

$$\begin{aligned} \sum_{n\in\mathbb{N}} h(X_{t\wedge\tau_{n+1}}) - h(X_{(t\wedge\tau_{n+1})^-}) &= \sum_{n=1}^{N_t} h(X_{\tau_n}) - h(X_{\tau_n^-}) \\ &= \sum_{n=1}^{N_t} h(X_{\tau_n^-} + K(\tau_n, Y_n) 1_E(Y_n)) - h(X_{\tau_n^-}) \\ &= \sum_{n=1}^{N_t} \big[h(X_{\tau_n^-} + K(\tau_n, Y_n)) - h(X_{\tau_n^-})\big] 1_E(Y_n) \\ &= \sum_{n=1}^{N_t} K'(\tau_n, Y_n) 1_E(Y_n) \\ &= \int_0^t \int_E K'(s, y) N(ds, dy) \\ &= \int_0^t \int_E \big[h(X_{s^-} + K(s, y)) - h(X_{s^-})\big] N(ds, dy) \quad a.s. \end{aligned}$$

Proposition 6.1 allows to write

$$\sum_{n\in\mathbb{N}} h(X_{(t\wedge\tau_{n+1})^-}) - h(X_{t\wedge\tau_n}) = \int_0^t \sum_{i=1}^m h_{x_i}(X_s)F_s^i ds$$
$$+ \int_0^t \frac{1}{2}\sum_{i=1}^m\sum_{l=1}^m\sum_{j=1}^r h_{x_ix_l}(X_s)G_s^{ij}G_s^{lj}ds$$
$$+ \int_0^t \sum_{i=1}^m\sum_{j=1}^r h_{x_i}(X_s)G_s^{ij}dB_s^j \quad a.s.$$

so that by (6.2) we finally get

$$h(X_t) = h(X_0) + \int_0^t \sum_{i=1}^m h_{x_i}(X_s)F_s^i ds + \int_0^t \frac{1}{2}\sum_{i=1}^m\sum_{l=1}^m\sum_{j=1}^r h_{x_ix_l}(X_s)G_s^{ij}G_s^{lj}ds$$
$$+ \int_0^t \sum_{i=1}^m\sum_{j=1}^r h_{x_i}(X_s)G_s^{ij}dB_s^j$$
$$+ \int_0^t\int_E \Big[h(X_{s^-} + K(s,y)) - h(X_{s^-})\Big]N(ds,dy) \quad a.s.$$

and the required result follows by the right continuity of the sample paths.

■

Chapter 7

Stochastic differential equations for jump-diffusion processes

In this chapter we present SDEs driven by brownian motion and compound Poisson process and give a result concerning existence and uniqueness of solutions.

7.1 Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses.
Let $(B_t)_{t\geq 0}$ be an $\mathbb{R}^r$-valued $\mathcal{F}_t$-brownian motion, $B = (B^1, ..., B^r)$.
Let $(Q_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued $\mathcal{F}_t$-compound Poisson process independent of $(B_t)_{t\geq 0}$, with the usual representation $Q_t = \sum_{n=1}^{N_t} Y_n \; \forall t \geq 0$, where $(N_t)_{t\geq 0}$ is the underlying Poisson process, $\{\tau_n\}_{n\in\mathbb{N}}$ the sequence of its arrival times, $\{Y_n\}_{n\geq 1}$ the sequence of the jump sizes of $(Q_t)_{t\geq 0}$.
Let $E = \mathbb{R}^d \setminus \{0\}$, $\mathcal{E} = \mathcal{B}(E)$: we associate to $(Q_t)_{t\geq 0}$ the random counting measure N as in the previous chapters.
Let $b : \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times r}$, $G : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^m$ be Borel measurable mappings. Let ξ be an $\mathcal{F}_0$-measurable $\mathbb{R}^m$-valued random variable.

Definition 7.1. *A process $(X_t)_{t\in[0,T]}$ is said to be a solution to*

$$\begin{cases} X_t = X_0 + \int_0^t b(X_{s^-})ds + \int_0^t \sigma(X_{s^-})dB_s + \int_0^t \int_E G(X_{s^-}, y)N(ds, dy) \quad t \in [0, T] \\ X_0 = \xi \end{cases} \tag{7.1}$$

if it is a jump-diffusion process such that $b(X_{s^-}) \in M^1_{loc}([0, T])$, $\sigma(X_{s^-}) \in M^2_{loc}([0, T])$, $G(X_{s^-}, y)$ $\mathcal{P}_{\mathbb{R}^m}$-measurable, and if it satisfies both the above

equalities.
A solution is said to be unique if it is indistinguishable from any other solution.

The final goal of this chapter is to find an existence and uniqueness result for solutions to (7.1).

Remark. As said in chapter 1 for theorem 1.11, since the processes $(B_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ are fixed and given as data, the above definition of solution matches the notion of what literature calls strong solution.

7.2 Existence and uniqueness theorem

In this section we establish an existence and uniqueness result for solutions to (7.1).
We will split the analysis in steps: first we extend theorem 1.11 to the case where the initial condition is not square-integrable and the starting time is a stopping time, then we build a solution to (7.1) and show that it is unique.
From now on we assume that b, σ are lipschitz continuous and that $x \mapsto G(x, y)$ is continuous for all $y \in E$.

The following result generalises theorem 1.11 to the case where the initial condition is not square-integrable.

Proposition 7.1. *Let $v \geq 0$ and ξ be an $\mathbb{R}^m$-valued $\mathcal{F}_v$-measurable random variable. Then there exists a unique solution to*

$$\begin{cases} X_t = X_v + \int_v^t b(X_s)ds + \int_v^t \sigma(X_s)dB_s \quad t \in [v, T] \\ X_v = \xi \end{cases} \tag{7.2}$$

The solution is $\mathcal{F}_t$-adapted and continuous.

Proof. We prove the thesis for $v = 0$.
The proof is based on the argument given in [AP], theorem 6.2.3. We carry it out in two steps: first we build a solution and show that it is $\mathcal{F}_t$-adapted and continuous, then we verify that it is unique.

Define $\Omega_n = \{\omega \in \Omega : |\xi| \leq n\}$ $\forall n \in \mathbb{N}$. We have that $\Omega_n \subseteq \Omega_{n+1}$ $\forall n \in \mathbb{N}$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$.
Define $\xi_n = \xi 1_{\Omega_n}$ $\forall n \in \mathbb{N}$: since $\xi_n \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ $\forall n \in \mathbb{N}$, then the equation

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s \quad t \in [0, T] \\ X_0 = \xi_n \end{cases} \tag{7.3}$$

has a unique solution $(X_t^n)_{t\geq 0}$ $\mathcal{F}_t$-adapted and continuous by theorem 1.11 for all $n \in \mathbb{N}$.
Fix $m \in \mathbb{N}$. Proposition 1.12 gives

$$X_t^m = X_t^n \quad \forall t \in [0,T] \; a.s. \; in \; \Omega_m \quad \forall n \geq m. \tag{7.4}$$

Define $A_{m,n} = \{\omega \in \Omega_m : X_t^m(\omega) = X_t^n(\omega) \; \forall t \in [0,T]\}$ and $N_{m,n} = \Omega_m \setminus A_{m,n}$. By (7.4) we find $\mathbb{P}(N_{m,n}) = 0 \; \forall n \geq m$ and thus, defining $N_m = \bigcup_{n\geq m} N_{m,n} \; \forall m \in \mathbb{N}$, we find $\mathbb{P}(\bigcup_{m\in\mathbb{N}} N_m) = 0$.
(7.4) also implies that

$$X_t^m(\omega) = X_t^n(\omega) \quad \forall t \in [0,T] \quad \forall n \geq m \quad \forall \omega \in \Omega_m \setminus N_m \quad \forall m \in \mathbb{N}.$$

This means that $\forall \omega \in \Omega_m \setminus N_m$ the sequence $\{X_t^n(\omega)\}_{n\geq m}$ is constant for all $t \in [0,T]$ and thus it converges uniformly in $t \in [0,T]$.
Define $\tilde{\Omega} = \bigcup_{m\in\mathbb{N}} \Omega_m \setminus N_m$: on $\tilde{\Omega}$ the sequence $\{X_t^n\}_{n\in\mathbb{N}}$ is definitively constant so that it converges uniformly in $t \in [0,T]$. Since $\bigcup_{m\in\mathbb{N}} \Omega_m \setminus \bigcup_{m\in\mathbb{N}} N_m \subseteq \tilde{\Omega}$ we find that

$$\mathbb{P}(\tilde{\Omega}) \geq \mathbb{P}(\bigcup_{m\in\mathbb{N}} \Omega_m) - \mathbb{P}(\bigcup_{m\in\mathbb{N}} N_m) = 1,$$

so that $\{X_t^n\}_{n\in\mathbb{N}}$ converges a.s. uniformly in $t \in [0,T]$.
Define the process $(X_t)_{t\geq 0}$ such that

$$X_t(\omega) = \lim_n X_t^n(\omega) \quad \forall t \in [0,T] \quad \forall \omega \in \tilde{\Omega}.$$

Since it is defined by a uniform convergence, $(X_t)_{t\geq 0}$ is $\mathcal{F}_t$-adapted and continuous.
Since for all $n \in \mathbb{N}$ the process $(X_t^n)_{t\geq 0}$ solves (7.3), then we can write

$$X_t^n = \xi_n + \int_0^t b(X_s^n)ds + \int_0^t \sigma(X_s^n)dB_s \quad t \in [0,T] \quad a.s. \quad \forall n \in \mathbb{N} \tag{7.5}$$

Now we take limits in (7.5).
Of course $\xi_n \to \xi$ a.s.
Fix $t \in [0,T]$: the uniform convergence implies $X_t^n \to X_t$ a.s. and together with the lipschitz continuity allows to write

$$\begin{aligned} \int_0^t |b(X_s^n) - b(X_s)|ds &\leq L \int_0^t |X_s^n - X_s|ds \\ &\leq LT \sup_{s\in[0,T]} |X_s^n - X_s| \to 0 \quad a.s. \end{aligned}$$

so that $\int_0^t b(X_s^n)ds \to \int_0^t b(X_s)ds$ a.s.
The uniform convergence, the lipschitz continuity and proposition 1.9 allow to write

$$\sup_{s\in[0,T]} \left| \int_0^s \sigma(X_u^n)dB_u - \int_0^s \sigma(X_u)dB_u \right| \to 0 \text{ in probability}$$

so that there exists a subsequence $\{X_t^{n_k}\}_{k\in\mathbb{N}}$ such that $\int_0^t \sigma(X_s^{n_k})dB_s \to \int_0^t \sigma(X_s)dB_s$ a.s.
Thus we can consider limits in (7.5) as $k \to \infty$ and get

$$X_t = \xi + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s \quad a.s.$$

Since t was fixed arbitrarily and by the continuity of the processes in the above equation we obtain the required result.

Now we prove that the solution is unique up to indistinguishability: let $(\hat{X}_t)_{t\geq 0}$ be another solution to (7.2).
We start by showing that, given $m \in \mathbb{N}$, we have

$$\hat{X}_t = X_t^n \quad \forall t \in [0,T] \ a.s.\ in\ \Omega_m \quad \forall n \geq m.$$

Take $n \geq m$ and define the process $(Z_t)_{t\geq 0}$ such that for all $t \in [0,T]$

$$Z_t = \begin{cases} \hat{X}_t & \omega \in \Omega_m \\ X_t^n & \omega \notin \Omega_m \end{cases}$$

Thus we have

$$\begin{aligned} Z_t &= \hat{X}_t 1_{\Omega_m} + X_t^n 1_{\Omega_m^c} \\ &= \xi 1_{\Omega_m} + 1_{\Omega_m} \int_0^t b(\hat{X}_s)ds + 1_{\Omega_m} \int_0^t \sigma(\hat{X}_s)dB_s \\ &+ \xi_n 1_{\Omega_m^c} + 1_{\Omega_m^c} \int_0^t b(X_s^n)ds + 1_{\Omega_m^c} \int_0^t \sigma(X_s^n)dB_s \quad \forall t \in [0,T] \quad a.s. \end{aligned}$$

Since $\xi 1_{\Omega_m} = \xi_n 1_{\Omega_m}$, then by the definition of $(Z_t)_{t\geq 0}$ and proposition 1.8 the above equation can be written as

$$Z_t = \xi_n + \int_0^t b(Z_s)ds + \int_0^t \sigma(Z_s)dB_s \quad \forall t \in [0,T] \quad a.s.$$

This means that the processes $(Z_t)_{t\geq 0}$, $(X_t^n)_{t\geq 0}$ solve (7.3) with the same initial condition ξ_n, so that by theorem 1.11 they are indistinguishable: the

required result follows by the definition of $(Z_t)_{t\geq 0}$.
By construction the processes $(X_t)_{t\geq 0}$, $(X_t^n)_{t\geq 0}$ are indistinguishable on Ω_m and by the above argument $(\hat{X}_t)_{t\geq 0}$, $(X_t^n)_{t\geq 0}$ are indistinguishable on Ω_m: hence $(\hat{X}_t)_{t\geq 0}$, $(X_t)_{t\geq 0}$ are indistinguishable on Ω_m, and the thesis follows since $\Omega = \bigcup_{m\in\mathbb{N}} \Omega_m$.

■

Let now τ be an $\mathcal{F}_t$-stopping time such that $\tau \leq T$ and $(\mathcal{F}_u^\tau)_{u\geq 0}$ be the τ-changed filtration.
We give some preliminary results.

Proposition 7.2. *Let W be an $\mathcal{F}_\tau$-measurable random variable. Then $W1_{\{\tau=t\}}$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.*

Proof. Notice that if τ is a continuous random variable then the result follows immediately from the completeness of $(\mathcal{F}_t)_{t\geq 0}$.
For the general case consider $C \in \mathcal{B}(\mathbb{R}^m)$: since W is $\mathcal{F}_\tau$-measurable we have $\{W \in C\} \in \mathcal{F}_\infty$ and

$$\{W \in C,\ \tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0$$

so that $\{W \in C,\ \tau = t\} = \{W \in C,\ \tau \leq t,\ \tau = t\} \in \mathcal{F}_t$ $\forall t \geq 0$.
Fix $t \geq 0$: if $0 \notin C$ then we have

$$\{W1_{\{\tau=t\}} \in C\} = \{W \in C,\ \tau = t\} \in \mathcal{F}_t.$$

On the other hand, if $0 \in C$ we get

$$\begin{aligned}\{W1_{\{\tau=t\}} \in C\} &= \{W1_{\{\tau=t\}} \in C \setminus 0\} \cup \{W1_{\{\tau=t\}} = 0\} \\ &= \{W \in C \setminus 0,\ \tau = t\} \cup \{W = 0\} \cup \{\tau \neq t\} \\ &= \{W \in C \setminus 0,\ \tau = t\} \cup \{W = 0, \tau = t\} \\ &\cup \{W = 0,\ \tau \neq t\} \cup \{\tau \neq t\} \\ &= \{W \in C,\ \tau = t\} \cup \{\tau \neq t\}.\end{aligned}$$

Since

$$\{\tau \neq t\} = \{\tau < t\} \cup \{\tau > t\} = \bigcup_{n\in\mathbb{N}} \left\{\tau \leq t - \frac{1}{n}\right\} \cup \{\tau \leq t\}^c,$$

then $\{\tau \neq t\} \in \mathcal{F}_t$ and we get the required result.

■

Proposition 7.3. *The random variable $(t - \tau) \vee 0$ is an $\mathcal{F}_u^\tau$-stopping time for all $t \geq 0$.*

Proof. Fix $t, u \geq 0$ and write

$$\begin{aligned}\{(t-\tau)\vee 0 \leq u\} &= \{t-\tau \leq u,\ \tau \leq t\} \cup \{u \geq 0,\ \tau > t\} \\ &= \{\tau + u \geq t,\ \tau \leq t\} \cup \{\tau > t\}.\end{aligned}$$

Since $\tau + u$ is an $\mathcal{F}_t$-stopping time and since $\{\tau \leq t\},\ \{\tau > t\} \in \mathcal{F}_\tau = \mathcal{F}_0^\tau$, then we get the required result.

■

The next result extends proposition 7.1 to the case where the starting time is a stopping time $\tau \leq T$.
Let ξ be an $\mathbb{R}^m$-valued $\mathcal{F}_\tau$-measurable random variable and consider another finite time horizon U.

Definition 7.2. *An $\mathcal{F}_t$-progressively measurable process $(X_t)_{t\in[0,T+U]}$ is said to be a solution to*

$$\begin{cases} X_{\tau+u} = X_\tau + \displaystyle\int_\tau^{\tau+u} b(X_s)ds + \int_\tau^{\tau+u} \sigma(X_s)dB_s \quad u \in [0,U] \\ X_\tau = \xi \end{cases} \tag{7.6}$$

if it is $\mathcal{F}_t$-adapted and such that

$$X_t = \xi + \int_\tau^t b(X_s)ds + \int_\tau^t \sigma(X_s)dB_s \quad \tau \leq t \leq \tau + U$$

Notice that in definition 7.2 we implicitly require that $b(X_t)1_{[\tau,\tau+U]}(t) \in M^1_{loc}([0, T+U])$ and $\sigma(X_t)1_{[\tau,\tau+U]}(t) \in M^2_{loc}([0, T+U])$.

Proposition 7.4. *Let τ be an $\mathcal{F}_t$-stopping time such that $\tau \leq T$ and let ξ be an $\mathbb{R}^m$-valued $\mathcal{F}_\tau$-measurable random variable.*
Then there exists a unique (up to indistinguishability) process $(X_t)_{t\in[0,T+U]}$ such that

$$X_t = \begin{cases} 0 & 0 \leq t < \tau \\ \xi + \int_\tau^t b(X_s)ds + \int_\tau^t \sigma(X_s)dB_s & \tau \leq t \leq \tau + U \\ X_{\tau+U} & \tau + U < t \leq T + U \end{cases}$$

The process $(X_t)_{t\in[0,T+U]}$ is càdlàg and solves (7.6).

Proof. Consider the τ-changed $\mathcal{F}_u^\tau$-brownian motion $(B_u^\tau)_{u\geq 0}$ and the equation

$$\begin{cases} Z_u = Z_0 + \displaystyle\int_0^u b(Z_s)ds + \int_0^u \sigma(Z_s)dB_s^\tau \quad u \in [0,U] \\ Z_0 = \xi \end{cases} \tag{7.7}$$

Since ξ is $\mathcal{F}_0^\tau$-measurable, then (7.7) admits a unique solution $(Z_u)_{u\in[0,U]}$ thanks to proposition 7.1; such solution is $\mathcal{F}_u^\tau$-adapted and continuous, thus it is also $\mathcal{F}_u^\tau$-progressively measurable.
Let us continue such solution beyond U by setting $Z_u = Z_U \ \forall u > U$ so to obtain $(Z_u)_{u\geq 0}$: the new process remains $\mathcal{F}_u^\tau$-adapted and continuous and thus $\mathcal{F}_u^\tau$-progressively measurable.
Define the process $(X_t)_{t\geq 0}$ such that

$$X_t = \begin{cases} 0 & 0 \leq t < \tau \\ Z_{t-\tau} & t \geq \tau \end{cases}$$

so that $X_{\tau+u} = Z_u \ \forall u \geq 0$.
Now we consider $(X_t)_{t\in[0,T+U]}$ and show that it is $\mathcal{F}_t$-adapted: fix $t \in [0, T+U]$, $C \in \mathcal{B}(\mathbb{R}^m)$ and write

$$\begin{aligned}\{X_t \in C\} &= \{X_t \in C,\ \tau > t\} \cup \{X_t \in C,\ \tau \leq t\} \\ &= \{0 \in C,\ \tau > t\} \cup \{Z_{t-\tau} \in C,\ \tau \leq t\}.\end{aligned}$$

Clearly $\{0 \in C,\ \tau > t\} \in \mathcal{F}_t$; for the other event we notice that $\{\tau \leq t\} = \{\tau \vee t = t\}$ and write

$$\{Z_{t-\tau} \in C,\ \tau \leq t\} = \{Z_{(t-\tau)\vee 0} 1_{\{\tau\vee t = t\}} \in C,\ \tau \leq t\},$$

so that it belongs to $\mathcal{F}_t$ thanks to the progressive measurability of $(Z_u)_{u\geq 0}$, proposition 1.6 and propositions 7.2 and 7.3.
Since the process $(X_t)_{t\in[0,T+U]}$ is also right-continuous, then it is also $\mathcal{F}_t$-progressively measurable.
Now rewrite (7.7) as follows

$$\begin{cases} X_{\tau+u} = X_\tau + \int_0^u b(X_{\tau+s})ds + \int_0^u \sigma(X_{\tau+s})dB_s^\tau & u \in [0, U] \\ X_\tau = \xi \end{cases}$$

The progressive measurability of $(X_t)_{t\in[0,T+U]}$ and proposition 1.16 allow to write

$$\int_0^u \sigma(X_{\tau+s})dB_s^\tau = \int_\tau^{\tau+u} \sigma(X_s)dB_s \quad \forall u \in [0, U] \quad a.s.$$

and substituting in the above equation we get that $(X_t)_{t\in[0,T+U]}$ is a solution to (7.6).
The uniqueness of $(X_t)_{t\in[0,T+U]}$ follows from that of $(Z_u)_{u\in[0,U]}$.

■

Now we slightly modify equation (7.6) as follows

$$\begin{cases} X_{\tau+u} = X_\tau + \int_\tau^{\tau+u} b(X_{s^-})ds + \int_\tau^{\tau+u} \sigma(X_{s^-})dB_s \quad u \in [0, U] \\ X_\tau = \xi \end{cases} \tag{7.8}$$

Since both the solutions to (7.6) and (7.8) have to be continuous on $[\tau, \tau+U]$, then (7.6) and (7.8) are actually the same equation written in a different way. Thus the result given in proposition 7.4 holds even for (7.8).

We have now generalised theorem 1.11 according to our need: we are ready to build a solution to (7.1) and verify its salient properties.
Let now ξ be an $\mathbb{R}^m$-valued $\mathcal{F}_0$-measurable random variable. Define $\tau_n' = \tau_n \wedge T$ $\forall n \in \mathbb{N}$ and consider the equations

$$\begin{cases} X_{\tau_n'+u} = X_{\tau_n'} + \int_{\tau_n'}^{\tau_n'+u} b(X_{s^-})ds + \int_{\tau_n'}^{\tau_n'+u} \sigma(X_{s^-})dB_s \quad u \in [0, T] \\ X_{\tau_n'} = \xi_n \end{cases} \tag{7.9}$$

where ξ_n is an $\mathcal{F}_{\tau_n'}$-measurable random variable for all $n \in \mathbb{N}$.
Notice that the potential solutions to (7.9) are defined on $[0, 2T] \times \Omega$.
We start by solving equation (7.9) for $n = 0$ and $\xi_0 = \xi$: by proposition 7.4 we obtain the solution $(X_t^0)_{t\in[0,2T]}$, then we set

$$\xi_1 = X_{\tau_1'}^0 + G(X_{\tau_1'}^0, Y_1)1_E(Y_1)$$

and solve (7.9) for $n = 1$, so that by proposition 7.4 we obtain the solution $(X_t^1)_{t\in[0,2T]}$. Then we proceed recursively by setting at the n-th step the initial condition

$$\xi_n = X_{\tau_n'}^{n-1} + G(X_{\tau_n'}^{n-1}, Y_n)1_E(Y_n).$$

Notice that the above procedure works since ξ_n is $\mathcal{F}_{\tau_n'}$-measurable for all $n \geq 1$: this can be shown by an induction arguments thanks to the progressive measurability of the solution to (7.8), the Borel measurability of G and proposition 2.6.
Thus, we get the sequence of stochastic processes $\{(X_t^n)_{t\in[0,2T]}\}_{n\in\mathbb{N}}$. In the following proposition we build a solution to (7.1) by means of these processes.

Theorem 7.5 (Existence and uniqueness). *Let ξ be an $\mathbb{R}^m$-valued $\mathcal{F}_0$-measurable random variable. Then there exists a unique solution to*

$$\begin{cases} X_t = X_0 + \int_0^t b(X_{s^-})ds + \int_0^t \sigma(X_{s^-})dB_s + \int_0^t \int_E G(X_{s^-}, y)N(ds, dy) \quad t \in [0, T] \\ X_0 = \xi \end{cases}$$

Proof. Define the process $(X_t)_{t\in[0,T]}$ such that for all $t \in [0,T]$

$$X_t = X_t^n \quad \tau'_n \le t < \tau'_{n+1} \quad \forall n \in \mathbb{N}.$$

Such process is of course càdlàg.
It is also $\mathcal{F}_t$-adapted since for all $t \in [0,T]$ we have

$$X_t = \sum_{n\in\mathbb{N}} X_t^n 1_{[\tau'_n,\tau'_{n+1})}(t).$$

The process $(X_t)_{t\in[0,T]}$ solves (7.1) since

$$\begin{aligned}
X_t &= \sum_{n\in\mathbb{N}} X_t^n 1_{[\tau'_n,\tau'_{n+1})}(t) \\
&= \xi + \sum_{n\in\mathbb{N}} \Bigg[\int_{\tau'_n\wedge t}^{\tau'_{n+1}\wedge t} b(X^n_{s^-})ds + \int_{\tau'_n\wedge t}^{\tau'_{n+1}\wedge t} \sigma(X^n_{s^-})dB_s \\
&+ G(X^n_{\tau'^-_{n+1}}, Y_{n+1}) 1_E(Y_{n+1}) 1_{[0,T]}(\tau'_{n+1}) \Bigg] \\
&= \xi + \sum_{n\in\mathbb{N}} \Bigg[\int_{\tau'_n\wedge t}^{\tau'_{n+1}\wedge t} b(X_{s^-})ds + \int_{\tau'_n\wedge t}^{\tau'_{n+1}\wedge t} \sigma(X_{s^-})dB_s \\
&+ G(X_{\tau'^-_{n+1}}, Y_{n+1}) 1_E(Y_{n+1}) 1_{[0,T]}(\tau'_{n+1}) \Bigg] \\
&= \xi + \int_0^t b(X_{s^-})ds + \int_0^t \sigma(X_{s^-})dB_s \\
&+ \int_0^t \int_E G(X_{s^-}, y) N(ds,dy) \quad a.s. \ \ \forall t \in [0,T]
\end{aligned}$$

and the required result follows by the right-continuity of $(X_t)_{t\in[0,T]}$.
For uniqueness, let $(\hat{X}_t)_{t\in[0,T]}$ be another solution to (7.1): we show that it is indistinguishable from $(X_t)_{t\in[0,T]}$.
We have

$$\begin{aligned}
\{\hat{X}_t = X_t \ \forall t \in [0,T]\} &= \bigcap_{n\in\mathbb{N}} \{\hat{X}_t = X_t \ \forall t \in [\tau'_n, \tau'_{n+1})\} \\
&= \bigcap_{n\in\mathbb{N}} \{\hat{X}_t = X_t^n \ \forall t \in [\tau'_n, \tau'_{n+1})\}
\end{aligned}$$

so that it is sufficient to show that $\mathbb{P}(\hat{X}_t = X_t^n \ \forall t \in [\tau'_n, \tau'_{n+1})) = 1$ for all $n \in \mathbb{N}$: we do it by an induction argument.

For $n = 0$ we have $X_0^0 = \hat{X}_0 = \xi$ by the construction of $(X_t^0)_{t\in[0,2T]}$ and since $(\hat{X}_t)_{t\in[0,T]}$ solves (7.1). For the same reason we get for $t \in [0, \tau_1')$

$$\begin{aligned} X_t^0 &= X_0^0 + \int_0^t b(X_{s^-}^0)ds + \int_0^t \sigma(X_{s^-}^0)dB_s \quad a.s. \\ \hat{X}_t &= \hat{X}_0 + \int_0^t b(\hat{X}_{s^-})ds + \int_0^t \sigma(\hat{X}_{s^-})dB_s \quad a.s. \end{aligned}$$

We introduce the process $(Z_t^0)_{t\in[0,2T]}$ and define it in steps: first we set $Z_t^0 = \hat{X}_t \ \forall t \in [0, \tau_1')$. Then we consider the solution $(Y_t^0)_{t\in[0,2T]}$ to the equation

$$\begin{cases} Y_{\tau_1'+u} = Y_{\tau_1'} + \displaystyle\int_{\tau_1'}^{\tau_1'+u} b(Y_{s^-})ds + \int_{\tau_1'}^{\tau_1'+u} \sigma(Y_{s^-})dB_s \quad u \in [0,T] \\ Y_{\tau_1'} = \hat{X}_{\tau_1'^-} \end{cases}$$

and set $Z_t^0 = Y_t^0 \ \forall t \in [\tau_1', T]$.
Finally set $Z_t^0 = Y_T^0 \ \forall t \in (T, 2T]$.
Now fix $t \in [0, 2T]$: by the definition of $(Z_t^0)_{t\in[0,2T]}$ and by proposition 1.8 we can write

$$\begin{aligned} Z_t^0 &= \hat{X}_t 1_{[0,\tau_1')}(t) + Y_t^0 1_{[\tau_1',T]}(t) + Y_T^0 1_{(T,2T]}(t) \\ &= \hat{X}_0 1_{[0,\tau_1')}(t) + 1_{[0,\tau_1')}(t)\int_0^t b(\hat{X}_{s^-})ds + 1_{[0,\tau_1')}(t)\int_0^t \sigma(\hat{X}_{s^-})dB_s \\ &+ Y_{\tau_1'}^0 1_{[\tau_1',T]}(t) + 1_{[\tau_1',T]}(t)\int_{\tau_1'}^t b(Y_{s^-}^0)ds + 1_{[\tau_1',T]}(t)\int_{\tau_1'}^t \sigma(Y_{s^-}^0)dB_s \\ &+ Y_T^0 1_{(T,2T]}(t) \\ &= Z_0^0 1_{[0,\tau_1')}(t) + 1_{[0,\tau_1')}(t)\int_0^t b(Z_{s^-}^0)ds + 1_{[0,\tau_1')}(t)\int_0^t \sigma(Z_{s^-}^0)dB_s \\ &+ Z_{\tau_1'}^0 1_{[\tau_1',T]}(t) + 1_{[\tau_1',T]}(t)\int_{\tau_1'}^t b(Z_{s^-}^0)ds + 1_{[\tau_1',T]}(t)\int_{\tau_1'}^t \sigma(Z_{s^-}^0)dB_s \\ &+ Z_T^0 1_{(T,2T]}(t) \\ &= 1_{[0,T]}(t)\bigg(Z_0^0 + \int_0^t b(Z_{s^-}^0)ds + \int_0^t \sigma(Z_{s^-}^0)dB_s\bigg) + Z_T^0 1_{(T,2T]}(t) \quad a.s. \end{aligned}$$

where the last equality holds since by the continuity of $(Z_t^0)_{t\in[0,2T]}$ and of the integrals we get

$$\begin{aligned}
Z^0_{\tau'_1} &= Z^0_{\tau'^-_1} \\
&= Z^0_0 + \int_0^{\tau'^-_1} b(Z^0_{s^-})ds + \int_0^{\tau'^-_1} \sigma(Z^0_{s^-})dB_s \\
&= Z^0_0 + \int_0^{\tau'_1} b(Z^0_{s^-})ds + \int_0^{\tau'_1} \sigma(Z^0_{s^-})dB_s \quad a.s.
\end{aligned}$$

By the continuity of $(Z^0_t)_{t\in[0,2T]}$ we can conclude

$$Z^0_t = Z^0_0 + \int_0^t b(Z^0_{s^-})ds + \int_0^t \sigma(Z^0_{s^-})dB_s \quad \forall t \in [0,T] \quad a.s.$$

so that the processes $(Z^0_t)_{t\in[0,2T]}$, $(X^0_t)_{t\in[0,2T]}$ solve (7.9) for $n = 0$: hence they must be indistinguishable.
Thus, we get

$$\begin{aligned}
\mathbb{P}(\hat{X}_t = X^0_t \ \forall t \in [0,\tau'_1)) &= \mathbb{P}(Z^0_t = X^0_t \ \forall t \in [0,\tau'_1)) \\
&\geq \mathbb{P}(Z^0_t = X^0_t \ \forall t \in [0,T]) = 1
\end{aligned}$$

For the inductive step, we consider true the thesis up to $n-1$ and prove it for n.
Since $(\hat{X}_t)_{t\in[0,T]}$ solves (7.1) and thanks to the induction hypothesis we get $X^n_{\tau'_n} = \hat{X}_{\tau'_n}$ a.s., that is we have the same initial condition at τ'_n. Then we set

$$Z^n_t = \begin{cases} 0 & 0 \leq t < \tau'_n \\ \hat{X}_t & \tau'_n \leq t < \tau'_{n+1} \\ Y^n_t & \tau'_{n+1} \leq t < T \\ Y^n_T & T \leq t \leq 2T, \end{cases}$$

where $(Y^n_t)_{t\in[0,2T]}$ is the solution to

$$\begin{cases} Y_{\tau'_{n+1}+u} = Y_{\tau'_{n+1}} + \displaystyle\int_{\tau'_{n+1}}^{\tau'_{n+1}+u} b(Y_{s^-})ds + \int_{\tau'_{n+1}}^{\tau'_{n+1}+u} \sigma(Y_{s^-})dB_s \quad u \in [0,T] \\ Y_{\tau'_{n+1}} = \hat{X}_{\tau'^-_{n+1}} \end{cases}$$

and the required result follows by an argument similar to the one for $n = 0$.
■

Chapter 8

Appendix

As said in the introduction, here we provide an alternative way to prove proposition 6.1 which makes use of dicrete valued stopping times and does not rely on the time changed brownian motion introduced in proposition 1.14.
To ease the notation we suppose to work with a process which is a jump-diffusion over all the compact subsets of the time line, but a similar argument holds if we set a finite time horizon.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses.
Let $(B_t)_{t\geq 0}$ be an $\mathbb{R}^r$-valued $\mathcal{F}_t$-brownian motion, $B = (B^1, ..., B^r)$.
Let $(Q_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued $\mathcal{F}_t$-compound Poisson process independent of $(B_t)_{t\geq 0}$, with the usual representation $Q_t = \sum_{n=1}^{N_t} Y_n$ $\forall t \geq 0$, where $(N_t)_{t\geq 0}$ is the underlying Poisson process, $\{\tau_n\}_{n\in\mathbb{N}}$ the sequence of its arrival times, $\{Y_n\}_{n\geq 1}$ the sequence of the jump sizes of $(Q_t)_{t\geq 0}$.
Let $E = \mathbb{R}^d \setminus \{0\}$, $\mathcal{E} = \mathcal{B}(E)$: we associate to $(Q_t)_{t\geq 0}$ the random counting measure N as in the previous chapters.
Consider the $\mathbb{R}^m$-valued process $(X_t)_{t\geq 0}$ such that

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t G_s dB_s + \int_0^t \int_E K(s,y) N(ds,dy) \quad a.s. \quad \forall t \geq 0$$

where $F \in M^1_{loc}([0,\infty))$, $G \in M^2_{loc}([0,\infty))$ and K is predictable.
Divide the continuous part of $(X_t)_{t\geq 0}$, call it $(X^c_t)_{t\geq 0}$, from its jump part,

call it $(X_t^j)_{t\geq 0}$, so that for all $t \geq 0$

$$X_t^c = X_0 + \int_0^t F_s ds + \int_0^t G_s dB_s$$
$$X_t^j = \int_0^t \int_E K(s,y) N(ds,dy).$$

Proposition 8.1. *Let τ be an a.s. finite $\mathcal{F}_t$-stopping time and $h : \mathbb{R}^m \to \mathbb{R}$ such that $h \in C^2(\mathbb{R}^m)$. Then for all $n \in \mathbb{N}$ we have*

$$\begin{aligned} h(X_{\tau_n+u}) &= h(X_{\tau_n}) + \int_{\tau_n}^{\tau_n+u} \sum_{i=1}^m h_{x_i}(X_s) F_s^i ds \\ &+ \int_{\tau_n}^{\tau_n+u} \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \sum_{j=1}^r h_{x_i x_l}(X_s) G_s^{ij} G_s^{lj} ds \\ &+ \int_{\tau_n}^{\tau_n+u} \sum_{i=1}^m \sum_{j=1}^r h_{x_i}(X_s) G_s^{ij} dB_s^j \quad \forall u \in [0, \tau_{n+1} - \tau_n) \quad a.s. \end{aligned}$$

Proof. Fix $n \in \mathbb{N}$. We have

$$X_t = \begin{cases} X_t & 0 \leq t \leq \tau_n \\ X_t^c + X_{\tau_n} - X_{\tau_n}^c & \tau_n < t < \tau_{n+1}. \end{cases}$$

Define the process $(X_t^n)_{t\geq 0}$ such that

$$X_t^n = \begin{cases} X_t & 0 \leq t \leq \tau_n \\ X_t^c + X_{\tau_n} - X_{\tau_n}^c & t > \tau_n \end{cases}$$

Then by definition we get

$$\begin{aligned} X_{\tau_n+u}^n &= X_{\tau_n+u}^c + X_{\tau_n} - X_{\tau_n}^c \\ &= X_{\tau_n}^n + \int_{\tau_n}^{\tau_n+u} F_s ds + \int_{\tau_n}^{\tau_n+u} G_s dB_s \quad \forall u \geq 0 \quad a.s. \end{aligned} \tag{8.1}$$

Now we introduce the sequence $\{\tau_k\}_{k\in\mathbb{N}}$ such that

$$\tau_k = \begin{cases} \frac{l+1}{2^k} & \frac{l}{2^k} < \tau_n \leq \frac{l+1}{2^k} \; l \in \mathbb{N} \\ +\infty & \tau_n = +\infty \end{cases}$$

so that $\tau_k \searrow \tau_n$ and each τ_k is a discrete valued $\mathcal{F}_t$-stopping time.
Fix $k \in \mathbb{N}$ and call $\{v_b^k\}_{b\in\mathbb{N}}$ the set of values assumed by τ_k, so that

$$\tau_k = \sum_{b\in\mathbb{N}} v_b^k 1_{\{\tau_k = v_b^k\}}.$$

Since $\tau_k > \tau_n$, then by (8.1) we get

$$X^n_{\tau_k} = X^n_{\tau_n} + \int_{\tau_n}^{\tau_k} F_{rs} ds + \int_{\tau_n}^{\tau_k} G_s dB_s \quad a.s.$$

$$X^n_{\tau_k+u} = X^n_{\tau_n} + \int_{\tau_n}^{\tau_k+u} F_s ds + \int_{\tau_n}^{\tau_k+u} G_s dB_s \quad \forall u \geq 0 \quad a.s.$$

so that

$$X^n_{\tau_k+u} = X^n_{\tau_k} + \int_{\tau_k}^{\tau_k+u} F_s ds + \int_{\tau_k}^{\tau_k+u} G_s dB_s \quad \forall u \geq 0 \quad a.s.$$

Now fix $b \in \mathbb{N}$, define $\Omega^k_b = \{\omega \in \Omega : \tau_k = v^k_b\}$ and multiply all the terms in the above equation with $1_{\Omega^k_b}$ to obtain

$$\begin{aligned} X^n_{v^k_b+u} 1_{\Omega^k_b} = X^n_{v^k_b} 1_{\Omega^k_b} + 1_{\Omega^k_b} \int_{v^k_b}^{v^k_b+u} F_s ds \\ + 1_{\Omega^k_b} \int_{v^k_b}^{v^k_b+u} G_s dB_s \quad \forall u \geq 0 \quad a.s. \end{aligned} \tag{8.2}$$

Notice that $\mathbb{P}(\Omega^k_b) > 0$ for all k, b.
Now by a pathwise argument we easily see that

$$1_{\Omega^k_b} \int_{v^k_b}^{v^k_b+u} F_s ds = \int_{v^k_b}^{v^k_b+u} F_s 1_{\Omega^k_b} ds \quad \forall u \geq 0$$

Furthermore, thanks to proposition 1.8 we have

$$1_{\Omega^k_b} \int_{v^k_b}^{v^k_b+u} G_s dB_s = \int_{v^k_b}^{v^k_b+u} G_s 1_{\Omega^k_b} dB_s \quad \forall u \geq 0 \quad a.s.$$

Thus by (8.2) we can write

$$\begin{aligned} X^n_{v^k_b+u} 1_{\Omega^k_b} = X^n_{v^k_b} 1_{\Omega^k_b} + \int_{v^k_b}^{v^k_b+u} F_s 1_{\Omega^k_b} ds \\ + \int_{v^k_b}^{v^k_b+u} G_s 1_{\Omega^k_b} dB_s \quad \forall u \geq 0 \quad a.s. \end{aligned}$$

Now since τ_k is a stopping time then we have that $(F_t 1_{\Omega^k_b})_{t\geq 0} \in M^1_{loc}([0,\infty))$ and $(G_t 1_{\Omega^k_b})_{t\geq 0} \in M^2_{loc}([0,\infty))$, so that $(X^n_{v^k_b+u} 1_{\Omega^k_b})_{u\geq 0}$ is an Itô process on

$[v_b^k, \infty)$ and it is adapted to $(\mathcal{F}_{v_b^k+u})_{u \geq 0}$.
Thus by theorem 1.10 we can write

$$\begin{aligned} h(X^n_{v_b^k+u} 1_{\Omega_b^k}) = h(X^n_{v_b^k} 1_{\Omega_b^k}) &+ \int_{v_b^k}^{v_b^k+u} \sum_{i=1}^m h_{x_i}(X^n_s 1_{\Omega_b^k}) F^i_s 1_{\Omega_b^k} ds \\ &+ \int_{v_b^k}^{v_b^k+u} \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \sum_{j=1}^r h_{x_i x_l}(X^n_s 1_{\Omega_b^k}) G^{ij}_s G^{lj}_s 1_{\Omega_b^k} ds \\ &+ \int_{v_b^k}^{v_b^k+u} \sum_{i=1}^m \sum_{j=1}^r h_{x_i}(X^n_s 1_{\Omega_b^k}) G^{ij}_s 1_{\Omega_b^k} dB^j_s \quad \forall u \geq 0 \quad a.s. \end{aligned}$$

Now we extract the indicators from the integrals by arguments similar to the ones above and multiply again with the same indicators to obtain

$$\begin{aligned} h(X^n_{v_b^k+u} 1_{\Omega_b^k}) 1_{\Omega_b^k} = h(X^n_{v_b^k} 1_{\Omega_b^k}) 1_{\Omega_b^k} &+ 1_{\Omega_b^k} \int_{v_b^k}^{v_b^k+u} \sum_{i=1}^m h_{x_i}(X^n_s 1_{\Omega_b^k}) F^i_s ds \\ &+ 1_{\Omega_b^k} \int_{v_b^k}^{v_b^k+u} \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \sum_{j=1}^r h_{x_i x_l}(X^n_s 1_{\Omega_b^k}) G^{ij}_s G^{lj}_s ds \\ &+ 1_{\Omega_b^k} \int_{v_b^k}^{v_b^k+u} \sum_{i=1}^m \sum_{j=1}^r h_{x_i}(X^n_s 1_{\Omega_b^k}) G^{ij}_s dB^j_s \quad \forall u \geq 0 \quad a.s. \quad (8.3) \end{aligned}$$

Now fix $u \geq 0$: by a pathwise argument we easily see that

$$\sum_{b \in \mathbb{N}} h(X^n_{v_b^k+u} 1_{\Omega_b^k}) 1_{\Omega_b^k} = h(X^n_{\tau_k+u}) \quad a.s.$$
$$\sum_{b \in \mathbb{N}} h(X^n_{v_b^k} 1_{\Omega_b^k}) 1_{\Omega_b^k} = h(X^n_{\tau_k}) \quad a.s.$$

and that

$$\begin{aligned} &\sum_{b \in \mathbb{N}} 1_{\Omega_b^k} \int_{v_b^k}^{v_b^k+u} \Big[\sum_{i=1}^m h_{x_i}(X^n_s 1_{\Omega_b^k}) F^i_s + \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \sum_{j=1}^r h_{x_i x_l}(X^n_s 1_{\Omega_b^k}) G^{ij}_s G^{lj}_s \Big] ds \\ &= \int_{\tau_k}^{\tau_k+u} \Big[\sum_{i=1}^m h_{x_i}(X^n_s) F^i_s + \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^m \sum_{j=1}^r h_{x_i x_l}(X^n_s) G^{ij}_s G^{lj}_s \Big] ds \quad a.s. \end{aligned}$$

Using again proposition 1.8 we get

$$\sum_{b \in \mathbb{N}} 1_{\Omega_b^k} \int_{v_b^k}^{v_b^k+u} \sum_{i=1}^m \sum_{j=1}^r h_{x_i}(X^n_s 1_{\Omega_b^k}) G^{ij}_s dB^j_s = \int_{\tau_k}^{\tau_k+u} \sum_{i=1}^m \sum_{j=1}^r h_{x_i}(X^n_s) G^{ij}_s dB^j_s \quad a.s.$$

Thus summing (8.3) over $b \in \mathbb{N}$ yields

$$h(X^n_{\tau_k+u}) = h(X^n_{\tau_k}) + \int_{\tau_k}^{\tau_k+u} \sum_{i=1}^{m} h_{x_i}(X^n_s) F^i_s ds + \int_{\tau_k}^{\tau_k+u} \frac{1}{2} \sum_{i=1}^{m} \sum_{l=1}^{m} \sum_{j=1}^{r} h_{x_i x_l}(X^n_s) G^{ij}_s G^{lj}_s ds + \int_{\tau_k}^{\tau_k+u} \sum_{i=1}^{m} \sum_{j=1}^{r} h_{x_i}(X^n_s) G^{ij}_s dB^j_s \quad a.s. \tag{8.4}$$

Since $u \geq 0$ was fixed arbitrarily and by the continuity of the paths, then (8.4) holds for all $u \geq 0$ a.s. We can repeat the argument for all $k \in \mathbb{N}$, so that (8.4) holds for all $u \geq 0$, $k \in \mathbb{N}$ a.s. Then taking the limit as $k \to \infty$, by the continuity of the paths we get

$$h(X^n_{\tau_n+u}) = h(X^n_{\tau_n}) + \int_{\tau_n}^{\tau_n+u} \sum_{i=1}^{m} h_{x_i}(X^n_s) F^i_s ds + \int_{\tau_n}^{\tau_n+u} \frac{1}{2} \sum_{i=1}^{m} \sum_{l=1}^{m} \sum_{j=1}^{r} h_{x_i x_l}(X^n_s) G^{ij}_s G^{lj}_s ds + \int_{\tau_n}^{\tau_n+u} \sum_{i=1}^{m} \sum_{j=1}^{r} h_{x_i}(X^n_s) G^{ij}_s dB^j_s \quad \forall u \geq 0 \quad a.s.$$

Since by definition $X^n_t = X_t$ for all $t \in [0, \tau_{n+1})$, then by a localization argument we get

$$h(X_{\tau_n+u}) = h(X_{\tau_n}) + \int_{\tau_n}^{\tau_n+u} \sum_{i=1}^{m} h_{x_i}(X_s) F^i_s ds + \int_{\tau_n}^{\tau_n+u} \frac{1}{2} \sum_{i=1}^{m} \sum_{l=1}^{m} \sum_{j=1}^{r} h_{x_i x_l}(X_s) G^{ij}_s G^{lj}_s ds + \int_{\tau_n}^{\tau_n+u} \sum_{i=1}^{m} \sum_{j=1}^{r} h_{x_i}(X_s) G^{ij}_s dB^j_s \quad \forall u \in [0, \tau_{n+1} - \tau_n) \quad a.s.$$

which is the required result.

■

Bibliography

[AP] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Second Edition, Cambridge Studies In Advanced Mathematics (2009)

[BA] P. Baldi, *Equazioni differenziali stocastiche e applicazioni*, Seconda Edizione, Quaderni dell'Unione Matematica Italiana 28, Pitagora Editrice (2000)

[BAn] P. Baldi, *Introduction to stochastic processes*, Scuola Matematica Interuniversitaria, Cortona (2007)

[BI] P. Billingsley, *Probability and Measures*, Third Edition, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons (1995)

[BR] P. Brémaud, *Point Processes and Queues, Martingale Dynamics*, Springer-Verlag (1981)

[CA] F.Caravenna, *Moto Browniano e Analisi Stocastica*, Versione 3.3, (2011)

[CT] R. Cont, P. Tankov, *Financial Modelling With Jumps Processes*, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC (2004)

[HWY] S. He, J. Wan, J. Yan, *Semimartingale Theory and Stochastic Calculus*, Science Press and CRC Press Inc (1992)

[IW] N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, Second Edition, Noth-Holland Mathematical Library (1989)

[JP] J. Jacod, P.E. Protter, *Probability Essentials*, Second Edition, Springer (2004)

[ME] M. Métivier, *Semimartingales: a Course on Stochastic Processes*, de Gruyter Studies in Mathematics 2, de Gruyter (1982)

[PR] P.E. Protter, *Stochastic Integration and Differential Equations*,Second Edition, Version 2.1, Stochastic Modelling and Applied Probability 21, Springer (2005)

[RY] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*, Corrected Third Printing of the Third Edition, Springer (2005)

[SH] S.E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer Finance, Springer (2004)

[SI] R. Situ, *Theory of Stochastic Differential Equations with Jumps and Applications*, Mathematical and Analytical Techniques with Applications to Engineering, Springer (2005)