POLITECNICO DI MILANO
DEPARTMENT OF MATHEMATICS
DOCTORAL PROGRAMME IN MATHEMATICAL MODELS AND METHODS IN ENGINEERING

SOME FOURTH ORDER DIFFERENTIAL EQUATIONS MODELING SUSPENSION BRIDGES

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2015 – XXVIII

Acknowledgements

FOREMOST, I would like to thank my advisor, Prof. Filippo Gazzola, who offered me the opportunity to study my PhD program at Politecnico di Milano. I especially want to express my deepest gratitude to him for the continuous support and tutoring of my PhD research. He patiently helped and guided me through these years of research and in writing this thesis. Without his guidance and persistent help, this dissertation would not have been possible.

My sincere thanks also go to all the faculties and staff of Politecnico di Milano, especially the Department of Mathematics and the PhD school for supplying me with all the necessary facilities for the research. I thank my colleagues at Politecnico di Milano for their help, for their cooperation and support during my PhD program. I also want to thank all my friends who made my time here a lot of fun.

Next, I would like to thank CSC (China Scholarship Council) who supported me in my study at Politecnico di Milano. Thanks to them for providing me with the scholarship, including the living stipend, during my PhD study period. Meanwhile, many thanks go to the Education Section Embassy of the People's Republic of China in the Republic of Italy, particularly, the First Secretary, Junhua Liu and the Counsellor, Linyi Zhang. They come to Milan at least one time every year for looking after the Chinese students studying here.

Last but not the least, many words of acknowledgement to my family members, who encouraged and supported me in my study. I would like to thank my parents for their constant love and support. I also want to express my thanks to my sisters and brothers, my nephews and my nieces for supporting me spiritually throughout my study.

Abstract

NOWADAYS, the complicated oscillations of suspension bridges are still not completely understood. In this thesis, we suggest several new reasonable mathematical models which may help to describe the oscillation behavior appearing in the actual suspension bridges.

First, in Introduction we survey several historical events on suspension bridges and we recall some existing mathematical models in literature. Most of these models fail to describe the static or dynamic behavior of the suspension bridges. Also in this part, we list the new mathematical models we suggested for describing the behavior of the bridges.

Chapter 2 is devoted to a kind of beam model for suspension bridge. The bridge is viewed as an elastic beam which is suspended to a sustaining cable, where the beam and the cable are connected by a large number of hangers. We analyze the energies in the system (the suspension bridge) after the deformation from the rest position to a new position due to a live load. The Euler-Lagrange equation is obtained by taking the critical points of the total energy. Together with the hinged boundary conditions, we deduce a nonlinear nonlocal problem and we prove that it admits at least one weak solution.

In Chapter 3, we view the suspension bridge as a long-narrow thin rectangular plate, which is suspended to two sustaining cables. We first recall the plate model suggested by Ferrero-Gazzola [27] for dynamical suspension bridge. Then we consider a non-coercive problem corresponding to the plate model and we analyze the asymptotic behavior of the unique lo-

cal solution of the problem for different initial conditions. Finally, in order to describe the boundary behavior of the plate, we set up a problem with dynamical boundary conditions that reflect the physical constraints on the boundaries. Assume that the restoring force due to the hangers is in a linear regime, we obtain a linear evolution problem with dynamical boundary conditions. We prove that the evolution problem admits a unique explicit solution.

In Chapter 4, the roadway of the suspension bridge is also considered as a thin rectangular plate. We suggest a quasilinear plate model based on the von Kármán plate equations when large deformations appear in the plate. In this case, the interaction with the stretching behavior of the plate should be analyzed. Then two fourth order differential equations are deduced by applying variational principles to the energy functional, where we introduce the so-called Airy stress function. By adding the restoring force due to the hangers to the equations, we obtain a system coupled by two fourth order differential equations. We prove existence and multiplicity results of the system with suitable boundary conditions.

The last chapter contains the conclusions on our mathematical models for suspension bridges. We do not claim that these models are perfect. This is just the beginning for reaching more challenging results in this field and much more work (both mathematical and engineering) is needed. We also list several open problems corresponding to our new models.

Contents

Notations

$D^k u = \frac{\partial^k u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \quad k \geq 0, \sum_{i=1}^n \alpha_i = k$ with $\alpha_i \in \mathbb{N}, 0 \leq \alpha_i \leq k$.
$C^l(\Omega) = \{u : \Omega \to \mathbb{R};\ u$ has continuous derivatives up to order l on $\Omega\}$.
$C_c^l(\Omega) = \big\{u \in C^l(\Omega) :\ \mathrm{supp}(u)$ is compact in $\Omega\big\}, \quad 0 \leq l \leq \infty$.
$L^p(\Omega) = \big\{u : \Omega \to \mathbb{R};\ \int_\Omega |u|^p < \infty\big\}, \quad 1 \leq p < \infty$.
$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R};\ \sup_\Omega |u| < \infty\}$.
$H^m(\Omega) = \big\{u : \Omega \to \mathbb{R};\ \sum_{i=0}^m \int_\Omega |D^i u|^2 < \infty\big\}, \quad m \geq 1$.
$H_0^1(\Omega) = \{u \in H^1(\Omega) :\ u = 0$ on $\partial\Omega\}$.
$\mathcal{H}(\Omega) =$ the dual space of $H^2 \cap H_0^1(\Omega)$.
$H^{m-j-1/2}(\partial\Omega) = \gamma_j[H^m(\Omega)]$ with $\gamma_j u = \frac{\partial^j u}{\partial^j \nu}$ for $j = 0, 1, \cdots, m-1$.
$[u, v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}$, the Monge-Ampère operator.
$C^0([0,T]; X) =$ space of functions $u = u(t) \in C^0([0,T])$ with respect to the X-norm in space; $u \in C^0([0,T]; X)$ iff $\|u(t) - u(t_0)\|_X \to 0$ as $t \to t_0$. Similarly, for the spaces $C^1([0,T]; X)$ and $C^2([0,T]; X)$.
TNB $=$ Tacoma Narrows Bridge.
(GP) $=$ General Principle of classical mechanics.

When $\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2$.
$H_*^2(\Omega) = \{u \in H^2(\Omega) :\ u = 0$ on $\{0, \pi\} \times (-\ell, \ell)\}$.
$\mathcal{H}_*(\Omega) =$ the dual space of $H_*^2(\Omega)$.
$H_{**}^2(\Omega) = \{u \in H_*^2(\Omega) :\ u = u_y = 0$ on $(0, \pi) \times \{\pm\ell\}\}$.
$\mathcal{H}_{**}(\Omega) =$ the dual space of $H_{**}^2(\Omega)$.
$H_*^4(\Omega) = \{u \in H^4(\Omega) :\ u = u_{xx} = 0$ on $\{0, \pi\} \times (-\ell, \ell)\}$.

CHAPTER 1

Introduction

A modern suspension bridge is a type of bridge where the deck (the roadway) is hung below the suspension cables by a large number of hangers, see Figure 1.1 for a simple sketch of the modern suspension bridge.

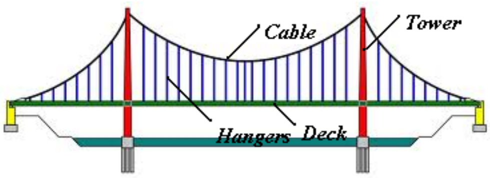

Figure 1.1: *Sketch of modern suspension bridge.*

1.1 Historical events and existing mathematical models

1.1.1 Historical events

Although the first design of a modern suspension bridge is made around the year 1595 by the Italian engineer Fausto Veranzio, see [77] and [62, p.7] or [43, p.16], only about two centuries later, the first bridge (Jacob's Creek

Bridge, 1801) was built in Pennsylvania by the Irish judge and engineer James Finley[1]. Finley patented his design in 1808, and published it in the Philadelphia journal, The Port Folio, in 1810, see [15].

In the about 200 years' history of suspension bridges, there are many dramatic events, such as the uncontrolled oscillations which, in some cases, led to collapses. Only in several decades between 1818 and 1889, ten suspension bridges suffered major damages or collapsed in windstorms, see [24, Table 1, p.13]. Furthermore, according to [42], around 400 recorded bridges (suspended or not) failed for different reasons (such as the wind, the traffic loads or mistakes in the project) and the ones who collapsed after the year 2000 are more than 70.

The Broughton Suspension Bridge (built in 1826) collapsed in 1831 due to the mechanical resonance caused by the marching soldiers matching over the bridge in step. Hence, from then on, the British Army issued an order that the troops should "break step" when crossing a bridge.

This kind of failures occurred due to an external response according to the classification by Gazzola [30] and this phenomenon is not the scope of this thesis. We next list several events due to some unexpected oscillations.

The Brighton Chain Pier (built in 1823) collapsed a first time in 1833 due to a violent windstorm. Then it was rebuilt and partially destroyed once again in 1836. For the collapse happened in 1836, a witness, William Reid [70, p.99], reported some valuable observations and sketched two pictures illustrating the destruction, see Figure 1.2, which is taken from [71].

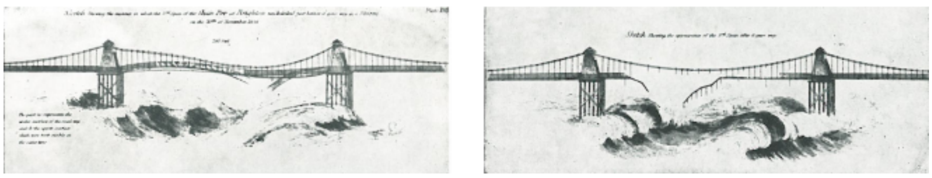

Figure 1.2: *Destruction of the Brighton Chain Pier.*

The Menai Straits Bridge, which was built in 1826, failed in 1839 due to a hurricane. In this occasion, unexpected oscillations appeared and Provis [69] provided the following description:

> ...the character of the motion of the platform was not that of a simple undulation, as had been anticipated, but the movement of the undulatory wave was oblique, both with respect to the lines of the bearers, and to the general direction of the bridge.

[1] Iron-Chain Suspension Bridge, https://en.wikipedia.org/wiki/Jacob%27s_Creek_Bridge_(Pennsylvania)

The Tacoma Narrows Bridge (TNB) collapse, occurred in November, 1940 just a few months after its opening, is the most celebrated bridge failure both because of the impressive video[2] and because of the large number of studies that it has inspired starting from the reports [4, 24–26, 73, 79].

It is not our purpose to give a detailed list of all the collapses for which we refer to [13, Section 1.1], to [71, Chapter IV], to [19,24,41,84], to recent monographs [2, 30, 43], and also to [42] for a complete database.

1.1.2 Existing mathematical models

The French engineer and mathematician Claude-Louis Navier in 1823 published a report [62], which has been the only mathematical treatise of suspension bridges for several decades. In the celebrated report, Navier mainly focused on the static of the cables and their interaction with the towers, and then several second order ordinary differential equations are derived and solved. At that time, no stiffening trusses had yet appeared and the models suggested by Navier are oversimplified in some aspects.

The monograph [59] published by the Austrian engineer Joseph Melan is another milestone contribution to suspension bridges. In his monograph, Melan viewed the suspension bridge as an elastic beam which is suspended to a sustaining cable by hangers. He made a detailed study of the static cables and beams through a careful analysis of different kinds of suspension bridges. By using the Castigliano Theorem repeatedly, in particular for the computation of the deflection [59, p.69], he suggested a fourth order ordinary differential equation to describe the behavior appearing in the actual suspension bridge

$$EIw''''(x) - (H + h(w))w''(x) + h(w)y''(x) = p, \quad x \in (0, L), \tag{1.1}$$

where L is the length of the beam which represents the roadway between the towers, $w = w(x)$ denotes the displacement of the beam, $y(x)$ is the position of the cable at rest, E and I are, respectively, the elastic modulus of the material and the moment of inertia of the cross section so that EI is called the flexural rigidity, H is the horizontal tension in the cable when subject to the dead load q, $h(w)$ represents the additional tension in the cable produced by the live load $p = p(x)$.

An excellent source to derive the equation of vertical oscillations in dynamical suspension bridges is [71, Chapter IV], where all the details are well explained. The equation derived in [71, p.132] reads

$$w_{tt} + EIw_{xxxx} - (H + h(w))w_{xx} + \frac{q}{H}h(w) = p, \quad x \in (0, L), t > 0.$$

[2] Tacoma Narrows Bridge collapse, http://www.youtube.com-watch?v=3mclp9QmCGs.

The function p here is in dependence of the time t.

After the TNB collapse, the engineering communities felt that it was necessary to find accurate equations in order to attempt explanations of what had occurred. Then some more mathematical models trying to describe the vertical-torsional oscillation appeared.

Assume that $y(x,t)$ is the vertical deflection of the bridge and $\theta(x,t)$ is the angle of torsion of the cross section, then the following system is derived in [20, (1)-(2)] for the linearized equations of the elastic combined vertical-torsional oscillation motion: for $x \in (0,L)$, $t > 0$,

$$
\begin{aligned}
y_{tt} + EIy_{xxxx} - Hy_{xx} + \frac{q^2EA}{H^2L}\int_0^L y(z,t)dz = f(x,t),\\
I_0\theta_{tt} + C_1\theta_{xxxx} - (C_2 + H\ell^2)\theta_{xx} + \frac{\ell^2q^2EA}{H^2L}\int_0^L \theta(z,t)dz = g(x,t),
\end{aligned}
$$

where q, H are as in (1.1), EI, C_1, C_2, EA are respectively the flexural, warping, torsional, extensional stiffness of the girder, I_0 is the polar moment of inertia of the girder section, 2ℓ is the roadway width, $f(x,t)$ and $g(x,t)$ are the lift and the moment for unit girder length of the self-excited forces.

Another source describing the vertical-torsional oscillations appearing in the suspension bridges is Pittel-Yakubovich [67], where a kind of fishbone beam model is suggested: for $x \in (0,L)$, $t > 0$,

$$
\begin{cases}
my_{tt} - \rho Iy_{xxtt} + EIy_{xxxx} - Hy_{xx} + \frac{W}{2}(x(L-x)\theta)_{xx} = 0,\\
\frac{m\ell^2}{3}\theta_{tt} - \gamma_1\theta_{xxtt} + \frac{W}{2}(x(L-x))\theta_{xx} + \gamma_2\theta_{xxxx} - \mu\theta_{xx} - W\ell\theta = 0,\\
y(0,t) = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0,\\
\theta(0,t) = \theta(L,t) = \theta_{xx}(0,t) = \theta_{xx}(L,t) = 0,
\end{cases}
$$

where L is the length of the roadway while 2ℓ is its width, m is the mass per unit length in the x-direction, ρ is the density of the material, W is the uniformly distributed horizontal wind load, γ_1 and γ_2 are two geometric parameters of the cross section, $\mu = GJ + H\ell^2$ (with $G =$ shear modulus, $J =$ moment of inertia of the pure torsion), I, EI, H as in (1.1). All these constants are positive.

McKenna claimed that the model for suspension bridges should be nonlinear and then a fourth order differential equation was suggested in [51,57, 58] as a one dimensional model for a suspension bridge:

$$
u_{tt} + u_{xxxx} + \gamma u^+ = W, \quad x \in (0,L), \;\; t > 0,
$$

where $u = u(x,t)$ denotes the vertical displacement of the bridge, $u^+ = \max\{u,0\}$ and γu^+ is the restoring force due to the hangers and cables which are considered as a linear spring with a one-sided restoring force, $W = W(x,t)$ represents the forcing term acting on the bridge.

McKenna-Tuama [56] numerically showed that a purely vertical forcing may create a torsional response and then they suggested a slightly different model coupled by two second order differential equations:

$$\begin{cases} my'' = -\left(f(y-\ell\sin\theta) + f(y+\ell\sin\theta)\right), \\ \frac{m\ell^2}{3}\theta'' = \ell\cos\theta\left(f(y-\ell\sin\theta) + f(y+\ell\sin\theta)\right), \end{cases}$$

where y represents the vertical displacement of the barycenter B of the cross section of the roadway and θ is the deflection from horizontal, see Figure 1.3. Here, 2ℓ is the width of the roadway whereas C_1 and C_2 de-

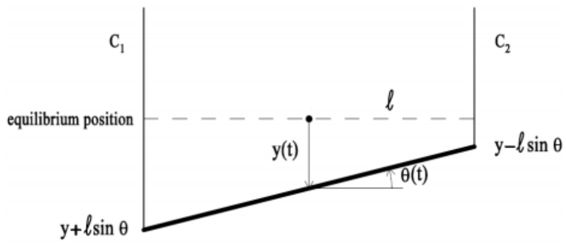

Figure 1.3: *Vertical and torsional displacements of the cross section of the roadway.*

note the two lateral hangers which have opposite extension behaviors. This model was revisited and complemented with multiple cross-sections in [6].

It is not the scope of this thesis to list all the existing mathematical models for suspension bridges, for which we refer to [29,30] and the references therein. However, most of these models fail to satisfy the general principle (GP) of classical mechanics by Goldstein-Poole-Safko [36, Section 11.7]:

> neither linear differential equations nor systems of less than three first order equations can exhibit chaos.

This principle suggests that **any model aiming to describe oscillating bridges should be nonlinear and with enough (mathematical) degrees of freedom,** which is also the opinion of Gazzola [29]. By "(mathematical) degrees of freedom" we mean here the order of a partial differential equation, or the number of initial conditions for an ordinary differential equation or a system of ordinary differential equations. Hence, the previous models, in our opinion, must be modified.

Some of the existing mathematical models which followed the TNB collapse are usually derived by approximating factors, by linearizing equations or by neglecting higher order terms. This leaves several doubts [30]:

Are these so obtained equations reliable?

Do these equations give satisfactory response?

On the other hand, the nonlinear behavior of suspension bridges is by now well established, see [16,29,48,68]. Therefore, the necessity of dealing with nonlinear models for suspension bridges is by now quite clear.

1.2 The new models

Quite recently, some attempts to improve suspension bridges performances can be found in [39] where, in particular, a careful analysis of the role played by the hangers is made. While from [39, p.1624], we quote that

> Research on the robustness of suspension bridges is at the very beginning.

The main scope of the present dissertation is to suggest and discuss several new mathematical models for suspension bridges.

Let us describe in detail the models considered in the sequel.

1.2.1 A beam model

As in [59, 81], the suspension bridge is viewed as an elastic beam which is suspended to a sustaining cable, where the beam and the cable are connected by a large number of hangers. Although this point of view rules out (GP), it appears to be a reasonable approximation since the width of the roadway is much smaller than its length.

In this case, we suggest a kind of beam model for suspension bridge. By analyzing the energies of the suspension bridge for displacement from the equilibrium position to a new position under a live load, one gets the total energy in the system. Then the corresponding Euler-Lagrange equation of the system is obtained by taking the critical points of the total energy. Combing with the hinged boundary condition, we obtain a fourth order ordinary differential problem:

$$\begin{cases} EIw''''(x) - H\left(\frac{w'(x)}{1+(y'(x))^2}\right)' \\ \qquad\qquad - \frac{w''(x)+y''(x)}{(1+(w'(x)+y'(x))^2)^{3/2}}h(w) = p, \quad x \in (0,L) \\ w(0) = w(L) = w''(0) = w''(L) = 0, \end{cases} \tag{1.2}$$

where L is the length of the roadway between the two towers, $w = w(x)$ denotes the displacement of the beam, $y = y(x)$ is the position of the cable at rest, E and I are, respectively, the elastic modulus of the material and the moment of inertia of the cross section so that EI is called the flexural rigidity, H is the horizontal tension in the cable when subject to the dead load $q = q(x)$, $h(w)$ represents the additional tension in the cable produced by the live load $p = p(x)$.

The additional tension $h(w)$ needs a special attention for solving the problem (1.2) due to the fact that it is a nonlinear nonlocal term. In literature, there are several different ways to approximate $h(w)$, see [32, 74, 75, 81]. Note that if one omits the higher order terms with respect to w' and y', then the problem (1.2) is the Melan problem (1.1), see [59].

Concerning this nonlinear nonlocal problem (1.2), we show that there exists at least one equilibrium position of the bridge. Moreover, the problem (1.2) admits a unique solution under suitable assumptions on the live load and the coefficients in the equation, see Theorem 2.1.

1.2.2 A plate model with small deformations

The most natural way is to view the roadway of the bridge as a thin rectangular plate Ω. This is also the opinion of Rocard [71, p.150]:

> The plate as a model is perfectly correct and corresponds mechanically to a vibrating suspension bridges.

In this case, Ferrero-Gazzola [27] suggested a plate model for dynamical suspension bridge based on the classical Kirchhoff-Love theory [44, 54]. The two short edges of the plate are assumed to be hinged whereas the two long edges are assumed to be free. They discussed the material nonlinearities, such as the behavior of the restoring force due to the hangers and the sustaining cables and analyzed the energies, such as the kinetic energy and the potential energy, appearing in the bridge. Then the evolution problem corresponding to the plate model reads: for $t > 0$

$$\begin{cases} u_{tt} + \mu u_t + \Delta^2 u + h(x,y,u) = f & (x,y) \in \Omega, \\ u(0,y,t) = u_{xx}(0,y,t) = 0 & y \in (-\ell,\ell), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0 & y \in (-\ell,\ell), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0 & x \in (0,\pi), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma) u_{xxy}(x,\pm\ell,t) = 0 & x \in (0,\pi), \\ u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y) & (x,y) \in \Omega, \end{cases} \tag{1.3}$$

where $\Omega = (0,\pi) \times (-\ell,\ell)$ ($2\ell \ll \pi$) represents the plate, $u = u(x,y,t)$ (downwards positive) is the vertical displacement of the plate, μu_t with $\mu > 0$ is the damping term due to the internal friction, $\sigma > 0$ denotes the Poisson ratio depends on the material, f is an external forcing term, $h(x,y,u)$ is the restoring force due to the hangers, $u_0(x,y)$ is the initial position of the plate and $u_1(x,y)$ is the initial vertical velocity of the plate.

Following Micheletti-Pistoia [60,61], we assume that the restoring force $h = au - |u|^{p-2}u$ with $2 < p < \infty$. Here $a = a(x,y,t)$ is a sign-changing and bounded measurable function. Then we have by supposing that $f \equiv 0$: for $t > 0$

$$\begin{cases} u_{tt} + \mu u_t + \Delta^2 u + au = |u|^{p-2}u & (x,y) \in \Omega, \\ u(0,y,t) = u_{xx}(0,y,t) = 0 & y \in (-\ell,\ell), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0 & y \in (-\ell,\ell), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0 & x \in (0,\pi), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma)u_{xxy}(x,\pm\ell,t) = 0 & x \in (0,\pi), \\ u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y) & (x,y) \in \Omega. \end{cases} \tag{1.4}$$

This is a non-coercive problem, which admits a unique local solution. We investigate the asymptotic behavior of the unique local solution of (1.4), see Theorem 3.2 and Theorem 3.3.

In Section 3.3, in order to describe the boundary behavior appearing in the actual bridges more realistically, we set up a model with dynamical boundary conditions which depend on the energy of the system: for every $t > 0$, $x \in (0,\pi)$ and $y \in (-\ell,\ell)$,

$$\begin{cases} u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \\ u_y(x,-\ell,t) - u_y(x,\ell,t) = 0, \\ u_y(x,-\ell,t) + u_y(x,\ell,t) = \alpha, \\ u(x,-\ell,t) + u(x,\ell,t) = \beta, \\ u_t(x,-\ell,t) - u_t(x,\ell,t) - \eta(t)(u(x,-\ell,t) - u(x,\ell,t)) = \theta(t)\gamma. \end{cases} \tag{1.5}$$

Assume that the restoring force h acts on every point of the plate and has a linear form, i.e. $h = ku$ with $k > 0$ the Hooke constant of the elastic hangers. Then by replacing the boundary conditions in (1.3) with (1.5), one gets a linear plate model with dynamical boundary conditions, which admits a unique explicit solution, see Section 3.3.3.

1.2.3 A plate model with large deformations

For small deformations, a linear plate theory is accurate enough to describe the behavior of the roadway. While in some cases, large deformations maybe appear and then geometric nonlinearities arise. Hence, one should stick to a nonlinear theory of the plate.

In 1910, the Hungarian physicist and engineer Theodore von Kármán [80] suggested a two-dimensional system in order to describe large deformations of a thin plate. This theory was considered as a breakthrough in several scientific communities, including in the National Advisory Committee for Aeronautics, an American federal agency during the 20th century: the purpose of this agency was to undertake, to promote, and to institutionalize aeronautical research and the von Kármán equations were studied for a comparison between theoretical and experimental results, see [52, 53]. In his report, Levy [52] writes that

> In the design of thin plates that bend under lateral and edge loading, formulas based on the Kirchhoff theory which neglects stretching and shearing in the middle surface are quite satisfactory provided that the deflections are small compared with the thickness. If deflections are of the same order as the thickness, the Kirchhoff theory may yield results that are considerably in error and a more rigorous theory that takes account of deformations in the middle surface should therefore be applied. The fundamental equations for the more exact theory have been derived by von Kármán.

In order to describe its structural behavior, we view the roadway of the bridge as a thin plate subject to the restoring force due to the hangers and behaving nonlinearly: we adapt the quasilinear von Kármán [80] model to a suspension bridge. Ciarlet [17] provided an important justification of the von Kármán equations. He made an asymptotic expansion with respect to the thickness of a three-dimensional class of elastic plates under suitable loads. He then showed that the leading term of the expansion solves a system of equations equivalent to those of von Kármán. Davet [21] pursued further and proved that the von Kármán equations may be justified by asymptotic expansion methods starting from very general 3-dimensional constitutive laws.

An important contribution of Berger-Fife [11] reduces the von Kármán system to a variational problem and tackles it with critical point and bifurcation theories. Subsequently, Berger [10] made a full analysis of the unloaded clamped plate problem (Dirichlet boundary conditions) which is somehow the simplest one but does not model the physical situation of a bridge. The loaded clamped plate was analyzed in [45,46] where existence and possible nonuniqueness results were obtained. Different boundary conditions for the hinged plate (named after Navier) and for free boundaries were then analyzed with the same tools by Berger-Fife [12]. Since free edges of the plate are considered, this last paper is of particular interest for our purposes. As clearly stated by Ciarlet [17, p.353] the boundary conditions for the Airy function *are often left fairly vague in the literature*; we take them in a "dual form", that is, more restrictions for the edges yield less restrictions for the Airy function and viceversa.

Following the setting in [27] (see also [3,82,83]), we consider a thin rectangular plate where the two short edges are assumed to be hinged whereas the two long edges are assumed to be free. The plate is subject to three actions:
• normal dead and live loads acting orthogonally on the plate;
• edge loading, also called buckling loads, namely compressive forces along its edges;
• the restoring force due to the hangers, which acts in a neighborhood of the long edges.

Then the model describing the suspension bridge involves a fourth order quasilinear elliptic system:

$$\begin{cases} \Delta^2\Phi = -[u,u] & \text{in } \Omega \\ \Delta^2 u + \Upsilon(y)g(u) = [\Phi,u] + f - \lambda u_{xx} & \text{in } \Omega \\ u = \Phi = u_{xx} = \Phi_{xx} = 0 & \text{on } \{0,\pi\}\times(-\ell,\ell) \\ u_{yy} + \sigma u_{xx} = u_{yyy} + (2-\sigma)u_{xxy} = 0 & \text{on } (0,\pi)\times\{\pm\ell\} \\ \Phi = \Phi_y = 0 & \text{on } (0,\pi)\times\{\pm\ell\}\,, \end{cases} \tag{1.6}$$

where $\Omega = (0,\pi)\times(-\ell,\ell)$ ($2\ell \ll \pi$) represents the plate, $u = u(x,y,t)$ (downwards positive) is the vertical displacement of the plate, Φ is the Airy stress function, the parameter $\lambda \geq 0$ measures the magnitude of the compressive forces acting on $\partial\Omega$, $\sigma > 0$ denotes the Poisson ratio depends on the material, f is an external forcing term, $\Upsilon(y)g(u)$ is the restoring force due to the hangers.

For the quasilinear system (1.6) modeling the suspension bridge, we prove that there exists at least one solution. Moreover, we show the uniqueness and multiplicity of the equilibrium positions of the bridge in suitable situation, see Theorem 4.1.

CHAPTER 2

A nonlinear beam model for suspension bridge

This chapter is devoted to a beam model for suspension bridge. The bridge is viewed as an elastic beam which is suspended to a sustaining cable, where the beam and the cable are connected by a large number of hangers. Although this point of view does not follow the general principle (GP), it appears reasonable since the width of the roadway is much smaller than its length.

Assuming that the beam is hinged at the endpoints, we obtain a nonlinear nonlocal problem by analyzing the energy in the bridge and by applying a variational principle to the total energy. The length increment of the cable plays an important role in this model. After analyzing the behavior of the energy functional, we show that there exists at least one equilibrium position of the beam.

2.1 The beam model

As in [81, Section VII.5], see also [32, 59], the suspension bridge is viewed as an elastic beam which is suspended to a sustaining cable, where the beam and the cable are connected by a large number of hangers, see Figure 2.1.

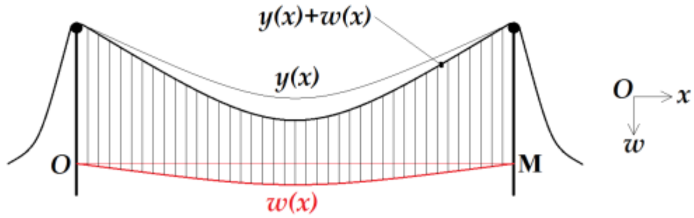

Figure 2.1: *Beam sustained by a cable through parallel hangers.*

The point O is the origin of the orthogonal coordinate system and positive displacements are oriental downwards. The point M has the coordinate $(0, L)$, where L is the length of the roadway between two towers. The cable is modeled as a perfectly flexible string and the hangers are assumed to be inextensible.

2.1.1 The additional tension

Assume that the system (the suspension bridge) is subject to an action of dead load $q(x)$, including the weight of the cable, the weight of the hangers and the dead weight of the roadway without producing a bending moment in the beam. At this moment, the cable is in the position $y(x)$, while the unloaded beam is the segment connecting O and M, see Figure 2.1. Then the horizontal component $H > 0$ of the tension remains constant. Therefore, there is an equilibrium position in the system and this deduces an equation (see [81, (1.3), Section VII]):

$$Hy''(x) = -q(x), \qquad \forall x \in (0, L). \tag{2.1}$$

If the endpoints of the cable are at the same level η and if the dead load $q(x)$ is constant, i.e. $q(x) = q$, then the solution of (2.1) is

$$y(x) = \eta + \frac{q}{2H}x(L - x).$$

Hence, the cable takes the shape of a parabolic function. Since y is positive downwards, it has a $\cup$-shaped graph. Moreover,

$$y'(x) = \frac{q}{H}\left(\frac{L}{2} - x\right), \quad y''(x) = -\frac{q}{H}, \qquad \forall x \in (0, L). \tag{2.2}$$

Then the length of the cable at this moment is

$$
\begin{aligned}
L_c &= \int_0^L \sqrt{1+y'(x)^2}dx \\
&= \frac{L}{2}\sqrt{1+\frac{q^2L^2}{4H^2}} + \frac{H}{q}\log\left(\frac{qL}{2H}+\sqrt{1+\frac{q^2L^2}{4H^2}}\right). \qquad (2.3)
\end{aligned}
$$

When a live load $p = p(x)$ is added on the roadway of the bridge, the beam may be out of the horizontal position and it produces a displacement $w = w(x)$ (the positive displacement are oriental downwards). Note that the hangers which connect the cable and the beam are assumed to be inextensible, then the deflection of the cable should also be added by w, i.e. the cable is in a new position $y + w$. In this case, the length of the cable is

$$
L_c^* = \int_0^L \sqrt{1+(w'(x)+y'(x))^2}dx.
$$

Hence, there is an increment length of the cable due to the deformation w. We denote it by

$$
\Gamma(w) := \int_0^L \left[\sqrt{1+(w'(x)+y'(x))^2} - \sqrt{1+y'(x)^2}\right]dx.
$$

According to (2.2) and (2.3), the exact value of $\Gamma(w)$ is

$$
\Gamma(w) = L_c^* - L_c = \int_0^L \sqrt{1+\left(w'(x)+\frac{q}{H}\left(\frac{L}{2}-x\right)\right)^2}dx - L_c. \qquad (2.4)
$$

If A denotes the cross-sectional area of the cable and E_c is the modulus of elasticity of the cable, then the additional tension in the cable produced by the live load p is given by

$$
h(w) = \frac{E_cA}{L_c}\Gamma(w).
$$

2.1.2 The energies in the bridge

In this subsection, we analyze the energies of the system from the original position to the new position as Arioli-Gazzola did in [5]. First, note that the beam representing the bridge is assumed to be hinged at its endpoints. Hence, the boundary conditions are

$$
w(0) = w(L) = w''(0) = w''(L) = 0. \qquad (2.5)
$$

• **Energy produced by the live load and dead load.** Under the live load p, the system deforms from the original position to a new position. In this case, it generates a kind of energy into the system due to the live load. Moreover, the gravitational energy produced by the dead load q also appears. Hence, the energy due to the live and dead load is given by

$$\mathcal{E}_L = \int_0^L (p+q)w dx.$$

• **Bending and stretching energy of the beam.** The elastic energy stored in a deformed beam consists of terms that can be described by bending and by stretching. The bending energy of a beam depends on its curvature, see [81] and also [31] for a more recent approach and further references. Assume that E is the elastic modulus of the material and Idx denotes the moment of inertia of a cross section of length dx, then the constant quantity EI is the flexural rigidity. The energy necessary to bend the beam is the square of the curvature times half the flexural rigidity:

$$\begin{aligned}\mathcal{E}_{B_1} &= \frac{EI}{2}\int_0^L \frac{(w'')^2}{(1+(w')^2)^3}\sqrt{1+(w')^2}dx \\ &= \frac{EI}{2}\int_0^L \frac{(w'')^2}{(1+(w')^2)^{5/2}}dx.\end{aligned}$$

If large deformations are involved, the strain-displacement relation is not linear and a possible nonlinear model was suggested by Woinowsky-Krieger [86]: he modified the classical Bernoulli-Euler beam theory by assuming a nonlinear dependence of the axial strain on the deformation gradient and by taking into account the stretching of the beam due to its elongation in the longitudinal direction. In this situation there is a coupling between bending and stretching and the stretching energy is proportional to the elongation of the beam which results in

$$\mathcal{E}_{B_2} = \frac{\gamma}{2}\left(\int_0^L \sqrt{1+(w')^2} - 1dx\right)^2 \approx \frac{\gamma}{8}\left(\int_0^L (w')^2 dx\right)^2,$$

where $\gamma > 0$ is the elastic constant of the beam.

Assuming that w' is small, an asymptotic expansion yields that

$$\begin{aligned}\mathcal{E}_{B_1} + \mathcal{E}_{B_2} &\approx \frac{EI}{2}\int_0^L (w'')^2\left(1 - \frac{5}{2}(w')^2\right)dx + \frac{\gamma}{8}\left(\int_0^L (w')^2 dx\right)^2 \\ &\approx \frac{EI}{2}\int_0^L (w'')^2 dx.\end{aligned}$$

Therefore, from now on, we simply take the energy necessary to bend and stretch a beam by

$$\mathcal{E}_B = \frac{EI}{2}\int_0^L (w'')^2 dx.$$

• **Stretching energy of the cable.** Since the cable is assumed to be perfectly flexible, there is no resistance to bending. Hence, the only internal force is the tension of the cable. The tension of the cable consists of two parts, the tension at rest

$$H(x) = H\sqrt{1+(y'(x))^2}$$

and the additional tension

$$h(w) = \frac{E_c A}{L_c}\Gamma(w)$$

due to the increment length of the cable. The latter requires the energy

$$\mathcal{E}_{C_1}(w) = \frac{E_c A}{2L_c}\Gamma(w)^2.$$

On the other hand, the amount of energy needed to deform the cable at rest under the tension $H(x)$ in the infinitesimal interval $[x, x+dx]$ from the original position $y(x)$ to the new position $y(x)+w(x)$ is the variation of length times the tension, that is

$$\mathcal{E}_{C_2} dx = H\sqrt{1+(y'(x))^2}\Delta\Gamma dx,$$

where $\Delta\Gamma = \sqrt{1+(w'(x)+y'(x))^2} - \sqrt{1+y'(x)^2}$. Hence, the energy necessary to deform the whole cable at rest under the tension $H(x)$ is

$$\begin{aligned}\mathcal{E}_{C_2} &= H\int_0^L \sqrt{1+(y'(x))^2}\Delta\Gamma dx\\ &= H\int_0^L \left[1+(y'(x))^2\right]\left(\sqrt{1+\frac{2w'(x)y'(x)+(w'(x))^2}{1+(y'(x))^2}}-1\right)dx.\end{aligned}$$

Following the idea of Timoshenko [74, 75], i.e., $\sqrt{1+\varepsilon}\approx 1+\frac{\varepsilon}{2}$, one has

$$\mathcal{E}_{C_2} dx \approx H\int_0^L w'(x)y'(x)dx + \frac{H}{2}\int_0^L (w'(x))^2 dx.$$

However, this approximation is not correct due to the fact that the term $\left(\frac{2w'(x)y'(x)}{1+(y'(x))^2}\right)^2$ needs to be considered. Hence, it is suitable to adopt the approximation $\sqrt{1+\varepsilon} \approx 1+\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}$ for small ε. Then we have by assuming that $o((w'(x))^2)=0$

$$\begin{aligned}\mathcal{E}_{C_2} &\approx H\int_0^L \left[1+(y'(x))^2\right]\left(\frac{w'(x)y'(x)}{1+(y'(x))^2}+\frac{(w'(x))^2}{2(1+(y'(x))^2)^2}\right)dx\\ &= \frac{H}{2}\int_0^L \frac{(w'(x))^2}{1+(y'(x))^2}dx + H\int_0^L w'(x)y'(x)dx,\end{aligned}$$

By integration by part, it yields that

$$\mathcal{E}_{C_2} = \frac{H}{2}\int_0^L \frac{(w')^2}{1+(y')^2}dx + q\int_0^L wdx.$$

Therefore, the energy necessary to stretch the cable is given by

$$\mathcal{E}_C = \mathcal{E}_{C_1}+\mathcal{E}_{C_2} = \frac{H}{2}\int_0^L \frac{(w')^2}{1+(y')^2}dx + q\int_0^L wdx + \frac{E_cA}{2L_c}\Gamma(w)^2.$$

Summarizing, the total energy in the system after the deformation w due to the live load p is

$$\begin{aligned}\mathcal{E} =& \mathcal{E}_B+\mathcal{E}_C-\mathcal{E}_L\\ =&\frac{EI}{2}\int_0^L (w'')^2dx + \frac{H}{2}\int_0^L \frac{(w')^2}{1+(y')^2}dx + \frac{E_cA}{2L_c}\Gamma(w)^2 - \int_0^L pwdx.\end{aligned}$$

2.1.3 The nonlinear problem

The Euler-Lagrange equation is obtained by taking the critical points of the energy $\mathcal{E}$. Then by recalling (2.2), we have

$$\begin{aligned}&EIw''''(x) - H\left(\frac{w'(x)}{1+(y'(x))^2}\right)'\\ &\qquad - \frac{w''(x)-q/H}{(1+(w'(x)+y'(x))^2)^{3/2}}h(w) = p, \quad x\in(0,L). \qquad (2.6)\end{aligned}$$

Therefore, together with the boundary condition (2.5), the new beam model for suspension bridges is

$$\begin{cases} EIw''''(x) - H\left(\frac{w'(x)}{1+(y'(x))^2}\right)' \\ \qquad -\frac{w''(x)-q/H}{(1+(w'(x)+y'(x))^2)^{3/2}}h(w) = p, \quad x\in(0,L) \\ w(0)=w(L)=w''(0)=w''(L)=0. \end{cases} \qquad (2.7)$$

Note that if one omits the higher order of $w'(x)$ and $y'(x)$, then (2.6) reads

$$EIw''''(x) - (H + h(w))w''(x) + h(w)y''(x) = p,$$

which is the Melan equation (1.1).

2.2 Existence and uniqueness results

For simplicity, let $a = EI$, $b = H$ and $c = \frac{E_c A}{L_c}$. Then we consider the problem

$$\begin{cases} aw''''(x) - b\left(\frac{w'(x)}{1+(y'(x))^2}\right)' - c\frac{w''(x)-q/H}{(1+(w'(x)+y'(x))^2)^{3/2}}\Gamma(w) = p, \quad x \in (0, L) \\ w(0) = w(L) = w''(0) = w''(L) = 0, \end{cases} \tag{2.8}$$

where $a, b, c > 0$ and the functional $\Gamma(w)$ is as in (2.4), which is nonlinear nonlocal and of indefinite sign.

Given $k \in [1, \infty]$, we denote the L^k-norm by $\|u\|_k$ for any $u \in L^k(0, L)$ and the standard $H^2 \cap H_0^1$-norm by $\|u\|_{H^2\cap H_0^1}$ for any $u \in H^2 \cap H_0^1(0, L)$. Recalling (2.2) and $a, b > 0$, one may define a new scalar product on the space $H^2 \cap H_0^1(0, L)$ by

$$(v, w)_y := a\int_0^L w''v''dx + b\int_0^L \frac{w'v'}{1 + (y')^2}dx \quad v, w \in H^2 \cap H_0^1(0, L). \tag{2.9}$$

This scalar product induces a new norm on $H^2 \cap H_0^1(0, L)$ denoted by

$$\|w\|_y := \left(a\int_0^L (w'')^2dx + b\int_0^L \frac{(w')^2}{1 + (y')^2}dx\right)^{1/2} \quad w \in H^2 \cap H_0^1(0, L), \tag{2.10}$$

which is equivalent to $\|w\|_{H^2\cap H_0^1}$ due to the facts that the function y' is bounded and that $a, b > 0$. Let $\mathcal{H}$ be the dual space of $H^2 \cap H_0^1(0, L)$. We denote by $\|\cdot\|_{\mathcal{H}}$ the $\mathcal{H}$-norm and by $\langle\cdot,\cdot\rangle$ the corresponding duality between $H^2 \cap H_0^1(0, L)$ and $\mathcal{H}$.

Since $u \in H^2 \cap H_0^1(0, L) \subset C^1([0, L])$, there exists $x_0 \in (0, L)$ such that $u'(x_0) = 0$. Hence, we have

$$u'(x) = \int_{x_0}^x u''(z)dz \le \|u''\|_1 \le \sqrt{L}\|u''\|_2,$$

which yields that

$$\|u'\|_1^2 \le L^3\|u''\|_2^2, \qquad \|u'\|_2^2 \le L^2\|u''\|_2^2.$$

Moreover,

$$\begin{aligned}\|u'\|_1^2 &\le \int_0^L (1+(y')^2)dx \int_0^L \frac{(u')^2}{1+(y')^2}dx \\ &= \left[1+\frac{q^2L^2}{12H^2}\right] L \int_0^L \frac{(u')^2}{1+(y')^2}dx\end{aligned}$$

and

$$\begin{aligned}\|u'\|_2^2 &\le \|1+(y')^2\|_\infty \int_0^L \frac{(u')^2}{1+(y')^2}dx \\ &= \left[1+\frac{q^2L^2}{4H^2}\right] \int_0^L \frac{(u')^2}{1+(y')^2}dx.\end{aligned}$$

Therefore,

$$\|u'\|_1^2 \le \alpha^2 \|u\|_y^2, \qquad \|u'\|_2^2 \le \beta^2 \|u\|_y^2 \tag{2.11}$$

with $\alpha, \beta > 0$ and

$$\alpha^2 = \frac{(12H^2+q^2L^2)L^3}{(12H^2+q^2L^2)a + 12H^2L^2b}, \quad \beta^2 = \frac{(4H^2+q^2L^2)L^2}{(4H^2+q^2L^2)a+4H^2L^2b}.$$

In addition, the simple inequality

$$\left|\sqrt{1+(\lambda+\mu)^2} - \sqrt{1+\mu^2}\right| \le |\lambda| \qquad \forall \lambda, \mu \in \mathbb{R} \tag{2.12}$$

implies that for any $u, v \in H^2 \cap H_0^1(0,L)$

$$|\Gamma(u)| \le \|u'\|_1, \qquad |\Gamma(u) - \Gamma(v)| \le \|u' - v'\|_1. \tag{2.13}$$

Assume that $p \in \mathcal{H}$, we say that $w \in H^2 \cap H_0^1(0,L)$ is a weak solution of (2.8) if for any $v \in H^2 \cap H_0^1(0,L)$, we have

$$(w,v)_y + \Gamma(w) \int_0^L \frac{(w'+y')v'}{\sqrt{1+(w'+y')^2}}dx = \langle p, v\rangle, \tag{2.14}$$

where $(\cdot,\cdot)_y$ is the scalar product (2.9) and the function y' is as in (2.2).

Then we prove

Theorem 2.1. *For all $p \in \mathcal{H}$, there exists a weak solution of the problem* (2.8). *Moreover, assume that*

$$0 < c < \frac{1}{\alpha^2}. \tag{2.15}$$

Then for any $p \in \mathcal{H}$ satisfying

$$\|p\|_{\mathcal{H}} < \frac{(1-c\alpha^2)^2}{2c\alpha\beta^2}, \tag{2.16}$$

the problem (2.8) *admits a unique weak solution $w \in H^2 \cap H_0^1(0,L)$.*

Proof. We divide several steps to prove Theorem 2.1. The first step is to introduce an equivalent definition of the weak solution of (2.8). To this aim we need to check the continuity and differentiability of the functional $\Gamma^2(w)$.

Lemma 2.1. *Let $\Gamma(w)$ be as in* (2.4)*. Then $\Gamma(w)^2$ is weakly continuous and differentiable on $H^2 \cap H_0^1(0,L)$.*

Proof. Let the sequence $\{w_n\}$ weakly converge to w in $H^2 \cap H_0^1(0,L)$. Then by the compact embedding we have

$$\begin{aligned} |\Gamma(w_n)^2 - \Gamma(w)^2| &= |\Gamma(w_n) + \Gamma(w)||\Gamma(w_n) - \Gamma(w)| \\ &\le C\|w_n' - w'\|_1 \to 0, \end{aligned}$$

which proves $\Gamma(w)^2$ is weakly continuous.

Now we show that it is differentiable. For any $w, v \in H^2 \cap H_0^1(0,L)$, we obtain

$$\begin{aligned} \langle (\Gamma(w)^2)', v\rangle &= \lim_{t\to 0} \frac{1}{t}\left(\Gamma(w+tv)^2 - \Gamma(w)^2\right) \\ &= \lim_{t\to 0} \frac{1}{t}\int_0^L \frac{(\Gamma(w+tv)+\Gamma(w))(2w'+2y'+tv')tv'}{\sqrt{1+(w'+tv'+y')^2}+\sqrt{1+(w'+y')^2}}dx \\ &= 2\Gamma(w)\int_0^L \frac{(w'+y')v'}{\sqrt{1+(w'+y')^2}}dx \\ &= -2\Gamma(w)\int_0^L \frac{(w''-q/H)v}{(1+(w'+y')^2)^{3/2}}dx. \end{aligned}$$

Then for any $w \in H^2 \cap H_0^1(0,L)$ and for $v \in H^2 \cap H_0^1(0,L)$ with $\|v\|_y = 1$, we have

$$\lim_{t\to 0} \frac{\Gamma(w+tv)^2 - \Gamma(w)^2 + 2t\Gamma(w)\int_0^L \frac{(w''-q/H)v}{(1+(w'+y')^2)^{3/2}}dx}{t} = 0,$$

which finishes the proof of Lemma 2.1. $\square$

Then the energy functional corresponding to the problem (2.8) is given by

$$J_p = J_p(w) = \frac{1}{2}\|w\|_y^2 + \frac{1}{2}\Gamma(w)^2 - \langle p, w\rangle, \qquad \text{for any } w \in H^2 \cap H_0^1(0, L).$$

According to Lemma 2.1, one has a one-to-one correspondence between the weak solutions of (2.8) and the critical points of the functional J_p:

Definition 2.1. *Let $p \in \mathcal{H}$. The function $w \in H^2 \cap H_0^1(0, L)$ is a weak solution of* (2.8) *if and only if w is a critical point of J_p.*

The next step is to prove the geometrical properties (coercivity) and compactness properties (Palais-Smale (PS) condition) of J_p.

Lemma 2.2. *For any $p \in \mathcal{H}$, the functional J_p is coercive and bounded below in $H^2 \cap H_0^1(0, L)$. Moreover, it satisfies the (PS) condition.*

Proof. Since $p \in \mathcal{H}$, we have for any $w \in H^2 \cap H_0^1(0, L)$

$$J_p \geq \frac{1}{2}\|w\|_y^2 - \langle p, w\rangle \geq \frac{1}{2}\|w\|_y^2 - \|p\|_{\mathcal{H}}\|w\|_y \geq -\frac{\|p\|_{\mathcal{H}}^2}{2},$$

which implies that the functional J_p is coercive and bounded below.

Next we prove J_p satisfies the (PS) condition. Let $\{w_n\}$ be a sequence such that $J_p(w_n)$ is bounded and $J_p'(w_n) \to 0$. Then there exists $M > 0$ such that

$$M \geq \frac{1}{2}\|w_n\|_y^2 + \frac{1}{2}\Gamma(w_n)^2 - \langle p, w_n\rangle \geq \frac{1}{2}\|w_n\|_y^2 - \|p\|_{\mathcal{H}}\|w_n\|_y.$$

Hence, $\|w_n\|_y$ is bounded and there exists some $\overline{w} \in H^2 \cap H_0^1(0, L)$ such that w_n weakly converges to $\overline{w}$ in $H^2 \cap H_0^1(0, L)$.

We claim that $J_p'(\overline{w}) = 0$. In fact, by Lemma 2.1, it is easy to verify that the functional J_p is weakly continuous and differentiable. Therefore, for any $v \in H^2 \cap H_0^1(0, L)$ there results that

$$\langle J_p'(w_n), v\rangle \to \langle J_p'(\overline{w}), v\rangle.$$

Hence, $\langle J_p'(\overline{w}), v\rangle = 0$ for any $v \in H^2 \cap H_0^1(0, L)$, i.e. $J_p'(\overline{w}) = 0$.

To complete the proof of Lemma 2.2, we need to prove that $w_n \to \overline{w}$ strongly in $H^2 \cap H_0^1(0, L)$. Thanks to $J_p'(\overline{w}) = 0$, it follows that by denoting $(\Gamma(w)^2)' = 2\Gamma(w)\langle\Gamma'(w), w\rangle$ for any $w \in H^2 \cap H_0^1(0, L)$

$$\begin{aligned}\langle J_p'(w_n), w_n\rangle &= \|w_n\|_y^2 + c\Gamma(w_n)\langle\Gamma'(w_n), w_n\rangle - \langle p, w_n\rangle \\ &\to 0 = \langle J_p'(\overline{w}), \overline{w}\rangle = \|\overline{w}\|_y^2 + c\Gamma(\overline{w})\langle\Gamma'(\overline{w}), \overline{w}\rangle - \langle p, \overline{w}\rangle.\end{aligned}$$

Now we estimate

$$\begin{aligned}
&|\Gamma(w_n)\langle\Gamma'(w_n),w_n\rangle-\Gamma(\overline{w})\langle\Gamma'(\overline{w}),\overline{w}\rangle|\\
&\quad=|(\Gamma(w_n)-\Gamma(\overline{w}))\langle\Gamma'(\overline{w}),\overline{w}\rangle+\Gamma(\overline{w})(\langle\Gamma'(w_n),w_n\rangle-\langle\Gamma'(\overline{w}),\overline{w}\rangle)|\\
&\quad\le|(\Gamma(w_n)-\Gamma(\overline{w}))\langle\Gamma'(\overline{w}),\overline{w}\rangle|+|\Gamma(\overline{w})(\langle\Gamma'(w_n),w_n\rangle-\langle\Gamma'(\overline{w}),\overline{w}\rangle)|\\
&\quad:=I_1+I_2.
\end{aligned}$$

Since $\|w_n\|_y$ and $\|\overline{w}\|_y$ are bounded, by recalling Lemma 2.1 and (2.12)-(2.13) we have

$$\begin{aligned}
I_1&\le|\Gamma(w_n)-\Gamma(\overline{w})|\left|\int_0^L\frac{(\overline{w}'+y')\overline{w}'}{\sqrt{1+(\overline{w}'+y')^2}}dx\right|\\
&\le\|\overline{w}'\|_1\|w_n'-\overline{w}'\|_1\le C\|w_n'-\overline{w}'\|_1.
\end{aligned}$$

and

$$\begin{aligned}
I_2=&\left|\Gamma(\overline{w})\int_0^L\left(\frac{(w_n'+y')w_n'}{\sqrt{1+(w_n'+y')^2}}-\frac{(\overline{w}'+y')\overline{w}'}{\sqrt{1+(\overline{w}'+y')^2}}\right)dx\right|\\
\le&\left|\Gamma(\overline{w})\int_0^L\frac{[(w_n'+y')w_n'-(\overline{w}'+y')\overline{w}']\sqrt{1+(\overline{w}'+y')^2}}{\sqrt{1+(w_n'+y')^2}\sqrt{1+(\overline{w}'+y')^2}}dx\right|\\
&+\left|\Gamma(\overline{w})\int_0^L\frac{(\overline{w}'+y')\overline{w}'[\sqrt{1+(\overline{w}'+y')^2}-\sqrt{1+(w_n'+y')^2}]}{\sqrt{1+(w_n'+y')^2}\sqrt{1+(\overline{w}'+y')^2}}dx\right|\\
\le&\|\overline{w}'\|_1\left(\int_0^L|w_n'+\overline{w}'+y'|\cdot|w_n'-\overline{w}'|dx+\int_0^L|\overline{w}'|\cdot|w_n'-\overline{w}'|dx\right)\\
\le&C\|w_n'-\overline{w}'\|_2.
\end{aligned}$$

Then it yields that by compact embedding

$$|\Gamma(w_n)\langle\Gamma'(w_n),w_n\rangle-\Gamma(\overline{w})\langle\Gamma'(\overline{w}),\overline{w}\rangle|\to 0,$$

which together with $\langle p,w_n\rangle\to\langle p,\overline{w}\rangle$ by the compact embedding leads to $\|w_n\|_y\to\|\overline{w}\|_y$. By the weakly convergence $w_n\rightharpoonup\overline{w}$ in $H^2\cap H_0^1(0,L)$, it holds that

$$w_n\to\overline{w}\qquad\text{in } H^2\cap H_0^1(0,L).$$

And then the proof of Lemma 2.2 is finished. □

Step 3. By Lemma 2.2, the functional J_p admits a global minimum in $H^2\cap H_0^1(0,L)$ for any $p\in\mathcal{H}$. This minimum point is a critical point for J_p

and hence, by Definition 2.1, it gives a weak solution of (2.8). This proves the first part of Theorem 2.1.

Step 4. This step is to prove the uniqueness result. We first show that for any $c > 0$ and any $p \in \mathcal{H}$, the weak solutions of (2.8) are bounded in $H^2 \cap H_0^1(0, L)$.

Let w be a weak solution of (2.8), then by (2.14) we have

$$\|w\|_y^2 + c\Gamma(w) \int_0^L \frac{(w' + y')w'}{\sqrt{1 + (w' + y')^2}} dx = \langle p, w \rangle \leq \|p\|_{\mathcal{H}} \|w\|_y. \quad (2.17)$$

Denote $\Upsilon = \Gamma(w) \int_0^L \frac{(w'+y')w'}{\sqrt{1+(w'+y')^2}} dx$, then we have by recalling the definition of $\Gamma(w)$

$$\begin{aligned}
\Upsilon =& \int_0^L \sqrt{1 + (w' + y')^2} dx \int_0^L \frac{(w' + y')w'}{\sqrt{1 + (w' + y')^2}} dx \\
& - L_c \int_0^L \frac{(w' + y')w'}{\sqrt{1 + (w' + y')^2}} dx \\
\geq & \int_0^L \sqrt{1 + (w' + y')^2} dx \int_0^L \frac{w'y'}{\sqrt{1 + (w' + y')^2}} dx \\
& - L_c \int_0^L \frac{(w' + y')w'}{\sqrt{1 + (w' + y')^2}} dx \\
=& \Gamma(w) \int_0^L \frac{(w' + y')y'}{\sqrt{1 + (w' + y')^2}} dx - \Gamma(w) \int_0^L \frac{(y')^2}{\sqrt{1 + (w' + y')^2}} dx \\
& + L_c \int_0^L \frac{w'y'}{\sqrt{1 + (w' + y')^2}} dx - L_c \int_0^L \frac{(w' + y')w'}{\sqrt{1 + (w' + y')^2}} dx \\
\geq & - \|w'\|_1 \|y'\|_1 - \|w'\|_1 \|y'\|_2^2 - \frac{qLL_c}{2H} \|w'\|_1 - L_c \|w'\|_1 \\
\geq & - \left(\frac{qL^2}{4H} + \frac{q^2 L^3}{12H^2} + \frac{qL + 2H}{2H} L_c \right) \alpha \|w\|_y.
\end{aligned}$$

Therefore, it yields that

$$\|w\|_y \leq \left(\frac{qL^2}{4H} + \frac{q^2 L^3}{12H^2} + \frac{qL + 2H}{2H} L_c \right) c\alpha + \|p\|_{\mathcal{H}}. \quad (2.18)$$

Now we prove the uniqueness. If $0 < c < \alpha^{-2}$, we are able to show that (2.18) has a different bound, which is helpful to the proof in the sequel. By

(2.11) and (2.13), it deduces that

$$|\Gamma(w)| \left| \int_0^L \frac{(w'+y')w'}{\sqrt{1+(w'+y')^2}} dx \right| \le \|w'\|_1^2 \le \alpha^2 \|w\|_y^2.$$

Since $0 < c < \alpha^{-2}$, we obtain from (2.17)

$$\|w\|_y \le (1 - c\alpha^2)^{-1} \|p\|_{\mathcal{H}}. \tag{2.19}$$

For any fixed $p \in \mathcal{H}$, we denote

$$R = (1 - c\alpha^2)^{-1} \|p\|_{\mathcal{H}}. \tag{2.20}$$

Then one can define the closed ball by

$$B_R := \{ w \in H^2 \cap H_0^1(0, L); \|w\|_y \le R \}$$

and (2.19) shows that all the solutions of (2.8) with $0 < c < \alpha^{-2}$ are in B_R.

Next we consider the linear problem for any $v \in H^2 \cap H_0^1(0, L)$,

$$\begin{cases} aw''''(x) - b\left(\frac{w'(x)}{1+(y'(x))^2}\right)' = c\frac{v''(x)-q/H}{(1+(v'(x)+y'(x))^2)^{3/2}}\Gamma(v) + p, \quad x \in (0, L) \\ w(0) = w(L) = w''(0) = w''(L) = 0, \end{cases} \tag{2.21}$$

where $0 < c < \alpha^{-2}$.

It is easy to see that $\frac{v''(x)-q/H}{(1+(v'(x)+y'(x))^2)^{3/2}}\Gamma(v) + p \in \mathcal{H}$. Hence, there exists a unique solution of (2.21) due to the Lax-Milgram theorem. This allows us to define a map by

$$\Phi : B_R \to H^2 \cap H_0^1(0, L); \quad \Phi(v) = w$$

with w being the unique solution of (2.21).

Lemma 2.3. $\Phi(B_R) \subseteq B_R$.

Proof. For any fixed $v \in B_R$, the solution $w = \Phi(v)$ of (2.21) satisfies

$$\begin{aligned} \|w\|_y^2 &= c\Gamma(v) \int_0^L \frac{(v'' - q/H)w}{(1 + (v' + y')^2)^{3/2}} dx + \langle p, w \rangle \\ &\le -c\Gamma(v) \int_0^L \frac{(v' + y')w'}{\sqrt{1 + (v' + y')^2}} dx + \|p\|_{\mathcal{H}} \|w\|_y \\ \text{by (2.13)} \quad &\le c\|v'\|_1 \|w'\|_1 + \|p\|_{\mathcal{H}} \|w\|_y \\ \text{by (2.11)} \quad &\le \left(c\alpha^2 \|v\|_y + \|p\|_{\mathcal{H}} \right) \|w\|_y \le \left(c\alpha^2 R + \|p\|_{\mathcal{H}} \right) \|w\|_y \\ &= R\|w\|_y. \end{aligned}$$

Hence, $\|w\|_y \le R$, which shows that $\Phi(B_R) \subseteq B_R$. $\square$

Lemma 2.4. Φ *is a contractive map.*

Proof. Take $v_1, v_2 \in B_R$ and let $w_1 = \Phi(v_1)$, $w_2 = \Phi(v_2)$, then we have for all $u \in H^2 \cap H_0^1(0,L)$

$$(w_i, u)_y = c\Gamma(v_i)\int_0^L \frac{(v_i'' - q/H)u}{(1+(v_i'+y')^2)^{3/2}}dx + \langle p, u\rangle \qquad i = 1, 2.$$

Subtracting these two equations, taking $u = w = w_1 - w_2$ and recalling (2.2), we obtain that by applying integration by part

$$\begin{aligned}
\|w\|_y^2 =& c\Gamma(v_2)\int_0^L \frac{(v_2'+y')w'}{\sqrt{1+(v_2'+y')^2}}dx - c\Gamma(v_1)\int_0^L \frac{(v_1'+y')w'}{\sqrt{1+(v_1'+y')^2}}dx \\
=& c\Gamma(v_2)\left(\int_0^L \frac{(v_2'+y')w'}{\sqrt{1+(v_2'+y')^2}}dx - \int_0^L \frac{(v_1'+y')w'}{\sqrt{1+(v_1'+y')^2}}dx\right) \\
&+ c\left(\Gamma(v_2) - \Gamma(v_1)\right)\int_0^L \frac{(v_1'+y')w'}{\sqrt{1+(v_1'+y')^2}}dx \\
:=& \Lambda_1 + \Lambda_2,
\end{aligned}$$

where $w = w_1 - w_2$. Now we estimate Λ_1, Λ_2. By (2.11)-(2.13), one has

$$\begin{aligned}
\Lambda_1 =& c\Gamma(v_2)\int_0^L \frac{(v_2'+y')\sqrt{1+(v_1'+y')^2}}{\sqrt{1+(v_1'+y')^2}\sqrt{1+(v_2'+y')^2}}w'dx \\
&- c\Gamma(v_2)\int_0^L \frac{(v_1'+y')\sqrt{1+(v_2'+y')^2}}{\sqrt{1+(v_1'+y')^2}\sqrt{1+(v_2'+y')^2}}w'dx \\
=& c\Gamma(v_2)\int_0^L \frac{(v_2'+y')\left[\sqrt{1+(v_1'+y')^2} - \sqrt{1+(v_2'+y')^2}\right]}{\sqrt{1+(v_1'+y')^2}\sqrt{1+(v_2'+y')^2}}w'dx \\
&+ c\Gamma(v_2)\int_0^L \frac{(v_2'-v_1')}{\sqrt{1+(v_1'+y')^2}}w'dx \\
\le& 2c\|v_2'\|_1\int_0^L |v_1'-v_2'||w'|dx \le 2c\|v_2'\|_1\|v_1'-v_2'\|_2\|w'\|_2 \\
\le& 2c\alpha\beta^2 R\|v_1 - v_2\|_y\|w\|_y.
\end{aligned}$$

For Λ_2, we have by (2.11) and (2.13)

$$\Lambda_2 \le c\|v_1' - v_2'\|_1\|w'\|_1 \le c\alpha^2\|v_1 - v_2\|_y\|w\|_y.$$

Hence, by (2.20)

$$
\begin{aligned}
\|\Phi(v_1) - \Phi(v_2)\|_y &\le c\alpha \left(\alpha + 2\beta^2 R\right) \|v_1 - v_2\|_y \\
&= c\alpha \left(\alpha + \frac{2\beta^2 \|p\|_{\mathcal{H}}}{(1 - c\alpha^2)}\right) \|v_1 - v_2\|_y := \rho \|v_1 - v_2\|_y.
\end{aligned}
$$

Note that the condition (2.16) yields that $0 < \rho < 1$. Then

$$
\|\Phi(v_1) - \Phi(v_2)\|_y = \|w\|_y \le \rho \|v_1 - v_2\|_y \qquad \text{with } 0 < \rho < 1
$$

and we prove the Lemma 2.4. □

Then by the Banach Contraction principle (see [37, Section 1, Theorem 1.1]), there exists a unique fixed point in B_R, which is the unique solution to (2.8) with $0 < c < \alpha^{-2}$ if $p \in \mathcal{H}$ satisfies (2.16) and we finish the proof of Theorem 2.1. □

As a complement to Theorem 2.1, let us discuss what happens in the opposite "limit" case, i.e. $c \to +\infty$. The problem (2.8) degenerates to:

$$
\begin{cases}
\frac{w'' - q/H}{(1 + (w' + y')^2)^{3/2}} \Gamma(w) = 0, & x \in (0, L) \\
w(0) = w(L) = 0,
\end{cases}
\tag{2.22}
$$

which admits infinitely many solutions in $H^2 \cap H_0^1(0, L)$. Clearly, $w = 0$ and $w = -\frac{q}{2H}(L - x)x \in H^2 \cap H_0^1(0, L)$ are two solutions.

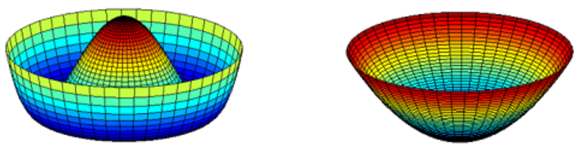

Figure 2.2: *Qualitative shape of the graphs of the action functionals $w \mapsto \frac{c}{2}\Gamma^2(w)$ (left) and $w \mapsto J_p(w) - \frac{c}{2}\Gamma^2(w)$ (right).*

Now we show that problem (2.22) has more solutions in $H^2 \cap H_0^1(0, L)$. If w belongs to $H^2 \cap H_0^1(0, L)$ such that $\Gamma(w) = 0$, then it is a solution to (2.22). Then we consider the functional $\Gamma(w)$.

We shift it by $Y(x) := \frac{q}{2H} x(L - x)$ (so that $Y \in H^2 \cap H_0^1(0, L)$ and $Y' = y'$) and, for all $0 \not\equiv w \in H^2 \cap H_0^1(0, L)$, we define the real function

$$
\gamma_w(t) := \Gamma(tw - Y) = \int_0^L \left[\sqrt{1 + (tw')^2} - \sqrt{1 + (y')^2}\right] dx, \quad t \in \mathbb{R}.
$$

Clearly, $\gamma_w(\pm\infty) = +\infty$ and γ_w is strictly convex in $\mathbb{R}$. Since $\gamma_w(0) < 0$, there exist $T_w^- < 0 < T_w^+$ such that $\gamma_w(T_w^\pm) = 0$. Hence, for any $w \neq 0$ we have $\Gamma(T_w^\pm w - Y) = 0$, that is, $T_w^\pm w - Y$ solves (2.22); therefore, (2.22) admits infinitely many solutions.

The qualitative graph of the functional $\frac{c}{2}\Gamma^2$ is depicted in Figure 2.2; since the functional $J_p - \frac{c}{2}\Gamma^2$ is convex (see again Figure 2.2), if c is large then the behavior of the functional J_p is not clear; in this situation, the uniqueness and/or multiplicity for (2.8) is an open problem.

CHAPTER 3

A plate model for suspension bridge with small deformations

In this chapter, we focus our attention on the fourth order partial differential equation modeling the dynamical suspension bridges, suggested by Ferrero-Gazzola [27]. This model is based on the linear Kirchhoff-Love plate theory, see [44, 54]. Ferrero-Gazzola [27] analyzed the energies, such as the kinetic energy and the potential energy, appearing in the bridge, and obtained the equation by applying variational principles (see [22]) to the difference between the kinetic energy and the total potential energy.

Assume that the external force vanishes and the restoring force due to the hangers is in a special nonlinear case. Then we have a non-coercive evolution problem, which admits a unique local solution. The asymptotic behavior of the unique local solution is investigated for different initial conditions, see Section 3.2.

In Section 3.3, in order to describe the boundary behavior of the roadway more likely, we introduce dynamical boundary conditions which depend on the energy of the bridge. Then we obtain a linear evolution problem with dynamical boundary conditions modeling the suspension bridge by assuming that the restoring force due to the hangers is in a linear case. By transferring this linear evolution problem to a simplified variational problem, we prove that there exists a unique explicit solution of the original linear problem.

3.1 The plate model

In this section, we recall the plate model for dynamical suspension bridges suggested by Ferrero-Gazzola [27]. There they discussed the material nonlinearity, such as the restoring force due to the hangers and the sustaining cables.

3.1.1 The energy for a plate modeling bridge

Let $\Omega = (0, L) \times (-\ell, \ell) \subset \mathbb{R}^2$ represent the roadway of the suspension bridge, where L is the length of the roadway and 2ℓ is its width. For simplicity, we take $L = \pi$, then the plate $\Omega = (0, \pi) \times (-\ell, \ell)$, see Figure 3.1. In this case, a realistic assumption is that $2\ell \ll \pi$.

Figure 3.1: *The rectangular plate Ω.*

It is well known that the bending energy of a plate involves the curvatures of the surface. Let κ_1 and κ_2 denote the principal curvatures of the graph of a smooth function $u = u(x, y)$, which represents the vertical displacement of the plate in the downwards direction, then according to [28, 44] (see also [31, Section 1.1.2]), a simple model for the bending energy of the deformed plate Ω due to an external force f is

$$\mathbb{E}_B = \frac{Ed^3}{12(1-\sigma^2)} \int_\Omega \left(\frac{\kappa_1^2}{2} + \frac{\kappa_2^2}{2} + \sigma\kappa_1\kappa_2 - fu \right) dxdy,$$

where d denotes the thickness of the plate, σ is the Poisson ratio defined by $\sigma = \lambda/(2(\lambda + \mu))$ and E is the Young modulus defined by $E = 2\mu(1 + \sigma)$ with the so-called Lamé constants λ, μ that depend on the material. For physical reasons it holds that $\mu > 0$ and usually $\lambda > 0$ so that

$$0 < \sigma < \frac{1}{2}. \tag{3.1}$$

For small deformations u, one has the approximations

$$(\kappa_1+\kappa_2)^2 \approx (\Delta u)^2, \quad \kappa_1\kappa_2 \approx \det(D^2u) = u_{xx}u_{yy} - u_{xy}^2$$

and then

$$\frac{\kappa_1^2}{2} + \frac{\kappa_2^2}{2} + \sigma\kappa_1\kappa_2 \approx \frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}).$$

Therefore, from now on we denote the bending energy by

$$\mathbb{E}_B = \frac{Ed^3}{12(1-\sigma^2)} \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) - fu\right) dxdy. \tag{3.2}$$

Since the roadway is suspended to the hangers, there exists an action, which is concentrated in the union of two thin strips parallel to the two horizontal edges of the plate Ω, i.e. in a set of the type (see Figure 3.1):

$$\omega := (0,\pi) \times [(-\ell, -\ell+\epsilon) \cup (\ell-\epsilon, \ell)] \quad \text{with } \epsilon > 0 \text{ small.}$$

In order to describe the action of the hangers, Ferrero-Gazzola [27] introduced a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$g(s) = 0 \text{ for any } s \leq 0, \quad g'(0^+) > 0, \quad g'(s) > 0 \text{ for any } s > 0$$

and the restoring force due to the hangers is suggested to take the form

$$h(x,y,u) = \Upsilon(y)g(u + \gamma x(\pi - x)), \tag{3.3}$$

where Υ is the characteristic function of $(-\ell, -\ell+\epsilon) \cup (\ell-\epsilon, \ell)$ and $\gamma > 0$.

More generally, one can consider a force h satisfying the following:

$$h : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function,} \tag{3.4}$$

$$s \mapsto h(\cdot,\cdot,s) \text{ is nondecreasing in } \mathbb{R}, \quad \exists \overline{s} \in \mathbb{R}, h(\cdot,\cdot,\overline{s}) = 0. \tag{3.5}$$

The restoring force h admits an elastic potential energy given by

$$\mathbb{E}_H = \int_\Omega H(x,y,u)dxdy, \quad H(x,y,s) = \int_{\overline{s}}^{s} h(x,y,t)dt \text{ for any } s \in \mathbb{R}.$$

Let f denote the external force vertical load acting on the plate Ω, then the total potential energy of the bridge is

$$\begin{aligned}\mathbb{E}_P =& \mathbb{E}_B + \mathbb{E}_H \\ =& \frac{Ed^3}{12(1-\sigma^2)} \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})\right) dxdy \\ &+ \int_\Omega H(x,y,u)dxdy - \frac{Ed^3}{12(1-\sigma^2)} \int_\Omega fudxdy.\end{aligned}$$

If the external force f also depends on time t, i.e. $f = f(x, y, t)$, then the corresponding deformation u has a kinetic energy given by the integral

$$\mathbb{E}_K = \frac{m}{2|\Omega|} \int_\Omega u_t^2 dxdy,$$

where m is the mass of the plate.

Therefore, the total energy for the plate modeling the dynamical suspension bridge is

$$\begin{aligned}\mathbb{E}_T =& \mathbb{E}_K + \mathbb{E}_P \\ =& \frac{m}{2|\Omega|} \int_\Omega u_t^2 dxdy + \int_\Omega H(x, y, u)dxdy - \frac{Ed^3}{12(1-\sigma^2)} \int_\Omega fudxdy \\ &+ \frac{Ed^3}{12(1-\sigma^2)} \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) \right) dxdy.\end{aligned}$$

3.1.2 Boundary conditions

In this subsection, we introduce the suitable boundary conditions representing the physical situation of a plate modeling the suspension bridge.

Due to the connection with the ground, the plate Ω is assumed to be hinged on its short edges and hence

$$u(x, y, t) = u_{xx}(x, y, t) = 0, \quad (x, y) \in \{0, \pi\} \times (-\ell, \ell),\ t > 0. \tag{3.6}$$

On the other two edges $y = \pm\ell$, they are assumed to be free without any physical constraints on them and then the boundary conditions there become (see e.g. [76, (2.40)])

$$\begin{cases} u_{yy}(x, \pm\ell, t) + \sigma u_{xx}(x, \pm\ell, t) = 0, & x \in (0, \pi),\ t > 0 \\ u_{yyy}(x, \pm\ell, t) + (2-\sigma)u_{xxy}(x, \pm\ell, t) = 0, & x \in (0, \pi),\ t > 0. \end{cases} \tag{3.7}$$

3.1.3 The semilinear problem

The natural functional space where to set up the problem is

$$H_*^2(\Omega) := \left\{ u \in H^2(\Omega);\ u = 0 \text{ on } \{0, \pi\} \times (-\ell, \ell) \right\} .$$

Clearly, $H_*^2(\Omega)$ satisfies $H_0^2(\Omega) \subset H_*^2(\Omega) \subset H^2(\Omega)$. Since we are in the plane, $H^2(\Omega) \subset C^0(\overline{\Omega})$ (see [1]) so that the condition on $\{0, \pi\} \times (-\ell, \ell)$ introduced in the definition of $H_*^2(\Omega)$ makes sense.

As for the action, one has to take the difference between the kinetic energy $\mathbb{E}_K$ and the total potential energy $\mathbb{E}_P$. By several suitable scaling and integrating the difference on an interval $(0, T)$ with $T > 0$, we have:

$$\mathcal{A} = \frac{1}{2}\int_0^T \left[\int_\Omega u_t^2 dxdy\right] dt + \int_0^T \left[\int_\Omega fudxdy\right] dt$$
$$- \int_0^T \left[\int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) + H(x, y, u)\right) dxdy\right] dt.$$

Then the equation modeling the bridge is obtained by taking the critical points of the functional $\mathcal{A}$:

$$u_{tt} + \Delta^2 u + h(x, y, u) = f \qquad \text{in } \Omega \times (0, T). \tag{3.8}$$

Due to the internal friction in the plate, one needs to add a damping term μu_t with $\mu > 0$ in (3.8) and then

$$u_{tt} + \mu u_t + \Delta^2 u + h(x, y, u) = f \qquad \text{in } \Omega \times (0, T). \tag{3.9}$$

Assume that $u_0(x, y)$ is the initial position of the plate and $u_1(x, y)$ is the initial vertical velocity of the plate. Then by combing the equation (3.9) with the boundary conditions (3.6)-(3.7), one has: for $t > 0$,

$$\begin{cases} u_{tt} + \mu u_t + \Delta^2 u + h(x, y, u) = f & (x, y) \in \Omega, \\ u(0, y, t) = u_{xx}(0, y, t) = 0 & y \in (-\ell, \ell), \\ u(\pi, y, t) = u_{xx}(\pi, y, t) = 0 & y \in (-\ell, \ell), \\ u_{yy}(x, \pm\ell, t) + \sigma u_{xx}(x, \pm\ell, t) = 0 & x \in (0, \pi), \\ u_{yyy}(x, \pm\ell, t) + (2-\sigma)u_{xxy}(x, \pm\ell, t) = 0 & x \in (0, \pi), \\ u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = u_1(x, y) & (x, y) \in \Omega, \end{cases} \tag{3.10}$$

which admits a unique solution, see [27, Theorem 3.6].

3.2 The asymptotic behavior of the solution

This section is devoted to a non-coercive problem, which admits a unique local solution. We discuss the asymptotic behavior of the unique solution for different suitable initial data. This part comes from [82].

Motivated by the works of Micheletti-Pistoia [60, 61] where they considered some biharmonic problems with Navier boundary conditions, we take the restoring force h due to the hangers in the form

$$h = au - |u|^{p-2} \quad \text{with } 2 < p < \infty,$$

where $a = a(x, y, t)$ is a sigh-changing and bounded measurable function and so h is of indefinite sign. Clearly, h satisfies the assumptions (3.4)-(3.5). If the external force $f \equiv 0$, then from (3.10) we deduce the following non-coercive problem: for $t > 0$,

$$\begin{cases} u_{tt} + \mu u_t + \Delta^2 u + au = |u|^{p-2}u & (x,y) \in \Omega, \\ u(0,y,t) = u_{xx}(0,y,t) = 0 & y \in (-\ell,\ell), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0 & y \in (-\ell,\ell), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0 & x \in (0,\pi), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma)u_{xxy}(x,\pm\ell,t) = 0 & x \in (0,\pi), \\ u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y) & (x,y) \in \Omega. \end{cases} \tag{3.11}$$

3.2.1 Preliminaries

Let $\Omega = (0,\pi) \times (-\ell,\ell) \subset \mathbb{R}^2$. Denote the standard $L^q(\Omega)$ norm by $\|\cdot\|_q$ for $1 \leq q \leq \infty$ and the standard $H^2(\Omega)$ norm by

$$\|u\|_{H^2} = \left(\int_\Omega \left[|u|^2 + \left|D^2 u\right|^2\right] dxdy\right)^{1/2}, \quad u \in H^2(\Omega).$$

Recall the space $H^2_*(\Omega)$ defined in the previous section,

$$H^2_*(\Omega) := \left\{u \in H^2(\Omega) : u = 0 \text{ on } \{0,\pi\} \times (-\ell,\ell)\right\},$$

which is a Hilbert space when endowed with the following scalar product for any $u, v \in H^2_*(\Omega)$

$$\begin{aligned}(u,v)_{H^2_*} = &\int_\Omega \Delta u \Delta v dxdy \\ &+ (1-\sigma)\int_\Omega (2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}) dxdy.\end{aligned} \tag{3.12}$$

This scalar product induces a new norm on $H^2_*(\Omega)$ by recalling (3.1)

$$\|u\|_{H^2_*} = \left(\int_\Omega |\Delta u|^2 + 2(1-\sigma)\int_\Omega u_{xy}^2 - u_{xx}u_{yy} dxdy\right)^{1/2}, \tag{3.13}$$

which is equivalent to $u \mapsto \|D^2 u\|_2$ for $0 < \sigma < 1/2$, see [27, Lemma 4.1]. We define

$$\mathcal{H}_*(\Omega) := \text{ the dual space of } H^2_*(\Omega)$$

and we denote by $\langle \cdot, \cdot \rangle$ the corresponding duality.

For this case, there is a Sobolev embedding inequality:

Lemma 3.1. *Let $1 \le q < \infty$. Then for any $u \in H_*^2(\Omega)$, the inequality*

$$\|u\|_q \le S_q \|u\|_{H_*^2}$$

holds, where $S_q = \left(\frac{\pi}{2\ell} + \frac{\sqrt{2}}{2}\right)(2\pi\ell)^{(q+2)/2q}\left(\frac{1}{1-\sigma}\right)^{1/2}$.

Proof. Take any $u \in H_*^2(\Omega) \subset C^0(\overline{\Omega})$, then we have

$$\begin{aligned}
|u(x,y)| &= \left|\int_0^x u_x(t,y)dt\right| \le \int_0^\pi |u_x(x,y)|\,dx \\
&\le \sqrt{\pi}\left(\int_0^\pi (u_x(x,y))^2\,dx\right)^{1/2} \\
&= \sqrt{\pi}\left(\int_0^\pi -u_{xx}(x,y)u(x,y)dx\right)^{1/2} \\
&\le \sqrt{\pi}\left(\int_0^\pi |u_{xx}(x,y)|^2\,dx\right)^{1/4}\left(\int_0^\pi |u(x,y)|^2\,dx\right)^{1/4},
\end{aligned}$$

which yields that

$$\int_0^\pi |u(x,y)|^2\,dx \le \pi^4 \int_0^\pi |u_{xx}(x,y)|^2\,dx.$$

Then

$$|u(x,y)| \le \pi^{3/2}\left(\int_0^\pi |u_{xx}(x,y)|^2\,dx\right)^{1/2}. \tag{3.14}$$

Furthermore, for any $u \in H_*^2(\Omega) \subset C^0(\overline{\Omega})$, one has $u_y(\cdot,y) \subset C^0([0,\pi])$ for any fixed $y \in (-\ell,\ell)$. Hence,

$$|u_y(x,y)| = \left|\int_0^x u_{yx}(t,y)dt\right| \le \int_0^\pi |u_{yx}(x,y)|\,dx, \tag{3.15}$$

and for any $z \in (-\ell,\ell)$

$$\begin{aligned}
|u(x,y)| &= \left|u(x,z) + \int_z^y u_y(x,s)ds\right| \\
&\le |u(x,z)| + \int_{-\ell}^{\ell} |u_y(x,y)|\,dy.
\end{aligned} \tag{3.16}$$

Integrating the inequality (3.16) about z on $(-\ell,\ell)$, we get

$$2\ell\,|u(x,y)| \le \int_{-\ell}^{\ell} |u(x,z)|\,dz + 2\ell\int_{-\ell}^{\ell} |u_y(x,y)|\,dy,$$

which together with (3.14) and (3.15) yields that

$$
\begin{aligned}
|u(x,y)| \leq & \frac{1}{2\ell}\pi^{3/2}\int_{-\ell}^{\ell}\left(\int_0^{\pi}|u_{xx}(x,z)|^2\,dx\right)^{1/2}dz \\
& + \int_{-\ell}^{\ell}\int_0^{\pi}|u_{yx}(x,y)|\,dxdy \\
\leq & \frac{\pi^{3/2}}{(2\ell)^{1/2}}\left(\int_{-\ell}^{\ell}\int_0^{\pi}|u_{xx}(x,y)|^2\,dxdy\right)^{1/2} \\
& + (\pi\ell)^{1/2}\left(\int_{-\ell}^{\ell}\int_0^{\pi}2\,|u_{yx}(x,y)|^2\,dxdy\right)^{1/2} \\
\leq & \left(\frac{\pi}{2\ell}+\frac{\sqrt{2}}{2}\right)(2\pi\ell)^{1/2}\left(\int_{\Omega}\left|D^2u\right|^2\,dxdy\right)^{1/2}.
\end{aligned}
$$

From [27, Lemma 4.1], we know that

$$
(1-\sigma)\|D^2u\|_2^2 \leq \|u\|_{H_*^2}^2.
$$

Therefore,

$$
|u(x,y)| \leq \left(\frac{\pi}{2\ell}+\frac{\sqrt{2}}{2}\right)(2\pi\ell)^{1/2}\left(\frac{1}{1-\sigma}\right)^{1/2}\|u\|_{H_*^2},
$$

which implies that for $1 \leq q < \infty$,

$$
\|u\|_q \leq \left(\frac{\pi}{2\ell}+\frac{\sqrt{2}}{2}\right)(2\pi\ell)^{(q+2)/2q}\left(\frac{1}{1-\sigma}\right)^{1/2}\|u\|_{H_*^2},
$$

which completes the proof. □

Assume that $\{\Lambda_i\}_{i=1}^{\infty}$ is the eigenvalue sequence of the following linear problem

$$
\begin{cases}
\Delta^2 u = \Lambda u & (x,y)\in\Omega, \\
u(0,y,t) = u_{xx}(0,y,t) = 0 & y\in(-\ell,\ell), \\
u(\pi,y,t) = u_{xx}(\pi,y,t) = 0 & y\in(-\ell,\ell), \\
u_{yy}(x,\pm\ell,t)+\sigma u_{xx}(x,\pm\ell,t) = 0 & x\in(0,\pi), \\
u_{yyy}(x,\pm\ell,t)+(2-\sigma)u_{xxy}(x,\pm\ell,t) = 0 & x\in(0,\pi),
\end{cases}
$$

which has been solved in [27]. Particularly, $\Lambda_1 < 1$ and we have an elementary result.

Lemma 3.2. *Assume that* $-\Lambda_1 < a_1 \le a \le a_2$ *with* $a_1, a_2 \in \mathbb{R}$*. Then for any* $u \in H_*^2(\Omega)$*, there holds*

$$A_1\|u\|_{H_*^2}^2 \le \|u\|_{H_*^2}^2 + (au, u)_2 \le A_2\|u\|_{H_*^2}^2,$$

where $(\cdot,\cdot)_2$ *is the* L^2 *inner product and* A_1, A_2 *are given by*

$$A_1 = \begin{cases} 1 + \frac{a_1}{\Lambda_1}, & a_1 < 0, \\ 1, & a_1 \ge 0. \end{cases} \quad A_2 = \begin{cases} 1, & a_2 < 0, \\ 1 + \frac{a_2}{\Lambda_1}, & a_2 \ge 0. \end{cases}$$

3.2.2 Local existence

In this subsection we are concerned with the existence and uniqueness results of the problem (3.11).

Definition 3.1. *The function* $u \in C([0,T], H_*^2(\Omega)) \cap C^1([0,T], L^2(\Omega)) \cap C^2([0,T], \mathcal{H}_*(\Omega))$ *with* $u_t \in L^2([0,T], L^2(\Omega))$ *is said a weak solution to* (3.11)*, if* $u(0) = u_0,\ u_t(0) = u_1$ *and for all* $\eta \in H_*^2(\Omega)$ *and a.e.* $t \in [0,T]$

$$\langle u_{tt}, \eta\rangle + (u,\eta)_{H_*^2} + \mu(u_t,\eta)_2 + (au,\eta)_2 = \left(|u|^{p-2}\, u, \eta\right)_2,$$

where $(\cdot,\cdot)_2$ *is the* L^2 *scalar product.*

Then we prove

Theorem 3.1. *Assume that* (3.1)*. Let* $\mu > 0$*,* $2 < p < \infty$ *and* $-\Lambda_1 < a_1 \le a \le a_2$*. Then for any* $u_0 \in H_*^2(\Omega),\ u_1 \in L^2(\Omega)$*, there exists* $T > 0$ *such that* (3.11) *has a unique local weak solution* u *on* $[0,T]$*. Moreover, if*

$$T_{\max} = \sup\{T > 0 : u(t) = u(\cdot,t) \text{ exists on } [0,T]\} < \infty,$$

then

$$\lim_{t\to T_{\max}} \|u(t)\|_q = \infty, \quad \text{for } q \ge 1 \text{ such that } q > (p-2)/2.$$

Proof. We start with several definitions. For every $T > 0$, define the space by

$$H := C([0,T], H_*^2(\Omega)) \cap C^1([0,T], L^2(\Omega))$$

with the norm

$$\|u\|_H = \left(\max_{t\in[0,T]} \left(A_1 \|u(t)\|_{H_*^2}^2 + \|u_t(t)\|_2^2\right)\right)^{1/2},$$

where A_1 is given in Lemma 3.2. For $u_0 \in H_*^2(\Omega)$, $u_1 \in L^2(\Omega)$, denote

$$M_T = \left\{u \in H : u(0) = u_0,\ u_t(0) = u_1 \text{ and } \|u\|_H^2 \le R^2\right\},$$

where $R^2 \geq 2\left(A_2\|u_0\|^2_{H^2_*} + \|u_1\|^2_2\right)$ with A_2 being given in Lemma 3.2.

Then we consider an initial-boundary value problem: for $t \in (0, T]$,

$$\begin{cases} v_{tt} + \Delta^2 v + \mu v_t + av = |u|^{p-2}u, & (x, y, t) \in \Omega, \\ v(0, y, t) = v_{xx}(0, y, t) = 0, & y \in (-\ell, \ell), \\ v(\pi, y, t) = v_{xx}(\pi, y, t) = 0, & y \in (-\ell, \ell), \\ v_{yy}(x, \pm\ell, t) + \sigma v_{xx}(x, \pm\ell, t) = 0, & x \in (0, \pi), \\ v_{yyy}(x, \pm\ell, t) + (2-\sigma)v_{xxy}(x, \pm\ell, t) = 0, & x \in (0, \pi), \\ v(x, y, 0) = u_0(x, y), \quad v_t(x, y, 0) = u_1(x, y), & (x, y) \in \Omega. \end{cases} \tag{3.17}$$

Following the procedure of [27, Section 8], by several estimates we have known that there exists a unique solution $v \in H \cap C^2([0, T], \mathcal{H}_*(\Omega))$ with $v_t \in L^2([0, T], L^2(\Omega))$ to (3.17), see for details [82].

For any fixed $u \in M_T$, one can introduce a map $\Phi : H \to H$ defined by $v = \Phi(u)$, here v is the unique solution to (3.17).

Claim. Φ *is a contractive map satisfying* $\Phi(M_T) \subseteq M_T$ *for small* $T > 0$.

In fact, for any $u \in M_T$, the corresponding solution $v = \Phi(u)$ satisfies

$$\begin{aligned} \|v_t\|_2^2 + A_1 \|v\|^2_{H^2_*} &\leq \|u_1\|_2^2 + A_2 \|u_0\|^2_{H^2_*} + CR^{2p-2}T \\ &\leq \frac{R^2}{2} + CR^{2p-2}T. \end{aligned}$$

If T is small enough, then $\|v\|_H \leq R$, which implies that $\Phi(M_T) \subseteq M_T$.

Let $v_1 = \Phi(w_1), v_2 = \Phi(w_2)$ with $w_1, w_2 \in M_T$. Putting v_1, v_2 in the equation (3.17), subtracting the two equations and setting $v = v_1 - v_2$, we obtain for all $\eta \in H^2_*(\Omega)$ and a.e. $t \in [0, T]$,

$$\begin{aligned} \langle v_{tt}, \eta\rangle + (v, \eta)_{H^2_*} + (\mu v_t + av, \eta)_2 &= \left(|w_1|^{p-2} w_1 - |w_2|^{p-2} w_2, \eta\right)_2 \\ &= \int_\Omega \gamma(t)\,(w_1 - w_2)\,\eta dxdy, \end{aligned}$$

where $\gamma(t) \leq C(p)(|w_1| + |w_2|)^{p-2}$.

Taking $\eta = v_t$ and arguing similarly as above, we have

$$\|\Phi(w_1) - \Phi(w_2)\|^2_H = \|v\|^2_H \leq CR^{2p-4}T\|w_1 - w_2\|^2_H.$$

If T is small enough, then there exists a constant $0 < \delta < 1$ such that

$$\|\Phi(w_1) - \Phi(w_2)\|^2_H \leq \delta\|w_1 - w_2\|^2_H.$$

Therefore, Φ is a contract map, and the claim is proved.

By the Contracting Mapping Principle (see [37]), there exists a unique function $u \in M_T$ such that $u = \Phi(u)$, which is the unique solution of the problem (3.11). Moreover, $u \in C^2([0,T], \mathcal{H}_*(\Omega))$. This proves the first part of Theorem 3.1.

To complete the proof, we define an energy functional by

$$\begin{aligned}&\mathrm{E} : H^2_*(\Omega) \times L^2(\Omega) \to \mathbb{R}\\ &\mathrm{E}(v,w) = \frac{1}{2}\|v(t)\|^2_{H^2_*} + \frac{1}{2}(av(t),v(t))_2 - \frac{1}{p}\|v(t)\|^p_p + \frac{1}{2}\|w\|^2_2.\end{aligned}$$

Then the Lyapunov function defined for the solution $u(t)$ of the problem (3.11) is

$$\begin{aligned}E(t) &= \mathrm{E}(u(t),u_t(t))\\ &= \frac{1}{2}\|u(t)\|^2_{H^2_*} + \frac{1}{2}\|u_t(t)\|^2_2 + \frac{1}{2}(au(t),u(t))_2 - \frac{1}{p}\|u(t)\|^p_p.\end{aligned}$$

According to the continuation principle (see [40]), if $\|u(t)\|_H < \infty$, then the solution $u(t)$ should be continued, see also [65, p.158] for a similar argument. Hence, if $T_{\max} < \infty$, then it follows

$$\lim_{t\to T_{\max}} \left(A_1\|u(t)\|^2_{H^2_*} + \|u_t(t)\|^2_2\right) = \lim_{t\to T_{\max}} \|u(t)\|^2_H = \infty. \tag{3.18}$$

Since the Lyapunov function $E(t)$ satisfies

$$E(t) + \mu\int_s^t \|u_t(\tau)\|^2_2\,d\tau = E(s), \qquad \text{for every } 0 \le s \le t < T_{\max}, \tag{3.19}$$

we have for all $t \in [0, T_{\max})$

$$\frac{1}{2}\|u(t)\|^2_{H^2_*} + \frac{1}{2}\|u_t(t)\|^2_2 + \frac{1}{2}(au(t),u(t))_2 \le \frac{1}{p}\|u(t)\|^p_p + E(0).$$

By Lemma 3.2, there holds for all $t \in [0, T_{\max})$

$$\frac{A_1}{2}\|u(t)\|^2_{H^2_*} + \frac{1}{2}\|u_t(t)\|^2_2 \le \frac{1}{p}\|u(t)\|^p_p + E(0). \tag{3.20}$$

From (3.18), we have

$$\lim_{t\to T_{\max}} \|u(t)\|_p = \infty$$

and then by Lemma 3.1

$$\lim_{t\to T_{\max}} \|u(t)\|_{H^2_*} = \infty. \tag{3.21}$$

Moreover, by (3.20)

$$\frac{A_1}{2}\|u(t)\|_{H_*^2}^2 \leq \frac{1}{p}\|u(t)\|_p^p + E(0),$$

which combined with Gagliardo-Nirenberg inequality yields that

$$C\|u(t)\|_{H_*^2}^2 - C \leq \|u(t)\|_p^p \leq C\|u(t)\|_q^{p(1-\alpha)}\|u(t)\|_{H_*^2}^{p\alpha} \tag{3.22}$$

with $\alpha = 2(p-q)/p(q+2)$.

If $\alpha \in (0,1)$ such that $p\alpha < 2$, i.e. $(p-2)/2 < q < p$, then (3.21) and (3.22) immediately yield

$$\lim_{t\to T_{\max}} \|u(t)\|_q = \infty, \text{ for } q \geq 1 \text{ such that } q > (p-2)/2.$$

This finally completes the proof. □

3.2.3 Global existence

In this subsection we will show that the unique local solution of (3.11) is global for some suitable initial data.

First we list several notations and lemmas. Define the Nehari functional I and the energy functional J by

$$\begin{aligned} I(u) &= \|u\|_{H_*^2}^2 + (au,u)_2 - \|u\|_p^p, \quad \text{ for every } u \in H_*^2(\Omega), \\ J(u) &= \frac{1}{2}\|u\|_{H_*^2}^2 + \frac{1}{2}(au,u)_2 - \frac{1}{p}\|u\|_p^p, \quad \text{ for every } u \in H_*^2(\Omega). \end{aligned}$$

We consider a real value function defined for any $0 \not\equiv u \in H_*^2(\Omega)$ by

$$j(\lambda) = J(\lambda u), \quad \lambda \geq 0.$$

Then

$$\begin{aligned} j'(\lambda) &= \lambda\|u\|_{H_*^2}^2 + \lambda(au,u)_2 - \lambda^{p-1}\|u\|_p^p, \\ j''(\lambda) &= \|u\|_{H_*^2}^2 + (au,u)_2 - (p-1)\lambda^{p-2}\|u\|_p^p. \end{aligned}$$

Clearly, $j(0) = j'(0) = 0$ and $j''(0) = \|u\|_{H_*^2}^2 + (au,u)_2 > 0$ for $a > -\Lambda_1$. Hence, for any $0 \neq u \in H_*^2(\Omega)$, $j(\lambda)$ is a convex function for small $\lambda > 0$ and has the following behaviors:

Lemma 3.3. *Assume that $-\Lambda_1 < a_1 \leq a \leq a_2$. Then for any nontrivial $u \in H_*^2(\Omega)$, one has*
i). $\lim_{\lambda\to\infty} j(\lambda) = -\infty$;
ii). there exists a unique $\bar{\lambda} = \bar{\lambda}(u) > 0$ such that $j'(\bar{\lambda}) = 0$;
iii). $j''(\bar{\lambda}) < 0$.

Proof. The proof is almost similar to that of [66, Lemma 2.2], the only difference is to deal with the term $(au, u)_2$, which can be treated by applying Lemma 3.2, so we omit it. $\square$

Then one may define the potential well depth of the functional J (also known as mountain pass level) by

$$d = \inf_{u \in H_*^2(\Omega) \setminus \{0\}} \max_{\lambda > 0} J(\lambda u). \tag{3.23}$$

Denote the set of all nontrivial stationary solutions of (3.11) by

$$N = \{u \in H_*^2(\Omega) \setminus \{0\} : I(u) = 0\},$$

which is the so-called Nehari manifold, see [64, 85]. By considering a map $s \mapsto I(su)$ for all u such that $\|u\|_{H_*^2}^2 = 1$ and Lemma 3.3, it is easy to check that each half line starting from the origin of $H_*^2(\Omega)$ intersects only once the manifold N and N separates the two open sets

$$N_+ = \{u \in H_*^2(\Omega) : I(u) > 0\} \cup \{0\}, \quad N_- = \{u \in H_*^2(\Omega) : I(u) < 0\}.$$

Then the stable set W and unstable set U may be defined by

$$W = \{u \in N_+ : J(u) < d\}, \quad U = \{u \in N_- : J(u) < d\},$$

which have the following properties:

Lemma 3.4. *i). W is a neighborhood of the origin of $H_*^2(\Omega)$;*
ii). $0 \notin \overline{U}$ (closure in $H_*^2(\Omega)$).

As Payne and Sattinger did in [66], the potential well depth d defined in (3.23) can be also characterized as

$$d = \inf_{u \in N} J(u). \tag{3.24}$$

Then we prove

Theorem 3.2. *Assume that* (3.1). *Let $u(t) = u(x, y, t)$ be the unique local weak solution of* (3.11). *Assume that $u_0 \in H_*^2(\Omega)$, $u_1 \in L^2(\Omega)$. Then $u(t)$ is a global solution and*

$$\lim_{t \to \infty} \left(\|u(t)\|_{H_*^2} + \|u_t(t)\|_2 \right) = 0$$

if and only if there exists a real number $t_0 \in [0, T_{\max})$ such that

$$u(t_0) \in W \text{ and } E(t_0) < D$$

with $D = \min \left\{ d, \frac{p-2}{2p} A_1^{\frac{p}{p-2}} S_p^{-\frac{2p}{p-2}} \right\}$.

Proof. First we prove that $T_{\max} = \infty$ and $\lim_{t\to\infty} \left(\|u(t)\|_{H_*^2} + \|u_t(t)\|_2 \right) = 0$. Without loss of generality, assume that $t_0 = 0$. If $u(0) \in W$, $E(0) < D$, then we claim that

$$u(t) \in W \text{ and } E(t) < D, \qquad \text{for every } t \in [0, T_{\max}).$$

In fact, since $E(t)$ is a nonincreasing function, $E(t) \leq E(0) < D$ for all $t \in [0, T_{\max})$. Suppose that there exists $\bar{t} > 0$ such that $u(\bar{t}) \in N$. By (3.24), it follows that

$$d \leq J(u(\bar{t})) \leq E(\bar{t}) < D,$$

which is impossible. Thus, $u(t) \in W$ for all $t \in [0, T_{\max})$.

For all $t \in [0, T_{\max})$, one has by recalling the functionals I and J

$$\begin{aligned} J(u(t)) &= \frac{p-2}{2p} \left(\|u(t)\|_{H_*^2}^2 + (au(t), u(t))_2 \right) + \frac{I(u(t))}{p} \\ &\geq \frac{p-2}{2p} \left(\|u(t)\|_{H_*^2}^2 + (au(t), u(t))_2 \right). \end{aligned} \tag{3.25}$$

By (3.19), we have

$$\frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)) + \mu \int_0^t \|u_t(\tau)\|_2^2 d\tau = E(0) < D.$$

Hence,

$$\frac{1}{2}\|u_t(t)\|_2^2 + \frac{p-2}{2p} \left(\|u(t)\|_{H_*^2}^2 + (au(t), u(t))_2 \right) \leq C, \tag{3.26}$$

which implies that $T_{\max} = \infty$ by the continue principle. Moreover,

$$\int_0^t \|u_t(\tau)\|_2^2 d\tau \leq \frac{C}{\mu}, \qquad \text{for every } t \in [0, \infty). \tag{3.27}$$

Note that the following trivial inequality holds

$$\frac{d}{dt}((1+t)E(t)) \leq E(t). \tag{3.28}$$

Integrating (3.28) on $[0, t]$, we get for any $t \in [0, \infty)$

$$\begin{aligned} (1+t)E(t) \leq & E(0) + \int_0^t J(u(\tau)) d\tau + \frac{1}{2} \int_0^t \|u_t(\tau)\|_2^2 d\tau \\ =& E(0) + \frac{1}{p} \int_0^t I(u(\tau)) d\tau + \frac{1}{2} \int_0^t \|u_t(\tau)\|_2^2 d\tau \\ & + \frac{p-2}{2p} \int_0^t \left(\|u(\tau)\|_{H_*^2}^2 + (au(\tau), u(\tau))_2 \right) d\tau, \end{aligned} \tag{3.29}$$

where we use $J(u(t)) = \frac{p-2}{2p}\left(\|u(t)\|^2_{H^2_*} + (au(t), u(t))_2\right) + \frac{1}{p}I(u(t))$.

A simple computation induces that

$$\langle u_{tt}(t), u(t)\rangle = \frac{d}{dt}\int_\Omega u_t(t)u(t) - \|u_t(t)\|_2^2, \qquad \text{for a.e. } t \in [0, \infty).$$

Testing the equation in (3.11) with $u(t)$, we have for a.e. $t \in [0, \infty)$

$$\begin{aligned}\langle u_{tt}(t), u(t)\rangle &+ \|u(t)\|^2_{H^2_*} \\ &+ \mu(u_t(t), u(t))_2 + (au(t), u(t))_2 = \|u(t)\|_p^p. \end{aligned} \tag{3.30}$$

Thus,

$$\frac{d}{dt}\left(\int_\Omega u_t(t)u(t) + \frac{\mu}{2}\|u(t)\|_2^2\right) = \|u_t(t)\|_2^2 - I(u(t)). \tag{3.31}$$

Integrating (3.31) on $[0, t]$, by (3.25)-(3.26) and Lemmas 3.1 and 3.2, we obtain for any $t \in [0, \infty)$

$$\begin{aligned}\int_0^t I(u(\tau))d\tau =& \int_0^t \|u_t(\tau)\|_2^2 d\tau + \int_\Omega u_0u_1 + \frac{\mu}{2}\|u_0\|_2^2 - \int_\Omega u_t(t)u(t) \\ &- \frac{\mu}{2}\|u(t)\|_2^2 \\ \le& \int_0^t \|u_t(\tau)\|_2^2 d\tau + \|u_0\|_2\|u_1\|_2 + \frac{\mu}{2}\|u_0\|_2^2 + \frac{1}{2}\|u_t(t)\|_2^2 \\ &+ \frac{1}{2}\|u(t)\|_2^2 \\ \le& C(\mu). \end{aligned} \tag{3.32}$$

By Lemmas 3.1, 3.2 and (3.25)

$$\begin{aligned}\|u\|_p^p \le S_p^p\|u\|^p_{H^2_*} &\le S_p^pA_1^{-p/2}\left(\|u\|^2_{H^2_*} + (au, u)_2\right)^{p/2} \\ &= S_p^pA_1^{-p/2}\left(\|u\|^2_{H^2_*} + (au, u)_2\right)^{(p-2)/2}\left(\|u\|^2_{H^2_*} + (au, u)_2\right) \\ &\le S_p^pA_1^{-p/2}\left(\frac{2p}{p-2}J(u(t))\right)^{(p-2)/2}\left(\|u\|^2_{H^2_*} + (au, u)_2\right) \\ &\le S_p^pA_1^{-p/2}\left(\frac{2p}{p-2}\right)^{(p-2)/2}E(0)^{(p-2)/2}\left(\|u\|^2_{H^2_*} + (au, u)_2\right), \end{aligned}$$

which implies that by the definition of $I(u)$

$$\gamma\left(\|u(t)\|^2_{H^2_*} + (au(t), u(t))_2\right) \le I(u(t)), \tag{3.33}$$

where $\gamma = 1 - S_p^p A_1^{-p/2} \left(\frac{2p}{p-2}\right)^{(p-2)/2} E(0)^{(p-2)/2} > 0$.

Combining (3.27), (3.29), (3.32) and (3.33), we get

$$E(t) \leq \frac{C}{t}, \quad \text{for every } t \in [0, \infty).$$

Consequently, by (3.25) we immediately have

$$\|u(t)\|_{H_*^2}^2 + (au(t), u(t))_2 + \|u_t(t)\|_2^2 \leq \frac{C}{t}, \quad \text{for every } t \in [0, \infty),$$

which combining with Lemma 3.2 tells us that

$$\lim_{t\to\infty} \left(\|u(t)\|_{H_*^2} + \|u_t(t)\|_2 \right) = 0.$$

Conversely, if $T_{\max} = \infty$ and $\|u(t)\|_{H_*^2}^2 + \|u_t(t)\|_2^2 \to 0$ as $t \to \infty$, then it follows that by Lemmas 3.1, 3.2

$$\lim_{t\to\infty} \|u(t)\|_p = 0 \text{ and } \lim_{t\to\infty} \left(\|u(t)\|_{H_*^2} + (au(t), u(t))_2 + \|u_t(t)\|_2 \right) = 0,$$

which imply that

$$\lim_{t\to\infty} E(t) = 0.$$

Therefore, by Lemma 3.4 and the above mentioned results, there must exist some $t_0 > 0$ such that $E(t_0) < D$ and $u(t_0) \in W$. $\square$

3.2.4 Finite time blow-up

This subsection is devoted to the blow-up behavior of the unique local solution of (3.11).

Theorem 3.3. *Assume that* (3.1). *Let* $u(t) = u(x, y, t)$ *be the unique local weak solution of* (3.11). *Assume that* $u_0 \in H_*^2(\Omega)$, $u_1 \in L^2(\Omega)$. *Then* $u(t)$ *blows up in finite time, that is,*

$$T_{\max} < \infty$$

if and only if there exists a real number $t_0 \in [0, T_{\max})$ *such that*

$$u(t_0) \in U, \quad E(t_0) < D.$$

Proof. Firstly, we assume that there exists $t_0 \geq 0$ such that $u(t_0) \in U$ and $E(t_0) < D$. Without loss of generality, let $t_0 = 0$, then we claim that

$$u(t) \in U, \quad E(t) < D, \quad \text{for every } t \in [0, T_{\max}).$$

Indeed, (3.19) implies that

$$E(t) \leq E(0) < d, \quad \text{for all } t \in [0, T_{\max}). \tag{3.34}$$

Suppose that there exists $\bar{t} > 0$ such that $u(\bar{t}) \in N$, then by (3.24) we have

$$d \leq J(u(\bar{t})) \leq E(\bar{t}) < D,$$

which is contradict to (3.34), and therefore $u(t) \in U$ for all $t \in [0, T_{\max})$.

From (3.23), we obtain that

$$d = \inf_{u \in H_*^2 \setminus \{0\}} \frac{p-2}{2p} \frac{\left(\|u\|_{H_*^2}^2 + (au, u)_2\right)^{p/(p-2)}}{\left(\|u\|_p^p\right)^{2/(p-2)}}$$

and then

$$\frac{2pd}{p-2} \leq \frac{\left(\|u\|_{H_*^2}^2 + (au, u)_2\right)^{p/(p-2)}}{\left(\|u\|_p^p\right)^{2/(p-2)}}.$$

Since $u(t) \in U$ for all $t \in [0, T_{\max})$, we have $I(u(t)) < 0$ and then

$$\frac{2pd}{p-2} < \|u(t)\|_{H_*^2}^2 + (au(t), u(t))_2, \quad \text{for every } t \in [0, T_{\max}).$$

Now we prove $T_{\max} < \infty$ by following the proof of [33, Theorem 3.11]. Assume by contradiction that $T_{\max} = \infty$. For any $T > 0$, we define a continuous positive function for $t \in [0, T]$ by

$$\theta(t) = \|u(t)\|_2^2 + \mu \int_0^t \|u(\tau)\|_2^2 \, d\tau + \mu(T - t) \|u_0\|_2^2.$$

With the same spirit (but it needs to be careful for the computations since the presence of the term au) as in [33], we have

$$\theta(t)\theta''(t) - \frac{p+2}{4}\theta'(t)^2 \geq C > 0, \quad \text{for a.e. } t \in [0, T].$$

Let $y(t) = \theta(t)^{-(p-2)/4}$, then the above differential inequality becomes

$$y''(t) \leq -\frac{p-2}{4} C y(t), \quad \text{for a.e. } t \in [0, T],$$

which together with $y'(t) = -\frac{p-2}{4}\theta(t)^{-(p+2)/4} < 0$ proves that $y(t) = 0$ at some time, say as $t = T^*$ (independent of T). This tells us that

$$\lim_{t \to T^*} \theta(t) = \infty.$$

Hence,

$$\lim_{t\to T^*} \|u(t)\|_2^2 = \infty \text{ or } \lim_{t\to T^*} \int_0^t \|u(\tau)\|_2^2\, d\tau = \infty.$$

For both cases, we always have

$$\lim_{t\to T^*} \|u(t)\|_{H_*^2}^2 = \infty$$

Therefore, $u(t)$ cannot be a global solution. That is

$$T_{\max} < \infty.$$

Conversely, assume that there exists no $t \geq 0$ such that $u(t) \in U$ or $E(t) < D$, we must have either $u(t) \in W$ and $E(t) < D$ (this is impossible by Theorem 3.2) or $E(t) \geq D$ for all $t \geq 0$. Then from (3.19),

$$\mu \int_0^t \|u_t(\tau)\|_2^2\, d\tau \leq E(0) - D.$$

By Jensen's inequality, for any $t \geq 0$ there holds

$$t \int_0^t \|u_t(\tau)\|_2^2\, d\tau \geq \int_\Omega \left(\int_0^t u_t(\tau) d\tau \right)^2 \geq \|u(t)\|_2^2 - \|u_0\|_2^2,$$

hence, for any finite $T > 0$ we have

$$\|u(t)\|_2^2 \leq C_T, \quad \text{for all } t \in [0, T). \tag{3.35}$$

Assume that $T_{\max} < \infty$, we know that from Theorem 3.1

$$\lim_{t\to T_{\max}} \|u(t)\|_p = \infty.$$

Hence, by Lemmas 3.1, 3.2, for every $M > E(0)$, there exists some $0 < \bar{t} < T_{\max}$ such that

$$M < \frac{p-2}{2p} \left(\|u(t)\|_{H_*^2} + (au(t), u(t))_2 \right), \quad \text{for every } t \geq \bar{t}. \tag{3.36}$$

Denote

$$\mathcal{V}(t) = M - E(t), \quad \text{for every } t \geq \bar{t}, \tag{3.37}$$

and then from (3.19)

$$\mathcal{V}(t) = \mathcal{V}_0 + \mu \int_0^t \|u_t(\tau)\|_2^2 d\tau, \tag{3.38}$$

where $\mathcal{V}_0 = M - E(0)$. Therefore, by (3.36)-(3.38)

$$0 < \mathcal{V}(t) \leq M - J(u(t)) \leq M - \frac{p}{p-2}M + \frac{1}{p}\|u(t)\|_p^p,$$

then

$$\|u(t)\|_p^p \geq p\left(\frac{2M}{p-2} + \mathcal{V}(t)\right) > p\mathcal{V}(t). \tag{3.39}$$

Since $p > 2$, by Hölder, Young inequalities and (3.38)-(3.39), we have

$$\begin{aligned}
|\mu(u(t), u_t(t))_2| &\leq C\mu\|u(t)\|_p^{1-k}\|u(t)\|_p^k\|u_t(t)\|_2 \\
&\leq C\|u(t)\|_p^{1-k}\left(\mu\nu\|u(t)\|_p^{2k} + C(\nu)^{-1}\|u_t(t)\|_2^2\right) \\
&< C\mathcal{V}^{(1-k)/p}(t)\left(\mu\nu\|u(t)\|_p^{2k} + C(\nu)^{-1}\mathcal{V}'(t)\right) \\
&\leq C\nu\mathcal{V}_0^{(1-k)/p}\|u(t)\|_p^p + C(\nu)^{-1}\left(\mathcal{V}^{1/\kappa}(t)\right)', \qquad (3.40)
\end{aligned}$$

where $\nu > 0$, $k \in (1, p/2)$ and $\kappa = (1 + (1-k)/p)^{-1} \in (1, 2p/(p+2))$.

Next, we estimate $I(u(t))$. Since $\mathcal{V}_0 < M$, then from (3.38)-(3.39)

$$(\mathcal{V}_0 - M)\|u(t)\|_p^p \leq (\mathcal{V}_0 - M)p\left(\frac{2M + (p-2)\mathcal{V}_0}{p-2}\right),$$

that is,

$$2(\mathcal{V}_0 - M) \geq \frac{p-2}{p}\frac{2(\mathcal{V}_0 - M)}{2M + (p-2)\mathcal{V}_0}\|u(t)\|_p^p. \tag{3.41}$$

Hence, by (3.37) and (3.41)

$$\begin{aligned}
-I(u(t)) &= -2J(u(t)) + \frac{p-2}{p}\|u(t)\|_p^p \geq -2E(t) + \frac{p-2}{p}\|u(t)\|_p^p \\
&\geq 2(\mathcal{V}(t) - M) + \frac{p-2}{p}\|u(t)\|_p^p \geq 2(\mathcal{V}_0 - M) + \frac{p-2}{p}\|u(t)\|_p^p \\
&\geq \frac{p-2}{p}\frac{p\mathcal{V}_0}{2M + (p-2)\mathcal{V}_0}\|u(t)\|_p^p. \qquad (3.42)
\end{aligned}$$

Now we consider a function defined by

$$\mathcal{F}(t) \equiv \mathcal{V}(t)^{1/\kappa} + \varepsilon(u(t), u_t(t))_2, \qquad \text{for every } t \geq \bar{t}.$$

By (3.40) and (3.42), taking $\nu > 0$ sufficiently small, from (3.31) we have

$$\begin{aligned}\frac{d}{dt}(u(t),u_t(t))_2 =& \|u_t(t)\|_2^2 - I(u(t)) - \mu(u(t),u_t(t))_2\\ \geq& \|u_t(t)\|_2^2 + \left(C - C\nu\mathcal{V}_0^{(1-k)/p}\right)\|u(t)\|_p^p\\ &- C(\nu)^{-1}\left(\mathcal{V}^{1/\kappa}(t)\right)'\\ \geq& \|u_t(t)\|_2^2 + C\|u(t)\|_p^p - C(\nu)^{-1}\left(\mathcal{V}^{1/\kappa}(t)\right)'.\end{aligned}$$

Therefore, if $\varepsilon > 0$ is small enough,

$$\mathcal{F}'(t) \geq C\left(\|u_t(t)\|_2^2 + \|u(t)\|_p^p\right) > 0. \tag{3.43}$$

Then choosing $\varepsilon > 0$ even smaller if needed, we get

$$\mathcal{F}(t) \geq \mathcal{F}_0 \equiv \mathcal{V}_0^{1/\kappa} + \varepsilon(u(0),u_t(0))_2 > 0.$$

Utilizing (3.39) and Hölder, Young inequalities, we have

$$\begin{aligned}\mathcal{F}^\kappa(t) &\leq 2^{\kappa-1}\left(\mathcal{V}(t) + \varepsilon^\kappa|(u(t),u_t(t))_2|^\kappa\right)\\ &\leq C\left(\|u(t)\|_p^p + \|u_t(t)\|_2^2\right),\end{aligned}$$

which together (3.43) yields that

$$\mathcal{F}'(t) \geq C\mathcal{F}^\kappa(t),$$

then by [23, Lemma 2.10], $\mathcal{F}(t)$ blows up at $T^* > \bar{t}$ and we have

$$\mathcal{F}(t) \geq \frac{C}{(T^*-t)^{1/(\kappa-1)}}, \tag{3.44}$$

By (3.40), (3.42) and since $E(t) \geq D$, $\mathcal{V}(t) \leq M - D$,

$$\begin{aligned}\|u(t)\|_2^2 =& \|u(\bar{t})\|_2^2 + \frac{2}{\varepsilon}\int_{\bar{t}}^t \left(\mathcal{F}(\tau) - \mathcal{V}^{1/\kappa}(\tau)\right)d\tau\\ \geq& \|u(\bar{t})\|_2^2 + \frac{2}{\varepsilon}\int_{\bar{t}}^t \left(\frac{C}{(T^*-\tau)^{1/(\kappa-1)}} - (M-d)^{1/\kappa}\right)d\tau\\ \geq& \|u(\bar{t})\|_2^2 - C\\ &+ C\left(\left(\frac{1}{T^*-t}\right)^{(2-\kappa)/(\kappa-1)} - \left(\frac{1}{T^*-\bar{t}}\right)^{(2-\kappa)/(\kappa-1)}\right)\end{aligned}$$

Consequently,

$$\lim_{t\to T^*}\|u(t)\|_2^2 = \infty,$$

which contradicts (3.35). The proof is complete. □

3.3 A linear model with dynamical boundary conditions

In this section, we consider a linear plate model with dynamical boundary conditions for a dynamical suspension bridge. For describing the boundary behavior of the bridge, we set up dynamical boundary conditions which are dependent on the energy of the system. This part comes from [83].

3.3.1 The linear model

Since the plate is very narrow (compared to its length), we may assume that the restoring elastic force due to the hangers acts on every point of the plate and has the linear form, i.e. $h = ku$ with an elasticity constant $k > 0$, then the equation in problem (3.10) reads

$$u_{tt} + \mu u_t + \Delta^2 u + ku = f.$$

Hence, we have a linear initial value problem:

$$\begin{cases} u_{tt} + \mu u_t + \Delta^2 u + ku = f & (x, y) \in \Omega,\ t > 0, \\ u(x, y, 0) = u_0(x, y) & (x, y) \in \Omega, \\ u_t(x, y, 0) = u_1(x, y) & (x, y) \in \Omega, \end{cases} \tag{3.45}$$

where $\Omega = (0, \pi) \times (-\ell, \ell)$, $u_0 = u_0(x, y)$ is the initial position of the plate, $u_1 = u_1(x, y)$ is the initial vertical velocity of the plate.

Now we seek the boundary conditions which describe the physical situation appearing in the actual suspension bridges. On the two short edges $x = 0$ and $x = \pi$, which are connected with the ground, we assume that they are hinged and then

$$u(x, y, t) = u_{xx}(x, y, t) = 0, \quad (x, y) \in \{0, \pi\} \times (-\ell, \ell), t > 0, \tag{3.46}$$

which is the same as any other model we met.

While on the other two sides $y = \pm\ell$, due to the continuous impact from the external forces occurring on them, it is necessary to choose a kind of dynamical boundary conditions. The external force f acting on the bridge inserts an energy into the structure. We denote it by

$$\mathcal{E}(t) = \int_\Omega f^2 dxdy.$$

Let $\overline{E}_\mu > 0$ be the critical energy threshold (see Arioli-Gazzola [6]) above which the bridge displays self-excited oscillations and it increasingly depends on the damping parameter μ: for instance, $\overline{E}_\mu = \overline{E}_0 + c\mu$ for some $c > 0$ and $\overline{E}_0 > 0$ being the threshold of the undamped problem.

Since the assumption $2\ell \ll \pi$, it is natural to assume that the cross section of the plate tends to remain straight. While due to the appearance of the torsional oscillation, the cross section cannot always be in a horizontal position. Therefore, these two boundaries satisfy

$$\begin{cases} u_y(x,-\ell,t) - u_y(x,\ell,t) = 0, & x \in (0,\pi), t > 0, \\ u_y(x,-\ell,t) + u_y(x,\ell,t) = \alpha, & x \in (0,\pi),\ t > 0, \end{cases} \tag{3.47}$$

where $\alpha = \alpha(x,t)$ depends on f and u_0 for $y = \pm\ell$. Furthermore, for any cross section of the bridge, the vertical displacements at the two endpoints also depend on the initial position and the external forces acting on these two points. Thus, it is reasonable to assume that the sum of the two displacements fulfills

$$u(x,-\ell,t) + u(x,\ell,t) = \beta, \quad \text{for } x \in (0,\pi), t > 0, \tag{3.48}$$

where $\beta = \beta(x,t)$ depends on $\overline{f}(x,t)$ and $\overline{u}_0(x)$ with

$$\overline{f}(x,t) = \frac{f(x,\ell,t) + f(x,-\ell,t)}{2}, \quad \overline{u}_0(x) = \frac{u_0(x,\ell) + u_0(x,-\ell)}{2}.$$

Moreover, the vibration of the cross section will switch to some different kind of oscillations, such as the torsional oscillation, when the inserted energy $\mathcal{E}(t)$ exceeds $\overline{E}_\mu$. Therefore, as long as $\mathcal{E}(t) \leq \overline{E}_\mu$, we assume that there is only vertical vibration appearing in the motion, then the difference between the two displacements should be zero, that is,

$$u(x,-\ell,t) - u(x,\ell,t) = 0, \quad \text{for } x \in (0,\pi). \tag{3.49}$$

While if $\mathcal{E}(t) > \overline{E}_\mu$, the torsional oscillation begins to arise and its amplitude increases as the energy $\mathcal{E}(t) \to \infty$, which means that the difference between the two displacements is related to the energy $\mathcal{E}(t)$. Concretely, if $\mathcal{E}(t) \downarrow \overline{E}_\mu$, then the motion tends to vertical-type so that

$$u(x,-\ell,t) - u(x,\ell,t) \to 0,$$

while if $\mathcal{E}(t) \to \infty$, then the difference also increases. But it cannot increase to infinity because the bridge will collapse earlier. Therefore, as long as $\mathcal{E}(t) > \overline{E}_\mu$, the boundaries are assumed to satisfy the equation for $x \in (0,\pi)$

$$\frac{d}{dt}\Big\{[u(x,-\ell,t) - u(x,\ell,t)]\frac{\mathcal{E}(0) - \overline{E}_\mu}{\mathcal{E}(t) - \overline{E}_\mu}\exp\Big(\frac{\mathcal{E}(0)}{\overline{E}_\mu} - \frac{\mathcal{E}(t)}{\overline{E}_\mu}\Big)\Big\} = \gamma, \tag{3.50}$$

here we assume $\mathcal{E}(0) > \overline{E}_\mu$ and $\gamma = \gamma(x,t)$ is a real function satisfying

$$\gamma \text{ has the same sign as } (u_0(x,-\ell) - u_0(x,\ell)).$$

In this case, the conditions (3.49)-(3.50) can be combined into

$$u_t(x,-\ell,t) - u_t(x,\ell,t) - \eta(t)(u(x,-\ell,t) - u(x,\ell,t)) = \theta(t)\gamma, \quad (3.51)$$

where

$$\theta(t) = \frac{(\mathcal{E}(t) - \overline{E}_\mu)^+}{\mathcal{E}(0) - \overline{E}_\mu} \exp\Big(\frac{\mathcal{E}(t)}{\overline{E}_\mu} - \frac{\mathcal{E}(0)}{\overline{E}_\mu}\Big), \quad (3.52)$$

$$\eta(t) = \mathcal{E}'(t)E(t)\Big(\frac{1}{\overline{E}_\mu} + \frac{1}{(\mathcal{E}(t) - \overline{E}_\mu)^+}\Big) \quad (3.53)$$

with $(\mathcal{E}(t) - \overline{E}_\mu)^+ = \max\{\mathcal{E}(t) - \overline{E}_\mu, 0\}$, $\mathcal{E}'(t) = \frac{d}{dt}\mathcal{E}(t)$ and

$$E(t) = \begin{cases} -1, & \mathcal{E}(t) \le \overline{E}_\mu, \\ 1, & \mathcal{E}(t) > \overline{E}_\mu. \end{cases} \quad (3.54)$$

Therefore, the boundary conditions for a rectangular plate Ω modeling dynamical suspension bridges are (3.46)-(3.48) and (3.51). Then together with (3.45), we deduce an evolution problem

$$\begin{cases} u_{tt} + \Delta^2 u + \mu u_t + ku = f, & (x,y) \in \Omega, t > 0, \\ u(x,y,0) = u_0, & (x,y) \in \Omega, \\ u_t(x,y,0) = u_1, & (x,y) \in \Omega, \end{cases} \quad (3.55)$$

with dynamical boundary conditions: for $t > 0$, $x \in (0,\pi)$ and $y \in (-\ell,\ell)$,

$$\begin{cases} u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \\ u_y(x,-\ell,t) - u_y(x,\ell,t) = 0, \\ u_y(x,-\ell,t) + u_y(x,\ell,t) = \alpha, \\ u(x,-\ell,t) + u(x,\ell,t) = \beta, \\ u_t(x,-\ell,t) - u_t(x,\ell,t) - \eta(t)(u(x,-\ell,t) - u(x,\ell,t)) = \theta(t)\gamma. \end{cases} \quad (3.56)$$

3.3.2 An auxiliary problem

In this section we consider an auxiliary problem which plays a crucial role in solving the original problem (3.55)-(3.56). We first introduce a subspace of $H^2_*(\Omega)$ denoted by

$$H^2_{**}(\Omega) := \{u \in H^2_*(\Omega) : u = u_y = 0 \text{ on } (0,\pi)\times\{\pm\ell\}\}.$$

Clearly, $H^2_0(\Omega) \subset H^2_{**}(\Omega) \subset H^1_0(\Omega) \cap H^2(\Omega)$. Hence, one may define a scalar product on the space $H^2_{**}(\Omega)$ by

$$(u,v)_{H^2_{**}} = \int_\Omega \Delta u \Delta v \, dx\, dy, \quad \text{for any } u, v \in H^2_{**}(\Omega),$$

which induces the norm

$$\|u\|_{H^2_{**}} = \Big(\int_\Omega |\Delta u|^2 \, dx\, dy\Big)^{1/2}, \quad \text{for all } u \in H^2_{**}(\Omega).$$

Now we consider a nonhomogeneous linear problem: for $t > 0$,

$$\begin{cases} v_{tt} + \Delta^2 v + \mu v_t + kv = \varphi, & (x,y)\in\Omega, \\ v(x,y,t) = v_{xx}(x,y,t) = 0, & (x,y)\in\{0,\pi\}\times(-\ell,\ell), \\ v(x,y,t) = v_y(x,y,t) = 0, & (x,y)\in(0,\pi)\times\{\pm\ell\}, \\ v(x,y,0) = v_0, & (x,y)\in\Omega, \\ v_t(x,y,0) = v_1, & (x,y)\in\Omega, \end{cases} \tag{3.57}$$

where $\varphi = \varphi(x,y,t) \in C^0([0,\infty); L^2(\Omega))$, $v_0 = v_0(x,y) \in H^2_{**}(\Omega)$ and $v_1 = v_1(x,y) \in L^2(\Omega)$.

Claim. The problem (3.57) admits a unique weak solution

$$v \in C^0([0,\infty); H^2_{**}(\Omega)) \cap C^1([0,\infty); L^2(\Omega)).$$

Actually, the variational problem (3.57) is similar to [27, problem (22)] if we take the nonlinear term $h = kv$. The only difference lies in the boundary conditions on $(0,\pi)\times\{\pm\ell\}$, but this has no influence when we solve it by following the procedure in [27]. Hence, we only list the key steps, for details see [27, Section 8].

Step 1. Consider the approximated problems for $m \geq 1$

$$\begin{cases} v''_m + Lv_m + \mu v'_m + P_m(kv_m) = P_m(\varphi), & t\in[0,\tau_m) \\ v(0) = v^m_0, \quad v'(0) = v^m_1 \end{cases} \tag{3.58}$$

where L is defined by $\langle Lu, v\rangle := (u,v)_{H^2_{**}}$ for any $u, v \in H^2_{**}(\Omega)$, $P_m : H^2_{**}(\Omega) \to W_m$ is the orthogonal projection onto W_m, which is spanned by the eigenfunctions $\{w_m\}_{m\geq 1}$ of the problem

$$\begin{cases} \Delta^2 w = \lambda w, & (x,y) \in \Omega, \\ w(x,y) = w_{xx}(x,y) = 0, & (x,y) \in \{0,\pi\} \times (-\ell,\ell), \\ w(x,y) = w_y(x,y) = 0, & (x,y) \in (0,\pi) \times \{-\ell,\ell\}. \end{cases}$$

By Galerkin-type procedure, we obtain that problem (3.58) has a unique local solution $v_m \in C^2([0,\tau_m); H^2_{**}(\Omega))$, where $[0,\tau_m)$ is the maximal interval of the continuation of v_m.

Step 2. The solution sequence $\{v_m\}$ is uniform bounded.

Testing (3.58) with v_m' and integrating over $(0,t)$, we have by several estimates

$$\|v_m\|^2_{H^2_{**}} + \|v_m'\|^2_{L^2} \leq C, \quad \text{for any } t \in [0,\tau_m) \text{ and } m \geq 1,$$

where C is independent of m and t. Hence, the solution v_m is globally defined and $\{v_m\}$ is bounded in $C^0([0,\infty); H^2_{**}(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$.

Step 3. $\{v_m\}$ admits a strongly convergent subsequence in the function space $C^0([0,\infty); H^2_{**}(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$.

By Ascoli-Arzelà Theorem, we deduce that, up to subsequences, there exists $v \in C^0([0,\infty); L^2(\Omega))$ such that $v_m \to v$ in $C^0([0,\infty); L^2(\Omega))$.

On the other hand, the sequence $\{v_m\}$ is a Cauchy sequence in the space $C^0([0,\infty); H^2_{**}(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$. Hence, up to subsequences,

$$v_m \to v \quad \text{in } C^0([0,\infty); H^2_{**}(\Omega)) \cap C^1([0,\infty); L^2(\Omega)) \quad \text{as } m \to \infty.$$

Step 4. Take the limit in (3.58) and then we prove the existence of solution to (3.57).

Step 5. The solution to (3.57) is unique.

Assume that v_1, v_2 are two solutions of (3.57), denote $v = v_1 - v_2$. Then using v' as a test function, we obtain after integration over $(0,t)$

$$\|v'\|^2_{L^2} + \|v\|^2_{H^2_{**}} = -2\mu \int_0^t \|v'(s)\|^2_{L^2} ds \leq 0,$$

from which it immediately follows that $v = 0$. Hence, the problem (3.57) admits a unique solution $v \in C^0([0,\infty); H^2_{**}(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$.

3.3.3 Existence and uniqueness result

Due to the dynamical boundary conditions (3.56), it is not straightforward to solve the original problem (3.55)-(3.56) directly. Hence, we first transfer it to a simpler case, which uses the auxiliary problem. Recalling the boundary conditions (3.47), one has

$$u_y(x,\ell,t) = u_y(x,-\ell,t) = \alpha/2, \quad \text{for any } x \in (0,\pi),\ t > 0.$$

Moreover, let β be a C^1 function in t, then we obtain by (3.48) and (3.51) for any $x \in (0,\pi)$

$$\begin{cases} u_t(x,\ell,t) - \eta(t)u(x,\ell,t) = g_1(x,t), & t > 0,\\ u(x,\ell,0) = u_0(x,\ell) \end{cases}$$

and

$$\begin{cases} u_t(x,-\ell,t) - \eta(t)u(x,-\ell,t) = g_2(x,t), & t > 0,\\ u(x,-\ell,0) = u_0(x,-\ell), \end{cases}$$

where

$$\begin{cases} g_1(x,t) = \frac{1}{2}(\beta_t(x,t) - \eta(t)\beta(x,t)) - \frac{1}{2}\theta(t)\gamma(x,t),\\ g_2(x,t) = \frac{1}{2}(\beta_t(x,t) - \eta(t)\beta(x,t)) + \frac{1}{2}\theta(t)\gamma(x,t). \end{cases}$$

For any fixed $x \in (0,\pi)$, they are first order ordinary differential problems in t. Hence, one can get the explicit representation of the values of u on $(0,\pi) \times \{\pm\ell\}$, which we denote by $(h_1,h_2) = (u(x,\ell,t), u(x,-\ell,t))$. Then the problem (3.55)-(3.56) reduces to an evolution linear problem

$$\begin{cases} u_{tt} + \Delta^2 u + \mu u_t + ku = f, & (x,y) \in \Omega,\ t > 0,\\ u(x,y,0) = u_0, & (x,y) \in \Omega,\\ u_t(x,y,0) = u_1, & (x,y) \in \Omega \end{cases} \tag{3.59}$$

with nonhomogeneous boundary conditions: for $t > 0$

$$\begin{cases} u(x,y,t) = u_{xx}(x,y,t) = 0, & (x,y) \in \{0,\pi\} \times (-\ell,\ell),\\ u(x,y,t) = h_1, \quad u_y(x,y,t) = \alpha/2, & (x,y) \in (0,\pi) \times \{\ell\},\\ u(x,y,t) = h_2, \quad u_y(x,y,t) = \alpha/2, & (x,y) \in (0,\pi) \times \{-\ell\}. \end{cases} \tag{3.60}$$

Now we want to reduce the boundary conditions (3.60) to the homogeneous case. To this end, it is necessary to construct a suitable inverse trace operator. Note that $\Omega \subset \mathbb{R}^2$ is a rectangular domain. Let us define the space

$$H^4_*(\Omega) := \{u \in H^4(\Omega) : u = u_{xx} = 0 \text{ on } \{0,\pi\} \times (-\ell,\ell)\}$$

and a continuous map

$$T : H^4_*(\Omega) \to \prod_{i=1}^{2} \left(H^{7/2}(\Sigma_i) \times H^{5/2}(\Sigma_i) \times H^{3/2}(\Sigma_i) \times H^{1/2}(\Sigma_i)\right)$$
$$\phi \mapsto \prod_{i=1}^{2} \left(\phi_{|\Sigma_i}, (\phi_y)_{|\Sigma_i}, (\phi_{yy})_{|\Sigma_i}, (\phi_{yyy})_{|\Sigma_i}\right)$$

where $\Sigma_1 := (0,\pi) \times \{\ell\}$, $\Sigma_2 := (0,\pi) \times \{-\ell\}$. For more details, see [63, Section 2.5] or [38, Chapter 1].

Denote the range of T by $R(T) := T(H^4_*(\Omega))$ and define the following norm on $R(T)$ by

$$\|v\|_R := \inf\{\|w\|_{H^4(\Omega)} : w \in H^4_*(\Omega), T(w) = v\} \quad \text{for any } v \in R(T),$$

then $R(T)$ is a Banach space with the norm $\|\cdot\|_R$. Therefore, the restriction of the previous map T to $(\ker(T))^\perp$, i.e, $T_{|(\ker(T))^\perp} : (\ker(T))^\perp \to R(T)$ is an isometric isomorphism.

Since $R(T) \subset \prod_{i=1}^{2} \left(H^{7/2}(\Sigma_i) \times H^{5/2}(\Sigma_i) \times H^{3/2}(\Sigma_i) \times H^{1/2}(\Sigma_i)\right)$, one may represent T as (T_1, T_2, T_3, T_4) with

$$T_1 : H^4_*(\Omega) \to \prod_{i=1}^{2} H^{7/2}(\Sigma_i), \quad T_2 : H^4_*(\Omega) \to \prod_{i=1}^{2} H^{5/2}(\Sigma_i),$$
$$T_3 : H^4_*(\Omega) \to \prod_{i=1}^{2} H^{3/2}(\Sigma_i), \quad T_4 : H^4_*(\Omega) \to \prod_{i=1}^{2} H^{1/2}(\Sigma_i),$$

which are continuous from $H^4_*(\Omega)$ to the respective spaces each of them endowed with its normal norm. Then one may define four subspaces

$$V_1 := \{v \in R(T) : v = (T_1(w), 0, 0, 0), w \in H^4_*(\Omega)\},$$
$$V_2 := \{v \in R(T) : v = (0, T_2(w), 0, 0), w \in H^4_*(\Omega)\},$$
$$V_3 := \{v \in R(T) : v = (0, 0, T_3(w), 0), w \in H^4_*(\Omega)\},$$
$$V_4 := \{v \in R(T) : v = (0, 0, 0, T_4(w)), w \in H^4_*(\Omega)\}.$$

Concerning the fact that $T_{|(\ker(T))^\perp}$ is an isometric isomorphism and the continuity of the map T, one can show that V_i $(i = 1, 2, 3, 4)$ are closed in $R(T)$ endowed with $\|\cdot\|_R$.

Now, let $\alpha \in C^2([0,\infty); V_2)$, $\beta \in C^1([0,\infty); V_1)$, $\gamma \in C^0([0,\infty); V_1)$ and h_1, h_2 be in $C^2([0,\infty); V_1)$. Then one may define the map

$$w := T^{-1}_{|(\ker T)^\perp}\left((h_1, \alpha/2, 0, 0) \times (h_2, \alpha/2, 0, 0)\right)$$

with $w = h_i$, $w_y = \alpha/2$, $w_{yy} = 0$ and $w_{yyy} = 0$ on the boundary Σ_i. In this way we have that $w \in C^2([0,\infty); H^4_*(\Omega))$. Then putting $u = v + w$ (v to be fixed) into the problem (3.59)-(3.60), we obtain the following variational problem

$$
\begin{cases}
v_{tt} + \Delta^2 v + \mu v_t + kv = \widetilde{f}, \quad (x,y) \in \Omega, \\
v(x,y,t) = v_{xx}(x,y,t) = 0, \quad (x,y) \in \{0,\pi\} \times (-\ell,\ell), \\
v(x,y,t) = v_y(x,y,t) = 0, \quad (x,y) \in (0,\pi) \times \{\pm\ell\}, \\
v(x,y,0) = v_0, \quad (x,y) \in \Omega, \\
v_t(x,y,0) = v_1, \quad (x,y) \in \Omega,
\end{cases}
\tag{3.61}
$$

where $\widetilde{f} = f - w_{tt} - \Delta^2 w - \mu w_t - kw$, $v_0 = u_0(x,y) - w(x,y,0)$ and $v_1 = u_1(x,y) - w_t(x,y,0)$.

Recalling the function space

$$H^2_*(\Omega) := \{u \in H^2(\Omega) : u = 0 \text{ on } \{0,\pi\} \times (-\ell,\ell)\},$$

which is defined in Section 3.1.3, we have the uniqueness result.

Theorem 3.4. *Assume that $f \in C^0([0,\infty); L^2(\Omega))$, $u_0 \in H^2_*(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there exists a unique solution to the problem* (3.55)-(3.56).

Proof. It is easy to see that the functions $\widetilde{f}$, v_0 and v_1 satisfy

$$\widetilde{f} \in C^0([0,\infty); L^2(\Omega)), \quad v_0 \in H^2_{**}(\Omega), \quad v_1 \in L^2(\Omega).$$

Hence, we obtain from Section 3.3.2 that (3.61) has a unique solution

$$v \in C^0([0,\infty); H^2_{**}(\Omega)) \cap C^1([0,\infty); L^2(\Omega)).$$

Then, according to the arguments above,

$$u = v + w \in C^0([0,\infty); H^2_*(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$$

is the unique solution of the original problem (3.55)-(3.56) with the initial conditions $u(x,y,0) = u_0$, $u_t(x,y,0) = u_1$ and the boundary conditions (3.46), (3.47), (3.48), (3.51). □

Since the equation in (3.55) is linear, it is possible to find an explicit form of the unique solution. We first consider an initial value problem for $S = S(t)$

$$
\begin{cases}
S'' + \mu S' + aS = g(t), \qquad t > 0, \\
S(0) = A, \quad S'(0) = B
\end{cases}
\tag{3.62}
$$

where $g(t)$ is a given function, μ and a are positive constants, A, B are constants.

We know that (3.62) is a second-order ordinary differential problem. According to the theory of ordinary differential equations, we have the following cases if we denote $\delta = \mu^2 - 4a$:

Case 1. If $\delta \neq 0$, then the two eigenvalues are $\lambda_1 \neq \lambda_2$. Hence, we have

$$\begin{aligned} S(t) =& \frac{B - A\lambda_2}{\lambda_1 - \lambda_2} \exp(\lambda_1 t) + \frac{\exp(\lambda_1 t)}{\lambda_1 - \lambda_2} \int_0^t \exp(-\lambda_1 s) g(s) ds \\ &+ \frac{A\lambda_1 - B}{\lambda_1 - \lambda_2} \exp(\lambda_2 t) + \frac{\exp(\lambda_2 t)}{\lambda_1 - \lambda_2} \int_0^t \exp(-\lambda_2 s) g(s) ds; \end{aligned}$$

Case 2. If $\delta = 0$, then the two eigenvalues are $\lambda_1 = \lambda_2$. Therefore,

$$\begin{aligned} S(t) =& (A + (B - \lambda_1 A)\, t) \exp(\lambda_1 t) \\ &- \exp(\lambda_1 t) \int_0^t (s - t) \exp(-\lambda_1 s) g(s) ds; \end{aligned}$$

To obtain a Fourier series type of the unique solution, we introduce the following

$$\begin{aligned} u_{0m}(y) &= \frac{2}{\pi} \int_0^\pi u_0 \sin(mx) dx, \quad u_{1m}(y) = \frac{2}{\pi} \int_0^\pi u_1 \sin(mx) dx, \\ \alpha_m(t) &= \frac{2}{\pi} \int_0^\pi \alpha \sin(mx) dx, \quad \beta_m(t) = \frac{2}{\pi} \int_0^\pi \beta \sin(mx) dx \\ \gamma_m(t) &= \frac{2}{\pi} \int_0^\pi \gamma \sin(mx) dx, \quad f_m(y,t) = \frac{2}{\pi} \int_0^\pi f \sin(mx) dx \end{aligned} \tag{3.63}$$

and define a function by

$$R_m(t) = \theta(t) \Big(\frac{u_{0m}(\ell) - u_{0m}(-\ell)}{2\ell} - \int_0^t \frac{\gamma_m(s)}{2\ell} ds \Big), \quad t > 0, \tag{3.64}$$

here $\theta(t)$ is as in (3.52).

Then the unique solution to the original problem (3.55)-(3.56) can be explicitly represented.

Theorem 3.5. *Assume that $f \in C^0([0,\infty); L^2(\Omega))$, $u_0 \in H^2_*(\Omega)$ and $u_1 \in L^2(\Omega)$. Then the unique solution to* (3.55)-(3.56) *is given by*

$$u(x,y,t) = \sum_{m=1}^{\infty} U_m(y,t) \sin(mx) \tag{3.65}$$

with

$$U_m(y,t) = \sum_{n=1}^{\infty}(T_m)_n(t)\sin\Big(\frac{n\pi}{\ell}y\Big) + \sum_{n=1}^{\infty}(S_m)_n(t)\cos\Big(\frac{n\pi}{\ell}y\Big) + C_m(t)y + D_m(t), \tag{3.66}$$

where $(T_m)_n(t)$*,* $(S_m)_n(t)$ *and* $D_m(t)$ *are in the form of* $S(t)$*, the solution of* (3.62)*,* $C_m(t)$ *is as in* (3.64)*.*

Proof. According to the boundary conditions for $x = 0$ and $x = \pi$, we seek the solution u of the problem (3.55)-(3.56) in the form

$$u(x,y,t) = \sum_{m=1}^{\infty} U_m(y,t)\sin(mx). \tag{3.67}$$

Inserting (3.67) in (3.55)-(3.56) and recalling (3.63), we have for every $m \geq 1$

$$\begin{cases} (U_m)_{tt} + (U_m)_{yyyy} - 2m^2(U_m)_{yy} + (m^4+k)U_m + \mu(U_m)_t = f_m(y,t), \\ U_m(y,0) = u_{0m}(y), \\ (U_m)_t(y,0) = u_{1m}(y), \end{cases} \tag{3.68}$$

with the boundary conditions: for $t > 0$

$$\begin{cases} (U_m)_y(-\ell,t) - (U_m)_y(\ell,t) = 0, \\ (U_m)_y(-\ell,t) + (U_m)_y(\ell,t) = \alpha_m(t), \\ U_m(-\ell,t) + U_m(\ell,t) = \beta_m(t), \\ (U_m)_t(-\ell,t) - (U_m)_t(\ell,t) - \eta(t)(U_m(-\ell,t) - U_m(\ell,t)) = \gamma_m(t)\theta(t). \end{cases} \tag{3.69}$$

Now, we look for the solution to (3.68)-(3.69) in the form of (3.66). Putting (3.66) into the equation in (3.68), we obtain

$$\sum_{n=1}^{\infty}\Big((T_m)''_n(t) + \mu(T_m)'_n(t) + \delta(T_m)_n(t)\Big)\sin\big(\frac{n\pi}{\ell}y\big) + \sum_{n=1}^{\infty}\Big((S_m)''_n(t) + \mu(S_m)'_n(t) + \delta(S_m)_n(t)\Big)\cos\big(\frac{n\pi}{\ell}y\big) + C''_m(t)y + \mu C'_m(t)y + \sigma C_m(t)y + D''_m(t) + \mu D'_m(t) + \sigma D_m(t) = f_m(y,t)$$

with $\delta = \Big(m^2 + \big(\frac{n\pi}{l}\big)^2\Big)^2 + k$ and $\sigma = m^4 + k$.

For $C_m(t)$ being as in the form in (3.64), we define the function

$$\phi_m(y,t) = C_m''(t)y + (m^4 + k + \mu\eta(t))C_m(t)y - \frac{\mu\gamma_m(t)\theta(t)}{2\ell}y$$

and let $f_m(y,t) = f_m(y,t) - \phi_m(y,t) + \phi_m(y,t)$, then it follows that

$$C_m'(t) - \eta(t)C_m(t) = -\frac{\gamma_m(t)\theta(t)}{2\ell},$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big((T_m)_n''(t) + \mu(T_m)_n'(t) + \delta(T_m)_n(t)\Big)\sin\big(\frac{n\pi}{\ell}y\big) \\
&\quad + \sum_{n=1}^{\infty}((S_m)_n''(t) + \mu(S_m)_n'(t) + \delta(S_m)_n(t))\cos\Big(\frac{n\pi}{\ell}y\Big) \\
&\quad + D_m''(t) + \mu D_m'(t) + \sigma D_m(t) \\
&= \sum_{n=1}^{\infty}(\varphi_{1m})_n(t)\sin\Big(\frac{n\pi}{\ell}y\Big) + \sum_{n=1}^{\infty}(\varphi_{2m})_n(t)\cos\Big(\frac{n\pi}{\ell}y\Big) + f_m(t),
\end{aligned}$$

where

$$\begin{aligned}
f_m(t) &= \frac{1}{2\ell}\int_{-\ell}^{\ell} f_m(y,t)dy, \\
(f_{1m})_n(t) &= \frac{1}{\ell}\int_{-\ell}^{\ell}(f_m(y,t) - \phi_m(y,t))\sin\Big(\frac{n\pi}{\ell}y\Big)dy, \\
(f_{2m})_n(t) &= \frac{1}{\ell}\int_{-\ell}^{\ell}(f_m(y,t) - \phi_m(y,t))\cos\Big(\frac{n\pi}{\ell}y\Big)dy.
\end{aligned}$$

The initial conditions yield, for $y \in (-\ell, \ell)$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}(T_m)_n(0)\sin\Big(\frac{n\pi}{\ell}y\Big) + \sum_{n=1}^{\infty}(S_m)_n(0)\cos\Big(\frac{n\pi}{\ell}y\Big) + D_m(0) \\
&= \sum_{n=1}^{\infty}(A_{1m})_n\sin\Big(\frac{n\pi}{\ell}y\Big) + \sum_{n=1}^{\infty}(A_{2m})_n\cos\Big(\frac{n\pi}{\ell}y\Big) + A_m, \\
&\sum_{n=1}^{\infty}(T_m)_n'(0)\sin\Big(\frac{n\pi}{\ell}y\Big) + \sum_{n=1}^{\infty}(S_m)_n'(0)\cos\Big(\frac{n\pi}{\ell}y\Big) + D_m'(0) \\
&= \sum_{n=1}^{\infty}(B_{1m})_n\sin\Big(\frac{n\pi}{\ell}y\Big) + \sum_{n=1}^{\infty}(B_{2m})_n\cos\Big(\frac{n\pi}{\ell}y\Big) + B_m,
\end{aligned}$$

where

$$
\begin{aligned}
(A_{1m})_n &= \frac{1}{\ell}\int_{-\ell}^{\ell}(u_{0m}(y)-C_m(0)y)\sin\Big(\frac{n\pi}{\ell}y\Big)dy,\\
(A_{2m})_n &= \frac{1}{\ell}\int_{-\ell}^{\ell}(u_{0m}(y))\cos\Big(\frac{n\pi}{\ell}y\Big)dy,\\
(B_{1m})_n &= \frac{1}{\ell}\int_{-\ell}^{\ell}(u_{1m}(y)-C_m'(0)y)\sin\Big(\frac{n\pi}{\ell}y\Big)dy,\\
(B_{2m})_n &= \frac{1}{\ell}\int_{-\ell}^{\ell}(u_{1m}(y))\cos\Big(\frac{n\pi}{\ell}y\Big)dy,\\
A_m = \frac{1}{2\ell}\int_{-\ell}^{\ell}&u_{0m}(y)dy, \quad B_m = \frac{1}{2\ell}\int_{-\ell}^{\ell}u_{1m}(y)dy.
\end{aligned}
$$

Assume that the functions $\alpha_m(t)$, $\beta_m(t)$ and $\gamma_m(t)$ are given by

$$
\begin{aligned}
&\alpha_m(t) = 2\sum_{n=1}^{\infty}\frac{n\pi}{\ell}(T_m)_n(t)\cos(n\pi), \quad t>0,\\
&\beta_m(t) = 2\sum_{n=1}^{\infty}(S_m)_n(t)\cos(n\pi) + 2D_m(t), \quad t>0,\\
&\gamma_m(t) = -2\ell(C_m'(t)-\eta(t)C_m(t))/\theta(t), \quad t>0,
\end{aligned}
$$

then the boundary conditions (3.56) are satisfied. Moreover, we are led to several ordinary differential problems

$$
\begin{cases}
C_m'(t)-\eta(t)C_m(t) = -\frac{\gamma_m(t)\theta(t)}{2\ell}, & t>0,\\
C_m(t) = \frac{u_{0m}(\ell)-u_{0m}(-\ell)}{2\ell}, & t=0\,;
\end{cases} \tag{3.70}
$$

$$
\begin{cases}
D_m''(t)+\mu D_m'(t)+\sigma D_m(t) = f_m(t), & t>0,\\
D_m(t) = A_m, \quad D_m'(t) = B_m, & t=0\,;
\end{cases} \tag{3.71}
$$

$$
\begin{cases}
(T_m)_n''(t)+\mu(T_m)_n'(t)+\delta(T_m)_n(t) = (f_{1m})_n(t), & t>0,\\
(T_m)_n(t) = (A_{1m})_n, \quad (T_m)_n'(t) = (B_{1m})_n, & t=0\,;
\end{cases} \tag{3.72}
$$

$$
\begin{cases}
(S_m)_n''(t)+\mu(S_m)_n'(t)+\delta(S_m)_n(t) = (f_{2m})_n(t), & t>0,\\
(S_m)_n(t) = (A_{2m})_n, \quad (S_m)_n'(t) = (B_{2m})_n, & t=0.
\end{cases} \tag{3.73}
$$

To complete the proof, we need to check that problem (3.70) admits a solution in the form $R_m(t)$ and that the problems (3.71)-(3.73) have solutions in the form of $S(t)$. Since the last three problems depend on $C_m(t)$, we first consider the problem (3.70). In fact, this is a first order ordinary differential problem and we know that the solution is

$$C_m(t) = C_m(0)\exp\Big(\int_0^t \eta(s)ds\Big)$$
$$-\exp\Big(\int_0^t \eta(s)ds\Big)\int_0^t \frac{\gamma_m(s)\theta(s)}{2\ell}\exp\Big(-\int_0^s \eta(\tau)d\tau\Big)ds.$$

Since $\eta(t)$ is not a continuous function, see (3.53), to compute conveniently, we denote $E(t)$ given in (3.54) here by

$$E(t) = \begin{cases} -1, & \mathcal{E}(t) \le \overline{E}_\mu - \varepsilon, \\ \frac{1}{\varepsilon}\left(\mathcal{E}(t) - \overline{E}_\mu\right), & \overline{E}_\mu - \varepsilon < \mathcal{E}(t) \le \overline{E}_\mu + \varepsilon, \\ 1, & \mathcal{E}(t) > \overline{E}_\mu + \varepsilon. \end{cases}$$

Then we consider the exponential function $\xi(t) = \exp\big(\int_0^t \eta(s)ds\big)$. There are three cases:

(1) If $\mathcal{E}(t) > \overline{E}_\mu + \varepsilon$, then

$$\xi(t) = \exp\Big(\frac{\mathcal{E}(t)}{\overline{E}_\mu} - \frac{\mathcal{E}(0)}{\overline{E}_\mu} + \ln\frac{\mathcal{E}(t) - \overline{E}_\mu}{\mathcal{E}(0) - \overline{E}_\mu}\Big);$$

(2) If $\overline{E}_\mu < \mathcal{E}(t) \le \overline{E}_\mu + \varepsilon$, then

$$\xi(t) = \exp\Big(\frac{\mathcal{E}^2(t) - (\overline{E}_\mu)^2}{2\varepsilon\overline{E}_\mu} + \frac{\varepsilon - 2\mathcal{E}(0)}{2\overline{E}_\mu} + \ln\frac{\varepsilon}{\mathcal{E}(0) - \overline{E}_\mu}\Big);$$

(3) If $\mathcal{E}(t) \le \overline{E}_\mu$, then

$$\xi(t) = \exp\Big(\frac{\varepsilon - 2\mathcal{E}(0)}{2\overline{E}_\mu} + \ln\frac{\varepsilon}{\mathcal{E}(0) - \overline{E}_\mu} + \int_{t_\mu}^t \eta(s)ds\Big),$$

where t_μ satisfies $\mathcal{E}(t_\mu) = \overline{E}_\mu$.

Let $\varepsilon \to 0$, we have

$$\xi(t) = \begin{cases} 0, & \mathcal{E}(t) \le \overline{E}_\mu, \\ \frac{\mathcal{E}(t) - \overline{E}_\mu}{\mathcal{E}(0) - \overline{E}_\mu}\exp\left(\frac{\mathcal{E}(t)}{\overline{E}_\mu} - \frac{\mathcal{E}(0)}{\overline{E}_\mu}\right), & \mathcal{E}(t) > \overline{E}_\mu. \end{cases}$$

Therefore,

$$C_m(t) = \begin{cases} 0, & \mathcal{E}(t) \le \overline{E}_\mu, \\ \theta(t)\left(\frac{u_{0m}(\ell)-u_{0m}(-\ell)}{2\ell} - \int_0^t \frac{\gamma_m(s)}{2\ell} ds\right), & \mathcal{E}(t) > \overline{E}_\mu, \end{cases}$$

which is as in (3.64) if one recalls that (3.52).

Next, for every $n \ge 1$ and $m \ge 1$, we solve the other three ordinary differential problems (3.71), (3.72) and (3.73). Recalling the initial value problem (3.62) and let $a = \sigma$ for (3.71), $a = \delta$ for (3.72) and (3.73), we obtain that there exists a unique solution for (3.71), (3.72) and (3.73) separately in the form of $S(t)$ with the constants A, B replaced by $A_m, B_m, (A_{1m})_n, (B_{1m})_n, (A_{2m})_n, (B_{2m})_n$.

Therefore, the unique solution to (3.55)-(3.56) has the form of (3.65) with $U_m(y,t)$ being given in (3.66). And then the proof is finished. □

By using the explicit form of the solution to (3.55)-(3.56), we are able to analyze the amplitude of the torsional oscillation appearing in suspension bridges.

Corollary 3.1. *Assume that u is the unique solution to the original problem* (3.55)-(3.56). *Then the amplitude of the torsional oscillation on the two sides $y = \pm\ell$ reads*

$$|u(x,-\ell,t) - u(x,\ell,t)| = \theta(t)\big|u_0(x,-\ell) - u_0(x,\ell) + 2\ell \int_0^t \gamma(x,s)ds\big|,$$

where $\theta(t)$ is given in (3.52).

By (3.52), if $\mathcal{E}(t) \le \overline{E}_\mu$, then $\theta(t) = 0$, which yields that from Corollary 3.1

$$|u(x,\ell,t) - u(x,-\ell,t)| = 0.$$

That is, when $\mathcal{E}(t) \le \overline{E}_\mu$, there is no torsional oscillation appearing in the bridge structure.

However, once the energy $\mathcal{E}(t)$ exceeds $\overline{E}_\mu$, then $\theta(t) > 0$ and

$$|u(x,\ell,t) - u(x,-\ell,t)| \ne 0,$$

which show that the torsional oscillation appears. Since $\theta(t)$ is an increasing function with respect to $\mathcal{E}(t)$ and by Corollary 3.1, if the energy $\mathcal{E}(t)$ ($\mathcal{E}(t) > \overline{E}_\mu$) increases, then the amplitude of the torsional oscillation will go up till the bridges collapse.

CHAPTER 4

A plate model for suspension bridge with large deformations

This chapter is devoted to a system (composed by two coupled fourth order partial differential equations) modeling the suspension bridge. This model is based on the von Kármán plate equations and this part is from [34].

In the previous chapter, we considered the plate model by assuming that the deflection of the plate is small. There the material nonlinearities of the restoring force due to the hangers and the cables were discussed. While in some cases a kind of wide oscillations may appear in the roadway of suspension bridge due to some reasons, such as the strong wind. In this case, the linear Kirchhoff-Love plate theory (see [44, 54]) cannot describe the behavior of the oscillations and hence, one needs to consider the geometric nonlinearities of the plate due to the wide oscillations.

4.1 The plate model

4.1.1 The energy in the plate

As in Chapter 3, assume that $\Omega = (0,\pi)\times(-\ell,\ell)\subset\mathbb{R}^2$ represents the roadway of the suspension bridge. π is the length of the roadway and 2ℓ is its width. The realistic assumption is that $2\ell \ll \pi$.

In Section 3.1.1, we have seen that the bending energy of a deformed plate Ω due to an external force f is $\mathbb{E}_B$, see (3.2). While if large deformations in the plate are involved, one does not have a linear strain-displacement relation. For a plate of uniform thickness $d > 0$, one assumes that the plate has a middle surface midway between its two parallel faces that, in equilibrium, occupies the region Ω in the plane $z = 0$. Let $w = w(x, y)$, $v = v(x, y)$, $u = u(x, y)$ denote the components (respectively in the x, y, z directions) of the displacement vector of the particle of the middle surface which, when the plate is in equilibrium, occupies the position $(x, y) \in \Omega$: u is the component in the vertical z-direction which is related to bending while w and v are the in-plane stretching components.

For large deformations of the plate Ω there is a coupling between u and (w, v). In order to describe it, we compute the stretching in the x and y directions (see e.g. [76, (7.80)]):

$$\begin{cases} \epsilon_x = \sqrt{1 + 2w_x + u_x^2} - 1 \approx w_x + \frac{u_x^2}{2}, \\ \epsilon_y = \sqrt{1 + 2v_y + u_y^2} - 1 \approx v_y + \frac{u_y^2}{2}, \end{cases} \tag{4.1}$$

where the approximation is due to the fact that, compared to unity, all the components are small in the horizontal directions x and y. One can also compute the shear strain (see e.g. [76, (7.81)]):

$$\gamma_{xy} \approx w_y + v_x + u_x u_y \,. \tag{4.2}$$

Moreover, it is convenient to introduce the so-called stress resultants which are the integrals of suitable components of the strain tensor (see e.g. [49, (1.22)]), namely,

$$\begin{cases} N^x = \frac{Ed}{1-\sigma^2}\left(w_x + \sigma v_y + \frac{1}{2}u_x^2 + \frac{\sigma}{2}u_y^2\right), \\ N^y = \frac{Ed}{1-\sigma^2}\left(v_y + \sigma w_x + \frac{1}{2}u_y^2 + \frac{\sigma}{2}u_x^2\right), \\ N^{xy} = \frac{Ed}{2(1+\sigma)}\left(w_y + v_x + u_x u_y\right), \end{cases} \tag{4.3}$$

so that

$$\epsilon_x = \frac{N^x - \sigma N^y}{Ed}\,, \quad \epsilon_y = \frac{N^y - \sigma N^x}{Ed}\,, \quad \gamma_{xy} = \frac{2(1+\sigma)}{Ed}\,N^{xy}\,.$$

We are now in a position to define the energy functional. The first term of the energy is due to pure bending and to the external load f and is given by $\mathbb{E}_B$, see (3.2). For large deformations, one needs to consider also the interaction with the stretching components v and w and the total energy reads (see [50, (1.7)]):

$$J(u,v,w) = \mathbb{E}_B(u) + \frac{E\,d}{2(1-\sigma^2)} \int_\Omega \left(\epsilon_x^2 + \epsilon_y^2 + 2\sigma\,\epsilon_x\epsilon_y + \frac{1-\sigma}{2}\,\gamma_{xy}^2\right) dxdy, \tag{4.4}$$

which has to be compared with (3.2). In view of (4.1)-(4.3), the additional term $I := J - \mathbb{E}_B$ may also be written as

$$I(u,v,w) = \frac{E\,d}{4(1+\sigma)} \int_\Omega (w_y+v_x+u_xu_y)^2 dxdy$$
$$+\Lambda \int_\Omega \left\{ \left(w_x+\frac{u_x^2}{2}\right)^2 + \left(v_y+\frac{u_y^2}{2}\right)^2 + 2\sigma \left(w_x+\frac{u_x^2}{2}\right)\left(v_y+\frac{u_y^2}{2}\right) \right\} dxdy$$

with $\Lambda = \frac{E\,d}{2(1-\sigma^2)}$.

4.1.2 The Euler-Lagrange equations

Assume that the plate $\Omega = (0,\pi) \times (-\ell,\ell) \subset \mathbb{R}^2$ with $\ell \ll \pi$. Recall the Hilbert space

$$H_*^2(\Omega) := \big\{ w \in H^2(\Omega);\ w = 0 \text{ on } \{0,\pi\} \times (-\ell,\ell) \big\}$$

and the dual space $\mathcal{H}_*$. We also denote by $\langle\cdot,\cdot\rangle$ the corresponding duality.

On the space $H^2(\Omega)$, we define the Monge-Ampère operator

$$[\phi,\psi] := \phi_{xx}\psi_{yy} + \phi_{yy}\psi_{xx} - 2\phi_{xy}\psi_{xy}, \quad \forall \phi,\psi \in H^2(\Omega), \tag{4.5}$$

so that, in particular, $[\phi,\phi] = 2\det(D^2\phi)$ where $D^2\phi$ is the Hessian matrix of ϕ. Then the scalar product (3.12) and the corresponding norm (3.13) read

$$(u,v)_{H_*^2} = \int_\Omega \Big(\Delta u \Delta v - (1-\sigma)[u,v]\Big)\, dxdy, \quad \forall u,v \in H_*^2(\Omega) \tag{4.6}$$

and

$$\|u\|_{H_*^2} = \left(\int_\Omega \Big(|\Delta u|^2 - (1-\sigma)[u,u]\Big)\, dxdy \right)^{1/2}, \quad \forall u \in H_*^2(\Omega)\,. \tag{4.7}$$

Therefore, the unique minimiser u of the convex functional $\mathbb{E}_B$ in (3.2) over the space $H_*^2(\Omega)$ satisfies the Euler-Lagrange equation

$$\frac{E\,d^3}{12(1-\sigma^2)}\,\Delta^2 u = f(x,y) \qquad \text{in } \Omega\,.$$

On the other hand, the Euler-Lagrange equation for the energy J in (4.4) characterises the critical points of J: we need to compute the variation δJ of J and to find triples (u, v, w) such that for any $\phi, \psi, \xi \in C_c^\infty(\Omega)$

$$\langle \delta J, (\phi, \psi, \xi) \rangle = \lim_{t \to 0} \frac{J(u + t\phi, v + t\psi, w + t\xi) - J(u, v, w)}{t} = 0\,.$$

Replacing N^x, N^y, N^{xy}, see (4.3), we have for any $\phi, \psi, \xi \in C_c^\infty(\Omega)$

$$\begin{cases} \frac{E\,d^3}{12(1-\sigma^2)} \int_\Omega \big(\Delta u \Delta \phi + (\sigma - 1)[u, \phi]\big)\, dxdy \\ + \int_\Omega \big((N^x u_x + N^{xy} u_y)\phi_x + (N^y u_y + N^{xy} u_x)\phi_y\big)\, dxdy = \int_\Omega f\phi\,, \\ \int_\Omega \big(N^y \psi_y + N^{xy} \psi_x\big)\, dxdy = 0, \\ \int_\Omega \big(N^x \xi_x + N^{xy} \xi_y\big)\, dxdy = 0. \end{cases}$$

Thanks to some integration by parts and by arbitrariness of the test functions, we may rewrite the above identities in strong form

$$\begin{cases} \frac{E\,d^3}{12(1-\sigma^2)} \Delta^2 u - (N^x u_x + N^{xy} u_y)_x - (N^y u_y + N^{xy} u_x)_y = f & \text{in } \Omega\,, \\ N^y_y + N^{xy}_x = 0\,, \quad N^x_x + N^{xy}_y = 0 & \text{in } \Omega\,. \end{cases} \tag{4.8}$$

The last two equations in (4.8) show that there exists a function Φ (called Airy stress function, see [72, p.363]), unique up to an affine function, such that

$$\Phi_{yy} = N^x, \quad \Phi_{xx} = N^y, \quad \Phi_{xy} = -N^{xy}\,. \tag{4.9}$$

Then, after some tedious computations, by using the Monge-Ampère operator (4.5) and by normalizing the coefficients, the system (4.8) may be written as

$$\begin{cases} \Delta^2 \Phi = -[u, u] & \text{in } \Omega \\ \Delta^2 u = [\Phi, u] + f & \text{in } \Omega\,. \end{cases} \tag{4.10}$$

In a plate subjected to compressive forces along its edges, one should consider a prestressing constraint which may lead to buckling. Then the system (4.10) becomes

$$\begin{cases} \Delta^2 \Phi = -[u, u] & \text{in } \Omega \\ \Delta^2 u = [\Phi, u] + f + \lambda[F, u] & \text{in } \Omega\,. \end{cases} \tag{4.11}$$

The term $\lambda[F, u]$ in the right hand side of (4.11) represents the boundary stress, where the parameter $\lambda \geq 0$ measures the magnitude of the compressive forces acting on $\partial\Omega$ while the smooth function F satisfies

$$F \in C^4(\overline{\Omega})\,, \quad \Delta^2 F = 0 \text{ in } \Omega\,, \quad F_{xx} = F_{xy} = 0 \text{ on } (0, \pi) \times \{\pm \ell\}\,, \tag{4.12}$$

see [11, pp.228-229]: the term λF represents the stress function in the plate resulting from the applied force if the plate were artificially prevented from deflecting and the boundary constraints in (4.12) physically mean that no external stresses are applied on the free edges of the plate. Following Knightly-Sather [47], we take

$$F(x,y) = \frac{\ell^2 - y^2}{2} \qquad \text{so that} \quad [F,u] = -u_{xx}\,.$$

Therefore, (4.11) becomes

$$\begin{cases} \Delta^2\Phi = -[u,u] & \text{in } \Omega \\ \Delta^2 u = [\Phi,u] + f - \lambda u_{xx} & \text{in } \Omega\,. \end{cases} \tag{4.13}$$

4.1.3 Boundary conditions

The purpose of this subsection is to determine the boundary conditions for a plate modeling the roadway of suspension bridge.

In literature the system (4.13) is usually considered under the Dirichlet boundary conditions, see [18, § 1.5] and [78, p.514]. But since we aim to model a suspension bridge, these conditions (Dirichlet boundary conditions) are not the correct ones. Following [27] (see also [3,82]) we view the deck of a suspension bridge as a long narrow rectangular thin plate hinged at its two opposite short edges and free on the remaining two long edges.

Let us first consider the two short edges $\{0,\pi\} \times (-\ell,\ell)$. Due to the connection with the ground, u is assumed to be hinged there and hence it satisfies the Navier boundary conditions:

$$u = u_{xx} = 0 \qquad \text{on } \{0,\pi\} \times (-\ell,\ell)\,. \tag{4.14}$$

In this case, Ventsel-Krauthammer [76, Example 7.4] suggest that $N^x = v = 0$ on $\{0,\pi\}\times(-\ell,\ell)$. In view of (4.3) this yields that on $\{0,\pi\}\times(-\ell,\ell)$

$$0 = w_x + \sigma v_y + \frac{1}{2}u_x^2 + \frac{\sigma}{2}u_y^2 = w_x + \frac{1}{2}u_x^2 = \frac{Ed}{(1-\sigma^2)\sigma}N^y,$$

where the condition $u_y = 0$ comes from the first of (4.14). In turn, by (4.9) this implies that

$$\Phi_{xx} = 0 \quad \text{on } \{0,\pi\} \times (-\ell,\ell).$$

For the second boundary condition we recall that $N^x = 0$ so that, by (4.9), also $\Phi_{yy} = 0$: since the Airy function Φ is defined up to the addition of an affine function, we may take $\Phi = 0$. Summarising, we have

$$\Phi = \Phi_{xx} = 0 \qquad \text{on } \{0,\pi\} \times (-\ell,\ell)\,. \tag{4.15}$$

On the long edges $(0,\pi)\times\{\pm\ell\}$ the plate is assumed to be free, which results in

$$u_{yy}+\sigma u_{xx}=u_{yyy}+(2-\sigma)u_{xxy}=0 \qquad \text{on } (0,\pi)\times\{\pm\ell\}\,, \tag{4.16}$$

see e.g. [76, (2.40)] or [27]. Note that here the boundary conditions (4.16) do not depend on the parameter λ. For the Airy stress function Φ, we follow the usual Dirichlet boundary conditions on $(0,\pi)\times\{\pm\ell\}$, see e.g. [11,12]. Then

$$\Phi=\Phi_y=0 \qquad \text{on } (0,\pi)\times\{\pm\ell\}\,. \tag{4.17}$$

These boundary conditions suggest to introduce the following subspace of $H^2_*(\Omega)$

$$H^2_{**}(\Omega):=\{u\in H^2_*(\Omega): u=u_y=0 \text{ on } (0,\pi)\times\{\pm\ell\}\},$$

which is also a Hilbert space when endowed with the scalar product

$$(u,v)_{H^2_{**}}:=\int_\Omega \Delta u\Delta v dxdy\,.$$

This scalar product introduces the norm denoted by

$$\|u\|_{H^2_{**}}:=\left(\int_\Omega |\Delta u|^2 dxdy\right)^{1/2}.$$

We denote the dual space of $H^2_{**}(\Omega)$ by $\mathcal{H}_{**}(\Omega)$.

4.1.4 The model for suspension bridge

In a plate modeling a suspension bridge, one should also add the nonlinear restoring action due to the hangers and the cables.

The Official Report [4, p.11] states that the region of interaction of the hangers with the plate was of approximately 2 ft on each side: this means that $\epsilon\approx\frac{\pi}{1500}$, see Fighre 3.1. Augusti-Sepe [8] (see also [7]) view the restoring force at the endpoints of a cross-section of the deck as composed by two connected springs, see Figure 4.1. The top one represents the action of the sustaining cables and the bottom one (connected with the deck) represents the hangers. The action of the cables is considered by Bartoli-Spinelli [9, p.180] the main cause of the nonlinearity of the restoring force: they suggest quadratic and cubic perturbations of a linear behavior. Assuming that **the vertical axis is oriented downwards**, the restoring force acts in those parts of the deck which are below the equilibrium position (where

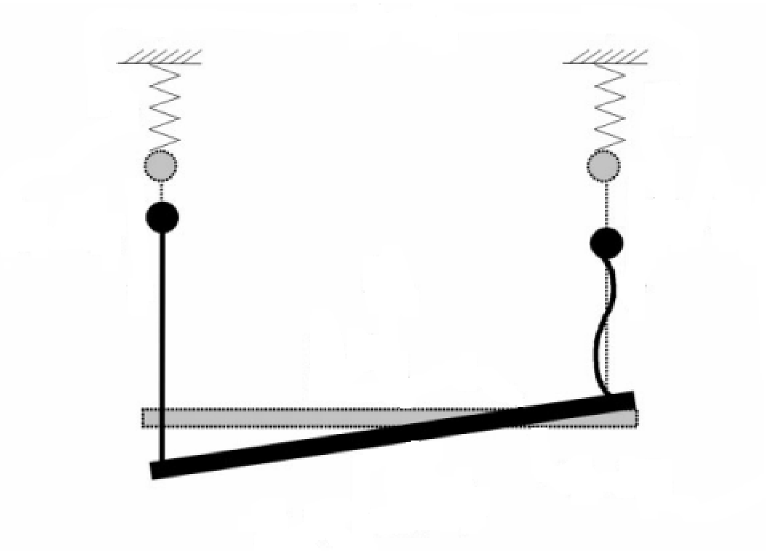

Figure 4.1: *The restoring force on a cross-section.*

$u > 0$) while it exerts no action where the deck is above the equilibrium position ($u < 0$).

Taking into account all these facts, for the explicit action of the restoring force, we take

$$g(u) = ku^+ + \delta(u^+)^3 = (ku + \delta u^3)^+, \quad k, \delta > 0,$$

which is a compromise between the nonlinearities suggested by McKenna-Walter [57] and Plaut-Davis [68] and follows the idea of Ferrero-Gazzola [27]. Here $k > 0$ denotes the Hooke constant of elasticity of steel (hangers) while $\delta > 0$ is a small parameter reflecting the nonlinear behavior of the sustaining cables. Only the positive part is taken into account due to possible slackening, see [4, V-12]: the hangers behave as a restoring force if extended (when $u > 0$) and give no contribution when they lose tension (when $u \leq 0$). Let Υ be the characteristic function of $(-\ell, -\ell+\epsilon) \cup (\ell-\epsilon, \ell)$ for some small $\epsilon > 0$. In Section 3.1.1, we have known that the restoring force due to the hangers is concentrated in two tiny parallel strips adjacent to the long edges (the free part of the boundary), see Figure 3.1. The the restoring force is

$$h = \Upsilon(y)g(u). \tag{4.18}$$

Assume that (4.18), then together with the boundary conditions (4.14)-

(4.17), we obtain the system modeling the suspension bridges:

$$\begin{cases} \Delta^2\Phi = -[u,u] & \text{in } \Omega \\ \Delta^2 u + \Upsilon(y)g(u) = [\Phi,u] + f - \lambda u_{xx} & \text{in } \Omega \\ u = \Phi = u_{xx} = \Phi_{xx} = 0 & \text{on } \{0,\pi\} \times (-\ell,\ell) \\ u_{yy} + \sigma u_{xx} = u_{yyy} + (2-\sigma)u_{xxy} = 0 & \text{on } (0,\pi) \times \{\pm\ell\} \\ \Phi = \Phi_y = 0 & \text{on } (0,\pi) \times \{\pm\ell\}\,. \end{cases} \tag{4.19}$$

Finally, we go back to the original unknowns u, v, w. After that a solution (u,Φ) of (4.19) is found, (4.1)-(4.3) and (4.9) yield

$$\begin{aligned} w_x + \sigma v_y &= \frac{1-\sigma^2}{E\,d}\,\Phi_{yy} - \frac{1}{2}u_x^2 - \frac{\sigma}{2}u_y^2\,, \\ \sigma w_x + v_y &= \frac{1-\sigma^2}{E\,d}\,\Phi_{xx} - \frac{1}{2}u_y^2 - \frac{\sigma}{2}u_x^2 \end{aligned}$$

which immediately gives w_x and v_y.

Upon integration, it gives $w = w(x,y)$ up to the addition of a function only depending on y and $v = v(x,y)$ up to the addition of a function depending only on x. These two additive functions are determined by solving the last constraint given by (4.1)-(4.9), that is,

$$w_y + v_x = -\frac{2(1+\sigma)}{E\,d}\,\Phi_{xy} - u_x - u_y\,.$$

4.2 Preliminaries

4.2.1 Some useful operators and functionals

For any $v, w \in H^2_*(\Omega)$, consider the linear problem

$$\begin{cases} \Delta^2\Phi = -[v,w] & \text{in } \Omega \\ \Phi = \Phi_{xx} = 0 & \text{on } \{0,\pi\} \times (-\ell,\ell) \\ \Phi = \Phi_y = 0 & \text{on } (0,\pi) \times \{\pm\ell\}\,. \end{cases} \tag{4.20}$$

We claim that (4.20) has a unique solution $\Phi = \Phi(v,w)$ and $\Phi \in H^2_{**}(\Omega)$.

In fact, since $\Omega \subset \mathbb{R}^2$, we have $H^{1+\varepsilon}(\Omega) \Subset L^\infty(\Omega) = (L^1(\Omega))'$, for all $\varepsilon > 0$. On the other hand, $L^1(\Omega) \subset (L^\infty(\Omega))' \Subset H^{-(1+\varepsilon)}(\Omega)$. If $v, w \in H^2_*(\Omega) \subset H^2(\Omega)$, then $[v,w] \in L^1(\Omega)$. Therefore,

$$[v,w] \in H^{-(1+\varepsilon)}(\Omega) \qquad \forall \varepsilon > 0\,.$$

Then by the Lax-Milgram Theorem and the regularity theory of elliptic equations, there exists a unique solution of (4.20) and $\Phi \in H^{3-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. An embedding and the boundary conditions show that $\Phi \in H^2_{**}(\Omega)$, which completes the proof of the claim.

This result enables us to define a bilinear form

$$B = B(v, w) = -\Phi,$$

where Φ is the unique solution of (4.20); this form is implicitly characterized for any $v, w \in H^2_*(\Omega)$ and any $\varphi \in H^2_{**}(\Omega)$ by

$$B : (H^2_*(\Omega))^2 \to H^2_{**}(\Omega)\,, \quad (B(v,w), \varphi)_{H^2_{**}} = \int_\Omega [v, w]\varphi\,.$$

Similarly, one can prove that for any $v \in H^2_*(\Omega)$ and any $\varphi \in H^2_{**}(\Omega)$ there exists a unique solution $\Psi \in H^2_*(\Omega)$ of the problem

$$\begin{cases} \Delta^2\Psi = -[v, \varphi] & \text{in } \Omega \\ \Psi = \Psi_{xx} = 0 & \text{on } \{0, \pi\} \times (-\ell, \ell) \\ \Psi_{yy} + \sigma\Psi_{xx} = \Psi_{yyy} + (2-\sigma)\Psi_{xxy} = 0 & \text{on } (0, \pi) \times \{\pm\ell\}\,. \end{cases}$$

This defines another bilinear form

$$C = C(v, \varphi) = -\Psi,$$

which is implicitly characterized for any $v, w \in H^2_*(\Omega)$ and any $\varphi \in H^2_{**}(\Omega)$ by

$$C : H^2_*(\Omega) \times H^2_{**}(\Omega) \to H^2_*(\Omega)\,, \quad (C(v,\varphi), w)_{H^2_*} = \int_\Omega [v, \varphi]w\,.$$

Then we prove

Lemma 4.1. *The trilinear form*

$$(H^2_*(\Omega))^3 \ni (v, w, \varphi) \mapsto \int_\Omega [v, w]\varphi \tag{4.21}$$

is independent of the order of v, w, φ if at least one of them is in $H^2_{**}(\Omega)$. Moreover, if $\varphi \in H^2_{**}(\Omega), v, w \in H^2_*(\Omega)$, then

$$\begin{aligned}(B(v,w), \varphi)_{H^2_{**}} &= (B(w,v), \varphi)_{H^2_{**}} \\ &= (C(v,\varphi), w)_{H^2_*} = (C(w,\varphi), v)_{H^2_*}.\end{aligned} \tag{4.22}$$

Finally, the operators B and C are compact.

Proof. By a density argument and by continuity it suffices to prove all the identities for smooth functions v, w, φ, in such a way that third interior derivatives and second boundary derivatives are well defined and integration by parts is allowed. In the trilinear form (4.21) one can exchange the order of v and w by exploiting the symmetry of the Monge-Ampère operator, that is, $[v, w] = [w, v]$ for all v and w. So, we may assume that one among w, φ is in $H^2_{**}(\Omega)$: note that this function also has vanishing x-derivative on $(0, \pi) \times \{\pm\ell\}$. Then some integration by parts enable to switch the position of w and φ.

From the just proved symmetry of the trilinear form (4.21) we immediately infer (4.22).

If $\varphi \in H^2_{**}(\Omega)$, then $\varphi_{xx} = \varphi_{xy} = 0$ on $(0, \pi) \times \{\pm\ell\}$ and an integration by parts yields

$$\begin{aligned}(B(v, w), \varphi)_{H^2_{**}} &= \int_\Omega [v, w]\varphi = \int_\Omega [\varphi, w]v \\ &= \int_\Omega \varphi_{xy}(w_x v_y + w_y v_x) - \int_\Omega (\varphi_{xx} w_y v_y + \varphi_{yy} w_x v_x).\end{aligned}$$

In turn, this shows that for any $v, w \in H^2_*(\Omega)$ and $\varphi \in H^2_{**}(\Omega)$

$$|(B(v, w), \varphi)_{H^2_{**}}| \le c\|\varphi\|_{H^2_{**}} \|v\|_{W^{1,4}} \|w\|_{W^{1,4}}.$$

Therefore,

$$\begin{aligned}\|B(v, w)\|_{H^2_{**}} &= \sup_{0 \ne \varphi \in H^2_{**}(\Omega)} \frac{(B(v, w), \varphi)_{H^2_{**}}}{\|\varphi\|_{H^2_{**}}} \\ &\le c\|v\|_{W^{1,4}} \|w\|_{W^{1,4}}. \end{aligned} \tag{4.23}$$

Assume that the sequence $\{(v_n, w_n)\} \subset H^2_*(\Omega) \times H^2_*(\Omega)$ weakly converges to $(v, w) \in H^2_*(\Omega) \times H^2_*(\Omega)$. Then the triangle inequality and the just proved estimate yield

$$\begin{aligned}&\|B(v_n, w_n) - B(v, w)\|_{H^2_{**}} \\ &\quad \le \|B(v_n - v, w_n)\|_{H^2_{**}} + \|B(v, w_n - w)\|_{H^2_{**}} \\ &\quad \le c\|v_n - v\|_{W^{1,4}} \|w_n\|_{W^{1,4}} + c\|v\|_{W^{1,4}} \|w_n - w\|_{W^{1,4}}\,.\end{aligned}$$

The compact embedding $H^2_*(\Omega) \Subset W^{1,4}(\Omega)$ then shows that

$$\|B(v_n, w_n) - B(v, w)\|_{H^2_{**}} \to 0$$

and hence that B is a compact operator. The proof for C is similar. □

We now define another operator $D : H^2_*(\Omega) \to H^2_*(\Omega)$ by

$$D(v) = C(v, B(v, v)) \qquad \forall v \in H^2_*(\Omega)$$

and we prove

Lemma 4.2. *The operator D is compact.*

Proof. Assume that the sequence $\{v_n\} \subset H^2_*(\Omega)$ weakly converges to $v \in H^2_*(\Omega)$. Then, by Lemma 4.1, $B(v_n, v_n) \to B(v, v)$ in $H^2_{**}(\Omega)$ and $C(v_n, B(v_n, v_n)) \to C(v, B(v, v))$ in $H^2_*(\Omega)$. This proves that $D(v_n) \to D(v)$ in $H^2_*(\Omega)$ and that D is a compact operator. □

In turn, the operator D enables us to define a functional $d : H^2_*(\Omega) \to \mathbb{R}$ by

$$d(v) = \frac{1}{4}(D(v), v)_{H^2_*} \quad \forall v \in H^2_*(\Omega)\,.$$

In the next statement we prove some of its properties.

Lemma 4.3. *The functional $d : H^2_*(\Omega) \to \mathbb{R}$ has the following properties:*
(i) d is nonnegative and $d(v) = 0$ if and only if $v = 0$ in Ω. Moreover,

$$d(v) = \frac{1}{4}\|B(v, v)\|^2_{H^2_{**}};$$

(ii) d is quartic, i.e.,

$$d(rv) = r^4 d(v), \quad \forall r \in \mathbb{R}, \forall v \in H^2_*(\Omega);$$

(iii) d is differentiable in $H^2_*(\Omega)$ and

$$\langle d'(v), w\rangle = (D(v), w)_{H^2_*}, \quad v, w \in H^2_*(\Omega);$$

(iv) d is weakly continuous on $H^2_*(\Omega)$.

Proof. (i) By (4.22) we know that for any $v \in H^2_*(\Omega)$,

$$\begin{aligned}(D(v), v)_{H^2_*} &= (C(v, B(v, v)), v)_{H^2_*} \\ &= (B(v, v), B(v, v))_{H^2_{**}} = \|B(v, v)\|^2_{H^2_{**}}.\end{aligned}$$

Whence, if $d(v) = 0$, then $B(v, v) = 0$ and $[v, v] = 0$, see (4.20). But $[v, v]$ is proportional to the Gaussian curvature and since it vanishes identically this implies that the surface $v = v(x, y)$ is covered by straight lines. By using the boundary condition (4.14) we finally infer that $v \equiv 0$. This idea of the last part of this proof is taken from [12, Lemma 3.2’].

(ii) The functional d is quartic as a trivial consequence of its definition.

(iii) From (4.22) we infer that for all $v, w \in H^2_*(\Omega)$

$$(C(v, B(v, w)), v)_{H^2_*} = (B(v, v), B(v, w))_{H^2_{**}} \\ = (C(v, B(v, v)), w)_{H^2_*}. \tag{4.24}$$

Then we compute

$$\begin{aligned}\langle d'(v), w\rangle =& \lim_{\varepsilon \to 0} \frac{1}{4\varepsilon}\{(D(v + \varepsilon w), v + \varepsilon w)_{H^2_*} - (D(v), v)_{H^2_*}\} \\ =& \frac{1}{4}\{(C(w, B(v, v)), v)_{H^2_*} + (C(v, B(v, v)), w)_{H^2_*}\} \\ &+ \frac{1}{2}\{(C(v, B(v, w)), v)_{H^2_*}\} \\ =& \frac{1}{2}\{(C(v, B(v, v)), w)_{H^2_*} + (C(v, B(v, w)), v)_{H^2_*}\} \\ =& (D(v), w)_{H^2_*},\end{aligned}$$

which proves (iii).

(iv) Assume that the sequence $\{v_n\} \subset H^2_*(\Omega)$ weakly converges to $v \in H^2_*(\Omega)$. Then by Lemma 4.2 we know that

$$\lim_{n\to\infty} \|D(v_n) - D(v)\|_{H^2_*} = 0.$$

This shows that

$$\lim_{n\to\infty} (D(v_n) - D(v), v_n)_{H^2_*} = 0.$$

Finally, this yields

$$d(v_n) - d(v) = \frac{1}{4}(D(v_n) - D(v), v_n)_{H^2_*} + \frac{1}{4}(D(v), v_n - v)_{H^2_*} \to 0$$

which proves (iv). □

4.2.2 An linear problem

Assume that (3.1), i.e. $0 < \sigma < 1/2$, we analyze the spectrum of the linear problem:

$$\begin{cases} \Delta^2 u + \lambda u_{xx} = 0 & \text{in } \Omega \\ u = u_{xx} = 0 & \text{on } \{0, \pi\} \times (-\ell, \ell) \\ u_{yy} + \sigma u_{xx} = u_{yyy} + (2 - \sigma)u_{xxy} = 0 & \text{on } (0, \pi) \times \{\pm\ell\}\,. \end{cases} \tag{4.25}$$

Lemma 4.4. *The problem (4.25) admits a sequence of divergent eigenvalues*

$$\lambda_1 < \lambda_2 \leq ... \leq \lambda_k \leq ...$$

whose corresponding eigenfunctions $\{\overline{e}_k\}$ form a complete orthonormal system in $H^2_*(\Omega)$.

Moreover, the least eigenvalue λ_1 is simple and is the unique value of $\lambda \in ((1-\sigma)^2, 1)$ such that

$$\sqrt{1-\lambda^{1/2}}\left(\lambda^{1/2}+1-\sigma\right)^2 \tanh(\ell\sqrt{1-\lambda^{1/2}}\,)$$
$$= \sqrt{1+\lambda^{1/2}}\left(\lambda^{1/2}-1+\sigma\right)^2 \tanh(\ell\sqrt{1+\lambda^{1/2}}\,)\,;$$

the corresponding eigenspace is generated by the positive eigenfunction

$$\overline{e}_1(x,y) = \left\{(\lambda^{1/2}+1-\sigma)\,\frac{\cosh\left(y\sqrt{1-\lambda^{1/2}}\right)}{\cosh\left(\ell\sqrt{1-\lambda^{1/2}}\right)}\right\}\sin x$$
$$+\left\{(\lambda^{1/2}-1+\sigma)\,\frac{\cosh\left(y\sqrt{1+\lambda^{1/2}}\right)}{\cosh\left(\ell\sqrt{1+\lambda^{1/2}}\right)}\right\}\sin x\,.$$

Proof. We proceed as in [27, Theorem 3.4], see also [3, Theorem 4], with some changes due to the presence of the buckling term. We write the eigenvalue problem (4.25) as

$$(u_x, v_x)_{L^2} = \frac{1}{\lambda}(u,v)_{H^2_*} \quad \forall v \in H^2_*(\Omega).$$

Define the linear operator $T : H^2_*(\Omega) \to H^2_*(\Omega)$ such that

$$(Tu, v)_{H^2_*} = (u_x, v_x)_{L^2} \quad \forall v \in H^2_*(\Omega).$$

The operator T is self-adjoint since for any $u, v \in H^2_*(\Omega)$

$$(Tu, v)_{H^2_*} = (u_x, v_x)_{L^2} = (v_x, u_x)_{L^2} = (u, Tv)_{H^2_*}\,.$$

Moreover, by the compact embedding $H^2_*(\Omega) \Subset H^1(\Omega)$ and the definition of T, the following implications hold:

$$\begin{aligned} u_n \rightharpoonup u \text{ in } H^2_*(\Omega) &\Longrightarrow (u_n)_x \to u_x \text{ in } L^2(\Omega) \\ &\Longrightarrow \sup_{\|v\|_{H^2_*}=1} ((u_n-u)_x, v_x)_{L^2} \to 0 \\ &\Longrightarrow \sup_{\|v\|_{H^2_*}=1} (T(u_n-u), v)_{H^2_*} \to 0 \\ &\Longrightarrow Tu_n \to Tu \text{ in } H^2_*(\Omega), \end{aligned}$$

which shows that T is also compact. Then the spectral theory of linear compact self-adjoint operator yields that (4.25) admits an ordered increasing sequence of eigenvalues and the corresponding eigenfunctions form an Hilbertian basis of $H^2_*(\Omega)$. This proves the first part of Lemma 4.4.

According to the boundary conditions on $x = 0, \pi$, we seek eigenfunctions in the form:

$$u(x,y) = \sum_{m=1}^{+\infty} h_m(y)\sin(mx) \qquad \text{for } (x,y) \in (0,\pi)\times(-\ell,\ell)\,. \tag{4.26}$$

Then we are led to find nontrivial solutions of the ordinary differential equation

$$h_m''''(y) - 2m^2 h_m''(y) + (m^4 - m^2\lambda)h_m(y) = 0\,, \qquad (\lambda > 0) \tag{4.27}$$

with the boundary conditions

$$\begin{cases} h_m''(\pm\ell) - \sigma m^2 h_m(\pm\ell) = 0\,, \\ h_m'''(\pm\ell) + (\sigma - 2)m^2 h_m'(\pm\ell) = 0\,. \end{cases} \tag{4.28}$$

The characteristic equation related to (4.27) is $\alpha^4 - 2m^2\alpha^2 + m^4 - m^2\lambda = 0$ and then

$$\alpha^2 = m^2 \pm m\sqrt{\lambda}\,. \tag{4.29}$$

For a given $\lambda > 0$ three cases have to be distinguished.

• **The case** $m^2 > \lambda$**.** By (4.29) we infer

$$\alpha = \pm\beta \text{ or } \alpha = \pm\gamma \quad \text{with } \sqrt{m^2 - m\sqrt{\lambda}} =: \gamma < \beta := \sqrt{m^2 + m\sqrt{\lambda}}\,.$$

Nontrivial solutions of (4.27) have the form

$$h_m(y) = a\cosh(\beta y) + b\sinh(\beta y) + c\cosh(\gamma y) + d\sinh(\gamma y)\,, \tag{4.30}$$

where $a, b, c, d \in \mathbb{R}$. By imposing the boundary conditions (4.28) and arguing as in [27] we see that a nontrivial solution of (4.27) exists if and only if one of the two following equalities holds:

$$\frac{\gamma}{(\gamma^2 - m^2\sigma)^2}\tanh(\ell\gamma) = \frac{\beta}{(\beta^2 - m^2\sigma)^2}\tanh(\ell\beta)\,, \tag{4.31}$$

$$\frac{\beta}{(\beta^2 - m^2\sigma)^2}\coth(\ell\beta) = \frac{\gamma}{(\gamma^2 - m^2\sigma)^2}\coth(\ell\gamma)\,. \tag{4.32}$$

For any integer $m > \sqrt{\lambda}$ such that (4.31) holds, the function h_m in (4.30) with $b = d = 0$ and suitable $a = a_m \neq 0$ and $c = c_m \neq 0$ yields the eigenfunction $h_m(y)\sin(mx)$ associated to the eigenvalue λ. Similarly, for any integer $m > \sqrt{\lambda}$ such that (4.32) holds, the function h_m in (4.30) with $a = c = 0$ and suitable $b = b_m \neq 0$ and $d = d_m \neq 0$ yields the eigenfunction $h_m(y)\sin(mx)$ associated to the eigenvalue λ. Clearly, the number of both such integers is finite. In particular, when $m = 1$ the equation (4.27) coincides with [27, (57)]. Therefore, the statement about the least eigenvalue and the explicit form of the corresponding eigenfunction hold.

• **The case $m^2 = \lambda$.** This case is completely similar to the second case in [27]. By (4.29) we infer that possible nontrivial solutions of (4.27)-(4.28) have the form

$$h_m(y) = a\cosh(\sqrt{2}my) + b\sinh(\sqrt{2}my) + c + dy \qquad (a, b, c, d \in \mathbb{R}).$$

Then one sees that $a = c = 0$ if $0 < \sigma < 1/2$. Moreover, let $\overline{s} > 0$ be the unique solution of $\tanh(s) = \left(\frac{\sigma}{2-\sigma}\right)^2 s$. If $m_* := \overline{s}/\ell\sqrt{2}$ is an integer, and only in this case, then $\lambda = m_*^2$ is an eigenvalue and the corresponding eigenfunction is

$$\left[\sigma\ell\sinh(\sqrt{2}m_* y) + (2-\sigma)\sinh(\sqrt{2}m_*\ell)\, y\right]\, \sin(m_* x).$$

• **The case $m^2 < \lambda$.** By (4.29) we infer that

$$\alpha = \pm\beta \text{ or } \alpha = \pm i\gamma \;\text{ with }\; \sqrt{m\sqrt{\lambda} - m^2} = \gamma < \beta = \sqrt{m\sqrt{\lambda} + m^2}.$$

Therefore, possible nontrivial solutions of (4.27) have the form

$$h_m(y) = a\cosh(\beta y) + b\sinh(\beta y) + c\cos(\gamma y) + d\sin(\gamma y),$$

where $a, b, c, d \in \mathbb{R}$. Differentiating h_m and imposing the boundary conditions (4.28) yields the two systems:

$$\begin{cases} (\beta^2 - m^2\sigma)\cosh(\beta\ell)a - (\gamma^2 + m^2\sigma)\cos(\gamma\ell)c = 0 \\ (\beta^3 - m^2(2-\sigma)\beta)\sinh(\beta\ell)a + (\gamma^3 + m^2(2-\sigma)\gamma)\sin(\gamma\ell)c = 0, \end{cases}$$

$$\begin{cases} (\beta^2 - m^2\sigma)\sinh(\beta\ell)b - (\gamma^2 + m^2\sigma)\sin(\gamma\ell)d = 0 \\ (\beta^3 - m^2(2-\sigma)\beta)\cosh(\beta\ell)b - (\gamma^3 + m^2(2-\sigma)\gamma)\cos(\gamma\ell)d = 0. \end{cases}$$

Due to the presence of trigonometric sine and cosine, for any integer m there exists a sequence $\zeta_k^m \uparrow +\infty$ such that $\zeta_k^m > m^2$ for all $k \in \mathbb{N}$ and such that if $\lambda = \zeta_k^m$ for some k then one of the above systems admits a nontrivial solution. On the other hand, for any eigenvalue λ there exists at most a finite number of integers m such that $m^2 < \lambda$; if these integers yield non-trivial solutions h_m, then the function $h_m(y)\sin(mx)$ is an eigenfunction corresponding to λ. $\square$

The simplicity of the least eigenvalue was not to be expected. It is shown in [47, §3] that the eigenvalue problem (4.25) for a fully hinged (simply supported) rectangular plate, that is with $u = \Delta u = 0$ on the four edges, may admit a least eigenvalue of multiplicity 2.

The least eigenvalue λ_1 represents the critical buckling load and may be characterised variationally by

$$\lambda_1 := \min_{v\in H^2_*(\Omega)} \frac{\|v\|^2_{H^2_*}}{\|v_x\|^2_{L^2}} .$$

Ferrero-Gazzola [27] studied the eigenvalue problem $\Delta^2 u = \lambda u$ under the boundary conditions in (4.25): by comparing [27, Theorem 3.4] with the above Lemma 4.4 we observe that the least eigenvalues (and eigenfunctions) of the two problems coincide, that is,

$$\lambda_1 = \min_{v\in H^2_*(\Omega)} \frac{\|v\|^2_{H^2_*}}{\|v_x\|^2_{L^2}} = \min_{v\in H^2_*(\Omega)} \frac{\|v\|^2_{H^2_*}}{\|v\|^2_{L^2}} . \tag{4.33}$$

Therefore, the critical buckling load for a rectangular plate equals the eigenvalue relative to the first eigenmode of the plate. In turn, the first eigenmode is also the first buckling deformation of the plate. From (4.33) we readily infer the Poincaré-type inequalities

$$\lambda_1\|v_x\|^2_{L^2} \le \|v\|^2_{H^2_*} , \quad \lambda_1\|v\|^2_{L^2} \le \|v\|^2_{H^2_*} \quad \forall v \in H^2_*(\Omega) \tag{4.34}$$

with strict inequality unless v minimises the ratio in (4.33), that is, v is a real multiple of $\bar{e}_1$. Note also that by taking $v(x,y) = \sin x$ one finds that $\lambda_1 < 1$.

Finally, let us mention that Lemma 4.4 may be complemented with the explicit form of all the eigenfunctions: they are $\sin(mx)$ ($m \in \mathbb{N}$) multiplied by trigonometric or hyperbolic functions with respect to y.

4.3 The equilibrium positions

In this section, we study the nonlinear plate model (4.19) for the suspension bridge, that is, with the action of the hangers. First define the constants

$$\alpha := \int_\Omega \Upsilon(y)\bar{e}_1^2 , \qquad \overline{\lambda} := (\alpha k + 1)\lambda_1 > \lambda_1 , \tag{4.35}$$

where λ_1 denotes the least eigenvalue and $\overline{e}_1$ denotes here the positive least eigenfunction normalised in $H^2_*(\Omega)$, see Lemma 4.4. Then we have

Theorem 4.1. *For all $f \in L^2(\Omega)$, $\lambda \geq 0$ and $k, \delta > 0$, the problem* (4.19) *admits a solution $(u, \Phi) \in H^2_*(\Omega) \times H^2_{**}(\Omega)$. Moreover:*
(i) if $\lambda < \lambda_1$ there exists $K > 0$ such that if $\|f\|_{L^2} < K$ then (4.19) *admits a unique solution $(u, \Phi) \in H^2_*(\Omega) \times H^2_{**}(\Omega)$;*
(ii) if $\lambda > \lambda_1$ and $f = 0$ then (4.19) *admits at least two solutions $(u, \Phi) \in H^2_*(\Omega) \times H^2_{**}(\Omega)$ and one of them is trivial and unstable;*
(iii) if $\overline{\lambda} < \lambda_2$ and $\overline{\lambda} < \lambda < \lambda_2$, there exists $K > 0$ such that if $\|f\|_{L^2} < K$ then (4.19) *admits at least three solutions $(u, \Phi) \in H^2_*(\Omega) \times H^2_{**}(\Omega)$, two being stable and one being unstable.*

Proof. By Lemma 4.3, we know that the energy functional corresponding to the problem (4.19) reads

$$\begin{aligned} J(u) =& \frac{1}{2}\|u\|^2_{H^2_*} - \frac{\lambda}{2}\|u_x\|^2_{L^2} \\ &+ \int_\Omega \Upsilon(y)\left(\frac{k}{2}(u^+)^2 + \frac{\delta}{4}(u^+)^4\right) + d(u) - \int_\Omega fu\,. \end{aligned}$$

Combining with Lemmas 4.1-4.3, we obtain a one-to-one correspondence between solutions of (4.19) and critical points of the functional J:

Lemma 4.5. *Let $f \in L^2(\Omega)$. The couple $(u, \Phi) \in H^2_*(\Omega) \times H^2_{**}(\Omega)$ is a weak solution of* (4.19) *if and only if $u \in H^2_*(\Omega)$ is a critical point of J and if $\Phi \in H^2_{**}(\Omega)$ weakly solves $\Delta^2\Phi = -[u, u]$ in Ω.*

Next, we prove the geometrical properties (coercivity) and compactness properties (Palais-Smale condition) of J. Although the former may appear straightforward, it requires delicate arguments. The reason is that no useful lower bound for $d(u)$ is available. We prove

Lemma 4.6. *For any $f \in L^2(\Omega)$ and any $\lambda \geq 0$, the functional J is coercive in $H^2_*(\Omega)$ and it is bounded from below. Moreover, it satisfies the Palais-Smale (PS) condition.*

Proof. Assume for contradiction that there exists a sequence $\{v_n\} \subset H^2_*(\Omega)$ and $M > 0$ such that

$$\lim_{n\to\infty} \|v_n\|_{H^2_*} \to \infty, \quad J(v_n) \leq M.$$

Put $w_n = \frac{v_n}{\|v_n\|_{H^2_*}}$ so that $v_n = \|v_n\|_{H^2_*} w_n$ and

$$\|w_n\|_{H^2_*} = 1 \quad \forall n\,. \tag{4.36}$$

Since

$$\int_\Omega \Upsilon(y)\left(\frac{k}{2}(u^+)^2+\frac{\delta}{4}(u^+)^4\right)\geq 0, \qquad \text{for all } u\in H^2_*(\Omega), \tag{4.37}$$

we infer that by combining the Hölder inequality with (4.34)

$$\begin{aligned} M\geq J(v_n)\geq&\frac{1}{2}\|v_n\|^2_{H^2_*}+\|v_n\|^4_{H^2_*}d(w_n)\\ &-\frac{\lambda}{2}\|v_n\|^2_{H^2_*}\|(w_n)_x\|^2_{L^2}-\frac{\|f\|_{L^2}}{\sqrt{\lambda_1}}\|v_n\|_{H^2_*}, \end{aligned} \tag{4.38}$$

where we also used Lemma 4.3 (ii).

By letting $n\to\infty$, we have $d(w_n)\to 0$ which, combined with Lemma 4.3 and (4.36), shows that $w_n\rightharpoonup 0$ in $H^2_*(\Omega)$; then, $(w_n)_x\to 0$ in $L^2(\Omega)$ by compact embedding. Hence, since $d(w_n)\geq 0$, (4.38) yields

$$\begin{aligned} o(1)=\frac{M}{\|v_n\|^2_{H^2_*}}\geq&\frac{1}{2}+\|v_n\|^2_{H^2_*}d(w_n)-\frac{\lambda}{2}\|(w_n)_x\|^2_{L^2}\\ &-\frac{\|f\|_{L^2}}{\|v_n\|_{H^2_*}\sqrt{\lambda_1}}\\ \geq&\frac{1}{2}+o(1), \end{aligned}$$

which leads to a contradiction by letting $n\to\infty$. Therefore J is coercive. Since the lower bound for $J(v_n)$ in (4.38) only depends on $\|v_n\|_{H^2_*}$, we also know that J is bounded from below.

In order to prove that J satisfies the (PS) condition we consider a sequence $\{u_n\}\subset H^2_*(\Omega)$ such that $J(u_n)$ is bounded and $J'(u_n)\to 0$ in $\mathcal{H}_*(\Omega)$. By what we just proved, we know that $\{u_n\}$ is bounded and therefore, there exists $\overline{u}\in H^2_*(\Omega)$ such that $u_n\rightharpoonup\overline{u}$ and, by weak continuity, $J'(\overline{u})=0$. Moreover, by Lemma 4.3,

$$\begin{aligned} \langle J'(u_n),u_n\rangle=&\|u_n\|^2_{H^2_*}+(D(u_n),u_n)_{H^2_*}-\lambda\|(u_n)_x\|^2_{L^2}\\ &+\int_\Omega\Upsilon(y)\left(k(u_n^+)u_n+\delta(u_n^+)^3u_n\right)-\int_\Omega fu_n\\ \to&0=\langle J'(\overline{u}),\overline{u}\rangle=\|\overline{u}\|^2_{H^2_*}+(D(\overline{u}),\overline{u})_{H^2_*}-\lambda\|\overline{u}_x\|^2_{L^2}\\ &+\int_\Omega\Upsilon(y)\left(k(u^+)u+\delta(u^+)^3u\right)-\int_\Omega f\overline{u}\,. \end{aligned}$$

By compact embedding, we have

$$\|(u_n)_x\|^2_{L^2}\to\|\overline{u}_x\|^2_{L^2},\qquad \int_\Omega fu_n\to\int_\Omega f\overline{u}$$

and

$$\int_\Omega \Upsilon(y)\,\big(k(u_n^+)u_n + \delta(u_n^+)^3 u_n\big) \to \int_\Omega \Upsilon(y)\,\big(k(u^+)u + \delta(u^+)^3 u\big).$$

Moreover, by Lemma 4.2, $(D(u_n), u_n)_{H^2_*} \to (D(\overline{u}), \overline{u})_{H^2_*}$. Then it follows that

$$\|u_n\|_{H^2_*} \to \|\overline{u}\|_{H^2_*}.$$

This fact, together with the weak convergence $u_n \rightharpoonup \overline{u}$ proves that

$$u_n \to \overline{u} \quad \text{strongly;}$$

this proves (PS). □

Lemma 4.6 shows that the (smooth) functional J admits a global minimum in $H^2_*(\Omega)$ for any $f \in L^2(\Omega)$ and any $\lambda \geq 0$. This minimum is a critical point for J and hence, by Lemma 4.5, it gives a weak solution of (4.19). This proves the first part of Theorem 4.1. Let us now prove the items.

(i) For any $f \in L^2(\Omega)$, if u is a critical point of the functional J, then it satisfies $\langle J'(u), u\rangle = 0$ and therefore, by (4.37) and the Hölder inequality,

$$\|u\|^2_{H^2_*} + 4d(u) - \lambda\|u_x\|^2_{L^2} + \int_\Omega \Upsilon(y)\,\big(k(u^+)^2 + \delta(u^+)^4\big) \leq \|f\|_{L^2}\|u\|_{L^2}\,.$$

In turn, by using Lemma 4.3 (i) and twice (4.34), we obtain

$$\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2_{H^2_*} \leq \frac{\|f\|_{L^2}}{\sqrt{\lambda_1}}\|u\|_{H^2_*}\,.$$

This gives the a priori bound

$$\|u\|_{H^2_*} \leq \frac{\sqrt{\lambda_1}}{\lambda_1 - \lambda}\,\|f\|_{L^2}\,. \tag{4.39}$$

Next, we prove a local convexity property of the functional J. Let

$$Q(u) := \|u\|^2_{H^2_*} - \lambda\|u_x\|^2_{L^2} \quad \forall u \in H^2_*(\Omega)\,.$$

Then, for all $u, v \in H^2_*(\Omega)$ and all $t \in [0,1]$, we have

$$Q\Big(tu + (1-t)v\Big) - tQ(u) - (1-t)Q(v) = -t(1-t)\Big(\|u-v\|^2_{H^2_*} - \lambda\|u_x - v_x\|^2_{L^2}\Big)\,. \tag{4.40}$$

Moreover, for all $u, v \in H^2_*(\Omega)$ and all $t \in [0,1]$, some tedious computations show that

$$d\Big(tu+(1-t)v\Big)-td(u)-(1-t)d(v)=$$

$$\begin{aligned}
=-\frac{t(1-t)}{4}\Big\{&(t^2-3t+1)(\|B(v,u-v)\|^2_{H^2_{**}}-\|B(u,u-v)\|^2_{H^2_{**}})\\
&+2(t^2-t+1)\big(B(u,u-v),B(u+v,u-v)\big)_{H^2_{**}}\\
&+2\big(B(v,v),B(v-u,v-u)\big)_{H^2_{**}}\\
&-4t(1-t)\big(B(u-v,u),B(v-u,v)\big)_{H^2_{**}}\Big\}\\
\text{by (4.23)}\quad &\le C\,t(1-t)\,(\|u\|^2_{H^2_*}+\|v\|^2_{H^2_*})\,\|u-v\|^2_{H^2_*}\,; \qquad (4.41)
\end{aligned}$$

here $C>0$ is a constant independent of t, u, v.

Consider the functional defined by

$$I(u)=\frac{1}{2}\|u\|^2_{H^2_*}+d(u)-\frac{\lambda}{2}\|u_x\|^2_{L^2}=\frac{Q(u)}{2}+d(u)\,. \qquad (4.42)$$

By putting together (4.40) and (4.41) we see that

$$\begin{aligned}
&I\Big(tu+(1-t)v\Big)-tI(u)-(1-t)I(v)\\
&\le-\frac{t(1-t)}{2}\Big(\|u-v\|^2_{H^2_*}-\lambda\|u_x-v_x\|^2_{L^2}\Big)\\
&\quad+C\,t(1-t)\,(\|u\|^2_{H^2_*}+\|v\|^2_{H^2_*})\,\|u-v\|^2_{H^2_*}\\
&\le t(1-t)\Big(C\,(\|u\|^2_{H^2_*}+\|v\|^2_{H^2_*})-\frac{\lambda_1-\lambda}{2\lambda_1}\Big)\|u-v\|^2_{H^2_*}\,. \qquad (4.43)
\end{aligned}$$

Take f sufficiently small such that

$$\|f\|^2_{L^2}<K^2:=\frac{(\lambda_1-\lambda)^3}{4C\,\lambda_1^2}\,. \qquad (4.44)$$

By (4.39) and (4.44) we know that any critical point of J satisfies

$$\|u\|^2_{H^2_*}\le\frac{\lambda_1}{(\lambda_1-\lambda)^2}\,K^2=\frac{\lambda_1-\lambda}{4C\,\lambda_1}=:\rho^2\,;$$

put $B_\rho=\{u\in H^2_*(\Omega);\ \|u\|_{H^2_*}\le\rho\}$. Moreover, from (4.43) we know that

$$I\Big(tu+(1-t)v\Big)-tI(u)-(1-t)I(v)\le 0 \qquad \forall u,v\in B_\rho\,,$$

with strict inequality if $u \neq v$ and $t \notin \{0,1\}$. This proves that I is strictly convex in B_ρ. Hence, the functional $I_f = I - \int_\Omega fu$ is also strictly convex in B_ρ.

Moreover, by $(u+v)^+ \leq u^+ + v^+$ for any $u, v \in H^2_*(\Omega)$, we have for any $t \in [0,1]$

$$\begin{aligned}\int_\Omega \Upsilon(y)\left((tu+(1-t)v)^+\right)^2 &\leq \int_\Omega \Upsilon(y)\left(tu^+ + (1-t)v^+\right)^2 \\ &\leq t\int_\Omega \Upsilon(y)\left(u^+\right)^2 + (1-t)\int_\Omega \Upsilon(y)\left(v^+\right)^2\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \Upsilon(y)\left((tu+(1-t)v)^+\right)^4 &\leq \int_\Omega \Upsilon(y)\left((tu^+ + (1-t)v^+)^2\right)^2 \\ &\leq \int_\Omega \Upsilon(y)\left(t(u^+)^2 + (1-t)(v^+)^2\right)^2 \\ &\leq t\int_\Omega \Upsilon(y)\left(u^+\right)^4 + (1-t)\int_\Omega \Upsilon(y)\left(v^+\right)^4.\end{aligned}$$

Hence, $\int_\Omega \Upsilon(y)\left(\frac{k}{2}(u^+)^2 + \frac{\delta}{4}(u^+)^4\right)$ is convex and then the functional $J(u)$ is strictly convex in B_ρ.

Summarising, if (4.44) holds, then we know that:

- by (4.39) all the critical points of J belong to B_ρ;
- by the first part of the proof we then know that there exists at least a critical point in B_ρ;
- J is strictly convex in B_ρ.

We then deduce that J admits a unique critical point in B_ρ (its absolute minimum) and no other critical points elsewhere. Together with Lemma 4.5, this completes the proof of item (i).

(ii) If $f = 0$, then $u = 0$ is a solution for any $\lambda \geq 0$. We just need to show that it is not the global minimum which we know to exist. Let $\overline{e}_1$ and α be as in (4.35) and consider the function for $t \in \mathbb{R}$

$$\begin{aligned}g(t) &= J(t\overline{e}_1) \\ &= -\frac{\lambda-\lambda_1}{2\lambda_1}\,t^2 + \frac{k\,\alpha}{2}\,(t^+)^2 + \frac{\delta\,(t^+)^4}{4}\int_\Omega \Upsilon(y)\overline{e}_1^4 + t^4\,d(\overline{e}_1)\,. \quad (4.45)\end{aligned}$$

Since $\lambda > \lambda_1$, the coefficient of $(t^-)^2$ is negative and the qualitative graph of g is as in Figure 4.2 (on the left the case where $\lambda < \overline{\lambda}$ so that the coefficient of $(t^+)^2$ is nonnegative, on the right the case where also the coefficient

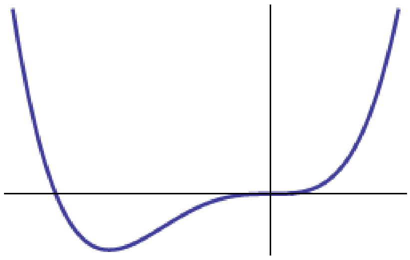

Figure 4.2: *Qualitative graphs of the functions g.*

of $(t^+)^2$ is negative). It is clear that there exists $\bar{t} < 0$ such that $g(\bar{t}) < 0$. This means that $J(\bar{t}\bar{e}_1) < 0$ and that 0 is not the absolute minimum of J. This completes the proof of item (ii).

(iii) We study first the case where $f = 0$ and we name J_0 the unforced functional, that is,

$$J_0(u) = \frac{1}{2}\|u\|^2_{H^2_*} - \frac{\lambda}{2}\|u_x\|^2_{L^2} + d(u) + \int_\Omega \Upsilon(y)\left(\frac{k}{2}(u^+)^2 + \frac{\delta}{4}(u^+)^4\right).$$

We consider again the function g in (4.45) that we name here h in order to distinguish their graphs, $h(t) = g(t)$ as in (4.45). Since $\lambda > \overline{\lambda}$, the coefficient of $(t^+)^2$ is now also negative and the qualitative graph of h is as in the right picture of Figure 4.3. Then the function h has a nondegenerate local

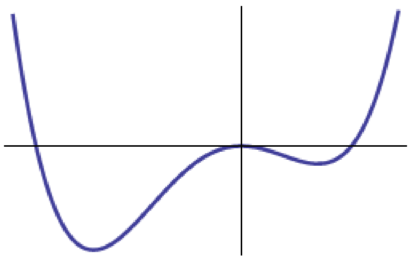

Figure 4.3: *Qualitative graphs of the functions h.*

maximum at $t = 0$ which means that also the map $t \mapsto J_0(t\bar{e}_1)$ has a local maximum at $t = 0$ and it is strictly negative in a punctured interval containing $t = 0$. Let $E = \text{span}\{\bar{e}_k;\ k \geq 2\}$ denote the infinite dimensional space of codimension 1 being the orthogonal complement of $\text{span}\{\bar{e}_1\}$. By the improved Poincaré inequality

$$\lambda_2\|v_x\|^2_{L^2} \leq \|v\|^2_{H^2_*} \quad \forall v \in E$$

and by taking into account Lemma 4.3 (i) and $\lambda \leq \lambda_2$, we see that for any $u \in E$

$$J_0(u) \geq \frac{\lambda_2 - \lambda}{2\lambda_2} \|u\|^2_{H^2_*} + \int_\Omega \Upsilon(y) \left(\frac{k}{2}(u^+)^2 + \frac{\delta}{4}(u^+)^4 \right) \geq 0 \,.$$

Therefore, the two open sets

$$A^+ = \{u \in H^2_*(\Omega);\ (u, \overline{e}_1)_{H^2_*} > 0,\ J_0(u) < 0\}\,,$$

$$A^- = \{u \in H^2_*(\Omega);\ (u, \overline{e}_1)_{H^2_*} < 0,\ J_0(u) < 0\}$$

are disconnected. Since J_0 satisfies the (PS) condition and is bounded from below, J_0 admits a global minimum u^+ (resp. u^-) in A^+ (resp. A^-) and $J_0(u^\pm) < 0$.

A sufficiently small linear perturbation of J_0 then has a local minimum in a neighborhood of both $u^\pm$. Whence, if f is sufficiently small, say $\|f\|_{L^2} < K$, then the functional J defined by $J(u) = J_0(u) - \int_\Omega f u dx dy$ admits a local minimum in two neighborhoods of both $u^\pm$. These local minima, which we name u_1 and u_2, are the first two critical points of J. A minimax procedure then yields an additional (mountain-pass) solution. Indeed, consider the set of continuous paths connecting u_1 and u_2:

$$\Gamma := \Big\{ p \in C^0([0,1], H^2_*(\Omega));\ p(0) = u_1,\ p(1) = u_2 \Big\}\,.$$

Since the functional J satisfies the (PS) condition, the mountain-pass Theorem guarantees that the level

$$\min_{p \in \Gamma} \max_{t \in [0,1]} J\big(p(t)\big) \ > \ \max \Big\{ J(u_1), J(u_2) \Big\}$$

is a critical level for J; this yields a third critical point. By Lemma 4.5 this proves the existence of (at least) three weak solutions of (4.19). □

Theorem 4.1 gives both uniqueness and multiplicity results. Item (ii) states that even in absence of an external load ($f = 0$), if the buckling load λ is sufficiently large then there exists at least two equilibrium positions; we conjecture that if we further assume that $\lambda < \overline{\lambda}$ then there exist no other solutions and that the equilibrium positions look like in Figure 4.4. In the left picture we see the trivial equilibrium $u = 0$ which is unstable due to the buckling load. In the right picture we see the stable equilibrium for some $u < 0$ (above the horizontal position). We conjecture that it is a negative multiple of the first eigenfunction $\overline{e}_1$, see Lemma 4.4; since ℓ is very small, a rough approximation shows that this negative multiple looks

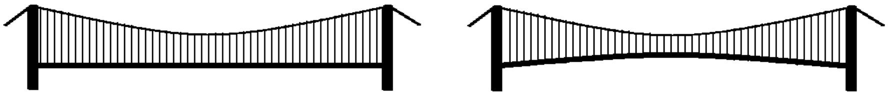

Figure 4.4: *Equilibrium positions of the buckled bridge.*

like $\approx C\sin(x)$ for some $C < 0$, which is the shape represented in the right picture. The reason of this conjecture will become clear in the proof, see in particular the plots in Figures 4.2 and 4.3: in this pattern, a crucial role is played by the positivity of $\overline{e}_1$.

Our feeling is that the action functional corresponding to this case has a qualitative shape as described in Figure 4.5, where O is the trivial unstable

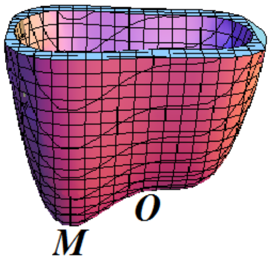

Figure 4.5: *Qualitative shape of the action functional for Theorem 4.1 (ii) when $\lambda < \overline{\lambda}$.*

equilibrium and M is the stable equilibrium. If there were no hangers also the opposite position would be a stable equilibrium. But the presence of the restoring force requires a larger buckling term in order to generate a positive (downwards) displacement. Indeed, item (iii) states, in particular, that if $f = 0$ and the buckling load is large then there exist three equilibria: one is trivial and unstable, the second is the enlarged negative one already found in item (ii), the third should precisely be the positive one which appears because the buckling load λ is stronger than the restoring force due to the hangers. All these conjectures and qualitative explanations are supported by similar results for a simplified (one dimensional) beam equation, see [14, Theorem 3.2].

CHAPTER 5

Conclusions and open problems

In this thesis, we considered several new mathematical models (one beam model, two plate models) for suspension bridges due to the lack of fully reliable models in the literature. Certainly, we do not claim that our models are perfect. This is just the beginning in order to reach more challenging results in this field and much more work is still necessary.

We studied the nonlinear nonlocal beam model in Chapter 2 and obtained an existence result of the problem (2.8). Since it is not clear that the behavior of the energy functional J_p for lager c, we are not able to ensure that the problem (2.8) (with larger c) admits a unique solution or multiplicity of solutions.

Since a beam cannot display the torsional oscillations appearing in the suspension bridges for the one dimensional beam model, we analyzed in Chapters 3 and 4 two different plate models. In Chapter 3, we mainly focused on the dynamical suspension bridges and recalled the plate model suggested by Ferrero-Gazzola [27]. For a non-coercive problem corresponding to the plate model, we investigated the asymptotic behavior of the unique solution for different initial conditions. Then in order to describe the boundary behavior of the plate, we set up a kind of dynamical boundary conditions and showed that there exists a unique explicit solution of the linear evolution problem with the dynamical boundary conditions.

If wide deformation appears in the plate, the plate equation based on the linear Kirchhoff-Love plate theory is not adequate to describe the oscillation behavior. Hence, we suggested in Chapter 4 a system (coupled by two fourth order partial differential equations based on the von Kármán plate equations) to model the suspension bridges. Then we proved that stable equilibrium positions or unstable equilibrium positions of the plate for different assumptions.

5.1 Open problems

Although we obtained several results about our new mathematical models (see Chapters 2-4), some more problems related to these models are still open.

5.1.1 The beam model

In Chapter 2, we suggested a kind of nonlinear beam model for suspension bridge and derived the nonlinear nonlocal problem:

$$\begin{cases} aw''''(x) - b\left(\frac{w'(x)}{1+(y'(x))^2}\right)' - c\frac{w''(x)-q/H}{(1+(w'(x)+y'(x))^2)^{3/2}}\Gamma(w) = p, \\ w(0) = w(L) = w''(0) = w''(L) = 0, \end{cases} \tag{5.1}$$

which admits at least one weak solution in $H^2 \cap H_0^1(0, L)$. We also gave a uniqueness result under suitable assumptions, see Theorem 2.1. However, the assumption (2.15) does not hold if one takes the values of parameters from the actual suspension bridge, see [87]. Hence, one cannot ensure the uniqueness result in this case.

As mentioned in Chapter 2, in the “limit” case ($c \to +\infty$), the problem (5.1) degenerates to:

$$\begin{cases} \frac{w''-q/H}{(1+(w'+y')^2)^{3/2}}\Gamma(w) = 0, & x \in (0, L) \\ w(0) = w(L) = 0, \end{cases} \tag{5.2}$$

which admits infinitely many solutions in $H^2 \cap H_0^1(0, L)$.

Moreover, it is not clear about the behavior of the total functional J_p for lager c. Hence, the uniqueness or multiplicity result of (5.1) with larger c is still open. Namely,

Remark 5.1. *Does the problem* (5.1) *with larger c admit a unique solution or at least two solutions if $p \in \mathcal{H}$ is small in the sense of $\|p\|_{\mathcal{H}}$?*

This problem will be addressed in [35].

5.1.2 The plate models

In this thesis, we introduced two plate models for suspension bridges, one is for suspension bridge with small deformation, the other one is for suspension bridge with large deformation.

In Section 3.3, we suggested a linear plate model (see (3.55)-(3.56)) for dynamical suspension bridges with small deformations and we derived a linear evolution problem with dynamical boundary conditions, which admits a unique explicit solution.

For simplicity, we assume that in Section 3.3 the restoring force h due to the hangers is in linear case, i.e. $h = ku$ with an elasticity constant $k > 0$. However, as pointed in Chapter 1, the restoring force h due to the hangers should be "more than" linear, which is also the opinion of McKenna [55]:

> We doubt that a bridge oscillating up and down by about 10 meters every 4 seconds obeys Hooke's law.

Therefore, one should consider the restoring force $h = h(x, y, u)$ in a nonlinear case, such as (3.3) or (4.18) and then one gets a nonlinear evolution problem

$$\begin{cases} u_{tt} + \Delta^2 u + \mu u_t + h(x, y, u) = f, & (x, y) \in \Omega, t > 0, \\ u(x, y, 0) = u_0, \quad u_t(x, y, 0) = u_1, & (x, y) \in \Omega \end{cases} \tag{5.3}$$

with the dynamical boundary conditions (3.56), see Section 3.3.1.

While the method we used in Section 3.3 is not applied to this nonlinear problem (5.3) with the dynamical boundary conditions (3.56) and this problem seems much more difficult. Hence, there is no answers to the following open problem:

Remark 5.2. *Does the problem* (5.3) *with the dynamical boundary conditions* (3.56) *admit solutions?*

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