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Mathematical analysis of some diffuse interface models for binary fluids

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Abstract

The subject of this dissertation is the mathematical analysis of some diffuse interface models which arise in the realm of Fluid Dynamics to describe the motion of two globally immiscible, incompressible and viscous fluids. Depending on the interplay between inertial and viscous forces, we consider two classes of equations governing the velocity field, known in literature as:

- the model H,
- the Hele–Shaw approximation.

The interface separating two fluids is assumed to be a region with non-zero thickness. Over this interfacial region the surface tension (also called Korteweg stress) is distributed. In these models a crucial role is played by the choice of the free energy. In the first part of this contribution we study *local* models originating from the Ginzburg–Landau free energy. In the second part we consider *nonlocal* models related to the Helmholtz free energy taking more general long-range interactions into account. Both of them penalize concentration variations. This twofold choice is motivated by the classical literature and leads to different Cahn–Hilliard type equations for the order parameter.

The common denominator throughout our investigation is the presence of the physically relevant free energy density which consists of a logarithmic function. The main advantage is the possibility to show the existence of a *physical* solution, meaning that the order parameter (i.e. the difference of concentrations) is forced to take physically admissible values. Thus, the order parameter maintains its original meaning. On the other hand, the study of such logarithmic potential requires non-classical mathematical methods. Indeed, by virtue of the different behaviour between the logarithmic potential and its derivatives close to the singular points, high order estimates involving the order parameter are hard to get.

The main results herein concern the uniqueness and regularity of weak solutions as well as the existence of strong solutions. Particular attention is given to the so-called *separation property*. The latter means that, if the initial datum is not a pure phase, then the order parameter eventually stays away from the pure states with a uniform in time

displacement. In the two dimensional setting, we present two different methods in order to handle local or nonlocal models leading to the *instantaneous* separation property, namely the separation occurs for any positive time with a parameter depending (explicitly) only on the initial energy value and the total mass of the initial datum. As an interesting application of the regularity properties, we discuss the asymptotic behaviour of solutions.

Let us now describe a summary of the main results contained in this thesis. First, the Navier–Stokes–Cahn–Hilliard–Oono system is studied in dimension two. This model is a generalization of the classical model H accounting for (reversible) chemical reactions. We show the uniqueness and the instantaneous regularization of weak solutions as well as the validity of the separation property. The latter has been obtained by combining high order Sobolev estimates with a regularity theory for an elliptic problem with logarithmic nonlinear term, for which the Trudinger–Moser inequality plays an essential role. The same result also goes for the Navier–Stokes–Cahn–Hilliard system.

The Hele–Shaw–Cahn–Hilliard system is analyzed in both two and three dimensions. We first prove the existence of a global weak solution. Then, in dimension two we demonstrate the uniqueness of weak solutions, their regularity propagation in time and the separation property. Instead, in dimension three we show the global existence of strong solution provided that the initial datum is regular enough and sufficiently close to any local minimizer of the Gindzburg–Landau free energy.

We also investigate the Brinkman–Cahn–Hilliard system in dimension two. In particular, we address the unmatched viscosities case. We show the existence and uniqueness of weak solutions, their regularity properties and the separation property.

Next, we study the nonlocal model H. First, we provide a comprehensive analysis of the nonlocal Cahn–Hilliard equation. In particular, we introduce a novel technique for the separation property which differs from the one employed in the local case. The proposed argument is base on an Alikakos–Moser iteration argument combined with the Trudinger–Moser inequality. Then, the analysis has been extended to the nonlocal Navier–Stokes–Cahn–Hilliard system is studied in dimension two.

Finally, the nonlocal Hele–Shaw–Cahn–Hilliard system is considered in two and three space dimensions. In both cases we show existence of weak solutions, uniqueness, existence of strong solutions and their regularity properties.

Thesis Details

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The main body of this thesis consists of the following publications and submitted papers:

[W.1] C.G. Gal, A. Giorgini, M. Grasselli, *The nonlocal Cahn–Hilliard equation with singular potential: well-posedness, regularity and strict separation property*, J. Differential Equations, **263** (2017), 5253–5297.

[W.2] A. Giorgini, M. Grasselli, A. Miranville, *The Cahn-Hilliard-Oono equation with singular potential*, Math. Models Meth. Appl. Sci., **27** (2017), 2485–2510.

[W.3] M. Conti, A. Giorgini, *On the Cahn–Hilliard–Brinkman system with singular potential and nonconstant viscosity*, submitted.

[W.4] A. Giorgini, M. Grasselli, H. Wu, *The Cahn–Hilliard–Hele–Shaw system with singular potential*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.

[W.5] F. Della Porta, A. Giorgini, M. Grasselli, *The nonlocal Cahn–Hilliard–Hele–Shaw system with singular potential*, submitted.

[W.6] A. Giorgini, A. Miranville, R. Temam, *Uniqueness and regularity results for the Navier–Stokes–Cahn–Hilliard–Oono system with singular potential*, in preparation.

This thesis has been submitted for assessment in partial fulfillment of the PhD degree. The thesis is based on the submitted or published scientific papers which are listed above. Parts of the papers are used directly or indirectly in the extended summary of the thesis.

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CHAPTER 1

Introduction

THE mathematical description of the motion of immiscible viscous fluids is a long-standing problem in Fluid Dynamics starting at the beginning of the 19th century. Since then, a vast literature has been devoted to find accurate models which comply the physical laws and lead to efficient numerical calculations. The mutual interaction between the interface dynamics and the surrounding fluid motion is indeed a complex phenomenon, depending also on surface tension effects, topological changes, viscosity ratios, temperature gradients and imposed flow at the boundary. The main common goal among these investigations has been understanding the nature of the interface separating the binary mixture. In the classical attempt, it is assumed to be an evolving in time surface with zero thickness, across which physical quantities must satisfy suitable boundary conditions. Instead, a more recent and powerful approach treats the interface as a narrow zone with finite thickness. In the latter the interface evolution is described through the concentration variable which is uniform in bulk phases and varies steeply but continuously across the interfaces. Whilst considerable progresses have been made towards more efficient models, its mathematical analysis involving well-posedness as well as regularity still remains challenging.

1.1 Fluids in motion

A wide class of processes in engineering applications rely on the interaction between fluids within multicomponent systems. Due to the complexity of moving fluid structures and their mutual interplay, the study of interfacial dynamics has played an increasingly crucial role. This non-trivial behavior already arises in simple experiments that are considered as real benchmarks in literature.

- **Breakup or coalescence of drops**. A fluid drop is immersed in a surrounding fluid filling the domain. If the viscous forces prevail on the surface tension acting across the interface, viscous stresses bend out the drop shape. As a result, the drop pulls outward and breakup into two drops.

- **Mixing in a driven cavity**. Two fluids are initially separated by a flat interface in a bounded geometry. The flow is driven by steady and uniform boundary velocities of one wall alone (or two symmetric walls). As the time goes by, the fluids start to stretch and fold as long as eddy structures arise. Chaotic pattern motions, such as corner eddies, may also occur.

- **Moving contact lines**. When two viscous fluids interact each other and are also in contact with a solid wall, the interface between the fluids which intersect the solid surface is said contact line. A phenomenological equation (Young's law) rules the static angle between the solid surface and the fluid/fluid surface relating three different coefficients of surface tension. However, the angle can be altered whether the contact line is moving by the flow.

- **Thermocapillary or Marangoni flow**. The effect of a temperature gradient imposed at the interface between the two immiscible viscous fluids consists in a local variation of the interfacial tension. In turn, the instability generates tangential shear stresses and so the motion of the fluids.

- **Fingering instability in Hele-Shaw cell**. The injection of a viscous fluid between two flat and parallel plates at a small distance drives a more viscous one. The unstable interface takes the shape of several fingers. One finger then grows up at the expense of its neighbours and the flow reaches a steady state with the propagation of a single finger.

All of the above mentioned examples have been studied to make comparisons between theory and experiments in the simplest geometrical settings, to find common features and develop models for more complicated flows. In particular, these processes own a common feature. Being driven by the action of an imposed flow or a temperature gradient or by an intrinsic mechanism like the surface tension, the spatial regions occupied by a single flow is deformed. This leads to an evolution of the interface area, which decreases its characteristic length scale and can even change its topology.

1.2 Sharp interface method

In a fixed bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the motion of an incompressible viscous (Newtonian) fluid is described by the celebrated Navier–Stokes equations. Neglecting external forces and assuming constant density $\rho = 1$ and viscosity ν, the velocity field $\boldsymbol{u} : \Omega \times (0, \infty) \to \mathbb{R}^d$ and the pressure $\pi : \Omega \times (0, \infty) \to \mathbb{R}$ are ruled by

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla \pi = 0, \\ \operatorname{div} \boldsymbol{u} = 0, \end{cases} \qquad \text{in } \Omega \times (0, \infty), \tag{1.2.1}$$

subject, for instance, to the no-slip boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = 0, & \text{on } \partial\Omega \times (0,\infty), \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot), & \text{in } \Omega. \end{cases}$$

From Leray's seminal work until nowadays, the mathematical analysis of Navier–Stokes equations (1.2.1) has been developed by many authors but it is still far from being complete. We refer the reader to, e.g., [150] and references therein for a review on the theory and connected problems.

Instead, if two globally immiscible, incompressible and viscous fluids lie into a fixed domain, they additionally interact at the interface among them. To describe their motion, the time variation of physical quantities such as velocity, pressure and stresses is formulated within a certain spatial regions. In the sharp interface approach the Navier–Stokes equations are written separately for each phases, supposing that density and viscosity are equal to their equilibrium values. In addition, the system is closed via boundary conditions involving velocity and strain tensor at the interface, which is unknown and has to be determined as well.

To define the free boundary problem, Ω is separated into two subdomains $\Omega_A(t)$ and $\Omega_B(t)$, for $t \geq 0$, occupied by each fluid. They are separated by a surface $\Sigma(t)$, such that $\Omega = \Omega_A(t) \cup \Omega_B(t) \cup \Sigma(t)$. We introduce the velocity $\boldsymbol{u} : \Omega \times (0,\infty) \to \mathbb{R}^d$ and the pressure $\pi : \Omega \times (0,\infty) \to \mathbb{R}$. Assuming the two Newtonian fluids have different constant viscosities ν_i, $i = A, B$, neglecting densities difference $\rho_A = \rho_B = 1$ (the so-called Boussinesq approximation) and gravity or external forces, the free surface problem for $t > 0$ reads as

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \operatorname{div}\big(-\pi I + 2\nu_i D\boldsymbol{u}\big) = 0, & \text{in } \Omega_i(t), \\ \operatorname{div} \boldsymbol{u} = 0, & \text{in } \Omega_i(t), \end{cases} \tag{1.2.2}$$

subject to the boundary and initial conditions

$$\begin{cases} [\boldsymbol{u}]_\Sigma = 0, & \text{on } \Sigma(t), \\ \boldsymbol{u}\cdot\boldsymbol{n} = V, & \text{on } \Sigma(t) \\ [-\pi I + \nu_i D\boldsymbol{u}]_\Sigma \cdot \boldsymbol{n} = \sigma\kappa\boldsymbol{n}, & \text{on } \Sigma(t), \\ \boldsymbol{u} = 0, & \text{on } \partial\Omega, \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot), & \text{in } \Omega. \end{cases} \tag{1.2.3}$$

Here the notation I stands for the identity tensor, D is the strain rate tensor such that $D\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$, $[f]_\Sigma$ denotes the jump of limiting values across the surface Σ, $\boldsymbol{n}$ is the normal vector to Σ pointing into Ω_2, V is the normal velocity, σ is the surface tension and κ the total curvature. On the interface Σ, the above boundary condition on $\boldsymbol{u}$ means that the interface is transported by the flow and the velocity is continuous across the interface (no jump condition), while the one for the stress tensor is the so-called Young-Laplace condition taking the capillary force into account.

From the mathematical viewpoint, only few results are available for system (1.2.2)-(1.2.3) with surface tension. The existence of a local in time strong solution is proved under the assumption that Ω_A and Ω_B do not change their topology. Alternatively, a global in time solution exists provided that the initial condition is closed to a (regular) equilibrium. Such results can be found in [7], [17], [18] for the motion in an infinite layer and [48], [148] , [129], [140], [143] for bounded domains. In computational fluid dynamics, numerical methods dealing with the free boundary formulation can be divided into two classes: front tracking and front capturing. In the first case the flow is approximated over a stationary mesh to which a separate unstructured grid is added for the moving interface. However, tracking the moving mesh entails a large computational overhead. In the second approach the interface is instead caught as the zero-level set of a scalar function which satisfies a transport equation driven by the velocity field. Classical references are d [153] and [154] for the former, [135] and [136] for the latter (see also [102] and [114] and references therein).

The motion of fluids in simple geometries, where key mechanisms of more complicated systems are already evident, has been the subject of many works. The starting point of one of those branches was the experiment proposed by Hele–Shaw in [91]. A fluid (originally a gas) is injected into a system consisting of two parallel plates separated by a narrow gap to avoid the gravity (see [156] for a historical review). The particular structure of the cell leads to the assumption that viscous forces prevail over inertial ones, under which Navier–Stokes equations reduce to a linear relation. Indeed, from (1.2.1) with no slip boundary conditions on two flat plates, assuming first the flow is steady and parallel ($\partial_t \boldsymbol{u} = 0$ and $\boldsymbol{u} = (u_1, u_2, 0)$), then neglecting derivatives with respect to x_1 and x_2, the fluid evolution is described by

$$u_1 = \frac{1}{2}\frac{\partial \pi}{\partial x_1}\Big(\frac{x_3^2}{\nu} - \frac{hx_3}{\nu}\Big), \quad u_2 = \frac{1}{2}\frac{\partial \pi}{\partial x_2}\Big(\frac{x_3^2}{\nu} - \frac{hx_3}{\nu}\Big).$$

Introducing the gap-averaged velocity

$$\overline{u}_i = \frac{1}{h}\int_0^h u_i \, \mathrm{d}x_3, \quad i = 1, 2,$$

the two dimensional vector field $\overline{\boldsymbol{u}} = (\overline{u}_1, \overline{u}_2)$ satisfies

$$\overline{\boldsymbol{u}} = -\frac{h^2}{12\nu}\nabla\pi, \tag{1.2.4}$$

where h is the cell gap. This is the so-called Hele–Shaw equation. It has the same form of the Darcy's law employed for saturated flow in porous media in three dimensions. Computing the divergence of (1.2.4), a free boundary problem for the original Hele–Shaw experiment is formulated in term of the pressure π with pointwise source/sink in $\Omega(t)$. An equivalent formulation can be introduced leading to a boundary-value problem for a conformal map from the unit disk to the phase domain $\Omega(t)$. This gave the possibility to construct explicit solutions from a suitable ansatz (see [76], [127], [130], [132], [142]). Regarding a general initial domain $\Omega(0)$, the system is well posed only in the case of fluid injection. In particular, given a domain $\Omega(0)$ with smooth boundary, it has been proved the existence of local in time strong solutions (i.e. a smooth family of $\Omega(t)$, $t > 0$). On the other hand, existence of a unique global weak solution is reached reducing the problem to a variational inequality. We refer the reder to [89] and references therein for a fuller treatment of this subject.

Going back to the motion of two immiscible fluids, trapped in a Hele–Shaw cell, the related free boundary problem for the gap-averaged velocity reads as follows

$$\begin{cases} \boldsymbol{u} = -\frac{h}{12\nu_i}\nabla\pi, & \text{in } \Omega_i(t), \\ \operatorname{div}\boldsymbol{u} = 0, & \text{in } \Omega_i(t), \end{cases} \tag{1.2.5}$$

subject to the boundary and initial conditions

$$\begin{cases} [\boldsymbol{u}]_\Sigma = 0, & \text{on } \Sigma(t), \\ \boldsymbol{u}\cdot\boldsymbol{n} = V, & \text{on } \Sigma(t) \\ [-\pi\cdot\boldsymbol{n}]_\Sigma = \sigma\kappa\boldsymbol{n}, & \text{on } \Sigma(t), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \text{on } \partial\Omega, \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot), & \text{in } \Omega. \end{cases} \tag{1.2.6}$$

In contrast to the one-phase problem, the techniques to find exact solutions usually fail in two-phase Hele–Shaw flows. Several papers have been devoted to the well-posedness of system (1.2.5)-(1.2.6) with zero and non-zero surface tension, also considering the gravity field with applications to porous media flows. The same also goes for applications and numerical simulations. A good source of reference on two-phase Hele-Shaw flows can be found in the introduction of [44].

1.3 Diffuse interface method

A radical change of view in the theory dates back to Van der Waals which postulated in [155] the notion of diffuse interface rather than a sharp one. This idea inspired many physicists during the last century leading to the development of the so-called phase field method. The origin of the first equations can be attributed to Cahn and Hilliard in [29] (see also [28]), whose goal was to describe the spinodal decomposition in alloy mixtures. Afterwards the method has been used by Allen and Cahn in [8] for antiphase domain coarsening. Since then, the approach has been employed in many areas of Materials Science such as solidification of pure and binary materials, grain boundary, nucleation, solid-solid or liquid-liquid phase transition and crystallization. We refer the reader to, e.g., [52] for a general overview.

The key concept of diffuse interface methods consists in treating the interface as a finite-width region in which the physical quantities have a rapid but smooth variation. The evolution of thick interfaces is taken into account by means of an additional variable upon which the free energy of the systems depends. This is the so-called phase order parameter or phase field φ which distinguishes one constituent (or one phase) from the other. Hence, the interface can be recovered as level-sets of φ. The order parameter is ruled by an additional equation based on the mass balance of the mixture and assuming Fick's law for the mass flux. In addition, the form of the total free energy is deduced from Statistical Mechanics. In this approach interface topological changes are naturally allowed by the formulation of the system. This is one of the main reasons that made this approach widespread in numerical simulations. On the other hand, letting the interface thickness go to zero, the relating free boundary (or sharp interface) problems can be formally recovered from diffuse interface models.

The model H

The so-called model H was introduced by Hohenberg and Halperin in [93] to study critical points of single and binary fluids. A detailed derivation of the model was proposed in [88] and [144] for the motion of fluids driven by capillarity forces (neglecting densities difference). More precisely, in [88] the balance of mass and momentum are combined with constitutive laws compatible with a version of the second law of thermodynamics. This model has been employed in several numerical studies for concrete applications. Main examples are interface stretching during mixing [31], thermocapillary flow [98], drop breakup, moving contact lines and large-deformation sloshing flow [96]. For a review on these topics see [10]. Later on suitable generalizations of the model H have been discussed for fluid mixtures with unmatched densities in [2], [11], [50], [114]. A diffuse interface model accounting for two-phase flow with soluble surface agents has been proposed in [3]. Further generalizations to contact angles problem, ternary fluids and numerical methods can be founded in [102] and references therein.

We consider two globally immiscible, incompressible and viscous fluids labelled by A and B that occupy a domain $\Omega \subset \mathbb{R}^d$. Following the diffuse interface approach, we assume a partial mixing of the fluids molecules. Additionally, we accept the so-called Boussinesq approximation, namely the fluids have the same density $\rho_A = \rho_B = 1$. We indicate with c the molar fraction (or concentration) of fluid A ($0 \leq c \leq 1$). Obviously, $1-c$ is the concentration of fluid B. From the conservation of mass of the fluid mixture, we can write the equation of continuity for the fluid A as

$$\partial_t c + \operatorname{div}(c\boldsymbol{u}) = \operatorname{div} \mathbf{J_A},$$

where $\boldsymbol{u}$ is the mean velocity of the fluid flow and $\mathbf{J_A}$ is the mass flux of fluid A. We recall that the mass flux of fluid B is $\mathbf{J}_B = -\mathbf{J}_A$. We now rewrite the equation for the difference of mass concentrations which plays the role of order parameter. Introducing $\varphi = 2c - 1$, we have

$$\partial_t \varphi + \operatorname{div}(\varphi\boldsymbol{u}) = \operatorname{div} \mathbf{J},$$

where $\mathbf{J} = 2\mathbf{J}_A$. This is a convective Cahn–Hilliard type equation with mass flux $\mathbf{J}$. The balance of momentum ruling the mean velocity reads as

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \operatorname{div} \mathbf{T},$$

where $\mathbf{T}$ is the stress tensor such that $\mathbf{T} = \mathbf{T}^T$, together with the incompressibility constraint

$$\operatorname{div} \boldsymbol{u} = 0.$$

We proceed by stating the constitutive equations. The mass flux $\mathbf{J}$ is defined by Fick's law

$$\mathbf{J} = m\nabla\mu,$$

where m is the chemical mass diffusivity (also called mobility) and μ is the chemical potential. The chemical potential μ is then related to the phase parameter by the variational derivative of the total free energy

$$\mu = \frac{\delta\mathcal{E}}{\delta\varphi}.$$

In the theory proposed by Cahn and Hilliard in [29], also employed for fluid mixtures, the form of the free energy is of Ginzburg–Landau type

$$\mathcal{E}_{GL}(\varphi) = \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi(x)|^2 + \frac{1}{\varepsilon}F(\varphi(x))\,\mathrm{d}x, \tag{1.3.1}$$

where $\varepsilon > 0$ is related to the thickness of the interface and the homogeneous free energy density F is given by

$$F(s) = \Psi(s) - \frac{\Theta_0}{2}s^2, \quad s \in (-1,1), \tag{1.3.2}$$

where its convex part Ψ is defined as

$$\Psi(s) = \frac{\Theta}{2}\left[(1+s)\log(1+s) + (1-s)\log(1-s)\right]. \tag{1.3.3}$$

The form of F is the sum of the free energy densities of the system before and after mixing. In particular, the latter is deduced from the Boltzmann equation for the mixing entropy. The constant Θ denotes the absolute temperature of the mixture, while Θ_0 is the so-called critical temperature, depending on interaction potentials of same and different phases, Avogadro's number and Boltzmann's constant (see [105] for a recent review). The relation between Θ and Θ_0 determines the mathematical features of F. The interesting case is the double well one when $0 < \Theta < \Theta_0$, so that the segregation takes place.

The stress tensor $\mathbf{T}$ is the sum of the Cauchy stress tensor and a further contribution which models capillarity effects

$$\mathbf{T} = \mathbf{T}^c + \mathbf{T}^s.$$

For incompressible Newtonian fluids, the Cauchy stress tensor accounts for the effects due to pressure and the viscous stress tensor, this is

$$\mathbf{T}^c = -\tilde{\pi}I + 2\nu D\boldsymbol{u} = -\tilde{\pi}I + \nu\big(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\big),$$

where ν is the viscosity of the fluid. From the mechanical version of the second law of thermodynamics (see [88]), the capillary stress has the form

$$\mathbf{T}^s = -\varepsilon\nabla\varphi \otimes \nabla\varphi.$$

Summing up, the resulting system reads as follows

$$\begin{cases} \partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \mathrm{div}\,(\nu D\boldsymbol{u}) + \nabla\tilde{\pi} = -\varepsilon\mathrm{div}\,(\nabla\varphi\otimes\nabla\varphi), \\ \mathrm{div}\,\boldsymbol{u} = 0, \\ \partial_t\varphi + \mathrm{div}\,(\varphi\boldsymbol{u}) = \mathrm{div}\,(m\nabla\mu), \\ \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi), \end{cases} \qquad \text{in } \Omega\times(0,T).$$

The system is subject to the following natural boundary conditions

$$\boldsymbol{u} = \mathbf{0}, \quad \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}}\varphi = 0, \quad \text{on } \partial\Omega\times(0,T), \tag{1.3.4}$$

that is, a no-slip boundary condition for $\boldsymbol{u}$ and homogeneous Neumann boundary conditions for the chemical potential μ and for φ. The latter ones entail that there is no mass flux and the interface separating the two fluids is orthogonal to the boundary. The system is closed with the initial conditions

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot), \ \varphi(\cdot,0) = \varphi_0(\cdot), \quad \text{in } \Omega. \tag{1.3.5}$$

We observe that, due to the equality

$$-\varepsilon\mathrm{div}\,(\nabla\varphi \otimes \nabla\varphi) = \mu\nabla\varphi - \nabla\Big(\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}F(\varphi)\Big), \tag{1.3.6}$$

the system can be rewritten in the following form

$$\begin{cases} \partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \mathrm{div}\,(\nu D\boldsymbol{u}) + \nabla\pi = \mu\nabla\varphi, \\ \mathrm{div}\,\boldsymbol{u} = 0, \\ \partial_t\varphi + \mathrm{div}\,(\varphi\boldsymbol{u}) = \mathrm{div}\,(m\nabla\mu), \\ \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi), \end{cases} \quad \text{in } \Omega\times(0,T), \tag{1.3.7}$$

where π is the modified pressure, namely,

$$\pi = \tilde{\pi} + \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}F(\varphi).$$

Here it is more evident the role of the capillary force. In the diffuse interface approach the singular force on the free surface in (1.2.2)-(1.2.3) is smoothed and acts in the finite transition region between the pure phases. Indeed, the term on the right hand side $\mu\nabla\varphi$ is not zero only in the mixing zones. The connection with the capillarity force has been explained in, e.g., [31] and [110] (see also [114]).

Let us now comment two coefficients of system (1.3.7) which need to be specified. The viscosity of the mixture ν is a strictly positive function of the concentrations, namely

$$\nu = \nu(\varphi) \geq \nu^* > 0.$$

A possible choice of the local viscosity is a linear combination of the bulk viscosities,

$$\nu(s) = \nu_A\frac{1+s}{2} + \nu_B\frac{1-s}{2}, \tag{1.3.8}$$

where ν_A and ν_B are the positive viscosities of the two fluids. Otherwise, if the two fluids have the same viscosity, it is usually called matched viscosities case. Another parameter of the system is the mobility function m. Throughout this thesis it is assumed to be a positive constant taken equal to the unity.

The total energy associated to the system is defined as the sum of kinetic and free energies, namely,

$$E_{GL}(\boldsymbol{u},\varphi) = \int_\Omega |\boldsymbol{u}(x)|^2 + \frac{\varepsilon}{2}|\nabla\varphi(x)|^2 + \frac{1}{\varepsilon}F(\varphi(x))\,\mathrm{d}x.$$

Exploiting the balance of momentum and concentrations together with their boundary conditions, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{GL}(\boldsymbol{u},\varphi) = -\int_\Omega \nu(\varphi)|D\boldsymbol{u}|^2 + |\nabla\mu|^2\,\mathrm{d}x.$$

The above energy identity (formally) entails the dissipation of the energy being the terms on the right hand side negative. In particular, the term $\nabla\mu$, which acts to sharpen interfacial gradients, is a diffusive term on the whole domain. It is worth mentioning that a further significant example of mobility is the so-called degenerate case, namely

$$m(s) = M(1 - s^2).$$

In this case the dissipative term $m(\varphi)\nabla\mu$ plays an effective role only in the mixing zones (see [22] and [54]).

An important field of application of the model H is the description of phase separation in alloys or polymer mixtures. In this area the study of patterns formation is of great interest for morphology selection because of the influence on macroscopic properties of the system. In particular, pattern tunability has been studied by adding chemical reaction mechanisms to hydrodynamic effects in [94] and [95]. We mention that these reversible reactions were still included in the phase segregation process occurring after spinodal decomposition in [85] and in microphase separation of diblock copolymers in [19] and [152] (see also [126]). The proposed model is called Navier–Stokes–Cahn–Hilliard–Oono system and it reads as follows

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \operatorname{div}(\nu D\boldsymbol{u}) + \nabla\pi = \mu\nabla\varphi, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \partial_t\varphi + \operatorname{div}(\varphi\boldsymbol{u}) + \beta(\varphi - c) = \Delta\mu, \\ \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi), \end{cases} \quad \text{in } \Omega\times(0,T). \tag{1.3.9}$$

with

$$\beta = \gamma_1 + \gamma_2, \quad c = \frac{\gamma_2 - \gamma_1}{\gamma_2 + \gamma_1},$$

where γ_1 and γ_2 are the forward and backward reaction rates, respectively. The system is closed with boundary and initial conditions (1.3.4)-(1.3.5). A relevant feature of system (1.3.9) is that, contrary to the model H, it does not necessarily preserve the total mass. Indeed, on account of the boundary conditions, a formal integration of the convective Cahn–Hilliard–Oono equation over Ω gives

$$\overline{\varphi}(t) = c + \mathrm{e}^{-\beta t}(\overline{\varphi}_0 - c), \quad \forall\, t \geq 0, \tag{1.3.10}$$

having set the total mass

$$\overline{\varphi} = \frac{1}{|\Omega|}\int_\Omega \varphi\,\mathrm{d}x.$$

Accordingly, the mass is conserved only if $\overline{\varphi}_0 = c$. Otherwise, noting that $c \in (-1,1)$ and $\beta > 0$ by definition, $\overline{\varphi}(t)$ converges exponentially fast to c (the so-called off-critical case). In turn, the linear reaction term accounts for long-range interactions. Indeed, the convective Cahn–Hilliard–Oono equation (cf. also (1.3.10)) is equivalent to

$$\partial_t\varphi + \operatorname{div}(\varphi\boldsymbol{u}) + \beta\mathrm{e}^{-\beta t}(\overline{\varphi}_0 - c) = \Delta\Big(-\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) + \beta G * (\varphi - \overline{\varphi})\Big).$$

Here the symbol $*$ stands for the spatial convolution over Ω while G denotes the Green function associated to the Laplacian with homogeneous Neumann boundary condition. In the critical case ($\overline{\varphi}_0 = c$), the Cahn–Hilliard–Oono equation (neglecting the velocity field) can be viewed as the conserved gradient flow of the so-called Ohta–Kawasaki energy functional

$$\mathcal{F}(\varphi) = \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}F(\varphi)\,\mathrm{d}x \\ + \frac{\beta}{2}\int_{\Omega\times\Omega} G(x-y)(\varphi(x)-\overline{\varphi})(\varphi(y)-\overline{\varphi})\,\mathrm{d}x\mathrm{d}y. \tag{1.3.11}$$

This functional was introduced in [125] for diblock copolymers (see also [37]) with smooth potential $F(s) = \frac{1}{4}(1-s^2)^2$. The nature of the minimizers of $\mathcal{F}$ is more complicated than the classical Ginzburg–Landau case ($\beta = 0$) due to the competition between local and nonlocal interactions. Several properties such as structure of minimizers and scaling properties of the Ohta–Kawasaki functional have been the subject of a number of papers (see, for instance, [35], [36], [86], [104], [121], [124] and references therein). In the same way, the Navier–Stokes–Cahn–Hilliard–Oono system has not a decreasing in time energy (in the off-critical case) with respect to the Ginzburg–Landau or Ohta–Kawasaki free energies. Its dissipation rate respect to the former energy is

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{GL}(\boldsymbol{u},\varphi) = -\int_\Omega \nu(\varphi)|D\boldsymbol{u}|^2 + |\nabla\mu|^2\,\mathrm{d}x - \beta\int_\Omega (\varphi - c)\mu\,\mathrm{d}x,$$

where the last term on the right hand side has not a definite sign.

Simplified models in a Hele–Shaw cell

Towards the purpose of modeling flow in elementary geometries, a simplification of the model H was proposed in [106] and [107] (see also [111]). A diffuse interface models allowing topological changes in binary mixtures is introduced to study pinchoff and reconnection in a Hele–Shaw cell. More precisely, starting from the unmatched densities version of model H derived in [114] and assuming a Poiseuille flow, a Hele–Shaw–Cahn–Hilliard model have been obtained. Lately, a different unmatched densities Hele–Shaw–Cahn–Hilliard model has been studied in [46] and applied to rising bubbles and fingering instabilities. This has been derived from the model H with unmatched densities proposed by [2]. Even though these models are different in the unmatched densities case, both models in the Boussinesq approximation share the same form (up to nondimensional factors). Numerical simulations have been provided in [32] for rotating Hele–Shaw flow including inertial effect due to Coriolis force (see [160] for spinodal decomposition). A further generalization is due to [146] and applied to buoyancy-driven two phase flow involving Rayleigh-Taylor instability. Sharp interface limits have been carried out in [46] and [111]. In these last years, the Hele–Shaw–Cahn–Hilliard model has also had a considerable impact in modeling tumor growth. Indeed, this system has been coupled with reaction-diffusion equations to take chemotaxis, active transport and nutrients into account. Among the large literature devoted to this subject, we mention [33], [42], [43], [61], [67], [78], [79], [161].

The Hele–Shaw–Cahn–Hilliard system is deduced performing a rescaled procedure on (1.3). Analogously to the sharp interface approximation, we consider a flat thin domain $\Omega = \Omega' \times [0, h]$, where Ω' is a smooth bounded domain in $\mathbb{R}^2$ and h is a small gap. We introduce the characteristic length L, the characteristic velocity V and the parameter $\delta = \frac{h}{L} << 1$. Following the detailed argument in [46], neglecting the gravity field and assuming the Boussinesq approximation, the resulting system reads as

$$\begin{cases} \boldsymbol{u} = \frac{1}{12\nu}\big(- \nabla\tilde{\pi} - \varepsilon \mathrm{div}\,(\nabla\varphi \otimes \nabla\varphi)\big), \\ \mathrm{div}\,\boldsymbol{u} = 0, \\ \partial_t\varphi + \mathrm{div}\,(\varphi\boldsymbol{u}) = \Delta\mu, \\ \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi), \end{cases} \quad \text{in } \Omega \times (0,T), \tag{1.3.12}$$

where $\boldsymbol{u}$ is the two dimensional gap-averaged velocity, $\nu = \nu(\varphi)$ is the nondimensional viscosity. Here other nondimensional parameters, such as the Péclet or the capillary number, have been fixed equal to one. By using the relation (1.3.6), it is possible to rewrite the above system as

$$\begin{cases} \boldsymbol{u} = \frac{1}{12\nu}\big(- \nabla\pi + \mu\nabla\varphi\big), \\ \mathrm{div}\,\boldsymbol{u} = 0, \\ \partial_t\varphi + \mathrm{div}\,(\varphi\boldsymbol{u}) = \Delta\mu, \\ \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi), \end{cases} \quad \text{in } \Omega \times (0,T), \tag{1.3.13}$$

where π is the modified pressure as in the model H. This system is closed with the following boundary and initial conditions

$$\begin{cases} \boldsymbol{u}\cdot\boldsymbol{n} = \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}}\varphi = 0, & \text{on } \partial\Omega \times (0,T), \\ \varphi(\cdot,0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{1.3.14}$$

Even though the system has a derivation that naturally leads to a two-dimensional space-domain, it can be generalized to porous media flow in three dimensions. The associated energy to the Hele–Shaw–Cahn–Hilliard system is dissipated by the evolution. According to (1.3.13)-(1.3.14), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}(\varphi) = -\int_\Omega \nu(\varphi)|\boldsymbol{u}|^2 + |\nabla\mu|^2 \,\mathrm{d}x.$$

A relevant modification of the Hele–Shaw–Cahn–Hilliard model is the Brinkman–Cahn–Hilliard system. This model have been first employed in [157] to model thermocapillary flow in a Hele–Shaw cell. More recently, it has been derived also in the setting of porous media in [123] and [134]. Numerical simulations are provided in [41] for spinodal decomposition of viscous fluid and boundary-driven flows. The Brinkman–Cahn–Hilliard system in its general form read as follows

$$\begin{cases} -\mathrm{div}\,(\nu D\boldsymbol{u}) + \eta\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi, \\ \mathrm{div}\,\boldsymbol{u} = 0, \\ \partial_t\varphi + \mathrm{div}\,(\varphi\boldsymbol{u}) = \Delta\mu, \\ \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi), \end{cases} \quad \text{in } \Omega \times (0,T), \tag{1.3.15}$$

where $\nu = \nu(\varphi) > 0$ is the viscosity, $\eta = \eta(\varphi) > 0$ is the permeability. The system is subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0}, \quad \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}}\varphi = 0, & \text{on } \partial\Omega \times (0,T), \\ \varphi(\cdot,0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{1.3.16}$$

It is worth mentioning that the above model can be considered as a suitable regularization of the Hele–Shaw–Cahn–Hilliard system. In particular, we observe that this model (with $\eta = 0$) is exactly the one used in [31] for numerical calculations of mixing fluids in a driven cavity. In addition, the Brinkman–Cahn–Hilliard model satisfies the following energy dissipation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}(\varphi) = -\int_\Omega \nu(\varphi)|D\boldsymbol{u}|^2 + \eta(\varphi)|\boldsymbol{u}|^2 + |\nabla\mu|^2 \,\mathrm{d}x.$$

1.4 Local versus nonlocal interaction free energies

In multicomponent fluid mixtures the total (Helmholtz) free energy $\mathcal{E}$ depends on intermolecular forces resulting from the competition between molecular collisions and long-range attractions. A fundamental approach to derive its form is based on Statistical Mechanics which connects the macroscopic description of the system (thermodynamic potentials) and microscopic single-molecule energy states. The method consists in a thermodynamic limit starting from microscopic models with a finite number of particles interacting each other. The general form of the free energy includes not only local concentrations but also a dependence on concentration at neighbouring points. However, in the literature it is generally accepted to describe these nonlocal interactions through concentration gradients. This is the main assumption in the derivation of Ginzburg–Landau free energy. In a region of nonuniform composition it is assumed that

$$\mathcal{E}(\varphi) = \int_\Omega \psi(\varphi)\,\mathrm{d}x, \quad \text{where} \quad \psi(\varphi) = \psi(\varphi, \nabla\varphi, \nabla^2\varphi, ...).$$

The free energy density ψ is then expanded in a multivariable Taylor series of density gradients around the homogeneous free energy, namely

$$\psi(\varphi) = F(\varphi) + \sum_{i=1}^{3}\psi_i(\varphi)\partial_i\varphi + \sum_{i,j=1}^{3}\psi^1_{i,j}(\varphi)\partial_{ij}\varphi + \frac{1}{2}\sum_{i,j=1}^{3}\psi^2_{i,j}(\varphi)\partial_i\varphi\partial_j\varphi + \text{h.o.t.}$$

Here $F(\varphi)$ is the free energy density of an homogeneous system defined as (1.3.2) and ψ_i, $\psi^1_{i,j}$ and $\psi^2_{i,j}$ are coefficient depending on φ. Ignoring higher terms than second-order derivative and imposing that the system is invariant under rotations and reflections, the free energy density is approximated by

$$\psi(\varphi) = F(\varphi) + \tilde{\psi}^1(\varphi)\Delta\varphi + \frac{1}{2}\tilde{\psi}^2(\varphi)|\nabla\varphi|^2.$$

Observing that

$$\tilde{\psi}^1(\varphi)\Delta\varphi = \mathrm{div}\,(\tilde{\psi}^1(\varphi)\nabla\varphi) - \frac{\mathrm{d}\tilde{\psi}^1(\varphi)}{\mathrm{d}\varphi}|\nabla\varphi|^2,$$

the first term on the right hand side vanishes due to the boundary condition. Thus, by setting $\tilde{\psi}^2(\varphi) - 2\frac{\mathrm{d}\tilde{\psi}^1(\varphi)}{\mathrm{d}\varphi} = 1$, the free energy density has the form

$$\psi(\varphi) = F(\varphi) + \frac{1}{2}|\nabla\varphi|^2.$$

Therefore, the gradient term accounting for spatial inhomogeneities in the concentration comes out from the assumption of short range interaction between molecules.

A different form of the free energy relies on the approach of Statistical Mechanics to describe mutual interactions through convolution integrals weighted by a scale of density kernels. In a basic case the Helmholtz free energy reads as

$$\mathcal{E}_H(\varphi) = \int_\Omega \Psi(\varphi(x))\,\mathrm{d}x - \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\varphi(x)\varphi(y)\,\mathrm{d}x\mathrm{d}y, \tag{1.4.1}$$

where Ψ is the singular convex part of F given in (1.3.3) and J is the interaction kernel such that $J(x) = J(-x)$. We have fixed for simplicity the thickness of the interface equal to the unity $\varepsilon = 1$. The free energy (1.4.1) was already known in Van der Waals formalism. However, the contribution deriving from the dependence on concentration in neighboring points was considered to be therein negligible. This form has been more recently rederived by [81] and [82] (see also [80]) in the context of phase segregation dynamics in alloys with long range interactions. Nonetheless, the global Helmholtz free energy $\mathcal{E}_H$ is deeply connected with the local one (1.3.1). Indeed, the Helmholtz free energy (1.4.1) is equivalent to

$$\mathcal{E}_H(\varphi) = \int_\Omega \tilde{F}(x,\varphi)\,\mathrm{d}x + \frac{1}{4}\int_\Omega\int_\Omega J(x-y)\big(\varphi(x)-\varphi(y)\big)^2\,\mathrm{d}x\mathrm{d}y, \tag{1.4.2}$$

where

$$\tilde{F}(x,s) = \Psi(s) - \frac{1}{2}(J*1)(x)s^2. \tag{1.4.3}$$

Note that $a = J*1$ is a constant in specific domains (e.g. a $d-$dimensional torus) and can be interpret as the critical temperature θ_c with a suitable scaling. Thus, the first approximation of the nonlocal interaction in (1.4.2) is (formally) $\frac{1}{2}|\nabla\varphi|^2$ provided that J is sufficiently peaked around 0 (i.e. close to the delta function). Thus, the Ginzburg–Landau free energy can be seen as an approximation of the nonlocal one.

We now reformulate the model H and the Hele–Shaw approximation in the setting of the nonlocal Helmholtz free energy. According to the above formulations the only difference concerns with the chemical potential. In this case, the first variation of the total free energy is given by

$$\mu = \Psi'(\varphi) - J*\varphi.$$

then, the nonlocal model H can be written as

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \operatorname{div}(\nu D\boldsymbol{u}) + \nabla\pi = \mu\nabla\varphi, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \partial_t\varphi + \operatorname{div}(\varphi\boldsymbol{u}) = \Delta\mu, \\ \mu = \Psi'(\varphi) - J*\varphi, \end{cases} \quad \text{in } \Omega\times(0,T), \tag{1.4.4}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0}, \quad \partial_{\boldsymbol{n}}\mu = 0, & \text{on } \partial\Omega\times(0,T), \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot),\ \varphi(\cdot,0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{1.4.5}$$

The system still conserves a dissipative nature. Introducing the total energy as the sum of kinetic and free energy

$$E_H(\boldsymbol{u},\varphi) = \int_\Omega |\boldsymbol{u}(x)|^2 + \Psi(\varphi(x))\,\mathrm{d}x - \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\varphi(x)\varphi(y)\,\mathrm{d}x\mathrm{d}y,$$

we have the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}E_H(\boldsymbol{u},\varphi) = -\int_\Omega \nu(\varphi)|D\boldsymbol{u}|^2 + |\nabla\mu|^2\,\mathrm{d}x.$$

On the other hand, the nonlocal formulation of the Hele–Shaw–Cahn–Hilliard system is

$$\begin{cases} \boldsymbol{u} = \frac{1}{12\nu}(-\nabla\pi + \mu\nabla\varphi), \\ \operatorname{div}\boldsymbol{u} = 0, \\ \partial_t\varphi + \operatorname{div}(\varphi\boldsymbol{u}) = \Delta\mu, \\ \mu = \Psi'(\varphi) - J*\varphi, \end{cases} \quad \text{in } \Omega\times(0,T), \tag{1.4.6}$$

subject to the following natural boundary and initial conditions

$$\begin{cases} \boldsymbol{u}\cdot\boldsymbol{n} = \partial_{\boldsymbol{n}}\mu = 0, & \text{on } \partial\Omega\times(0,T), \\ \varphi(\cdot,0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{1.4.7}$$

In this case the energy dissipation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_H(\varphi) = -\int_\Omega \nu(\varphi)|\boldsymbol{u}|^2 + |\nabla\mu|^2\,\mathrm{d}x.$$

It is worth mentioning that, passing from a local to a nonlocal free energy, the Cahn–Hilliard equation reduces from a fourth to a second order equation in space in the unknown φ. Therefore, only one boundary condition is needed in the nonlocal setting in order to guarantee the conservation of mass.

1.5 The mathematical point of view

The mathematical theory of diffuse interface models arising in fluid dynamics is quite challenging. This is motivated by the interplay between nonlinear terms and the logarithmic potential that also affects the regularity of the velocity field. Anyway, for the cases presented above, a rather satisfactory mathematical analysis can be carried out under suitable assumptions on the coefficients. More precisely, we will prove well-posedness of weak and strong solutions, regularity theory and longtime behavior in the next chapters. We now provide an informal overview on the main feature of the results.

In order to investigate the well-posedness, the first task is the existence of weak solution, namely solutions with finite total energy for any time. Such existence is reasonably expected by virtue of the dissipative nature entailed by the (formal) identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varphi) + \big(\text{dissipation terms}\big) = 0. \tag{1.5.1}$$

We combine this feature with a two steps approximation technique. Our approach consists in approximating the logarithmic potential via a family of regular functions defined on the whole real line. Then, we introduce approximate problems by means of a Galerkin procedure. Since (1.5.1) is satisfied by approximating solutions, we find uniform estimates with respect to the approximation parameters. This allows us to recover a solution to the original problem through a compactness procedure and a passage to the limit. The advantage of this approach is the construction of approximating (Galerkin) solutions which are regular enough to perform high order estimates rigorously. We also remind other approaches based on the theory of maximal monotone operator on Banach spaces and fixed point methods (see, e.g., [1] and [75]). It is worth remarking that we end up with *physical* solutions, namely the relative difference of concentrations φ maintains its physical meaning. More precisely, we have

$$\varphi \in L^\infty(\Omega \times (0,\infty)) \text{ with } |\varphi(x,t)| < 1 \text{ a.e. } (x,t) \in \Omega \times (0,\infty). \tag{1.5.2}$$

Note that this is an immediate consequence of the composition between Ψ' and φ.

In the literature, the homogeneous free energy density (1.3.3) is very often approximated by a fourth–order polynomial, namely

$$F_0(s) = \frac{k}{4}(s^2-1)^2, \quad s \in \mathbb{R}, \tag{1.5.3}$$

where $k > 0$ is a constant related to Θ_0. This regular approximation is justified whenever Θ is close to Θ_0. However, whether F is replaced with the regular potential F_0, it is impossible to ensure that φ takes value within the physically admissible interval $[-1,1]$.

Before proceeding to the next step, let us comment on the main difficulties to obtain further results. A first problem concerns the velocity field due to the still open issue regarding uniqueness and regularity of Navier–Stokes equations in three space dimensions. In addition, the elliptic regularity theory of the Stokes operator with a nonconstant viscosity $-\mathrm{div}(\nu(\varphi)D\boldsymbol{u})$ differs from the classical one related to the Stokes operator, requiring more regularity on φ. Also, the Korteweg force on the right hand side is nonlinear. In the local case, up to a gradient term (cf. (1.3.6)), it reads as

$$f_K = -\mathrm{div}\,(\nabla\varphi \otimes \nabla\varphi).$$

On the other hand, in the nonlocal case, the different form of the chemical potential entails

$$\mu\nabla\varphi = \nabla\Big(F(\varphi) - (J * \varphi)\varphi\Big) + (\nabla J * \varphi)\varphi.$$

In turn, the forcing term

$$f_K = (\nabla J * \varphi)\varphi$$

seems to be easier to deal with (cf. (1.5.2)). It is worth mentioning that a nonlocal free energy involves the study of a second order problem (in φ) contrary to the local case (fourth order problem). This has some advantages (cf. Korteweg forces) provided one knows how to handle the lack of regularity. Besides, even analyzing the Hele–Shaw problem is not an easy task. Indeed, the Darcy's law (in the matched viscosities case) can be seen as the Helmholtz decomposition of f_K. Thus, the regularity of the velocity is the same of the Korteweg force.

A second problem regards the physically relevant form of the potential. The logarithmic function (1.3.3) is difficult to treat due to the different growth of their derivatives when they approach $\pm\infty$ as s goes to ± 1. Indeed, if the chemical potential μ is regular enough, we aim to deduce higher order estimates on φ with respect to the ones obtained from the boundedness of the energy. Consequently, performing (even formally) spatial derivatives on μ, it naturally requires to control terms such as $\Psi''(\varphi)$ and $\Psi'''(\varphi)$. However, derivatives of Ψ' fulfil the following growth bound conditions

$$\Psi''(s) \leq \mathrm{e}^{C(\Psi'(s)+1)}, \quad |\Psi'''(s)| \leq C\Psi''(s)^2. \tag{1.5.4}$$

This prevents the possibility to control $\Psi''(\varphi)$ or $\Psi'''(\varphi)$ in L^p-spaces in terms of $\Psi'(\varphi)$.

Coming back to our plan, the basic issues are uniqueness of weak solutions as well as their regularity propagation in time (at least in two space dimensions). The instantaneous smoothing effect of weak solutions is somehow expected due to the parabolic dissipative nature of the systems (cf. (1.5.1)). Besides, this property is closely related to the existence of strong solutions, namely solutions that satisfy the equations for almost every (x,t) in $\Omega \times (0,T)$. To this purpose, either estimates for the difference of solutions or higher order energy estimates involves coupling terms which require some regularity properties on $\boldsymbol{u}$ and φ, usually stronger than the ones obtained from energy estimates. Therefore, these questions are really connected with the difficulties mentioned above. Nonetheless, this obstacle will be overcome by controlling the difference of solutions in dual spaces, by deducing further bounds on $\boldsymbol{u}$ from the regularity $\varphi \in L^4(0,T;H^2(\Omega))$ (local case) or by exploiting the exact form of $\boldsymbol{u}$ (Hele–Shaw case).

A natural and important question from both the physical and mathematical viewpoints is whether the separation property from the pure phases takes place. This means that the phase parameter φ stays eventually within a suitable closed subset in $(-1,1)$. Accordingly, this implies that we have a complete mixing of the two fluids. More precisely, we investigate whether there exist $\delta > 0$ and $t^* > 0$ such that

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1-\delta, \quad \forall\, t \geq t^*.$$

The following distinction can be made:

- *Instantaneous* separation property. For any $\sigma > 0$, there exists $\delta = \delta(\sigma) > 0$ such that

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1-\delta, \quad \forall\, t \geq \sigma. \tag{1.5.5}$$

  This is the separation property occurs instantaneously. The aim is to prove that there exists $C = C(\sigma)$ such that

$$\|F'(\varphi(t))\|_{L^\infty(\Omega)} \leq C, \quad \forall\, t \geq \sigma.$$

  In both local and nonlocal case, this estimate can be obtained once the chemical potential μ is uniformly in time bounded in $L^\infty(\Omega)$. Unfortunately, after a first smoothing effect, the achieved regularity for μ is only $H^1(\Omega)$ uniformly in time. Therefore higher order estimates are needed. Despite the foregoing difficulties, this is possible in two space dimensions. Indeed, L^p-estimates of $\Psi''(\varphi)$ (cf. (1.5.4)) can be obtained taking advantage of the critical Trudinger–Moser inequality in $\mathbb{R}^2$. This argument might be extended to some particular three-dimensional geometries (see, e.g., [38] for rings). In addition, we mention that a quantitative estimate (from above) of δ can be found in terms of the parameters of the system such as the form of Ψ, the initial energy $\mathcal{E}(\varphi_0)$ and the total mass $\overline{\varphi}_0$. As a byproduct, dealing with dissipative systems, it is possible to infer that (1.5.5) holds uniformly with respect to the initial condition. More precisely, for any $m \in (-1, 1)$ and $R > 0$, there exist $\delta = \delta(m, R)$ and $t_0 = t_0(m, R)$ such that for any initial condition φ_0 satisfying $|\overline{\varphi}_0| \leq m$ and $\mathcal{E}(\varphi_0) \leq R$, we have

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall\, t \geq t_0. \tag{1.5.6}$$

- *Asymptotic* separation property. There exist $t_a > 0$ and $\delta > 0$ such that

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall\, t \geq t_a. \tag{1.5.7}$$

  In this case, the time t_a can be eventually large and it is not estimated in terms of the initial condition or the parameters of the system. The method consist in a compactness argument combined with the gradient structure of the system and the regularity of stationary points φ_∞ (which are separated from the pure phases). In particular, since a bound on $\|F'(\varphi_\infty)\|_{L^\infty}$ depends on μ_∞, that is a constant, this value can be estimated in terms of the initial energy $\mathcal{E}(\varphi_0)$ and $\overline{\varphi}_0$. In turn, as in the previous case, the parameter δ can be controlled from above in a quantitative way.

- *Local* separation property. There exist $t_f > 0$ and $\delta > 0$ such that

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall\, t \in [0, t_f]. \tag{1.5.8}$$

  In comparison with the previous cases, the separation property (1.5.8) is only valid on a finite time interval. To obtain this, the continuity of the solutions is needed. Indeed, assuming that φ_0 is such that $\|\varphi_0\|_{C(\overline{\Omega})} \leq 1 - \delta$ and φ is continuous in time and space, then there exists $t_f = t_f(\varphi_0)$ such that $\|\varphi(t)\|_{C(\overline{\Omega})} \leq 1 - \frac{\delta}{2}$, for all $t \in [0, t_f]$.

In literature the *instantaneous* separation property has been firstly proven for the local Cahn–Hilliard equation in dimension two in [118]. A different proof will be proposed in the next chapters. This argument is more flexible than the one used in [118] and can be easily extended to equations with convection. Concerning nonlocal problems, the *instantaneous* separation property has not been achieved before. Herein we will provide a first proof for the nonlocal convective Cahn–Hilliard equation. The *asymptotic* separation property has been shown for the local Cahn–Hilliard equation in [4] and then for the model H in [1]. It has been mainly used to show the convergence of a single trajectory to its equilibrium point. We mention that a nonlocal version will be treated in [70]. On the other hand, the *local* separation property has been used to prove the existence of exponential attractors for the local Cahn–Hilliard equation in [118] and [119]. Instead, we will use it to show the existence of strong solution (for the Hele–Shaw case) provided that the initial datum is regular enough and sufficiently close to a local minimizer. We remark that the asymptotic and the local separation properties usually hold in three space dimensions.

The final question we want to address is the longtime behavior of solutions. We will divide the analysis into two main classes. The first one involves the behavior of bunch of trajectories starting from a given bounded set of initial conditions. This will be treated within the theory of Infinite Dimensional Dissipative Dynamical System (see [120] and [149]). More precisely, we will show the existence of compact invariant attracting sets in the phase space. On the other hand, the second one deals with the behavior of the single solution as t approaches ∞. We will prove the convergence of solutions to equilibrium points of the system. In particular, both of these settings rely on the regularity properties and on the separation property which allows us to handle the nonlinearity as a globally Lipschitz function of the order parameter.

CHAPTER 2

Contribution of this thesis

In this chapter we provide an overview of the results accomplished in this thesis. We first explain the basic assumptions that we will use throughout this contribution. Next, the main achievements on the diffuse interface models presented in the Introduction are stated and compared with the existing literature.

2.1 Main assumptions

We consider a bounded domain (open and connected) $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$, with a smooth boundary $\partial\Omega$. Unless otherwise stated, a sufficient requirement is a $\mathcal{C}^4$-boundary. As anticipated in the Introduction, we assume in the sequel that the mobility function m and the interface thickness parameter ε are fixed equal to one, since they do not play a role in the subsequent analysis. Also, we accept the Boussinesq approximation, namely differences on fluid densities are neglected having set $\rho = 1$. Further requirements on the viscosity coefficient will be specified for any model under investigation. We recall that the variables are defined as follows:

$\boldsymbol{u} = \boldsymbol{u}(x,t) \in \mathbb{R}^d$ is the mean velocity,

$\pi = \pi(x,t) \in \mathbb{R}$ denotes the pressure,

$\varphi = \varphi(x,t) \in \mathbb{R}$ is the difference of concentrations (phase parameter),

$\mu = \mu(x,t) \in \mathbb{R}$ represents the chemical potential,

where $x \in \Omega$ and $t \in \mathbb{R}^+$.

Motivated by the physically relevant potential with logarithmic convex part, we suppose hereafter that a singular potential F is a function which fulfils the following assumption:

(H) F can be decomposed into the form

$$F(s) = \Psi(s) - \frac{\Theta_0}{2}s^2, \quad \forall\, s \in [-1,1], \tag{2.1.1}$$

where the function $\Psi : [-1,1] \mapsto \mathbb{R}$ satisfies $\Psi \in \mathcal{C}([-1,1]) \cap \mathcal{C}^2(-1,1)$,

$$\lim_{s\to -1^+} \Psi'(s) = -\infty, \quad \lim_{s\to 1^-} \Psi' s) = +\infty,$$
$$\Psi''(s) \geq \Theta > 0, \quad \forall\, s \in (-1,1),$$

with the constants Θ_0, Θ satisfying

$$\Theta_0 - \Theta := \alpha > 0.$$

Without loss of generality, we suppose $\Psi(0) = \Psi'(0) = 0$. We also make the extension that

$$\Psi(s) = +\infty, \quad \text{for all } |s| > 1.$$

As far as the interaction kernel is concerned, we assume that

(K) $J \in W^{1,1}(\mathbb{R}^d)$ such that

$$J(x) = J(-x).$$

In the sequel we will also refer to the following further requirements only when needed:

$(H.1)$ There exists $\kappa \in (0,1)$ such that Ψ'' is non-decreasing in $[1-\kappa, 1)$ and non-increasing in $(-1, -1+\kappa]$;

(H.2) $\Psi''(s)$ is a convex function in $(-1,1)$;

(H.3) There exists a positive constant C such that

$$\Psi''(s) \leq \mathrm{e}^{C|\Psi'(s)|+C}, \quad \forall\, s \in (-1,1);$$

(H.4) $\Psi \in \mathcal{C}^3(-1,1)$ and there exists a positive constant C such that

$$\Psi'(s)\Psi'''(s) \geq 0 \quad \text{and} \quad |\Psi'''(s)| \leq C\Psi''(s)^2, \quad \forall\, s \in (-1,1);$$

(H.5) $\Psi \in \mathcal{C}^4(-1,1)$ and there exists $\kappa \in (0,1)$ such that

$$\Psi'''(s)s \geq 0 \quad \text{and} \quad \Psi^{iv}(s) > 0, \quad \forall\, s \in (-1,-1+\kappa] \cap [1-\kappa, 1);$$

(K.1) $J \in W^{2,1}(B_\rho)$, where $B_\rho = \{x \in \mathbb{R}^d : |x| < \rho\}$ with ρ sufficiently large such that $\overline{\Omega} \subset B_\rho$ or J is admissible in the sense of [16].

Remark 2.1.1. All the assumptions $(\mathrm{H}.1) - (\mathrm{H}.5)$ are satisfied by the logarithmic potential

$$\Psi(s) = \frac{\Theta}{2}[(1+s)\log(1+s) + (1-s)\log(1-s)], \quad \forall\, s \in [-1,1].$$

Main examples of interaction kernels are Newtonian and Bessel potentials.

2.2 Results on the Navier–Stokes–Cahn–Hilliard–Oono system

We consider the Navier–Stokes–Cahn–Hilliard–Oono system with matched viscosities ($\nu = 1$) which takes the form

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \mu \nabla \varphi, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi + \beta(\varphi - c) = \Delta \mu, \\ \mu = -\Delta \varphi + F'(\varphi), \end{cases} \quad \text{in } \Omega \times (0,T), \tag{2.2.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = \mathbf{0}, \quad \partial_{\boldsymbol{n}} \mu = \partial_{\boldsymbol{n}} \varphi = 0, & \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot), \quad \varphi(\cdot,0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{2.2.2}$$

Here the parameter β and c satisfies the structural requirements

$$\beta \geq 0 \quad \text{and } c \in (-1,1).$$

In the literature, the Navier–Stokes–Cahn–Hilliard system ($\beta = 0$) has been extensively studied in the case of regular potentials. The potential F is replaced by polynomial-like double well functions such as, e.g., $F_0(s) = \frac{k}{4}(s^2 - 1)^2$. The well-posedness of strong solution in the whole space $\mathbb{R}^2$ was firstly proven in [145]. In a box domain of $\mathbb{R}^3$, the existence of weak solutions has been shown in [22] assuming unmatched viscosities. The author also proved the well-posedness of strong solutions which are global in time in two dimensions and local in time in three dimensions (see also [110] and [164]). In a general bounded domain of $\mathbb{R}^2$, the well-posedness and regularity of weak solutions have been shown in [71]. Strong solutions in two space dimension have been also obtained in [30] for the system with mixed partial viscosity and mobility. The existence of global and exponential attractors is discussed in [71]. A lower bound for the Hausdorff dimension of the global attractor is established in [73]. The same authors have been studied the system in a bounded domain of $\mathbb{R}^3$ within the framework of trajectory attractors in [72]. Regarding the longtime behavior of the single trajectory and, in particular, its convergence to equilibrium points, we mention [22], [71], [145], [164]. On the other hand, the Navier–Stokes–Cahn–Hilliard–Oono system ($\beta > 0$) in two dimensions two with regular potentials has been considered in [21]. The well-posedness and regularity of weak solutions as well as the existence of global and exponential attractors are provided. We also refer the reader to e.g. [12], [14], [58], [59], [77], [92], [97], [100], [103], [110], [133], [137], [138], [139], [147] and [163] for numerical analysis and simulations. Conversely, only few results are available on the model H with the physically relevant logarithmic potential. The Navier–Stokes–Cahn–Hilliard system with unmatched viscosities has been studied in [1]. First, the existence of weak solutions is shown. In particular, the phase parameter becomes instantaneously more regular. The weak solutions are unique under extra regularity assumptions on the solutions (conditional uniqueness) and become regular on the time interval (T, ∞), for some time $T > 0$. In two dimensions, the existence of global in time strong solution is demonstrated provided that $\varphi_0 \in H^2(\Omega)$ such that $-\Delta \varphi_0 + F'(\varphi_0) \in H^1(\Omega)$ and $\boldsymbol{u}_0$ is a bit more regular than

$\mathbf{H}^1(\Omega)$. An analogous result for local in time strong solutions hold in three-dimension. Finally, the author establishes the *asymptotic* separation property which allows to show that any single trajectory converges to an equilibrium. More recently, the existence of a weak solution has been extended to the Navier–Stokes–Cahn–Hilliard system with moving contact lines in [75] and to the Navier–Stokes–Cahn–Hilliard–Oono system in [117].

Our main result on system (2.2.1)-(2.2.2) is

Theorem 2.2.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Assume that $\boldsymbol{u}_0 \in \mathbf{L}^2(\Omega)$ such that $\operatorname{div} \boldsymbol{u}_0 = 0$, $\boldsymbol{u}_0 \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ and $\varphi_0 \in H^1(\Omega)$ with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Then, there exists a unique weak solution $(\boldsymbol{u}, \varphi)$ to* (2.2.1)-(2.2.2) *such that*

$$\begin{aligned}
&\boldsymbol{u} \in \mathcal{C}([0,\infty), \mathbf{L}^2(\Omega)),\\
&\varphi \in \mathcal{C}([0,\infty), H^1(\Omega)) \cap L^2_{loc}(0,\infty; W^{2,p}(\Omega)),
\end{aligned}$$

for any $2 \leq p < \infty$. Moreover, we have the following further results:

- *For any $\sigma > 0$, the weak solution satisfies*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(\sigma,\infty; \mathbf{H}^1(\Omega)) \cap L^2_{loc}(\sigma,\infty; \mathbf{H}^2(\Omega)) \cap H^1_{loc}(\sigma,\infty; \mathbf{L}^2(\Omega)),\\
&\varphi \in L^\infty(\sigma,\infty; W^{2,p}(\Omega)) \cap H^1_{loc}(\sigma,\infty; H^1(\Omega)),
\end{aligned}$$

  for any $2 \leq p < \infty$.

- *Suppose that $\Psi \in \mathcal{C}^3(-1,1)$ and* (H.2), (H.3) *hold. Then, for any $\sigma > 0$, there exists $\delta > 0$ such that*

$$\sup_{t \geq 2\sigma} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta,$$

  and

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(2\sigma,\infty; \mathbf{H}^2(\Omega)) \cap H^1_{loc}(2\sigma,\infty; \mathbf{H}^1(\Omega)),\\
&\varphi \in L^\infty(2\sigma,\infty; H^4(\Omega)) \cap H^1_{loc}(2\sigma,\infty; H^2(\Omega)).
\end{aligned}$$

In summary, Theorem 2.2.1 asserts uniqueness and regularity properties of weak solutions for the Navier–Stokes–Cahn–Hilliard–Oono system in two dimensions. The key idea for the proof of uniqueness is a continuous dependence estimate in a dual norm which allows us to handle the so-called Korteweg force. Taking $\beta = 0$, we also obtain a uniqueness result for the Navier–Stokes–Cahn–Hilliard system (model H with matched densities). For any $\sigma > 0$, we show that any weak solution is more regular on (σ, ∞), and the *instantaneous* separation property holds on $[2\sigma, \infty)$. In particular, the abovementioned regularity properties derive from a priori higher order estimates, which are uniform with respect to the initial datum and independent of β. Therefore, they can be employed to characterize the longtime behavior.

2.3 Results on the Hele–Shaw–Cahn–Hilliard system

The Hele–Shaw–Cahn–Hilliard system with matched viscosities ($\nu = 1$) reads as follows

$$\begin{cases} \boldsymbol{u} = -\nabla\pi + \mu\nabla\varphi, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi = \Delta\mu, \\ \mu = -\Delta\varphi + F'(\varphi), \end{cases} \qquad \text{in } \Omega\times(0,T), \tag{2.3.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u}\cdot\boldsymbol{n} = \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}}\varphi = 0, & \text{on } \partial\Omega\times(0,T), \\ \varphi(\cdot,0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{2.3.2}$$

Although the Hele–Shaw–Cahn–Hilliard system has been proposed as a simplification of the Navier–Stokes–Cahn–Hilliard system, it does not appear to be much simpler from the mathematical viewpoint. Most of the available papers are rather recent and mainly treat the regular potential case. In a periodic setting, existence and uniqueness of global in time classical solutions in dimension two and the existence of local in time strong solutions along with certain blow-up criteria in dimension three were proven in [159]. In a rectangle or a box, existence, uniqueness and regularity of global in time two-dimensional (or local in time three-dimensional) strong solutions were established in [113]. The longtime behavior of global solutions and the stability of local energy minimizers in both two and three dimensions were analyzed in [158]. The sharp-interface limit of (2.3.1) has been investigated quite recently in [57] and [116]. More recently, in [99] the authors analyse a variant of (2.3.1) with regular potential and $\operatorname{div} \boldsymbol{u} = S$, where S is a given space-time dependent mass source that also appears as a forcing term in the Cahn–Hilliard equation. The existence of global in time weak solutions and local in time strong solutions is proven. In addition, the authors investigate the longtime behavior in dimension two (pullback attractor and convergence to single equilibrium). In the case of singular potential, some results on the existence of weak solutions for tumor growth systems can be found in [45] and [67]. The system (2.3.1)-(2.3.2) is therein coupled with other equations describing proliferating tumor cells and nutrient concentrations. Regarding the numerical analysis, we mention [60] and [160] (cf. also [46]).

Our main result on system (2.3.1)-(2.3.2) is

Theorem 2.3.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^d$, $d = 2,3$. Assume that* (H.1) *holds and $\varphi_0 \in H^1(\Omega)$ with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Then, there exists at least a weak solution $(\boldsymbol{u},\varphi)$ to* (2.3.1)-(2.3.2) *such that*

$$\begin{aligned} &\boldsymbol{u} \in L^r_{loc}(0,\infty;\mathbf{H}^1(\Omega)) \cap L^2_{loc}(0,\infty;\mathbf{L}^2(\Omega)), \\ &\varphi \in \mathcal{C}([0,\infty),H^1(\Omega)) \cap L^4_{loc}(0,\infty;H^2(\Omega)) \cap L^2_{loc}(0,\infty;W^{2,p}(\Omega)), \end{aligned}$$

where $r = \frac{6}{5}$ if $d = 3$ and for any $r \in [1,\frac{4}{3})$ if $d = 2$, $p = 6$ if $d = 3$ and for any $2 \le p < \infty$ if $d = 2$. Moreover, we have the following further results:

- *Let* $d = 2$. *Then, the weak solution is unique and, for any* $\sigma > 0$, *we have*

$$\begin{aligned}&\boldsymbol{u} \in L^\infty(\sigma, \infty; \mathbf{H}^1(\Omega)),\\&\varphi \in L^\infty(\sigma, \infty; W^{2,p}(\Omega)) \cap H^1_{loc}(\sigma, \infty; H^1(\Omega)),\end{aligned}$$

where $2 \le p < \infty$. *In addition, suppose that* $\Psi \in \mathcal{C}^3(-1, 1)$ *and* (H.2), (H.3) *hold. Then, for any* $\sigma > 0$, *there exists* $\delta > 0$ *such that*

$$\sup_{t \ge 2\sigma} \|\varphi(t)\|_{L^\infty(\Omega)} \le 1 - \delta,$$

and

$$\begin{aligned}&\boldsymbol{u} \in L^\infty(2\sigma, \infty; \mathbf{H}^2(\Omega)),\\&\varphi \in L^\infty(2\sigma, \infty; H^4(\Omega)) \cap H^1_{loc}(2\sigma, \infty; H^2(\Omega)).\end{aligned}$$

- *Let* $d = 3$ *and* (H.5) *hold. Suppose that* $\psi \in H^1(\Omega)$ *such that* $\Psi(\psi) \in L^1(\Omega)$, $|\overline{\psi}| < 1$ *and is a local energy minimizer of* $\mathcal{E}_{GL}$. *Then, for any* $\epsilon > 0$, *there exists a constant* $\eta \in (0, 1)$ *such that for an arbitrary initial datum* $\varphi_0 \in H^3(\Omega)$ *satisfying* $\partial_{\boldsymbol{n}}\varphi_0 = 0$ *on* $\partial\Omega$, $\overline{\varphi}_0 = \overline{\psi}$ *and* $\|\varphi_0 - \psi\|_{H^2(\Omega)} \le \eta$, *problem* (2.3.1)–(2.3.2) *has a unique global strong solution* $(\boldsymbol{u}, \varphi)$ *such that*

$$\begin{aligned}&\boldsymbol{u} \in \mathcal{C}([0, +\infty), \mathbf{L}^2) \cap L^2_{loc}(0, +\infty; \mathbf{H}^3(\Omega)),\\&\varphi \in \mathcal{C}([0, +\infty), H^3(\Omega)) \cap L^2_{loc}(0, +\infty; H^5(\Omega)) \cap H^1_{loc}(0, +\infty; V).\end{aligned}$$

and

$$\|\varphi(t) - \psi\|_{H^2(\Omega)} \le \epsilon, \quad \forall\, t \ge 0.$$

Theorem (2.3.1) provides a fairly complete analysis of the initial boundary value problem (2.3.1)–(2.3.2) with singular potential. First, we establish the existence of a global in time weak solution with finite energy. In two dimensions, we demonstrate the uniqueness of weak solutions $(\boldsymbol{u}, \varphi)$ along with a continuous dependence estimate. The goal is achieved due to the integrability properties $\varphi \in L^\infty(\Omega \times (0, \infty))$ and $\varphi \in L^4_{loc}(0, \infty; H^2(\Omega))$ within the class of global weak solutions. We recall that the same problem remains an open issue for the case with regular potentials. Then, we show that weak solutions become more regular, in particular, we prove the validity of the *instantaneous* separation property. Moreover, a similar argument easily yields the existence of a unique global strong solution to (2.3.1)–(2.3.2) for arbitrary regular initial datum $\varphi_0 \in H^2(\Omega)$ such that $-\Delta\varphi_0 + F'(\varphi_0) \in H^1(\Omega)$. On the other hand, the existence of strong solutions in three dimensions is a hard task due to the low regularity of the velocity field that satisfies a Darcy's law. This complexity has been partially overcome by requiring that the initial datum is regular and sufficiently close to a local energy minimizer of the Ginzburg–Landau free energy. Combining this choice on the initial datum with the Łojasiewicz–Simon approach, we prove the existence of a global in time unique strong solution $(\boldsymbol{u}, \varphi)$ such that φ stays in a small neighborhood of that minimizer for all $t \ge 0$. This immediately yields a local Lyapunov stability property for any local energy minimizer of $\mathcal{E}_{GL}$.

2.4 Results on the Brinkman–Cahn–Hilliard system

We consider the Brinkman–Cahn–Hilliard system with unmatched viscosities

$$\begin{cases} -\operatorname{div}(\nu(\varphi)D\boldsymbol{u}) + \boldsymbol{u} + \nabla\pi = \mu\nabla\varphi, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \partial_t\varphi + \boldsymbol{u}\cdot\varphi = \Delta\mu, \\ \mu = -\Delta\varphi + F'(\varphi), \end{cases} \qquad \text{in } \Omega\times(0,T), \tag{2.4.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = \mathbf{0}, \quad \partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\mu = 0, & \text{on } \partial\Omega\times(0,T), \\ \varphi(\cdot,0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{2.4.2}$$

Here, the viscosity coefficient $\nu \in \mathcal{C}^2(\mathbb{R})$ (depending on φ) is a bounded function that satisfies

$$\nu(s) \geq 2\nu_1 > 0, \quad \forall\, s \in \mathbb{R}.$$

In addition, the permeability coefficient η is taken equal to one for simplicity.

Remark 2.4.1. We recall that the local viscosity form is a linear combination of viscosity coefficients defined by

$$\nu(s) = \nu_A\frac{1+s}{2} + \nu_B\frac{1-s}{2}, \quad \forall\, s \in [-1,1], \tag{2.4.3}$$

where ν_A and ν_B are positive. It is clear that, up to a suitable extension on $[-1,1]^c$, the expression (2.4.3) complies with our assumptions.

In the literature, the Brinkman–Cahn–Hilliard system has been studied with a regular potential of the form $F_0(s) = \frac{k}{4}(s^2-1)^2$. In both two and three dimensions, the well-posedness of weak solutions and their regularity properties have been discussed in [20], setting ν as a positive constant. The authors also address the longtime behavior of the system. Then, the finite dimensionality of the global attractor has been proved in [108]. Similar results have been obtained in [162] taking into account a dynamic boundary condition for φ. Finally, the case of nonconstant ν, η and m has been analyzed from the numerical viewpoint in [41] (see also [49]).

Our main result on system (2.4.1)-(2.4.2) reads as follows

Theorem 2.4.2. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Assume that $\varphi_0 \in H^1(\Omega)$ with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Then, there exists a unique weak solution $(\boldsymbol{u},\varphi)$ to* (2.4.1)-(2.4.2) *such that*

$$\boldsymbol{u} \in L^4_{loc}(0,\infty;\mathbf{H}^1(\Omega)) \cap L^2_{loc}(0,\infty;\mathbf{W}^{1,3}(\Omega)),$$
$$\varphi \in \mathcal{C}([0,\infty),H^1(\Omega)) \cap L^4_{loc}(0,\infty;H^2(\Omega)) \cap L^2_{loc}(0,\infty;W^{2,p}(\Omega)),$$

where $2 \leq p < \infty$. Moreover, we have the following further results:

- *For any $\sigma > 0$, the weak solution satisfies*

$$\boldsymbol{u} \in L^\infty(\sigma,\infty;\mathbf{H}^1(\Omega)) \cap L^2_{loc}(0,\infty;\mathbf{H}^2(\Omega)),$$
$$\varphi \in L^\infty(\sigma,\infty;W^{2,p}(\Omega)) \cap H^1_{loc}(\sigma,\infty;H^1(\Omega)),$$

  where $2 \leq p < \infty$.

- *Suppose that* $\Psi \in \mathcal{C}^3(-1,1)$ *and* (H.2), (H.3) *hold. Then, for any* $\sigma > 0$*, there exists* $\delta > 0$ *such that*
$$\sup_{t \geq 2\sigma} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta,$$
and
$$\begin{aligned}&\boldsymbol{u} \in L^\infty(2\sigma, \infty; \mathbf{H}^2(\Omega)),\\ &\varphi \in L^\infty(2\sigma, \infty; H^4(\Omega)) \cap H^1_{loc}(2\sigma, \infty; H^2(\Omega)).\end{aligned}$$

A complete analysis of well-posedness and regularity for the system (2.4.1)–(2.4.2) in dimension two is provided by Theorem 2.4.2. First, we show the existence of weak solutions $(\boldsymbol{u}, \varphi)$ such that $\varphi \in L^4_{loc}(0, \infty; H^2(\Omega))$. As a consequence, by studying the Stokes problem with variable viscosity (see [1]), we obtain higher regularity properties for $\boldsymbol{u}$, such as $\boldsymbol{u} \in L^4_{loc}(0, \infty; H^1(\Omega)) \cap L^2_{loc}(0, \infty; W^{1,3}(\Omega))$. These integrability properties are the key tools to obtain a continuous dependence estimate with respect to the initial datum and higher order estimates on the solution. Finally, we prove the *instantaneous* separation property and some regularity consequences. It is worth mentioning that the statement of Theorem 2.4.2 can be easily generalized to the case with positive permeability depending on φ.

2.5 Results on the nonlocal Navier–Stokes–Cahn–Hilliard system

We consider the nonlocal Navier–Stokes–Cahn–Hilliard system with matched viscosities ($\nu = 1$) that reads as
$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \mu \nabla \varphi, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi = \Delta \mu, \\ \mu = \Psi'(\varphi) - J * \varphi, \end{cases} \quad \text{in } \Omega \times (0,T), \tag{2.5.1}$$
subject to the boundary and initial conditions
$$\begin{cases} \boldsymbol{u} = \mathbf{0}, \quad \partial_{\boldsymbol{n}} \mu = 0, & \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0(\cdot), \quad \varphi(\cdot, 0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{2.5.2}$$

The nonlocal version of the Navier–Stokes–Cahn–Hilliard system has been investigated in recent years. First of all, in the case of regular potentials, the existence of a weak solution in two and three dimensions has been shown in [39]. The same result has been proven for nonconstant mobilities in [66]. In two dimensional domains, uniqueness of weak solutions has been achieved in [62]. Regularity of weak solutions and existence of global in time strong solutions have been established in [65]. Moreover, the asymptotic behavior has been studied in [62], [63] and [65]. In the case of unmatched densities, we mention that the existence of global in time strong solutions and the weak-strong uniqueness have been demonstrated in [62]. On the other hand, in the case of a singular potential, the existence of weak solutions has been shown in [64]. Uniqueness of weak solutions in two dimensions hes been recently proven in [62].

Our main result on system (2.5.1)-(2.5.2) is

Theorem 2.5.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Assume that $\boldsymbol{u}_0 \in L^2(\Omega)$ such that $\operatorname{div} \boldsymbol{u}_0 = 0$, $\boldsymbol{u}_0 \cdot \boldsymbol{n} = 0$ and φ_0 is a measurable function such that $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Then, for any $\sigma > 0$, the (unique) weak solution $(\boldsymbol{u}, \varphi)$ to* (2.5.1)-(2.5.2) *satisfies*

$$\boldsymbol{u} \in L^\infty(\sigma, \infty; \mathbf{H}^1(\Omega)) \cap L^2_{loc}(\sigma, \infty; \mathbf{H}^2(\Omega)) \cap H^1_{loc}(\sigma, \infty; \mathbf{L}^2(\Omega)),$$
$$\varphi \in L^\infty(\sigma, \infty; H^1(\Omega)) \cap L^q_{loc}(0, \infty; W^{1,p}(\Omega)) \cap H^1_{loc}(\sigma, \infty; L^2(\Omega)),$$

where $\frac{p-2}{p} = \frac{2}{q}$. In addition, suppose that (H.3) *and* (H.4) *hold. Then, for any $\sigma > 0$, there exists $\delta > 0$ such that*

$$\sup_{t \geq 2\sigma} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta,$$

and

$$\boldsymbol{u} \in L^\infty(3\sigma, \infty; \mathbf{W}^{1,4}(\Omega)),$$
$$\varphi \in \mathcal{C}^{\beta, \frac{\beta}{2}}(\overline{\Omega} \times (4\sigma, \infty)),$$

for some $\beta \in (0,1)$.

The regularity properties and, in particular, the *instantaneous* separation property are provided by Theorem 2.5.1 in dimension two. Due to the second order nature of the system, we employ a different argument from the one proposed for local models. Indeed, even if we deduce a L^p-bounds on $\Psi''(\varphi)$ and $\Psi'''(\varphi)$ via the Trudinger–Moser inequality, higher order energy estimates seem out of reach without a control of $\Psi''(\varphi)$ in $L^\infty(\Omega)$. Nonetheless, we overcome this obstacle by performing a suitable Alikakos–Moser iterative argument. This is the first result regarding the separation property in the nonlocal setting. As a byproduct, we reach further regularity properties on $\boldsymbol{u}$ and φ. We mention that the existence of global in time strong solutions can be easily inferred by Theorem 2.5.1 provided that the initial condition also satisfies $\nabla \Psi'(\varphi_0) \in L^2(\Omega)$.

2.6 Results on the nonlocal Hele–Shaw–Cahn–Hilliard system

We recall that the nonlocal Hele–Shaw–Cahn–Hilliard system with matched viscosities ($\nu = 1$) reads as follows

$$\begin{cases} \boldsymbol{u} = -\nabla \pi + \mu \nabla \varphi, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi = \Delta \mu, \\ \mu = \Psi'(\varphi) - J * \varphi, \end{cases} \quad \text{in } \Omega \times (0,T), \tag{2.6.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = \partial_{\boldsymbol{n}} \mu = 0, & \text{on } \partial\Omega \times (0,T), \\ \varphi(\cdot, 0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{2.6.2}$$

In the literature, this system has been only studied in the case of regular potential in [47]. A nonlocal version of the Brinkman–Cahn–Hilliard system is also considered n the same paper. The authors prove the existence of a weak solution (unmatched densities case) provided that the initial condition belongs to $L^\infty(\Omega)$. Assuming that the viscosity coefficient is constant, the uniqueness of weak solutions is also achieved.

Our main result on system (2.6.1)-(2.6.2) is

Theorem 2.6.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^d$, $d = 2, 3$. Assume that φ_0 is a measurable function such that $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Then, there exists a unique weak solution $(\boldsymbol{u}, \varphi)$ to* (2.6.1)-(2.6.2) *such that*

$$
\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,\infty; \mathbf{L}^p(\Omega)) \cap L^2_{loc}(0,\infty; \mathbf{H}^1(\Omega)),\\
&\varphi \in L^2_{loc}(0,\infty; H^1(\Omega)).
\end{aligned}
$$

In addition, we have the following further results:

- *Let $\nabla\Psi'(\varphi_0) \in L^2(\Omega)$. Then, there exists a unique strong solution $(\boldsymbol{u}, \varphi)$ to* (2.6.1)-(2.6.2) *such that*

$$
\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,\infty; \mathbf{H}^1(\Omega)) \cap L^{\frac{8}{d}}_{loc}(0,\infty; \mathbf{W}^{1,4}(\Omega)) \cap L^2_{loc}(0,\infty; \mathbf{W}^{1,p}(\Omega)),\\
&\varphi \in L^\infty(0,\infty; H^1(\Omega)) \cap L^{\frac{8}{d}}_{loc}(0,\infty; W^{1,4}(\Omega)) \cap L^2_{loc}(0,\infty; W^{1,p}(\Omega)),
\end{aligned}
$$

  where $4 < p < \infty$ if $d = 2$ and $4 < p \leq 6$ if $d = 3$.

- *For any $\sigma > 0$, the weak solution is a strong solution on (σ, ∞).*

- *Let $d = 2$ and* (H.3), (H.4) *and* (K.1) *hold. Then, for every $\sigma > 0$, there exists $\delta > 0$ such that*

$$
\sup_{t \geq 2\sigma} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta,
$$

  and

$$
\begin{aligned}
&\boldsymbol{u} \in L^\infty(3\sigma,\infty; \mathbf{H}^2(\Omega)) \cap H^1_{loc}(3\sigma,\infty; \mathbf{L}^2(\Omega)),\\
&\varphi \in L^\infty(3\sigma,\infty; H^2(\Omega)) \cap H^1_{loc}(3\sigma,\infty; L^2(\Omega)).
\end{aligned}
$$

The above result states the well-posedness of weak and strong solutions. In comparison with the local Hele–Shaw–Cahn–Hilliard system, some regularity results and existence of strong solutions for general initial conditions can be proven also in three dimensions. Besides, we have the validity of the *instantaneous* separation property in dimension two. This is proved by exploiting the Alikakos–Moser argument employed for the nonlocal Navier–Stokes–Cahn–Hilliard system.

CHAPTER 3

Mathematical preliminaries

This chapter is devoted to some basic mathematical tools we will use repeatedly in the rest of this contribution.

3.1 Framework of Sobolev spaces

Let X be a (real) Banach or Hilbert space, whose norm is denoted by $\|\cdot\|_X$. X' denotes the dual space of X and $\langle\cdot,\cdot\rangle$ the corresponding duality product. The boldface letter $\mathbf{X}$ stands for the vectorial space X^d endowed with the product structure (d is the spatial dimension). We denote by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, $k\in\mathbb{N}$ and $p\in[1,+\infty]$, the Lebesgue spaces and Sobolev spaces of real measurable functions on the domain Ω. We introduce the Hilbert spaces $H^k(\Omega)=W^{k,2}(\Omega)$ with respect to the inner product

$$(u,v)_k=\sum_{|\zeta|\leq k}\int_\Omega D^\zeta u(x)D^\zeta v(x)\,\mathrm{d}x,$$

where $\zeta=(\zeta_1,...,\zeta_d)$ is a multi-index, and the induced norm $\|u\|_{H^k(\Omega)}=\sqrt{(u,u)_k}$. We set $H=L^2(\Omega)$ with inner product and associate norm denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We denote $V=H^1(\Omega)$ endowed with the norm

$$\|f\|_V^2=\|\nabla f\|^2+\|f\|^2.$$

Its dual space $V'=(H^1(\Omega))'$ is endowed with the standard dual nor. Given an interval I of $\mathbb{R}^+$, we introduce the function space $L^p(I;X)$ with $p\in[1,+\infty]$, which consists of Bochner measurable p-integrable functions with values in the Banach space X.

We report classical results within the theory of Sobolev spaces concerning embeddings, differentiation of products and compositions, interpolation inequalities. These can be found in literature in [5], [26], [27], [115], [122], [151].

◇ **Embeddings**. Let $u \in W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$. We have the continuous embeddings

$$\begin{aligned}
&W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), &&\text{where } \frac{1}{q} = \frac{1}{p} - \frac{1}{d}, &&\text{if } p < d;\\
&W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), &&\forall\, q \in [p, \infty), &&\text{if } p = d;\\
&W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^\alpha(\overline{\Omega}), &&\text{where } \alpha = 1 - \frac{d}{p}, &&\text{if } p > d.
\end{aligned}$$

◇ **Products**. Let $f, g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $1 \leq p \leq \infty$. Then, the product $fg \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}, \quad i = 1, ..., d.$$

In particular, there exists $C > 0$ such that

$$\|fg\|_V \leq C\Big(\|f\|_{L^\infty(\Omega)}\|g\|_V + \|f\|_V\|g\|_{L^\infty(\Omega)}\Big), \tag{3.1.1}$$

for all $f, g \in V \cap L^\infty(\Omega)$, and

$$\|fg\|_{H^2(\Omega)} \leq C\Big(\|f\|_{L^\infty(\Omega)}\|g\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)}\|g\|_{L^\infty(\Omega)}\Big), \tag{3.1.2}$$

for all $f, g \in H^2(\Omega)$ ($d = 2, 3$). Moreover, we have the following inequality

$$\|fg\|_{W^{1,p}(\Omega)} \leq C\Big(\|f\|_{L^\infty(\Omega)}\|g\|_{W^{1,p}(\Omega)} + \|f\|_{W^{1,q}(\Omega)}\|g\|_{L^r(\Omega)}\Big), \tag{3.1.3}$$

for all $f \in W^{1,q} \cap L^\infty(\Omega), g \in W^{1,p}(\Omega) \cap L^r(\Omega)$ provided that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Accordingly, there exists $C > 0$ such that

$$\|fg\|_V \leq C\Big(\|f\|_{L^\infty(\Omega)}\|g\|_V + \|f\|_{W^{1,4}(\Omega)}\|g\|_{L^4(\Omega)}\Big), \tag{3.1.4}$$

for all $f \in W^{1,4}(\Omega)$, $g \in V$.

◇ **Composition**. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly Lipschitz function and $u \in W^{1,p}(\Omega)$, with $1 \leq p < \infty$. Then, the composition $f \circ u \in W^{1,p}(\Omega)$ and

$$\nabla(f \circ u) = (f' \circ u)\nabla u, \quad \text{a.e. in } \Omega.$$

◇ **Poincaré–Wirtinger inequality**. For every $f \in V'$, we denote by $\overline{f}$ the average of the function f over Ω such that

$$\overline{f} = \frac{1}{|\Omega|}\langle f, 1\rangle.$$

Then, there exists $C > 0$ such that

$$\|f - \overline{f}\| \leq C\|\nabla f\|, \quad \forall\, f \in V. \tag{3.1.5}$$

As a byproduct, we have that $f \to (\|\nabla f\|^2 + |\overline{f}|^2)^{\frac{1}{2}}$ is an equivalent norm on V.

- ◇ **Trudinger–Moser inequality**. Let $d = 2$. Then, there exists $C > 0$ such that

$$\int_\Omega e^{|f|}\,\mathrm{d}x \le C e^{C\|f\|_V^2}, \quad \forall\, f \in V. \tag{3.1.6}$$

- ◇ **Ladyzhenskaya's inequality**. Let $d = 2$. Then, there exists $C > 0$ such that

$$\|f\|_{L^4(\Omega)} \le C\|f\|^{\frac12}\|f\|_V^{\frac12}, \quad \forall\, f \in V. \tag{3.1.7}$$

- ◇ **Agmon's inequality**. There exists $C > 0$ such that, in the case $d = 2$,

$$\|f\|_{L^\infty(\Omega)} \le C\|f\|^{\frac12}\|f\|_{H^2(\Omega)}^{\frac12}, \quad \forall\, f \in H^2(\Omega), \tag{3.1.8}$$

and, in the case $d = 3$,

$$\|f\|_{L^\infty(\Omega)} \le C\|f\|_V^{\frac12}\|f\|_{H^2(\Omega)}^{\frac12}, \quad \forall\, f \in H^2(\Omega). \tag{3.1.9}$$

- ◇ **Gagliardo–Nirenberg inequality**. If j, m are arbitrary integers satisfying $0 \le j < m$ and $\frac{j}{m} \le a \le 1$, and $1 \le q, r \le +\infty$ such that

$$\frac1p - \frac{j}{d} = a\left(\frac1r - \frac{m}{d}\right) + (1-a)\frac1q.$$

Then, there exists $C > 0$ such that

$$\|D^j f\|_{L^p(\Omega)} \le C\|f\|_{L^q(\Omega)}^{1-a}\|f\|_{W^{m,r}(\Omega)}^a, \quad \forall\, f \in W^{m,r}(\Omega) \cap L^q(\Omega).$$

If $1 < r < +\infty$ and $m - j - \frac{d}{r}$ is a nonnegative integer, then the above inequality holds only for $\frac{j}{m} \le a < 1$. Particular cases are

$$\|f\|_{L^p(\Omega)} \le C\|f\|^{\frac2p}\|f\|_V^{1-\frac2p}, \quad \forall f \in V, \tag{3.1.10}$$

with $p \ge 2$ and $d = 2$,

$$\|f\|_{L^p(\Omega)} \le C\|f\|^{\frac{6-p}{2p}}\|f\|_V^{\frac{3p-6}{2p}}, \quad \forall f \in V, \tag{3.1.11}$$

with $2 \le p \le 6$ and $d = 3$.

An application of the Gagliardo–Nirenberg inequality in dimension two is the following

Lemma 3.1.1. *Let $d = 2$ and $f \in V$. Then, for any $0 < \varepsilon \le 1$ and any $1 < s < \infty$, there exists $C = C(s) > 0$ such that*

$$\|u\|_{L^s(\Omega)}^2 \le \varepsilon\|\nabla u\|^2 + \frac{C}{\varepsilon^{s-1}}\|u\|_{L^1(\Omega)}^2. \tag{3.1.12}$$

Proof. We start from the following particular case of Gagliardo–Nirenberg inequality in dimension two

$$\|f\|_{L^s(\Omega)} \le C\|f\|_{L^1(\Omega)}^{\frac1s}\|f\|_V^{1-\frac1s},$$

for any $1 < s < \infty$. Exploiting (3.1.5), for any $\varepsilon > 0$, we deduce that

$$\|f\|_{L^s(\Omega)} \leq C\varepsilon^{\frac{s-1}{s}} \left(\|\nabla f\| + \|f\|_{L^1(\Omega)}\right)^{1-\frac{1}{s}} \left(\varepsilon^{\frac{1-s}{s}} \|f\|_{L^1(\Omega)}^{\frac{1}{s}}\right).$$

Applying the Young inequality with exponents $(s/(s-1), s)$, we obtain

$$\begin{aligned}\|f\|_{L^s(\Omega)} &\leq \varepsilon \left(\|\nabla f\| + \|f\|_{L^1(\Omega)}\right) + \frac{C}{\varepsilon^{s-1}}\|\|\|_{L^1(\Omega)} \\ &= \varepsilon\|\nabla f\| + \frac{C}{\varepsilon^{s-1}}\|f\|_{L^1(\Omega)}.\end{aligned}$$

Then, by rescaling ε with $\sqrt{\varepsilon}$, we easily infer the claim. $\square$

3.2 Approximation of the logarithmic potential

Let us consider the singular potential Ψ. According to (H), it is immediate to prove that Ψ is proper, convex and lower semicontinuous with domain $D(\Psi) = [-1, 1]$. Appealing to theory of maximal monotone operators (see, for instance, [15], [25], [141] and references therein), we define the subgradient of Ψ as

$$\mathbb{A} = \partial\Psi : D(\mathbb{A}) \subset \mathbb{R} \to \mathbb{R}.$$

We report here below a result which establishes the action of the subgradient operator on regular points (see [15, Chapter 1, Example 3]).

Lemma 3.2.1. *Let $\varphi : \mathbb{R} \to (-\infty, +\infty]$ be convex and differentiable at a point $s \in \mathbb{R}$. Then $\partial\varphi(s) = \varphi'(s)$.*

Since Ψ is continuously differentiable in $(-1, 1)$, we infer that

$$\mathbb{A}(s) = \Psi'(s), \quad \forall\, s \in (-1, 1), \tag{3.2.1}$$

where Ψ' stands for the standard derivative of Ψ. Moreover, we also have the following characterization.

Lemma 3.2.2. *Let the potential Ψ satisfy* (H). *Then,* $D(\mathbb{A}) \equiv (-1, 1)$.

Proof. From [141, Corollary 1.4, Chapter 4], we notice that

$$(-1, 1) \subset D(\mathbb{A}) \subset D(\Psi) = [-1, 1].$$

We suppose by contradiction that $1 \in D(\mathbb{A})$ and we consider $z \in \mathbb{A}(1) \subset \mathbb{R}$. It is immediate to see that $1 + z \in 1 + \mathbb{A}(1) = (I + \mathbb{A})(1)$. Besides, the map

$$g : (-1, 1) \to \mathbb{R}, \quad g(s) = (I + \mathbb{A})(s) = s + \Psi'(s)$$

is continuous, $\lim_{s\to 1^-} g(s) = +\infty$ and $\lim_{s\to -1^+} g(s) = -\infty$. Then, the range of g is $\mathbb{R}$. Thus, there exists $\overline{s} \in (-1, 1)$ such that $g(\overline{s}) = 1 + z$. Since $\mathbb{A}$ is a maximal monotone operator, the inclusion $1+z \in (I+\mathbb{A})s$ has at most one solution, so $1 \notin D(\mathbb{A})$. Repeating the same argument for -1, we conclude that $D(\mathbb{A}) = (-1, 1)$. $\square$

Thanks to the properties of maximal monotone operators (see for instance [25] and [141]), we approximate Ψ by means of the sequence of everywhere defined non-negative functions

$$\Psi_\lambda(s) = \frac{\lambda}{2}|\mathbb{A}_\lambda s|^2 + \Psi(J_\lambda(s)), \quad \forall\, s \in \mathbb{R},\ \lambda > 0, \tag{3.2.2}$$

where $J_\lambda = (I + \lambda\mathbb{A})^{-1}$ is the resolvent operator and $\mathbb{A}_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ is the Yosida approximation of $\mathbb{A}$. According to the general theory, the following main properties holds:

(i) Ψ_λ is convex and $\Psi_\lambda(s) \nearrow \Psi(s)$, for all $s \in \mathbb{R}$, as λ goes to 0;

(ii) $\Psi'_\lambda(s) = \mathbb{A}_\lambda(s)$ and Ψ'_λ is Lipschitz on $\mathbb{R}$ with constant $\frac{1}{\lambda}$;

(iii) $|\Psi'_\lambda(s)| \nearrow |\Psi'(s)|$ for all $s \in (-1, 1)$ and $|\Psi'_\lambda(s)| \nearrow \infty$, for all $|s| \geq 1$, as λ goes to 0;

(iv) $\Psi_\lambda(0) = \Psi'_\lambda(0) = 0$, for all $\lambda > 0$.

Remark 3.2.3. We recall that, due to the convexity of Ψ_λ (cf. (i)), we have

$$\Psi_\lambda(s) \leq \Psi_\lambda(r) + (s - r)\Psi'_\lambda(s), \quad \forall\, s, r \in \mathbb{R}. \tag{3.2.3}$$

Now we prove some properties of Ψ_λ which are uniform with respect to λ.

Lemma 3.2.4. *For any $0 < \lambda \leq 1$ and for any $s \in \mathbb{R}$, $\Psi''_\lambda(s)$ exists and satisfies*

$$\Psi''_\lambda(s) \geq \frac{\Theta}{1 + \Theta}. \tag{3.2.4}$$

Proof. We preliminarily note that J_λ is the inverse function of $g_\lambda(s) = (I + \lambda\mathbb{A})(s) : (-1, 1) \to \mathbb{R}$ which is differentiable with $g'_\lambda(s) \geq 1 + \lambda\Theta > 0$. This entails that $\mathbb{A}_\lambda$ is differentiable in $\mathbb{R}$. From the differentiation formula of the inverse function and the assumption (H), we deduce that

$$\Psi''_\lambda(s) = \frac{1}{\lambda}\left[1 - \frac{1}{1 + \lambda\Psi''(J_\lambda(s))}\right] \geq \frac{\Theta}{1 + \lambda\Theta}, \tag{3.2.5}$$

which, in turn, implies (3.2.4). □

Lemma 3.2.5. *For any $0 < \lambda^* \leq 1$, we have*

$$\Psi_\lambda(s) \geq \frac{1}{4\lambda^*}s^2 - C, \quad \forall\, s \in \mathbb{R},\ 0 < \lambda \leq \lambda^*, \tag{3.2.6}$$

where C depends only on λ^* but is independent of λ.

Proof. We infer from de L'Hôpital's rule that

$$\lim_{s \to \pm\infty} \frac{\Psi_\lambda(s)}{s^2} = \lim_{s \to \pm\infty} \frac{\Psi'_\lambda(s)}{2s} = \lim_{s \to \pm\infty} \frac{s - J_\lambda(s)}{2\lambda s} = \frac{1}{2\lambda} - \lim_{s \to \pm\infty} \frac{J_\lambda(s)}{2\lambda s} = \frac{1}{2\lambda},$$

where we have used that $Range(J_\lambda) = (-1, 1)$. Setting $0 < \lambda^* \leq 1$, the above limit entails that there exists M_{λ^*} such that

$$\Psi_{\lambda^*}(s) \geq \frac{1}{4\lambda^*}s^2, \quad \forall\, |s| \geq M_{\lambda^*}$$

On account of the monotonicity of Ψ_λ with respect to λ, we have

$$\Psi_\lambda(s) \geq \frac{1}{4\lambda^*}s^2, \quad \forall\, |s| \geq M_{\lambda^*},\ 0 < \lambda \leq \lambda^*.$$

Since Ψ_λ is non-negative, according to the last inequality, we conclude that

$$\Psi_\lambda(s) \geq \frac{1}{4\lambda^*}s^2 - C, \quad \forall\, s \in \mathbb{R},\, 0 < \lambda \leq \lambda^*,$$

where $C = M_{\lambda^*}^2/(4\lambda^*)$ is independent of λ. □

Lastly, we state an immediate result of convergence.

Lemma 3.2.6. *For any set* $[a,b] \subset (-1,1)$, Ψ'_λ *converges uniformly to* Ψ' *on* $[a,b]$.

3.3 The Neumann problem

We consider the homogeneous Neumann problem for the Laplace equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \partial_{\boldsymbol{n}} u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.3.1}$$

and the associated linear operator $A : V \to V'$ defined by

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \quad \forall\, u, v \in V.$$

Recalling the definition of total mass

$$\overline{u} = \frac{1}{|\Omega|}\langle u, 1\rangle,$$

we introduce the Hilbert spaces

$$V_0 = \{u \in V :\ \overline{u} = 0\}, \quad L_0^2(\Omega) = \{u \in H :\ \overline{u} = 0\}, \quad V_0' = \{u \in V' :\ \overline{u} = 0\}.$$

By the Poincaré's inequality (3.1.5), the restriction of $A : V_0 \to V_0'$ is an isomorphism. In particular, A is positively defined on V_0 and self-adjoint. We denote its inverse map by $\mathcal{N} = A^{-1} : V_0' \to V_0$. Notice that for every $f \in V_0'$, $u = \mathcal{N} f \in V_0$ is the unique weak solution of the Neumann problem (3.3.1) such that

$$(\nabla u, \nabla v) = \langle f, v\rangle, \quad \forall\, v \in V_0.$$

In accordance with these definitions, we have

$$\langle Au, \mathcal{N} f\rangle = \langle u, f\rangle, \quad \forall\, u \in V,\ \forall\, f \in V_0', \tag{3.3.2}$$

$$\langle f, \mathcal{N} g\rangle = \langle \mathcal{N} f, g\rangle = \int_\Omega \nabla(\mathcal{N} f) \cdot \nabla(\mathcal{N} g)\,\mathrm{d}x, \quad \forall\, f, g \in V_0'. \tag{3.3.3}$$

We equip the Hilbert spaces V_0' with inner product and norm

$$\langle f, g\rangle_{V_0'} = (\nabla\mathcal{N} f, \nabla\mathcal{N} g), \quad \|f\|_{V_0'} = \|\nabla\mathcal{N} f\|.$$

In particular, we obtain that $f \to \|f\|_* := (\|f - \overline{f}\|_{V_0'}^2 + |\overline{f}|^2)^{\frac{1}{2}}$ is an equivalent norm on V'. In addition, it follows from (3.3.2) the Hilbert interpolation inequality

$$\|f\| \leq \|f\|_{V_0'}^{\frac{1}{2}} \|\nabla f\|^{\frac{1}{2}}, \quad \forall\, f \in V_0, \tag{3.3.4}$$

and the chain rule

$$\langle f_t, \mathcal{N} f\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{V_0'}^2, \quad \text{a.e.}\, t \in (0,T),\, \forall\, f \in H^1(0,T; V_0'). \tag{3.3.5}$$

Let us now report some facts from the elliptic regularity theory of the Laplace operator $A_N = -\Delta + I$ with homogeneous Neumann boundary conditions. Assume that u satisfies $A_N u = f$ in weak sense. Then, we have the following results (see [109]):

- ⋄ Let $f \in (W^{1,p'}(\Omega))'$, with $1 < p' < \infty$. Then, $u \in W^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, and there exists $C > 0$ such that
$$\|u\|_{W^{1,p}(\Omega)} \leq C\|f\|_{(W^{1,p'})'}.$$
- ⋄ Let $f \in L^p(\Omega)$, with $1 < p < \infty$. Then, $u \in W^{2,p}(\Omega)$, $-\Delta u + u = f$ for a.e. $x \in \Omega$, $\partial_{\boldsymbol{n}} u = 0$ on $\partial\Omega$ in the sense of traces and there exists $C > 0$ such that
$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$
- ⋄ Let $f \in W^{1,p}(\Omega)$, with $1 < p < \infty$. Then, $u \in W^{3,p}(\Omega)$ and there exists $C > 0$ such that
$$\|u\|_{W^{3,p}(\Omega)} \leq C\|f\|_{W^{1,p}}.$$

The above positive constants C depend on p, p' and Ω. Moreover, we deduce the following estimates for the Neumann problem

$$\|\nabla \mathcal{N} f\|_{\mathbf{H}^k(\Omega)} \leq C\|f\|_{H^{k-1}(\Omega)}, \quad \forall\, f \in H^{k-1}(\Omega) \cap L_0^2(\Omega),\, k = 1, 2. \tag{3.3.6}$$

3.4 The Neumann problem with logarithmic nonlinearity

We introduce the homogeneous Neumann problem with a singular nonlinear term

$$\begin{cases} -\Delta u + \Psi'(u) = f, & \text{in } \Omega \\ \partial_{\boldsymbol{n}} u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.4.1}$$

Given $f \in H$, the existence of a (unique) solution to (3.4.1) can be proved by exploiting the convexity of Ψ. In the sequel, we assume that u is a solution to (3.4.1) such that $u \in H^2(\Omega)$ with $\Psi'(u) \in H$, $\partial_{\boldsymbol{n}} u = 0$ on $\partial\Omega$ in the sense of traces and satisfies $-\Delta u + \Psi'(u) = f$ for a.e. $x \in \Omega$. In particular, we observe that $\|u\|_{L^\infty(\Omega)} \leq 1$. Now we deduce some elliptic regularity estimates which are motivated by [1] and [118]. To this aim, for $k \in \mathbb{N}$, we define the globally Lipschitz function $h_k : \mathbb{R} \to \mathbb{R}$ such that

$$h_k(s) = \begin{cases} -1 + \frac{1}{k}, & s < -1 + \frac{1}{k}, \\ s, & s \in [-1 + \frac{1}{k}, 1 - \frac{1}{k}], \\ 1 - \frac{1}{k}, & s > 1 - \frac{1}{k}. \end{cases} \tag{3.4.2}$$

Now we consider $u_k = h_k \circ u$. Since $u \in V$, the result on compositions in Sobolev spaces yields $u_k \in V$, for any $k > 0$, and

$$\nabla u_k = \nabla u \cdot \chi_{[-1+\frac{1}{k},1-\frac{1}{k}]}(u).$$

This cutoff function will be repeatedly employ in the course of this section.

Lemma 3.4.1. *Let $f \in L^2(\Omega)$. Then, we have*

$$\|\Delta u\| \leq \|f - \overline{f}\|.$$

Proof. Testing the problem by $-\Delta u$ and noticing that $\overline{\Delta u} = 0$, we obtain

$$\|\Delta u\|^2 - (\Psi'(u), \Delta u) = -(f - \overline{f}, \Delta u).$$

We rewrite the above equality by using $\Psi'(u_k)$ as follows

$$\|\Delta u\|^2 - (\Psi'(u_k), \Delta u) = (\Psi'(u) - \Psi'(u_k), \Delta u) - (f - \overline{f}, \Delta u).$$

Observing that $\Psi'(u_k) \in V$ with $\Psi''(u_k) > 0$, from an integration by parts we deduce that the second term on the left hand side is positive. In addition, we have $\Psi'(u_k) \to \Psi'(u)$ for a.e. $x \in \Omega$ and $|\Psi'(u_k)| \leq |\Psi'(u)|$. Since $\Psi'(u) \in H$, the dominated convergence theorem implies

$$(\Psi'(u) - \Psi'(u_k), \Delta u) \to 0,$$

as k goes to ∞. This entails

$$\|\Delta u\|^2 \leq -(f - \overline{f}, \Delta u),$$

which, in turn, gives our claim. □

Lemma 3.4.2. *Let $f \in L^p(\Omega)$, where $2 \leq p \leq \infty$. Then, we have*

$$\|\Psi'(u)\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

Proof. Let us consider $f \in L^p(\Omega)$ with $2 \leq p < \infty$. We take the test function $|\Psi'(u_k)|^{p-2}\Psi'(u_k)$, which belongs to V for any k. Since $\Psi(u_k)''$ is well-defined and positive, we learn that

$$\begin{aligned}(-\Delta u, &|\Psi'(u_k)|^{p-2}\Psi'(u_k)) \\ &= (p-1)(|\Psi'(u_k)|^{p-2}\Psi''(u_k)\nabla u \cdot \chi_{[-1+\frac{1}{k},1-\frac{1}{k}]}(u), \nabla u) \geq 0.\end{aligned}$$

Then, we deduce that

$$(\Psi'(u), |\Psi'(u_k)|^{p-2}\Psi'(u_k)) \leq (f, |\Psi'(u_k)|^{p-2}\Psi'(u_k)).$$

Noticing that Ψ' is increasing and $\Psi'(s)s \geq 0$, we are lead to

$$\|\Psi'(u_k)\|^p_{L^p(\Omega)} \leq (\Psi'(u), |\Psi'(u_k)|^{p-2}\Psi'(u_k)).$$

By the Hölder inequality

$$(f, |\Psi'(u_k)|^{p-2}\Psi'(u_k)) \leq \|\Psi'(u_k)\|^{p-1}_{L^p(\Omega)}\|f\|_{L^p(\Omega)}.$$

Therefore, we arrive at

$$\|\Psi'(u_k)\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

By Fatou's lemma, we end up with

$$\|\Psi'(u)\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

If $f \in L^\infty$, we infer from the above estimate that

$$\|\Psi'(u)\|_{L^p(\Omega)} \leq C,$$

where C is independent of p. Thus, the claim follows from [5][Theorem 2.14]. $\square$

Corollary 3.4.3. *Let $f \in V$. Then, there exists a positive constant $C = C(p)$ such that*

$$\|u\|_{W^{2,p}(\Omega)} + \|\Psi'(u)\|_{L^p(\Omega)} \leq C\big(1 + \|f\|_V\big),$$

where $p = 6$ if $d = 3$ and for any $p \geq 2$ if $d = 2$.

Proof. On account of the Sobolev embeddings, an application of Lemma 3.4.2 implies

$$\|\Psi'(u)\|_{L^p(\Omega)} \leq C\|f\|_V,$$

where $p = 6$ if $d = 3$ and for any $p \geq 2$ if $d = 2$. We now interpret u as the solution to $-\Delta u + u = g$, where $g = f + u + \Psi'(u)$. It is clear that

$$\|g\|_{L^p(\Omega)} \leq C(1 + \|f\|_V).$$

Then, the desired conclusion follows from the elliptic regularity of the Neumann problem. $\square$

Lemma 3.4.4. *Let $f \in V$. Given $R > 0$, assume that $\|\nabla u\| \leq R$. Then, we have*

$$\|\Delta u\| \leq R^{\frac{1}{2}}\|\nabla f\|^{\frac{1}{2}}.$$

Proof. Arguing as in the proof of Lemma 3.4.1, we have

$$\|\Delta u\|^2 \leq -(f, \Delta u).$$

Thus, by virtue of the homogeneous boundary condition, an integration by parts gives

$$\|\Delta u\|^2 \leq R\|\nabla f\|.$$

$\square$

We prove a generalized version of Young's inequality.

Lemma 3.4.5. *Let $L > 0$ be given. Then, there exists $N = N(L) > 0$ such that*

$$xy\mathrm{e}^{Ly} \leq \mathrm{e}^{Nx-1} + \frac{1}{2}y^2\mathrm{e}^{Ly}, \quad \forall\, x, y \geq 0. \tag{3.4.3}$$

Proof. Let us first show that, for every $a, b \geq 0$,

$$ab \leq b \ln b + \mathrm{e}^{a-1}. \tag{3.4.4}$$

The function $f(b) = b \ln b + \mathrm{e}^{a-1} - ab$ satisfies $f(0) = \mathrm{e}^{a-1} > 0$ and $\lim_{b\to\infty} f(b) = \infty$. Besides $f'(b) = \ln b + 1 - a$, hence $\overline{b} = \mathrm{e}^{a-1}$ is the absolute minimum of f. Then, we have

$$f(b) \geq f(\overline{b}) = \mathrm{e}^{a-1} \ln \mathrm{e}^{a-1} + \mathrm{e}^{a-1} - a\mathrm{e}^{a-1} = 0$$

for every $b \geq 0$, which implies (3.4.4). Letting $a = Nx$ and $b = \frac{y}{N}\mathrm{e}^{Ly}$ in (3.4.4) for any given $N, L > 0$, we easily find

$$\begin{aligned} xy\mathrm{e}^{Ly} &\leq \mathrm{e}^{Nx-1} + \frac{y}{N}\mathrm{e}^{Ly}\Big(\ln \frac{y}{N} + \ln \mathrm{e}^{Ly}\Big) \\ &\leq \mathrm{e}^{Nx-1} + \frac{L+1}{N} y^2 \mathrm{e}^{Ly}, \end{aligned}$$

and the claim follows with $N > 2(L+1)$. □

Lemma 3.4.6. *Let $d = 2$ and $f \in V$. Suppose that* (H.3) *holds. Then, for any $p \geq 1$, there exists a positive constant $C = C(p)$ such that*

$$\|\Psi''(u)\|_{L^p(\Omega)} \leq C\Big(1 + e^{C\|f\|_V^2}\Big).$$

Proof. For $k \in \mathbb{N}$, let u_k be the cutoff function introduced at the beginning. Given $L > 0$, we consider the test function $\Psi'(u_k)\mathrm{e}^{\mathrm{L}|\Psi'(\mathrm{u_k})|}$. Since the function $s \to \Psi'(s)\mathrm{e}^{L|\Psi'(s)|}$ is monotone, by arguing as in the proof of Lemma 3.4.2, we find

$$\int_\Omega |\Psi'(u_k)|^2 \mathrm{e}^{L|\Psi'(u_k)|}\,\mathrm{d}x \leq \int_\Omega |f||\Psi'(u_k)|\mathrm{e}^{L|\Psi'(u_k)|}\,\mathrm{d}x.$$

We estimate the right-hand side by the generalized Young inequality (3.4.3) with the choice

$$x = |f|, \quad y = |\Psi'(u_k)|.$$

Accordingly, we find $N = N(L)$ such that

$$\int_\Omega |f||\Psi'(u_k)|\mathrm{e}^{L|\Psi'(u_k)|}\,\mathrm{d}x \leq \int_\Omega \frac{1}{2}|\Psi'(u_k)|^2 \mathrm{e}^{L|\Psi'(u_k)|}\,\mathrm{d}x + \int_\Omega \mathrm{e}^{N|f|}\,\mathrm{d}x,$$

and we obtain

$$\frac{1}{2}\int_\Omega |\Psi'(u_k)|^2 \mathrm{e}^{L|\Psi'(u_k)|}\,\mathrm{d}x \leq \int_\Omega \mathrm{e}^{N|f|}\,\mathrm{d}x.$$

Due to the Trudinger-Moser inequality in dimension two, we have the following control

$$\frac{1}{2}\int_\Omega |\Psi'(u_k)|^2 \mathrm{e}^{L|\Psi'(u_k)|}\,\mathrm{d}x \leq C\Big(1 + \mathrm{e}^{CN^2\|f\|_V^2}\Big). \tag{3.4.5}$$

On the other hand, by (H.3) we observe that

$$\Psi''(s)^p \leq pC\Big(1 + |\Psi'(s)|^2 \mathrm{e}^{pC|\Psi'(s)|}\Big), \quad \forall\, s \in (-1, 1).$$

Thus, taking $L = pC$ in (3.4.5), we end up with

$$\int_\Omega |\Psi''(u_k)|^p\,\mathrm{d}x \leq C\Big(1 + \mathrm{e}^{C\|f\|_V^2}\Big),$$

where $C > 0$ depends on p. □

3.5 Solenoidal vector fields

We introduce the Hilbert space of solenoidal vector fields

$$\mathbf{H}_\sigma = \{\boldsymbol{u} \in \mathbf{L}^2(\Omega) : \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \ \boldsymbol{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

endowed with the usual inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let Π be the orthogonal Leray projection in $\mathbf{H} = \mathbf{L}^2(\Omega)$ onto $\mathbf{H}_\sigma$. By the Helmholtz–Leray decomposition, every vector field $\boldsymbol{f} \in \mathbf{H}$ can be uniquely represented as

$$\boldsymbol{f} = \boldsymbol{u} + \nabla\pi,$$

where $\boldsymbol{u} = \Pi\boldsymbol{f} \in \mathbf{H}_\sigma$ and $\pi \in V_0$. We recall that Π is a bounded operator from $\mathbf{W}^{k,p}(\Omega)$ ($1 < p < \infty$, $k \geq 0$) into itself (cf. [83, Lemma 3.3]), namely

$$\|\Pi\boldsymbol{f}\|_{\mathbf{W}^{k,p}} \leq C\|\boldsymbol{f}\|_{\mathbf{W}^{k,p}}, \quad \forall\, \boldsymbol{f} \in \mathbf{W}^{k,p}(\Omega), \tag{3.5.1}$$

where the constant C depends on k and p. On the other hand, if $\boldsymbol{f} \in \mathbf{W}^{k,p}(\Omega)$, with $1 < p < \infty$, then from $\nabla\pi = \boldsymbol{f} - \Pi\boldsymbol{f}$ we see that π is the unique solution to the Neumann problem

$$\begin{cases} -\Delta\pi = \operatorname{div} \boldsymbol{f}, & \text{in } \Omega, \\ \partial_{\boldsymbol{n}}\pi = \boldsymbol{f} \cdot \boldsymbol{n}, & \text{on } \partial\Omega, \end{cases} \tag{3.5.2}$$

satisfying $\overline{\pi} = 0$. Then, it follows that $\pi \in W^{k+1,p}(\Omega)$ by the elliptic regularity results for the nonhomogeneous Neumann problem. In addition, solenoidal vector fields in $\mathbf{V} = \mathbf{H}^1(\Omega)$ satisfy the following inequality (see, e.g., [84, Theorem 3.8])

$$\|\boldsymbol{u}\|_{\mathbf{V}} \leq C\left(\|\nabla \times \boldsymbol{u}\| + \|\boldsymbol{u}\|\right), \quad \forall\, \boldsymbol{u} \in \mathbf{V} \cap \mathbf{H}_\sigma. \tag{3.5.3}$$

We also introduce the higher order Hilbert space of solenoidal vector fields

$$\mathbf{V}_\sigma = \{\boldsymbol{u} \in \mathbf{V} : \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}$$

equipped with inner product and norm

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathbf{V}_\sigma} = (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}), \quad \|\boldsymbol{u}\|_{\mathbf{V}_\sigma} = \|\nabla\boldsymbol{u}\|.$$

Since $\mathbf{V}_\sigma \subset \mathbf{H}_0^1(\Omega)$, which consists of vector fields in $\mathbf{V}$ with null trace on $\partial\Omega$, the classical Poincaré inequality is valid in $\mathbf{V}_\sigma$. That is, there exists $C > 0$ such that

$$\|\boldsymbol{u}\| \leq C\|\nabla\boldsymbol{u}\|, \quad \forall\, \boldsymbol{u} \in \mathbf{V}_\sigma.$$

We also recall the Korn inequality

$$\|\nabla\boldsymbol{u}\|^2 \leq 2\|D\boldsymbol{u}\|^2 \leq 2\|\nabla\boldsymbol{u}\|^2, \quad \forall\, \boldsymbol{u} \in \mathbf{V}_\sigma. \tag{3.5.4}$$

In turn, $\|D\boldsymbol{u}\|$ is an equivalent norm on $\mathbf{V}_\sigma$. As customary, we define the trilinear form on $\mathbf{V}_\sigma \times \mathbf{V}_\sigma \times \mathbf{V}_\sigma$

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w}\, \mathrm{d}x = \sum_{i,j=1}^{d} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j\, \mathrm{d}x,$$

satisfying the relation

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0. \tag{3.5.5}$$

3.6 The Stokes problem

Let us consider the Stokes problem

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, & \text{in } \Omega, \\ \boldsymbol{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.6.1}$$

We introduce the Stokes operator $\mathbf{A} : \mathbf{V}_\sigma \to \mathbf{V}_\sigma'$ defined by

$$\langle \mathbf{A}\boldsymbol{u}, \boldsymbol{v} \rangle = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V},$$

namely $\mathbf{A}$ is the canonical isomorphism from $\mathbf{V}_\sigma$ onto $\mathbf{V}_\sigma'$. We denote by $\mathbf{A}^{-1} : \mathbf{V}_\sigma' \to \mathbf{V}_\sigma'$ its inverse map. Given $\boldsymbol{f} \in \mathbf{V}_\sigma'$, $\mathbf{A}^{-1}\boldsymbol{f}$ is the unique function $\boldsymbol{u} \in \mathbf{V}_\sigma$ such that

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \quad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma.$$

By definition, it follows that

$$\|\boldsymbol{f}\|_\sharp := \|\nabla \mathbf{A}^{-1}\boldsymbol{f}\| = \langle \boldsymbol{f}, \mathbf{A}^{-1}\boldsymbol{f} \rangle^{\frac{1}{2}}$$

is an equivalent norm in $\mathbf{V}_\sigma'$ and

$$\langle \boldsymbol{f}_t, \mathbf{A}^{-1}\boldsymbol{f} \rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{f}\|_\sharp^2, \quad \text{a.e. } t \in (0,T),\ \forall\, \boldsymbol{f} \in H^1(0,T;\mathbf{V}_\sigma').$$

Notice that, in order to recover the pressure and interpret $(\boldsymbol{u}, \pi)$ as the solution to (3.6.1), the forcing term $\boldsymbol{f}$ need to be in $(\mathbf{H}_0^1(\Omega))'$. In this case, we recall the following regularity results for the Stokes problem (3.6.1) (see [24] and [151]):

- ⋄ Let $\boldsymbol{f} \in (\mathbf{H}_0^1(\Omega))'$. Then, these exists a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{V}_\sigma \cap L_0^2(\Omega)$ to (3.6.1) and there exists $C > 0$ such that

$$\|\boldsymbol{u}\|_{\mathbf{V}_\sigma} + \|\pi\|_{L_0^2(\Omega)} \le C\|\boldsymbol{f}\|_{(\mathbf{H}_0^1(\Omega))'}.$$

- ⋄ Let $\boldsymbol{f} \in \mathbf{W}^{k,p}(\Omega)$, with $k = 0, 1$ and $1 < p < \infty$. Then, there exists $C > 0$ depending on k, p and Ω such that

$$\|\boldsymbol{u}\|_{\mathbf{W}^{k+2,p}(\Omega)} + \|\pi\|_{W^{k+1,p}(\Omega)} \le C\|\boldsymbol{f}\|_{\mathbf{W}^{k,p}(\Omega)}.$$

We now consider $\mathbf{A}$ as an unbounded operator on $\mathbf{H}_\sigma$ with domain

$$D(\mathbf{A}) = \{\boldsymbol{u} \in \mathbf{V}_\sigma : \mathbf{A}\boldsymbol{u} \in \mathbf{H}_\sigma\}.$$

It is well known that $\mathbf{A}$ is a positive self-adjoint operator on $\mathbf{H}_\sigma$. On account of the regularity theory for (3.6.1), we have a precise description of $\mathbf{A}$ on its domain, namely

$$D(\mathbf{A}) = \mathbf{V}_\sigma \cap \mathbf{H}^2(\Omega), \quad \mathbf{A}\boldsymbol{u} = \Pi(-\Delta \boldsymbol{u}), \quad \forall\, \boldsymbol{u} \in D(\mathbf{A}).$$

Then, we define the Hilbert space

$$\mathbf{W}_\sigma = \mathbf{V}_\sigma \cap \mathbf{H}^2(\Omega)$$

endowed with inner product and norm

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{W}_\sigma} = (\mathbf{A}\boldsymbol{u}, \mathbf{A}\boldsymbol{v}), \quad \|\boldsymbol{u}\|_{\mathbf{W}_\sigma} = \|\mathbf{A}\boldsymbol{u}\|.$$

By virtue of the above regularity, there exists $C > 0$ such that

$$\frac{1}{C}\|\boldsymbol{u}\|_{\mathbf{W}_\sigma} \leq \|\boldsymbol{u}\|_{\mathbf{H}^2(\Omega)} \leq C\|\boldsymbol{u}\|_{\mathbf{W}_\sigma}, \quad \forall\, \boldsymbol{u} \in \mathbf{W}_\sigma. \tag{3.6.2}$$

We conclude this section with a regularity result on the Stokes problem with non-constant viscosity depending on concentration. Let us consider the following Stokes problem

$$\begin{cases} -\operatorname{div}(\nu(\varphi)D\boldsymbol{u}) + \nabla\pi = f, & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0, & \text{in } \Omega, \\ \boldsymbol{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.6.3}$$

Here, the viscosity coefficient $\nu \in \mathcal{C}^2(\mathbb{R})$ satisfies

$$0 < \nu_* \leq \nu(s) \leq \nu^*, \quad \forall\, s \in \mathbb{R}.$$

We recall an elliptic estimate on system (3.6.3) (see [1] for the proof).

Lemma 3.6.1. *Let $\varphi \in W^{1,r}(\Omega)$, with $r > d \geq 2$, and $\boldsymbol{f} \in \mathbf{H}$. Assume that $\boldsymbol{u} \in \mathbf{V}_\sigma$ is a weak solution to* (3.6.3)*, namely*

$$(\nu(\varphi)D\boldsymbol{u}, D\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma.$$

Then, there exists $C > 0$, depending on ν_*, ν^*, r, such that

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C\big(1 + \|\nabla\varphi\|_{\mathbf{L}^r(\Omega)}\big)\big(\|\boldsymbol{f}\| + \|\nabla\boldsymbol{u}\|\big),$$

where $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$.

3.7 Gronwall type lemmas

We report in this section some Gronwall type results that will be needed in the course of the investigation. The proofs can be found in [25] and [149].

Lemma 3.7.1 (Gronwall lemma)**.** *Let f be an absolutely continuous function on $[0, T]$ and g, h two summable functions on $[0, T]$ which satisfy the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \leq g(t)f(t) + h(t)$$

for almost every $t \in [0, T]$. Then, we have

$$f(t) \leq f(0)\mathrm{e}^{\int_0^t g(\tau)\,\mathrm{d}\tau} + \int_0^t \mathrm{e}^{\int_\tau^t g(s)\,\mathrm{d}s} h(\tau)\,\mathrm{d}\tau, \quad \forall\, t \in [0, T].$$

Lemma 3.7.2 (Integral Gronwall lemma)**.** *Let f be a continuous function on $[0,T]$, g a positive summable function and R a positive constant which satisfy the integral inequality*

$$\frac{1}{2}f^2(t) \leq \frac{1}{2}R^2 + \int_0^t g(\tau)f(\tau)\,\mathrm{d}\tau, \quad \forall\, t \in [0,T].$$

Then, we have

$$|f(t)| \leq R + \int_0^t g(\tau)\,\mathrm{d}\tau, \quad \forall\, t \in [0,T].$$

Lemma 3.7.3 (Uniform Gronwall lemma)**.** *Let f be an absolutely continuous positive function on $[0,\infty)$ and g, h two positive locally summable functions on $[0,\infty)$ which satisfy the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \leq g(t)f(t) + h(t),$$

for almost every $t \geq 0$, and the uniform bounds

$$\int_t^{t+r} f(\tau)\,\mathrm{d}\tau \leq a_1, \quad \int_t^{t+r} g(\tau)\,\mathrm{d}\tau \leq a_2, \quad \int_t^{t+r} h(\tau)\,\mathrm{d}\tau \leq a_3, \quad \forall\, t \geq 0,$$

for some r, a_1, a_2, a_3 positive. Then, we have

$$f(t) \leq (\frac{a_1}{r} + a_3)\mathrm{e}^{a_2}, \quad \forall\, t \geq r.$$

Part I

Local interaction models

CHAPTER 4

The Navier–Stokes–Cahn–Hilliard–Oono system

In this chapter we consider the Navier–Stokes–Cahn–Hilliard–Oono (NSCHO) system with matched viscosities in two space dimensions. We will address the uniqueness of weak solutions, the regularity propagation in time and the separation property from the pure phases. In the last part we will discuss some consequences regarding the longtime behavior of solutions. As mentioned earlier, all the results presented in this chapter are also valid for the Navier–Stokes–Cahn–Hilliard (NSCH) system.

In a bounded domain $\Omega \subset \mathbb{R}^2$, the Navier–Stokes–Cahn–Hilliard–Oono system with matched viscosities ($\nu = 1$) reads as follows

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \mu \nabla \varphi, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi + \beta(\varphi - c) = \Delta \mu, \\ \mu = -\Delta \varphi + F'(\varphi), \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{4.0.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0}, \quad \partial_{\boldsymbol{n}} \mu = \partial_{\boldsymbol{n}} \varphi = 0, & \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0(\cdot), \quad \varphi(\cdot, 0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{4.0.2}$$

General agreement. Throughout this chapter, if it is not otherwise stated, we indicate by C a generic positive constant depending only on the domain and on structural quantities. The constant C may vary from line to line and even within the same line. Any further dependence will be explicitly pointed out if necessary.

4.1 Existence of Weak Solutions and Dissipativity

In the sequel the parameter β and c satisfy the structural requirements

$$\beta \geq 0 \quad \text{and } c \in (-1,1).$$

In addition, we remind that the singular potential F fulfils the assumption (H). As a byproduct, we have the basic inequality

$$F(s) \leq F(w) + F'(s)(s-w) + \frac{\alpha}{2}(s-w)^2, \quad \forall\, s, w \in (-1,1). \tag{4.1.1}$$

We begin by stating the definition of weak solution.

Definition 4.1.1. *Given $\boldsymbol{u}_0 \in \mathbf{H}_\sigma$, $\varphi_0 \in V$, with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$, a pair $(\boldsymbol{u}, \varphi)$ is a weak solution to* (4.0.1)-(4.0.2) *on $[0,T]$ if*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;\mathbf{H}_\sigma) \cap L^2(0,T;\mathbf{V}_\sigma) \cap H^1(0,T;\mathbf{V}_\sigma'),\\
&\varphi \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;V'),\\
&\varphi \in L^\infty(\Omega \times (0,T)), \quad \textit{with} \quad |\varphi(x,t)| < 1 \textit{ a.e. } (x,t) \in \Omega \times (0,T),\\
&\Psi'(\varphi) \in L^2(0,T;H),
\end{aligned}$$

and

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = (\mu \nabla \varphi, \boldsymbol{v}), \qquad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma, \tag{4.1.2}$$

$$\langle \partial_t \varphi, v\rangle + (\boldsymbol{u} \cdot \nabla \varphi, v) + \beta(\varphi - c, v) + (\nabla \mu, \nabla v) = 0, \qquad \forall\, v \in V, \tag{4.1.3}$$

for almost every $t \in (0,T)$, where $\mu \in L^2(0,T;V)$ is given by

$$\mu = -\Delta \varphi + F'(\varphi), \tag{4.1.4}$$

for almost every $(x,t) \in \Omega \times (0,T)$. Moreover, $\partial_{\boldsymbol{n}} \varphi = 0$ a.e. on $\partial\Omega \times (0,T)$, $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0$ and $\varphi(\cdot,0) = \varphi_0$ a.e. in Ω.

Remark 4.1.2. We observe that any admissible initial condition φ_0 belongs to $V \cap L^\infty(\Omega)$ with $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$. However, due to $|\overline{\varphi}_0| < 1$, it can not be a pure phase, i.e. $\varphi_0 \equiv \pm 1$.

Remark 4.1.3. It is straightforward to see that any energy solution satisfies the relation

$$\overline{\varphi}(t) = c + \mathrm{e}^{-\beta t}(\overline{\varphi}_0 - c), \quad \forall\, t \geq 0, \tag{4.1.5}$$

Remark 4.1.4. Note that the weak formulation for the velocity field is equivalent to

$$\langle \boldsymbol{u}_t, \boldsymbol{v}\rangle - (\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{v}) + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = (\nabla \varphi \otimes \nabla \varphi, \nabla \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbf{V},$$

where $(v \otimes w)_{ij} = v_i w_j$, $i, j = 1, 2$, in light of the equalities

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \mathrm{div}(\boldsymbol{u} \otimes \boldsymbol{u}) \tag{4.1.6}$$

and

$$\mu \nabla \varphi = \nabla\Big(\frac{1}{2}|\nabla \varphi|^2 + F(\varphi)\Big) - \mathrm{div}(\nabla \varphi \otimes \nabla \varphi).$$

Remark 4.1.5. As customary, the pressure π disappears in the weak formulations. Indeed, once we have a weak solution in the sense of Definition 4.1.1, the pressure is recovered by using de Rham's theorem (see [151]).

Theorem 4.1.6. *Let $\boldsymbol{u}_0 \in \mathbf{H}_\sigma$, $\varphi_0 \in V$, with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Then, for any $T > 0$, there exists a weak solution $(\boldsymbol{u}, \varphi)$ to problem* (4.0.1)-(4.0.2) *on $[0, T]$ such that*

$$\boldsymbol{u} \in \mathcal{C}([0,T], \mathbf{H}_\sigma), \quad \varphi \in \mathcal{C}([0,T], V).$$

In addition, the energy identity

$$E_{GL}(\boldsymbol{u}(t), \varphi(t)) + \int_s^t \|\nabla \boldsymbol{u}(\tau)\|^2 + \|\nabla \mu(\tau)\|^2 + \beta(\varphi(\tau) - c, \mu(\tau)) \,\mathrm{d}\tau = E_{GL}(\boldsymbol{u}(s), \varphi(s))$$

is satisfied for all $0 \le s < t < \infty$.

Proof. The existence of a weak solution has been proved in [117] through a standard procedure involving a Galerkin scheme together with an approximation of the singular potential. Moreover, the regularity of the velocity field $\boldsymbol{u} \in \mathcal{C}([0,T], \mathbf{H})$ is an immediate consequence of Definition 4.1.1. In order to show the energy identity, we consider the convex part of the Ginzburg–Landau free energy

$$\mathcal{E}^*_{GL}(\varphi) = \frac{1}{2}\|\nabla \varphi\|^2 + \int_\Omega \Psi(\varphi)\,\mathrm{d}x, \quad \forall\, \varphi \in H.$$

On account of the continuity and the convexity of Ψ, $\mathcal{E}^*_{GL}$ is a proper, lower semicontinuous and convex functional. Let φ be a weak solution in the sense of Definition 4.1.1. In view of $-\Delta\varphi + \Psi'(\varphi) \in L^2(0,T;V)$ and $\varphi_t \in L^2(0,T;V')$, we infer from [131, Lemma 4.1] that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^*_{GL}(\varphi) = \langle \partial_t \varphi, \mu + \Theta_0 \varphi\rangle, \quad \text{a.e. } t \in [0,T].$$

We also observe that, as a byproduct, the map $t \mapsto \mathcal{E}^*_{GL}(\varphi(t))$ is absolutely continuous on $[0,T]$. In addition, through a standard argument, we also learn that $\varphi \in \mathcal{C}([0,T], V)$. Besides, using the standard chain rule in $L^2(0,T;V) \cap H^1(0,T;V')$ and the weak formulation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}(\varphi) + \|\nabla\mu\|^2 + (\boldsymbol{u} \cdot \nabla\varphi, \mu) + \beta(\varphi - c, \mu) = 0, \quad \text{a.e. } t \in [0,T].$$

Now, taking $\boldsymbol{v} = \boldsymbol{u}$ as test function and using (3.5.5), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2 + \|\nabla \boldsymbol{u}\|^2 = (\mu\nabla\varphi, \boldsymbol{u}), \quad \text{a.e. } t \in [0,T].$$

In summary, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{GL}(\boldsymbol{u}, \varphi) + \|\nabla \boldsymbol{u}\|^2 + \|\nabla\mu\|^2 + \beta(\varphi - c, \mu) = 0, \quad \text{a.e. } t \in [0,T].$$

Accordingly, $\frac{\mathrm{d}}{\mathrm{d}t}E_{GL}(\boldsymbol{u}, \varphi)$ is the sum of functions in $L^1(0,T)$ so that E_{GL} is an absolutely continuous function on $[0,T]$. A final integration in time on (s,t) of the above equality entails the energy identity. $\square$

We show the dissipative nature of the NSCHO system in the next result.

Theorem 4.1.7. *Let $(\boldsymbol{u}, \varphi)$ be a weak solution with initial datum $(\boldsymbol{u}_0, \varphi_0)$. Then, we have the dissipative estimate*

$$E_{GL}(\boldsymbol{u}(t), \varphi(t)) + \int_t^{t+1} \|\nabla \boldsymbol{u}(\tau)\|^2 + \|\nabla \mu(\tau)\|^2 \, \mathrm{d}\tau \le C E_{GL}(\boldsymbol{u}_0, \varphi_0) \mathrm{e}^{-\omega t} + C,$$

for every $t \ge 0$. Here, ω and C are positive constants independent of the initial datum. In addition, let $m \in [0, 1)$ be such that $c \in [-m, m]$ and $\overline{\varphi}_0 \in [-m, m]$. Then, for any $p \ge 2$, there exists a positive constant $C = C(m, p)$ such that

$$\int_t^{t+1} \|\varphi(\tau)\|^2_{W^{2,p}(\Omega)} + \|\Psi'(\varphi(\tau))\|^2_{L^p(\Omega)} \, \mathrm{d}\tau \le C E_{GL}(\boldsymbol{u}_0, \varphi_0) \mathrm{e}^{-\omega t} + C, \qquad (4.1.7)$$

and

$$\int_t^{t+1} \|\varphi(\tau)\|^4_{H^2(\Omega)} \, \mathrm{d}\tau \le C E_{GL}(\boldsymbol{u}_0, \varphi_0)^2 \mathrm{e}^{-\omega t} + C, \qquad (4.1.8)$$

for every $t \ge 0$.

Proof. We introduce the functional

$$\Gamma(t) = (\varphi(t) - \overline{\varphi}(t), \mu(t)).$$

Due to the definition of the chemical potential, after an integration by parts, we have

$$\Gamma = \|\nabla \varphi\|^2 + (F'(\varphi), \varphi - \overline{\varphi}).$$

In light of the assumptions on F, the inequalities (4.1.1) and $\|\varphi(t)\|_{L^\infty(\Omega)} \le 1$, for almost every $t \ge 0$, we obtain

$$\begin{aligned}(F'(\varphi), \varphi - \overline{\varphi}) &\ge \int_\Omega F(\varphi) \, \mathrm{d}x - F(\overline{\varphi})|\Omega| - \frac{\alpha}{2}\|\varphi - \overline{\varphi}\|^2 \\ &\ge \int_\Omega F(\varphi) \, \mathrm{d}x - C.\end{aligned}$$

At the same time, the uniform control in $L^\infty(\Omega)$ of φ and the Poincaré–Wirtinger inequality (3.1.5) yield

$$\Gamma = (\varphi - \overline{\varphi}, \mu - \overline{\mu}) \le C\|\nabla \mu\|. \qquad (4.1.9)$$

Hence, we reach

$$\|\nabla \varphi\|^2 + \int_\Omega \Psi(\varphi) \, \mathrm{d}x \le \frac{1}{2}\|\nabla \mu\|^2 + C.$$

Adding the above inequality to the energy identity (cf. Theorem 4.1.6) and using the Poincaré inequality (3.1.5), there exists $\omega > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{GL}(\boldsymbol{u}, \varphi) + \omega E_{GL}(\boldsymbol{u}, \varphi) + \frac{1}{2}\|\nabla \boldsymbol{u}\|^2 + \frac{1}{2}\|\nabla \mu\|^2 + \beta(\varphi - c, \mu) \le C.$$

In a similar way, we have

$$\beta(\varphi - c, \mu) \ge \beta\|\nabla \varphi\|^2 + \beta \int_\Omega F(\varphi) \, \mathrm{d}x - \beta F(c)|\Omega| - \beta\frac{\alpha}{2}\|\varphi - c\|^2 \ge -C.$$

Therefore, we find the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{GL}(\boldsymbol{u},\varphi)+\omega E_{GL}(\boldsymbol{u},\varphi)+\frac{1}{2}\|\nabla\boldsymbol{u}\|^2+\frac{1}{2}\|\nabla\mu\|^2\leq C.$$

An application of the Gronwall lemma entails the dissipative estimate. Next, in order to prove (4.1.7), the first task is to provide a uniform estimate on μ in V. Owing to (3.1.5), it is sufficient to control its total mass, that is

$$|\overline{\mu}|=|\overline{F'(\varphi)}|.$$

According to the assumptions on c and $\overline{\varphi}_0$, we infer from (4.1.5) that

$$|\overline{\varphi}(t)|\leq m<1,\quad \forall\, t\geq 0.$$

Consequently, thanks to the hypotheses on F, we have the well-known inequality (see, e.g., [118, Proposition A.2] and Chapter 7)

$$\|F'(\varphi)\|_{L^1(\Omega)}\leq C(F'(\varphi)-\overline{F'(\varphi)},\varphi-\overline{\varphi})+C.$$

Here C depends on m. Then, by definition of Γ, and exploiting (4.1.9), we deduce that

$$\|F'(\varphi)\|_{L^1(\Omega)}\leq C\|\nabla\mu\|+C.$$

Hence, we arrive at

$$\|\mu\|_V\leq C\left(1+\|\nabla\mu\|\right). \tag{4.1.10}$$

Now, by taking $f=\mu+\Theta_0\varphi$, an application of Corollary 3.4.3 yields

$$\|\varphi\|^2_{W^{2,p}(\Omega)}+\|\Psi'(\varphi)\|^2_{L^p(\Omega)}\leq C\big(1+\|\nabla\mu\|\big)^2. \tag{4.1.11}$$

In addition, Lemma 3.4.4 together with the elliptic regularity of the Neumann problems entails

$$\|\varphi\|^4_{H^2(\Omega)}\leq C\big(1+\|\nabla\varphi\|\|\nabla\mu\|\big)^2. \tag{4.1.12}$$

Integrating in time the above inequalities (4.1.11) and (4.1.12), and by using the dissipative estimate, we infer 4.1.7 and 4.1.8, respectively. This completes the proof. □

Remark 4.1.8. We observe that the dissipative estimate has been proved by working directly with the weak solution in the sense of Definition 4.1.1. In particular, we have used the global boundedness $\|\varphi\|_{L^\infty(\Omega\times(0,T))}\leq 1$. On the other hand, the same dissipative estimate can be proved within a Galerkin approximating sequence and replacing the singular potential with a suitable approximation (see [117]). Besides, the same goes for (4.1.7) with $p=2$ and (5.3.15). Nonetheless, this is not the case for (4.1.7) for $p>2$.

4.2 Uniqueness of Weak Solutions

In this section we prove the uniqueness of weak solutions to the NSCHO system. This is a direct consequence of the continuous dependence estimate with respect to the initial data stated below. An analogous result holds for the NSCH system. In fact the argument used here applies equally well in that case.

Theorem 4.2.1. *Let* $(\boldsymbol{u}_{01}, \varphi_{01})$, $(\boldsymbol{u}_{02}, \varphi_{02})$ *be such that* $\boldsymbol{u}_{0i} \in \mathbf{H}_\sigma$, $\varphi_{0i} \in V$, $\Psi(\varphi_{0i}) \in L^1(\Omega)$ *and* $|\overline{\varphi}_{0i}| < 1$, $i = 1, 2$. *Assume that* $(\boldsymbol{u}_1, \varphi_1)$ *and* $(\boldsymbol{u}_2, \varphi_2)$ *are two pairs of weak solutions to* (4.0.1)-(4.0.2) *on* $[0, T]$ *with initial data* $(\boldsymbol{u}_{01}, \varphi_{01})$ *and* $(\boldsymbol{u}_{02}, \varphi_{02})$, *respectively. Then, there exists a positive constant* $C = C(T)$ *such that*

$$\begin{aligned}&\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\mathbf{V}_\sigma'} + \|\varphi_1(t) - \varphi_2(t)\|_{V'} \\ &\qquad \le C\|\boldsymbol{u}_{01} - \boldsymbol{u}_{02}\|_{\mathbf{V}_\sigma'} + C\|\varphi_{01} - \varphi_{02}\|_{V'} + C|\overline{\varphi}_{01} - \overline{\varphi}_{02}|^{\frac{1}{2}}, \quad \forall\, t \in [0, T].\end{aligned}$$

Proof. Let us define $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\varphi = \varphi_1 - \varphi_2$, where $(\boldsymbol{u}_1, \varphi_1)$, $(\boldsymbol{u}_2, \varphi_2)$ are two weak solutions corresponding to the initial data $(\boldsymbol{u}_{01}, \varphi_{01})$, $(\boldsymbol{u}_{02}, \varphi_{02})$, respectively. According to Remark 6.1.3, $\boldsymbol{u}$ and φ solve

$$\begin{aligned}&\langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle - (\boldsymbol{u}_1 \otimes \boldsymbol{u}, \nabla \boldsymbol{v}) - (\boldsymbol{u} \otimes \boldsymbol{u}_2, \nabla \boldsymbol{v}) + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) \\ &\qquad\qquad = (\nabla\varphi_1 \otimes \nabla\varphi, \nabla v) + (\nabla\varphi \otimes \nabla\varphi_2, \nabla \boldsymbol{v}), \qquad (4.2.1)\\ &\langle \partial_t \varphi, v\rangle + (\boldsymbol{u}_1 \cdot \nabla\varphi, v) + (\boldsymbol{u} \cdot \nabla\varphi_2, v) + \beta(\varphi, v) + (\nabla\mu, \nabla v) = 0, \qquad (4.2.2)\end{aligned}$$

for all $\boldsymbol{v} \in \mathbf{V}_\sigma$ and $v \in V$, where $\mu = \mu_1 - \mu_2$ satisfies

$$\mu = -\Delta\varphi + F'(\varphi_1) - F'(\varphi_2).$$

Taking $v = 1$ in (4.2.2) and recalling that $\boldsymbol{u}_1 \cdot \nabla\varphi$ and $\boldsymbol{u} \cdot \nabla\varphi_2$ have zero total mass, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\varphi} + \beta\overline{\varphi} = \overline{\partial_t\varphi} + \beta\overline{\varphi} = 0. \qquad (4.2.3)$$

Hence, we deduce that

$$\overline{\varphi}(t) = \overline{\varphi}_0 \mathrm{e}^{-\beta t}, \quad \forall\, t \ge 0, \qquad (4.2.4)$$

and, in turn, we rewrite (4.2.2) as

$$\langle \partial_t\varphi - \overline{\partial_t\varphi}, v\rangle + (\boldsymbol{u}_1 \cdot \nabla\varphi, v) + (\boldsymbol{u} \cdot \nabla\varphi_2, v) + \beta(\varphi - \overline{\varphi}, v) + (\nabla\mu, \nabla v) = 0. \quad (4.2.5)$$

Now, taking $v = \mathcal{N}(\varphi - \overline{\varphi})$ in (4.2.5) and using (3.3.2)-(3.3.5), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \overline{\varphi}\|_{V_0'}^2 + \beta\|\varphi - \overline{\varphi}\|_{V_0'}^2 + (\mu, \varphi - \overline{\varphi}) = I_1 + I_2,$$

where

$$I_1 = -(\boldsymbol{u}_1 \cdot \nabla\varphi, \mathcal{N}(\varphi - \overline{\varphi})), \quad I_2 = -(\boldsymbol{u} \cdot \nabla\varphi_2, \mathcal{N}(\varphi - \overline{\varphi})).$$

In light of (4.2.3),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\overline{\varphi}|^2 + \beta|\overline{\varphi}|^2 = 0$$

and we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2 + \beta\|\varphi\|_*^2 + (\mu, \varphi - \overline{\varphi}) = I_1 + I_2.$$

By the assumptions on F, we get

$$\begin{aligned}(\mu, \varphi - \overline{\varphi}) &= \|\nabla\varphi\|^2 + (F'(\varphi_1) - F'(\varphi_2), \varphi) - (F'(\varphi_1) - F'(\varphi_2), \overline{\varphi}) \\ &\ge \|\nabla\varphi\|^2 - \alpha\|\varphi\|^2 - (F'(\varphi_1) - F'(\varphi_2), \overline{\varphi}).\end{aligned}$$

By (3.3.4), we have

$$\alpha\|\varphi\|^2 \leq \frac{1}{2}\|\nabla\varphi\|^2 + C\|\varphi\|_*^2. \tag{4.2.6}$$

Then, we end up with

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2 + \frac{1}{2}\|\varphi\|_V^2 + \beta\|\varphi\|_*^2 \\ &\qquad \leq C\|\varphi\|_*^2 + C\Big(1 + \|F'(\varphi_1)\|_{L^1(\Omega)} + \|F'(\varphi_2)\|_{L^1(\Omega)}\Big)|\overline{\varphi}| + I_1 + I_2. \end{aligned} \tag{4.2.7}$$

Taking $\boldsymbol{v} = \mathbf{A}^{-1}\boldsymbol{u}$ in (4.2.1), we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_\sharp^2 + \|\boldsymbol{u}\|^2 = I_3 + I_4, \tag{4.2.8}$$

where

$$\begin{aligned} I_3 &= (\boldsymbol{u}_1 \otimes \boldsymbol{u}, \nabla\mathbf{A}^{-1}\boldsymbol{u}) + (\boldsymbol{u} \otimes \boldsymbol{u}_2, \nabla\mathbf{A}^{-1}\boldsymbol{u}), \\ I_4 &= (\nabla\varphi_1 \otimes \nabla\varphi, \nabla\mathbf{A}^{-1}\boldsymbol{u}) + (\nabla\varphi \otimes \nabla\varphi_2, \nabla\mathbf{A}^{-1}\boldsymbol{u}). \end{aligned}$$

Now, setting

$$\Phi = \frac{1}{2}\|\boldsymbol{u}\|_\sharp^2 + \frac{1}{2}\|\varphi\|_*^2$$

and summing (4.2.7) and (4.2.8), we are led to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \|\boldsymbol{u}\|^2 + \frac{1}{2}\|\varphi\|_V^2 + \beta\|\varphi\|_*^2 \\ &\qquad \leq C\|\varphi\|_*^2 + C\Big(1 + \|\Psi'(\varphi_1)\|_{L^1(\Omega)} + \|\Psi'(\varphi_2)\|_{L^1(\Omega)}\Big)|\overline{\varphi}| \\ &\qquad\quad + I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We proceed estimating all the remaining terms on the right-hand side. By (3.1.7), (3.6.2), the embedding $V \hookrightarrow L^p(\Omega)$, and the uniform bound of φ_2 in $L^\infty(\Omega)$, we have

$$\begin{aligned} I_1 &= (\boldsymbol{u}_1\varphi, \nabla\mathcal{N}(\varphi - \overline{\varphi})) \\ &\leq \|\boldsymbol{u}_1\|_{L^3(\Omega)}\|\varphi\|_{L^6(\Omega)}\|\varphi - \overline{\varphi}\|_{V_0'} \\ &\leq \frac{1}{4}\|\varphi\|_V^2 + C\|\boldsymbol{u}_1\|_{L^3(\Omega)}^2\|\varphi\|_*^2, \\ I_2 &= (\boldsymbol{u}\varphi_2, \nabla\mathcal{N}(\varphi - \overline{\varphi})) \\ &\leq \|\boldsymbol{u}\|\|\varphi - \overline{\varphi}\|_{V_0'} \\ &\leq \frac{1}{2}\|\boldsymbol{u}\|^2 + C\|\varphi\|_*^2, \\ I_3 &\leq \Big(\|\boldsymbol{u}_1\|_{L^4(\Omega)} + \|\boldsymbol{u}_2\|_{L^4(\Omega)}\Big)\|\boldsymbol{u}\|\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\|_{L^4(\Omega)} \\ &\leq C\Big(\|\boldsymbol{u}_1\|_{L^4(\Omega)} + \|\boldsymbol{u}_2\|_{L^4(\Omega)}\Big)\|\boldsymbol{u}\|_\sharp^{\frac{1}{2}}\|\boldsymbol{u}\|^{\frac{3}{2}} \\ &\leq \frac{1}{2}\|\boldsymbol{u}\|^2 + C\Big(\|\boldsymbol{u}_1\|_{L^4(\Omega)}^4 + \|\boldsymbol{u}_2\|_{L^4(\Omega)}^4\Big)\|\boldsymbol{u}\|_\sharp^2, \\ I_4 &\leq \Big(\|\nabla\varphi_1\|_{L^\infty(\Omega)} + \|\nabla\varphi_2\|_{L^\infty(\Omega)}\Big)\|\nabla\varphi\|\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\| \\ &\leq \frac{1}{4}\|\varphi\|_V^2 + \Big(\|\nabla\varphi_1\|_{L^\infty(\Omega)}^2 + \|\nabla\varphi_2\|_{L^\infty(\Omega)}^2\Big)\|\boldsymbol{u}\|_\sharp^2. \end{aligned}$$

Collecting the above estimates, we find the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi \leq C\Upsilon_1\Phi + C\Upsilon_2|\overline{\varphi}|,$$

where

$$\begin{aligned}\Upsilon_1(t) = 1 &+ \|\boldsymbol{u}_1(t)\|^2_{L^3(\Omega)} + \|\boldsymbol{u}_1(t)\|^4_{L^4(\Omega)} + \|\boldsymbol{u}_2(t)\|^4_{L^4(\Omega)} \\ &+ \|\nabla\varphi_1(t)\|^2_{L^\infty(\Omega)} + \|\nabla\varphi_2(t)\|^2_{L^\infty(\Omega)},\end{aligned}$$

and

$$\Upsilon_2(t) = 1 + \|F'(\varphi_1(t))\|_{L^1(\Omega)} + \|F'(\varphi_2(t))\|_{L^1(\Omega)}.$$

Thanks to Theorem (4.1.7) and the Sobolev embedding $W^{2,3}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we deduce that Υ_1 and Υ_2 belong to $L^1(0,T)$. Therefore, an application of the Gronwall lemma together with (4.2.4) entails

$$\|\boldsymbol{u}(t)\|^2_\sharp + \|\varphi(t)\|^2_* \leq C\|\boldsymbol{u}(0)\|^2_\sharp + C\|\varphi(0)\|^2_* + C|\overline{\varphi}(0)|, \quad \forall\, t \in [0,T],$$

where C depends on T. Due to the equivalence of norms, the above inequality concludes the proof. $\square$

4.3 Regularity Properties and Separation Property

In this section we show that weak solutions regularize instantaneously by virtue of the intrinsic parabolic nature of the system. Accordingly, any weak solution is indeed a strong solution on $\Omega \times (\sigma, \infty)$, for any $\sigma > 0$, namely system (4.0.1) is satisfied almost everywhere. To this aim, we provide higher order regularity estimates which are independent of the specific choice of the initial datum, but only depend on its total mass and the value of the energy. Hence, we fix $R > 0$ and $m \in [0,1)$ such that $-m \leq c \leq m$ and we consider bundles of trajectories $(\boldsymbol{u}, \varphi)$ departing from $(\boldsymbol{u}_0, \varphi_0)$ with

$$\mathcal{E}(\boldsymbol{u}_0, \varphi_0) \leq R \quad \text{and} \quad -m \leq \overline{\varphi}_0 \leq m.$$

In particular, due to (4.1.5), we deduce that

$$|\overline{\varphi}(t)| \leq m, \quad \forall\, t \geq 0.$$

In what follows, the generic constant $C > 0$ may depend on R and m.

Theorem 4.3.1. *For any $\sigma > 0$, there exists a positive constant $C = C(\sigma)$ such that*

$$\|\boldsymbol{u}\|_{L^\infty(\sigma,\infty;\mathbf{V}_\sigma)} + \|\mu\|_{L^\infty(\sigma,\infty;V)} \leq C \tag{4.3.1}$$

and

$$\int_t^{t+1} \|\boldsymbol{u}(\tau)\|^2_{\mathbf{W}_\sigma} + \|\partial_t\boldsymbol{u}(\tau)\|^2 + \|\partial_t\varphi(\tau)\|^2_V \,\mathrm{d}\tau \leq C, \quad \forall\, t \geq \sigma. \tag{4.3.2}$$

In addition, for any $p \geq 2$, there exists a positive constant $C = C(\sigma, p)$ such that

$$\|\varphi\|_{L^\infty(\sigma,\infty;W^{2,p}(\Omega))} + \|\Psi'(\varphi)\|_{L^\infty(\sigma,\infty;L^p(\Omega))} \leq C. \tag{4.3.3}$$

The estimates presented in the following proof are formal. However, they can be rigorously justified by establishing them first for Galerkin approximating solutions (see for instance [117]) and then passing to the limit in the usual way.

Proof of Theorem 4.3.1. We start by recalling that the dissipative inequalities in Theorem 4.1.7 (cf. Remark 4.1.8) yield, for any $t \geq 0$,

$$E_{GL}(\boldsymbol{u}(t), \varphi(t)) + \int_t^{t+1} \|\nabla \boldsymbol{u}(\tau)\|^2 + \|\nabla \mu(\tau)\|^2 + \|\varphi(\tau)\|_{H^2(\Omega)}^4 \,\mathrm{d}\tau \leq C. \tag{4.3.4}$$

In particular, we have for any $t \geq 0$

$$\|\boldsymbol{u}(t)\| + \|\varphi(t)\|_V \leq C, \tag{4.3.5}$$

and

$$\|\mu(t)\|_V \leq C(1 + \|\nabla \mu(t)\|). \tag{4.3.6}$$

Taking $v = \mu_t$ in (4.1.3), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \mu\|^2 + \langle \partial_t \varphi, \partial_t \mu \rangle + (\boldsymbol{u} \cdot \nabla \varphi, \partial_t \mu) + \beta(\varphi - c, \partial_t \mu) = 0.$$

In light of

$$\alpha \|\partial_t \varphi\|^2 \leq \frac{1}{2}\|\nabla \partial_t \varphi\|^2 + C\|\partial_t \varphi\|_{V'}^2,$$

we infer from the assumptions on F that

$$\begin{aligned}
\langle \partial_t \varphi, \partial_t \mu \rangle &= (\partial_t \varphi, -\Delta \partial_t \varphi) + (\partial_t \varphi, F''(\varphi)\partial_t \varphi) \\
&\geq \|\nabla \partial_t \varphi\|^2 - \alpha \|\partial_t \varphi\|^2 \\
&\geq \frac{1}{2}\|\nabla \partial_t \varphi\|^2 + C\|\partial_t \varphi\|_{V'}^2.
\end{aligned}$$

Besides, we observe that

$$\begin{aligned}
&(\boldsymbol{u} \cdot \nabla \varphi, \partial_t \mu) + \beta(\varphi - c, \partial_t \mu) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big\{(\boldsymbol{u} \cdot \nabla \varphi, \mu) + \beta(\varphi - c, \mu)\Big\} - (\partial_t \boldsymbol{u} \cdot \nabla \varphi, \mu) - (\boldsymbol{u} \cdot \nabla \partial_t \varphi, \mu) - \beta(\partial_t \varphi, \mu) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big\{(\boldsymbol{u} \cdot \nabla \varphi, \mu) + \beta(\varphi - c, \mu) - \frac{\beta}{2}\|\nabla \varphi\|^2 - \beta \int_\Omega F(\varphi)\,\mathrm{d}x\Big\} \\
&\quad - (\partial_t \boldsymbol{u} \cdot \nabla \varphi, \mu) - (\boldsymbol{u} \cdot \nabla \partial_t \varphi, \mu).
\end{aligned}$$

By (4.3.6), we estimate the last two terms as

$$\begin{aligned}
(\partial_t \boldsymbol{u} \cdot \nabla \varphi, \mu) &\leq \|\partial_t \boldsymbol{u}\| \|\nabla \varphi\|_{L^3(\Omega)} \|\mu\|_{L^6(\Omega)} \\
&\leq \frac{1}{4}\|\partial_t \boldsymbol{u}\|^2 + C\|\nabla \varphi\|_{L^3(\Omega)}^2 \|\mu\|_V^2 \\
&\leq \frac{1}{4}\|\partial_t \boldsymbol{u}\|^2 + C\|\nabla \varphi\|_{L^3(\Omega)}^2 \big(1 + \|\nabla \mu\|^2\big)
\end{aligned}$$

and

$$\begin{aligned}(\boldsymbol{u}\cdot\nabla\partial_t\varphi,\mu) &\le \|\boldsymbol{u}\|_{L^3(\Omega)}\|\nabla\partial_t\varphi\|\|\mu\|_{L^6(\Omega)}\\ &\le \frac{1}{4}\|\nabla\partial_t\varphi\|^2 + C\|\boldsymbol{u}\|^2_{L^3(\Omega)}\|\mu\|_V\\ &\le \frac{1}{4}\|\nabla\partial_t\varphi\|^2 + C\|\boldsymbol{u}\|^2_{L^3(\Omega)}\big(1+\|\nabla\mu\|^2\big).\end{aligned}$$

Accordingly, we arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\|\nabla\mu\|^2 + (\boldsymbol{u}\cdot\nabla\varphi,\mu) + \beta(\varphi-c,\mu) - \frac{\beta}{2}\|\nabla\varphi\|^2 - \beta\int_\Omega F(\varphi)\,\mathrm{d}x\Big\} + \frac{1}{4}\|\nabla\partial_t\varphi\|^2\\ &\le \frac{1}{4}\|\partial_t\boldsymbol{u}\|^2 + C\|\partial_t\varphi\|^2_{V'} + C\Big(\|\nabla\varphi\|^2_{L^3(\Omega)} + \|\boldsymbol{u}\|^2_{L^3(\Omega)}\Big)\|\nabla\mu\|^2\\ &\quad + C\|\nabla\varphi\|^2_{L^3(\Omega)} + C\|\boldsymbol{u}\|^2_{L^3(\Omega)}. \qquad (4.3.7)\end{aligned}$$

Taking $\boldsymbol{v}=\mathbf{A}\boldsymbol{u}$ and $\boldsymbol{v}=\partial_t\boldsymbol{u}$ in (4.1.2) and summing the resulting equations, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{u}\|^2 + \|\mathbf{A}\boldsymbol{u}\|^2 + \|\partial_t\boldsymbol{u}\|^2\\ &\qquad = -b(\boldsymbol{u},\boldsymbol{u},\mathbf{A}\boldsymbol{u}) - b(\boldsymbol{u},\boldsymbol{u},\partial_t\boldsymbol{u}) + (\mu\nabla\varphi,\mathbf{A}\boldsymbol{u}) + (\mu\nabla\varphi,\partial_t\boldsymbol{u}).\end{aligned}$$

By (3.1.7) and using (4.3.4)-(4.3.6), we get

$$\begin{aligned}-b(\boldsymbol{u},\boldsymbol{u},\mathbf{A}\boldsymbol{u}) - b(\boldsymbol{u},\boldsymbol{u},\partial_t\boldsymbol{u}) &\le \Big(\|\mathbf{A}\boldsymbol{u}\| + \|\partial_t\boldsymbol{u}\|\Big)\|\boldsymbol{u}\|_{L^4(\Omega)}\|\nabla\boldsymbol{u}\|_{L^4(\Omega)}\\ &\le C\Big(\|\mathbf{A}\boldsymbol{u}\| + \|\partial_t\boldsymbol{u}\|\Big)\|\nabla\boldsymbol{u}\|\|\mathbf{A}\boldsymbol{u}\|^{\frac{1}{2}}\\ &\le \frac{1}{4}\|\mathbf{A}\boldsymbol{u}\|^2 + \frac{1}{8}\|\partial_t\boldsymbol{u}\|^2 + C\|\nabla\boldsymbol{u}\|^4\end{aligned}$$

and

$$\begin{aligned}(\mu\nabla\varphi,\mathbf{A}\boldsymbol{u}) + (\mu\nabla\varphi,\partial_t\boldsymbol{u}) &\le \|\mu\|_V\|\nabla\varphi\|_{L^3(\Omega)}\Big(\|\mathbf{A}\boldsymbol{u}\| + +\|\partial_t\boldsymbol{u}\|\Big)\\ &\le \frac{1}{4}\|\mathbf{A}\boldsymbol{u}\|^2 + \frac{1}{8}\|\partial_t\boldsymbol{u}\|^2 + C\|\nabla\varphi\|^2_{L^3(\Omega)}\big(1+\|\nabla\mu\|^2\big).\end{aligned}$$

Hence, we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{u}\|^2 + \frac{1}{2}\|\mathbf{A}\boldsymbol{u}\|^2 + \frac{3}{4}\|\partial_t\boldsymbol{u}\|^2 \le C\|\nabla\boldsymbol{u}\|^4 + C\|\nabla\varphi\|^2_{L^3(\Omega)}\big(1+\|\nabla\mu\|^2\big). \qquad (4.3.8)$$

Collecting (4.3.7) and (4.3.8), we find the differential inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \frac{1}{2}\|\mathbf{A}\boldsymbol{u}\|^2 + \frac{1}{2}\|\partial_t\boldsymbol{u}\|^2 + \frac{1}{4}\|\nabla\partial_t\varphi\|^2 \qquad (4.3.9)\\ &\quad\le C\|\partial_t\varphi\|^2_{V'} + C\|\nabla\boldsymbol{u}\|^4 + C\Big(\|\nabla\varphi\|^2_{L^3(\Omega)} + \|\boldsymbol{u}\|^2_{L^3(\Omega)}\Big)\|\nabla\mu\|^2\\ &\qquad + C\|\nabla\varphi\|^2_{L^3(\Omega)} + C\|\boldsymbol{u}\|^2_{L^3(\Omega)},\end{aligned}$$

where

$$\begin{aligned}\Lambda(t) &= \|\nabla\boldsymbol{u}(t)\|^2 + \frac{1}{2}\|\nabla\mu(t)\|^2 + (\boldsymbol{u}(t)\cdot\nabla\varphi(t),\mu(t))\\ &\quad + \beta(\varphi(t)-c,\mu(t)) - \frac{\beta}{2}\|\nabla\varphi(t)\|^2 - \beta\int_\Omega F(\varphi(t))\,\mathrm{d}x.\end{aligned}$$

Next, we show that $\Lambda \geq -C$, for some positive constant C. Arguing as in Theorem 4.1.7,

$$\beta(\varphi - c, \mu) \geq \beta \|\nabla \varphi\|^2 + \beta \int_\Omega F(\varphi)\, \mathrm{d}x - \beta F(c)|\Omega| - \beta \Theta_0 \|\varphi - c\|^2.$$

Furthermore, by (3.1.7) and exploiting (4.3.4)-(4.3.6), we have

$$\begin{aligned}
(\boldsymbol{u} \cdot \nabla \varphi, \mu) &\leq \|\boldsymbol{u}\|_{L^4(\Omega)} \|\nabla \varphi\| \|\mu\|_{L^4(\Omega)} \\
&\leq C \|\boldsymbol{u}\|^{\frac{1}{2}} \|\nabla \boldsymbol{u}\|^{\frac{1}{2}} \|\mu\|_V \\
&\leq C \|\nabla \boldsymbol{u}\|^{\frac{1}{2}} + C \|\nabla \boldsymbol{u}\|^{\frac{1}{2}} \|\nabla \mu\| \\
&\leq \frac{1}{2} \|\nabla \boldsymbol{u}\|^2 + \frac{1}{4} \|\nabla \mu\|^2 + C.
\end{aligned}$$

In summary, we infer that

$$\Lambda \geq \frac{1}{2} \|\nabla \boldsymbol{u}\|^2 + \frac{1}{4} \|\nabla \mu\|^2 - C.$$

Moreover, it is easily seen that

$$\Lambda \leq C \|\nabla \boldsymbol{u}\|^2 + C \|\nabla \mu\|^2 + C,$$

which leads us to rewrite (4.3.9) as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \Lambda + \frac{1}{2} \|\mathbf{A} \boldsymbol{u}\|^2 + \frac{1}{2} \|\partial_t \boldsymbol{u}\|^2 + \frac{1}{4} \|\nabla \partial_t \varphi\|^2 \\
&\quad \leq C \Lambda^2 + C \Big(\|\nabla \varphi\|^2_{L^3(\Omega)} + \|\boldsymbol{u}\|^2_{L^3(\Omega)} \Big) \Lambda + C \\
&\qquad + C \|\partial_t \varphi\|^2_{V'} + C \|\nabla \varphi\|^2_{L^3(\Omega)} + C \|\boldsymbol{u}\|^2_{L^3(\Omega)}.
\end{aligned} \tag{4.3.10}$$

Owing to (4.3.4) and exploiting Sobolev embeddings, we infer that

$$\int_t^{t+1} \Lambda(\tau) + \|\boldsymbol{u}(\tau)\|^2_{L^3(\Omega)} + \|\nabla \varphi(\tau)\|^2_{L^3(\Omega)} \,\mathrm{d}\tau \leq C.$$

Also, by comparison,

$$\int_t^{t+1} \|\partial_t \varphi(\tau)\|^2_{V'} \,\mathrm{d}\tau \leq C.$$

Therefore, the uniform Gronwall lemma entails

$$\|\nabla \boldsymbol{u}(t)\| + \|\nabla \mu(t)\| \leq C, \quad \forall\, t \geq \sigma,$$

where C depends on σ. A further integration in time of (4.3.10) on any interval $[t, t+1]$, for $t \geq \sigma$, gives

$$\int_t^{t+1} \|\mathbf{A} \boldsymbol{u}(\tau)\|^2 + \|\partial_t \boldsymbol{u}(\tau)\|^2 + \|\nabla \partial_t \varphi(\tau)\|^2 \,\mathrm{d}\tau \leq C.$$

In turn, together with (4.1.5) and (3.6.2), this implies (4.3.2). Finally, having in mind the Neumann problem (3.4.1), we deduce the desired control (4.3.3) from Corollary 3.4.3 and $\mu \in L^\infty(\sigma, \infty; V)$. $\square$

Remark 4.3.2. As a consequence of Theorem 4.3.1, we learn from the regularity theory of the Neumann problem that $\mu \in L^2(t, t+1; H^3(\Omega))$ for every $t \geq \sigma$ and

$$\frac{\partial \mu}{\partial \boldsymbol{n}} = 0, \quad \text{a.e. } (x,t) \in \partial\Omega \times (\sigma, \infty).$$

The regularity attained in Theorem 4.3.1 does not entail the separation property from the pure phases. Indeed, a uniform-in-time control of $\Psi'(\varphi)$ in $L^\infty(\Omega)$ is needed. To this aim, we first learn from Lemma 3.4.6 that $\Psi''(\varphi)$ is bounded in $L^p(\Omega \times (t, t+1))$, for any $t \geq \sigma$. Then, the achieved regularity allows us to perform further higher order estimates which will guarantee the validity of the *instantaneous* separation property. More precisely, we have

Theorem 4.3.3. *Assume that* $\Psi \in \mathcal{C}^3(-1,1)$ *and* (H.2), (H.3) *hold. Then, for any* $\sigma > 0$*, there exists a positive constant* $C = C(\sigma)$ *such that*

$$\|\partial_t \boldsymbol{u}\|_{L^\infty(2\sigma,\infty;\mathbf{H}_\sigma)} + \|\partial_t \varphi\|_{L^\infty(2\sigma,\infty;H)} \leq C \tag{4.3.11}$$

and

$$\int_t^{t+1} \|\partial_t \boldsymbol{u}(\tau)\|_{\mathbf{V}_\sigma}^2 + \|\partial_t \varphi(\tau)\|_{H^2(\Omega)}^2 \,\mathrm{d}\tau \leq C, \quad \forall\, t \geq 2\sigma. \tag{4.3.12}$$

In addition, there exist $\delta = \delta(\sigma, R, m) \in (0,1)$ *and* $C = C(\sigma) > 0$ *such that*

$$\sup_{t \geq 2\sigma} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta,$$

and

$$\sup_{t \geq 2\sigma} \|\boldsymbol{u}(t)\|_{\mathbf{W}_\sigma} + \|\varphi(t)\|_{H^4(\Omega)} \leq C. \tag{4.3.13}$$

Remark 4.3.4. Note that $\varphi \in \mathcal{C}(\overline{\Omega} \times [\sigma, \infty))$, $\sigma > 0$, by Theorem 4.3.1 and the Aubin embedding theorem. Hence, we have, in particular,

$$|\varphi(x,t)| \leq 1 - \delta, \quad \forall\, (x,t) \in \overline{\Omega} \times [2\sigma, \infty).$$

Proof of Theorem 4.3.3. Let us define $f = \mu + \Theta_0 \varphi$. An application of Lemma 3.4.6 entails for any $p \geq 2$ that

$$\|\Psi''(\varphi)\|_{L^\infty(\sigma,\infty;L^p(\Omega))} \leq C, \tag{4.3.14}$$

where C depends on σ and p. We proceed by showing additional a priori estimates on the solutions. To this end, given $h > 0$, we denote the difference quotient of a function v by

$$\partial_t^h v = \frac{1}{h}\Big(v(t+h) - v(t)\Big).$$

On account of Definition 4.1.1, $\partial_t^h \varphi$ solves

$$\langle \partial_t \partial_t^h \varphi, v\rangle + (\partial_t^h \boldsymbol{u} \cdot \nabla \varphi(t+h), v) + (\boldsymbol{u} \cdot \nabla \partial_t^h \varphi, v) + \beta(\partial_t^h \varphi, v) + (\nabla \partial_t^h \mu, \nabla v) = 0,$$

for all $v \in V$. Taking $v = \partial_t^h \varphi$ in the above equation, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2 + \beta\|\partial_t^h\varphi\|^2 + (\nabla\partial_t^h\mu, \nabla\partial_t^h\varphi) = J_1 + J_2, \tag{4.3.15}$$

having set

$$J_1 = -(\partial_t^h\boldsymbol{u}\cdot\nabla\varphi(t+h), \partial_t^h\varphi), \quad J_2 = -(\boldsymbol{u}\cdot\nabla\partial_t^h\varphi, \partial_t^h\varphi).$$

Integrating by parts, and using the boundary conditions (cf. Remark 4.3.2), we have

$$\begin{aligned}
&(\nabla\partial_t^h\mu, \nabla\partial_t^h\varphi) = -(\partial_t^h\mu, \Delta\partial_t^h\varphi)\\
&\quad = \|\Delta\partial_t^h\varphi\|^2 - \Theta_0\|\nabla\partial_t^h\varphi\|^2 - (\frac{1}{h}[\Psi'(\varphi(t+h)) - \Psi'(\varphi(t))], \Delta\partial_t^h\varphi).
\end{aligned}$$

By the convexity of Ψ'', we find the control

$$\begin{aligned}
\frac{1}{h}\Big|\Psi'(\varphi(t+h)) - \Psi'(\varphi(t))\Big| &\le \int_0^1 \Psi''(\tau\varphi(t+h) + (1-\tau)\varphi(t))\,|\partial_t^h\varphi|\,\mathrm{d}\tau\\
&\le \int_0^1 \Big(\tau\Psi''(\varphi(t+h)) + (1-\tau)\Psi''(\varphi(t))\Big)\,|\partial_t^h\varphi|\,\mathrm{d}\tau\\
&\le \Big(\Psi''(\varphi(t+h)) + \Psi''(\varphi(t))\Big)|\partial_t^h\varphi|,
\end{aligned}$$

which, in turn, leads us to

$$\begin{aligned}
&\Big|(\frac{1}{h}[\Psi'(\varphi(t+h)) - \Psi'(\varphi(t))], \Delta\partial_t^h\varphi)\Big|\\
&\quad \le \frac{1}{4}\|\Delta\partial_t^h\varphi\|^2 + C\Big(\|\Psi''(\varphi(t+h))\|_{L^3(\Omega)}^2 + \|\Psi''(\varphi(t))\|_{L^3(\Omega)}^2\Big)\|\partial_t^h\varphi\|_{L^6(\Omega)}^2.
\end{aligned}$$

We notice that

$$\begin{aligned}
\|\partial_t^h\varphi\|_{L^6(\Omega)}^2 &\le C\|\nabla\partial_t^h\varphi\|^2 + C|\overline{\partial_t^h\varphi}|^2\\
&\le C\|\partial_t^h\varphi\|\|\Delta\partial_t^h\varphi\| + C|\overline{\partial_t^h\varphi}|^2.
\end{aligned}$$

Then, we arrive at

$$\begin{aligned}
(\nabla\partial_t^h\mu, \nabla\partial_t^h\varphi) \ge \frac{1}{2}\|\Delta\partial_t^h\varphi\|^2 &- C\Big(1 + \|\Psi''(\varphi(t+h))\|_{L^3(\Omega)}^4 + \|\Psi''(\varphi(t))\|_{L^3(\Omega)}^4\Big)\|\partial_t^h\varphi\|^2\\
&- C\Big(\|\Psi''(\varphi(t+h))\|_{L^3(\Omega)}^2 + \|\Psi''(\varphi(t))\|_{L^3(\Omega)}^2\Big)|\overline{\partial_t^h\varphi}|^2.
\end{aligned}$$

Moreover, by (4.3.1) and (4.3.3), we have

$$\begin{aligned}
J_1 &\le \|\partial_t^h\boldsymbol{u}\|\|\nabla\varphi(t+h)\|_{L^\infty(\Omega)}\|\partial_t^h\varphi\|\\
&\le C\|\partial_t^h\boldsymbol{u}\|^2 + C\|\partial_t^h\varphi\|^2
\end{aligned}$$

and

$$\begin{aligned}
J_2 &\le \|\boldsymbol{u}(t)\|_{L^3(\Omega)}\|\nabla\partial_t^h\varphi\|\|\partial_t^h\varphi\|_{L^6(\Omega)}\\
&\le C\|\nabla\partial_t^h\varphi\|^2 + |\overline{\partial_t^h\varphi}|\|\nabla\partial_t^h\varphi\|\\
&\le \frac{1}{4}\|\Delta\partial_t^h\varphi\|^2 + C\|\partial_t^h\varphi\|^2 + C|\overline{\partial_t^h\varphi}|^2.
\end{aligned}$$

Collecting the above estimates, we end up with the differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2+\frac{1}{4}\|\Delta\partial_t^h\varphi\|^2+\beta\|\partial_t^h\varphi\|^2\le\Upsilon\|\partial_t^h\varphi\|^2+\Upsilon^{\frac{1}{2}}|\overline{\partial_t^h\varphi}|^2+C\|\partial_t^h\boldsymbol{u}\|^2, \quad (4.3.16)$$

where

$$\Upsilon(t)=C(1+\|\Psi''(\varphi(t+h))\|^4_{L^3(\Omega)}+\|\Psi''(\varphi(t))\|^4_{L^3(\Omega)}).$$

We observe that Υ is summable in light of (4.3.14). Now, we rewrite (4.1.2) for $\partial_t^h\boldsymbol{u}$ getting

$$\begin{aligned}\langle\partial_t\partial_t^h\boldsymbol{u},\boldsymbol{v}\rangle-(\boldsymbol{u}(t+h)\otimes\partial_t^h\boldsymbol{u},\nabla\boldsymbol{v})-(\partial_t^h\boldsymbol{u}\otimes\boldsymbol{u},\nabla\boldsymbol{v})+(\nabla\partial_t^h\boldsymbol{u},\nabla\boldsymbol{v})\\=(\nabla\varphi(t+h)\otimes\nabla\partial_t^h\varphi,\nabla\boldsymbol{v})+(\nabla\partial_t^h\varphi\otimes\nabla\varphi,\nabla\boldsymbol{v}),\end{aligned}$$

for all $\boldsymbol{v}\in\mathbf{V}_\sigma$. Taking $\boldsymbol{v}=\partial_t^h\boldsymbol{u}$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\boldsymbol{u}\|^2+\|\nabla\partial_t^h\boldsymbol{u}\|^2=J_3+J_4,$$

having set

$$\begin{aligned}J_3&=(\boldsymbol{u}(t+h)\otimes\partial_t^h\boldsymbol{u},\nabla\partial_t^h\boldsymbol{u})+(\partial_t^h\boldsymbol{u}\otimes\boldsymbol{u},\nabla\partial_t^h\boldsymbol{u}),\\J_4&=(\nabla\varphi(t+h)\otimes\nabla\partial_t^h\varphi,\nabla\partial_t^h\boldsymbol{u})+(\nabla\partial_t^h\varphi\otimes\nabla\varphi,\nabla\partial_t^h\boldsymbol{u}).\end{aligned}$$

Exploiting the estimates (4.3.1) and (4.3.3), and using (3.1.7), we control J_i, $i=3,4$, as follows

$$\begin{aligned}J_3&\le\Big(\|\boldsymbol{u}(t+h)\|_{L^4(\Omega)}+\|\boldsymbol{u}(t)\|_{L^4(\Omega)}\Big)\|\partial_t^h\boldsymbol{u}\|_{L^4(\Omega)}\|\nabla\partial_t^h\boldsymbol{u}\|\\&\le\frac{1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2+C\|\partial_t^h\boldsymbol{u}\|^2\end{aligned}$$

and

$$\begin{aligned}J_4&=\Big(\|\nabla\varphi(t+h)\|_{L^\infty(\Omega)}+\|\nabla\varphi(t)\|_{L^\infty(\Omega)}\Big)\|\nabla\partial_t^h\varphi\|\|\nabla\partial_t^h\boldsymbol{u}\|\\&\le C\|\partial_t^h\varphi\|^{\frac{1}{2}}\|\Delta\partial_t^h\varphi\|^{\frac{1}{2}}\|\nabla\partial_t^h\boldsymbol{u}\|\\&\le\frac{1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2+\frac{1}{8}\|\Delta\partial_t^h\varphi\|^2+C\|\partial_t^h\varphi\|^2.\end{aligned}$$

In summary, setting

$$\Phi(t)=\frac{1}{2}\|\partial_t^h\varphi(t)\|^2+\frac{1}{2}\|\partial_t^h\boldsymbol{u}(t)\|^2,$$

we find the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi+\frac{1}{8}\|\Delta\partial_t^h\varphi\|^2+\beta\|\partial_t^h\varphi\|^2+\frac{1}{2}\|\nabla\partial_t^h\boldsymbol{u}\|^2\le C(1+\Upsilon)\Phi+\Upsilon^{\frac{1}{2}}|\overline{\partial_t^h\varphi}|^2.$$

Observing that

$$\|\partial_t^h v\|_{L^2(t,t+1;H)}\le\|\partial_t v\|_{L^2(t,t+1+h;H)},$$

we easily deduce from (4.3.2) and (4.3.14) that

$$\int_t^{t+1}\Phi(\tau)+\Upsilon(\tau)\,\mathrm{d}\tau\le C,$$

where C is independent of h. Hence, the uniform Gronwall lemma yields

$$\|\partial_t^h \boldsymbol{u}\|_{L^\infty(2\sigma,\infty;\mathbf{H}_\sigma)} + \|\partial_t^h \varphi\|_{L^\infty(2\sigma,\infty;H)} \le C$$

and

$$\int_t^{t+1} \|\nabla \partial_t^h \boldsymbol{u}(\tau)\|^2 + \|\Delta \partial_t^h \varphi(\tau)\|^2 \, \mathrm{d}\tau \le C, \quad \forall\, t \ge 2\sigma.$$

A final passage to the limit as $h \to 0$ entails (4.3.11) and (4.3.12). We are now in a position to prove the separation property. By the elliptic regularity of the Neumann problem, we obtain

$$\|\mu\|_{L^\infty(2\sigma,\infty;H^2(\Omega))} \le C. \tag{4.3.17}$$

Accordingly, using the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we infer from Lemma 3.4.2 with $p = \infty$ that

$$\|\Psi'(\varphi)\|_{L^\infty(\Omega\times(t,t+1))} \le C, \quad \forall\, t \ge 2\sigma.$$

Hence, there exists $\delta > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega\times(t,t+1))} \le 1 - \delta, \quad \forall\, t \ge 2\sigma.$$

Besides, due to (4.3.2) and (4.3.12), an application of [9, Theorem 1.1] gives us

$$\|\boldsymbol{u}\|_{L^\infty(2\sigma,\infty;\mathbf{W}^{1,4}(\Omega))} \le C.$$

Therefore, $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ is bounded in $L^\infty(2\sigma, \infty; H)$. In turn, we infer from the regularity theory of the Stokes problem together with (4.3.11) that

$$\|\boldsymbol{u}\|_{L^\infty(2\sigma,\infty;\mathbf{W}_\sigma)} \le C.$$

Next, in light of (4.3.3) and (4.3.17), we derive that

$$\|\varphi(t)\|_{L^\infty(2\sigma,\infty;H^4(\Omega))} \le C.$$

Finally, due to the continuity in time of the solution, the above inequalities hold for any $t \ge 2\sigma$, this is

$$\sup_{t \ge 2\sigma} \|\boldsymbol{u}(t)\|_{\mathbf{W}_\sigma} + \|\varphi(t)\|_{H^4(\Omega)} \le C.$$

□

Remark 4.3.5. It is worth mentioning that the regularity results stated in Section 4.3 are also valid if $\varepsilon = 0$. In that case, the novelty compared to [1] is the validity of the separation property in dimension two on the interval (σ, ∞), for any $\sigma > 0$.

4.4 Longtime Behavior

In this section we discuss the longtime behavior of the NSCH system ($\varepsilon = 0$) and the NSCHO system ($\varepsilon > 0$) within the theory of infinite-dimensional dynamical systems. Given $m \in [0, 1)$ such that $-m \le c \le m$, we introduce the complete metric space (cf. Remark 4.1.2)

$$\mathcal{V}_m = \Big\{ \varphi \in V \cap L^\infty(\Omega) : \|\varphi\|_{L^\infty(\Omega)} \le 1 \text{ and } -m \le \overline{\varphi} \le m \Big\},$$

endowed with the metric

$$\mathbf{d}(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|_V.$$

We also define the phase space

$$\mathcal{H}_m = \mathbf{H}_\sigma \times \mathcal{V}_m$$

equipped with the corresponding graph metric. For any $\beta \geq 0$, on account of Theorems 4.1.6 and 4.2.1, problem (4.0.1)-(4.0.2) generates a strongly continuous semigroup (dynamical system)

$$\mathcal{S}_\beta(t) : \mathcal{H}_m \to \mathcal{H}_m,$$

via the rule

$$\mathcal{S}_\beta(t)(\boldsymbol{u}_0, \varphi_0) = (\boldsymbol{u}(t), \varphi(t)), \quad \forall\, t \geq 0,$$

$(\boldsymbol{u}, \varphi)$ being the unique weak solution to the NSCHO system with initial condition $(\boldsymbol{u}_0, \varphi_0)$. This is a one-parameter family of maps $\mathcal{S}_\beta(t)$ on $\mathcal{H}_m$ satisfying the properties:

- $\mathcal{S}_\beta(0) = \mathrm{Id}$;
- $\mathcal{S}_\beta(t + \tau) = \mathcal{S}_\beta(t)\mathcal{S}_\beta(\tau)$, for every $t, \tau \geq 0$;
- $t \mapsto \mathcal{S}_\beta(t)(\boldsymbol{u}_0, \varphi_0) \in \mathcal{C}([0, \infty), \mathcal{H}_m)$, for every $(\boldsymbol{u}_0, \varphi_0) \in \mathcal{H}_m$.

Moreover, thanks to Theorem 4.3.1 and arguing by interpolation, we have the following further property.

Proposition 4.4.1. *For every $t \geq 0$, $\mathcal{S}_\beta(t) \in \mathcal{C}(\mathcal{H}_m, \mathcal{H}_m)$.*

Proof. The case $t = 0$ is trivial. Let us fix $t > 0$. We consider a sequence $\{(\boldsymbol{u}_{0n}, \varphi_{0n})\} \subset \mathcal{H}_m$ and $(\boldsymbol{u}_0, \varphi_0) \in \mathcal{H}_m$ such that $\|\boldsymbol{u}_{0n} - \boldsymbol{u}_0\| \to 0$, $\mathbf{d}(\varphi_{0n}, \varphi_0) \to 0$ as $n \to \infty$. By (4.3.1) and (4.3.3), there exists a constant $C > 0$, independent of n, such that

$$\|\boldsymbol{u}_n(t)\|_{\mathbf{V}_\sigma} + \|\varphi_n(t)\|_{H^2(\Omega)} + \|\boldsymbol{u}(t)\|_{\mathbf{V}_\sigma} + \|\varphi(t)\|_{H^2(\Omega)} \leq C, \tag{4.4.1}$$

where $(\boldsymbol{u}_n(t), \varphi_n(t)) = \mathcal{S}_\beta(t)(\boldsymbol{u}_{0n}, \varphi_{0n})$. Then, by virtue of Theorem 4.2.1, by standard interpolation we find

$$\begin{aligned}
&\|\boldsymbol{u}(t) - \boldsymbol{u}_n(t)\| + \|\varphi(t) - \varphi_n(t)\|_V \\
&\leq \|\boldsymbol{u}(t) - \boldsymbol{u}_n(t)\|_{\mathbf{V}'_\sigma}^{\frac{1}{2}} \|\boldsymbol{u}(t) - \boldsymbol{u}_n(t)\|_{\mathbf{V}_\sigma}^{\frac{1}{2}} + \|\varphi(t) - \varphi_n(t)\|_{V'}^{\frac{1}{3}} \|\varphi(t) - \varphi_n(t)\|_{H^2(\Omega)}^{\frac{2}{3}} \\
&\leq C\|\boldsymbol{u}(t) - \boldsymbol{u}_n(t)\|_{\mathbf{V}'_\sigma}^{\frac{1}{3}} + C\|\varphi(t) - \varphi_n(t)\|_{V'}^{\frac{1}{3}} \\
&\leq C\Big(\|\boldsymbol{u}_0 - \boldsymbol{u}_{0n}\|_{\mathbf{V}'_\sigma} + \|\varphi_0 - \varphi_{0n}\|_{V'} + |\overline{\varphi}_0 - \overline{\varphi}_{0n}|^{\frac{1}{2}}\Big)^{\frac{1}{3}},
\end{aligned}$$

which, in turn, gives the desired conclusion. $\square$

On account of Theorem 4.1.7, the semigroup $\mathcal{S}_\beta(t)$ is dissipative. Namely, there exists $R > 0$ such that the ball $\mathcal{B}_0$ of radius R centered at 0 in $\mathcal{H}_m$ is an absorbing set, i.e. for every bounded set $\mathcal{B} \subset \mathcal{H}_m$, there exists $t_{\mathcal{B}}$ such that

$$\mathcal{S}_\beta(t)\mathcal{B} \subset \mathcal{B}_0, \quad \forall\, t \geq t_{\mathcal{B}}.$$

Hence, the trajectories originating from any given bounded set eventually fall into a bounded set of the phase space. Actually, the trajectories are in fact attracted by thinner invariant subsets of the phase space $\mathcal{H}_m$. For a complete dissertation of the theory, we refer the reader to [120] and [149]. In order to show this, in addition to the general requirements, we assume in what follows that Ψ complies with the hypotheses of Theorem 4.3.3.

Theorem 4.4.2. *For any $\beta \geq 0$, the dynamical system $\mathcal{S}_\beta(t)$ on $\mathcal{H}_m$ has the global attractor $\mathcal{A}_\beta \subset \mathbf{W}_\sigma \times H^4(\Omega)$, that is*

- *$\mathcal{A}_\beta$ is compact in $\mathcal{H}_m$;*
- *$\mathcal{A}_\beta$ is invariant, i.e. $\mathcal{S}_\beta(t)\mathcal{A}_\beta = \mathcal{A}_\beta$, for all $t \geq 0$;*
- *$\mathcal{A}_\beta$ is an attracting set, i.e. for every bounded set $\mathcal{B} \subset \mathcal{H}_m$*

$$\lim_{t\to\infty} \operatorname{dist}(\mathcal{S}(t)\mathcal{B}, \mathcal{A}_\beta) = 0,$$

where $\operatorname{dist}(\mathcal{A}, \mathcal{B})$ is the Hausdorff semidistance between $\mathcal{A}$ and $\mathcal{B}$.

The proof of Theorem 4.4.2 immediately follows from an application of the general result [149, Theorem 1.1]. Indeed, the uniform a priori estimate (4.3.13) entails the existence of a compact absorbing set in the phase space $\mathcal{H}_m$.

Theorem 4.4.3. *For any $\beta \geq 0$, the dynamical system $\mathcal{S}_\beta(t)$ on $\mathcal{H}_m$ possesses an exponential attractor $\mathcal{M}_\beta \subset \mathbf{W}_\sigma \times H^4(\Omega)$, that is*

- *$\mathcal{M}_\beta$ is compact in $\mathcal{H}_m$;*
- *$\mathcal{M}_\beta$ is semi-invariant, i.e. $\mathcal{S}_\beta(t)\mathcal{M}_\beta \subset \mathcal{M}_\beta$, for all $t \geq 0$;*
- *$\mathcal{M}_\beta$ has finite fractal dimension in $\mathcal{H}_m$;*
- *$\mathcal{M}_\beta$ is an exponentially attracting set, i.e. there exists $\omega > 0$ such that, for every bounded set $\mathcal{B} \subset \mathcal{H}_m$, there exists a constant C such that*

$$\operatorname{dist}(\mathcal{S}_\beta(t)\mathcal{B}, \mathcal{M}_\beta) \leq C\mathrm{e}^{-\omega t}, \quad \forall\, t \geq 0.$$

By virtue of the strict separation property, the proof of Theorem 4.4.3 can be done arguing as in [71, Section 3.3] if $\beta = 0$ and [21] if $\beta > 0$. In particular, the construction of a family of exponential attractors, which is robust (i.e. continuous) with respect to β, showed in [21] can be easily generalized to system (4.0.1)-(4.0.2).

We conclude by noticing that the convergence to an equilibrium for the NSCHO system is an interesting open issue. In the two-dimensional case, one would argue as in Section 7 of [W.2], taking advantage of the strict separation property. However, in three dimensions, the problem is connected with the lack of a Lyapunov function (cf. [1, Lemma 11] for $\varepsilon = 0$).

CHAPTER 5

The Hele-Shaw-Cahn-Hilliard system

In this chapter we study the Hele–Shaw–Cahn–Hilliard (HSCH) system with matched viscosities in two and three space dimensions. First, the existence of a global weak solution that satisfies a dissipative property is proven. Then, in dimension two, we obtain the uniqueness and regularity of global weak solutions. In particular, we show that any two-dimensional weak solution satisfies the *instantaneous* separation property. When the spatial dimension is three, we prove the existence of a unique global strong solution, provided that the initial datum is regular enough and sufficiently close to any local minimizer of the free energy. This also yields the local Lyapunov stability of the local minimizer itself. Finally, we discuss the convergence of any global solution to a single stationary state as time goes to infinity.

In a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the Hele–Shaw–Cahn–Hilliard system with matched viscosities ($\nu = 1$) reads as follows

$$\begin{cases} \boldsymbol{u} = -\nabla \pi + \mu \nabla \varphi, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi = \Delta \mu, \\ \mu = -\Delta \varphi + F'(\varphi), \end{cases} \qquad \text{in } \Omega \times (0, T), \tag{5.0.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = \partial_{\boldsymbol{n}} \mu = \partial_{\boldsymbol{n}} \varphi = 0, & \text{on } \partial\Omega \times (0, T), \\ \varphi(\cdot, 0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{5.0.2}$$

General agreement. Throughout this chapter, if it is not otherwise stated, we indicate by C a generic positive constant depending only on the domain and on structural quantities. The constant C may vary from line to line and even within the same line. Any further dependence will be explicitly pointed out if necessary.

5.1 Existence of Weak Solutions and Dissipativity

We assume throughout this chapter that the singular potential F satisfies the assumptions (H) and (H.1).

We introduce the notion of weak solution to the initial boundary value problem (5.0.1)-(5.0.2).

Definition 5.1.1. *Let $\varphi_0 \in V$ be such that $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. A triple $(\boldsymbol{u}, \pi, \varphi)$ is a weak solution to problem* (5.0.1)-(5.0.2) *on* $[0,T]$ *if*

$$
\begin{aligned}
&\boldsymbol{u} \in L^2(0,T;\mathbf{H}_\sigma), \quad \pi \in L^{\frac{8}{5}}(0,T;V_0),\\
&\varphi \in L^\infty(0,T;V) \cap L^4(0,T;H^2(\Omega)) \cap H^1(0,T;V'),\\
&\varphi \in L^\infty(\Omega \times (0,T)) \text{ with } |\varphi(x,t)| < 1 \text{ a.e. } (x,t) \in \Omega \times (0,T),\\
&\Psi(\varphi) \in L^\infty(0,T;L^1(\Omega)), \quad \Psi'(\varphi) \in L^2(0,T;H),\\
&\mu \in L^2(0,T;V),
\end{aligned}
$$

and

$$
\langle \partial_t \varphi, v \rangle + (\boldsymbol{u} \cdot \nabla \varphi, v) + (\nabla \mu, \nabla v) = 0, \quad \forall\, v \in V, \tag{5.1.1}
$$

for almost every $t \in (0,T)$, where

$$
\mu = -\Delta \varphi + F'(\varphi), \quad \boldsymbol{u} = -\nabla \pi + \mu \nabla \varphi, \tag{5.1.2}
$$

for almost every $(x,t) \in \Omega \times (0,T)$. Moreover, $\partial_{\boldsymbol{n}} \varphi = 0$ a.e. on $\partial\Omega \times (0,T)$, $\varphi(\cdot,0) = \varphi_0$ a.e. in Ω.

Remark 5.1.2. According to the Darcy's equation, the above Definition 5.1.1 is equivalent to the following weak formulation

$$
\begin{aligned}
\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x &= \int_\Omega \mu \nabla \varphi \cdot \boldsymbol{v} \, \mathrm{d}x,\\
\int_\Omega \nabla \pi \cdot \nabla \psi \, \mathrm{d}x &= \int_\Omega \mu \nabla \varphi \cdot \nabla \psi \, \mathrm{d}x,
\end{aligned}
$$

for almost every $t \in (0,T)$ and for any $\boldsymbol{v} \in \mathbf{H}_\sigma$, $\psi \in V$. Thus, the pressure π is recovered by the second equation. In addition, in light of the boundary conditions and the identity

$$
\mu \nabla \varphi = \nabla \Big(\frac{1}{2} |\nabla \varphi|^2 + \Psi(\varphi) \Big) - \mathrm{div}(\nabla \varphi \otimes \nabla \varphi),
$$

we can rewrite the weak formulation of Darcy's equation as

$$
\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x = \int_\Omega \varphi \nabla \mu \cdot \boldsymbol{v} \, \mathrm{d}x = -\int_\Omega \mathrm{div}(\nabla \varphi \otimes \nabla \varphi) \cdot \boldsymbol{v} \, \mathrm{d}x. \tag{5.1.3}
$$

The first result concerns the existence of global weak solutions, in both two and three dimensions.

Theorem 5.1.3. *Let $d = 2, 3$. Assume that $\varphi_0 \in V$ with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Then, for any $T > 0$, there exists at least one weak solution $(\boldsymbol{u}, \pi, \varphi)$ to problem* (5.0.1)-(5.0.2) *on $[0, T]$ such that*

$$\begin{aligned}
&\boldsymbol{u} \in L^s(0,T;\mathbf{V}), \quad \pi \in L^q(0,T;H^2(\Omega)),\\
&\varphi \in \mathcal{C}([0,T],V) \cap L^2(0,T;W^{2,p}(\Omega)),\\
&\Psi'(\varphi) \in L^2(0,T;L^p(\Omega)),
\end{aligned}$$

where $s = \frac{6}{5}$ if $d = 3$ or $s \in [1, \frac{4}{3})$ if $d = 2$; $q = \frac{8}{7}$ is $d = 3$ or $1 \le q < \frac{6}{5}$ if $d = 2$; $p = 6$ if $d = 3$ or $2 \le p < \infty$ if $d = 2$. Moreover, every weak solution satisfies the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}(\varphi(t)) + \|\boldsymbol{u}(t)\|^2 + \|\nabla\mu(t)\|^2 = 0, \quad \text{for a.e. } t \in (0,T), \tag{5.1.4}$$

as well as the mass conservation

$$\int_\Omega \varphi(t)\,\mathrm{d}x = \int_\Omega \varphi_0\,\mathrm{d}x, \quad \forall\, t \in [0,T]. \tag{5.1.5}$$

Remark 5.1.4. The assumption $|\overline{\varphi}_0| < 1$ indicates that the initial datum is not allowed to be a pure state (i.e. ± 1). On the other hand, we observe that if the initial datum is a pure state then no separation process will take place because we now have a single fluid.

The strategy to prove Theorem 5.1.3 is based on a standard approximation procedure. First, we introduce a family of regular potentials $\{F_\varepsilon\}$ that suitably approximates the singular potential F. Then we establish an existence result to the approximating problem with the regular potential F_ε, by means of the Galerkin method. Finally, for the approximate solutions $(\boldsymbol{u}_\varepsilon, \pi_\varepsilon, \varphi_\varepsilon)$ related to the family of regular potentials $\{F_\varepsilon\}$, we recover compactness by means of uniform energy estimates with respect to the approximation parameter ε and we show that as $\varepsilon \to 0$ the limit triple $(\boldsymbol{u}, p, \varphi)$ is indeed a global weak solution with weak to problem (5.0.1)-(5.0.2).

The approximating problem

For any $\varepsilon \in (0, \kappa)$ with κ being the constant given in (H.1), we introduce a family of regular potentials $\{F_\varepsilon\}$ that approximates the original singular potential F by setting

$$F_\varepsilon(s) = \Psi_\varepsilon(s) - \frac{\Theta_0}{2}s^2, \quad \forall\, s \in \mathbb{R}, \tag{5.1.6}$$

where

$$\Psi_\varepsilon(s) = \begin{cases} \displaystyle\sum_{j=0}^{2} \frac{1}{j!}\Psi^{(j)}(1-\varepsilon)\left[s-(1-\varepsilon)\right]^j, & \forall\, s \ge 1-\varepsilon,\\ \Psi(s), & \forall\, s \in [-1+\varepsilon, 1-\varepsilon],\\ \displaystyle\sum_{j=0}^{2} \frac{1}{j!}\Psi^{(j)}(-1+\varepsilon)\left[s-(-1+\varepsilon)\right]^j, & \forall\, s \le -1+\varepsilon. \end{cases} \tag{5.1.7}$$

By the above construction of F_ε, we obtain the following properties that will be useful in the proof of Theorem 5.1.3.

Lemma 5.1.5. *Assume that* (H.1) *is satisfied. Then, there exists* $\overline{\kappa} \in (0, \kappa]$ *such that for any* $\varepsilon \in (0, \overline{\kappa})$*, the approximating function* F_ε *given by* (5.1.6) *satisfies*

(AH) $F_\varepsilon \in \mathcal{C}^2(\mathbb{R})$ *and*

$$-\widetilde{\alpha} \le F_\varepsilon(s), \quad -\alpha \le F''_\varepsilon(s) \le L, \quad \forall\, s \in \mathbb{R},$$

where $\widetilde{\alpha}$ *is a positive constant independent of* ε*, the constant* α *is given in* (H) *while* L *is a positive constant that may depend on* ε*.*

For every $\varepsilon \in (0, \overline{\kappa})$ and F_ε being the regular potential constructed in (5.1.6), we consider the approximating problem (**AP1**):

$$\begin{cases} \boldsymbol{u}_\varepsilon = -\nabla \pi_\varepsilon + \mu_\varepsilon \nabla \varphi_\varepsilon, \\ \operatorname{div} \boldsymbol{u}_\varepsilon = 0, \\ \partial_t \varphi_\varepsilon + \boldsymbol{u}_\varepsilon \cdot \nabla \varphi_\varepsilon = \Delta \mu_\varepsilon, \\ \mu_\varepsilon = -\Delta \varphi_\varepsilon + \Psi'_\varepsilon(\varphi_\varepsilon), \end{cases} \quad \text{in } \Omega \times (0,T), \tag{5.1.8}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} = \partial_{\boldsymbol{n}} \mu_\varepsilon = \partial_{\boldsymbol{n}} \varphi_\varepsilon = 0, & \text{on } \partial\Omega \times (0,T), \\ \varphi_\varepsilon(\cdot, 0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{5.1.9}$$

We state the global existence of weak solutions to the approximating problem.

Proposition 5.1.6. *Let* $d = 2, 3$ *and* $\varepsilon \in (0, \overline{\kappa})$*. Suppose that* $\varphi_0 \in V$ *with* $\Psi(\varphi_0) \in L^1(\Omega)$ *and* $|\overline{\varphi}_0| < 1$*. Then, we have:*

(1) For every $T > 0$*, there exists at least one weak solution* $(\boldsymbol{u}_\varepsilon, p_\varepsilon, \varphi_\varepsilon)$ *to the approximating problem (**AP1**) on* $[0, T]$ *such that*

$$\begin{aligned} &\boldsymbol{u}_\varepsilon \in L^2(0,T; \mathbf{H}_\sigma), \quad \pi_\varepsilon \in L^{\frac{8}{5}}(0,T; V_0), \\ &\varphi_\varepsilon \in L^\infty(0,T;V) \cap L^2(0,T; H^3(\Omega)) \cap W^{1,\frac{8}{5}}(0,T;V'), \\ &\mu_\varepsilon \in L^2(0,T;V). \end{aligned}$$

Such a solution satisfies the weak formulation

$$\langle \partial_t \varphi_\varepsilon, v \rangle + (\boldsymbol{u}_\varepsilon \cdot \nabla \varphi_\varepsilon, v) + (\nabla \mu_\varepsilon, \nabla v) = 0, \quad \forall\, v \in V, \tag{5.1.10}$$

for almost every $t \in (0,T)$*, where*

$$\mu_\varepsilon = -\Delta \varphi_\varepsilon + \Psi'_\varepsilon(\varphi_\varepsilon), \quad \boldsymbol{u}_\varepsilon = -\nabla \pi_\varepsilon + \mu_\varepsilon \nabla \varphi_\varepsilon, \tag{5.1.11}$$

for almost every $(x,t) \in \Omega \times (0,T)$*.*

(2) The total mass is conserved

$$\int_\Omega \varphi_\varepsilon(t)\,\mathrm{d}x = \int_\Omega \varphi_0\,\mathrm{d}x, \quad \forall\, t \in [0,T].$$

(3) The pair $(\boldsymbol{u}_\varepsilon, \varphi_\varepsilon)$ *satisfies the* energy inequality

$$\frac{1}{2}\|\nabla\varphi_\varepsilon(t)\|^2 + \int_\Omega F_\varepsilon(\varphi_\varepsilon(t))\,\mathrm{d}x \le \frac{1}{2}\|\nabla\varphi_0\|^2 + \int_\Omega F_\varepsilon(\varphi_0)\,\mathrm{d}x, \qquad (5.1.12)$$

for almost every $t \in (0,T)$*, and*

$$\int_0^T \|\boldsymbol{u}_\varepsilon(\tau)\|^2 + \|\nabla\mu_\varepsilon(\tau)\|^2\,\mathrm{d}\tau \le \frac{1}{2}\|\nabla\varphi_0\|^2 + \int_\Omega F_\varepsilon(\varphi_0)\,\mathrm{d}x + C, \qquad (5.1.13)$$

where C *is a constant depending on* $\widetilde{\alpha}$ *(cf. Lemma 5.1.5), but is independent of the parameter* ε*.*

The existence of a weak solution to the approximating problem (**AP1**) on $[0,T]$ can be easily proven by employing a Galerkin approximation scheme (see, e.g., [99, Section 3] and [113]). Indeed, according to the property (AH) in Lemma 5.1.5, for any $\varepsilon \in (0,\overline{\kappa}]$, the approximating potential F_ε has a quadratic growth as $|s| \to +\infty$ and F'_ε is globally Lipschitz on $\mathbb{R}$.

Remark 5.1.7. We note that it is sufficient to assume $\varphi_0 \in V$ to reach the conclusions of Proposition 5.1.6. Indeed, the additional assumptions such that $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi_0}| < 1$ will be necessary to derive uniform estimates with respect to ε in the subsequent section. Moreover, the estimates for p_ε and $\partial_t\varphi_\varepsilon$ when $d = 2$ can be improved by arguing as in [90, Section 3.4]. Nonetheless, the regularity properties stated above are enough to pass to the limit as $\varepsilon \to 0^+$.

ε-independent a priori estimates

In order to pass to the limit as $\varepsilon \to 0^+$, it is necessary to obtain suitable uniform estimates for the approximating solutions $(\boldsymbol{u}_\varepsilon, \pi_\varepsilon, \varphi_\varepsilon)$ that are independent of $\varepsilon \in (0,\overline{\kappa}]$.

First, we report the following lemma which turns out to be useful in the sequel (see, e.g., [63] for a proof).

Lemma 5.1.8. *Suppose that* (H.1) *is satisfied. For* $\varepsilon \in (0,\overline{\kappa}]$*, the approximating function* F_ε *given by* (5.1.6) *satisfies the following properties:*

$$\begin{aligned} &|F'_\varepsilon(s)| \le |F'(s)|, && \forall\, s \in (-1,1),\\ &F_\varepsilon(s) \le F(s), && \forall\, s \in [-1,1]. \end{aligned}$$

Now, we are in a position to derive uniform estimates with respect to the approximate parameter ε.

First estimate. Since $\Psi(\varphi_0) \in L^1(\Omega)$, it holds $\|\varphi_0\|_{L^\infty(\Omega)} \le 1$. Then it follows from Lemma 5.1.8 and the energy inequality (5.1.12) that, for almost every $t \in (0,T)$,

$$\frac{1}{2}\|\nabla\varphi_\varepsilon(t)\|^2 + \int_\Omega \Psi_\varepsilon(\varphi_\varepsilon(t))\,\mathrm{d}x \le \frac{1}{2}\|\nabla\varphi_0\|^2 + \int_\Omega \Psi(\varphi_0)\,\mathrm{d}x = \mathcal{E}_{GL}(\varphi_0). \qquad (5.1.14)$$

Similarly, we infer from (5.1.13) that

$$\int_0^T \|\boldsymbol{u}_\varepsilon(\tau)\|^2 + \|\nabla\mu_\varepsilon(\tau)\|^2\,\mathrm{d}\tau \le \mathcal{E}_{GL}(\varphi_0) + C. \qquad (5.1.15)$$

Second estimate. Testing the fourth equation in (5.1.8) by $-\Delta\varphi_\varepsilon$, we obtain

$$\|\Delta\varphi_\varepsilon\|^2 - (\Psi_\varepsilon'(\varphi_\varepsilon), \Delta\varphi_\varepsilon) = -(\mu_\varepsilon, \Delta\varphi_\varepsilon).$$

Exploiting the integration by parts and the homogeneous Neumann boundary condition for φ_ε, we get

$$\|\Delta\varphi_\varepsilon\|^2 + (\Psi_\varepsilon''(\varphi_\varepsilon)\nabla\varphi_\varepsilon, \nabla\varphi_\varepsilon) = (\nabla\mu_\varepsilon, \nabla\varphi_\varepsilon).$$

Hence, we deduce from (AH) that

$$\|\Delta\varphi_\varepsilon\|^2 \le \alpha\|\nabla\varphi_\varepsilon\|^2 + \|\nabla\mu_\varepsilon\|\|\nabla\varphi_\varepsilon\|.$$

Taking the square of both sides and integrating in time, we have

$$\int_0^T \|\Delta\varphi_\varepsilon(\tau)\|^4 \,\mathrm{d}\tau \le C(1+T)\big(1+\mathcal{E}_{GL}(\varphi_0)\big)^2. \tag{5.1.16}$$

Third estimate. We provide a uniform estimate for $\partial_t\varphi_\varepsilon$. By comparison, we easily obtain

$$\|\partial_t\varphi_\varepsilon\|_{V'} \le \|\nabla\mu_\varepsilon\| + \|\boldsymbol{u}_\varepsilon\|\|\varphi_\varepsilon\|_{L^\infty(\Omega)}.$$

Applying the Hölder and Agmon inequalities (3.1.9), we infer from (5.1.14)–(5.1.16) that

$$\begin{aligned}
&\int_0^T \|\partial_t\varphi_\varepsilon(\tau)\|_{V'}^{\frac{8}{5}} \,\mathrm{d}\tau \\
&\le C\int_0^T \|\nabla\mu_\varepsilon(\tau)\|^{\frac{8}{5}} \,\mathrm{d}\tau + C\int_0^T \|\boldsymbol{u}_\varepsilon(\tau)\|^{\frac{8}{5}}\|\varphi_\varepsilon(\tau)\|_{L^\infty(\Omega)}^{\frac{8}{5}} \,\mathrm{d}\tau \\
&\le C\int_0^T \|\nabla\mu_\varepsilon(\tau)\|^{\frac{8}{5}} \,\mathrm{d}\tau + C\int_0^T \|\boldsymbol{u}_\varepsilon(\tau)\|^{\frac{8}{5}}\|\varphi_\varepsilon(\tau)\|_V^{\frac{4}{5}}\|\varphi_\varepsilon(\tau)\|_{H^2(\Omega)}^{\frac{4}{5}} \,\mathrm{d}\tau \\
&\le C\int_0^T (1+\|\nabla\mu_\varepsilon(\tau)\|^2) \,\mathrm{d}\tau \\
&\quad + C\|\varphi_\varepsilon\|_{L^\infty(0,T;V)}^{\frac{4}{5}} \left(\int_0^T \|\boldsymbol{u}_\varepsilon(\tau)\|^2 \,\mathrm{d}\tau\right)^{\frac{4}{5}} \left(\int_0^T \|\varphi_\varepsilon(\tau)\|_{H^2(\Omega)}^4 \,\mathrm{d}\tau\right)^{\frac{1}{5}} \\
&\le C(1+T)\big(1+\mathcal{E}_{GL}(\varphi_0)\big)^{\frac{8}{5}}.
\end{aligned} \tag{5.1.17}$$

Fourth estimate. We derive a uniform estimate for $\|\mu_\varepsilon\|_V$. On account of the Poincaré–Wirtinger inequality

$$\|\mu_\varepsilon\| \le C(\|\nabla\mu_\varepsilon\| + |\overline{\mu_\varepsilon}|) \tag{5.1.18}$$

and in light of (5.1.15), it is sufficient to estimate the mean value $\overline{\mu_\varepsilon}$. On the other hand, since

$$\int_\Omega \mu_\varepsilon \,\mathrm{d}x = \int_\Omega F_\varepsilon'(\varphi_\varepsilon) \,\mathrm{d}x,$$

it remains to find a uniform control of $F_\varepsilon'(\varphi_\varepsilon)$ in $L^1(\Omega)$. To this aim, we recall the well-known inequality for approximating functions of singular potentials satisfying the assumption (H) (see, e.g., [63], [118] and Chapter 7 for the proof)

$$\|\Psi_\varepsilon'(\varphi_\varepsilon)\|_{L^1(\Omega)} \le C\int_\Omega (\varphi_\varepsilon - \overline{\varphi}_0)\big(\Psi_\varepsilon'(\varphi_\varepsilon) - \overline{\Psi_\varepsilon'(\varphi_\varepsilon)}\big) \,\mathrm{d}x + C, \tag{5.1.19}$$

where C may depend on $\overline{\varphi}_0$ and Ψ but is independent of ε. Then, testing the fourth equation in (5.1.8) by $\varphi_\varepsilon - \overline{\varphi}_0$, and using the integration by parts together with the boundary condition on φ_ε and Poincaré's inequality (3.1.5), we find

$$\begin{aligned}
&\|\nabla\varphi_\varepsilon\|^2 + \int_\Omega (\varphi_\varepsilon - \overline{\varphi}_0)\Psi_\varepsilon'(\varphi_\varepsilon)\,\mathrm{d}x \\
&\quad = \int_\Omega \mu_\varepsilon(\varphi_\varepsilon - \overline{\varphi}_0)\,\mathrm{d}x + \Theta_0 \int_\Omega \varphi_\varepsilon(\varphi_\varepsilon - \overline{\varphi}_0)\,\mathrm{d}x \\
&\quad = \int_\Omega (\mu_\varepsilon - \overline{\mu_\varepsilon})(\varphi_\varepsilon - \overline{\varphi}_0)\,\mathrm{d}x + C\|\nabla\varphi_\varepsilon\|^2 \\
&\quad \le C\left(\|\nabla\mu_\varepsilon\|\|\nabla\varphi_\varepsilon\| + \|\nabla\varphi_\varepsilon\|^2\right).
\end{aligned} \tag{5.1.20}$$

Hence, collecting (5.1.19) and (5.1.20), and using (5.1.14) and (5.1.15), after an integration in time we get

$$\begin{aligned}
\int_0^T \|\Psi_\varepsilon'(\varphi_\varepsilon(\tau))\|_{L^1(\Omega)}^2\,\mathrm{d}\tau &\le C\int_0^T (\|\nabla\mu_\varepsilon(\tau)\|^2\|\nabla\varphi_\varepsilon(\tau)\|^2 + \|\nabla\varphi_\varepsilon(\tau)\|^4)\,\mathrm{d}\tau \\
&\le C(1+T)\big(1+\mathcal{E}_{GL}(\varphi_0)\big)^2,
\end{aligned} \tag{5.1.21}$$

which together with (5.1.14) yields

$$\begin{aligned}
\int_0^T |\overline{\mu_\varepsilon}(\tau)|^2\,\mathrm{d}\tau &\le C\int_0^T \Big(\|\Psi_\varepsilon'(\varphi_\varepsilon(\tau))\|_{L^1(\Omega)}^2 + \Theta_0^2\|\varphi_\varepsilon(\tau)\|_{L^1(\Omega)}^2\Big)\mathrm{d}\tau \\
&\le C(1+T)\big(1+\mathcal{E}_{GL}(\varphi_0)\big)^2.
\end{aligned}$$

The above estimate together with (5.1.15) and (5.1.18) implies

$$\int_0^T \|\mu_\varepsilon(\tau)\|_V^2\,\mathrm{d}\tau \le C(1+T)\big(1+\mathcal{E}_{GL}(\varphi_0)\big)^2. \tag{5.1.22}$$

Fifth estimate. We aim to derive a uniform estimate for the pressure π_ε. It follows from the Darcy's equation for $\boldsymbol{u}_\varepsilon$, the Gagliardo–Nirenberg inequality ($d = 3$) and the estimates (5.1.14), (5.1.15), (5.1.16) and (5.1.22) that

$$\begin{aligned}
&\int_0^T \|\nabla\pi_\varepsilon(\tau)\|^{\frac{8}{5}}\,\mathrm{d}\tau \\
&\quad \le C\int_0^T \|\boldsymbol{u}_\varepsilon(\tau)\|^{\frac{8}{5}} + \|\nabla\varphi_\varepsilon(\tau)\|_{\mathbf{L}^3(\Omega)}^{\frac{8}{5}}\|\mu_\varepsilon(\tau)\|_{L^6(\Omega)}^{\frac{8}{5}}\,\mathrm{d}\tau \\
&\quad \le C\int_0^T (1+\|\boldsymbol{u}_\varepsilon(\tau)\|^2)\,\mathrm{d}\tau + C\|\varphi_\varepsilon(t)\|_{L^\infty(0,T;V)}^{\frac{4}{5}}\int_0^T \|\varphi_\varepsilon(\tau)\|_{H^2(\Omega)}^{\frac{4}{5}}\|\mu_\varepsilon\|_V^{\frac{8}{5}}\,\mathrm{d}\tau \\
&\quad \le C\int_0^T (1+\|\boldsymbol{u}_\varepsilon(\tau)\|^2)\,\mathrm{d}\tau \\
&\qquad + C\|\varphi_\varepsilon\|_{L^\infty(0,T;V)}^{\frac{4}{5}}\Big(\int_0^T \|\varphi_\varepsilon(\tau)\|_{H^2(\Omega)}^4\,\mathrm{d}\tau\Big)^{\frac{1}{5}}\Big(\int_0^T \|\mu_\varepsilon(\tau)\|_V^2\,\mathrm{d}\tau\Big)^{\frac{4}{5}} \\
&\quad \le C(1+T)\big(1+\mathcal{E}_{GL}(\varphi_0)\big)^{\frac{12}{5}}.
\end{aligned} \tag{5.1.23}$$

Collecting all the above estimates, we conclude that

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(0,T;\mathbf{H}_\sigma)} \le C, \tag{5.1.24}$$
$$\|\pi_\varepsilon\|_{L^{\frac{8}{5}}(0,T;V_0)} \le C, \tag{5.1.25}$$
$$\|\varphi_\varepsilon\|_{L^\infty(0,T;V)} \le C, \tag{5.1.26}$$
$$\|\varphi_\varepsilon\|_{L^4(0,T;H^2(\Omega))} \le C, \tag{5.1.27}$$
$$\|\partial_t\varphi_\varepsilon\|_{L^{\frac{8}{5}}(0,T;V')} \le C, \tag{5.1.28}$$
$$\|\mu_\varepsilon\|_{L^2(0,T;V)} \le C, \tag{5.1.29}$$

where the constant $C > 0$ depends on the initial energy $\mathcal{E}_{GL}(\varphi_0)$, the form of Ψ, Ω and coefficients of the system, but is independent of ε.

Proof of Theorem 5.1.3

We are now in a position to prove Theorem 5.1.3. The proof consists of several steps.

Step 1. Preliminary convergence results. Thanks to the uniform estimates (5.1.24)–(5.1.29), letting $\varepsilon \to 0^+$, the following weak convergence results hold (up to a subsequence):

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}, \qquad \text{weakly in } L^2(0,T;\mathbf{H}_\sigma), \tag{5.1.30}$$
$$\pi_\varepsilon \rightharpoonup \pi, \qquad \text{weakly in } L^{\frac{8}{5}}(0,T;V_0), \tag{5.1.31}$$
$$\varphi_\varepsilon \rightharpoonup \varphi, \qquad \text{weakly star in } L^\infty(0,T;V), \tag{5.1.32}$$
$$\varphi_\varepsilon \rightharpoonup \varphi, \qquad \text{weakly in } L^4(0,T;H^2(\Omega)), \tag{5.1.33}$$
$$\partial_t\varphi_\varepsilon \rightharpoonup \partial_t\varphi, \quad \text{weakly in } L^{\frac{8}{5}}(0,T;V'), \tag{5.1.34}$$
$$\mu_\varepsilon \rightharpoonup \mu, \qquad \text{weakly in } L^2(0,T;V), \tag{5.1.35}$$

Besides, on account of the Aubin–Lions compactness lemma, we have

$$\varphi_\varepsilon \to \varphi, \quad \text{strongly in } \mathcal{C}([0,T],H) \cap L^4(0,T;W^{1,r}(\Omega)), \tag{5.1.36}$$

for $r \in [1,6)$ when $d = 3$ and $r \in [1,+\infty)$ when $d = 2$, which also implies the pointwise convergence

$$\varphi_\varepsilon \to \varphi, \quad \text{a.e. in } \Omega \times (0,T). \tag{5.1.37}$$

Step 2. L^∞-estimate for φ. On account of the singular potential Ψ, we shall prove that the limit function φ fulfills

$$\varphi \in L^\infty(\Omega \times (0,T)) \quad \text{and} \quad |\varphi(x,t)| < 1 \quad \text{a.e. in } \Omega \times (0,T). \tag{5.1.38}$$

It follows from (5.1.21) that

$$\|\Psi_\varepsilon'(\varphi_\varepsilon)\|_{L^1(\Omega\times(0,T))} \le C, \tag{5.1.39}$$

with C independent of ε. By the definition of Ψ_ε and the assumptions $(\mathrm{H})-(\mathrm{H.1})$, there exists a constant $\varrho \in (0,\overline{\kappa})$ such that for all $\varepsilon \in (0,\varrho]$, $\Psi_\varepsilon'(s) \ge 1$ for $s \in [1-\varrho,+\infty)$

and $\Psi_\varepsilon'(s) \le -1$ for $s \in (-\infty, -1+\varrho]$ and $\Psi_\varepsilon'(s)$ is monotone increasing for $s \in \mathbb{R}$. Then we introduce the sets

$$\begin{aligned} E_\varrho^\varepsilon &= \{(x,t) \in \Omega \times (0,T): \ |\varphi_\varepsilon(x,t)| > 1-\varrho\}, \quad \varepsilon \in (0,\varrho], \\ E_\varrho &= \{(x,t) \in \Omega \times (0,T): \ |\varphi(x,t)| > 1-\varrho\}. \end{aligned}$$

From the pointwise convergence of φ_ε and Fatou's Lemma, we have for any fixed ϱ

$$\mathrm{meas}(E_\varrho) \le \liminf_{\varepsilon\to 0^+} \mathrm{meas}(E_\varrho^\varepsilon).$$

At the same time, when $\varepsilon \in (0,\varrho]$, we infer from (5.1.39) that

$$\min\{\Psi'(1-\varrho), -\Psi'(-1+\varrho)\}\,\mathrm{meas}(E_\varrho^\varepsilon) \le \|\Psi_\varepsilon'(\varphi_\varepsilon)\|_{L^1(\Omega\times(0,T))} \le C,$$

where the constant C does not depend on ϱ and ε. Therefore, we have

$$\mathrm{meas}(E_\varrho) \le \frac{C}{\min\{\Psi'(1-\varrho), -\Psi'(-1+\varrho)\}}.$$

Passing to the limit as $\varrho \to 0^+$, we deduce that

$$\mathrm{meas}(\{(x,t) \in \Omega \times (0,T) : |\varphi(x,t)| \ge 1\}) = 0,$$

which yields the conclusion (5.1.38).

Step 3. Passage to the limit as $\varepsilon \to 0^+$. The L^∞-estimate (5.1.38) together with the pointwise convergence of φ_ε and the uniform convergence of Ψ_ε' to Ψ' on every compact set in $(-1,1)$ entails that

$$\Psi_\varepsilon'(\varphi_\varepsilon) \to \Psi'(\varphi) \quad \text{a.e. } (x,t) \in \Omega \times (0,T),$$

as $\varepsilon \to 0^+$. Besides, by comparison in the equation for μ_ε (see (5.1.11)) and owing to the estimates (5.1.27) and (5.1.29), we have

$$\|\Psi_\varepsilon'(\varphi_\varepsilon)\|_{L^2(0,T;H)} \le C,$$

uniformly in ε. Hence, up to a subsequence, it holds

$$\Psi_\varepsilon'(\varphi_\varepsilon) \to \Psi'(\varphi), \quad \text{weakly in } L^2(0,T;H). \tag{5.1.40}$$

On the other hand, it follows from (5.1.30), (5.1.35), and (5.1.36) with $r=4$ that

$$\mu_\varepsilon \nabla\varphi_\varepsilon \rightharpoonup \mu\nabla\varphi, \quad \text{weakly in } L^{\frac{4}{3}}(\Omega \times (0,T)). \tag{5.1.41}$$

In a similar manner, we have

$$\boldsymbol{u}_\varepsilon \cdot \nabla\varphi_\varepsilon \rightharpoonup \boldsymbol{u} \cdot \nabla\varphi, \quad \text{weakly in } L^{\frac{4}{3}}(\Omega \times (0,T)). \tag{5.1.42}$$

On account of (5.1.30)-(5.1.35) and (5.1.40)-(5.1.42), we are able to pass to the limit as $\varepsilon \to 0^+$ (up to a subsequence) in the weak formulation (5.1.10)-(5.1.11) for $(\boldsymbol{u}_\varepsilon, \pi_\varepsilon, \varphi_\varepsilon)$ and conclude that the limit triple $(\boldsymbol{u}, \pi, \varphi)$ fulfills (5.1.1)-(5.1.2).

Step 4. Further regularity properties. We establish some further regularity results for the global weak solution $(\boldsymbol{u}, \pi, \varphi)$ by making use of the estimates (5.1.27) and (5.1.38). First, it follows from (5.1.30) and (5.1.38) that

$$\boldsymbol{u} \cdot \nabla\varphi \in L^2(0,T;V'). \tag{5.1.43}$$

Then by comparison in (5.1.1) we immediately see that

$$\varphi_t \in L^2(0,T;V'). \tag{5.1.44}$$

Next, an application of Corollary 3.4.3 with $f = \mu + \Theta_0\varphi$ gives

$$\begin{aligned}\int_0^T \|\varphi(\tau)\|^2_{W^{2,p}(\Omega)}\,\mathrm{d}\tau + \int_0^T \|\Psi'(\varphi(\tau))\|^2_{L^p(\Omega)}\,\mathrm{d}\tau \\ \le C\int_0^T 1 + \|\mu(\tau)\|_V^2 + \|\varphi(\tau)\|_V^2\,\mathrm{d}\tau,\end{aligned} \tag{5.1.45}$$

where $p = 6$ if $d = 3$ and $2 \le p < \infty$ if $d = 2$. We recall that by the Gagliardo–Nirenberg inequality

$$\|\nabla\varphi\|_{\mathbf{L}^\infty(\Omega)} \le \begin{cases} C\|\varphi\|_{L^\infty(\Omega)}^{\frac{r-2}{2(r-1)}}\|\varphi\|_{W^{2,r}(\Omega)}^{\frac{r}{2(r-1)}}, & \text{for } d=2,\ r>2,\\ C\|\varphi\|_{L^\infty(\Omega)}^{\frac{1}{3}}\|\varphi\|_{W^{2,6}(\Omega)}^{\frac{2}{3}}, & \text{for } d=3,\end{cases} \tag{5.1.46}$$

and by interpolation between L^p-spaces

$$\|\Delta\varphi\|_{L^3(\Omega)} \le C\|\varphi\|_{H^2(\Omega)}^{\frac{2(r-3)}{3(r-2)}}\|\varphi\|_{W^{2,r}(\Omega)}^{\frac{r}{3(r-2)}}, \quad \text{for } r \ge 3. \tag{5.1.47}$$

Applying the curl operator to the Darcy's equation for $\boldsymbol{u}$ and exploiting the particular form of the Korteweg force, we deduce that

$$\|\nabla\times\boldsymbol{u}\| = \|\nabla\mu\times\nabla\varphi\| \le C\|\nabla\mu\|\|\nabla\varphi\|_{\mathbf{L}^\infty(\Omega)}. \tag{5.1.48}$$

Then by (5.1.38) and (5.1.46), we find

$$\begin{aligned}\|\nabla\times\boldsymbol{u}\|^{\frac{4(r-1)}{3r-2}} &\le C\|\nabla\mu\|^{\frac{4(r-1)}{3r-2}}\|\varphi\|_{W^{2,r}(\Omega)}^{\frac{2r}{3r-2}}\\ &\le C\|\nabla\mu\|^2 + C\|\varphi\|^2_{W^{2,r}(\Omega)}, \qquad \text{for } d=2,\ r>2,\\ \|\nabla\times\boldsymbol{u}\|^{\frac{6}{5}} &\le C\|\nabla\mu\|^{\frac{6}{5}}\|\varphi\|_{W^{2,6}(\Omega)}^{\frac{4}{5}}\\ &\le C\|\nabla\mu\|^2 + C\|\varphi\|^2_{W^{2,6}(\Omega)}, \qquad \text{for } d=3.\end{aligned}$$

Besides, according to the Neumann problem (3.5.2) with $\boldsymbol{f} = \mu\nabla\varphi$ and by using (3.1.3), we have

$$\|\Delta\pi\| = \|\nabla\cdot(\mu\nabla\varphi)\| \le C\|\nabla\mu\|\|\nabla\varphi\|_{\mathbf{L}^\infty(\Omega)} + C\|\mu\|_{L^6(\Omega)}\|\Delta\varphi\|_{L^3(\Omega)}. \tag{5.1.49}$$

Hence, we get

$$\begin{aligned}\|\Delta\pi\|^{\frac{6(r-2)}{5r-9}} &\le C\|\nabla\mu\|^{\frac{6(r-2)}{5r-9}}\|\varphi\|_{W^{2,r}(\Omega)}^{\frac{3r(r-2)}{(5r-9)(r-1)}} + C\|\mu\|_V^{\frac{6(r-2)}{5r-9}}\|\varphi\|_{H^2(\Omega)}^{\frac{4(r-3)}{5r-9}}\|\varphi\|_{W^{2,r}(\Omega)}^{\frac{2r}{5r-9}}\\ &\le C\|\mu\|_V^2 + C\|\varphi\|^2_{W^{2,r}(\Omega)} + C\|\varphi\|^4_{H^2(\Omega)} + C, \qquad \text{for } d=2,\ r\ge 3,\\ \|\Delta\pi\|^{\frac{8}{7}} &\le C\|\nabla\mu\|^{\frac{8}{7}}\|\varphi\|_{W^{2,6}(\Omega)}^{\frac{16}{21}} + C\|\mu\|_V^{\frac{8}{7}}\|\varphi\|_{W^{2,6}(\Omega)}^{\frac{4}{7}}\|\varphi\|_{H^2(\Omega)}^{\frac{4}{7}}\\ &\le C\|\mu\|_V^2 + C\|\varphi\|^2_{W^{2,6}(\Omega)} + C\|\varphi\|^4_{H^2(\Omega)} + C, \qquad \text{for } d=3.\end{aligned}$$

The above estimates together with (5.1.26), (5.1.29), (5.1.45) and the inequality (3.5.3) yield that

$$\|\boldsymbol{u}\|_{L^s(0,T;\mathbf{V})} \le C, \quad \|\pi\|_{L^q(0,T;H^2(\Omega))} \le C, \tag{5.1.50}$$

where $s = \frac{6}{5}$ if $d = 3$ and $1 \le s < \frac{4}{3}$ if $d = 2$, $q = \frac{8}{7}$ is $d = 3$ and $1 \le q < \frac{6}{5}$ if $d = 2$.

Step 5. Mass conservation and energy identity. For every global weak solution, taking $v = 1$ in (5.1.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi \,\mathrm{d}x = \langle \varphi_t, 1\rangle = 0,$$

which implies the mass conservation (5.1.5).

Next, due to the regularity properties of weak solution (see (5.1.30), (5.1.43), (5.1.44) and (5.1.50)), for a.e. $t \in (0, T)$ we are able to test the Darcy's equation by $\boldsymbol{u}$ and to take $v = \mu$ in (5.1.2) to get

$$\|\boldsymbol{u}\|^2 = \int_\Omega (\mu\nabla\varphi)\cdot \boldsymbol{u}\,\mathrm{d}x, \tag{5.1.51}$$

$$\langle \varphi_t, \mu\rangle + \int_\Omega (\boldsymbol{u}\cdot\nabla\varphi)\mu\,\mathrm{d}x + \|\nabla\mu\|^2 = 0. \tag{5.1.52}$$

Adding (5.1.51) and (5.1.52) together, we get

$$\langle \varphi_t, \mu\rangle + \|\boldsymbol{u}\|^2 + \|\nabla\mu\|^2 = 0, \quad \text{for a.e. } t \in (0, T). \tag{5.1.53}$$

On the other hand, we consider the functional

$$\mathcal{E}^*_{GL}(\varphi) = \frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega \Psi(\varphi)\,\mathrm{d}x$$

defined on H. It is well-known that $\mathcal{E}^*_{GL}$ is proper, lower semicontinuous, convex with domain

$$D(\mathcal{E}^*_{GL}) = \{\psi \in V \cap L^\infty(\Omega) : \psi(x) \in [-1, 1] \text{ a.e. } x \in \Omega\}.$$

Being the subgradient of $\mathcal{E}^*_{GL}$ equal to $\partial\mathcal{E}^*_{GL}(\varphi) = -\Delta\varphi + \Psi'(\varphi)$, and on account of the regularity $-\Delta\varphi + \Psi'(\varphi) \in L^2(0, T; V)$ and $\partial_t\varphi \in L^2(0, T; V')$ of a weak solution we learn from [131, Lemma 4.1] that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}(\varphi) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^*_{GL}(\varphi) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\Theta_0}{2}\|\varphi\|^2\\ &= \langle \varphi_t, -\Delta\varphi + \Psi'(\varphi)\rangle - \Theta_0\langle\varphi_t, \varphi\rangle\\ &= \langle \varphi_t, \mu\rangle, \quad \text{for a.e. } t \in (0, T).\end{aligned}$$

Here, we have also used the standard chain rule in $L^2(0, T; V) \cap H^1(0, T; V')$. As a consequence, the required energy identity (5.1.4) holds, which yields that $\mathcal{E}_{GL}(\varphi(t))$ is absolutely continuous on $[0, T]$ and fulfills

$$\mathcal{E}_{GL}(\varphi(t)) + \int_0^t \|\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2\,\mathrm{d}\tau = \mathcal{E}_{GL}(\varphi_0), \quad \forall\, t \ge 0. \tag{5.1.54}$$

Lastly, we see from (5.1.32) and (5.1.34) that $\varphi \in \mathcal{C}_w([0, T], V)$ as well as $\varphi \in \mathcal{C}([0, T], H)$. At the same time, the convexity of the function Ψ, we obtain that $t \to \|\nabla\varphi(t)\|^2$ is continuous (cf. [1, Theorem 6]). As a result, this gives $\varphi \in \mathcal{C}([0, T], V)$. The proof of Theorem 5.1.3 is complete.

Remark 5.1.9. The energy identity (5.1.54) and $\varphi \in \mathcal{C}([0,T],V)$ entail that $\int_\Omega \Psi(\varphi(t))\,\mathrm{d}x$ is bounded for all $t \geq 0$. Then, from assumption (H), we get

$$\sup_{t\geq 0} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1. \tag{5.1.55}$$

Remark 5.1.10. It is worth pointing out that the hypothesis on the growth of Ψ'' assumed in $(\mathrm{H}.1)$ can be removed by working with the approximation family of singular potentials proposed in Section 3.2.

Next, we show that any global weak solution is dissipative, namely,

Theorem 5.1.11. *Let the assumptions of Theorem 5.1.3 hold. Then, any global weak solution* $(\boldsymbol{u},\pi,\varphi)$ *satisfies the following dissipative estimate*

$$\mathcal{E}_{GL}(\varphi(t)) + \int_t^{t+1} \|\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2\,\mathrm{d}\tau \leq C\mathcal{E}_{GL}(\varphi_0)\mathrm{e}^{-\omega t} + C, \quad \forall\, t \geq 0,$$

where ω *and* C *are positive constants independent of the initial datum. Moreover, we have for all* $t \geq 0$

$$\int_t^{t+1} \|\varphi(\tau)\|_{H^2(\Omega)}^4 + \|\varphi(\tau)\|_{W^{2,p}(\Omega)}^2 + \|\Psi'(\varphi(\tau))\|_{L^p(\Omega)}^2\,\mathrm{d}\tau \leq C\mathcal{E}_{GL}(\varphi_0)^2\mathrm{e}^{-\omega t} + C,$$

where $p=6$ *if* $d=3$ *or* $p \in [2,+\infty)$ *if* $d=2$*. Here, the positive constant* C *depends on* $\overline{\varphi}_0 \in (-1,1)$ *and the parameter* p*, but is independent of other norms of the initial datum.*

Proof. Testing the equation for μ in (5.1.2) by $\varphi - \overline{\varphi}_0$ and using the mass conservation (5.1.5), we get

$$\|\nabla\varphi\|^2 + \int_\Omega F'(\varphi)(\varphi-\overline{\varphi}_0)\,\mathrm{d}x = \int_\Omega (\mu-\overline{\mu})(\varphi-\overline{\varphi}_0)\,\mathrm{d}x. \tag{5.1.56}$$

Recalling the basic inequality for a singular potential satisfying (H)

$$F(s) \leq F(w) + F'(s)(s-w) + \frac{\alpha}{2}(s-w)^2, \quad \forall\, s,w \in (-1,1),$$

and exploiting the estimate $|\varphi(x,t)| < 1$ for almost every $(x,t) \in \Omega\times(0,+\infty)$, we find

$$\begin{aligned}\int_\Omega F'(\varphi)(\varphi-\overline{\varphi}_0)\,\mathrm{d}x &\geq \int_\Omega F(\varphi)\,\mathrm{d}x - F(\overline{\varphi}_0)|\Omega| - \frac{\alpha}{2}\|\varphi-\overline{\varphi}_0\|^2\\ &\geq \int_\Omega F(\varphi)\,\mathrm{d}x - C,\end{aligned}$$

where $C>0$ depends on Ω but is independent of φ_0. Inserting the above inequality into (5.1.56) and applying Poincaré's inequality (3.1.5), we infer that

$$\frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega \Psi(\varphi)\,\mathrm{d}x \leq C\|\nabla\mu\|^2 + C. \tag{5.1.57}$$

Hence, in light of the energy identity (5.1.4), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}(\varphi) + \omega\mathcal{E}_{GL}(\varphi) + \|\boldsymbol{u}\|^2 + \frac{1}{2}\|\nabla\mu\|^2 \leq C,$$

where ω, C are positive constants independent of φ_0. An application of the Gronwall lemma yields

$$\mathcal{E}_{GL}(\varphi(t)) + \int_t^{t+1} \|\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \, \mathrm{d}\tau \leq C\mathcal{E}_{GL}(\varphi_0)\mathrm{e}^{-\omega t} + C. \tag{5.1.58}$$

for all $t \geq 0$. In particular, this gives

$$\|\varphi(t)\|_V^2 \leq C\mathcal{E}_{GL}(\varphi_0)\mathrm{e}^{-\omega t} + C, \quad \forall\, t \geq 0. \tag{5.1.59}$$

Next, by Lemma 3.4.4 with $f = \mu + \Theta_0\varphi$, we learn from (5.1.58) and (5.1.59) that

$$\int_t^{t+1} \|\Delta\varphi(\tau)\|^4 \, \mathrm{d}\tau \leq C\mathcal{E}_{GL}(\varphi_0)^2\mathrm{e}^{-\omega t} + C, \quad \forall\, t \geq 0.$$

Then, by repeating the same argument exploited in the proof of Theorem 5.1.3 to get a uniform control of μ in V (cf. (5.1.22)), we obtain

$$\int_t^{t+1} \|\mu(\tau)\|_V^2 \, \mathrm{d}\tau \leq C\mathcal{E}_{GL}(\varphi_0)^2\mathrm{e}^{-\omega t} + C, \quad \forall\, t \geq 0.$$

Here, the constant C depends on the total mass of the initial datum φ_0. Therefore, by Corollary 3.4.3 with the same choice of $f = \mu + \Theta_0\varphi$, we find

$$\int_t^{t+1} \|\varphi(\tau)\|_{W^{2,p}(\Omega)}^2 + \|\Psi'(\varphi)\|_{L^p(\Omega)}^2 \, \mathrm{d}\tau \leq C\mathcal{E}(\varphi_0)^2\mathrm{e}^{-\omega t} + C, \quad \forall\, t \geq 0.$$

where $p = 6$ if $d = 3$ and for any $p \geq 2$ if $d = 2$. □

5.2 Uniqueness of Weak Solutions in Two Dimensions

In this section, we address the uniqueness of weak solutions to problem (5.0.1)-(5.0.2) when the spatial dimension is two. In general, uniqueness of weak solutions for the Hele–Shaw–Cahn–Hilliard system (5.0.1) turns out to be a rather hard task, due to the low regularity of the velocity field $\boldsymbol{u}$ (cf. [99, 158] for the case of regular potentials, where the uniqueness remains an open question even in two dimensions). However, in the case of physically relevant singular potential, we are able to prove the following continuous dependence result on initial data in a lower-order function space (i.e., V_0') when $d = 2$.

Theorem 5.2.1. *Let $d = 2$. Assume that $\varphi_{0i} \in V$ with $\Psi(\varphi_{0i}) \in L^1(\Omega)$, $i = 1, 2$, and $\overline{\varphi}_{01} = \overline{\varphi}_{02} = m \in (-1, 1)$. Then, any pair of global weak solutions $(\boldsymbol{u}_1, \pi_1, \varphi_1)$ and $(\boldsymbol{u}_2, \pi_2, \varphi_2)$ to problem* (5.0.1)-(5.0.2) *on $[0, T]$ with initial data φ_{01} and φ_{02}, respectively, fulfills the following estimate*

$$\|\varphi_1(t) - \varphi_2(t)\|_{V_0'}^2 + \int_0^t \|\varphi_1(\tau) - \varphi_2(\tau)\|_V^2 \, \mathrm{d}\tau \leq C\|\varphi_{01} - \varphi_{02}\|_{V_0'}^2, \tag{5.2.1}$$

for every $t \in [0, T]$. Here, the positive constant C depends on T as well as on the initial energy $\mathcal{E}_{GL}(\varphi_{0i})$, $i = 1, 2$. In particular, the global weak solution to problem (5.0.1)-(5.0.2) *is unique.*

Proof. Let $(\boldsymbol{u}_1, \pi_1, \varphi_1)$ and $(\boldsymbol{u}_2, \pi_2, \varphi_2)$ be two global weak solutions to problem (5.0.1)-(5.0.2) on $[0, T]$ with initial data φ_{01} and φ_{02}, respectively. Their difference denoted by $(\boldsymbol{u}, \pi, \varphi) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, \pi_1 - \pi_2, \varphi_1 - \varphi_2)$ solves

$$\langle \partial_t \varphi, v \rangle - (\boldsymbol{u}_1 \varphi, \nabla v) - (\boldsymbol{u} \varphi_2, \nabla v) + (\nabla \mu, \nabla v) = 0, \quad \forall\, v \in V, \tag{5.2.2}$$

for almost every $t \in (0, T)$, where $\boldsymbol{u}$ and the difference of chemical potentials $\mu := \mu_1 - \mu_2$ satisfy (cf. Remark 5.1.2)

$$\begin{cases} \boldsymbol{u} = -\Pi\big(\operatorname{div}(\nabla \varphi_1 \otimes \nabla \varphi) + \operatorname{div}(\nabla \varphi \otimes \nabla \varphi_2)\big), \\ \mu = -\Delta \varphi + F'(\varphi_1) - F'(\varphi_2). \end{cases} \tag{5.2.3}$$

Thanks to the mass conservation and $\overline{\varphi}_{01} = \overline{\varphi}_{02}$, we observe that $\overline{\varphi} = 0$ for all $t \geq 0$. By Theorem 5.1.3 (see (5.1.26), (5.1.55)), we also know that

$$\|\varphi_i(t)\|_V + \|\varphi_i(t)\|_{L^\infty(\Omega)} \leq C, \quad \forall\, t \in [0, T], \quad i = 1, 2. \tag{5.2.4}$$

Taking $v = \mathcal{N}\varphi$ in (5.2.2), and using (3.3.2), we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi\|_{V_0'}^2 + (\mu, \varphi) = (\boldsymbol{u}_1 \varphi, \nabla \mathcal{N}\varphi) + (\boldsymbol{u} \varphi_2, \nabla \mathcal{N}\varphi). \tag{5.2.5}$$

Using integration by parts and the homogeneous Neumann boundary condition for φ, and making use of (H) and (3.3.4), we have

$$\begin{aligned} (\mu, \varphi) &= \|\nabla \varphi\|^2 + (F'(\varphi_1) - F'(\varphi_2), \varphi) \\ &\geq \|\nabla \varphi\|^2 - \alpha \|\varphi\|^2 \\ &\geq \frac{1}{2} \|\nabla \varphi\|^2 - C \|\varphi\|_{V_0'}^2. \end{aligned}$$

Then, the differential equality (5.2.5) turns into

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi\|_{V_0'}^2 + \frac{1}{2} \|\nabla \varphi\|^2 \leq C \|\varphi\|_{V_0'}^2 + I + J, \tag{5.2.6}$$

where

$$I = (\boldsymbol{u}_1 \varphi, \nabla \mathcal{N}\varphi) \quad \text{and} \quad J = (\boldsymbol{u} \varphi_2, \nabla \mathcal{N}\varphi).$$

Firstly, by (3.1.5), (3.1.7), (3.3.4) and (3.3.6), we control I as follows

$$\begin{aligned} I &\leq \|\boldsymbol{u}_1\| \|\varphi\|_{L^4(\Omega)} \|\nabla \mathcal{N}\varphi\|_{\mathbf{L}^4(\Omega)} \\ &\leq C \|\boldsymbol{u}_1\| \|\varphi\|^{\frac{1}{2}} \|\varphi\|_V^{\frac{1}{2}} \|\nabla \mathcal{N}\varphi\|^{\frac{1}{2}} \|\nabla \mathcal{N}\varphi\|_{\mathbf{V}}^{\frac{1}{2}} \\ &\leq C \|\boldsymbol{u}_1\| \|\varphi\|_{V_0'}^{\frac{1}{2}} \|\varphi\| \|\nabla \varphi\|^{\frac{1}{2}} \\ &\leq C \|\boldsymbol{u}_1\| \|\varphi\|_{V_0'} \|\nabla \varphi\| \\ &\leq \frac{1}{8} \|\nabla \varphi\|^2 + C \|\boldsymbol{u}_1\|^2 \|\varphi\|_{V_0'}^2. \end{aligned}$$

Next, we take care of J. To this aim, by means of (5.2.3), J can be rewritten as

$$\begin{aligned} J &= -\big(\Pi(\operatorname{div}(\nabla \varphi_1 \otimes \nabla \varphi) + \operatorname{div}(\nabla \varphi \otimes \nabla \varphi_2)),\ \varphi_2 \nabla \mathcal{N}\varphi\big) \\ &= -\big(\operatorname{div}(\nabla \varphi_1 \otimes \nabla \varphi) + \operatorname{div}(\nabla \varphi \otimes \nabla \varphi_2),\ \Pi(\varphi_2 \nabla \mathcal{N}\varphi)\big). \end{aligned}$$

A further integration by parts together with the homogeneous Neumann boundary condition for φ entails

$$
\begin{aligned}
&-\big(\mathrm{div}(\nabla\varphi_1 \otimes \nabla\varphi),\, \Pi(\varphi_2 \nabla\mathcal{N}\varphi)\big) \\
&\quad = \big(\nabla\varphi_1 \otimes \nabla\varphi,\, \nabla\Pi(\varphi_2\nabla\mathcal{N}\varphi)\big) - \int_{\partial\Omega} (\nabla\varphi \otimes \nabla\varphi_1)\Pi(\varphi_2\nabla\mathcal{N}\varphi)\cdot \boldsymbol{n}\,\mathrm{d}\sigma \\
&\quad = \big(\nabla\varphi_1 \otimes \nabla\varphi,\, \nabla\Pi(\varphi_2\nabla\mathcal{N}\varphi)\big) - \int_{\partial\Omega} (\nabla\varphi \cdot \boldsymbol{n})(\nabla\varphi_1 \cdot \Pi(\varphi_2\nabla\mathcal{N}\varphi))\,\mathrm{d}\sigma \\
&\quad = \big(\nabla\varphi_1 \otimes \nabla\varphi,\, \nabla\Pi(\varphi_2\nabla\mathcal{N}\varphi)\big) := J_1.
\end{aligned}
$$

Similarly, we infer that

$$
\begin{aligned}
&-\big(\mathrm{div}(\nabla\varphi \otimes \nabla\varphi_2), \Pi(\varphi_2 \nabla\mathcal{N}\varphi)\big) \\
&\quad = \big(\nabla\varphi \otimes \nabla\varphi_2,\, \nabla\Pi(\varphi_2\nabla\mathcal{N}\varphi)\big) - \int_{\partial\Omega} (\nabla\varphi_2 \cdot \boldsymbol{n})(\nabla\varphi \cdot \Pi(\varphi_2\nabla\mathcal{N}\varphi))\,\mathrm{d}\sigma \\
&\quad = \big(\nabla\varphi \otimes \nabla\varphi_2,\, \nabla\Pi(\varphi_2\nabla\mathcal{N}\varphi)\big) := J_2.
\end{aligned}
$$

We now estimate J_1 and J_2. Exploiting (3.1.7), (3.5.1) and (5.2.4), we obtain

$$
\begin{aligned}
J_1 &\le C\|\nabla\varphi_1\|_{\mathbf{L}^4(\Omega)}\|\nabla\varphi\|\|\nabla\Pi(\varphi_2\nabla\mathcal{N}\varphi)\|_{\mathbf{L}^4(\Omega)} \\
&\le C\|\nabla\varphi_1\|^{\frac12}\|\nabla\varphi_1\|_{\mathbf{V}}^{\frac12}\|\nabla\varphi\|\|\nabla\Pi(\varphi_2\nabla\mathcal{N}\varphi)\|^{\frac12}\|\nabla\Pi(\varphi_2\nabla\mathcal{N}\varphi)\|_{\mathbf{V}}^{\frac12} \\
&\le C\|\varphi_1\|_{H^2(\Omega)}^{\frac12}\|\nabla\varphi\|\|\Pi(\varphi_2\nabla\mathcal{N}\varphi)\|_{\mathbf{V}}^{\frac12}\|\Pi(\varphi_2\nabla\mathcal{N}\varphi)\|_{\mathbf{H}^2(\Omega)}^{\frac12} \\
&\le C\|\varphi_1\|_{H^2(\Omega)}^{\frac12}\|\nabla\varphi\|\|\varphi_2\nabla\mathcal{N}\varphi\|_{\mathbf{V}}^{\frac12}\|\varphi_2\nabla\mathcal{N}\varphi\|_{\mathbf{H}^2(\Omega)}^{\frac12}.
\end{aligned}
\tag{5.2.7}
$$

It follows from (3.1.8), (3.1.1), (3.3.4), (3.3.6) and (5.2.4), that

$$
\begin{aligned}
\|\varphi_2\nabla\mathcal{N}\varphi\|_{\mathbf{V}} &\le C\|\varphi_2\|_{L^\infty(\Omega)}\|\nabla\mathcal{N}\varphi\|_{\mathbf{V}} + C\|\varphi_2\|_V\|\nabla\mathcal{N}\varphi\|_{\mathbf{L}^\infty(\Omega)} \\
&\le C\|\nabla\mathcal{N}\varphi\|_{\mathbf{V}} + C\|\nabla\mathcal{N}\varphi\|_{\mathbf{L}^\infty(\Omega)} \\
&\le C\|\varphi\| + C\|\nabla\mathcal{N}\varphi\|^{\frac12}\|\nabla\mathcal{N}\varphi\|_{\mathbf{H}^2(\Omega)}^{\frac12} \\
&\le C\|\varphi\|_{V_0'}^{\frac12}\|\nabla\varphi\|^{\frac12}.
\end{aligned}
$$

On the other hand, by (3.1.8), (3.1.2), (3.3.6) and (5.2.4), we deduce that

$$
\begin{aligned}
\|\varphi_2\nabla\mathcal{N}\varphi\|_{\mathbf{H}^2(\Omega)} &\le C\|\varphi_2\|_{L^\infty(\Omega)}\|\nabla\mathcal{N}\varphi\|_{\mathbf{H}^2(\Omega)} + C\|\varphi_2\|_{H^2(\Omega)}\|\nabla\mathcal{N}\varphi\|_{\mathbf{L}^\infty(\Omega)} \\
&\le C\|\nabla\mathcal{N}\varphi\|_{\mathbf{H}^2(\Omega)} + C\|\varphi_2\|_{H^2(\Omega)}\|\nabla\mathcal{N}\varphi\|^{\frac12}\|\nabla\mathcal{N}\varphi\|_{\mathbf{H}^2(\Omega)}^{\frac12} \\
&\le C\|\nabla\varphi\| + C\|\varphi_2\|_{H^2(\Omega)}\|\varphi\|_{V_0'}^{\frac12}\|\nabla\varphi\|^{\frac12}.
\end{aligned}
$$

Thus, from (5.2.7) we obtain

$$
\begin{aligned}
J_1 &\le C\|\varphi_1\|_{H^2(\Omega)}^{\frac12}\|\varphi\|_{V_0'}^{\frac14}\|\nabla\varphi\|^{\frac74} + C\|\varphi_1\|_{H^2(\Omega)}^{\frac12}\|\varphi_2\|_{H^2(\Omega)}^{\frac12}\|\varphi\|_{V_0'}^{\frac12}\|\nabla\varphi\|^{\frac32} \\
&:= R_1 + R_2.
\end{aligned}
$$

Using Young's inequality, the reminder terms R_1 and R_2 can be controlled as follows

$$\begin{aligned} R_1 &= C\|\varphi_1\|_{H^2(\Omega)}^{\frac{1}{2}}\|\varphi\|_{V_0'}^{\frac{1}{4}}\|\nabla\varphi\|^{\frac{7}{4}} \\ &\le \frac{1}{32}\|\nabla\varphi\|^2 + C\|\varphi_1\|_{H^2(\Omega)}^4\|\varphi\|_{V_0'}^2, \end{aligned}$$

and

$$\begin{aligned} R_2 &= C\|\varphi_1\|_{H^2(\Omega)}^{\frac{1}{2}}\|\varphi_2\|_{H^2(\Omega)}^{\frac{1}{2}}\|\varphi\|_{V_0'}^{\frac{1}{2}}\|\nabla\varphi\|^{\frac{3}{2}} \\ &\le \frac{1}{32}\|\nabla\varphi\|^2 + C\|\varphi_1\|_{H^2(\Omega)}^2\|\varphi_2\|_{H^2(\Omega)}^2\|\varphi\|_{V_0'}^2. \end{aligned}$$

Collecting the above estimates and using again Young's inequality, we end up with

$$J_1 \le \frac{1}{16}\|\nabla\varphi\|^2 + C\Big(\|\varphi_1\|_{H^2(\Omega)}^4 + \|\varphi_2\|_{H^2(\Omega)}^4\Big)\|\varphi\|_{V_0'}^2.$$

Repeating the same calculations for J_2 line by line, we get

$$J_2 \le \frac{1}{16}\|\nabla\varphi\|^2 + C\|\varphi_2\|_{H^2(\Omega)}^4\|\varphi\|_{V_0'}^2.$$

Finally, combining (5.2.6) with the above controls of I, J_1 and J_2, we find the differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{V_0'}^2 + \frac{1}{4}\|\nabla\varphi\|^2 \le C\Big(1 + \|\boldsymbol{u}_1\|^2 + \|\varphi_1\|_{H^2(\Omega)}^4 + \|\varphi_2\|_{H^2(\Omega)}^4\Big)\|\varphi\|_{V_0'}^2. \tag{5.2.8}$$

On the other hand, thanks to Theorem 5.1.3, we have

$$\int_0^T \|\boldsymbol{u}_1(\tau)\|^2 + \|\varphi_1(\tau)\|_{H^2(\Omega)}^4 + \|\varphi_2(\tau)\|_{H^2(\Omega)}^4\,\mathrm{d}\tau \le C. \tag{5.2.9}$$

Thus, an application of Gronwall lemma together with (5.2.8) and (5.2.9) gives (5.2.1). □

5.3 Regularity Properties and Separation Property in Two Dimensions

In this section, we show that global weak solutions become instantaneously more regular when the spatial space dimension is two. Furthermore, we are able to prove the validity of the *instantaneous* separation property. The goal will be achieved by obtaining some higher order estimates for weak solutions that only depend on the initial energy $\mathcal{E}(\varphi_0)$ and on the average of total mass $\overline{\varphi}_0$. In particular, these estimates will be independent of any other norm of φ_0. To this end, given arbitrary but fixed numbers $R > 0$ and $m \in (-1,1)$, we consider weak solutions $(\boldsymbol{u}, \pi, \varphi)$ departing from φ_0 with

$$\mathcal{E}(\varphi_0) \le R \quad \text{and} \quad \overline{\varphi}_0 = m.$$

Consequently, in this section the generic constant $C > 0$ depends on R, m and possibly on Ω.

Our result on regularity properties of weak solutions reads as follows

Theorem 5.3.1. *Let $d = 2$. Then, for every $\sigma > 0$ and $p \geq 1$, there exists a positive constant $C = C(\sigma, p)$ such that*

$$\|\boldsymbol{u}\|_{L^\infty(\sigma,\infty;\mathbf{V})} + \|\pi\|_{L^\infty(\sigma,\infty;H^2(\Omega))} + \|\varphi\|_{L^\infty(\sigma,\infty;W^{2,p}(\Omega))} \leq C.$$

In addition, suppose that $\Psi \in \mathcal{C}^3(-1,1)$ and (H.2)-(H.3) hold. Then, for every $\sigma > 0$, there exist $\delta = \delta(\sigma, R, m) \in (0,1)$ and a positive constant $C = C(\sigma)$ such that

$$\sup_{t \geq 2\sigma} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \tag{5.3.1}$$

and

$$\|\boldsymbol{u}\|_{L^\infty(2\sigma,\infty;\mathbf{H}^2(\Omega))} + \|\pi\|_{L^\infty(2\sigma,\infty;H^3(\Omega))} + \|\varphi\|_{L^\infty(2\sigma,\infty;H^4(\Omega))} \leq C. \tag{5.3.2}$$

Remark 5.3.2. The proof of Theorem 5.3.1 easily implies the existence of a unique global strong solution to problem (5.0.1)-(5.0.2) in two dimensions, provided that the initial datum φ_0 is more regular, e.g., $\mu(0) = -\Delta\varphi_0 + F'(\varphi_0) \in V$ with $\partial_{\mathbf{n}}\varphi_0 = 0$ on $\partial\Omega$.

The proof of Theorem 5.3.1 is carried out by means of several lemmas. Our first regularity result is the following estimate for $\partial_t\varphi$.

Lemma 5.3.3. *Let the assumptions of Theorem 5.3.1 hold. For any $\sigma > 0$, there exists a positive constant $C = C(\sigma)$ such that*

$$\|\partial_t\varphi\|_{L^\infty(\sigma,t;V')} + \|\partial_t\varphi\|_{L^2(t,t+1;V)} \leq C, \quad \forall\, t \geq \sigma. \tag{5.3.3}$$

Proof. We first note that, by virtue of Theorem 5.1.11 (see also (5.1.55)),

$$\|\varphi(t)\|_V + \|\varphi(t)\|_{L^\infty(\Omega)} \leq C, \quad \forall\, t \geq 0. \tag{5.3.4}$$

Given $h > 0$, let us introduce the difference quotient of a function v by

$$\partial_t^h v = \frac{1}{h}\Big(v(t+h) - v(t)\Big).$$

Owing to Definition 5.1.1, the difference quotient of a weak solution satisfies

$$\langle \partial_t \partial_t^h \varphi, v\rangle + (\boldsymbol{u}(t+h)\cdot\nabla\partial_t^h\varphi, v) + (\partial_t^h\boldsymbol{u}\cdot\nabla\varphi, v) + (\nabla\partial_t^h\mu, \nabla v) = 0, \quad \forall\, v \in V,$$

for almost every $t \in (0, +\infty)$, where

$$\partial_t^h\mu = -\Delta\partial_t^h\varphi + \frac{1}{h}\Big(F'(\varphi(t+h)) - F'(\varphi(t))\Big) \tag{5.3.5}$$

and

$$\partial_t^h\boldsymbol{u} = -\Pi\Big(\mathrm{div}(\nabla\varphi(t+h)\otimes\nabla\partial_t^h\varphi) + \mathrm{div}(\nabla\partial_t^h\varphi\otimes\nabla\varphi)\Big). \tag{5.3.6}$$

We observe from the mass conservation that

$$\overline{\partial_t^h\varphi} = \partial_t^h\overline{\varphi} = 0.$$

Taking $v = \mathcal{N}\partial_t^h\varphi$ in the above weak formulation, and exploiting (3.3.2)-(3.3.5), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|_{V_0'}^2 &+ (\partial_t^h\mu, \partial_t^h\varphi)\\ &= -(\boldsymbol{u}(t+h)\cdot\nabla\partial_t^h\varphi, \mathcal{N}\partial_t^h\varphi) - (\partial_t^h\boldsymbol{u}\cdot\nabla\varphi, \mathcal{N}\partial_t^h\varphi).\end{aligned} \tag{5.3.7}$$

By the definition of $\partial_t^h \mu$ and making use of the homogeneous Neumann boundary condition for $\partial_t^h \varphi$ together with (H) and (3.3.4), we get

$$\begin{aligned}(\partial_t^h \mu, \partial_t^h \varphi) &= \|\nabla \partial_t^h \varphi\|^2 + \frac{1}{h}(F'(\varphi(t+h)) - F'(\varphi(t)), \partial_t^h \varphi) \\ &\geq \|\nabla \partial_t^h \varphi\|^2 - \alpha \|\partial_t^h \varphi\|^2 \\ &\geq \frac{1}{2}\|\nabla \partial_t^h \varphi\|^2 - C\|\partial_t^h \varphi\|_{V_0'}^2.\end{aligned}$$

Setting

$$K_1 = (\boldsymbol{u}(t+h)\partial_t^h \varphi, \nabla \mathcal{N} \partial_t^h \varphi) \quad \text{and} \quad K_2 = (\partial_t^h \boldsymbol{u} \varphi, \nabla \mathcal{N} \partial_t^h \varphi),$$

we find the differential inequality from (5.3.7) such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h \varphi\|_{V_0'}^2 + \frac{1}{2}\|\nabla \partial_t^h \varphi\|^2 \leq C\|\partial_t^h \varphi\|_{V_0'}^2 + K_1 + K_2. \tag{5.3.8}$$

In order to control K_1 and K_2, we argue similarly to the proof of Theorem 5.2.1. By (3.1.5), (3.1.7), (3.3.4) and (3.3.6), we estimate K_1 as follows

$$\begin{aligned}K_1 &\leq \|\boldsymbol{u}(t+h)\|\|\partial_t^h \varphi\|_{L^4(\Omega)}\|\nabla \mathcal{N}\partial_t^h \varphi\|_{\mathbf{L}^4(\Omega)} \\ &\leq C\|\boldsymbol{u}(t+h)\|\|\partial_t^h \varphi\|^{\frac{1}{2}}\|\partial_t^h \varphi\|_V^{\frac{1}{2}}\|\nabla \mathcal{N}\partial_t^h \varphi\|^{\frac{1}{2}}\|\nabla \mathcal{N}\partial_t^h \varphi\|_{\mathbf{V}}^{\frac{1}{2}} \\ &\leq C\|\boldsymbol{u}(t+h)\|\|\partial_t^h \varphi\|_{V_0'}^{\frac{1}{2}}\|\partial_t^h \varphi\|\|\nabla \partial_t^h \varphi\|^{\frac{1}{2}} \\ &\leq C\|\boldsymbol{u}(t+h)\|\|\partial_t^h \varphi\|_{V_0'}\|\nabla \partial_t^h \varphi\| \\ &\leq \frac{1}{8}\|\nabla \partial_t^h \varphi\|^2 + C\|\boldsymbol{u}(t+h)\|^2\|\partial_t^h \varphi\|_{V_0'}^2.\end{aligned}$$

Regarding K_2, in light of (5.3.6) we obtain

$$\begin{aligned}K_2 &= -\Big(\Pi(\mathrm{div}(\nabla \varphi(t+h) \otimes \nabla \partial_t^h \varphi)),\, \varphi \nabla \mathcal{N}\partial_t^h \varphi\Big) \\ &\quad - \Big(\Pi(\mathrm{div}(\nabla \partial_t^h \varphi \otimes \nabla \varphi)),\, \varphi \nabla \mathcal{N}\partial_t^h \varphi\Big) \\ &= -\Big(\mathrm{div}(\nabla \varphi(t+h) \otimes \nabla \partial_t^h \varphi),\, \Pi(\varphi \nabla \mathcal{N}\partial_t^h \varphi)\Big) \\ &\quad - \Big(\mathrm{div}(\nabla \partial_t^h \varphi \otimes \nabla \varphi),\, \Pi(\varphi \nabla \mathcal{N}\partial_t^h \varphi)\Big) \\ &= \Big(\nabla \varphi(t+h) \otimes \nabla \partial_t^h \varphi,\, \nabla \Pi(\varphi \nabla \mathcal{N}\partial_t^h \varphi)\Big) \\ &\quad + \Big(\nabla \partial_t^h \varphi \otimes \nabla \varphi,\, \nabla \Pi(\varphi \nabla \mathcal{N}\partial_t^h \varphi)\Big) \\ &:= Z_1 + Z_2.\end{aligned}$$

Let us proceed to estimate Z_1 and Z_2. By (3.1.7), (5.3.4) and (3.5.1), we deduce that

$$\begin{aligned}Z_1 &\leq C\|\nabla \varphi(t+h)\|_{\mathbf{L}^4(\Omega)}\|\nabla \partial_t^h \varphi\|\|\nabla \Pi(\varphi \nabla \mathcal{N}\partial_t^h \varphi)\|_{\mathbf{L}^4(\Omega)} \\ &\leq C\|\nabla \varphi(t+h)\|^{\frac{1}{2}}\|\varphi(t+h)\|_{H^2(\Omega)}^{\frac{1}{2}}\|\nabla \partial_t^h \varphi\|\|\Pi(\varphi \nabla \mathcal{N}\partial_t^h \varphi)\|_{\mathbf{W}^{1,4}(\Omega)} \\ &\leq C\|\varphi(t+h)\|_{H^2(\Omega)}^{\frac{1}{2}}\|\nabla \partial_t^h \varphi\|\|\varphi \nabla \mathcal{N}\partial_t^h \varphi\|_{\mathbf{V}}^{\frac{1}{2}}\|\varphi \nabla \mathcal{N}\partial_t^h \varphi\|_{\mathbf{H}^2(\Omega)}^{\frac{1}{2}}.\end{aligned}$$

A further application of (3.1.1), (3.1.2), (3.1.8), (3.3.4), (3.5.1), and (5.3.4) yields

$$\begin{aligned}\|\varphi\nabla\mathcal{N}\partial_t^h\varphi\|_{\mathbf{V}} &\le C\|\varphi\|_{L^\infty(\Omega)}\|\nabla\mathcal{N}\partial_t^h\varphi\|_{\mathbf{V}} + C\|\varphi\|_V\|\nabla\mathcal{N}\partial_t^h\varphi\|_{\mathbf{L}^\infty(\Omega)}\\ &\le C\|\partial_t^h\varphi\| + C\|\nabla\mathcal{N}\partial_t^h\varphi\|^{\frac12}\|\nabla\mathcal{N}\partial_t^h\varphi\|_{\mathbf{H}^2(\Omega)}^{\frac12}\\ &\le C\|\partial_t^h\varphi\|_{V_0'}^{\frac12}\|\nabla\partial_t^h\varphi\|^{\frac12},\end{aligned}$$

and

$$\begin{aligned}\|\varphi\nabla\mathcal{N}\partial_t^h\varphi\|_{\mathbf{H}^2(\Omega)} &\le C\|\varphi\|_{L^\infty(\Omega)}\|\nabla\mathcal{N}\partial_t^h\varphi\|_{\mathbf{H}^2(\Omega)} + C\|\varphi\|_{H^2(\Omega)}\|\nabla\mathcal{N}\partial_t^h\varphi\|_{\mathbf{L}^\infty(\Omega)}\\ &\le C\|\nabla\mathcal{N}\partial_t^h\varphi\|_{\mathbf{H}^2(\Omega)} + C\|\varphi\|_{H^2(\Omega)}\|\nabla\mathcal{N}\partial_t^h\varphi\|^{\frac12}\|\nabla\mathcal{N}\partial_t^h\varphi\|_{\mathbf{H}^2(\Omega)}^{\frac12}\\ &\le C\|\nabla\partial_t^h\varphi\| + C\|\varphi\|_{H^2(\Omega)}\|\partial_t^h\varphi\|_{V_0'}^{\frac12}\|\nabla\partial_t^h\varphi\|^{\frac12}.\end{aligned}$$

Therefore, we learn that

$$\begin{aligned}Z_1 &\le C\|\varphi(t+h)\|_{H^2(\Omega)}^{\frac12}\|\partial_t^h\varphi\|_{V_0'}^{\frac14}\|\nabla\partial_t^h\varphi\|^{\frac74}\\ &\quad C\|\varphi(t+h)\|_{H^2(\Omega)}^{\frac12}\|\varphi\|_{H^2(\Omega)}^{\frac12}\|\partial_t^h\varphi\|_{V_0'}^{\frac12}\|\nabla\partial_t^h\varphi\|^{\frac32}\\ &:= Y_1 + Y_2.\end{aligned}$$

By Young's inequality, we infer that

$$\begin{aligned}Y_1 &= C\|\varphi(t+h)\|_{H^2(\Omega)}^{\frac12}\|\partial_t^h\varphi\|_{V_0'}^{\frac14}\|\nabla\partial_t^h\varphi\|^{\frac74}\\ &\le \frac{1}{32}\|\nabla\partial_t^h\varphi\|^2 + C\|\varphi(t+h)\|_{H^2(\Omega)}^4\|\partial_t^h\varphi\|_{V_0'}^2,\end{aligned}$$

and

$$\begin{aligned}Y_2 &= C\|\varphi(t+h)\|_{H^2(\Omega)}^{\frac12}\|\varphi\|_{H^2(\Omega)}^{\frac12}\|\partial_t^h\varphi\|_{V_0'}^{\frac12}\|\nabla\partial_t^h\varphi\|^{\frac32}\\ &\le \frac{1}{32}\|\nabla\partial_t^h\varphi\|^2 + C\|\varphi(t+h)\|_{H^2(\Omega)}^2\|\varphi\|_{H^2(\Omega)}^2\|\partial_t^h\varphi\|_{V_0'}^2.\end{aligned}$$

Hence, combining the above estimates together, we end up with

$$Z_1 \le \frac{1}{16}\|\nabla\partial_t^h\varphi\|^2 + C\Big(\|\varphi(t+h)\|_{H^2(\Omega)}^4 + \|\varphi\|_{H^2(\Omega)}^4\Big)\|\partial_t^h\varphi\|_{V_0'}^2.$$

Arguing in the same way for Z_2, we also find

$$Z_2 \le \frac{1}{16}\|\nabla\partial_t^h\varphi\|^2 + C\|\varphi\|_{H^2(\Omega)}^4\|\partial_t^h\varphi\|_{V_0'}^2.$$

Then, collecting the estimates of K_1 and K_2, from (5.3.8) we deduce the following differential inequality

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|_{V_0'}^2 + \frac14\|\nabla\partial_t^h\varphi\|^2\\ &\quad\le C\Big(1 + \|\boldsymbol{u}(t+h)\|^2 + \|\varphi(t+h)\|_{H^2(\Omega)}^4 + \|\varphi\|_{H^2(\Omega)}^4\Big)\|\partial_t^h\varphi\|_{V_0'}^2.\end{aligned}$$

On account of

$$\|\partial_t^h \varphi\|_{L^2(t,t+1;V_0')} \leq \|\varphi_t\|_{L^2(t,t+1+h;V_0')}, \quad \forall\, t \geq 0,$$

and the dissipative estimate (cf. Theorem 5.1.11)

$$\int_t^{t+1} \|\varphi(\tau)\|_{H^2(\Omega)}^4 + \|\boldsymbol{u}(\tau)\|^2 + \|\varphi_t(\tau)\|_{V_0'}^2 \,\mathrm{d}\tau \leq C, \quad \forall\, t \geq 0, \tag{5.3.9}$$

an application of the uniform Gronwall lemma entails the uniform bounds

$$\|\partial_t^h \varphi\|_{L^\infty(\sigma,t;V_0')} + \|\partial_t^h \varphi\|_{L^2(t,t+1;V)} \leq C, \quad \forall\, t \geq \sigma.$$

Here, C is a positive constant which depends on $\sigma > 0$ but is independent of h. A final passage to the limit as $h \to 0^+$ completes the proof. $\square$

Thanks to Lemma 5.3.3, we derive a preliminary higher order estimate for φ with respect to the spatial variable.

Lemma 5.3.4. *Let the assumptions of Theorem 5.3.1 hold. For any $\sigma > 0$, there exists a positive constant $C = C(\sigma)$ such that*

$$\|\varphi\|_{L^\infty(\sigma,t;H^2(\Omega))} \leq C, \quad \forall\, t \geq \sigma. \tag{5.3.10}$$

Proof. By [9, Theorem 1.1], we have for an arbitrary $T \geq \sigma$

$$H^1(\sigma,T;V) \cap L^2(\sigma,T;H^2(\Omega)) \hookrightarrow \mathcal{C}([\sigma,T],W^{1,4}(\Omega)).$$

Hence, in light of Theorem 5.1.3 and Lemma 5.3.3, we infer that

$$\varphi \in \mathcal{C}([\sigma,+\infty),W^{1,4}(\Omega)).$$

In order to get a uniform-in-time estimate, we recall that (cf. Theorem 5.1.11)

$$\|\varphi\|_{H^1(t,t+1;V)} + \|\varphi\|_{L^2(t,t+1;H^2(\Omega))} \leq C, \quad \forall\, t \geq \sigma,$$

where C is independent of t. Then, by the above result, we have

$$\|\varphi(t)\|_{W^{1,4}(\Omega)} \leq C, \quad \forall\, t \in [\sigma,\sigma+1].$$

By the same argument replacing $\varphi(\cdot)$ with $\varphi(\cdot+n)$, for any $n \in \mathbb{N}$, we have

$$\|\varphi(t+n)\|_{W^{1,4}(\Omega)} \leq C, \quad \forall\, t \in [\sigma,\sigma+1] \text{ and } \forall\, n \in \mathbb{N},$$

where C is independent of n. This in turn gives the uniform estimate

$$\|\varphi(t)\|_{W^{1,4}(\Omega)} \leq C, \quad \forall\, t \geq \sigma. \tag{5.3.11}$$

Next, taking $v = \mathcal{N}(\mu - \overline{\mu})$ in the weak formulation (5.1.1), we get

$$\langle \varphi_t, \mathcal{N}(\mu-\overline{\mu})\rangle - (\boldsymbol{u}\varphi, \nabla\mathcal{N}(\mu-\overline{\mu})) + (\mu,\mu-\overline{\mu}) = 0.$$

We note that

$$(\mu,\mu-\overline{\mu}) = \|\mu-\overline{\mu}\|^2.$$

Besides, by (3.1.4), (3.3.6), (3.5.1) and (5.3.11), we control the other two terms as follows

$$\begin{aligned}\langle \varphi_t, \mathcal{N}(\mu-\overline{\mu})\rangle &\le \|\varphi_t\|_{V_0'}\|\mathcal{N}(\mu-\overline{\mu})\|_{V_0}\\ &\le C\|\varphi_t\|_{V'}\|\mu-\overline{\mu}\|\\ &\le \frac{1}{4}\|\mu-\overline{\mu}\|^2 + C\|\varphi_t\|_{V'}^2,\end{aligned}$$

and

$$\begin{aligned}&\big(\boldsymbol{u}\varphi, \nabla\mathcal{N}(\mu-\overline{\mu})\big)\\ &\quad= -\Big(\Pi(\mathrm{div}(\nabla\varphi\otimes\nabla\varphi)),\, \varphi\nabla\mathcal{N}(\mu-\overline{\mu})\Big)\\ &\quad= -\Big(\mathrm{div}(\nabla\varphi\otimes\nabla\varphi),\, \Pi(\varphi\nabla\mathcal{N}(\mu-\overline{\mu}))\Big)\\ &\quad= \Big(\nabla\varphi\otimes\nabla\varphi,\, \nabla\Pi(\varphi\nabla\mathcal{N}(\mu-\overline{\mu}))\Big)\\ &\quad\le C\|\nabla\varphi\|_{\mathbf{L}^4(\Omega)}^2\|\Pi(\varphi\nabla\mathcal{N}(\mu-\overline{\mu}))\|_{\mathbf{V}}\\ &\quad\le C\|\varphi\nabla\mathcal{N}(\mu-\overline{\mu})\|_{\mathbf{V}}\\ &\quad\le C\|\varphi\|_{L^\infty(\Omega)}\|\nabla\mathcal{N}(\mu-\overline{\mu})\|_{\mathbf{V}} + C\|\varphi\|_{W^{1,4}(\Omega)}\|\nabla\mathcal{N}(\mu-\overline{\mu})\|_{\mathbf{L}^4(\Omega)}\\ &\quad\le C\|\mu-\overline{\mu}\|\\ &\quad\le \frac{1}{4}\|\mu-\overline{\mu}\|^2 + C.\end{aligned}$$

Collecting the above estimates and using Lemma 5.3.3, we find

$$\|\mu-\overline{\mu}\|_{L^\infty(\sigma,t;H)} \le C, \quad \forall\, t\ge\sigma. \tag{5.3.12}$$

Applying Lemma 3.4.1 with $f=\mu+\Theta_0\varphi$, the above estimate further entails

$$\|\Delta\varphi\|_{L^\infty(\sigma,t;H)} \le C, \quad \forall\, t\ge\sigma.$$

Finally, due to classical elliptic regularity results for the Neumann problem, we conclude that (5.3.10) holds. The proof is complete. □

Now we can improve the regularity properties of global weak solutions $(\boldsymbol{u},\varphi)$ on the time interval $[\sigma,+\infty)$ for any $\sigma>0$.

Lemma 5.3.5. *Let the assumptions of Theorem 5.3.1 hold. For any $p>2$, there exists a positive constant $C=C(\sigma,p)$ such that*

$$\begin{aligned}&\|\boldsymbol{u}\|_{L^\infty(\sigma,t;\mathbf{V})} + \|\pi\|_{L^\infty(\sigma,t;H^2(\Omega))} \le C, \qquad \forall\, t\ge\sigma,\\ &\|\mu\|_{L^\infty(\sigma,t;V)} + \|\varphi\|_{L^\infty(\sigma,t;W^{2,p}(\Omega))} \le C, \quad \forall\, t\ge\sigma.\end{aligned}$$

Proof. First, we observe that

$$\boldsymbol{u} = -\nabla\pi^* - \Delta\varphi\nabla\varphi, \qquad \text{a.e. } (x,t)\in\Omega\times(0,+\infty),$$

where π^* is the modified pressure given by $\pi^* = \pi + \Psi(\varphi)$ (cf. Remark 5.1.2). Thus, we have $\boldsymbol{u} = \Pi(-\Delta\varphi\nabla\varphi)$ and by (3.5.1) together with the uniform estimates in Lemma 5.3.4, it follows that

$$\begin{aligned}\|\boldsymbol{u}\|_{L^\infty(\sigma,t;\mathbf{L}^{\frac{3}{2}}(\Omega))} &\leq C\|\Delta\varphi\nabla\varphi\|_{L^\infty(\sigma,t;\mathbf{L}^{\frac{3}{2}}(\Omega))}\\ &\leq C\|\varphi\|_{L^\infty(\sigma,t;H^2(\Omega))}\|\varphi\|_{L^\infty(\sigma,t;W^{1,6}(\Omega))}\\ &\leq C. \end{aligned} \tag{5.3.13}$$

Next, we prove a uniform bound for the V-norm of μ arguing as in the proof of Theorem 5.1.3. As customary, we need to control its average value over Ω. To this end, we recall that the singular potential F satisfies

$$\|F'(\varphi)\|_{L^1(\Omega)} \leq C\int_\Omega (\varphi - \overline{\varphi}_0)(F'(\varphi) - \overline{F'(\varphi)})\,\mathrm{d}x + C,$$

where C depends on m (cf. (5.1.19)). Testing μ by $\varphi - \overline{\varphi}_0$, integrating by parts and using (3.1.5) and (5.3.4), we easily get (cf. (5.1.20))

$$\int_\Omega (\varphi - \overline{\varphi_0})(F'(\varphi) - \overline{F'(\varphi)})\,\mathrm{d}x \leq C(1 + \|\nabla\mu\|).$$

Combining the above inequalities, we are led to the inequality

$$\|\Psi'(\varphi)\|_{L^1(\Omega)} \leq C\left(1 + \|\nabla\mu\|\right), \tag{5.3.14}$$

which together with (5.1.18) gives

$$\|\mu\|_V \leq C(1 + \|\nabla\mu\|). \tag{5.3.15}$$

Now, taking $v = \mu$ in (5.1.1), we have

$$\|\nabla\mu\|^2 = -\langle\varphi_t, \mu\rangle - (\boldsymbol{u}\cdot\nabla\varphi, \mu).$$

By (5.3.3), (5.3.10), (5.3.13) and (5.3.15), we get

$$\begin{aligned}\|\nabla\mu\|^2 &\leq \|\varphi_t\|_{V'}\|\mu\|_V + \|\boldsymbol{u}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\|\nabla\varphi\|_{\mathbf{L}^6(\Omega)}\|\mu\|_{L^6(\Omega)}\\ &\leq \|\varphi_t\|_{V'}\|\mu\|_V + \|\boldsymbol{u}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\|\varphi\|_{H^2(\Omega)}\|\mu\|_V\\ &\leq C(1 + \|\nabla\mu\|).\end{aligned}$$

Hence, we infer from the above estimate, Young's inequality and (5.3.15) that

$$\|\mu\|_{L^\infty(\sigma,t;V)} \leq C, \quad \forall\, t \geq \sigma. \tag{5.3.16}$$

Thus, we can apply Corollary 3.4.3 again with $f = \mu + \Theta_0\varphi$. As a consequence, for any $p > 2$, there exists $C > 0$ such that

$$\|\varphi\|_{L^\infty(\sigma,t;W^{2,p}(\Omega))} \leq C, \quad \forall\, t \geq \sigma. \tag{5.3.17}$$

Therefore, combining (5.1.48), (5.1.49), (5.3.16), (5.3.17) and using the Gagliardo–Nirenberg inequality (5.1.46) for $d = 2$, we have

$$\|\boldsymbol{u}\|_{L^\infty(\sigma,t;\mathbf{V})} + \|\pi\|_{L^\infty(\sigma,t;H^2(\Omega))} \leq C, \quad \forall\, t \geq \sigma.$$

The proof is complete. $\square$

Remark 5.3.6. Thanks to the regularity achieved in Lemma 5.3.5, it is easily seen that (5.1.1) holds almost everywhere in $\Omega \times (\sigma, +\infty)$ and, in particular, μ satisfies $\partial_{\boldsymbol{n}}\mu = 0$ almost everywhere on $\partial\Omega \times (\sigma, +\infty)$. Since $\sigma > 0$ is arbitrary, we infer that any global weak solution to problem (5.0.1)-(5.0.2) becomes a global strong solution instantaneously when $t > 0$.

We now have the necessary ingredients to prove the validity of the *instantaneous* separation property. The main task is to show that $\Psi'(\varphi)$ is essentially bounded in time and space. For this purpose, higher order estimates will be derived by using the further assumption (H.3) on the growth control between derivatives.

Lemma 5.3.7. *Let the assumptions of Theorem 5.3.1 hold. Then, for any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\partial_t\varphi\|_{L^\infty(2\sigma,t;H)} + \|\mu\|_{L^\infty(2\sigma,t;H^2(\Omega))} \le C, \quad \forall\, t \ge 2\sigma.$$

Moreover, there exist $\delta = \delta(\sigma, R, m) > 0$ and $C = C(\sigma)$ such that

$$\|\varphi(t)\|_{L^\infty(\Omega)} \le 1 - \delta, \quad \forall\, t \ge 2\sigma, \tag{5.3.18}$$

and

$$\|\boldsymbol{u}\|_{L^\infty(2\sigma,t;\mathbf{H}^2(\Omega))} + \|\pi\|_{L^\infty(2\sigma,t;H^3(\Omega))} + \|\varphi\|_{L^\infty(2\sigma,t;H^4(\Omega))} \le C, \quad \forall\, t \ge 2\sigma. \tag{5.3.19}$$

Proof. Under the assumption (H.2), we can apply Lemma 3.4.6 with $f = \mu + \Theta_0\varphi$. Hence, for any $p \ge 2$, using the estimates obtained in Lemma 5.3.5, there exists $C = C(p)$ such that

$$\|\Psi''(\varphi)\|_{L^\infty(\sigma,t;L^p(\Omega))} \le C, \quad \forall\, t \ge \sigma. \tag{5.3.20}$$

We recall that the finite difference $\partial_t^h\varphi$ solves

$$\langle \partial_t\partial_t^h\varphi, v\rangle + (\boldsymbol{u}(t+h)\cdot\nabla\partial_t^h\varphi, v) + (\partial_t^h\boldsymbol{u}\cdot\nabla\varphi, v) + (\nabla\partial_t^h\mu, \nabla v) = 0, \quad \forall\, v \in V,$$

where $\partial_t^h\mu$ and $\partial_t^h\boldsymbol{u}$ are given by (5.3.5) and (5.3.6), respectively. Taking $v = \partial_t^h\varphi$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2 + (\nabla\partial_t^h\mu, \nabla v) = H_1 + H_2, \tag{5.3.21}$$

having set

$$H_1 = -(\boldsymbol{u}(t+h)\cdot\nabla\partial_t^h\varphi, \partial_t^h\varphi), \quad H_2 = -(\partial_t^h\boldsymbol{u}\cdot\nabla\varphi, \partial_t^h\varphi).$$

By integration by parts and making use of the homogeneous Neumann boundary conditions for $\partial_t^h\varphi$ and $\partial_t^h\mu$, we find

$$\begin{aligned}
(\nabla\partial_t^h\mu,\, \nabla\partial_t^h\varphi) &= -(\partial_t^h\mu,\, \Delta\partial_t^h\varphi)\\
&= \|\Delta\partial_t^h\varphi\|^2 - \Theta_0\|\nabla\partial_t^h\varphi\|^2 - \big(\frac{1}{h}\big(\Psi'(\varphi(t+h)) - \Psi'(\varphi(t))\big),\, \Delta\partial_t^h\varphi\big).
\end{aligned}$$

Thanks to the convexity of Ψ'', we obtain

$$\begin{aligned}
\frac{1}{h}\Big|\Psi'(\varphi(t+h)) - \Psi'(\varphi(t))\Big| &\le \int_0^1 \Psi''(\tau\varphi(t+h) + (1-\tau)\varphi(t))\,|\partial_t^h\varphi|\,\mathrm{d}\tau\\
&\le \big(\Psi''(\varphi(t+h)) + \Psi''(\varphi(t))\big)|\partial_t^h\varphi|,
\end{aligned}$$

and as a consequence, it follows that

$$\begin{aligned}&\Big|\big(\frac{1}{h}\big(\Psi'(\varphi(t+h))-\Psi'(\varphi(t))\big),\, \Delta\partial_t^h\varphi\big)\Big|\\&\quad\le \frac{1}{2}\|\Delta\partial_t^h\varphi\|^2+C\Big(\|\Psi''(\varphi(t+h))\|^2_{L^3(\Omega)}+\|\Psi''(\varphi(t))\|^2_{L^3(\Omega)}\Big)\|\partial_t^h\varphi\|^2_{L^6(\Omega)}.\end{aligned}$$

By (3.1.5) and the boundary conditions, we infer from Poincaré's inequality and integration by parts that

$$\|\partial_t^h\varphi\|^2_{L^6(\Omega)}\le C\|\nabla\partial_t^h\varphi\|^2\le C\|\partial_t^h\varphi\|\|\Delta\partial_t^h\varphi\|.$$

Therefore, we easily derive from (5.3.21) the differential inequality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2+\frac{1}{4}\|\Delta\partial_t^h\varphi\|^2\\&\quad\le C\Big(1+\|\Psi''(\varphi(t+h))\|^4_{L^3(\Omega)}+\|\Psi''(\varphi(t))\|^4_{L^3(\Omega)}\Big)\|\partial_t^h\varphi\|^2+H_1+H_2.\end{aligned}\tag{5.3.22}$$

Regarding the term H_1, by Lemma 5.3.5, the Sobolev embedding $V\hookrightarrow L^p(\Omega)$ for any $p\ge 1$, and the elliptic estimate for the Neumann problem, we get

$$\begin{aligned}H_1&\le \|\boldsymbol{u}(t+h)\|_{\mathbf{L}^6(\Omega)}\|\nabla\partial_t^h\varphi\|_{\mathbf{L}^3(\Omega)}\|\partial_t^h\varphi\|\\&\le C\|\partial_t^h\varphi\|_{H^2(\Omega)}\|\partial_t^h\varphi\|\\&\le \frac{1}{16}\|\Delta\partial_t^h\varphi\|^2+C\|\partial_t^h\varphi\|^2.\end{aligned}$$

On the other hand, by the Darcy's equation (cf. Remark 5.1.2), Lemma 5.3.5 together with (3.1.5) and (3.1.4), we infer that

$$\begin{aligned}H_2&=\Big(\Pi(\mathrm{div}(\nabla\varphi(t+h)\otimes\nabla\partial_t^h\varphi+\nabla\partial_t^h\varphi\otimes\nabla\varphi)),\,\nabla\varphi\partial_t^h\varphi\Big)\\&=\Big(\nabla\varphi(t+h)\otimes\nabla\partial_t^h\varphi+\nabla\partial_t^h\varphi\otimes\nabla\varphi,\,\nabla\Pi(\nabla\varphi\partial_t^h\varphi)\Big)\\&\le\Big(\|\nabla\varphi(t+h)\|_{\mathbf{L}^\infty(\Omega)}+\|\nabla\varphi\|_{\mathbf{L}^\infty(\Omega)}\Big)\|\nabla\partial_t^h\varphi\|\|\nabla\Pi(\nabla\varphi\partial_t^h\varphi)\|\\&\le C\|\nabla\partial_t^h\varphi\|\|\nabla\varphi\partial_t^h\varphi\|_{\mathbf{V}}\\&\le C\|\nabla\partial_t^h\varphi\|\Big(\|\nabla\varphi\|_{\mathbf{L}^\infty(\Omega)}\|\nabla\partial_t^h\varphi\|+\|\varphi\|_{W^{2,4}(\Omega)}\|\partial_t^h\varphi\|_{L^4(\Omega)}\Big)\\&\le C\|\nabla\partial_t^h\varphi\|^2\\&\le\frac{1}{16}\|\Delta\partial_t^h\varphi\|^2+C\|\partial_t^h\varphi\|^2.\end{aligned}$$

Collecting the above estimates for H_1 and H_2, we end up with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2+\frac{1}{8}\|\Delta\partial_t^h\varphi\|^2\le W\|\partial_t^h\varphi\|^2,$$

where

$$W(t)=C\Big(1+\|S''(\varphi(t+h))\|^4_{L^3(\Omega)}+\|S''(\varphi(t))\|^4_{L^3(\Omega)}\Big).$$

On account of the estimate (5.3.20), we have

$$\int_t^{t+1}W(\tau)\,\mathrm{d}\tau\le C,\quad \forall\, t\ge\sigma.$$

Thus, an application of the uniform Gronwall lemma implies that

$$\|\partial_t^h \varphi\|_{L^\infty(2\sigma,t;H)} + \|\Delta \partial_t^h \varphi\|_{L^2(t,t+1;H)} \le C, \quad \forall\, t \ge 2\sigma,$$

where the constant C is independent of h. Passing to the limit as $h \to 0^+$, we obtain

$$\|\partial_t \varphi\|_{L^\infty(2\sigma,t;H)} \le C, \quad \forall\, t \ge 2\sigma. \tag{5.3.23}$$

Now, using Lemma 5.3.5 and (5.3.23), we deduce by comparison that

$$\|\mu\|_{L^\infty(2\sigma,t;H^2(\Omega))} \le C, \quad \forall\, t \ge 2\sigma. \tag{5.3.24}$$

Therefore, Lemma 3.4.2 with $p = \infty$ together with the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ yields

$$\|\Psi'(\varphi)\|_{L^\infty(2\sigma,t;L^\infty(\Omega))} \le C, \quad \forall\, t \ge 2\sigma.$$

Due to the singularity of Ψ' at the pure states ± 1, the above estimate immediately yields the conclusion (5.3.18). Thus, it is readily seen from (5.3.24) and the separation property (5.3.18) that

$$\|\varphi\|_{L^\infty(2\sigma,t;H^4(\Omega))} \le C, \quad \forall\, t \ge 2\sigma. \tag{5.3.25}$$

Finally, by (3.1.2), (5.3.24) and (5.3.25), we arrive at

$$\|\mu \nabla \varphi\|_{L^\infty(2\sigma,t;\mathbf{H}^2(\Omega))} \le C, \quad \forall\, t \ge 2\sigma.$$

In turn, due to (3.5.1), this gives the estimate (5.3.19). The proof of is complete. □

In summary, we have

Proof of Theorem 5.3.1. Combining the results obtained in Lemma 5.3.5 and Lemma 5.3.7, we immediately arrive at our conclusions in Theorem 5.3.1. □

Remark 5.3.8. The validity of the separation property (5.3.18) is crucial, since it entails further regularity of weak solutions to problem (5.0.1)-(5.0.2). Indeed, if (5.3.18) holds along the trajectory, the solution φ is confined to an interval that does not contain the pure states ± 1. Thus the term $\Psi'(\varphi)$ can be seen as a Lipschitz nonlinearity.

5.4 Strong Solutions and Lyapunov Stability in Three dimensions

When the spatial dimensional is three, the existence of a unique global strong solution to problem (5.0.1)-(5.0.2) with arbitrary large regular initial datum φ_0 can not be expected (cf. [158] for the case with regularity potential). In this section, we first prove the existence of a unique local strong solution $(\boldsymbol{u}, \pi, \varphi)$. Then, we show that if the initial datum φ_0 is sufficiently close to a local minimizer of the energy functional $\mathcal{E}$, then the local strong solution is indeed a global one and φ will stay close to that minimizer for all $t \ge 0$.

To this end, we recall the following definition

Definition 5.4.1. *Let us set*

$$\mathcal{V}_m = \Big\{ \varphi \in V \cap L^\infty(\Omega) : \|\varphi\|_{L^\infty(\Omega)} \leq 1 \text{ and } -m \leq \overline{\varphi} \leq m \Big\}, \tag{5.4.1}$$

A function $\psi \in \mathcal{V}_m$ is called a local energy minimizer *of the total energy $\mathcal{E}_{GL}$ if there exists a constant $\chi > 0$ such that $\mathcal{E}_{GL}(\psi) \leq \mathcal{E}_{GL}(\varphi)$ for all $\varphi \in \mathcal{V}_m$ satisfying $\|\varphi - \psi\|_V < \chi$. If $\chi = +\infty$, then ψ is called a* global energy minimizer *of $\mathcal{E}_{GL}$.*

Our result on the existence of strong solution in dimension three is the following

Theorem 5.4.2. *Let $d = 3$. Suppose that the assumptions* (H.1) *and* (H.5) *hold. In addition, assume that Ψ is real analytic in $(-1, 1)$ and $\psi \in \mathcal{V}_m$ is a local energy minimizer of the total energy $\mathcal{E}_{GL}$. Then, for any $\epsilon > 0$, there exists a constant $\eta \in (0, 1)$ such that for an arbitrary initial datum $\varphi_0 \in H^3(\Omega)$ satisfying $\partial_{\boldsymbol{n}}\varphi_0 = 0$ on $\partial\Omega$, $\overline{\varphi}_0 = \overline{\psi} = m$ and $\|\varphi_0 - \psi\|_{H^2(\Omega)} \leq \eta$, the problem* (5.0.1)-(5.0.2) *admits a unique global strong solution $(\boldsymbol{u}, \pi, \varphi)$ such that*

$$\begin{aligned}
&\boldsymbol{u} \in \mathcal{C}([0, +\infty), \mathbf{H}_\sigma) \cap L^2_{loc}(0, +\infty; \mathbf{H}^3(\Omega)),\\
&\pi \in \mathcal{C}([0, +\infty), V_0) \cap L^2_{loc}(0, +\infty; H^4(\Omega)),\\
&\varphi \in \mathcal{C}([0, +\infty), H^3(\Omega)) \cap L^2_{loc}(0, +\infty; H^5(\Omega)) \cap H^1_{loc}(0, +\infty; V),\\
&\mu \in \mathcal{C}([0, +\infty), V) \cap L^2_{loc}(0, +\infty; H^3(\Omega)) \cap H^1_{loc}(0, +\infty; V').
\end{aligned}$$

Moreover, the solution φ always stays close to the minimizer ψ such that

$$\|\varphi(t) - \psi\|_{H^2(\Omega)} \leq \epsilon, \quad \forall\, t \geq 0.$$

Namely, any local energy minimizer of $\mathcal{E}_{GL}$ is locally Lyapunov stable.

Remark 5.4.3. The conclusions of Theorem 5.4.2 (in particular, the Lyapunov stability for local energy minimizers) are still valid in two dimensions, with only minor modifications in the proof.

Remark 5.4.4. Actually the solution given by Theorem 5.4.2 is slightly more regular than the usual notion of strong solution (i.e., a solution which satisfies the equations and the initial and boundary conditions almost everywhere, cf. Theorem 5.3.1).

Local strong solutions in three dimensions

We first provide a result on the existence of local-in-time strong solutions.

Theorem 5.4.5. *Let $d = 3$. Assume that* (H.1) *and* (H.5) *hold and $\varphi_0 \in H^3(\Omega)$ satisfying $\|\varphi_0\|_{C(\overline{\Omega})} \leq 1 - \delta_0$, for an arbitrary but fixed $\delta_0 \in (0, 1)$, and $\partial_{\boldsymbol{n}}\varphi = 0$ on $\partial\Omega$. Then, there exists a unique local strong solution $(\boldsymbol{u}, \pi, \varphi)$ to problem* (5.0.1)-(5.0.2) *such that*

$$\begin{aligned}
&\boldsymbol{u} \in \mathcal{C}([0, T^*], \mathbf{H}_\sigma) \cap L^2(0, T^*; \mathbf{H}^3(\Omega)),\\
&\pi \in \mathcal{C}([0, T^*], V_0) \cap L^2(0, T^*; H^4(\Omega)),\\
&\varphi \in \mathcal{C}([0, T^*], H^3(\Omega)) \cap L^2(0, T^*; H^5(\Omega)) \cap H^1(0, T^*; V),\\
&\mu \in \mathcal{C}([0, T^*], V) \cap L^2(0, T^*; H^3(\Omega)) \cap H^1(0, T^*; V'),
\end{aligned}$$

and

$$\|\varphi(t)\|_{C(\overline{\Omega})} \leq 1 - \frac{1}{2}\delta_0, \quad \forall\, t \in [0, T^*], \tag{5.4.2}$$

for some $T^* \in (0, +\infty)$ *depending on* $\|\varphi_0\|_{H^3(\Omega)}$ *and* δ_0. *In particular, the strong solution satisfies* (5.0.1) *for almost every* $(x, t) \in \Omega \times (0, T^*)$ *and the boundary conditions* $\partial_{\boldsymbol{n}} p = \partial_{\boldsymbol{n}} \mu = 0$ *on* $\partial\Omega \times (0, T^*)$.

Proof. For any $\varepsilon \in (0, 1)$, we introduce a regular approximating potential $\widehat{F}_\varepsilon \in \mathcal{C}^4(\mathbb{R})$, namely,

$$\widehat{F}_\varepsilon(s) = \widehat{\Psi}_\varepsilon(s) - \frac{\Theta_0}{2}s^2, \quad \forall\, s \in \mathbb{R}, \tag{5.4.3}$$

where

$$\widehat{F}_\varepsilon(s) = \begin{cases} \displaystyle\sum_{j=0}^{4} \frac{1}{j!}\Psi^{(j)}(1-\varepsilon)\left[s - (1-\varepsilon)\right]^j, & \forall\, s \geq 1 - \varepsilon, \\ \Psi(s), & \forall\, s \in [-1+\varepsilon, 1-\varepsilon], \\ \displaystyle\sum_{j=0}^{4} \frac{1}{j!}\Psi^{(j)}(-1+\varepsilon)\left[s - (-1+\varepsilon)\right]^j, & \forall\, s \leq -1+\varepsilon. \end{cases} \tag{5.4.4}$$

Then we consider the following approximating problem (**AP2**)

$$\begin{cases} \boldsymbol{u}_\varepsilon = -\nabla\pi_\varepsilon + \mu_\varepsilon\nabla\varphi_\varepsilon, \\ \operatorname{div}\boldsymbol{u}_\varepsilon = 0, \\ \partial_t\varphi_\varepsilon + \boldsymbol{u}_\varepsilon \cdot \nabla\varphi_\varepsilon = \Delta\mu_\varepsilon, \\ \mu_\varepsilon = -\Delta\varphi_\varepsilon + \widehat{\Psi}'_\varepsilon(\varphi_\varepsilon), \end{cases} \quad \text{in } \Omega \times (0, T), \tag{5.4.5}$$

subject to the initial and boundary conditions (5.0.2) (with φ, μ being replaced by φ_ε and μ_ε, respectively).

For any given $\delta_0 \in (0, 1)$, by the assumption (H.5) we can choose a sufficiently small constant $\varepsilon \in (0, \min\{\kappa, \frac{1}{4}\delta_0\}]$ such that $\widehat{F}_\varepsilon \in \mathcal{C}^4(\mathbb{R})$ satisfies

$$\widehat{F}_\varepsilon(s) \geq \gamma s^4 - C, \quad \forall\, s \in \mathbb{R},$$

for some positive constants γ and C which are independent of ε. Local well-posedness of the approximating problem (**AP2**) easily follows by employing the Galerkin method as in [113,158]. In particular, by means of a differential inequality involving the $H^3(\Omega)$-norm of φ (see [158]), it follows that there exists a $T_\varepsilon \in (0, +\infty)$ depending on $\|\varphi_0\|_{H^3(\Omega)}$, ε and Ω such that problem (**AP2**) admits a unique local strong solution $(\boldsymbol{u}_\varepsilon, \pi_\varepsilon, \varphi_\varepsilon)$ on $[0, T_\varepsilon]$. Then, in light of the regularity $\varphi_\varepsilon \in L^2([0, T_\varepsilon]; H^5(\Omega)) \cap H^1(0, T_\varepsilon; V)$ (see [158, Proposition 2.1]), we deduce that $\varphi_\varepsilon \in \mathcal{C}^{\frac{1}{2}}([0, T_\varepsilon]; H^2(\Omega))$ (see [9]). Then, due to the Sobolev embedding $H^2(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$, the function $t \mapsto \|\varphi_\varepsilon(t)\|_{\mathcal{C}(\overline{\Omega})}$ is Hölder continuous and

$$\left|\|\varphi_\varepsilon(t)\|_{\mathcal{C}(\Omega)} - \|\varphi_0\|_{\mathcal{C}(\overline{\Omega})}\right| \leq Ct^{\frac{1}{2}},$$

where C only depends on the norm $\|\varphi_0\|_{H^3(\Omega)}$. Accordingly, we find $T^* \in (0, T_\varepsilon]$ such that

$$\|\varphi_\varepsilon(t)\|_{\mathcal{C}(\overline{\Omega})} \leq 1 - \frac{1}{2}\delta_0, \quad \forall\, t \in [0, T^*], \tag{5.4.6}$$

where T^* only depends on $\|\varphi_0\|_{H^3(\Omega)}$ and δ_0. Noting that, by the choice of ε and the definition of $\widehat{F}_\varepsilon$, it holds $\widehat{F}_\varepsilon|_{[-1+\frac{1}{2}\delta_0, 1-\frac{1}{2}\delta_0]} = F$. Hence, $(\boldsymbol{u}_\varepsilon, \pi_\varepsilon, \varphi_\varepsilon)$ actually is the strong solution $(\boldsymbol{u}, \pi, \varphi)$ to the original problem (5.0.1)-(5.0.2) on $[0, T^*]$. In turn, it is unique and satisfies the separation property (5.4.2). The proof is complete. □

Next, we derive a higher order differential inequality for (local) strong solutions.

Lemma 5.4.6. *Let $d = 3$ and let the assumptions of Theorem 5.4.5 hold. Assume that $(\boldsymbol{u}, \pi, \varphi)$ is a strong solution to problem* (5.0.1)-(5.0.2) *on $[0, T]$. Define the function*

$$\Lambda = \|\nabla \mu\|^2 + \|\boldsymbol{u}\|^2.$$

Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda \leq C_* \left(1 + \Lambda^{\frac{7}{3}}\right). \tag{5.4.7}$$

for almost every $t \in (0, T)$. Here, the constant C_* only depends on Ω, α, m and $\mathcal{E}_{GL}(\varphi_0)$.

Proof. By the regularity properties of a strong solution (see Theorem 5.4.5), we infer that $\boldsymbol{u} \cdot \nabla \varphi \in L^2(0, T; V)$. Thus, we test the third equation of (5.0.1) by μ_t such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \mu\|^2 + \|\nabla \varphi_t\|^2 + \int_\Omega F''(\varphi)|\varphi_t|^2 \, \mathrm{d}x = -\int_\Omega (\boldsymbol{u} \cdot \nabla \varphi)\mu_t \, \mathrm{d}x. \tag{5.4.8}$$

Next, in light of Remark 5.1.2, for any $\boldsymbol{v} \in \mathbf{H}_\sigma \cap \mathbf{V}$, we have

$$\begin{aligned}
\langle \boldsymbol{u}_t, \boldsymbol{v} \rangle &= \int_\Omega (-\nabla \pi + \mu \nabla \varphi)_t \cdot \boldsymbol{v} \, \mathrm{d}x \\
&= -2 \int_\Omega \operatorname{div}(\nabla \varphi \otimes \nabla \varphi_t) \cdot \boldsymbol{v} \, \mathrm{d}x \\
&= 2 \int_\Omega (\nabla \varphi \otimes \nabla \varphi_t) : \nabla \boldsymbol{v} \, \mathrm{d}x \\
&\leq 2\|\nabla \varphi\|_{\mathbf{L}^\infty(\Omega)} \|\nabla \varphi_t\| \|\boldsymbol{v}\|_{\mathbf{V}_\sigma} \\
&\leq C\|\varphi\|_{H^3(\Omega)} \|\nabla \varphi_t\| \|\boldsymbol{v}\|_{\mathbf{V}_\sigma},
\end{aligned}$$

which entails that $\boldsymbol{u}_t \in L^2(0, T; (\mathbf{H}_\sigma \cap \mathbf{V})')$. Thus, differentiating the first equation of (5.0.1) with respect to time and testing the resulting equation by $\boldsymbol{u}$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2 = \int_\Omega \mu_t \nabla \varphi \cdot \boldsymbol{u} \, \mathrm{d}x + \int_\Omega \mu \nabla \varphi_t \cdot \boldsymbol{u} \, \mathrm{d}x. \tag{5.4.9}$$

Noting that, by (3.3.4), we have

$$\begin{aligned}
\int_\Omega F''(\varphi)|\varphi_t|^2 \, \mathrm{d}x &\geq -\alpha \|\varphi_t\|_{V_0'} \|\nabla \varphi_t\| \\
&\geq -\frac{1}{2}\|\nabla \varphi_t\|^2 - C\|\varphi_t\|_{V_0'}^2.
\end{aligned}$$

Then adding (5.4.8) to (5.4.9), we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \|\nabla \varphi_t\|^2 \leq C\|\varphi_t\|_{V_0'}^2 + 2\int_\Omega \mu \nabla \varphi_t \cdot \boldsymbol{u} \, \mathrm{d}x. \tag{5.4.10}$$

By comparison, we have

$$\|\varphi_t\|_{V_0'} \le \|\boldsymbol{u}\| + \|\nabla\mu\|. \tag{5.4.11}$$

On the other hand, by (3.1.11) with $p = 3$, we obtain

$$\begin{aligned}
\int_\Omega \mu\nabla\varphi_t \cdot \boldsymbol{u}\,\mathrm{d}x &= -\int_\Omega \varphi_t \boldsymbol{u}\cdot\nabla\mu\,\mathrm{d}x \\
&\le \|\boldsymbol{u}\|\|\varphi_t\|_{L^3(\Omega)}\|\nabla\mu\|_{\mathbf{L}^6(\Omega)} \\
&\le C\|\boldsymbol{u}\|\|\varphi_t\|^{\frac12}\|\nabla\varphi_t\|^{\frac12}\Big(\|\Delta\mu\| + \|\mu\|\Big) \\
&\le C\|\boldsymbol{u}\|\|\varphi_t\|_{V_0'}^{\frac14}\|\nabla\varphi_t\|^{\frac34}\Big(\|\varphi_t\| + \|\boldsymbol{u}\cdot\nabla\varphi\| + \|\mu\|\Big) \\
&:= W_1 + W_2 + W_3.
\end{aligned} \tag{5.4.12}$$

The reminder terms W_i, $i = 1, 3$ can be controlled by (3.1.5), (3.3.4) and (5.4.11) as follows

$$\begin{aligned}
W_1 &= C\|\boldsymbol{u}\|\|\varphi_t\|_{V_0'}^{\frac14}\|\nabla\varphi_t\|^{\frac34}\|\varphi_t\| \\
&\le \|\boldsymbol{u}\|\|\varphi_t\|_{V_0'}^{\frac34}\|\nabla\varphi_t\|^{\frac54} \\
&\le \frac16\|\nabla\varphi_t\|^2 + C\|\boldsymbol{u}\|^{\frac83}\|\varphi_t\|_{V_0'}^2 \\
&\le \frac16\|\nabla\varphi_t\|^2 + C\Big(1 + \Lambda^{\frac73}\Big),
\end{aligned}$$

and

$$\begin{aligned}
W_3 &= C\|\boldsymbol{u}\|\|\varphi_t\|_{V_0'}^{\frac14}\|\nabla\varphi_t\|^{\frac34}\|\mu\| \\
&\le C\|\boldsymbol{u}\|\|\varphi_t\|_{V_0'}^{\frac14}\|\nabla\varphi_t\|^{\frac34}\Big(1 + \|\nabla\mu\|\Big) \\
&\le C\|\boldsymbol{u}\|\|\varphi_t\|_{V_0'}^{\frac14}\|\nabla\varphi_t\|^{\frac34}\Big(1 + \|\nabla\mu\|\Big) \\
&\le \frac16\|\nabla\varphi_t\|^2 + C\Big(1 + \Lambda^{\frac73}\Big).
\end{aligned}$$

Here, we have used the estimates (5.3.14) and Young's inequality. Concerning W_2, by the Gagliardo–Nirenberg inequality (5.1.46), Young's inequality and Lemma 3.4.6, we get

$$\begin{aligned}
W_2 &= C\|\boldsymbol{u}\|\|\varphi_t\|_{V_0'}^{\frac14}\|\nabla\varphi_t\|^{\frac34}\|\boldsymbol{u}\cdot\nabla\varphi\| \\
&\le C\|\boldsymbol{u}\|^2\|\varphi_t\|_{V_0'}^{\frac14}\|\nabla\varphi_t\|^{\frac34}\|\nabla\varphi\|_{\mathbf{L}^\infty(\Omega)} \\
&\le \frac16\|\nabla\varphi_t\|^2 + C\|\boldsymbol{u}\|^{\frac{16}{5}}\|\varphi_t\|_{V_0'}^{\frac25}\|\varphi\|_{L^\infty(\Omega)}^{\frac{8}{15}}\|\varphi\|_{W^{2,6}(\Omega)}^{\frac{16}{15}} \\
&\le \frac16\|\nabla\varphi_t\|^2 + C\Big(1 + \Lambda^{\frac73}\Big).
\end{aligned}$$

Collecting the above estimates together, we deduce (5.4.7). □

Remark 5.4.7. It is worth mentioning that the differential inequality (5.4.7) has been obtained without using the separation property (5.4.2) but only the regularity of strong solutions to problem (5.0.1)-(5.0.2) given by Theorem 5.4.5.

The stationary points

The stationary problem related to the evolution system (5.0.1)-(5.0.2) reads as follows

$$\begin{cases} -\Delta\psi + F'(\psi) = \overline{F'(\psi)}, & \text{in } \Omega \\ \partial_{\boldsymbol{n}}\psi = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.4.13}$$

where $\overline{\psi} = \overline{\varphi}_0 = m$. For any given $m \in (-1,1)$, we introduce the set of stationary points

$$\mathcal{Z}_m = \big\{\psi \in H^2(\Omega),\ \mathcal{E}_{GL}(\psi) < +\infty : \psi \text{ solves } (5.4.13)\big\}. \tag{5.4.14}$$

The following result has been proven in [4, Section 6].

Proposition 5.4.8. *The set $\mathcal{Z}_m$ is nonempty. Every element $\psi \in \mathcal{S}_m$ is a critical point of $\mathcal{E}_{GL}$. Moreover, for each $\psi \in \mathcal{Z}_m$, there is a constant $\xi \in (0,1)$ such that*

$$|\psi(x)| \le 1 - \xi, \quad \forall\, x \in \overline{\Omega}. \tag{5.4.15}$$

Next, on account of Proposition 5.4.8 and arguing as in [4, Proposition 6.3] (or [1, Proposition 3]), the following gradient inequality of Łojasiewicz–Simon type can be established. This inequality will be crucial in the study of stability and longtime behavior of problem (5.0.1)-(5.0.2).

Lemma 5.4.9 (Łojasiewicz–Simon inequality)**.** *Let $d = 2,3$. Assume that Ψ satisfies* (H) *and Ψ is real analytic on the open interval $(-1,1)$. For any $m \in (-1,1)$, let $\psi \in \mathcal{Z}_m$. Then, there exist constants $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ such that*

$$|\mathcal{E}_{GL}(\varphi) - \mathcal{E}_{GL}(\psi)|^{1-\theta} \le \left\| -\Delta\varphi + \Psi'(\varphi) - \overline{\Psi'(\varphi)} \right\|, \tag{5.4.16}$$

whenever $\varphi \in H^2(\Omega)$ satisfying $\|\varphi - \psi\|_{H^2(\Omega)} < \beta$, $\overline{\varphi} = m$ and $\partial_{\boldsymbol{n}}\varphi = 0$ on $\partial\Omega$.

Finally, we provide a characterization of local energy minimizers of the functional $\mathcal{E}$ (cf. Definition 5.4.1).

Lemma 5.4.10. *Let $\psi \in \mathcal{V}_m$ be a local energy minimizer of $\mathcal{E}_{GL}$. Then, $\psi \in \mathcal{Z}_m$ and it satisfies the separation property* (5.4.15).

Proof. We consider the Cahn–Hilliard equation with singular potential

$$\begin{cases} \varphi_t = \Delta\mu, & \text{in } \Omega \times (0,+\infty), \\ \mu = -\Delta\varphi + F'(\varphi), & \text{in } \Omega \times (0,+\infty), \\ \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}}\varphi = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ \varphi(\cdot,0) = \varphi_0, & \text{in } \Omega. \end{cases} \tag{5.4.17}$$

It has been proved in [4, Section 6] that for both $d = 2,3$ and any $\varphi_0 \in V$ with $\Psi(\varphi_0) \in L^1(\Omega)$, $\overline{\varphi}_0 = m \in (-1,1)$, problem (5.4.17) admits a unique solution $\varphi(t)$, which defines a family of operators $\{G(t)\}_{t\ge 0}$ such that $G(t) \in C([0,+\infty);\mathcal{V}_m)$, $G(t)\varphi_0 = \varphi(t)$, for all $t \ge 0$. Besides, $\varphi(t)$ regularizes instantaneously for positive time, e.g., $G(t)\varphi_0 \in H^2(\Omega)$ for every $t > 0$. Then, $\{G(t)\}_{t\ge 0}$ is a dynamical system on $\mathcal{V}_m$ in the sense of [149] and the energy functional $\mathcal{E}_{GL}(\varphi) : \mathcal{V}_m \to \mathbb{R}$ is a strict Lyapunov function for $\{G(t)\}_{t\geq 0}$ (due to an energy identity similar to (5.1.4) with $\boldsymbol{u} = \mathbf{0}$).

Therefore, every local energy minimizer $\psi \in \mathcal{V}_m$ must be a stationary point of the evolution problem (5.4.17), i.e., $G(t)\psi = \psi$ for all $t \geq 0$. On the other hand, due to the instantaneous regularity property of problem (5.4.17), it has been shown in [**?**, Section 6] that the set of *all* stationary points is characterized by $\mathcal{Z}_m$. As a consequence, we conclude that $\psi \in \mathcal{Z}_m$ and ψ fulfills (5.4.15) for some $\xi \in (0, 1)$. $\square$

Proof of Theorem 5.4.2

The following relations will be used in the subsequent proof.

Lemma 5.4.11. *Let $d = 3$ and let the assumptions of Theorem 5.4.5 hold. Suppose that $(\boldsymbol{u}, \pi, \varphi)$ is a strong solution to problem* (5.0.1)-(5.0.2) *on* $[0, T]$. *Then, we have*

$$
\begin{aligned}
&\|\boldsymbol{u}\| \leq \|\nabla\mu\|,\\
&\|\nabla\mu\| \leq \|\varphi\|_{H^3(\Omega)} + \|F''(\varphi)\|_{L^\infty(\Omega)}\|\nabla\varphi\|,\\
&\|\varphi\|_{H^3(\Omega)} \leq \overline{C}\Big(\|\nabla\mu\| + \|F''(\varphi)\|_{L^\infty(\Omega)}\|\varphi\|_V + \|\varphi\|_V\Big),
\end{aligned}
$$

where the positive constant $\overline{C}$ only depends on Ω.

Proof. The first two conclusions are obvious. Next, by elliptic regularity, we have

$$
\begin{aligned}
\|\varphi\|_{H^3(\Omega)} &\leq C(\|\Delta\varphi\|_V + \|\varphi\|_{L^2(\Omega)})\\
&\leq C\Big(\|\nabla\mu\| + \|F''(\varphi)\|_{L^\infty(\Omega)}\|\nabla\varphi\| + \|\varphi\|_{H^2(\Omega)}\Big)\\
&\leq C\Big(\|\nabla\mu\| + \|F''(\varphi)\|_{L^\infty(\Omega)}\|\varphi\|_V\Big) + C\|\varphi\|_V^{\frac{1}{2}}\|\varphi\|_{H^3(\Omega)}^{\frac{1}{2}}\\
&\leq \frac{1}{2}\|\varphi\|_{H^3(\Omega)} + C\Big(\|\nabla\mu\| + \|F''(\varphi)\|_{L^\infty(\Omega)}\|\varphi\|_V + \|\varphi\|_V\Big),
\end{aligned}
$$

where C only depends on Ω. $\square$

Proof of Theorem 5.4.2. The proof mainly follows the idea in [158], where problem (5.0.1)-(5.0.2) with the regular potential (1.5.3) was considered. However, here we meet an extra difficulty due the singular potential F. An essential step is to prove a separation property from the pure states ± 1 uniformly for $t \geq 0$ along the trajectory of $\varphi(t)$.

For any given $m \in (-1, 1)$, let $\psi \in \mathcal{Z}_m$ be an arbitrary local energy minimizer of the free energy $\mathcal{E}$ such that (cf. Definition 5.4.1 and Lemma 5.4.10)

$$
\|\psi\|_{\mathcal{C}(\overline{\Omega})} \leq 1 - \xi, \text{ and } \mathcal{E}_{GL}(\psi) \leq \mathcal{E}_{GL}(\varphi) \text{ for all } \varphi \in \mathcal{V}_m : \|\varphi - \psi\|_V < \chi. \quad (5.4.18)
$$

We note that the constants $\xi \in (0, 1)$ and $\chi > 0$ are fixed once ψ is given. Since Ψ is assumed to be real analytic on $(-1, 1)$, then by the separation property (5.4.15) and the classical elliptic regularity theory for the Neumann problem, we have $\psi \in H^k(\Omega)$ ($k \in \mathbb{N}$) provided that Ω is a domain of class $\mathcal{C}^k$.

Due to the Sobolev embedding theorem $H^2(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$ ($d = 3$), it holds

$$
\|\varphi\|_{\mathcal{C}(\overline{\Omega})} \leq C_S\|\varphi\|_{H^2(\Omega)}, \quad (5.4.19)
$$

meanwhile, by (5.1.18) we have

$$\|\varphi\|_V^2 \le C_P(\|\nabla\varphi\|^2 + |\overline{\varphi}|^2), \tag{5.4.20}$$

where C_S and C_P are positive constants that only depend on Ω.

Step 1. Bounds for the initial datum. We consider any initial datum $\varphi_0 \in H^3(\Omega)$ with $\partial_{\boldsymbol{n}}\varphi = 0$ on $\partial\Omega$ that satisfies

$$\|\varphi_0\|_{H^3(\Omega)} \le M, \quad \overline{\varphi}_0 = m, \tag{5.4.21}$$

$$\|\varphi_0 - \psi\|_{H^2(\Omega)} \le \eta, \tag{5.4.22}$$

where $\eta \in (0,1)$ will be determined later and $M > 0$ is given by (5.4.29) below. The fact $\eta < 1$ implies that

$$\|\varphi_0\|_{H^2(\Omega)} \le \|\psi\|_{H^2(\Omega)} + 1. \tag{5.4.23}$$

Moreover, if we further require

$$0 < \eta < \min\left\{1, \frac{\xi}{3C_S}\right\}, \tag{5.4.24}$$

it follows from (5.4.19) that

$$\|\varphi_0\|_{\mathcal{C}(\overline{\Omega})} \le \|\psi\|_{\mathcal{C}(\overline{\Omega})} + \|\varphi_0 - \psi\|_{\mathcal{C}(\overline{\Omega})} \le 1 - \frac{2\xi}{3}. \tag{5.4.25}$$

We define the constants

$$K_1 = \max_{s\in[-1+\frac{\xi}{3}, 1-\frac{\xi}{3}]} |F''(s)|, \qquad K_2 = \max_{s\in[-1,1]} |F(s)|. \tag{5.4.26}$$

Hence, it holds (cf. (5.4.25))

$$\begin{aligned} \mathcal{E}_{GL}(\varphi_0) &\le \frac{1}{2}\|\nabla\varphi_0\|^2 + |\Omega| \max_{s\in[-1,1]} |F(s)| \\ &\le \frac{1}{2}(\|\psi\|_V + 1)^2 + |\Omega| K_2 \\ &:= \gamma_1, \end{aligned} \tag{5.4.27}$$

where $\gamma_1 > 0$ only depends on $\|\psi\|_V$, Ω, F, but is independent of φ_0. Next, denote

$$\gamma_2 = C_P^{\frac{1}{2}}\big(2\gamma_1 + \Theta_0|\Omega| + 1\big)^{\frac{1}{2}}. \tag{5.4.28}$$

Then, we take the constant M in (5.4.21) to be a sufficiently large but fixed number such that

$$M \ge \overline{C}\big[(K_1 + 1)\gamma_2 + 2\big], \tag{5.4.29}$$

where the constant $\overline{C}$ is given in Lemma 5.4.11.

In the sequel we denote by C, C_i those constants that only depend on Ω, the averaged mass m, norms of the local minimizer ψ, the function Ψ and parameters like ξ, χ. Specific dependences will be pointed out explicitly.

Step 2. Strong solution on a finite interval. On account of the assumptions (5.4.21), (5.4.24) and (5.4.29), it follows from Theorem 5.4.5 that there exists a unique local strong solution $(\boldsymbol{u}, \pi, \varphi)$ to problem (5.0.1)-(5.0.2) on $[0, T_1]$ such that

$$\|\varphi(t)\|_{C(\overline{\Omega})} \leq 1 - \frac{\xi}{3}, \quad \forall\, t \in [0, T_1], \tag{5.4.30}$$

where T_1 depends, in particular, on M and ξ. We infer from (5.4.21) and Lemma 5.4.11 that

$$\Lambda(0) = \|\nabla \mu(0)\|^2 + \|\boldsymbol{u}(0)\|^2 \leq 2\Big[M + K_1(\|\psi\|_V + 1)\Big]^2 := M_1. \tag{5.4.31}$$

It follows from the higher order inequality (5.4.7) and (5.4.27) that there exists $T_2 \in (0, T_1]$ depending on M_1, α, Ω, m and $\|\psi\|_V$ such that

$$\Lambda(t) \leq 2M_1, \quad \forall\, t \in [0, T_2]. \tag{5.4.32}$$

Besides, we set

$$E_0 = \frac{\min\{1, M_1\} T_2}{2}, \tag{5.4.33}$$

which is a number that characterizes the energy drop along the trajectory $\varphi(t)$ (cf. (5.4.43)). By the energy identity (5.1.4), it holds $\mathcal{E}_{GL}(\varphi(t)) \leq \mathcal{E}_{GL}(\varphi_0)$ for $t \geq 0$. On the other hand, we know that

$$\mathcal{E}_{GL}(\varphi(t)) \geq \frac{1}{2}\|\nabla \varphi(t)\|^2 - \frac{\Theta_0}{2}|\Omega|, \quad \forall\, t \in [0, T_2].$$

Then, we deduce from (5.4.20), (5.4.27) and (5.4.28) that

$$\begin{aligned} \|\varphi(t)\|_V^2 &\leq C_\Omega(\|\nabla \varphi(t)\|^2 + m^2) \\ &\leq C_\Omega\big(2\mathcal{E}(\varphi_0) + \Theta_0|\Omega| + 1\big) \\ &\leq \gamma_2^2, \quad \forall\, t \in [0, T_2], \end{aligned} \tag{5.4.34}$$

which together with Lemma 5.4.11, (5.4.30) and (5.4.32) yields

$$\|\varphi(t)\|_{H^3(\Omega)} \leq \overline{C}\Big[\sqrt{2M_1} + (K_1 + 1)\gamma_2\Big] := M_2, \quad \forall\, t \in [0, T_2]. \tag{5.4.35}$$

Step 3. Refined estimates. Our aim is to find a sufficiently small $\eta > 0$ such that the local strong solution satisfies a uniform bound that is independent of the existence interval. If this is true, then we can extend the unique local strong solution obtained in Step 1 to be a global one on $[0, +\infty)$.

It follows from (5.4.30) and (5.4.34) that

$$\begin{aligned} \mathcal{E}_{GL}(\varphi_0) - \mathcal{E}_{GL}(\varphi(t)) &\leq \frac{1}{2}\|\nabla(\varphi(t) + \varphi_0)\|\|\nabla(\varphi(t) - \varphi_0)\| \\ &\quad + \max_{s \in [-1+\frac{\xi}{3}, 1-\frac{\xi}{3}]} |F'(s)|\|\varphi(t) - \varphi_0\|_{L^1(\Omega)} \\ &\leq M_3\|\varphi(t) - \varphi_0\|_V, \quad \forall\, t \in [0, T_2], \end{aligned} \tag{5.4.36}$$

where the constant M_3 depends on γ_2, $\|\psi\|_V$, Ψ, ξ and Ω.

For any $\epsilon > 0$, let us now set

$$\omega = \min\left\{1,\ \epsilon,\ \chi,\ \beta,\ \frac{\xi}{3C_S},\ \frac{2E_0}{3M_3}\right\}, \tag{5.4.37}$$

where $\beta > 0$ is determined by Lemma 5.4.9 and E_0 is given in (5.4.33). We define

$$T_\eta = \inf\{t > 0 :\ \|\varphi(t) - \psi\|_{H^2(\Omega)} \geq \omega\} \quad \text{for } \eta \in \left(0, \frac{\omega}{2}\right].$$

By (5.4.21) and continuity of the strong solution $\varphi(t)$ in $H^2(\Omega)$, it follows that $T_\eta > 0$. Next, we claim that there exists at least a value of η such that $T_\eta \geq T_2$. Indeed, by contradiction, we have that $T_\eta < T_2$ for all $\eta \in (0, \frac{\omega}{2}]$. As a consequence, we apply Lemma 5.4.9 to derive the following energy inequality on the interval $[0, T_\eta] \subset [0, T_2]$

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{E}_{GL}(\varphi(t)) - \mathcal{E}_{GL}(\psi)]^\theta &= -\theta[\mathcal{E}_{GL}(\varphi(t)) - \mathcal{E}_{GL}(\psi)]^{\theta-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varphi(t)) \\
&\geq \frac{\theta(\|\nabla\mu\|^2 + \|\boldsymbol{u}\|^2)}{\|\nabla\mu\|} \\
&\geq \frac{\theta}{2}(\|\boldsymbol{u}\| + \|\nabla\mu\|) \geq C_1\|\varphi_t\|_{V_0'},
\end{aligned} \tag{5.4.38}$$

where the constant C_1 depends on θ and Ω. Here and after, we shall always exclude the trivial case such that there is a $t_0 \in [0, T_\eta]$ such that $\mathcal{E}_{GL}(\varphi(t_0)) = \mathcal{E}_{GL}(\psi)$. In that case, $\|\boldsymbol{u}(t)\| = \|\nabla\mu(t)\| = 0$ for all $t \geq t_0$ by virtue of the energy identity (5.1.4) and the evolution stops.

Then, using (5.4.38) and recalling that $\mathcal{E}(\varphi(t))$ is nonincreasing and, by the choice of ω, $\mathcal{E}(\varphi(t)) \geq \mathcal{E}(\psi)$ on $[0, T_\eta]$ (cf. Definition 5.4.1), we infer that (cf. (5.4.36))

$$\int_0^{T_\eta} \|\varphi_t(\tau)\|_{V_0'}\,\mathrm{d}\tau \leq C_1\big(\mathcal{E}_{GL}(\varphi_0) - \mathcal{E}_{GL}(\psi)\big)^\theta \leq C_2\|\varphi_0 - \psi\|_V^\theta,$$

where C_2 depends on C_1, $\|\psi\|_V$, F, ξ and Ω. As a consequence, we obtain

$$\begin{aligned}
\|\varphi(T_\eta) - \psi\|_{H^2(\Omega)} &\leq \|\varphi_0 - \psi\|_{H^2(\Omega)} + \|\varphi(T_\eta) - \varphi_0\|_{H^2(\Omega)} \\
&\leq \|\varphi_0 - \psi\|_{H^2(\Omega)} + C_3\|\varphi(T_\eta) - \varphi_0\|_{H^3(\Omega)}^{\frac{3}{4}}\|\varphi(T_\eta) - \varphi_0\|_{V_0'}^{\frac{1}{4}} \\
&\leq \|\varphi_0 - \psi\|_{H^2(\Omega)} + C_3(M + M_2)^{\frac{3}{4}}\left(\int_0^{T_\eta} \|\varphi_t(\tau)\|_{V_0'}\,\mathrm{d}\tau\right)^{\frac{1}{4}} \\
&\leq \|\varphi_0 - \psi\|_{H^2(\Omega)} + C_3(M + M_2)^{\frac{3}{4}}C_2^{\frac{1}{4}}\|\varphi_0 - \psi\|_{H^2(\Omega)}^{\frac{\theta}{4}}.
\end{aligned}$$

Choosing now

$$\eta = \min\left\{\frac{\omega}{2}, \left(\frac{\omega}{4C_2^{\frac{1}{4}}C_3(M + M_2)^{\frac{3}{4}}}\right)^{\frac{4}{\theta}}\right\}, \tag{5.4.39}$$

we have $\|\varphi(T_\eta) - \psi\|_{H^2(\Omega)} \leq \frac{3}{4}\omega < \omega$, which yields a contradiction with the definition of T_η. As a consequence, for the above choice of η, it holds $T_\eta \geq T_2$ and we learn that

$$\|\varphi(t) - \psi\|_{H^2(\Omega)} \leq \omega, \quad \forall\, t \in [0, T_2]. \tag{5.4.40}$$

In turn, by (5.4.37) and (5.4.40) we obtain, for all $t \in [0, T_2]$,

$$\|\varphi(t)\|_{\mathcal{C}(\overline{\Omega})} \le \|\psi\|_{\mathcal{C}(\overline{\Omega})} + C_S\|\varphi(t) - \psi\|_{H^2(\Omega)} \le 1 - \frac{2\xi}{3}, \tag{5.4.41}$$

$$\|\varphi(t) - \varphi_0\|_{H^2(\Omega)} \le \|\varphi(t) - \psi\|_{H^2(\Omega)} + \|\varphi_0 - \psi\|_{H^2(\Omega)} \le \frac{E_0}{M_3}. \tag{5.4.42}$$

Then we infer from the energy identity (5.1.4), (5.4.36) and (5.4.42) that

$$\int_0^{T_2} \Lambda(\tau)\, \mathrm{d}\tau = \mathcal{E}_{GL}(\varphi_0) - \mathcal{E}_{GL}(\varphi(T_2)) \le E_0. \tag{5.4.43}$$

Step 4. Iteration argument. Due to the nonnegativity of the function Λ and (5.4.43), there exists $t^* \in [\frac{1}{2}T_2, T_2]$ such that

$$\Lambda(t^*) \le \min\{1, M_1\}. \tag{5.4.44}$$

Then it follows from Lemma 5.4.11, (5.4.29) and (5.4.34) that

$$\|\varphi(t^*)\|_{H^3(\Omega)} \le \overline{C}\Big[\sqrt{\min\{1,\ M_1\}} + (K_1+1)\gamma_2\Big] < M. \tag{5.4.45}$$

Now we easily see that $\varphi(t^*)$ satisfies the same bounds as for φ_0 (compare (5.4.21), (5.4.25) with (5.4.41), (5.4.45)). Thus, we can take $\varphi(t^*)$ as the new initial datum and solve the problem (5.0.1)-(5.0.2) as in Step 1 and Step 2 on $[t^*, t^* + T_2]$. Thanks to the uniqueness of strong solutions, this yields a local strong solution defined on the extended interval $[0, t^* + T_2]$. After that we repeat the argument in Step 3 on $[0, \frac{3}{2}T_2] \subset [0, t^* + T_2]$ to derive the same refined estimates (5.4.40)–(5.4.43) on $[0, \frac{3}{2}T_2]$ under exactly the same choice of η (i.e., (5.4.39)). Again, there exists $t^{**} \in [T_2, \frac{3}{2}T_2]$ such that $\Lambda(t^{**}) \le \min\{1, M_1\}$. Then we can take t^{**} as the initial time to repeat the above procedure and extend the unique local strong solution to the extended interval $[0, 2T_2]$ with uniform estimates (5.4.35) and (5.4.40) on $[0, 2T_2]$.

By iteration, we easily arrive at the conclusion of Theorem 5.4.2. □

5.5 Longtime Behavior

In this section, we investigate the longtime behavior of global solutions.

The Infinite Dimensional Dynamical System

We briefly discuss the infinite dimensional dynamical system associated to (5.0.1)-(5.0.2). For any $m \in (-1, 1)$, we consider the phase space $\mathcal{V}_m$ (see (5.4.1)) with the metric

$$\mathbf{d}(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|_V.$$

It is well-known that $\mathcal{V}_m$ is a complete metric space. The following result can be proven:

Theorem 5.5.1. *Let* $m \in (-1, 1)$. *Denote by* $\mathcal{G}_m$ *the family of all global weak solutions to problem* (5.0.1)-(5.0.2) *with initial condition* $\varphi_0 \in \mathcal{V}_m$. *Then,* $\mathcal{G}_m$ *defines a* generalized semiflow *on* $\mathcal{V}_m$ *in the sense of [13] and admits a unique global attractor.*

Thanks to the validity of Theorems 5.1.3 and 5.1.11, in particular, the energy identity (5.1.4) for global weak solutions, the proof of Theorem 5.5.1 can be carried out by a standard argument (see, e.g., [63]) with some minor modifications and thus we leave the details to the interested readers.

Next, in two spatial dimensions, thanks to the uniqueness and regularity results (cf. Theorem 5.2.1 and Theorem 5.3.1, we have a strongly continuous semigroup acting on the phase space $\mathcal{V}_m$ defined via the rule $\mathcal{S}_m(t)\varphi_0 = \varphi(t)$ (see Section 8.4). Moreover, Theorem 5.3.1 also entails that the global attractor is bounded in the more regular space $H^4(\Omega)$. Therefore, on account of known results for infinite dimensional dynamical systems, the global attractor obtained in Theorem 5.5.1 consists of a time-section of complete (i.e., defined on the whole $\mathbb{R}$) strong solutions. Besides, exploiting the separation property (5.3.1), one can proceed to establish the following result through the general approach described in [120].

Theorem 5.5.2. *Let $d = 2$ and $m \in (-1, 1)$. Assume that $\Psi \in \mathcal{C}^3(-1, 1)$ and $(\mathrm{H}.2) - (\mathrm{H}.3)$ hold. The dynamical system $(\mathcal{V}_m, \mathcal{S}_m(t))$ has an exponential attractor that is bounded in $H^4(\Omega)$. This further implies, in particular, the global attractor for problem* (5.0.1)-(5.0.2) *has finite fractal dimension.*

Convergence to Single Stationary State

Concerning the longtime behavior of single trajectory associated to (5.0.1)-(5.0.2), we have the uniqueness of the asymptotic limit as $t \to +\infty$. In order to show this, we need that the solution eventually satisfies the separation property. Accordingly, the following result deal with any global weak solutions in dimension two, whereas it considers global strong solutions in dimension three.

Theorem 5.5.3. *Assume that Ψ is real analytic in $(-1, 1)$. If $d = 2$, let $(\boldsymbol{u}, \pi, \varphi)$ be any global weak solution to problem* (5.0.1)-(5.0.2)*. If $d = 3$, let $(\boldsymbol{u}, \pi, \varphi)$ be a global strong solution given by Theorem 5.4.2. Then, for both cases, there exists $\varphi_\infty \in H^3(\Omega)$ which is a solution to the stationary Cahn–Hilliard equation*

$$\begin{cases} -\Delta\varphi_\infty + F'(\varphi_\infty) = \overline{F'(\varphi_\infty)}, & \text{in } \Omega \\ \partial_{\boldsymbol{n}}\varphi_\infty = 0, & \text{on } \partial\Omega, \\ \overline{\varphi}_\infty = \overline{\varphi}_0, \end{cases}$$

such that $(\boldsymbol{u}(t), \varphi(t))$ converges to $(\mathbf{0}, \varphi_\infty)$ as $t \to +\infty$ with the following convergence rate

$$\|\boldsymbol{u}(t)\| + \|\varphi(t) - \varphi_\infty\|_{H^3(\Omega)} \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \ge 1.$$

Here, $C \ge 0$ is a constant depending on $\|\varphi_0\|_V$ (if $d = 2$), $\|\varphi_0\|_{H^3(\Omega)}$ (if $d = 3$), $\|\varphi_\infty\|_{H^3(\Omega)}$ and F, while $\theta \in (0, \frac{1}{2})$ is a constant depending only on φ_∞ (cf. Lemma 5.4.9).

Remark 5.5.4. Theorem 5.5.3 implies that, for any global strong solution $(\boldsymbol{u}, \pi, \varphi)$ obtained in Theorem 5.4.2, φ will not only stay close to that local energy minimizer ψ, but also converge to a certain equilibrium φ_∞ that is near ψ. Furthermore, if ψ is an isolated minimizer, then it follows that $\varphi_\infty = \psi$, namely, ψ is locally asymptotically stable.

Before proving Theorem 5.5.3, we need the following preliminary convergence result.

Proposition 5.5.5. *Let $d = 2, 3$ and let the assumptions of Theorem 5.4.5 hold. Assume that the initial averaged mass satisfies $\overline{\varphi}_0 = m \in (-1, 1)$ and Ψ is real analytic in $(-1, 1)$. If $(\boldsymbol{u}, \pi, \varphi)$ is a global strong solution to problem* (5.0.1)-(5.0.2) *and there exist $M > 0$ and $\delta \in (0, 1)$ such that*

$$\|\varphi(t)\|_{H^3(\Omega)} \leq M, \quad \|\varphi(t)\|_{C(\overline{\Omega})} \leq 1 - \delta, \quad \forall\, t \geq 0, \tag{5.5.1}$$

then $(\boldsymbol{u}(t), \varphi(t))$ converges to an equilibrium $(\mathbf{0}, \varphi_\infty)$ as $t \to +\infty$ with the following convergence rate

$$\|\boldsymbol{u}(t)\| + \|\varphi(t) - \varphi_\infty\|_{H^3(\Omega)} \leq C(1 + t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \geq 0. \tag{5.5.2}$$

Here, $\varphi_\infty \in \mathcal{Z}_m \cap H^3(\Omega)$ is a solution to the stationary Cahn–Hilliard equation (5.4.13)*, $C > 0$ is a constant depending on M, $\|\varphi_\infty\|_{H^3(\Omega)}$, δ, Ψ and Ω, $\theta \in (0, \frac{1}{2})$ is a constant depending only on φ_∞.*

Proof. We observe that due to the assumption (5.5.1), $F(\varphi(t))$ is confined on $[-1 + \delta, 1 - \delta]$ along the trajectory $\varphi(t)$ for $t \geq 0$ so that Lemma 5.4.9 can apply. Then, the conclusion follows from the same argument as in [158] for the problem (5.0.1)-(5.0.2) with the regular potential (1.5.3). □

We can now proceed to prove Theorem 5.5.3.

Proof of Theorem 5.5.3. In light of Theorem 5.3.1 for any global weak solution $(\boldsymbol{u}, \pi, \varphi)$ in two dimensions, we can consider our solution from a certain positive time on to deal with a (global) strong solution. By Theorem 5.3.1 (resp. Theorem 5.4.2, with ϵ sufficiently small, cf. (5.4.41)) for global strong solutions in two (resp. three) dimensions, we see that the assumptions made in Proposition 5.5.5 are fulfilled. As an immediate consequence, the conclusion in Theorem 5.5.3 holds. □

CHAPTER 6

The Brinkman-Cahn-Hilliard system

In this chapter we consider the Brinkman–Cahn–Hilliard (BCH) system with unmatched viscosities in two space dimensions. We first show the existence of weak solutions. In particular, we are able to prove further regularities in the class of weak solutions. In turn, these properties allow us to address the uniqueness of weak solutions. Then, we study the regularity propagation in time and the *instantaneous* separation property from the pure phases.

In a bounded domain $\Omega \subset \mathbb{R}^2$, the Brinkman–Cahn–Hilliard system with unmatched viscosities reads as

$$\begin{cases} -\operatorname{div}(\nu(\varphi)D\boldsymbol{u}) + \boldsymbol{u} + \nabla\pi = \mu\nabla\varphi, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \partial_t\varphi + \boldsymbol{u}\cdot\varphi = \Delta\mu, \\ \mu = -\Delta\varphi + F'(\varphi), \end{cases} \qquad \text{in } \Omega\times(0,T), \tag{6.0.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = \mathbf{0}, \quad \partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\mu = 0, & \text{on } \partial\Omega\times(0,T), \\ \varphi(\cdot,0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{6.0.2}$$

General agreement. Throughout this chapter, if it is not otherwise stated, we indicate by C a generic positive constant depending only on the domain and on structural quantities. The constant C may vary from line to line and even within the same line. Any further dependence will be explicitly pointed out if necessary.

6.1 Existence of Weak Solutions

In the sequel the viscosity coefficient $\nu = \nu(s)$ is assumed to be a bounded function satisfying $\nu \in \mathcal{C}^2(\mathbb{R})$ and

$$\nu(s) \geq 2\nu_1 > 0, \quad \forall\, s \in \mathbb{R}. \tag{6.1.1}$$

We introduce the definition of weak solutions.

Definition 6.1.1. *Given $\varphi_0 \in V$ with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$, a pair $(\boldsymbol{u}, \varphi)$ is a weak solution to* (6.0.1)-(6.0.2) *on $[0,T]$ if*

$$\begin{aligned}
&\boldsymbol{u} \in L^2(0,T;\mathbf{V}_\sigma),\\
&\varphi \in L^\infty(0,T;V) \cap L^4(0,T;H^2(\Omega)) \cap H^1(0,T;V'),\\
&\varphi \in L^\infty(\Omega \times (0,T)) \quad \textit{with} \quad |\varphi(x,t)| < 1 \quad \textit{a.e. } (x,t) \in \Omega \times (0,T),\\
&\Psi'(\varphi) \in L^2(0,T;H),
\end{aligned}$$

and

$$(\nu(\varphi)D\boldsymbol{u}, D\boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}) = (\mu\nabla\varphi, \boldsymbol{v}), \qquad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma, \tag{6.1.2}$$

$$\langle \partial_t \varphi, v\rangle + (\boldsymbol{u} \cdot \nabla\varphi, v) + (\nabla\mu, \nabla v) = 0, \qquad \forall\, v \in V, \tag{6.1.3}$$

for almost every $t \in (0,T)$, where $\mu \in L^2(0,T;V)$ is given by

$$\mu = -\Delta\varphi + F'(\varphi), \tag{6.1.4}$$

for almost every $(x,t) \in \Omega \times (0,T)$. Moreover, $\partial_{\boldsymbol{n}}\varphi = 0$ a.e. on $\partial\Omega \times (0,T)$ and $\varphi(\cdot, 0) = \varphi_0$ a.e. in Ω.

Remark 6.1.2. It is straightforward to observe that any solution satisfies the mass conservation property, namely,

$$\overline{\varphi}(t) = \overline{\varphi}_0, \quad \forall\, t \geq 0.$$

Remark 6.1.3. Note that equation (6.1.2) is equivalent to

$$(\nu(\varphi)D\boldsymbol{u}, D\boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}) = (\nabla\varphi \otimes \nabla\varphi, \nabla\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma,$$

in light of the equality

$$\mu\nabla\varphi = \nabla\Big(\frac{1}{2}|\nabla\varphi|^2 + F(\varphi)\Big) - \operatorname{div}(\nabla\varphi \otimes \nabla\varphi).$$

Remark 6.1.4. As customary, the pressure term is dropped in the weak formulation of the Brinkman's law. Indeed, the pressure can be recovered (up to a constant) thanks to the classical de Rham's theorem (see, for instance, [151]). In particular, since

$$S = \nabla \cdot (\nu(\varphi)D\boldsymbol{u}) - \boldsymbol{u} - \mu\nabla\varphi$$

is orthogonal (in the dual sense) to any element of $\mathbf{H}_0^1(\Omega)$, then there exists a function $\pi \in L^2(0,T;L_0^2(\Omega))$ satisfying $\nabla\pi = S$.

We state our existence result.

Theorem 6.1.5. *Let $\varphi_0 \in V$ with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Then, for any $T > 0$, there exists a weak solution $(\boldsymbol{u}, \varphi)$ to problem* (6.0.1)-(6.0.2) *on $[0, T]$ such that*

$$\varphi \in \mathcal{C}([0, \infty), V),$$

and the energy identity

$$\mathcal{E}_{GL}(\varphi(t)) + \int_s^t \|\sqrt{\nu(\varphi(\tau))} D\boldsymbol{u}(\tau)\|^2 + \|\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \, \mathrm{d}\tau = \mathcal{E}_{GL}(\varphi(s))$$

is satisfied for all $0 \le s < t < \infty$. Furthermore, we have the dissipative estimates

$$\mathcal{E}_{GL}(\varphi(t)) + \int_t^{t+1} \|\nabla\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \, \mathrm{d}\tau \le C\mathcal{E}_{GL}(\varphi_0)\mathrm{e}^{-\omega t} + C, \tag{6.1.5}$$

and

$$\int_t^{t+1} \|\varphi(\tau)\|_{H^2(\Omega)}^4 + |\overline{\mu}(\tau)|^2 \, \mathrm{d}\tau \le C\mathcal{E}_{GL}(\varphi_0)^2\mathrm{e}^{-\omega \mathrm{t}} + \mathrm{C} \tag{6.1.6}$$

for every $t \ge 0$, where ω and C are positive constants independent of the initial datum.

The rest of the section is devoted to the proof of Theorem 6.1.5, which is obtained via an approximation procedure and energy estimates.

Approximation of the singular potential

We recall the sequence of regular functions Ψ_λ which approximate the singular potential Ψ introduced in Section 3.2. For any $\lambda > 0$, there exists

$$\Psi_\lambda : \mathbb{R} \to \mathbb{R}$$

such that

(i) Ψ_λ is convex and $\Psi_\lambda(s) \nearrow \Psi(s)$, for all $s \in \mathbb{R}$, as $\lambda \to 0$;

(ii) For any $0 < \overline{\lambda} \le 1$, there exists $C > 0$ such that

$$\Psi_\lambda(s) \ge \frac{1}{4\overline{\lambda}} s^2 - C, \quad \forall\, s \in \mathbb{R},\ \forall\, \lambda \in (0, \overline{\lambda}];$$

(iii) Ψ_λ' is Lipschitz on $\mathbb{R}$ with constant $\frac{1}{\lambda}$ and $\Psi_\lambda''(s)$ exists (see Lemma 5.1.5) for all $s \in \mathbb{R}$;

(iv) $|\Psi_\lambda'(s)| \nearrow |\Psi'(s)|$ for $s \in (-1, 1)$ and Ψ_λ' converges uniformly to Ψ' on any set $[a, b] \subset (-1, 1)$;

(v) $\Psi_\lambda(0) = \Psi_\lambda'(0) = 0$, for all $\lambda > 0$.

The approximating problems.

For any $\lambda \in (0, 1)$ fixed, we introduce the quadratic perturbation of Ψ_λ by

$$F_\lambda(s) = \Psi_\lambda(s) - \frac{\Theta_0}{2} s^2.$$

The corresponding regular BCH$_\lambda$ problem reads as

$$\begin{cases} -\mathrm{div}(\nu(\varphi)D\boldsymbol{u}) + \boldsymbol{u} + \nabla\pi = \mu\nabla\varphi, \\ \mathrm{div}\,\boldsymbol{u} = 0, \\ \varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \Delta\mu, \\ \mu = -\Delta\varphi + F'_\lambda(\varphi), \end{cases}$$

endowed with (6.0.2). Analogously to the singular case, given any $\varphi_0 \in V$, a pair $(\boldsymbol{u}, \varphi)$ is a solution of BCH$_\lambda$ on $[0, T]$ if

$$\boldsymbol{u} \in L^2(0, T; \mathbf{V}_\sigma),$$
$$\varphi \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)) \cap H^1(0, T; V'),$$

and

$$(\nu(\varphi)D\boldsymbol{u}, D\boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}) = (\mu\nabla\varphi, \boldsymbol{v}), \qquad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma, \tag{6.1.7}$$
$$\langle \partial_t\varphi, v\rangle + (\boldsymbol{u}\cdot\nabla\varphi, v) + (\nabla\mu, \nabla v) = 0, \qquad \forall\, v \in V, \tag{6.1.8}$$

for almost every $t \in (0, T)$, where

$$\mu = -\Delta\varphi + F'_\lambda(\varphi) \in L^2(0, T; V).$$

The Brinkman–Cahn–Hilliard system with regular potential having polynomial growth, satisfying also suitable dissipation assumptions, has been studied in [20]. Since F_λ complies with these requirements, for any $\lambda > 0$ we have the following

Theorem 6.1.6. *Let $\varphi_0 \in V$. Then, for any $T > 0$, the BCH$_\lambda$ problem has at least a weak solution $(\boldsymbol{u}, \varphi)$ on $[0, T]$ such that*

$$\varphi \in \mathcal{C}([0, T], V) \cap L^2(0, T; H^3(\Omega)),$$

which satisfies (6.1.7)-(6.1.8).

The proof of Theorem 6.1.6 is carried out by a standard Galerkin method and by exploiting the Lipschitz regularity of F_λ.

Energy estimates

Let $\lambda \in (0, 1)$ be fixed. We denote the energy of a solution to BCH$_\lambda$ by

$$\mathcal{E}^\lambda_{GL}(\varphi) = \frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega F_\lambda(\varphi)\,\mathrm{d}x.$$

In what follows, the generic positive constant C is independent of λ and of the initial datum.

Lemma 6.1.7. *There exist $\overline{\lambda} > 0$ and $\omega > 0$ such that, for any $\varphi_0 \in V$ and any $0 < \lambda \leq \overline{\lambda}$, we have the dissipative estimates*

$$\mathcal{E}^\lambda_{GL}(\varphi(t)) + \|\varphi(t)\|^2_V \leq C\mathcal{E}^\lambda_{GL}(\varphi_0)\mathrm{e}^{-\omega t} + C(\Psi_\lambda(\overline{\varphi}_0) + 1), \tag{6.1.9}$$

and

$$\int_t^{t+1} \|\nabla\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2\,\mathrm{d}\tau \leq C\mathcal{E}^\lambda_{GL}(\varphi_0)\mathrm{e}^{-\omega t} + C(\Psi_\lambda(\overline{\varphi}_0) + 1), \tag{6.1.10}$$

for every $t \geq 0$.

Proof. We take $\boldsymbol{v} = \boldsymbol{u}$ in (6.1.7) and $v = \mu$ in (6.1.8). Summing up the resulting equalities, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}^{\lambda}(\varphi) + \langle \nu(\varphi)D\boldsymbol{u}, D\boldsymbol{u}\rangle + \|\boldsymbol{u}\|^2 + \|\nabla\mu\|^2 = 0. \tag{6.1.11}$$

To reconstruct the energy in (6.1.11), we test μ by $\varphi - \overline{\varphi}$, getting

$$(\Psi_\lambda'(\varphi), \varphi - \overline{\varphi}) + \|\nabla\varphi\|^2 = \Theta_0(\varphi, \varphi - \overline{\varphi}) + (\mu - \overline{\mu}, \varphi - \overline{\varphi}). \tag{6.1.12}$$

By the convexity of Ψ_λ, we know that

$$\int_\Omega \Psi_\lambda(\varphi)\,\mathrm{d}x \le \int_\Omega \Psi_\lambda'(\varphi)(\varphi - \overline{\varphi})\,\mathrm{d}x + \int_\Omega \Psi_\lambda(\overline{\varphi})\,\mathrm{d}x.$$

By (3.1.5), we also get

$$\Theta_0(\varphi, \varphi - \overline{\varphi}) + (\mu - \overline{\mu}, \varphi - \overline{\varphi}) \le \frac{1}{2}\|\nabla\varphi\|^2 + C\|\nabla\mu\|^2 + C\Theta_0^2\|\varphi\|^2.$$

Then, we arrive at

$$\int_\Omega \Psi_\lambda(\varphi)\,\mathrm{d}x + \frac{1}{2}\|\nabla\varphi\|^2 \le C\|\nabla\mu\|^2 + C\Psi_\lambda(\overline{\varphi}) + C\Theta_0^2\|\varphi\|^2.$$

Now, exploiting (ii) with a small $\overline{\lambda} = \overline{\lambda}(\Theta_0)$, we find

$$\frac{1}{2}\mathcal{E}_{GL}^{\lambda}(\varphi) \le C\|\nabla\mu\|^2 + C\Psi_\lambda(\overline{\varphi}) + C.$$

Multiplying the above inequality by 2ω, where $\omega = \frac{1}{4C}$, and, summing up with (6.1.11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}^{\lambda}(\varphi) + \omega\mathcal{E}_{GL}^{\lambda}(\varphi) + \nu_1\|\nabla\boldsymbol{u}\|^2 + \frac{1}{2}\|\nabla\mu\|^2 \le C\Psi_\lambda(\overline{\varphi}) + C. \tag{6.1.13}$$

Here, we have also used (6.1.1) and the Korn inequality (3.5.4). An application of the Gronwall lemma, together with the mass conservation, yields

$$\mathcal{E}_{GL}^{\lambda}(\varphi(t)) \le \mathcal{E}_{GL}^{\lambda}(\varphi_0)\mathrm{e}^{-\omega t} + C(\Psi_\lambda(\overline{\varphi}_0) + 1),$$

for some $\omega, C > 0$ that are independent of λ. In addition, owing to (ii), for a possibly smaller $\overline{\lambda}$, there exists C such that

$$\mathcal{E}_{GL}^{\lambda}(\varphi) \ge \frac{1}{2}\|\varphi\|_V^2 - C,$$

for every $\lambda \in (0, \overline{\lambda}]$. Therefore, we infer that

$$\|\varphi(t)\|_V^2 \le C\mathcal{E}_{GL}^{\lambda}(\varphi_0)\mathrm{e}^{-\omega t} + C(\Psi_\lambda(\overline{\varphi}_0) + 1), \quad \forall\, t \ge 0.$$

A final integration of (6.1.13) on $[t, t+1]$ completes the proof. $\square$

We prove two consequences of the dissipative nature of the system, referring hereafter to $\overline{\lambda}$ and ω as the parameters defined in Lemma 6.1.7.

Lemma 6.1.8. *We have*

$$\int_t^{t+1} \|\partial_t\varphi(\tau)\|_{V'}^2 \,\mathrm{d}\tau \le C\Big(\mathcal{E}_{GL}^\lambda(\varphi_0)\mathrm{e}^{-\omega t} + \Psi_\lambda(\overline{\varphi}_0) + 1\Big)^2, \quad \forall\, t \ge 0. \tag{6.1.14}$$

Proof. We first observe that

$$(\boldsymbol{u}\cdot\nabla\varphi, v) \le \|\boldsymbol{u}\|_{L^3(\Omega)}\|\varphi\|_{L^6(\Omega)}\|\nabla v\| \le C\|\nabla\boldsymbol{u}\|\|\varphi\|_V\|\nabla v\|, \quad v\in V.$$

Then, exploiting (6.1.9), we have

$$\int_t^{t+1} \|\boldsymbol{u}(\tau)\cdot\nabla\varphi(\tau)\|_{V'}^2 \,\mathrm{d}\tau \le C\Big(\mathcal{E}_{GL}^\lambda(\varphi_0)\mathrm{e}^{-\omega t} + \Psi_\lambda(\overline{\varphi}_0) + 1\Big)\int_t^{t+1} \|\nabla\boldsymbol{u}(\tau)\|^2 \,\mathrm{d}\tau.$$

Therefore, by comparison

$$\begin{aligned}&\int_t^{t+1} \|\partial_t\varphi(\tau)\|_{V'}^2 \,\mathrm{d}\tau \\ &\qquad \le C\Big(\mathcal{E}_{GL}^\lambda(\varphi_0)\mathrm{e}^{-\omega t} + \Psi_\lambda(\overline{\varphi}_0) + 1\Big)\int_t^{t+1} \|\nabla\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \,\mathrm{d}\tau,\end{aligned}$$

which, in turn, entails the desired conclusion. □

Lemma 6.1.9. *Let* $\varphi_0 \in V$ *with* $\overline{\varphi}_0 = m \in (-1,1)$. *We have*

$$\int_t^{t+1} \|\Delta\varphi(\tau)\|^4 + \|\Psi'(\varphi(\tau))\|_{L^1(\Omega)}^2 + |\overline{\mu}(\tau)|^2 \,\mathrm{d}\tau \le C\Big(\mathcal{E}_{GL}^\lambda(\varphi_0)\mathrm{e}^{-\omega t} + \Psi_\lambda(\overline{\varphi}_0) + 1\Big)^2,$$

for every $t \ge 0$.

Proof. Testing μ by $-\Delta\varphi$ and integrating by parts, we get

$$(\nabla\mu, \nabla\varphi) = \|\Delta\varphi\|^2 + (F_\lambda'(\varphi), -\Delta\varphi).$$

An additional integration by parts, together with (H), yields

$$(F_\lambda'(\varphi), -\Delta\varphi) = (F_\lambda''(\varphi)\nabla\varphi, \nabla\varphi) \ge -\alpha\|\nabla\varphi\|^2.$$

Hence, we find

$$\|\Delta\varphi\|^2 \le C\|\nabla\varphi\|^2 + \|\nabla\mu\|\|\nabla\varphi\|.$$

Besides, we have

$$\overline{\mu} = (F_\lambda'(\varphi), 1).$$

In order to control the right-hand side, we recall that there exists $C>0$, independent of $\lambda \in (0, \overline{\lambda}]$, such that

$$\|F_\lambda'(\varphi)\|_{L^1(\Omega)} \le (F_\lambda'(\varphi) - \overline{F_\lambda'(\varphi)}, \varphi - \overline{\varphi}) + C.$$

where C depends on m (see, e.g., [118, Proposition A.2] and Chapter 7). Moreover, by virtue of (6.1.12), we know that

$$\begin{aligned}(F_\lambda'(\varphi), \varphi - \overline{\varphi}) &\le (\mu - \overline{\mu}, \varphi - \overline{\varphi}) \\ &\le C\|\nabla\mu\|\|\nabla\varphi\|.\end{aligned}$$

Combining the above controls, we arrive at

$$\|F'_\lambda(\varphi)\|_{L^1(\Omega)} + |\overline{\mu}| \leq C\|\nabla\mu\|\|\nabla\varphi\| + C.$$

Finally, we deduce that

$$\|\Delta\varphi\|^4 + \|F'_\lambda(\varphi)\|^2_{L^1(\Omega)} + |\overline{\mu}|^2 \leq C\big(\|\nabla\varphi\|^2 + \|\nabla\mu\|\|\nabla\varphi\| + 1\big)^2. \tag{6.1.15}$$

In light of (6.1.9)-(6.1.10), the claim follows from an integration in time on $[t, t+1]$. □

Remark 6.1.10. Note that $\varphi \in L^2(0,T;H^3(\Omega))$ is obtained by testing the equation of μ by $-\Delta^2\varphi$ (see [20]) and exploiting the Lipschitz regularity of F'_λ. In turn, $\varphi \in \mathcal{C}([0,T],V)$ immediately follows. Nonetheless, this argument does not work in presence of the singular potential.

Existence of a weak solution to the BCH system

Let us fix

$$\varphi_0 \in V \text{ with } \Psi(\varphi_0) \in L^1(\Omega) \text{ and } \overline{\varphi}_0 = m \in (-1,1).$$

Thanks to Theorem 6.1.6, for $\lambda \in (0, \overline{\lambda}]$, we consider the family of solutions $(\boldsymbol{u}_\lambda, \varphi_\lambda)$ to BCH$_\lambda$ departing from φ_0. On account of property (i), we observe that

$$\Psi_\lambda(s) \leq \Psi(s) \leq C, \quad \forall\, s \in [-1,1].$$

In turn, this gives $\mathcal{E}^\lambda_{GL}(\varphi_0) \leq \mathcal{E}_{GL}(\varphi_0)$. Thus, from Lemmas 6.1.7, 6.1.8 and 6.1.9, we infer the uniform estimates

$$\|\varphi_\lambda(t)\|^2_V \leq C,$$

and

$$\int_t^{t+1} \|\nabla\boldsymbol{u}_\lambda(\tau)\|^2 + \|\varphi_\lambda(\tau)\|^4_{H^2(\Omega)} + \|\partial_t\varphi_\lambda(\tau)\|^2_{V'} + \|\mu_\lambda(\tau)\|^2_V \,\mathrm{d}\tau \leq C,$$

for every $t \geq 0$, where the right-hand sides are independent of λ.
Now, in the limit $\lambda \to 0$, we have the following convergences (up to subsequences)

$$\begin{aligned}
&\boldsymbol{u}_\lambda \to \boldsymbol{u} \quad \text{weakly in } L^2(0,T;\mathbf{V}_\sigma),\\
&\varphi_\lambda \to \varphi \quad \text{weakly star in } L^\infty(0,T;V),\\
&\varphi_\lambda \to \varphi \quad \text{weakly in } L^4(0,T;H^2(\Omega)),\\
&\partial_t\varphi_\lambda \to \partial_t\varphi \quad \text{weakly in } L^2(0,T;V'),\\
&\mu_\lambda \to \mu \quad \text{weakly in } L^2(0,T;V).
\end{aligned}$$

By the classical Aubin-Lions compactness lemma, we also deduce that

$$\varphi_\lambda \to \varphi \text{ strongly in } L^2(0,T;V) \cap \mathcal{C}([0,T],H),$$

and

$$\varphi_\lambda(x,t) \to \varphi(x,t) \quad \text{a.e. } (x,t) \text{ in } \Omega \times (0,T).$$

We claim that the limit pair $(\boldsymbol{u}, \varphi)$ is a weak solution according to Definition 6.1.1. Indeed, the required regularity of $(\boldsymbol{u}, \varphi)$ immediately follows by the above convergences. Next, we show that φ fulfils

$$|\varphi(x,t)| < 1 \quad \text{a.e. } (x,t) \text{ in } \Omega \times (0,T).$$

To this aim, note that

$$\int_0^T \|\Psi_\lambda'(\varphi_\lambda(t))\|_{L^1(\Omega)}\,\mathrm{d}t \le C,$$

for some $C > 0$ depending on T and on the initial datum. Following a standard argument, for any fixed $\eta \in (0, 1/2)$ we introduce the set

$$E_\eta^\lambda = \{(x,t) \in \Omega \times [0,T] \,:\, |\varphi_\lambda(x,t)| > 1-\eta\}\,.$$

It is easy to see that

$$|E_\eta^\lambda| \le \frac{C}{\min\{\Psi_\lambda'(1-\eta), |\Psi_\lambda'(-1+\eta)|\}}.$$

Hence, passing to the limit as $\lambda \to 0$ and then letting $\eta \to 0$, we conclude

$$|\,\{(x,t) \in \Omega \times (0,T) : |\varphi(x,t)| \ge 1\}\,| = 0.$$

Regarding the nonlinear potential, using the pointwise convergence of φ_λ and the uniform convergence of Ψ_λ' to Ψ' on any closed subset of $(-1,1)$, we infer that

$$\Psi_\lambda'(\varphi_\lambda) \to \Psi'(\varphi) \quad \text{a.e. } (x,t) \in \Omega \times (0,T).$$

Moreover, using the definition of μ_λ, we get

$$\int_0^T \|\Psi_\lambda'(\varphi_\lambda(\tau))\|^2\,\mathrm{d}\tau \le C.$$

Then, we deduce that $\Psi_\lambda'(\varphi_\lambda) \to \Psi'(\varphi)$ weakly in $L^2(0,T;H)$, which allows us to identify

$$\mu = -\Delta\varphi + F'(\varphi) \in L^2(0,T;V).$$

Finally, in a standard way, we pass to the limit in the weak formulation of BCH$_\lambda$ proving the validity of (6.1.2)-(6.1.3).

Energy equality and dissipativity

Let us define the functional

$$\mathcal{E}_{GL}^*(\varphi) = \frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega \Psi(\varphi)\,\mathrm{d}x, \quad \forall \varphi \in H.$$

It is clear that $\mathcal{E}_{GL}^*$ is proper, convex and lower-semicontinuous. Hence, appealing to [131, Lemma 4.1], we infer that $t \mapsto \mathcal{E}_{GL}^*(\varphi)$ is absolutely continuous on $[0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}^*(\varphi) = \langle \partial_t\varphi, \mu + \Theta_0\varphi\rangle, \qquad \text{a.e. } t \in [0,T].$$

In particular, it follows by a standard argument that

$$\int_\Omega \Psi(\varphi(\cdot))\,\mathrm{d}x \in \mathcal{C}([0,T]),$$

which in turn gives $\varphi \in \mathcal{C}([0,T],V)$. Now, taking $v = \mu$ in (6.1.3) and exploiting the standard chain rule, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}(\varphi) + \|\nabla \mu\|^2 + (\boldsymbol{u}\nabla\varphi, \mu) = 0, \quad \text{a.e. } t \in [0,T]. \tag{6.1.16}$$

At this time, taking $\boldsymbol{v} = \boldsymbol{u}$ in (6.1.2) and summing up to (6.1.16), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{GL}(\varphi) + \|\sqrt{\nu(\varphi)}D\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2\|\nabla\mu\|^2 = 0, \quad \text{a.e. } t \in [0,T],$$

proving the energy equality claimed in Theorem 6.1.5. We are left to establish the dissipative estimates. As a matter of fact passing to the limit as $\lambda \to 0$ in (6.1.9)-(6.1.10), we deduce that

$$\mathcal{E}_{GL}(\varphi(t)) + \int_t^{t+1} \|\nabla \boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \,\mathrm{d}\tau \le C\mathcal{E}_{GL}(\varphi_0)\mathrm{e}^{-\omega t} + C,$$

for almost every $t \ge 0$. The continuity of the energy $\mathcal{E}_{GL} \in \mathcal{C}([0,T])$ allows us to conclude that the inequality holds true for all $t \ge 0$. Finally, the control (6.1.6) follows by Lemma 6.1.9. This finishes the proof of Theorem 6.1.5.

6.2 Further Regularity of Weak Solutions

We establish some further regularity properties of weak solutions $(\boldsymbol{u}, \varphi)$ given by Theorem 6.1.5. In the sequel, the generic constant $C > 0$ may depend on $\mathcal{E}(\varphi_0)$ and $\overline{\varphi}_0$.

Lemma 6.2.1. *For any $p \ge 2$, there exists $C = C(p)$ such that*

$$\int_t^{t+1} \|\varphi(\tau)\|^2_{W^{2,p}(\Omega)} + \|\Psi'(\varphi(\tau))\|^2_{L^p(\Omega)} \,\mathrm{d}\tau \le C, \quad \forall\, t \ge 0.$$

Proof. The claim immediately follows from Corollary 3.4.3 and Theorem 6.1.5. □

Lemma 6.2.2. *We have*

$$\int_t^{t+1} \|\nabla \boldsymbol{u}(\tau)\|^4 \,\mathrm{d}\tau \le C, \quad \forall\, t \ge 0.$$

Proof. We take $\boldsymbol{v} = \boldsymbol{u}$ in (6.1.2) (cfr. Remark 6.1.3)

$$(\nu(\varphi)D\boldsymbol{u}, D\boldsymbol{u}) + \|\boldsymbol{u}\|^2 = (\nabla\varphi \otimes \nabla\varphi, \nabla\boldsymbol{u}).$$

Hence, exploiting (6.1.1) and the Korn inequality (3.5.4), we have

$$\nu_1\|\nabla\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2 \le (\nabla\varphi \otimes \nabla\varphi, \nabla\boldsymbol{u}).$$

By (3.1.7), we deduce that

$$\begin{aligned}(\nabla\varphi \otimes \nabla\varphi, \nabla\boldsymbol{u}) &\le \|\nabla\boldsymbol{u}\|\|\nabla\varphi\|^2_{L^4(\Omega)} \\ &\le C\|\nabla\boldsymbol{u}\|\|\varphi\|_V\|\varphi\|_{H^2(\Omega)} \\ &\le \frac{\nu_1}{2}\|\nabla\boldsymbol{u}\|^2 + C\|\varphi\|^2_{H^2(\Omega)},\end{aligned}$$

so we end up with the control

$$\|\nabla \boldsymbol{u}\| \leq C\|\varphi\|_{H^2(\Omega)}.$$

The thesis follows from (6.1.6). □

Remark 6.2.3. It is worth noticing that the above regularity for $\boldsymbol{u}$ holds true also for the Galerkin approximating solutions. Unfortunately, this is not enough to ensure the uniqueness of weak solutions, and we need to gain some extra regularity properties for $\boldsymbol{u}$. To this aim, let us observe that (6.1.2) is also equivalent to

$$(\nu(\varphi)D\boldsymbol{u}, D\boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}) = -(\varphi\nabla\mu, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbf{V},$$

for almost every $t \in (0,T)$, where

$$\varphi\nabla\mu \in L^2(0,T;H),$$

according to the boundedness of φ stated in the Definition 6.1.1. However, due to the viscosity depending on concentrations, we do not expect the regularity $\boldsymbol{u} \in L^2(0,T;H^2(\Omega))$ as well as in the case of constant viscosity by the classical regularity result for the Stokes problem (see Section 3.6).

We prove the following crucial result.

Lemma 6.2.4. *We have*

$$\int_t^{t+1} \|\boldsymbol{u}(\tau)\|^{\frac{8}{5}}_{\mathbf{W}^{2,\frac{4}{3}}(\Omega)} \mathrm{d}\tau \leq C, \tag{6.2.1}$$

and

$$\int_t^{t+1} \|\boldsymbol{u}(\tau)\|^2_{\mathbf{W}^{1,3}(\Omega)} \mathrm{d}\tau \leq C, \tag{6.2.2}$$

for every $t \geq 0$.

Proof. Let us first observe that the velocity equivalently solves

$$\langle \nu(\varphi)D\boldsymbol{u}, D\boldsymbol{v}\rangle = \langle f, \boldsymbol{v}\rangle, \quad \boldsymbol{v} \in \mathbf{V}_\sigma,$$

where

$$f = -\varphi\nabla\mu - \boldsymbol{u}.$$

Now, we apply Lemma 3.6.1 with $p = \frac{4}{3}$ and $r = 4$ obtaining

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,\frac{4}{3}}(\Omega)} \leq C\big(1 + \|\nabla\varphi\|_{\mathbf{L}^4(\Omega)}\big)\big(\|\boldsymbol{f}\| + \|\nabla \boldsymbol{u}\|\big).$$

By (3.1.7) and the Young inequality, we have

$$\|\boldsymbol{u}\|^{\frac{8}{5}}_{\mathbf{W}^{2,\frac{4}{3}}(\Omega)} \leq C\big(1 + \|\boldsymbol{f}\|^2 + \|\nabla\boldsymbol{u}\|^2 + \|\varphi\|^4_{H^2(\Omega)}\big).$$

Recalling that

$$\int_t^{t+1} \|f(\tau)\|^2 + \|\nabla\boldsymbol{u}(\tau)\|^2 + \|\varphi(\tau)\|^4_{H^2(\Omega)}\,\mathrm{d}\tau \leq C,$$

we conclude

$$\int_t^{t+1} \|\boldsymbol{u}(\tau)\|^{\frac{8}{5}}_{\mathbf{W}^{2,\frac{4}{3}}(\Omega)} \mathrm{d}\tau \leq C.$$

In order to prove (6.2.4), we recall the following Gagliardo-Nirenberg inequality (see Section 3.1)

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,3}(\Omega)} \leq \|\boldsymbol{u}\|^{\frac{2}{3}}_{\mathbf{W}^{2,\frac{4}{3}}(\Omega)} \|\boldsymbol{u}\|^{\frac{1}{3}}_{\mathbf{W}^{1,2}(\Omega)}.$$

Hence, we deduce

$$\|\boldsymbol{u}\|^2_{\mathbf{W}^{1,3}(\Omega)} \leq C\|\boldsymbol{u}\|^{\frac{4}{3}}_{\mathbf{W}^{2,\frac{4}{3}}(\Omega)} \|\nabla \boldsymbol{u}\|^{\frac{2}{3}} \leq \|\boldsymbol{u}\|^{\frac{8}{5}}_{\mathbf{W}^{2,\frac{4}{3}}(\Omega)} + C\|\nabla \boldsymbol{u}\|^4,$$

and the conclusion follows by collecting (6.2.1) with Lemma 6.2.2. □

6.3 Uniqueness of Weak Solutions

We are now in a position to prove the following continuous dependece estimate with respect to the initial conditions. In turn, this implies the uniqueness of weak solutions.

Theorem 6.3.1. *Let φ_{01}, φ_{02} be such that $\varphi_{0i} \in V$, $\Psi(\varphi_{0i}) \in L^1(\Omega)$ and $|\varphi_{0i}| < 1$, $i = 1, 2$. Assume that $(\boldsymbol{u}_1, \varphi_1)$ and $(\boldsymbol{u}_2, \varphi_2)$ are two weak solutions to the BCH problem on $[0, T]$ with initial data φ_{01} and φ_{02}, respectively. Then, there exists a positive constant $C = C(T)$ such that*

$$\|\varphi_1(t) - \varphi_2(t)\|_{V'} \leq C\|\varphi_{01} - \varphi_{02}\|_{V'} + C|\overline{\varphi}_{01} - \overline{\varphi}_{02}|^{1/2},$$

for any $t \in [0, T]$. In particular, the energy solution to the BCH system is unique.

Proof. Let us consider $(\boldsymbol{u}_1, \varphi_1)$ and $(\boldsymbol{u}_2, \varphi_2)$ two weak solutions to the BCH system with total mass $\overline{\varphi}_{01}$ and $\overline{\varphi}_{02}$. Their difference $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\varphi = \varphi_1 - \varphi_2$ solves

$$\begin{aligned}(\nu(\varphi_1)D\boldsymbol{u}, D\boldsymbol{v}) &+ (\nu(\varphi_1) - \nu(\varphi_2)D\boldsymbol{u}_2, D\boldsymbol{v}) \\ &+(\boldsymbol{u}, \boldsymbol{v}) = (\nabla\varphi \otimes \nabla\varphi_1, \nabla\boldsymbol{v}) + (\nabla\varphi \otimes \nabla\varphi_2, \nabla\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma, \end{aligned} \tag{6.3.1}$$

and

$$\langle \partial_t\varphi, v\rangle + (\boldsymbol{u}_1 \cdot \nabla\varphi, v) + (\boldsymbol{u} \cdot \nabla\varphi_2, v) + (\nabla\mu, \nabla v) = 0, \quad \forall\, v \in V, \tag{6.3.2}$$

where $\mu = \mu_1 - \mu_2$ satisfies

$$\mu = -\Delta\varphi + F'(\varphi_1) - F'(\varphi_2).$$

We note that $\overline{\varphi}(t) = \overline{\varphi}_{01} - \overline{\varphi}_{02}$ for all $t \geq 0$. Taking $v = \mathcal{N}(\varphi - \overline{\varphi})$ in (6.3.2), we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \overline{\varphi}\|^2_{V_0'} + (\mu, \varphi - \overline{\varphi}) = I_1 + I_2,$$

having set

$$I_1 = (\varphi\boldsymbol{u}_1, \nabla\mathcal{N}(\varphi - \overline{\varphi})), \quad I_2 = (\varphi_2\boldsymbol{u}, \nabla\mathcal{N}(\varphi - \overline{\varphi})).$$

By the assumptions on F, we deduce

$$\begin{aligned}(\mu, \varphi - \overline{\varphi}) &= \|\nabla\varphi\|^2 + (F'(\varphi_1) - F'(\varphi_2), \varphi_1 - \varphi_2) + (F'(\varphi_1) - F'(\varphi_2), \overline{\varphi}) \\ &\geq \|\nabla\varphi\|^2 - \alpha\|\varphi\|^2 - |(F'(\varphi_1) - F'(\varphi_2), \overline{\varphi})| \\ &\geq \|\varphi\|_V^2 - (\alpha+1)\|\varphi\|^2 - \Big(\|F'(\varphi_1)\|_{L^1(\Omega)} + \|F'(\varphi_2)\|_{L^1(\Omega)}\Big)|\overline{\varphi}|.\end{aligned}$$

Besides, by (3.3.4), we have

$$\begin{aligned}(\alpha+1)\|\varphi\|^2 &\leq C\|\nabla\varphi\|\|\varphi - \overline{\varphi}\|_{V_0'} + C|\overline{\varphi}|^2 \\ &\leq \frac{1}{2}\|\varphi\|_V^2 + C\|\varphi\|_*^2.\end{aligned}$$

We set

$$\Upsilon(t) = C\Big(\|F'(\varphi_1(t))\|_{L^1(\Omega)} + \|F'(\varphi_2(t))\|_{L^1(\Omega)}\Big),$$

which is a summable on any $[0,T]$. Owing to the mass conservation, we thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2 + \frac{1}{2}\|\varphi\|_V^2 \leq C\|\varphi\|_*^2 + \Upsilon|\overline{\varphi}| + I_1 + I_2. \tag{6.3.3}$$

We proceed by estimating I_1 and I_2. First, we have

$$\begin{aligned}|I_1| &\leq \|\boldsymbol{u}_1\|_{\mathbf{L}^3(\Omega)}\|\varphi\|_{L^6(\Omega)}\|\varphi - \overline{\varphi}\|_{V_0'} \\ &\leq \frac{1}{4}\|\varphi\|_V^2 + C\|\boldsymbol{u}_1\|_{\mathbf{L}^3(\Omega)}^2\|\varphi\|_*^2.\end{aligned}$$

Next, since by definition of solution $\|\varphi_2\|_{L^\infty(\Omega)} \leq 1$,

$$\begin{aligned}|I_2| &\leq \|\boldsymbol{u}\|\|\varphi_2\|_{L^\infty(\Omega)}\|\varphi - \overline{\varphi}\|_{V_0'} \\ &\leq \|\boldsymbol{u}\|\|\varphi\|_*.\end{aligned}$$

Now, in order to find a control for the velocity field, we take $\boldsymbol{v} = \boldsymbol{u}$ in (6.3.1) yielding

$$(\nu(\varphi_1)D\boldsymbol{u}, D\boldsymbol{u}) + \|\boldsymbol{u}\|^2 + J = (\nabla\varphi \otimes \nabla\varphi_1, \nabla\boldsymbol{u}) + (\nabla\varphi \otimes \nabla\varphi_2, \nabla\boldsymbol{u}), \tag{6.3.4}$$

where

$$J = (\nu(\varphi_1) - \nu(\varphi_2)D\boldsymbol{u}_2, D\boldsymbol{u}).$$

By Sobolev embedding, we notice that

$$\begin{aligned}(\nabla\varphi \otimes \nabla\varphi_1, \nabla\boldsymbol{u}) &\leq \|\nabla\varphi\|\|\nabla\varphi\|_{\mathbf{L}^\infty(\Omega)}\|\nabla\boldsymbol{u}\| \\ &\leq \frac{\nu_1}{4}\|\nabla\boldsymbol{u}\|^2 + C\|\varphi_1\|_{W^{2,3}(\Omega)}^2\|\nabla\varphi\|^2.\end{aligned}$$

Dealing analogously with the last term, on account of (6.1.1) and (3.5.4), we arrive at

$$\frac{\nu_1}{2}\|\nabla\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2 + J \leq C\Big(\|\varphi_1\|_{W^{2,3}(\Omega)}^2 + \|\varphi_2\|_{W^{2,3}(\Omega)}^2\Big)\|\varphi\|_V^2. \tag{6.3.5}$$

Regarding J, by $\nu \in \mathcal{C}^1(\mathbb{R})$, we find the control

$$\begin{aligned}J &\leq C\|\varphi\|_{L^6(\Omega)}\|\nabla\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}\|\nabla\boldsymbol{u}\| \\ &\leq \frac{\nu_1}{2}\|\nabla\boldsymbol{u}\|^2 + C\|\nabla\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}^2\|\varphi\|_V^2.\end{aligned}$$

Thus, we learn by (6.3.5) that

$$\|\boldsymbol{u}\| \leq C\Big(\|\varphi_1\|_{W^{2,3}(\Omega)} + \|\varphi_2\|_{W^{2,3}(\Omega)} + \|\boldsymbol{u}_2\|_{\mathbf{W}^{1,3}(\Omega)}\Big)\|\varphi\|_V,$$

and, exploiting this estimate in I_2, we find

$$I_2 \leq \frac{1}{4}\|\varphi\|_V^2 + C\Big(\|\varphi_1\|^2_{W^{2,3}(\Omega)} + \|\varphi_2\|^2_{W^{2,3}(\Omega)} + \|\boldsymbol{u}_2\|^2_{\mathbf{W}^{1,3}(\Omega)}\Big)\|\varphi\|_*^2.$$

Collecting the above controls for I_1 and I_2 in (6.3.3), we obtain the final differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2 \leq \Gamma\|\varphi\|_*^2 + \Upsilon|\overline{\varphi}|,$$

having set

$$\Gamma(t) = C\Big(1 + \|\varphi_1(t)\|^2_{W^{2,3}(\Omega)} + \|\varphi_2(t)\|^2_{W^{2,3}(\Omega)} + \|\boldsymbol{u}_2(t)\|^2_{\mathbf{W}^{1,3}(\Omega)} + \|\boldsymbol{u}_1(t)\|^2_{\mathbf{V}_\sigma}\Big),$$

which is summable in light of Lemma 6.2.1 and Lemma 6.2.4. An application of the Gronwall lemma gives

$$\|\varphi(t)\|_*^2 \leq \|\varphi(0)\|_{-1}^2\mathrm{e}^C + C|\overline{\varphi}(0)|\mathrm{e}^C, \quad \forall\, t \in [0,T].$$

In particular, if $\varphi_{01} = \varphi_{02}$, then $\varphi_1 \equiv \varphi_2$ and by (6.3.4), $\boldsymbol{u}_1 \equiv \boldsymbol{u}_2$ as well, thus uniqueness follows. □

6.4 Regularity Properties and Separation Property

Let $R > 0$ and $m \in (-1,1)$ be given. In the sequel, we consider bundles of trajectories $(\boldsymbol{u},\varphi)$ departing from φ_0 such that

$$\mathcal{E}_{GL}(\varphi_0) \leq R \quad \text{ and } \quad \overline{\varphi}_0 = m.$$

The aim is proving higher order regularity estimates for the trajectories which depend on R and m but are independent of the specific choice of the initial datum. Accordingly, the generic constant $C > 0$ depends on R and m.

Theorem 6.4.1. *For every $\sigma > 0$, there exists $C = C(\sigma)$ such that*

$$\|\mu\|_{L^\infty(\sigma,\infty;V)} + \|\boldsymbol{u}\|_{L^\infty(\sigma,\infty;\mathbf{V}_\sigma)} \leq C,$$

and

$$\int_t^{t+1} \|\boldsymbol{u}(\tau)\|^2_{\mathbf{W}_\sigma} + \|\varphi_t(\tau)\|_V^2\,\mathrm{d}\tau \leq C, \quad \forall\, t \geq \sigma. \tag{6.4.1}$$

Moreover, for any $p \geq 2$, there exists $C = C(\sigma,p)$ such that

$$\|\varphi\|_{L^\infty(\sigma,\infty;W^{2,p}(\Omega))} + \|\Psi'(\varphi)\|_{L^\infty(\sigma,\infty;L^p(\Omega))} \leq C. \tag{6.4.2}$$

Remark 6.4.2. At this point, by the classical Aubin embedding lemma, we learn that

$$\varphi \in \mathcal{C}([\sigma,\infty), W^{1,p}(\Omega)),$$

for every $p \geq 2$ and $\sigma > 0$. In particular,

$$\varphi \in \mathcal{C}(\overline{\Omega} \times [\sigma,\infty)).$$

Proof of Theorem 6.4.1. Let us first recall that the dissipative inequalities (6.1.5) and (6.1.6) yield

$$\mathcal{E}(\varphi(t)) + \int_t^{t+1} \|\varphi(\tau)\|^4_{H^2(\Omega)} + \|\nabla\mu(\tau)\|^2 + \|\nabla \boldsymbol{u}(\tau)\|^4 \, \mathrm{d}\tau \leq C, \tag{6.4.3}$$

for every $t \geq 0$. Besides, arguing by comparison, we deduce

$$\|\varphi_t\|_{V'} \leq C(\|\nabla\mu\| + \|\nabla \boldsymbol{u}\|), \tag{6.4.4}$$

and, reasoning as in the proof of Lemma 6.1.9, we learn that

$$\|\mu\|_V \leq C(1 + \|\nabla\mu\|). \tag{6.4.5}$$

We take $v = \partial_t\mu$ in (6.1.3) getting

$$\langle \partial_t\varphi, \partial_t\mu\rangle + (\boldsymbol{u}\cdot\nabla\varphi, \partial_t\mu) + (\nabla\mu, \nabla\partial_t\mu) = 0.$$

Since

$$\begin{aligned}\langle \partial_t\varphi, \partial_t\mu\rangle &= (-\Delta\partial_t\varphi, \partial_t\varphi) + (F''(\varphi)\partial_t\varphi, \partial_t\varphi)\\ &\geq \frac{1}{2}\|\nabla\partial_t\varphi\|^2 - C\|\partial_t\varphi\|^2_{V'},\end{aligned}$$

we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mu\|^2 + \frac{1}{2}\|\nabla\partial_t\varphi\|^2 + (\boldsymbol{u}\cdot\nabla\varphi, \partial_t\mu) \leq C\|\partial_t\varphi\|^2_{V'}.$$

A differentiation in time of (6.1.2) entails

$$(\nu(\varphi)D\partial_t\boldsymbol{u}, D\boldsymbol{v}) + (\nu'(\varphi)\partial_t\varphi D\boldsymbol{u}, D\boldsymbol{v}) + (\partial_t\boldsymbol{u}, \boldsymbol{v}) = (\partial_t\mu\nabla\varphi, \boldsymbol{v}) + (\mu\nabla\partial_t\varphi, \boldsymbol{v}),$$

for all $\boldsymbol{v} \in \mathbf{V}_\sigma$. Taking $\boldsymbol{v} = \boldsymbol{u}$, this gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(\nu(\varphi)D\boldsymbol{u}, D\boldsymbol{u}) + \|\boldsymbol{u}\|^2\Big\} = (\partial_t\mu\nabla\varphi, \boldsymbol{u}) + (\mu\nabla\partial_t\varphi, \boldsymbol{u}) - \frac{1}{2}(\nu'(\varphi)\partial_t\varphi D\boldsymbol{u}, D\boldsymbol{u}).$$

Hence, setting

$$\boldsymbol{\Lambda} = (\nu(\varphi)D\boldsymbol{u}, D\boldsymbol{u}) + \|\boldsymbol{u}\|^2 + \|\nabla\mu\|^2,$$

we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Lambda} + \frac{1}{2}\|\nabla\partial_t\varphi\|^2 \leq C\|\partial_t\varphi\|^2_{V'} + (\mu\nabla\partial_t\varphi, \boldsymbol{u}) - \frac{1}{2}(\nu'(\varphi)\partial_t\varphi D\boldsymbol{u}, D\boldsymbol{u}).$$

Now, taking $\boldsymbol{v} = -\Delta\boldsymbol{u}$ in (6.1.2), we have

$$(\nu(\varphi)\Delta\boldsymbol{u}, \Delta\boldsymbol{u}) + \|\nabla\boldsymbol{u}\|^2 = -(\mu\nabla\varphi, \Delta\boldsymbol{u}) - (\nu'(\varphi)\nabla\varphi D\boldsymbol{u}, \Delta\boldsymbol{u}).$$

Recalling (6.1.1), we deduce

$$\begin{aligned}(\nu(\varphi)\Delta\boldsymbol{u},\Delta\boldsymbol{u})+\|\nabla\boldsymbol{u}\|^2&\geq\nu_1\|\Delta\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2\\&\geq\nu_1\|\boldsymbol{u}\|^2_{\mathbf{W}_\sigma}.\end{aligned}$$

Summing up with the above differential inequality for $\boldsymbol{\Lambda}$, we are lead to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Lambda}+\frac{1}{2}\|\nabla\partial_t\varphi\|^2+\nu_1\|\boldsymbol{u}|^2_{\mathbf{W}_\sigma)}&\leq C\|\partial_t\varphi\|^2_{V'}+|(\mu\nabla\partial_t\varphi,\boldsymbol{u})|\\&+C|(\nu'(\varphi)\partial_t\varphi D\boldsymbol{u},D\boldsymbol{u})|+|(\mu\nabla\varphi,\Delta\boldsymbol{u})|+|(\nu'(\varphi)\nabla\varphi D\boldsymbol{u},\Delta\boldsymbol{u})|.\end{aligned}$$

First, we estimate

$$|(\mu\nabla\partial_t\varphi,\boldsymbol{u})|\leq\frac{1}{8}\|\nabla\partial_t\varphi\|^2+C\|\boldsymbol{u}\|^2_{\mathbf{L}^3(\Omega)}\|\mu\|^2_V.$$

Next, by (3.1.7) and (3.3.4), we control

$$\begin{aligned}C|(\nu'(\varphi)\partial_t\varphi D\boldsymbol{u},D\boldsymbol{u})|&\leq C\|\partial_t\varphi\|\|D\boldsymbol{u}\|^2_{\mathbf{L}^4(\Omega)}\\&\leq C\|\partial_t\varphi\|^{\frac12}_{V'}\|\nabla\partial_t\varphi\|^{\frac12}\|\nabla\boldsymbol{u}\|\|\boldsymbol{u}\|_{\mathbf{W}_\sigma}\\&\leq\frac{\nu_1}{4}\|\boldsymbol{u}\|^2_{\mathbf{W}_\sigma}+\frac{1}{8}\|\nabla\partial_t\varphi\|^2+C\|\nabla\boldsymbol{u}\|^4\|\partial_t\varphi\|^2_{V'},\end{aligned}$$

and

$$\begin{aligned}|(\nu'(\varphi)\nabla\varphi D\boldsymbol{u},\Delta\boldsymbol{u})|&\leq C\|\nabla\varphi\|_{\mathbf{L}^4(\Omega)}\|D\boldsymbol{u}\|_{\mathbf{L}^4(\Omega)}\|\Delta\boldsymbol{u}\|\\&\leq C\|\nabla\varphi\|^{\frac12}\|\varphi\|^{\frac12}_{H^2(\Omega)}\|\nabla\boldsymbol{u}\|^{\frac12}\|\boldsymbol{u}\|^{\frac32}_{\mathbf{W}_\sigma}\\&\leq\frac{\nu_1}{4}\|\boldsymbol{u}\|^2_{\mathbf{W}_\sigma}+C\|\varphi\|^2_{H^2(\Omega)}\|\nabla\boldsymbol{u}\|^2.\end{aligned}$$

In addition,

$$|(\mu\nabla\varphi,\Delta\boldsymbol{u})|\leq\frac{\nu_1}{4}\|\boldsymbol{u}\|^2_{\mathbf{W}_\sigma}+C\|\mu\|^2_V\|\varphi\|^2_{H^2(\Omega)}.$$

Therefore, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Lambda}+\frac{1}{4}\|\nabla\partial_t\varphi\|^2+\frac{\nu_1}{4}\|\boldsymbol{u}\|^2_{\mathbf{W}_\sigma}\\&\leq C(1+\|\nabla\boldsymbol{u}\|^4)\|\partial_t\varphi\|^2_{V'}+C\|\boldsymbol{u}\|^2_{\mathbf{L}^3(\Omega)}\|\mu\|^2_V+C\|\varphi\|^2_{H^2(\Omega)}\big(\|\nabla\boldsymbol{u}\|^2+\|\mu\|^2_V\big).\end{aligned}$$

Keeping in mind (6.4.4) and (6.4.5), in light of the equivalence

$$\frac{1}{C}(\|\nabla\mu\|^2+\|\nabla\boldsymbol{u}\|^2)\leq\boldsymbol{\Lambda}\leq C(\|\nabla\mu\|^2+\|\nabla\boldsymbol{u}\|^2),$$

we finally obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Lambda}+\frac{1}{8}\|\nabla\partial_t\varphi\|^2+\frac{\nu_*}{4}\|\boldsymbol{u}\|^2_{\mathbf{W}_\sigma}\leq\Upsilon\boldsymbol{\Lambda}+\Upsilon,\tag{6.4.6}$$

where

$$\Upsilon(t)=C\Big(1+\|\boldsymbol{u}(t)\|^2_{\mathbf{L}^3(\Omega)}+\|\nabla\boldsymbol{u}(t)\|^4+\|\varphi(t)\|^2_{H^2(\Omega)}\Big).$$

Owing to (6.4.3), an application of the uniform Gronwall lemma yields

$$\|\nabla\mu(t)\| + \|\nabla \boldsymbol{u}(t)\| \le C, \quad \forall\, t \ge \sigma.$$

The inequality (6.4.1) follows by an integration of (6.4.6) on any interval $[t, t+1]$. Finally, from $\mu \in L^\infty(\sigma,\infty;V)$ and Corollary 3.4.3, we easily deduce the desired control (6.4.2). □

Remark 6.4.3. The proof of Theorem 6.4.1 is obtained by formal computations. However, they can be rigorously justified through the Galerkin scheme mentioned in Section 6.1.

Remark 6.4.4. As a consequence of Theorem 6.4.1, we learn that $\mu \in L^2(t, t{+}1; H^3(\Omega))$ for every $t \ge \sigma$. Then, it is immediate to deduce that

$$\varphi_t + \nabla\cdot(\boldsymbol{u}\varphi) = \Delta\mu, \qquad \text{a.e. } (x,t) \in \Omega\times(\sigma,\infty)$$

and $\partial_{\boldsymbol{n}}\mu = 0$ a.e. in $\partial\Omega\times(\sigma,\infty)$. Accordingly, the energy solution is indeed a strong solution on $\Omega\times(\sigma,\infty)$.

Our next aim is to prove the validity of the *instantaneous* separation property.

Theorem 6.4.5. *Assume that* $\Psi \in \mathcal{C}^3(-1,1)$ *and* (H.2), (H.3) *hold. Then, for any* $\sigma > 0$*, there exists a positive constant* $C = C(\sigma)$ *such that*

$$\|\partial_t\varphi\|_{L^\infty(2\sigma,\infty;H)} \le C.$$

In addition, there exists $\delta = \delta(\sigma, R, m) > 0$ *and* $C = C(\sigma)$ *such that*

$$\sup_{t\ge 2\sigma} \|\varphi(t)\|_{L^\infty(\Omega)} \le 1-\delta, \tag{6.4.7}$$

and

$$\sup_{t\ge 2\sigma} \|\varphi(t)\|_{H^4(\Omega)} \le C.$$

Proof. We first deduce integrability properties for $\Psi''(\varphi)$. To this aim, in light of Theorem 6.4.1, seeting $f = \mu + \Theta_0\varphi$, an application of Lemma 3.4.6 entails that, for any $p \ge 2$, there exists $C = C(\sigma, p)$, such that

$$\|\Psi''(\varphi)\|_{L^\infty(\sigma,\infty;L^p(\Omega))} \le C. \tag{6.4.8}$$

We are now in a position to prove higher order estimates. Given $h > 0$, let us introduce the difference quotient of a function v by

$$\partial_t^h v = \frac{1}{h}\Big(v(t+h) - v(t)\Big).$$

Owing to Remark 6.4.4, the solution solves

$$\partial_t\partial_t^h\varphi + \partial_t^h\boldsymbol{u}\cdot\nabla\varphi(t+h) + \boldsymbol{u}\cdot\nabla\partial_t^h\varphi = \Delta\partial_t^h\mu.$$

Testing the above equation by $\partial_t^h\varphi$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2 = (\Delta\partial_t^h\mu, \partial_t^h\varphi) + R_1 + R_2, \tag{6.4.9}$$

where
$$R_1 = -(\partial_t^h \boldsymbol{u} \cdot \nabla\varphi(t+h), \partial_t^h \varphi), \quad R_2 = -(\boldsymbol{u} \cdot \nabla \partial_t^h \varphi, \partial_t^h \varphi).$$
Integrating by parts and making use of the boundary conditions, we get
$$\begin{aligned}(\Delta\partial_t^h \mu, \partial_t^h \varphi) &= (\partial_t^h \mu, \Delta \partial_t^h \varphi)\\ &= -\|\Delta\partial_t^h\varphi\|^2 + \Theta_0\|\nabla\partial_t^h\varphi\|^2 + (\frac{1}{h}[\Psi'(\varphi(t+h)) - \Psi'(\varphi(t))], \Delta\partial_t^h\varphi).\end{aligned}$$
By using (H.2), we estimate
$$\begin{aligned}\frac{1}{h}\Big|\Psi'(\varphi(t+h)) - \Psi'(\varphi(t))\Big| &\le \int_0^1 \Psi''(\tau\varphi(t+h) + (1-\tau)\varphi(t))\, |\partial_t^h\varphi| \,\mathrm{d}\tau\\ &\le \int_0^1 \Big(\tau\Psi''(\varphi(t+h)) + (1-\tau)\Psi''(\varphi(t))\Big)\, |\partial_t^h\varphi| \,\mathrm{d}\tau\\ &\le \Big(\Psi''(\varphi(t+h)) + \Psi''(\varphi(t))\Big)|\partial_t^h\varphi|,\end{aligned}$$
and we deduce
$$\begin{aligned}&\Big|(\frac{1}{h}[\Psi'(\varphi(t+h)) - \Psi'(\varphi(t))], \Delta\partial_t^h\varphi)\Big|\\ &\quad\le \frac{1}{2}\|\Delta\partial_t^h\varphi\|^2 + C\Big(\|\Psi''(\varphi(t+h))\|_{L^3(\Omega)}^2 + \|\Psi''(\varphi(t))\|_{L^3(\Omega)}^2\Big)\|\partial_t^h\varphi\|_{L^6(\Omega)}^2.\end{aligned}$$
By interpolation
$$\begin{aligned}\|\partial_t^h\varphi\|_{L^6(\Omega)}^2 &\le C\|\nabla\partial_t^h\varphi\|^2\\ &\le C\|\partial_t^h\varphi\|\|\Delta\partial_t^h\varphi\|.\end{aligned}$$
Thus we easily derive from (6.4.9) the differential inequality
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2 + \frac{1}{4}\|\Delta\partial_t^h\varphi\|^2 \le \Upsilon\|\partial_t^h\varphi\|^2 + R_1 + R_2, \tag{6.4.10}$$
where
$$\Upsilon(t) = C\Big(1 + \|\Psi''(\varphi(t+h))\|_{L^3(\Omega)}^4 + \|\Psi''(\varphi(t))\|_{L^3(\Omega)}^4\Big).$$
Let us now consider the equation satisfied by $\partial_t^h\boldsymbol{u}$ in the equivalent formulation of Remark 6.1.3. Testing by $\partial_t^h\boldsymbol{u}$, we find
$$(\nu(\varphi(t+h))D\partial_t^h\boldsymbol{u}, D\partial_t^h\boldsymbol{u}) + \|\partial_t^h\boldsymbol{u}\|^2 = Z_1 + Z_2, \tag{6.4.11}$$
having set
$$Z_1 = -(\frac{1}{h}[\nu(\varphi(t+h)) - \nu(\varphi(t))]D\boldsymbol{u}, D\partial_t^h\boldsymbol{u}),$$
and
$$Z_2 = (\nabla\partial_t^h\varphi \otimes \nabla\varphi(t+h), \nabla\partial_t^h\boldsymbol{u}) + (\nabla\varphi \otimes \nabla\partial_t^h\varphi, \nabla\partial_t^h\boldsymbol{u}).$$
Note that by (6.1.1) and (3.5.4)
$$(\nu(\varphi(t+h))D\partial_t^h\boldsymbol{u}, D\partial_t^h\boldsymbol{u}) \ge \nu_1\|\nabla\partial_t^h\boldsymbol{u}\|^2,$$

hence, summing up (6.4.10) and (6.4.11), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2 + \frac{1}{4}\|\Delta\partial_t^h\varphi\|^2 + \nu_1\|\nabla\partial_t^h\boldsymbol{u}\|^2 \leq \Upsilon\|\partial_t^h\varphi\|^2 + R_1 + R_2 + Z_1 + Z_2.$$

We estimate the right-hand side term by term as follows. By Theorem 6.4.1, we have

$$\begin{aligned}|R_1| &\leq \|\partial_t^h\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)}\|\nabla\varphi(t+h)\|_{\mathbf{L}^3(\Omega)}\|\partial_t^h\varphi\|\\ &\leq \frac{\nu_1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2 + C\|\partial_t^h\varphi\|^2,\end{aligned}$$

and

$$\begin{aligned}|R_2| &\leq \|\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)}\|\nabla\partial_t^h\varphi\|_{\mathbf{L}^3(\Omega)}\|\partial_t^h\varphi\|\\ &\leq \frac{1}{24}\|\Delta\partial_t^h\varphi\|^2 + C\|\partial_t^h\varphi\|^2.\end{aligned}$$

Besides, recalling that $\nu \in \mathcal{C}^1(\mathbb{R})$, and by using (3.1.7), we find the control

$$\begin{aligned}|Z_1| &\leq C\|\partial_t^h\varphi\|_{L^4(\Omega)}\|\nabla\boldsymbol{u}\|_{\mathbf{L}^4(\Omega)}\|\nabla\partial_t^h\boldsymbol{u}\|\\ &\leq \frac{\nu_1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2 + C\|\nabla\boldsymbol{u}\|\|\boldsymbol{u}\|_{\mathbf{H}^2(\Omega)}\|\partial_t^h\varphi\|\|\Delta\partial_t^h\varphi\|\\ &\leq \frac{\nu_1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2 + \frac{1}{24}\|\Delta\partial_t^h\varphi\|^2 + C\|\boldsymbol{u}\|_{\mathbf{H}^2(\Omega)}^2\|\partial_t^h\varphi\|^2.\end{aligned}$$

Finally, the embedding $W^{1,3}(\Omega) \subset L^\infty(\Omega)$ together with (6.4.2) yields

$$\begin{aligned}|Z_2| &\leq \|\nabla\partial_t^h\varphi\|\Big(\|\nabla\varphi(t)\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla\varphi(t+h)\|_{\mathbf{L}^\infty(\Omega)}\Big)\|\nabla\partial_t^h\boldsymbol{u}\|\\ &\leq \frac{\nu_1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2 + \frac{1}{24}\|\Delta\partial_t^h\varphi\|^2 + C\|\partial_t^h\varphi\|^2.\end{aligned}$$

Collecting all the above estimates, we end up with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|^2 + \frac{1}{8}\|\Delta\partial_t^h\varphi\|^2 + \frac{\nu_1}{4}\|\nabla\partial_t^h\boldsymbol{u}\|^2 \leq C(1 + \|\boldsymbol{u}\|_{\mathbf{H}^2(\Omega)}^2 + \Upsilon)\|\partial_t^h\varphi\|^2.$$

Note that

$$\int_t^{t+1} \|\partial_t^h\varphi(\tau)\|^2 + \|\boldsymbol{u}(\tau)\|_{\mathbf{H}^2(\Omega)}^2 + \Upsilon(\tau)\,\mathrm{d}\tau \leq C, \quad \forall\, t \geq \sigma,$$

in light of Theorem 6.4.1 and (6.4.8). An application of the uniform Gronwall lemma and a final passage to the limit as $h \to 0$ imply that there exists $C = C(\sigma) > 0$ such that

$$\|\partial_t\varphi\|_{L^\infty(2\sigma,\infty;H)} \leq C,$$

and

$$\|\partial_t\varphi\|_{L^2(t,t+1;H^2(\Omega))} + \|\partial_t\boldsymbol{u}\|_{L^2(t,t+1;\mathbf{V})} \leq C, \quad \forall t \geq 2\sigma.$$

On the other hand, by Theorem 6.4.1, we observe that

$$\|\boldsymbol{u}\cdot\nabla\varphi\|_{L^\infty(2\sigma,\infty;H)} \leq C.$$

Thus, the elliptic regularity of the Neumann problem (see Section 3.3) entails

$$\|\mu\|_{L^\infty(2\sigma,\infty;H^2(\Omega))} \le C. \tag{6.4.12}$$

Therefore, by virtue of the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we deduce from Lemma 3.4.2 with $p = \infty$ that

$$\|\Psi'(\varphi)\|_{L^\infty(\Omega\times(t,t+1))} \le C, \quad \forall\, t \ge 2\sigma.$$

Since Ψ' diverges at ± 1 and φ is continuous as established in Remark 6.4.2, we immediately infer the existence of $\delta > 0$ such that

$$|\varphi(x,t)| \le 1-\delta, \quad \forall\,(x,t) \in \overline{\Omega} \times [2\sigma,\infty).$$

Finally, thanks to the separation property and (6.4.12), it is easily seen that

$$\|\varphi\|_{L^\infty(2\sigma,\infty;H^4(\Omega))} \le C,$$

which completes the proof. □

6.5 Further Comments

In this final section we collect some remarks and natural developments regarding the Brinkman–Cahn–Hilliard system.

• The longtime behavior of the BCH system can be characterized by virtue of the regularity properties here established. More specifically, for any $m \in (-1,1)$, we define the complete metric space

$$\mathcal{V}_m = \Big\{\varphi \in V \cap L^\infty(\Omega) : \|\varphi\|_{L^\infty(\Omega)} \le 1 \text{ and } -m \le \overline{\varphi} \le m\Big\},$$

On account of Theorem 6.1.5 and Theorem 6.3.1, system (6.0.1)-(6.0.2) generates a semigroup of operators

$$\mathcal{S}(t) : \mathcal{V}_m \to \mathcal{V}_m,$$

via the rule

$$\mathcal{S}(t)\varphi_0 = \varphi(t), \quad \forall\, t \ge 0,$$

being $(\varphi, \boldsymbol{u})$ the unique global energy solution to the BCH problem with initial condition φ_0. The semigroup turns out to be strongly continuous (see Proposition 4.4.1 for the proof) and dissipative due to (6.1.5). Then, in light of Theorem 6.4.1, the existence of a unique (compact and connected) global attractor $\mathcal{A}_m$ for $S(t)$ on $\mathcal{V}_m$ follows by the classical semigroup theory (see, e.g., [149]). Furthermore, once the strict separation is reached, a further investigation of the asymptotic behavior is possible. In particular, the existence of exponential attractors $\mathcal{M}_m$ (see [120]). Besides, the convergence of each trajectory to a single stationary state can be also proved.

• The results herein achieved can be easily generalized to the BCH model with permeability η depending on the concentration in dimension two (cf. (1.3.15)). The dependence of η on φ is similar to (1.3.8), namely

$$\eta(s) = \eta_A \frac{1+s}{2} + \eta_B \frac{1-s}{2},$$

where η_A, η_B are the positive fluid permeabilities. Being able to handle the higher nonlinear term $\nabla\cdot(\nu(\varphi)D(\boldsymbol{u}))$, we observe that the presence of $\eta(\varphi)\boldsymbol{u}$ in the Brinkman's law does not affect significantly the proofs of the present paper.

• A further interesting problem is the study of the BCH system with singular potential and matched viscosity in dimension three. In which case, the continuous dependence estimate and the regularization in finite time can be achieved by the same techniques exploited in this work. By virtue of the energy identity, the *asymptotic* separation property can be proved by exploiting the technique used in [4]. In turn, this would allow to show the convergence of weak solutions to single stationary state.

Part II

Nonlocal interaction models

CHAPTER 7

The nonlocal Navier-Stokes-Cahn-Hilliard system

This chapter is devoted to the regularity properties of the nonlocal Navier–Stokes–Cahn–Hilliard system with matched viscosities in two space dimensions. To this purpose, we first provide a comprehensive analysis of the nonlocal Cahn–Hilliard. In particular, we address the well-posedness of weak solutions, the regularity propagation in time and the longtime behavior. In particular, in two dimensions, we prove the *instantaneous* separation property. Thereafter, we extend these regularity results to the nonlocal Navier–Stokes–Cahn–Hilliard system.

In a bounded domain $\Omega \subset \mathbb{R}^2$, we consider the nonlocal Navier–Stokes–Cahn–Hilliard system with matched viscosities ($\nu = 1$)

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \mu \nabla \varphi, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi = \Delta \mu, \\ \mu = \Psi'(\varphi) - J * \varphi, \end{cases} \quad \text{in } \Omega \times (0,T), \tag{7.0.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0}, \quad \partial_{\boldsymbol{n}} \mu = 0, & \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot), \quad \varphi(\cdot,0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{7.0.2}$$

General agreement. Throughout this chapter, if it is not otherwise stated, we indicate by C a generic positive constant depending only on the domain and on structural quantities. The constant C may vary from line to line and even within the same line. Any further dependence will be explicitly pointed out if necessary.

7.1 The Nonlocal Cahn-Hilliard Equation: Well-posedness

In a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the nonlocal Cahn–Hilliard system reads as follows

$$\begin{cases} \partial_t \varphi = \Delta \mu, \\ \mu = \Psi'(\varphi) - J * \varphi, \end{cases} \qquad \text{in } \Omega \times (0, T), \tag{7.1.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \partial_{\boldsymbol{n}} \mu = 0, & \text{on } \partial\Omega \times (0, T), \\ \varphi(\cdot, 0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{7.1.2}$$

We remind that in the sequel the main assumptions on the singular potential and the interaction kernel are (H) and (K).

We are now ready to give the definition of a weak solution to problem (7.1.1)-(7.1.2).

Definition 7.1.1. *Let φ_0 be a measurable function with $\Psi(\varphi_0) \in L^1(\Omega)$. A function φ is a weak solution to* (7.1.1)-(7.1.2) *on* $[0, T]$ *if*

$$\begin{aligned} &\varphi \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap H^1(0, T; V'), \\ &\varphi \in L^\infty(\Omega \times (0, T)) \text{ with } |\varphi(x, t)| < 1 \text{ a.e. } (x, t) \in \Omega \times (0, T), \\ &\mu \in L^2(0, T; V), \end{aligned}$$

and

$$\langle \partial_t \varphi, v \rangle + (\nabla \mu, \nabla v) = 0, \quad \forall\, v \in V, \tag{7.1.3}$$

for almost every $t \in (0, T)$, where

$$\mu = \Psi'(\varphi) - J * \varphi$$

for almost every $(x, t) \in \Omega \times (0, T)$. Moreover, $\varphi(\cdot, 0) = \varphi_0$ a.e. in Ω.

Remark 7.1.2. Let us observe that:

1. From $\Psi(\varphi_0) \in L^1(\Omega)$ we deduce that $|\varphi_0(x)| \leq 1$, for almost every $x \in \Omega$.
2. The conservation of mass is a straightforward consequence of (7.1.3). Indeed, taking $v = 1$, we get $\langle \partial_t \varphi, 1 \rangle = 0$, so $\overline{\varphi}\,(t) = \overline{\varphi}_0$ for all $t \geq 0$.
3. Let $T > 0$ be arbitrary. Note that $\varphi \in L^\infty(\Omega \times (0, T))$ with $|\varphi(x, t)| < 1$ for almost any $(x, t) \in \Omega \times (0, T)$ implies $\varphi \in L^\infty(0, T; L^p(\Omega))$, for all $p \geq 1$, and $\|\varphi\|_{L^\infty(0,T;L^p(\Omega))} \leq |\Omega|^{\frac{1}{p}}$. Moreover, we observe that the function $t \mapsto \|\varphi(t)\|_{L^\infty(\Omega)}$ is measurable, essentially bounded and, for all $f \in L^1(0, T; L^1(\Omega))$, there holds
$$|(\varphi(t), f(t))| \leq \|f(t)\|_{L^1(\Omega)}, \qquad \text{a.e. } t \in (0, T).$$
We refer the reader to [56].
4. As a direct consequence of Definition 7.1.1, we have $\varphi \in \mathcal{C}([0, T], H)$ and $\Psi'(\varphi) \in L^2(0, T; V)$. The former property entails that the initial condition is well defined.

The well-posedness of system (7.1.1)-(7.1.2) is given by

Theorem 7.1.3. *Let φ_0 be a measurable function with $\Psi(\varphi_0) \in L^1(\Omega)$, $|\overline{\varphi}_0| < 1$ and $T > 0$ be given. Assume that hypotheses $(H.1) - (H.2)$ are satisfied. Then, for any $T > 0$, there exists a unique weak solution φ to problem* (7.1.1)-(7.1.2) *on* $[0, T]$. *In addition, the global weak solution satisfies the dissipative inequality*

$$\mathcal{E}_H(\varphi(t)) + \int_t^{t+1} \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \,\mathrm{d}\tau \leq C\mathcal{E}_H(\varphi_0)\mathrm{e}^{-\omega t} + C, \tag{7.1.4}$$

for every $t \geq 0$, where ω and C are positive constants independent of the initial condition. Moreover, for every two weak solutions φ_1 and φ_2 to (7.1.1)-(7.1.2) *on* $[0, T]$ *with initial data φ_{01} and φ_{02}, respectively, the following continuous dependence estimate holds*

$$\|\varphi_1(t) - \varphi_2(t)\|^2_{V'} \leq C\|\varphi_{01} - \varphi_{02}\|^2_{V'}\mathrm{e}^{CT} + C|\overline{\varphi}_{01} - \overline{\varphi}_{02}|\mathrm{e}^{CT}, \tag{7.1.5}$$

for every $t \in [0, T]$, where the positive constant C depends on T.

Remark 7.1.4. By virtue of the dissipative inequality (7.1.4) and $\varphi \in \mathcal{C}([0, T], H)$, the function $t \to \int_\Omega \Psi(\varphi(t))\, dx$ is bounded for all $t \geq 0$. This immediately entails that

$$\sup_{t\geq 0} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1.$$

As a consequence, we deduce by interpolation that $\varphi \in \mathcal{C}([0, T], L^p(\Omega))$, for any $p \geq 2$.

The proof of Theorem 7.1.3 is carried out via several steps. First, we provide a family of regular function defined on the whole $\mathbb{R}$ which approximates the singular potential. The existence of a weak solution to (7.1.1)-(7.1.2) with a regular potential is established via the Galerkin method (see [39]). Then, we show (uniform) estimates on the solutions of this approximate problem in order to pass to the limit via compactness. To the best of our knowledge, Theorem 7.1.3 ensures the existence and uniqueness of a weak solution in the most general framework. Indeed, it requires the convexity of the potential whereas other existence results (cf., for example, [63, Corollary 1]) require further monotonicity and sign conditions on higher derivatives (i.e. from the second one up) of Ψ. For related results obtained within a more abstract framework see also [40].

Remark 7.1.5. We highlight that our analysis relies on the assumption $\overline{\varphi}_0 \in (-1, 1)$ (see also, for instance, [101] for the standard Cahn–Hilliard equation). This is physically reasonable since $\overline{\varphi}_0 = 1$ (or $\overline{\varphi}_0 = -1$) means that the initial condition is a pure phase, so that no phase separation takes place in Ω.

Proof of Theorem 7.1.3

1. Approximation of Ψ. We consider the family of approximation functions Ψ_λ introduced in Section 3.2. We report herein the main properties:

(i) for any $0 < \overline{\lambda} \leq 1$, there exists $C > 0$ such that

$$\Psi_\lambda(s) \geq \frac{1}{4\overline{\lambda}} s^2 - C, \quad \forall\, s \in \mathbb{R},\, \forall\, \lambda \in (0, \overline{\lambda}]; \tag{7.1.6}$$

(ii) Ψ_λ is convex with

$$\Psi''_\lambda(s) \geq \frac{\alpha}{1+\alpha}, \quad \forall\, s \in \mathbb{R};$$

(iii) Ψ'_λ is Lipschitz on $\mathbb{R}$ with constant $\frac{1}{\lambda}$;

(iv) $\Psi_\lambda(s) \nearrow \Psi(s)$ and $|\Psi'_\lambda(s)| \nearrow |\Psi'(s)|$ for every $s \in \mathbb{R}$ as $\lambda \to 0$ and, in addition, Ψ'_λ converges uniformly to Ψ' on any interval $[a, b] \subset (-1, 1)$;

(v) $\Psi_\lambda(0) = \Psi'_\lambda(0) = 0$, for all $\lambda > 0$.

Remark 7.1.6. We recall that, due to the convexity of Ψ_λ (see (i)), we have

$$\Psi_\lambda(s) \leq \Psi_\lambda(w) + (s-w)\Psi'_\lambda(s), \quad \text{for all } s, w \in \mathbb{R}. \tag{7.1.7}$$

2. The approximating problem and the dissipative inequality. For any fixed $\lambda > 0$, we consider the problem (7.1.1)-(7.1.2) replacing Ψ with Ψ_λ. The corresponding problem reads as follows

$$\begin{cases} \partial_t \varphi = \Delta \mu, \\ \mu = \Psi'_\lambda(\varphi) - J * \varphi, \end{cases} \quad \text{in } \Omega \times (0, T), \tag{7.1.8}$$

subject to

$$\begin{cases} \partial_n \mu = 0, & \text{on } \partial\Omega \times (0, T), \\ \varphi(\cdot, 0) = \varphi_0, & \text{in } \Omega. \end{cases} \tag{7.1.9}$$

Here, we simply use φ instead of φ_λ for the sake of simplicity. We denote the energy functional $\mathcal{E}_H^\lambda : H \to \mathbb{R}$ by

$$\mathcal{E}_H^\lambda(v) = \int_\Omega \Psi_\lambda(v)\, dx - \frac{1}{2}(J * v, v)$$

and we show the dissipative nature of the system.

Lemma 7.1.7. *There exists $\overline{\lambda} > 0$ such that, for any $0 < \lambda \leq \overline{\lambda}$, any solution to* (7.1.8)-(7.1.9) *satisfies*

$$\mathcal{E}_H^\lambda(\varphi(t)) + \int_t^{t+1} \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \,\mathrm{d}\tau \\ \leq C\mathcal{E}_H^\lambda(\varphi_0)\mathrm{e}^{-\omega t} + C(1 + \Psi_\lambda(\overline{\varphi}_0)), \tag{7.1.10}$$

for every $t \geq 0$. Here, ω and C are positive constant that depend on J and α but are independent of the initial condition and λ.

We provide below a formal proof of Lemma 7.1.7. A rigorous argument can be done by performing the same computations within a Galerkin approximation scheme (see the proof of Theorem 7.1.9 reported below).

Proof. Let us consider $\mathcal{E}_H^\lambda$. By virtue of property (i) and the Young inequality for convolution, for any $\lambda < \overline{\lambda}$, we obtain

$$\begin{aligned} \mathcal{E}_H^\lambda(v) &\geq \frac{1}{4\overline{\lambda}}\|v\|^2 - C|\Omega| - \frac{1}{2}\|J * v\|\|v\| \\ &\geq \left(\frac{1}{4\overline{\lambda}} - \frac{\|J\|_{L^1(\Omega)}}{2}\right)\|v\|^2 - C|\Omega|. \end{aligned}$$

Therefore, for any $\gamma > 0$ there exists C such that

$$\mathcal{E}_H^\lambda(v) \geq \gamma \|v\|^2 - C|\Omega| \tag{7.1.11}$$

provided that $\overline{\lambda}$ is small enough. It is also apparent from (v) and (7.1.7) that

$$\Psi_\lambda(s) \leq s\Psi'_\lambda(s) \leq \frac{1}{\lambda}|s|^2.$$

Thus, we deduce that

$$\mathcal{E}_H^\lambda(v) \leq \left(\frac{2}{\lambda} + \frac{\|J\|_{L^1(\Omega)}}{2}\right)\|v\|^2 + \frac{2}{\lambda}|\Omega|. \tag{7.1.12}$$

Now, testing $(7.1.8)_1$ and $(7.1.8)_2$ by μ and φ_t, respectively, and adding the two equations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_H^\lambda(\varphi) + \|\nabla\mu\|^2 = 0. \tag{7.1.13}$$

In order to reconstruct the energy functional on the left-hand side, we take the gradient of $(7.1.8)_2$ and we test by $\nabla\varphi$ yielding

$$(\Psi''_\lambda(\varphi)\nabla\varphi, \nabla\varphi) = (\nabla\mu, \nabla\varphi) + (\nabla J * \varphi, \nabla\varphi).$$

According to (ii) and the Young inequality for convolution, we get

$$\frac{\beta}{2}\|\nabla\varphi\|^2 \leq \frac{1}{2\beta}\|\nabla\mu\|^2 + \frac{1}{2\beta}\|\nabla J\|^2_{L^1(\Omega)}\|\varphi\|^2, \tag{7.1.14}$$

where $\beta = \alpha/(1+\alpha)$. On the other hand, testing again $(7.1.8)_2$ by $\varphi - \overline{\varphi}$ and using the Poincaré inequality, we obtain

$$\begin{aligned}(\Psi'_\lambda(\varphi), \varphi - \overline{\varphi}) &= (J * \varphi, \varphi - \overline{\varphi}) + (\mu, \varphi - \overline{\varphi}) \\ &\leq C\|J * \varphi\|\|\nabla\varphi\| + C\|\nabla\mu\|\|\nabla\varphi\|.\end{aligned} \tag{7.1.15}$$

Exploiting (7.1.7) with $s = \varphi$, $w = \overline{\varphi}$, we find

$$\mathcal{E}_H^\lambda(\varphi) \leq \Psi_\lambda(\overline{\varphi})|\Omega| + (\Psi'_\lambda(\varphi), \varphi - \overline{\varphi}) - \frac{1}{2}(J * \varphi, \varphi). \tag{7.1.16}$$

Combining (7.1.15) with (7.1.16), and using the Young inequality, we infer that

$$\begin{aligned}\mathcal{E}_H^\lambda(\varphi) &\leq \Psi_\lambda(\overline{\varphi})|\Omega| + C\|J * \varphi\|\|\nabla\varphi\| + C\|\nabla\mu\|\|\nabla\varphi\| + \frac{1}{2}|(J * \varphi, \varphi)| \\ &\leq \Psi_\lambda(\overline{\varphi})|\Omega| + \frac{\beta}{4}\|\nabla\varphi\|^2 + \frac{C}{2\beta}\|\nabla\mu\|^2 + \Big(\frac{C}{2\beta}\|J\|^2_{L^1(\Omega)} + \frac{1}{2}\|J\|_{L^1(\Omega)}\Big)\|\varphi\|^2.\end{aligned}$$

Adding (7.1.14) to the above inequality, we reach

$$\begin{aligned}\mathcal{E}_H^\lambda(\varphi) + \frac{\beta}{4}\|\nabla\varphi\|^2 \leq{}& \left(\frac{C+1}{2\beta}\right)\|\nabla\mu\|^2 + \Psi_\lambda(\overline{\varphi})|\Omega| \\ &+ \Big(\frac{1}{2\beta}\|\nabla J\|^2_{\mathbf{L}^1(\Omega)} + \frac{C}{2\beta}\|J\|^2_{L^1(\Omega)} + \|J\|_{L^1(\Omega)}\Big)\|\varphi\|^2.\end{aligned}$$

In light of the control from below (7.1.11), there exists $\overline{\lambda} > 0$ such that for any $0 < \lambda < \overline{\lambda}$ we have

$$\frac{1}{2}\mathcal{E}_\lambda(\varphi) + \frac{\beta}{4}\|\nabla\varphi\|^2 \leq \frac{C+1}{2\beta}\|\nabla\mu\|^2 + \Psi_\lambda(\overline{\varphi})|\Omega| + \frac{C}{2}|\Omega|. \tag{7.1.17}$$

Summing up, by (7.1.13) and (7.1.17) we find the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_H^\lambda(\varphi) + \omega\Big(\mathcal{E}_H^\lambda(\varphi) + \|\nabla\varphi\|^2 + \|\nabla\mu\|^2\Big) \leq C\left(1 + \Psi_\lambda(\overline{\varphi})\right),$$

for some $\omega > 0$ independent of λ. Finally, an application of the Gronwall lemma completes the argument. $\square$

3. Existence of an approximate solution. By analogy with Definition 7.1.1, we recall the definition of weak solution to the approximating problem.

Definition 7.1.8. *Let φ_0 be a measurable function with $\Psi_\lambda(\varphi_0) \in L^1(\Omega)$. A function φ is a weak solution to problem* (7.1.8)-(7.1.9) *on $[0,T]$ if*

$$\begin{aligned}&\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V'),\\ &\mu \in L^2(0,T;V),\end{aligned}$$

and

$$\langle \partial_t\varphi, v\rangle + (\nabla\mu, \nabla v) = 0, \quad \forall\, v \in V,$$

for almost every $t \in (0,T)$, where

$$\mu = \Psi_\lambda'(\varphi) - J * \varphi,$$

for almost every $(x,t) \in \Omega \times (0,T)$. Moreover, $\varphi(\cdot,0) = \varphi_0$ a.e. in Ω.

It is immediate to see that point 2 and 4 of Remark 7.1.2 are valid in the regular potential case as well. We can thus prove the existence of a global weak approximating solution.

Theorem 7.1.9. *Let φ_0 be a measurable function with $\Psi_\lambda(\varphi_0) \in L^1(\Omega)$ and $0 < \lambda \leq \overline{\lambda}$. Then, there exists a global weak solution φ to problem* (7.1.8)-(7.1.9) *which fulfills the dissipative inequality* (7.1.10) *for all $t \geq 0$.*

Proof. The existence of a weak solution is established through a Galerkin scheme. Let us $n \in \mathbb{N}$ be fixed. We seek a function

$$\varphi_n(t) = \sum_{k=1}^{n} a_k(t)\psi_k$$

which solves for all $t \in (0,T)$

$$\langle \partial_t\varphi_n, v\rangle + \ (\nabla\mu_n, \nabla w) = 0, \quad \forall\, v \in V_n, \tag{7.1.18}$$

where

$$\mu_n = \Pi_n\left[\Psi_\lambda'(\varphi_n) - J * \varphi_n\right].$$

Here, $\{\psi_k\}_{k=1}^\infty$ are the eigenfunctions associated to the Neumann operator A+I, $V_n = \operatorname{span}\{\psi_1, ..., \psi_n\}$, Π_n is the projector operators from V onto V_n and $\varphi_{0n} = \Pi_n(\varphi_0)$. We observe that $\varphi_0 \in H$ due to $\Psi_\lambda(\varphi_0) \in L^1(\Omega)$. Equation (7.1.18) is equivalent to a system of ordinary differential equations $\dot{\mathbf{a}}_n(t) = \mathcal{G}(\mathbf{a}_n(t))$, where $\mathbf{a}_n(t) = [a_1(t), ..., a_n(t)]$ is the unknown and $\mathcal{G}$ is a locally Lipschitz continuous function. Then, the Cauchy–Lipschitz theorem entails the existence of a unique local solution $\mathbf{a}_n \in C^1([0, T^*), \mathbb{R}^n)$. Since $\Psi_1 = 1$ is the first eigenfunction of $A + I$, we note that the conservation of mass holds for the approximated problem, namely, $\overline{\varphi}_n(t) = \overline{\varphi}_{0n}$. Thanks to Lemma 7.1.7, we derive some uniform estimates in order to guarantee that $T^* = \infty$ and recover compactness properties of the sequence φ_n. Indeed, the Galerkin approximation φ_n fulfills the following inequality for all $t \geq 0$,

$$\mathcal{E}_H^\lambda(\varphi_n(t)) + \int_t^{t+1} \|\nabla\varphi_n(\tau)\|^2 + \|\nabla\mu_n(\tau)\|^2 \,\mathrm{d}\tau \leq C\mathcal{E}_H^\lambda(\varphi_{0n})\mathrm{e}^{-\omega t} + C(1 + \Psi_\lambda(\overline{\varphi}_{0n})).$$

By $\varphi_{0n} \to \varphi_0$ in H and (7.1.11) and (7.1.12), the right-hand side of can be controlled by a constant independent of n and we deduce that

$$\varphi_n \quad \text{is uniformly bounded in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{7.1.19}$$

$$\nabla\mu_n \quad \text{is uniformly bounded in } L^2(0,T;H). \tag{7.1.20}$$

On account of (iii) and the above boundedness properties, we have

$$|\overline{\mu}_n| \leq C\left(1 + \|\varphi_n\|_{L^1(\Omega)}\right) \leq C,$$

where C is independent by n. In turn, this combined with (3.1.5) entails that

$$\mu_n \quad \text{is uniformly bounded in } L^2(0,T;V). \tag{7.1.21}$$

By comparison, we find

$$\Psi'_\lambda(\varphi_n) \quad \text{is uniformly bounded in } L^2(\Omega \times (0,T)), \tag{7.1.22}$$

$$\partial_t\varphi_n \quad \text{is uniformly bounded in } L^2(0,T;V'). \tag{7.1.23}$$

Thanks to (7.1.19)-(7.1.23) and standard compactness arguments, we infer that, up to subsequences,

$$\begin{aligned}
&\varphi_n \to \varphi, && \text{weakly in } L^2(0,T;V),\\
&\varphi_n \to \varphi, && \text{weakly star in } L^\infty(0,T;H),\\
&\partial_t\varphi_n \to \partial_t\varphi, && \text{weakly in } L^2(0,T;V'),\\
&\mu_n \to \mu, && \text{weakly in } L^2(0,T;V),\\
&\Psi'_\lambda(\varphi_n) \to \Psi'_\lambda(\varphi), && \text{weakly in } L^2(\Omega \times (0,T)).
\end{aligned}$$

Hence, we can pass to the limit in the approximation problem achieving the existence of a weak solution to (7.1.8)-(7.1.9) in the sense of Definition 7.1.8. From $\varphi \in L^2(0,T;V)$ and $\partial_t\varphi \in L^2(0,T;V')$, we also deduce that $\varphi \in \mathcal{C}([0,T],H)$. Furthermore, according to the above convergences properties, and passing to the limit in the dissipative inequality, the weak solution satisfies

$$\mathcal{E}_H^\lambda(\varphi(t)) + \int_t^{t+1} \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \,\mathrm{d}\tau \leq C\mathcal{E}_H^\lambda(\varphi_0)\mathrm{e}^{-\omega t} + C(1 + \Psi_\lambda(\overline{\varphi}_0))$$

for almost every $t \geq 0$. In particular, we have used the fact that $\varphi_{0n} \to \varphi_0$ in H entails that $\mathcal{E}_H^\lambda(\varphi_{0n}) \to \mathcal{E}_H^\lambda(\varphi_0)$, which easily follows from

$$|\Psi_\lambda(s) - \Psi_\lambda(w)| \leq \frac{1}{\lambda}|s-w| \max\{|s|,|w|\}, \quad \forall\, s,w \in \mathbb{R}. \tag{7.1.24}$$

We conclude by observing that the above dissipation inequality holds for every $t \geq 0$ by virtue of $\varphi \in \mathcal{C}([0,T],H)$. $\square$

We can now prove Theorem 7.1.3.

4. Passage to the limit. First, we observe that $\Psi(\varphi_0) \in L^1(\Omega)$ implies that $\Psi_\lambda(\varphi_0) \in L^1(\Omega)$ for any $\lambda > 0$. Then, as a consequence of Theorem 7.1.9, for any $\lambda \in (0, \overline{\lambda}]$, there exists a weak solution φ_λ to problem (7.1.8)-(7.1.9) which satisfies

$$\mathcal{E}_H^\lambda(\varphi_\lambda(t)) + \int_t^{t+1} \|\nabla\varphi_\lambda(\tau)\|^2 + \|\nabla\mu_\lambda(\tau)\|^2 \,\mathrm{d}\tau \leq C\mathcal{E}_H(\varphi_0)\mathrm{e}^{-\omega t} + C(1+\Psi(\overline{\varphi}_0)),$$

for all $t \geq 0$. Here, we have used (i) to control the right-hand side. Hence, in light of (7.1.11), this entails that

$$\varphi_\lambda \quad \text{is uniformly bounded w.r.t. } \lambda \text{ in } L^\infty(0,T;H), \tag{7.1.25}$$

$$\varphi_\lambda \quad \text{is uniformly bounded w.r.t. } \lambda \text{ in } L^2(0,T;V), \tag{7.1.26}$$

$$\nabla\mu_\lambda \quad \text{is uniformly bounded w.r.t. } \lambda \text{ in } L^2(0,T;H). \tag{7.1.27}$$

By comparison we also obtain

$$\partial_t\varphi_\lambda \quad \text{is uniformly bounded w.r.t } \lambda \text{ in } L^2(0,T;V'). \tag{7.1.28}$$

In order to pass to the limit we need to recover a uniform estimate for μ_λ in V. To this aim, we first control the L^1-norm of $\Psi'_\lambda(\varphi_\lambda)$. We apply the argument devised in [101] (see also [63] for the details). Let us choose $m_1, m_2 \in (-1,1)$ in such a way that $m_1 < \overline{\varphi}_0 < m_2$. We also set $\delta := \min\{\overline{\varphi}_0 - m_1, m_2 - \overline{\varphi}_0\}$ and $\delta_1 := \max\{\overline{\varphi}_0 - m_1, m_2 - \overline{\varphi}_0\}$. Then, for almost every $t \in (0,T)$, we consider the sets

$$\Omega_0 := \{m_1 \leq \varphi_\lambda(x,t) \leq m_2\}, \quad \Omega_1 := \{\varphi_\lambda(x,t) < m_1\}, \quad \Omega_2 := \{\varphi_\lambda(x,t) > m_2\}.$$

Since Ψ'_λ is monotone and $\Psi'_\lambda(0) = 0$ for any λ, using the assumption $\overline{\varphi}_0 \in (-1,1)$ and property (iii), we get

$$\begin{aligned}
\delta\|\Psi'_\lambda(\varphi_\lambda)\|_{L^1(\Omega)} &= \delta\int_{\Omega_0} |\Psi'_\lambda(\varphi_\lambda)|\,\mathrm{d}x + \delta\int_{\Omega_1} |\Psi'_\lambda(\varphi_\lambda)|\,\mathrm{d}x + \delta\int_{\Omega_2} |\Psi'_\lambda(\varphi_\lambda)|\,\mathrm{d}x \\
&\leq \delta\int_{\Omega_0} |\Psi'_\lambda(\varphi_\lambda)|\,\mathrm{d}x + \int_{\Omega_1} (\varphi - \overline{\varphi}_0)\,\Psi'_\lambda(\varphi_\lambda)\,\mathrm{d}x \\
&\quad + \int_{\Omega_2} (\varphi - \overline{\varphi}_0)\,\Psi'_\lambda(\varphi_\lambda)\,\mathrm{d}x \\
&\leq (\delta + \delta_1)\int_{\Omega_0} |\Psi'(\varphi_\lambda)|\,\mathrm{d}x + \int_\Omega (\varphi_\lambda - \overline{\varphi}_0)\Psi'_\lambda(\varphi_\lambda)\,\mathrm{d}x \\
&\leq C + \int_\Omega (\varphi_\lambda - \overline{\varphi}_0)\Psi'_\lambda(\varphi_\lambda)\,\mathrm{d}x
\end{aligned}$$

where C is independent of λ. Now, testing μ_λ by $\varphi_\lambda - \overline{\varphi}_0$, we find

$$(\Psi_\lambda'(\varphi_\lambda), \varphi_\lambda - \overline{\varphi}_0) \leq C\|J * \varphi_\lambda\|\|\varphi_\lambda\| + C\|\nabla \mu_\lambda\|\|\varphi_\lambda\|. \tag{7.1.29}$$

Then, by (7.1.25), we obtain

$$\int_\Omega (\varphi_\lambda - \overline{\varphi}_0)\Psi_\lambda'(\varphi_\lambda)\,\mathrm{d}x \leq C\Big(1 + \|\nabla \mu_\lambda\|\Big),$$

where C is independent of λ. Therefore, combining the above inequalities, we deduce from (7.1.28) that

$$\int_0^T \|\Psi_\lambda'(\varphi_\lambda)(\tau)\|_{L^1(\Omega)}^2\,\mathrm{d}\tau \leq C \tag{7.1.30}$$

where C is independent of λ. In turn, by

$$\int_\Omega \mu_\lambda\,\mathrm{d}x = \int_\Omega \Psi_\lambda'(\varphi_\lambda)\,\mathrm{d}x + \int_\Omega J * \varphi_\lambda\,\mathrm{d}x,$$

we get

$$\|\overline{\mu}_\lambda\|_{L^2(0,T)} \leq C.$$

Thus, due to the Poincaré–Wirtinger inequality (3.1.5), we arrive at

$$\mu_\lambda \text{ is uniformly bounded w.r.t. } \lambda \text{ in } L^2(0,T;V). \tag{7.1.31}$$

Accordingly, up to subsequences, we have the following convergences

$$\varphi_\lambda \to \varphi, \qquad \text{weakly in } L^2(0,T;V), \tag{7.1.32}$$

$$\varphi_\lambda \to \varphi, \qquad \text{weakly star in } L^\infty(0,T;H), \tag{7.1.33}$$

$$\partial_t\varphi_\lambda \to \partial_t\varphi, \qquad \text{weakly in } L^2(0,T;V'), \tag{7.1.34}$$

$$\mu_\lambda \to \mu, \qquad \text{weakly in } L^2(0,T;V). \tag{7.1.35}$$

Furthermore, compactness yields

$$\varphi_\lambda \to \varphi, \quad \text{strongly in } L^2(0,T;H). \tag{7.1.36}$$

Also, (K) and (7.1.36) imply that

$$J * \varphi_\lambda \to J * \varphi, \quad \text{strongly in } L^2(0,T;V). \tag{7.1.37}$$

Concerning the nonlinear term, we prove that the limit function φ fulfills

$$|\varphi(x,t)| < 1 \quad \text{a.e. } (x,t) \text{ in } \Omega \times (0,T).$$

For a fixed $\eta \in (0,1)$, we introduce the sets

$$E_\eta^\lambda = \{(x,t) \in \Omega \times (0,T) : \ |\varphi_\lambda(x,t)| > 1 - \eta\},$$
$$E_\eta = \{(x,t) \in \Omega \times (0,T) : \ |\varphi(x,t)| > 1 - \eta\}.$$

Since $\varphi_\lambda \to \varphi$ a.e. $(x,t) \in \Omega \times (0,T)$, the Fatou's lemma entails

$$|E_\eta| \leq \liminf_{\lambda \to 0^+} |E_\eta^\lambda|.$$

Recalling that $\Psi'_\lambda(x) \geq 0$ for $x \in [0,1)$, $\Psi'_\lambda(x) \leq 0$ for $x \in (-1,0]$ and Ψ'_λ is monotone, we deduce

$$\min\{\Psi'(1-\eta), -\Psi'(-1+\eta)\}|E^\varepsilon_\varrho| \leq \|\Psi'_\lambda(\varphi_\lambda)\|_{L^1(\Omega\times(0,T))} \leq C,$$

where C does not depends on λ and η. Therefore, we have

$$|E_\eta| \leq \frac{C}{\min\{\Psi'(1-\eta), -\Psi'(-1+\eta)\}}.$$

Passing to the limit as $\eta \to 0^+$, we deduce that

$$|\,\{(x,t) \in \Omega \times (0,T) : |\varphi(x,t)| \geq 1\}\,| = 0$$

which yields the desired conclusion. As a byproduct,

$$\Psi'_\lambda(\varphi_\lambda) \to \Psi'(\varphi) \quad \text{a.e. } (x,t) \in \Omega \times (0,T),$$

where we have used the pointwise convergence of φ_λ and the uniform convergence of Ψ'_λ to Ψ'. Moreover, by the expression of μ_λ, we get

$$\Psi'_\lambda(\varphi_\lambda) \text{ is uniformly bounded w.r.t. } \lambda \text{ in } L^2(0,T;H). \tag{7.1.38}$$

A standard compactness argument implies that $\Psi'_\lambda(\varphi_\lambda) \to \Psi'(\varphi)$ weakly in $L^2(\Omega \times (0,T))$. On account of the above convergences, we easily find that

$$\langle \partial_t \varphi, v\rangle + (\nabla \mu, \nabla v) = 0, \quad \forall\, v \in V,$$

for almost every $t \geq 0$, with

$$\mu = \Psi'(\varphi) - J * \varphi \in L^2(0,T;V).$$

Now, by virtue of the regularity of φ and $\partial_t\varphi$, we have $\varphi \in \mathcal{C}([0,T],H)$. By the above convergences, we pass to limit in the dissipative inequality satisfied by φ_λ and we learn that, for almost every $t \geq 0$,

$$\mathcal{E}_H(\varphi(t)) + \int_t^{t+1} \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \,\mathrm{d}\tau \leq C\mathcal{E}_H(\varphi_0)\mathrm{e}^{-\omega t} + C.$$

Here we have used the boundedness of Ψ. On the other hand, the above inequality holds for any $t \geq 0$ since $\varphi \in \mathcal{C}([0,T],H)$. Indeed, $J*\varphi \in \mathcal{C}([0,T],H)$, the integral terms on the left-hand side are continuous as well as the right-hand side. Let $t > 0$, there exists a sequence $\{t_j\}$ which tends to t and for which the above inequality holds. We show that

$$\lim_{t_j \to t} \int_\Omega \Psi(\varphi(t_j))\,\mathrm{d}x = \int_\Omega \Psi(\varphi(t))\,\mathrm{d}x.$$

On account of the continuity of φ, $\varphi(t_j) \to \varphi(t)$ strongly in H, so there exists a subsequence which converges for almost every $x \in \Omega$ and the limit necessarily satisfies $|\varphi(x,t)| \leq 1$ for almost every $x \in \Omega$. Since Ψ is continuous on the compact set $[-1,1]$, using the Lebesgue theorem, we infer that (7.1.4) holds for all $t \geq 0$.

5. Continuous dependence on the initial data and uniqueness.

Let us consider two weak solutions φ_1 and φ_2 related to the initial conditions φ_{01} and φ_{02}, respectively. The function $\varphi(t) = \varphi_1(t) - \varphi_2(t)$ with $\varphi(0) = \varphi_{01} - \varphi_{02}$ solves

$$\langle \partial_t \varphi, v \rangle + (\nabla \mu, \nabla v) = 0, \quad \forall\, v \in V,$$

for almost every $t \in (0, T)$, where

$$\mu = \Psi'(\varphi_1) - \Psi'(\varphi_2) - J * \varphi.$$

Taking $v = \mathcal{N}(\varphi - \overline{\varphi})$ and exploiting (3.3.5), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \overline{\varphi}\|^2_{V_0'} + (\mu, \varphi - \overline{\varphi}) = 0. \tag{7.1.39}$$

According to the assumption (H) and the definition of the operator $\mathcal{N}$, we deduce that

$$(\mu, \varphi - \overline{\varphi}) \geq \ \alpha\|\varphi\|^2 - (\Psi'(\varphi_1) - \Psi'(\varphi_2), \overline{\varphi}) - (\nabla J * \varphi, \nabla \mathcal{N}(\varphi - \overline{\varphi})).$$

Moreover, we have

$$\begin{aligned} |(\nabla J * \varphi, \nabla \mathcal{N}(\varphi - \overline{\varphi}))| &\leq \|\nabla J * \varphi\| \|\varphi - \overline{\varphi}\|_{V_0'} \\ &\leq \|\nabla J\|_{\mathbf{L}^1(\Omega)} \|\varphi\| \|\varphi - \overline{\varphi}\|_{V_0'} \\ &\leq \frac{\alpha}{2}\|\varphi\|^2 + C\|\varphi - \overline{\varphi}\|^2_{V_0'}. \end{aligned}$$

Hence, we find the differential inequality for almost every $t \in [0, T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \overline{\varphi}\|^2_{V_0'} + \alpha\|\varphi\|^2 \leq C\|\varphi - \overline{\varphi}\|^2_{V_0'} + \Lambda|\overline{\varphi}|,$$

where

$$\Lambda = 2\|\Psi'(\varphi_1)\|_{L^1(\Omega)} + 2\|\Psi'(\varphi_2)\|_{L^1(\Omega)},$$

that is a summable function. Therefore, an application of the Gronwall lemma yields, for all $t \in [0, T]$,

$$\|\varphi(t) - \overline{\varphi}\,(t)\,\|^2_{V_0'} \leq \|\varphi(0) - \overline{\varphi}\,(0)\,\|^2_{V_0'}\mathrm{e}^{CT} + C|\overline{\varphi}\,(0)\,|\mathrm{e}^{CT} \tag{7.1.40}$$

Finally, by the conservation of mass, (7.1.5) follows. As a byproduct, we learn the uniqueness of weak solutions.

7.2 The Nonlocal Cahn-Hilliard Equation: Regularity Properties

In this section we study the regularity properties of the weak solutions which allow us, in particular, to establish the existence of the (smooth) global attractor for the dissipative dynamical system associated with (7.1.1)-(7.1.2).

We will derive some uniform higher order estimates which will be independent of the form of the initial datum, but only depend on its total mass and the value of the energy. Henceforth, the generic constant C may also depend on $m \in (0, 1)$ and R such that

$$-1 + m \leq \overline{\varphi}_0 \leq 1 - m, \quad \text{and} \quad \mathcal{E}_H(\varphi_0) \leq R.$$

As a consequence of the dissipative inequality (7.1.4), we have

$$\mathcal{E}_H(\varphi(t)) + \int_t^{t+1} \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 + \|\partial_t\varphi(\tau)\|_{V'}^2 \,\mathrm{d}\tau \le C, \tag{7.2.1}$$

for every $t \ge 0$.

Our first regularity result is

Theorem 7.2.1. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\partial_t\varphi\|_{L^\infty(\sigma,t;V')} + \|\nabla\mu\|_{L^\infty(\sigma,t;\mathbf{H})} + \|\partial_t\varphi\|_{L^2(t,t+1;H)} \le C, \quad \forall\, t \ge \sigma, \tag{7.2.2}$$

and

$$\sup_{t\ge\sigma} \|\varphi(t)\|_V \le C. \tag{7.2.3}$$

Proof. We provide below a formal estimate which can be easily justified by exploiting the difference quotient rather then differentiating with respect to time. We differentiate system (7.1.1) with respect to time and we obtain

$$\partial_{tt}\varphi = \Delta(\Psi''(\varphi)\partial_t\varphi - J * \partial_t\varphi).$$

Testing by $\mathcal{N}\partial_t\varphi$ and recalling that $\overline{\partial_t\varphi} = 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\varphi\|_{V_0'}^2 + (\Psi''(\varphi)\partial_t\varphi, \partial_t\varphi) = (J * \partial_t\varphi, \partial_t\varphi).$$

By (H),

$$(\Psi''(\varphi)\partial_t\varphi, \partial_t\varphi) \ge \alpha\|\partial_t\varphi\|^2.$$

Reasoning as in the proof of the continuous dependence estimate, the right-hand side is controlled as follows

$$\begin{aligned}(J * \partial_t\varphi, \partial_t\varphi) &= (\nabla J * \partial_t\varphi, \nabla\mathcal{N}\partial_t\varphi)\\ &\le \frac{\alpha}{2}\|\partial_t\varphi\|^2 + C\|\partial_t\varphi\|_*^2.\end{aligned}$$

Here we have used the Young inequality for convolution and (3.3.2). Summing up, we find differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\varphi\|_{V_0'}^2 + \alpha\|\partial_t\varphi\|^2 \le C\|\partial_t\varphi\|_{V_0'}^2.$$

Therefore, exploiting (7.2.1), an application of the uniform Gronwall Lemma gives

$$\|\partial_t\varphi(t)\|_{V_0'}^2 + \int_t^{t+1} \|\partial_t\varphi(\tau)\|^2 \,\mathrm{d}\tau \le C, \quad \forall\, t \ge \sigma. \tag{7.2.4}$$

By comparison, we easily deduce that

$$\|\nabla\mu(t)\| \le C, \quad \forall\, t \ge \sigma. \tag{7.2.5}$$

Let us recover a uniform estimate of the weak solution in V. Applying the gradient operator to the chemical potential, and testing by $\nabla\varphi$, we get

$$(\nabla\mu, \nabla\varphi) = (\Psi''(\varphi)\nabla\varphi, \nabla\varphi) - (\nabla J * \varphi, \nabla\varphi).$$

Recalling (H) and using Young and Cauchy–Schwarz inequalities, we arrive at

$$\alpha\|\nabla\varphi\|^2 \leq \|\nabla\mu\|\|\nabla\varphi\| + \|\nabla J\|_{\mathbf{L}^1(\Omega)}\|\varphi\|\|\nabla\varphi\|.$$

Then, on account of (7.2.1), the Young inequality gives

$$\|\nabla\varphi(t)\| \leq C, \quad \forall\, t \geq \sigma. \tag{7.2.6}$$

Since (7.2.4) and (7.2.5) hold for the Galerkin aprroximation, from the lower semicontinuity of the norm we deduce (7.2.2). Finally, we infer (7.2.3) from (7.2.6), the continuity $\varphi \in \mathcal{C}([0,T],H)$ and the mass conservation. □

Next, we establish further regularity results and, in particular, a uniform V-bound of μ. These properties will be helpful in the next section.

Proposition 7.2.2. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\Psi'(\varphi)\|_{L^\infty(\sigma,t;V)} + \|\mu\|_{L^\infty(\sigma,t;V)} + \|\mu\|_{L^2(t,t+1;H^2(\Omega))} \leq C, \quad \forall\, t \geq \sigma, \tag{7.2.7}$$

and

$$\|\nabla\mu\|_{L^q(t,t+1;\mathbf{L}^p(\Omega))} + \|\nabla\varphi\|_{L^q(t,t+1;\mathbf{L}^p(\Omega))} \leq C, \quad \text{if } \frac{p-2}{p} = \frac{2}{q},\ d=2, \tag{7.2.8}$$

$$\|\nabla\mu\|_{L^q(t,t+1;\mathbf{L}^p(\Omega))} + \|\nabla\varphi\|_{L^q(t,t+1;\mathbf{L}^p(\Omega))} \leq C, \quad \text{if } \frac{3p-6}{2p} = \frac{2}{q},\ d=3, \tag{7.2.9}$$

where $2 \leq p < \infty$ if $d = 2$ and $2 \leq p \leq 6$ if $d = 3$.

Proof. Let us consider the identity

$$\mu - \overline{\mu} = -J * \varphi + \overline{J * \varphi} + \Psi'(\varphi) - \overline{\Psi'(\varphi)}.$$

By (3.1.5), we deduce that

$$\|\Psi'(\varphi) - \overline{\Psi'(\varphi)}\|_V \leq C\|\nabla\mu\| + C\|\nabla J * \varphi\|.$$

Hence, according to Theorem 7.2.1, we have

$$\|\Psi'(\varphi) - \overline{\Psi'(\varphi)}\|_{L^\infty(\sigma,t;V)} \leq C, \quad \forall\, t \geq \sigma.$$

In order to control the missing term $\overline{F'(\varphi)}$, arguing as in the proof of Theorem 7.1.3, we find

$$\|\Psi'(\varphi)\|_{L^1(\Omega)} \leq C\int_\Omega (\varphi - \overline{\varphi}_0)\Psi'(\varphi)\,\mathrm{d}x + C.$$

Then, testing μ by $\varphi - \overline{\varphi}_0$ and using (3.1.5) and (7.2.3), we obtain

$$\int_\Omega (\varphi - \overline{\varphi}_0)\Psi'(\varphi)\,\mathrm{d}x \leq C(1 + \|\nabla\mu\|).$$

Therefore, the above inequalities yield

$$\|\Psi'(\varphi)\|_{L^\infty(\sigma,t;L^1(\Omega))} \leq C, \quad \forall\, t \geq \sigma,$$

which, in turn, gives

$$\|\overline{\mu}\|_{L^\infty(\sigma,t)} \leq C, \quad \forall\, t \geq \sigma.$$

Thus, we end up with

$$\|\Psi'(\varphi)\|_{L^\infty(\sigma,t;V)} + \|\mu(t)\|_{L^\infty(\sigma,t;V)} \leq C, \quad \forall\, t \geq \sigma. \tag{7.2.10}$$

Furthermore, notice that the regularity of φ_t in (7.2.2), (7.2.10) and the regularity theory of the Neumann problem entail that the first equation of problem (7.1.1) is satisfied for almost every $(x,t) \in \Omega \times (\sigma, \infty)$, $\partial_{\boldsymbol{n}}\mu = 0$ for almost every $(x,t) \in \partial\Omega \times (\sigma,\infty)$ and

$$\|\mu\|_{L^2(t,t+1;H^2(\Omega))} \leq C, \quad \forall\, t \geq \sigma. \tag{7.2.11}$$

Arguing now as in [65], we find a control of $\nabla\varphi$ in $L^p(\Omega)$ by means of the L^2-norm of φ_t. To this aim, we take the gradient of μ, multiply it by $|\nabla\varphi|^{p-2}\nabla\varphi$ and integrate over Ω. We observe that this estimate cannot be made rigorously within a Galerkin scheme. Nevertheless, the regularity of the weak solution is enough to compute it. Indeed, on account of (H) and (K), by (7.2.11) we deduce that

$$\|\Psi''(\varphi)\nabla\varphi\|_{L^2(t,t+1;\mathbf{L}^p(\Omega))} \leq C, \quad \forall\, t \geq \sigma, \tag{7.2.12}$$

where $2 \leq p < \infty$ if $d = 2$ and $2 \leq p \leq 6$ if $d = 3$. This allows us to multiply by $|\nabla\varphi|^{p-2}\nabla\varphi$ yielding

$$\int_\Omega \Psi''(\varphi)|\nabla\varphi|^p \,\mathrm{d}x \leq \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi \cdot \nabla\mu \,\mathrm{d}x + \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi \cdot \nabla J * \varphi \,\mathrm{d}x.$$

By (H) and Young's inequality, we have

$$\alpha\|\nabla\varphi\|^p_{\mathbf{L}^p(\Omega)} \leq \|\nabla\mu\|_{\mathbf{L}^p(\Omega)}\|\nabla\varphi\|^{p-1}_{\mathbf{L}^p(\Omega)} + \|\nabla J\|_{\mathbf{L}^1(\Omega)}\|\varphi\|_{L^p(\Omega)}\|\nabla\varphi\|^{p-1}_{\mathbf{L}^p(\Omega)}.$$

Then, by (K) and (7.2.3) we get

$$\|\nabla\varphi\|_{\mathbf{L}^p(\Omega)} \leq C\big(\|\nabla\mu\|_{\mathbf{L}^p(\Omega)} + \|\varphi\|_V\big). \tag{7.2.13}$$

In order to estimate $\nabla\mu$ in $L^p(\Omega)$, if $d = 2$, the Gagliardo–Nirenberg inequality (3.1.10), together with (7.2.7), entails

$$\begin{aligned}\|\nabla\mu\|_{\mathbf{L}^p(\Omega)} &\leq C\|\nabla\mu\|^{\frac{2}{p}}\|\nabla\mu\|_{\mathbf{V}}^{1-\frac{2}{p}}\\ &\leq C\Big(\|\Delta\mu\|^{1-\frac{2}{p}} + \|\mu\|^{1-\frac{2}{p}}\Big)\\ &\leq C\Big(1 + \|\varphi_t\|^{1-\frac{2}{p}}\Big).\end{aligned}$$

Hence, setting q such that $\frac{p-2}{p} = \frac{2}{q}$, using (7.2.2) and (7.2.13), the estimate (7.2.8) easily follows. On the other hand, if $d = 3$, applying the Gagliardo–Nirenberg inequality (3.1.11), we get

$$\begin{aligned}\|\nabla\mu\|_{\mathbf{L}^p(\Omega)} &\leq C\|\nabla\mu\|^{\frac{6-p}{2p}}\|\nabla\mu\|_{\mathbf{V}}^{\frac{3p-6}{2p}}\\ &\leq C\|\mu\|_{H^2(\Omega)}^{\frac{3p-6}{2p}}\\ &\leq C\Big(1 + \|\varphi_t\|^{\frac{3p-6}{2p}}\Big).\end{aligned}$$

Hence, (7.2.9) is obtained as a byproduct of (7.2.2) and (7.2.13). The proof is complete. □

Remark 7.2.3. An immediate consequence of (7.2.8), (7.2.9) is the regularity $\varphi \in L^\infty(\sigma, t; L^\infty(\Omega))$ with $\|\varphi\|_{L^\infty(\sigma,t;L^\infty(\Omega))} \leq 1$ for all $t \geq \sigma$ and $d = 2, 3$. Indeed, it follows from $\varphi \in L^6(\sigma, t; W^{1,3}(\Omega))$ for $d = 2$ and $\varphi \in L^{\frac{8}{3}}(\sigma, t; W^{1,4}(\Omega))$ for $d = 3$.

7.3 The Nonlocal Cahn-Hilliard Equation: The Separation Property

In this section we restrict our analysis to the two dimensional case, $d = 2$, and we prove the validity of the *instantaneous* separation property. Some consequences of this property will also be analyzed.

In the sequel, the generic constant C is allowed to depend on m and R as in the previous section.

Theorem 7.3.1. *Let $d = 2$. Assume that* (H.3) *and* (H.4) *hold. Then, for any $\sigma > 0$, there exists $\delta = \delta(m, R, \sigma) > 0$ such that*

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall\, t \geq 2\sigma. \tag{7.3.1}$$

Proof. We begin by proving some integrability properties of $\Psi''(\varphi)$ and $\Psi'''(\varphi)$. Let $p \geq 1$ be given. Thanks to the first assumption of (H.3), we have

$$\begin{aligned}\int_\Omega \Psi''(\varphi)^p \,\mathrm{d}x &\leq \int_\Omega \mathrm{e}^{p[C|\Psi'(\varphi)|+C]} \,\mathrm{d}x \\ &= \mathrm{e}^{Cp} \int_\Omega \mathrm{e}^{Cp|\Psi'(\varphi)|} \,\mathrm{d}x.\end{aligned}$$

Recalling that $\Psi'(\varphi) \in V$ for almost every $t \in [\sigma, \infty)$, an application of the Trudinger–Moser inequality (3.1.6) to $Cp\Psi'(\varphi)$ gives

$$\|\Psi''(\varphi)\|_{L^p(\Omega)}^p \leq \mathrm{e}^{Cp} \mathrm{e}^{Cp^2 \|\Psi'(\varphi)\|_V^2}.$$

Then, on account of (7.2.7), we infer

$$\|\Psi''(\varphi)\|_{L^\infty(\sigma,t;L^p(\Omega))} \leq C\mathrm{e}^{Cp}, \quad \forall\, t \geq \sigma. \tag{7.3.2}$$

In turn, by (7.2.2), (7.2.7) and (7.3.2), we get

$$\partial_t \Psi'(\varphi) = \Psi''(\varphi)\partial_t\varphi \in L^2(t, t+1; V'), \quad \forall\, t \geq \sigma.$$

Thus, we find $\Psi'(\varphi) \in \mathcal{C}([\sigma, t], H)$ for all $t \geq \sigma$ and

$$\|\Psi'(\varphi(t))\|_V \leq C, \quad \|\Psi''(\varphi(t))\|_{L^p(\Omega)} \leq C, \quad \forall\, t \geq \sigma. \tag{7.3.3}$$

Consequently, according to (H.4) and (7.3.3), we easily deduce that

$$\|\Psi'''(\varphi(t))\|_{L^p(\Omega)} \leq C\mathrm{e}^{Cp}, \quad \forall\, t \geq \sigma. \tag{7.3.4}$$

Now, our aim is to show a uniform in time control of the L^∞-norm of $\Psi'(\varphi)$. To this end, we perform a Alikakos–Moser iteration argument. Taking $v = |\Psi'(\varphi)|^{p-1}\Psi'(\varphi)\Psi''(\varphi)$ in (7.1.3), we have for almost every $t \geq \sigma$

$$\begin{aligned}\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Psi'(\varphi)|^{p+1} \,\mathrm{d}x + \int_\Omega \Psi''(\varphi)\nabla\varphi \cdot \nabla\big(|\Psi'(\varphi)|^{p-1}\Psi'(\varphi)\Psi''(\varphi)\big) \,\mathrm{d}x \\ = \int_\Omega (\nabla J * \varphi) \cdot \nabla\big(|\Psi'(\varphi)|^{p-1}\Psi'(\varphi)\Psi''(\varphi)\big) \,\mathrm{d}x. \quad (7.3.5)\end{aligned}$$

Observe that

$$\nabla\big(|\Psi'(\varphi)|^{p-1}\Psi'(\varphi)\Psi''(\varphi)\big) = p|\Psi'(\varphi)|^{p-1}\Psi''(\varphi)^2\nabla\varphi + |\Psi'(\varphi)|^{p-1}\Psi'(\varphi)\Psi'''(\varphi)\nabla\varphi.$$

Then, we can write

$$\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Psi'(\varphi)|^{p+1}\,\mathrm{d}x + \mathcal{I}_1 + \mathcal{I}_2 = \mathcal{I}_3 + \mathcal{I}_4, \tag{7.3.6}$$

where

$$\begin{aligned}
\mathcal{I}_1 &:= p\int_\Omega \Psi''(\varphi)\nabla\varphi\cdot|\Psi'(\varphi)|^{p-1}\Psi''(\varphi)^2\nabla\varphi\,\mathrm{d}x,\\
\mathcal{I}_2 &:= \int_\Omega \Psi''(\varphi)\nabla\varphi\cdot|\Psi'(\varphi)|^{p-1}\Psi'(\varphi)\Psi'''(\varphi)\nabla\varphi\,\mathrm{d}x,\\
\mathcal{I}_3 &:= p\int_\Omega (\nabla J*\varphi)\cdot|\Psi'(\varphi)|^{p-1}\Psi''(\varphi)^2\nabla\varphi\,\mathrm{d}x,\\
\mathcal{I}_4 &:= \int_\Omega (\nabla J*\varphi)\cdot|\Psi'(\varphi)|^{p-1}\Psi'(\varphi)\Psi'''(\varphi)\nabla\varphi\,\mathrm{d}x.
\end{aligned}$$

We point out that taking v in (7.1.3) is not formal. Indeed, it is easy to check that the regularities property $\nabla\varphi\in L^6(t,t+1;L^3(\Omega))$ in (7.2.8) and the uniform bounds (7.2.7), (7.3.2) and (7.3.4) entail $v\in L^2(t,t+1;V)$, for all $t\geq\sigma$. Then, since φ_t belong to $L^2(t,t+1;H)$, for any $t\geq\sigma$, and $s\mapsto|\Psi'(s)|^{p+1}$ is convex, an application of [141, Chap.IV, Lemma 4.3] gives

$$\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Psi'(\varphi)|^{p+1}\,\mathrm{d}x = \int_\Omega |\Psi'(\varphi)|^{p-1}\Psi'(\varphi)\Psi''(\varphi)\varphi_t\,\mathrm{d}x,$$

for almost every $t\geq\sigma$.

Now we have to estimate all the terms $\mathcal{I}_i$, $i=1,2,3,4$. By the identity

$$|\Psi'(\varphi)|^{p-1}\Psi''(\varphi)^2|\nabla\varphi|^2 = \frac{4}{(p+1)^2}|\nabla|\Psi'(\varphi)|^{\frac{p+1}{2}}|^2, \tag{7.3.7}$$

and recalling (H), we have

$$\mathcal{I}_1 \geq \alpha p\int_\Omega |\Psi'(\varphi)|^{p-1}\Psi''(\varphi)^2|\nabla\varphi|^2\,\mathrm{d}x \geq \frac{4\alpha p}{(p+1)^2}\int_\Omega |\nabla|\Psi'(\varphi)|^{\frac{p+1}{2}}|^2\,\mathrm{d}x. \tag{7.3.8}$$

On the other hand, from (H.4), we obtain

$$\mathcal{I}_2 = \int_\Omega |\Psi'(\varphi)|^{p-1}\Psi'(\varphi)\Psi'''(\varphi)\Psi''(\varphi)|\nabla\varphi|^2\,\mathrm{d}x \geq 0.$$

Hypotheses (K), (H.3) and (H.4) together with Young's inequality, Remark 7.1.4 and

(7.3.7) allow us to control $\mathcal{I}_3$ and $\mathcal{I}_4$ as follows

$$\begin{aligned}
\mathcal{I}_3 &\leq p \int_\Omega \Big(|\Psi'(\varphi)|^{\frac{p-1}{2}} \Psi''(\varphi) |\nabla \varphi| \Big) \Big(|\Psi'(\varphi)|^{\frac{p-1}{2}} \Psi''(\varphi) |\nabla J * \varphi| \Big) \, \mathrm{d}x \\
&\leq \varepsilon p \int_\Omega |\Psi'(\varphi)|^{p-1} \Psi''(\varphi)^2 |\nabla \varphi|^2 \, \mathrm{d}x + \frac{p}{4\varepsilon} \int_\Omega |\Psi'(\varphi)|^{p-1} \Psi''(\varphi)^2 |\nabla J * \varphi|^2 \, \mathrm{d}x \\
&\leq \frac{4\varepsilon p}{(p+1)^2} \int_\Omega |\nabla |\Psi'(\varphi)|^{\frac{p+1}{2}}|^2 \mathrm{d}x + \frac{p}{4\varepsilon} \|\nabla J * \varphi\|^2_{L^\infty(\Omega)} \int_\Omega |\Psi'(\varphi)|^{p-1} \Psi''(\varphi)^2 \, \mathrm{d}x \\
&\leq \frac{4\varepsilon p}{(p+1)^2} \int_\Omega |\nabla |\Psi'(\varphi)|^{\frac{p+1}{2}}|^2 \, \mathrm{d}x + \frac{Cp}{4\varepsilon} \int_\Omega |\Psi'(\varphi)|^{p-1} \Psi''(\varphi)^2 \, \mathrm{d}x \\
&\leq \frac{4\varepsilon p}{(p+1)^2} \int_\Omega |\nabla |\Psi'(\varphi)|^{\frac{p+1}{2}}|^2 \, \mathrm{d}x + \frac{Cp}{4\varepsilon} \|\Psi''(\varphi)\|^2 + \frac{Cp}{4\varepsilon} \int_\Omega |\Psi'(\varphi)|^{p+1} \Psi''(\varphi)^2 \, \mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_4 &\leq \int_\Omega \Big(|\Psi'(\varphi)|^{\frac{p-1}{2}} \Psi''(\varphi) |\nabla \varphi| \Big) \Big(|\Psi'(\varphi)|^{\frac{p+1}{2}} \frac{|\Psi'''(\varphi)|}{\Psi''(\varphi)} |\nabla J * \varphi| \Big) \, \mathrm{d}x \\
&\leq \varepsilon p \int_\Omega |\Psi'(\varphi)|^{p-1} \Psi''(\varphi)^2 |\nabla \varphi|^2 \, \mathrm{d}x + \frac{1}{4\varepsilon p} \int_\Omega |\Psi'(\varphi)|^{p+1} \frac{|\Psi'''(\varphi)|^2}{\Psi''(\varphi)^2} |\nabla J * \varphi|^2 \, \mathrm{d}x \\
&\leq \frac{4\varepsilon p}{(p+1)^2} \int_\Omega |\nabla |\Psi'(\varphi)|^{\frac{p+1}{2}}|^2 \mathrm{d}x + \frac{C}{4\varepsilon p} \|\nabla J * \varphi\|^2_{L^\infty(\Omega)} \int_\Omega |\Psi'(\varphi)|^{p+1} \Psi''(\varphi)^2 \, \mathrm{d}x \\
&\leq \frac{4\varepsilon p}{(p+1)^2} \int_\Omega |\nabla |\Psi'(\varphi)|^{\frac{p+1}{2}}|^2 \mathrm{d}x + \frac{C}{4\varepsilon p} \int_\Omega |\Psi'(\varphi)|^{p+1} \Psi''(\varphi)^2 \, \mathrm{d}x,
\end{aligned}$$

where $\varepsilon > 0$ is an arbitrary parameter. Choosing $\varepsilon = \frac{\alpha}{4}$ in the above estimates, we infer from (7.3.3) and (7.3.6) that

$$\begin{aligned}
\frac{1}{p+1} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Psi'(\varphi)|^{p+1} \, \mathrm{d}x &+ \frac{2\alpha p}{(p+1)^2} \int_\Omega |\nabla |\Psi'(\varphi)|^{\frac{p+1}{2}}|^2 \, \mathrm{d}x \\
&\leq Cp + Cp \int_\Omega |\Psi'(\varphi)|^{p+1} \Psi''(\varphi)^2 \, \mathrm{d}x, \qquad (7.3.9)
\end{aligned}$$

for almost every $t \geq \sigma$. Taking now

$$\mathcal{J} = \int_\Omega |\Psi'(\varphi)|^{p+1} |\Psi''(\varphi)|^2 \, \mathrm{d}x$$

and applying the Hölder inequality, we find

$$\begin{aligned}
\mathcal{J} &\leq \|\Psi''(\varphi)\|^2_{L^4(\Omega)} \|\Psi'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)} \\
&\leq C \|\Psi'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)},
\end{aligned}$$

where we have used (7.3.3) to control $\Psi''(\varphi)$. Hence, (7.3.9) turns into

$$\begin{aligned}
\frac{1}{p+1} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Psi'(\varphi)|^{p+1} \, \mathrm{d}x &+ \frac{2\alpha p}{(p+1)^2} \int_\Omega |\nabla |\Psi'(\varphi)|^{\frac{p+1}{2}}|^2 \, \mathrm{d}x \\
&\leq Cp \Big(1 + \|\Psi'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)} \Big).
\end{aligned}$$

Setting $w(t) = |\Psi'(\varphi(t))|^{\frac{p+1}{2}}$, we rewrite the above differential inequality in terms of w as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \frac{2\alpha p}{p+1}\|\nabla w\|^2 \leq Cp(p+1)\Big(1 + \|w\|^2_{L^4(\Omega)}\Big). \tag{7.3.10}$$

Exploiting Lemma 3.1.1 with $\varepsilon = \frac{\alpha}{C(p+1)^2}$

$$Cp(p+1)\|w\|^2_{L^4(\Omega)} \leq \frac{\alpha p}{p+1}\|\nabla w\|^2 + C\Big(1 + (p+1)^6\Big)\|w\|^2_{L^1(\Omega)}$$

and inserting the above estimate into (7.3.10), we obtain

$$\frac{d}{dt}\|w\|^2 + \frac{\alpha p}{p+1}\|\nabla w\|^2 \leq Cp^6\Big(1 + \|w\|^2_{L^1(\Omega)}\Big).$$

Then, noting that $\frac{p}{p+1} \geq \frac{1}{2}$, and using again Lemma 3.1.1 with $s = 2$ and $\varepsilon = \frac{\alpha}{2}$, we reach

$$\frac{d}{dt}\|w\|^2 + \|w\|^2 \leq Cp^6\Big(1 + \|w\|^2_{L^1(\Omega)}\Big). \tag{7.3.11}$$

for almost every $t \geq \sigma$ and any $p \geq 1$. We are now in a position to carry out an iterative argument (see [6]). To this aim, we observe that, by comparison, $\Psi'(\varphi) \in L^1(\sigma, 2\sigma; W^{1,3}(\Omega))$ (see (7.2.11)) with

$$\|\Psi'(\varphi)\|_{L^1(\sigma,2\sigma;W^{1,3}(\Omega))} \leq C,$$

where C only depends on σ. By the Sobolev embedding $W^{1,3}(\Omega) \hookrightarrow L^\infty(\Omega)$, we infer that there exists $\overline{\xi} \in (\sigma, 2\sigma)$ such that

$$\|\Psi'(\varphi(\overline{\xi}))\|_{L^\infty(\Omega)} \leq C.$$

Hence, denoting

$$\eta = \max\Big\{\|\Psi'(\varphi(\overline{\xi}))\|_{L^\infty(\Omega)}, \max_{t\geq\overline{\xi}} \int_\Omega |\Psi'(\varphi)|\, dx\Big\},$$

and according to (7.2.7), we find the estimate

$$1 \leq \eta \leq C. \tag{7.3.12}$$

Next, recalling the very definition of w, an application of the Gronwall Lemma to (7.3.11) gives

$$\max_{t\geq\overline{\xi}} \int_\Omega |\Psi'(\varphi(t))|^{p+1}\,\mathrm{d}x \leq \max\Big\{\eta^{p+1}, Cp^6 \max_{t\geq\overline{\xi}}\Big(1 + \int_\Omega |\Psi'(\varphi(t))|^{\frac{p+1}{2}}\,\mathrm{d}x\Big)^2\Big\}$$

for all $t \geq \overline{\xi}$ and $p \geq 1$. As customary, taking $p+1 = 2^k$, $k \in \mathbb{N}$, we rewrite the above inequality as

$$\begin{aligned}\max_{t\geq\overline{\xi}} \int_\Omega |\Psi'(\varphi(t))|^{2^k}\,\mathrm{d}x &\leq \max\Big\{\eta^{2^k}, C2^{6k} \max_{t\geq\overline{\xi}}\Big(1 + \int_\Omega |\Psi'(\varphi(t))|^{2^{k-1}}\mathrm{d}x\Big)^2\Big\}\\ &\leq \max\Big\{\eta^{2^k}, C2^{6k+2} \max_{t\geq\overline{\xi}}\Big(\int_\Omega |\Psi'(\varphi(t))|^{2^{k-1}}\mathrm{d}x\Big)^2\Big\}.\end{aligned}$$

Setting $A_k = C2^{6k+2}$ and arguing by iteration, we arrive at

$$\begin{aligned}\max_{t\geq\bar{\xi}} \int_\Omega |\Psi'(\varphi(t))|^{2^k}\, \mathrm{d}x &\leq \eta^{2^k} A_k\, A_{k-1}^2\, A_{k-2}^{2^2} \dots A_{k-(k-1)}^{2^{k-1}} \\ &\leq \eta^{2^k} C^{A2^k} 2^{B2^k},\end{aligned} \tag{7.3.13}$$

where

$$A = \sum_{i=1}^\infty \frac{1}{2^i} < \infty, \quad B = \sum_{i=1}^\infty \frac{6i+2}{2^i} < \infty.$$

Finally, taking the 2^{-k}-power on both sides of (7.3.13), passing to the limit as $k \to +\infty$, and using (7.3.12), we end up with

$$\max_{t\geq\bar{\xi}} \|\Psi'(\varphi(t))\|_{L^\infty(\Omega)} \leq C.$$

Therefore, (7.3.1) immediately follows from the above estimate. The proof is complete. □

Remark 7.3.2. Suppose the third condition in (H.4) is replaced by the more general one

$$|\Psi'''(s)| \leq C\Psi''(s)^q, \quad \forall\, s \in (-1,1), \tag{7.3.14}$$

for some $q \geq 1$. Then, following line by line the above proof, and setting now

$$\mathcal{J} = \int_\Omega |\Psi'(\varphi)|^{p+1}\Psi''(\varphi)^{2(q-1)}\, \mathrm{d}x,$$

we just need to control $\mathcal{J}$ in a slightly different way. Indeed, applying the Hölder inequality, we get

$$|\mathcal{J}| \leq \|\Psi'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)} \|\Psi''(\varphi)\|^{2(q-1)}_{L^{4(q-1)}(\Omega)} \leq C\|\Psi'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)}.$$

Thus the conclusion still follows arguing as above.

A first immediate consequence of Theorem 7.3.1 is

Corollary 7.3.3. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\mu(t)\|_{L^\infty(\Omega)} \leq C, \quad \forall\, t \geq 2\sigma.$$

Moreover, as a byproduct, we can also obtain the Hölder regularity of the weak solutions by means of [51, Corollary 4.2]. Indeed we have

Corollary 7.3.4. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ and $\beta = \beta(\sigma,\delta) \in (0,1)$ such that*

$$\begin{aligned}|\varphi(x_1,t_1) - \varphi(x_2,t_2)| &\leq C\left(|x_1-x_2|^\beta + |t_1-t_2|^{\frac{\beta}{2}}\right)\\ |\mu(x_1,t_1) - \mu(x_2,t_2)| &\leq C\left(|x_1-x_2|^\beta + |t_1-t_2|^{\frac{\beta}{2}}\right),\end{aligned}$$

for all $(x_1,t_1),(x_2,t_2) \in [t,t+1]\times\overline{\Omega}$ and $t \geq 3\sigma$.

Leaning on the strict separation property (7.3.1), we are able to interpret the weak solutions to problem (7.1.1)-(7.1.2) as the weak solutions to a similar problem where Ψ is replaced by a suitable regular potential. More precisely, we define the regular potential $\overline{\Psi} \in C^3(\mathbb{R})$, which extends Ψ outside of $[-1+\delta, 1-\delta]$, as follows

$$\begin{cases} \overline{\Psi}(s) = \sum_{k=0}^{3} \frac{\Psi^{(k)}(1-\delta)}{k!}(s-1+\delta)^k, & \forall\, s \geq 1-\delta, \\ \overline{\Psi}(s) = \Psi(s), & \forall\, s \in (-1+\delta, 1-\delta) \\ \overline{\Psi}(s) = \sum_{k=0}^{3} \frac{\Psi^{(k)}(-1+\delta)}{k!}(s+1-\delta)^k, & \forall\, s \leq -1+\delta. \end{cases} \tag{7.3.15}$$

According to the assumptions H.3 and H.4 and taking into account the sign of Ψ and its derivatives at $s = 1-\delta$ and $s = -1+\delta$, we deduce the following properties:

(A.1) for any $\Lambda > 0$, there exists $C > 0$ such that

$$\overline{\Psi}(s) \geq \Lambda s^2 - C, \quad \forall\, s \in \mathbb{R};$$

(A.2) there exists $N > 0$ such that

$$|\overline{\Psi}'(s)| \leq N(1+s^2), \quad \forall\, s \in \mathbb{R};$$

(A.3) there exists $N > 0$ such that

$$\alpha \leq \overline{\Psi}''(s) \leq N(1+|s|), \quad |\overline{\Psi}'''(s)| \leq N, \quad \forall\, s \in \mathbb{R}.$$

Here, α is the same value defined in assumption (H). Instead, C and N can be easily estimated in terms of δ.

Let us now set $\varphi_1 = \varphi(3\sigma)$, which is a function in V such that $\|\varphi_1\|_{L^\infty(\Omega)} \leq 1-\delta$, $\overline{\varphi}_1 \in [-1+m, 1-m]$. We consider the Cahn–Hilliard system

$$\begin{cases} \partial_t \tilde{\varphi} = \Delta \tilde{\mu}, \\ \tilde{\mu} = \overline{\Psi}'(\tilde{\varphi}) - J * \tilde{\varphi}, \end{cases} \quad \text{in } \Omega \times (0,T), \tag{7.3.16}$$

subject to the boundary and initial conditions

$$\begin{cases} \partial_{\boldsymbol{n}} \tilde{\mu} = 0, & \text{on } \partial\Omega \times (0,T) \\ \tilde{\varphi}(\cdot, 0) = \varphi_1, & \text{in } \Omega. \end{cases} \tag{7.3.17}$$

Combining Lemma 7.1.7 and Theorem 7.1.9, it follows immediately that problem (7.3.16)-(7.3.17) has a unique weak solution in the sense of Definition (7.1.8) obtained as a limit of a Galerkin sequence. On the other hand, from the separation property, the definition of $\overline{\Psi}$ and the uniqueness of (7.3.16)-(7.3.17), we easily infer that φ is also a weak solution to (7.3.16) so $\tilde{\varphi}(t) \equiv \varphi(t+3\sigma)$ for all $t \geq 0$. According to this equivalence, the idea is to compute some higher order estimates on the Galerkin sequence due to its regularity. Note that the Galerkin sequence does not satisfy the separation property. Nevertheless, we can take advantage of the specific form of $\overline{\Psi}$.

Lemma 7.3.5. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\partial_t \varphi\|_{L^\infty(5\sigma,t;H)} + \|\partial_t \varphi\|_{L^2(t,t+1;V)} + \|\nabla \partial_t \mu\|_{L^2(t,t+1;\mathbf{H})} \leq C, \quad \forall\, t \geq 5\sigma. \tag{7.3.18}$$

Proof. Let us consider the Galerkin sequence $\tilde{\varphi}_n$ which converges to $\tilde{\varphi}$. Due to the regularity of $\tilde{\varphi}_n$ and the properties of $\overline{\Psi}$, we can repeat line by line the proof of Theorem 7.2.1. In particular, we have

$$\|\tilde{\varphi}_n(t)\|_V^2 + \|\partial_t\tilde{\varphi}_n(t)\|_{V'}^2 + \int_t^{t+1} \|\partial_t\tilde{\varphi}_n(\tau)\|^2 \,\mathrm{d}\tau \leq C, \quad \forall\, t \geq \sigma, \tag{7.3.19}$$

where C is independent of n. Now, arguing as in [65], we differentiate the system with respect to time and we test by $\partial_t\tilde{\mu}_n$ getting

$$(\partial_{tt}\tilde{\varphi}_n, \partial_t\tilde{\mu}_n) + \|\nabla\partial_t\tilde{\mu}_n\|^2 = 0.$$

Hence, exploiting the form of $\tilde{\mu}_n$, we obtain

$$(\partial_{tt}\tilde{\varphi}_n, \overline{\Psi}''(\tilde{\varphi}_n)\partial_t\tilde{\varphi}_n) - (\partial_{tt}\tilde{\varphi}_n, J * \partial_t\tilde{\varphi}_n) + \|\nabla\partial_t\tilde{\mu}_n\|^2 = 0.$$

Using the first equation of (7.3.16), we can rewrite the above equality as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\partial_t\tilde{\varphi}_n|^2 \,\mathrm{d}x + \|\nabla\partial_t\tilde{\mu}_n\|^2 = (\Delta\partial_t\tilde{\mu}_n, J * \partial_t\tilde{\varphi}_n) + \frac{1}{2}\int_\Omega \overline{\Psi}'''(\tilde{\varphi}_n)\partial_t\tilde{\varphi}_n^3 \,\mathrm{d}x.$$

After an integration by parts in the right-hand side, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\partial_t\tilde{\varphi}_n|^2 \,\mathrm{d}x + \|\nabla\partial_t\tilde{\mu}_n\|^2 = -(\nabla\partial_t\tilde{\mu}_n, \nabla J * \partial_t\tilde{\varphi}_n) + \frac{1}{2}\int_\Omega \overline{\Psi}'''(\tilde{\varphi}_n)\partial_t\tilde{\varphi}_n^3 \,\mathrm{d}x.$$

By Young's inequality, assumption (K) and (A.3), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\partial_t\tilde{\varphi}_n|^2 \,\mathrm{d}x + \frac{1}{2}\|\nabla\partial_t\tilde{\mu}_n\|^2 \leq C\|\partial_t\tilde{\varphi}_n\|^2 + C\int_\Omega |\partial_t\tilde{\varphi}_n|^3 \,\mathrm{d}x. \tag{7.3.20}$$

On account of (3.1.10) with $p = 3$, we control the last term on the right-hand side as

$$\|\partial_t\tilde{\varphi}_n\|_{L^3(\Omega)}^3 \leq \gamma\|\nabla\partial_t\tilde{\varphi}_n\|^2 + C\|\partial_t\tilde{\varphi}_n\|^4,$$

for any $\gamma > 0$ and $C > 0$ depending on γ but independent of n. In order to reconstruct the L^2-norm of the gradient of $\partial_t\tilde{\varphi}_n$ on the left-hand side, we multiply the gradient of $\partial_t\tilde{\mu}_n$ by $\nabla\partial_t\tilde{\varphi}_n$ getting

$$\begin{aligned}\int_\Omega \nabla\partial_t\tilde{\mu}_n \cdot \nabla\partial_t\tilde{\varphi}_n \,\mathrm{d}x = &\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\nabla\partial_t\tilde{\varphi}_n|^2 \,\mathrm{d}x + \int_\Omega \overline{\Psi}'''(\tilde{\varphi}_n)\nabla\tilde{\varphi}_n \cdot \nabla\partial_t\tilde{\varphi}_n \,\mathrm{d}x \\ &- \int_\Omega \nabla J * \partial_t\tilde{\varphi}_n \cdot \nabla\partial_t\tilde{\varphi}_n \,\mathrm{d}x.\end{aligned}$$

Using again Young's inequality, assumption (K) and (A.3), we obtain

$$\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\nabla\partial_t\tilde{\varphi}_n|^2 \,\mathrm{d}x \leq \frac{\alpha}{2}\|\nabla\partial_t\tilde{\varphi}_n\|^2 + C\|\nabla\partial_t\tilde{\mu}_n\|^2 + C\|\nabla\tilde{\varphi}_n\|^2 + C\|\partial_t\tilde{\varphi}_n\|^2.$$

According to the bound from below of $\overline{\Psi}''$, the above inequality yields

$$\frac{1}{2}\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\nabla\partial_t\tilde{\varphi}_n|^2 \,\mathrm{d}x \leq C\|\nabla\partial_t\tilde{\mu}_n\|^2 + C\|\nabla\tilde{\varphi}_n\|^2 + C\|\partial_t\tilde{\varphi}_n\|^2. \tag{7.3.21}$$

Gathering together (7.3.20) and (7.3.21), there exists $\omega > 0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\partial_t\tilde{\varphi}_n|^2\,\mathrm{d}x + \omega\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\nabla\partial_t\tilde{\varphi}_{n,t}|^2\,\mathrm{d}x + \frac{1}{4}\|\nabla\partial_t\tilde{\mu}_n\|^2 \\ \le \gamma\|\nabla\partial_t\tilde{\varphi}_n\|^2 + C\|\partial_t\tilde{\varphi}_n\|^4 + C\|\nabla\tilde{\varphi}_n\|^2 + C\|\partial_t\tilde{\varphi}_n\|^2.\end{aligned}$$

Setting $\gamma = \frac{\omega\alpha}{2}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\partial_t\tilde{\varphi}_n|^2\,\mathrm{d}x + \gamma\int_\Omega |\nabla\partial_t\tilde{\varphi}_n|^2\,\mathrm{d}x + \frac{1}{4}\|\nabla\partial_t\tilde{\mu}_n\|^2 \\ \le C\|\partial_t\tilde{\varphi}_n\|^4 + C\|\nabla\tilde{\varphi}_n\|^2 + C\|\partial_t\tilde{\varphi}_n\|^2.\end{aligned}$$

Noting that the first term on the right-hand side can be controlled as follows

$$\|\partial_t\tilde{\varphi}_n\|^4 \le C\|\partial_t\tilde{\varphi}_n\|^2\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\partial_t\tilde{\varphi}_n|^2\,\mathrm{d}x,$$

we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\partial_t\tilde{\varphi}_n|^2\,\mathrm{d}x + \gamma\int_\Omega |\nabla\partial_t\tilde{\varphi}_n|^2\,\mathrm{d}x + \frac{1}{4}\|\nabla\partial_t\tilde{\mu}_n\|^2 \\ \le C\|\partial_t\tilde{\varphi}_n\|^2\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\partial_t\tilde{\varphi}_n|^2\,\mathrm{d}x + C\|\nabla\tilde{\varphi}_n\|^2 + C\|\partial_t\tilde{\varphi}_n\|^2. \qquad (7.3.22)\end{aligned}$$

In order to apply the uniform Gronwall lemma, we need to find a bound of

$$\int_t^{t+1}\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n(\tau))|\partial_t\tilde{\varphi}_n(\tau)|^2\,\mathrm{d}x\mathrm{d}\tau, \quad \forall\, t \ge \sigma.$$

To this aim, we observe that

$$\begin{aligned}\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n)|\partial_t\tilde{\varphi}_n|^2\,\mathrm{d}x &= (J * \partial_t\tilde{\varphi}_n, \partial_t\tilde{\varphi}_n) + (\partial_t\tilde{\mu}_n, \partial_t\tilde{\varphi}_n) \\ &= (J * \partial_t\tilde{\varphi}_n, \partial_t\tilde{\varphi}_n) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\tilde{\varphi}_n\|_*^2.\end{aligned}$$

Integrating in time from t to $t+1$ and exploiting (7.3.19), we get

$$\begin{aligned}\int_t^{t+1}\int_\Omega \overline{\Psi}''(\tilde{\varphi}_n(\tau))|\partial_t\tilde{\varphi}_n(\tau)|^2\,\mathrm{d}x\mathrm{d}\tau &\le C\int_t^{t+1}\|\partial_t\tilde{\varphi}_n(\tau)\|^2\,\mathrm{d}\tau + \frac{1}{2}\|\partial_t\tilde{\varphi}_n(t)\|_*^2 \\ &\le C.\end{aligned}$$

Therefore, due to the above estimate and (7.3.19), we apply the uniform Gronwall lemma to (7.3.22) deducing

$$\|\partial_t\tilde{\varphi}_n(t)\|^2 + \int_t^{t+1}\|\partial_t\tilde{\varphi}_t(\tau)\|_V^2 + \|\nabla\partial_t\tilde{\mu}_n(\tau)\|^2\,d\tau \le C, \quad \forall\, t \ge 2\sigma.$$

Passing to the limit as n goes to ∞, using the lower semicontinuity of the norm and the equivalence between $\tilde{\varphi}$ and φ, we obtain (7.3.18). □

Remark 7.3.6. Notice that by comparison, we also infer that for all $t \geq 5\sigma$,

$$\|\mu\|_{L^\infty(5\sigma,t;H^2(\Omega))} + \|\nabla\varphi\|_{L^\infty(5\sigma,t;\mathbf{L}^p(\Omega))} + \|\partial_{tt}\varphi\|_{L^2(t,t+1;V')} \leq C.$$

If we strengthen a bit the assumptions on the interaction kernel J, we can say more about the regularity of the solution.

Lemma 7.3.7. *Assume that J satisfies* K.1. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\sup_{t\geq 5\sigma} \|\varphi(t)\|_{H^2(\Omega)} \leq C.$$

The claim of Lemma 7.3.7 can deduced from the regularity $\mu \in L^\infty(5\sigma, \infty; H^2(\Omega))$ (see [62, Theorem 5]). Moreover, further regularity properties can be achieved by making use of the maximal regularity of the Neumann Laplacian (see [69]).

7.4 The Nonlocal Cahn-Hilliard Equation: Longtime behavior

In this section we discuss the longtime behavior of global solutions.

The Infinite Dimensional Dynamical System

Let us now analyze the dynamical system associated to problem (7.1.1)-(7.1.2). For any given $m \in (0,1)$, we introduce the phase space

$$\mathcal{V}_m = \left\{\varphi \in L^\infty(\Omega) : \|\varphi(x)\|_{L^\infty(\Omega)} \leq 1 \text{ and } -1+m \leq \overline{\varphi} \leq 1-m\right\}, \tag{7.4.1}$$

endowed with the metric

$$\mathbf{d}(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|. \tag{7.4.2}$$

It is easily seen that $\mathcal{V}_m$ is a complete metric space. Thanks to Theorem 7.1.3, we can set

$$\mathcal{S}(t) : \mathcal{V}_m \to \mathcal{V}_m, \quad \mathcal{S}(t)\varphi_0 = \varphi(t), \quad \forall\, t \geq 0,$$

where φ is the weak solution in the sense of Definition 7.1.1 corresponding to the i nitial condition φ_0. The dynamical system $(\mathcal{H}_\kappa, S(t))$ is dissipative owing to (7.1.4). Moreover, $\mathcal{S}(t)$ is a closed semigroup on the phase space $\mathcal{V}_m$ because of (7.1.5) (see [128]).

The following result concerns the existence of the global attractor.

Theorem 7.4.1. *The dynamical system $(\mathcal{V}_m, \mathcal{S}(t))$ has a connected global attractor $\mathcal{A}_m$ which is bounded in $\mathcal{V}_m \cap V$.*

Proof. Let us set

$$\mathcal{B} = B_V(0,R) \cap \mathcal{V}_m,$$

where $R > 0$ sufficiently large. We infer from Theorem 7.2.1 that $\mathcal{B}$ is a connected compact absorbing set for the dynamical system $(\mathcal{V}_m, \mathcal{S}(t))$. Hence, the existence of the global attractor is an immediate consequence of [128, Corollary 6]. □

Thanks to the validity of the separation property, we can deduce more information on the asymptotic behavior of the weak solutions in dimension two.

Theorem 7.4.2. *Let the assumptions of Theorem 7.3.1 hold. Then, for every* $m > 0$*, there exists an exponential attractor* $\mathcal{M}_m$ *bounded in* $V \cap \mathcal{C}^\beta(\overline{\Omega})$ *for the dynamical system* $(\mathcal{V}_m, \mathcal{S}(t))$*, namely,*

(i) $\mathcal{S}(t)\mathcal{M}_m \subset \mathcal{M}_m, \ \forall\, t \geq 0$*;*

(ii) $\mathcal{M}_m$ *exponentially attracts the bounded subsets of* $\mathcal{V}_m$*, i.e. there exist* C *and* ω *such that for every* $\mathcal{B}$ *bounded set of* $\mathcal{V}_m$

$$dist_{\mathcal{C}^\gamma(\overline{\Omega})}(\mathcal{S}(t)\mathcal{B}, \mathcal{M}_m) \leq Ce^{-\omega t}, \quad \forall\, t \geq 0,$$

for any $\gamma \in (0, \beta)$*;*

(iii) the fractal dimension of $\mathcal{M}_m$ *is finite, that is,*

$$dim_F(\mathcal{M}_m, \mathcal{C}^\beta(\overline{\Omega})) \leq C,$$

where C *depends on* β *and* m*.*

As consequences of Theorem 7.4.2 we have

Corollary 7.4.3. *Let the assumptions of Theorem 7.3.1 hold, then the global attractor is a bounded subset of* $V \cap \mathcal{C}^\beta(\overline{\Omega})$ *and has finite fractal dimension, that is,*

$$dim_F(\mathcal{A}_m, \mathcal{C}^\beta(\overline{\Omega})) \leq C.$$

Corollary 7.4.4. *Let the assumptions of Theorem 7.3.1 hold. If, in addition,* J *satisfies* (K.1)*, then the global attractor* $\mathcal{A}_m$ *and the exponential attractor* $\mathcal{M}_m$ *are bounded in* $\mathcal{H}_m \cap H^2(\Omega)$.

Theorem 7.4.2 and Corollaries 7.4.3 and 7.4.4 are byproducts of the separation property. Indeed, we recall that the weak solutions to (7.1.1)-(7.1.2) coincides in finite time with the weak solutions to (7.1.1)-(7.1.2) with a smooth potential. Hence we can use [74, Theorem 2.8] to guarantee the existence of an exponential attractor and its consequences.

Convergence to Single Stationary State

We conclude this section by stating a result on the convergence of single trajectories. More precisely, we have that any weak solution does converge to a single stationary state. This result also follows from the argument mentioned above which is based on the strict separation property. More precisely, it can be proven arguing as in [74, Theorem 2.21] where the regular potential case is considered. Thus, we also have

Corollary 7.4.5. *Let the assumptions of Theorem 7.3.1 hold. If* Ψ *is real analytic on* $[-1 + \delta(m), 1 - \delta(m)]$*. Then, any weak solution* φ *to problem* (7.1.1)-(7.1.2) *is such that*

$$\lim_{t\to\infty} \|\varphi\,(t) - \varphi_*\|_{L^\infty(\Omega)} = 0, \tag{7.4.3}$$

where $\varphi_* \in V \cap \mathcal{C}^\alpha(\overline{\Omega})$ *solves*

$$\Psi'(\varphi_*) - J * \varphi_* = \mu_*,$$

where $\mu_* \in \mathbb{R}$*, and* $\overline{\varphi}_0 = \overline{\varphi}_*$.

7.5 The Nonlocal Navier-Stokes-Cahn-Hilliard System: Regularity Properties

This section is devoted to the regularity results and the validity of the strict separation property to problem (7.0.1)-(7.0.2) in dimension two. Let us introduce first the definition of weak solution (see [63]).

Definition 7.5.1. *Let $\boldsymbol{u}_0 \in \mathbf{H}_\sigma$, φ_0 be a measurable function with $\Psi(\varphi_0) \in L^1(\Omega)$. A couple $(\boldsymbol{u}, \varphi)$ is a weak solution to problem* (7.0.1)-(7.0.2) *on $[0,T]$ if*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;\mathbf{H}_\sigma) \cap L^2(0,T;\mathbf{V}_\sigma) \cap H^1(0,T;\mathbf{V}_\sigma'),\\
&\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V'),\\
&\varphi \in L^\infty(\Omega \times (0,T)) \text{ with } |\varphi(x,t)| < 1 \text{ a.e. } (x,t) \in \Omega \times (0,T),\\
&\mu \in L^2(0,T;V)
\end{aligned}$$

such that

$$\begin{aligned}
\langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = (\mu \nabla \varphi, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma,\\
\langle \partial_t \varphi, v\rangle + (\boldsymbol{u}\cdot\nabla\varphi, v) + (\nabla \mu, \nabla v) = 0, \quad \forall\, v \in V,
\end{aligned}$$

for almost every $t \in (0,T)$, where

$$\mu = \Psi'(\varphi) - J * \varphi,$$

for almost every $(x,t) \in \Omega \times (0,T)$. Moreover, the initial conditions $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0$ and $\varphi(\cdot,0) = \varphi_0$ a.e. in $\Omega \times (0,T)$.

Recalling the energy associated to system (7.0.1)

$$E_H(\boldsymbol{u},\varphi) = \frac{1}{2}\|\boldsymbol{u}\|^2 + \int_\Omega \Psi(\varphi)dx - \frac{1}{2}(J*\varphi,\varphi),$$

we state the well-posedness result related to problem (7.0.1)-(7.0.2).

Theorem 7.5.2. *Let $\boldsymbol{u}_0 \in \mathbf{H}_\sigma$, φ_0 be a measurable function with $\Psi(\varphi_0) \in L^1(\Omega)$ and $\overline{\varphi}_0 \in (-1,1)$. Then, for any $T > 0$, there exists a unique weak solution $(\boldsymbol{u},\varphi)$ to problem* (7.0.1)-(7.0.2) *on $[0,T]$ which satisfies the dissipative estimate*

$$\begin{aligned}
E_H(\boldsymbol{u}(t),\varphi(t)) + \int_t^{t+1} \|\nabla \boldsymbol{u}(\tau)\|^2 + \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2\, d\tau&\\
\le C E_H(\boldsymbol{u}_0,\varphi_0)e^{-\omega t} + C&,
\end{aligned}$$

for all $t \geq 0$, where ω and C are positive constants independent of the initial condition.

The proof of Theorem 7.5.2 can be carried out by arguing as in Theorem 7.1.3 and by using the standard Galerkin scheme for the Navier–Stokes system (see [151]). We also refer to [63, Section 2, Theorem 1] for a different approximation technique. Instead, uniqueness has been proven arguing as in [62].

Let us fix $m \in (0,1)$ and $R \geq 0$. We consider trajectories such that

$$|\overline{\varphi}_0| \leq 1 - m \quad \text{and} \quad E_H(\boldsymbol{u}_0,\varphi_0) \leq R.$$

Accordingly, the generic constant C may depend on R and m but is independent of the specific form of the initial datum. Moreover, thanks to the above result, we have that any weak solution fulfills for all $t \geq 0$

$$E_H(\boldsymbol{u}(t), \varphi(t)) + \int_t^{t+1} \|\nabla \boldsymbol{u}(\tau)\|^2 + \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 + \|\partial_t\varphi(\tau)\|_{V'}^2 \,\mathrm{d}\tau \leq C. \tag{7.5.1}$$

We begin with a regularity result for the Navier–Stokes system in dimension two.

Lemma 7.5.3. *For any $\sigma > 0$, there exists $C = C(\sigma)$ such that*

$$\|\boldsymbol{u}\|_{L^\infty(\sigma,t;\mathbf{V}_\sigma)} + \|\boldsymbol{u}\|_{L^2(t,t+1;\mathbf{H}^2(\Omega))} + \|\partial_t\boldsymbol{u}\|_{L^2(t,t+1;\mathbf{L}^2(\Omega))} \leq C, \quad \forall\, t \geq \sigma. \tag{7.5.2}$$

Proof. We observe that the Korteweg force can be rewritten as

$$\mu\nabla\varphi = \nabla\pi^* - (J * \varphi)\nabla\varphi.$$

On the other hand, we have for all $t \geq 0$

$$\int_t^{t+1} \|(J * \varphi(\tau))\nabla\varphi(\tau)\|^2 \,\mathrm{d}\tau \leq C.$$

Thus (7.5.2) follows from [151, Theorem 3.10]. □

Thanks to Lemma 7.5.3, we can prove the following regularity result on the phase parameter.

Lemma 7.5.4. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\partial_t\varphi\|_{L^\infty(2\sigma,t;V')} + \|\partial_t\varphi\|_{L^2(t,t+1;H)} \leq C, \quad \forall\, t \geq 2\sigma. \tag{7.5.3}$$

Proof. We provide below a formal computation. A rigorous proof can be easily done by replacing the differentiation with respect to time with difference quotient. We differentiate the nonlocal Cahn–Hilliard equation with respect to time and we test the equation by $\mathcal{N}\varphi_t$. Then, arguing as in the proof of Lemma 7.2.1, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\varphi\|_{V_0'}^2 + \alpha\|\partial_t\varphi\|^2 \leq C\|\partial_t\varphi\|_{V_0'}^2 + 2|(\partial_t\boldsymbol{u}\varphi, \nabla\mathcal{N}\partial_t\varphi)| + 2|(\boldsymbol{u}\partial_t\varphi, \nabla\mathcal{N}\partial_t\varphi)|.$$

By the Hölder inequality and the properties of $\mathcal{N}$, we deduce that

$$|(\partial_t\boldsymbol{u}\varphi, \nabla\mathcal{N}\partial_t\varphi)| \leq \|\partial_t\boldsymbol{u}\|\|\partial_t\varphi\|_{V_0'}$$

and

$$|(\boldsymbol{u}\partial_t\varphi, \nabla\mathcal{N}\partial_t\varphi)| \leq \|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)}\|\partial_t\varphi\|\|\partial_t\varphi\|_{V_0'}.$$

Collecting together the above estimates and using the Young inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\varphi\|_{V_0'}^2 + \frac{\alpha}{2}\|\partial_t\varphi\|^2 \leq C\left(1 + \|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)}^2\right)\|\partial_t\varphi\|_{V_0'}^2 + \|\partial_t\boldsymbol{u}\|^2.$$

Now, exploiting the uniform Gronwall lemma together with (7.5.1) and (7.5.2), we easily infer (7.5.3). □

As an immediate consequence, we deduce two additional regularity results whose proofs can be performed following line by line the proofs of their previous counterparts (namely, Theorem 7.2.1 and Proposition 7.2.2).

Lemma 7.5.5. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\nabla\mu\|_{L^\infty(\sigma,t;\mathbf{H})} \leq C, \quad \forall\, t \geq 2\sigma, \tag{7.5.4}$$

and

$$\sup_{t\geq 2\sigma} \|\varphi(t)\|_V \leq C. \tag{7.5.5}$$

Lemma 7.5.6. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that for all $t \geq 2\sigma$*

$$\|\Psi'(\varphi)\|_{L^\infty(2\sigma,t;V)} + \|\mu\|_{L^\infty(2\sigma,t;V)} + \|\mu\|_{L^2(t,t+1;H^2(\Omega))} \leq C, \tag{7.5.6}$$

$$\|\nabla\mu\|_{L^q(t,t+1;\mathbf{L}^p(\Omega))} + \|\nabla\varphi\|_{L^q(t,t+1;\mathbf{L}^p(\Omega))} \leq C, \quad \text{if } \frac{p-2}{p} = \frac{2}{q} \text{ and } d = 2. \tag{7.5.7}$$

We now have all the ingredients to establish the separation property.

Theorem 7.5.7. *Given $\sigma > 0$. Suppose that Ψ also fulfills* (H.3)-(H.4). *Then, there exists $\delta = \delta(m, R, \sigma) > 0$ such that*

$$\sup_{t\geq 3\sigma} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta. \tag{7.5.8}$$

Proof. We apply the same argument of Theorem 7.3.1. We need to handle the following further term

$$\mathcal{Z} = \int_\Omega \boldsymbol{u}\varphi\nabla\left(|\Psi'(\varphi)|^{p-1}\Psi'(\varphi)\Psi''(\varphi)\right)\,\mathrm{d}x.$$

Using the boundedness of φ, we have

$$\begin{aligned}
|\mathcal{Z}| &\leq \int_\Omega |\boldsymbol{u}||\Psi'(\varphi)^{p-1}\Psi'(\varphi)\Psi'''(\varphi)\nabla\varphi|\,\mathrm{d}x + p\int_\Omega |\boldsymbol{u}|\Psi''(\varphi)^2|\Psi'(\varphi)|^{p-1}|\nabla\varphi|\,\mathrm{d}x \\
&\leq \mathcal{Z}_1 + \mathcal{Z}_2.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\mathcal{Z}_1 &\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|\Psi'(\varphi)|^{\frac{p+1}{2}}|^2\,\mathrm{d}x + \frac{1}{4Cp}\int_\Omega |\Psi'(\varphi)|^{p+1}|\frac{\Psi'''(\varphi)|^2}{\Psi''(\varphi)^2}\boldsymbol{u}^2\,\mathrm{d}x \\
&\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|\Psi'(\varphi)|^{\frac{p+1}{2}}|^2\,\mathrm{d}x + \frac{1}{4Cp}\int_\Omega |\Psi'(\varphi)|^{p+1}|\Psi''(\varphi)|^2\boldsymbol{u}^2\,\mathrm{d}x \\
&\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|\Psi'(\varphi)|^{\frac{p+1}{2}}|^2\,\mathrm{d}x + \|\Psi''(\varphi)^2\boldsymbol{u}^2\|_{\mathbf{H}}\|\Psi'(\varphi)\|_{L^{2(p+1)}(\Omega)}^{p+1} \\
&\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|\Psi'(\varphi)|^{\frac{p+1}{2}}|^2\,\mathrm{d}x + \|\Psi''(\varphi)\|_{L^8(\Omega)}^2\|\boldsymbol{u}\|_{\mathbf{L}^8(\Omega)}^2\|\Psi'(\varphi)\|_{L^{2(p+1)}(\Omega)}^{p+1} \\
&\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|\Psi'(\varphi)|^{\frac{p+1}{2}}|^2\,dx + C\|\Psi'(\varphi)\|_{L^{2(p+1)}(\Omega)}^{p+1}
\end{aligned}$$

and

$$\begin{aligned}\mathcal{Z}_2 &\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|\Psi'(\varphi)|^{\frac{p+1}{2}}|^2\,\mathrm{d}x + \frac{1}{4C}\|\boldsymbol{u}^2\Psi''(\varphi)^2\|_{\mathbf{H}}\|\Psi'(\varphi)\|^{p-1}_{L^{2(p+1)}(\Omega)}\\ &\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|\Psi'(\varphi)|^{\frac{p+1}{2}}|^2\,\mathrm{d}x + C\|\Psi'(\varphi)\|^{p-1}_{L^{2(p-1)}(\Omega)}.\end{aligned}$$

Arguing as in Theorem 7.3.1 and using an Alikakos–Moser type iteration procedure, we obtain (7.5.8). □

Thanks to the strict separation property we can also prove some Hölder continuity. Indeed we have

Lemma 7.5.8. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ and $\beta \in (0,1)$, depending on δ such that*

$$\sup_{t\in[4\sigma,\infty)} \|\boldsymbol{u}(t)\|_{\mathbf{W}^{1,4}(\Omega)} \leq C \tag{7.5.9}$$

and

$$|\varphi(x_1,t_1) - \varphi(x_2,t_2)| \leq C\left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}\right), \tag{7.5.10}$$

for all $(x_1,t_1),(x_2,t_2) \in [t,t+1]\times\overline{\Omega}$ and any $t \geq 5\sigma$.

Proof. We observe that the Korteweg force can be rewritten in the following form

$$\mu\nabla\varphi = \nabla\tilde{\pi} - (\nabla J * \varphi)\varphi.$$

Thanks to Lemma 7.5.4 and the boundedness of φ, we deduce that

$$\begin{aligned}&\|\partial_t((\nabla J*\varphi)\varphi)\|_{L^2(t,t+1;\mathbf{H})}\\ &\leq \|(\nabla J*\partial_t\varphi)\varphi\|_{L^2(t,t+1;\mathbf{H})} + \|(\nabla J*\varphi)\partial_t\varphi\|_{L^2(t,t+1;\mathbf{H})} \leq C, \quad \forall\, t\geq 3\sigma.\end{aligned}$$

Therefore, we can consider the Navier–Stokes equation

$$\langle\partial_t\boldsymbol{u},\boldsymbol{v}\rangle + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + (\nabla\boldsymbol{u},\nabla\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}), \quad \forall\,\boldsymbol{v}\in V_\sigma,$$

for almost every $t\geq 0$, where $\boldsymbol{f}$ is a vector-field bounded in $L^2(t,t+1;\mathbf{H})$ with $\partial_r\boldsymbol{f}$ bounded in $L^2(t,t+1;\mathbf{H})$. Setting

$$\partial_t^h v = \frac{1}{h}\Big(v(t+h) - v(t)\Big),$$

we take the difference of the above equation for $t+h$ and t and we test by $\partial_t^h\boldsymbol{u}$. This gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\boldsymbol{u}\|^2 + \|\nabla\partial_t^h\boldsymbol{u}\|^2 + b(\partial_t^h\boldsymbol{u},\boldsymbol{u}(t+h),\partial_t^h\boldsymbol{u}) + b(\boldsymbol{u},\partial_t^h\boldsymbol{u},\partial_t^h\boldsymbol{u}) = (\partial_t^h\boldsymbol{f},\partial_t^h\boldsymbol{u}).$$

Notice that the last term on the left-hand side is equal to zero. Exploiting (3.1.7) and the Young inequality, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\boldsymbol{u}\|^2 + \|\nabla\partial_t^h\boldsymbol{u}\|^2 \leq \|\boldsymbol{u}(t+h)\|_{\mathbf{V}_\sigma}\|\partial_t^h\boldsymbol{u}\|\|\partial_t^h\boldsymbol{u}\|_{\mathbf{V}_\sigma} + \|\partial_t^h\boldsymbol{f}\|\|\partial_t^h\boldsymbol{u}\|.$$

Due to the Poincaré inequality, we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h \boldsymbol{u}\|^2 + \|\nabla \partial_t^h \boldsymbol{u}\|^2 &\le C\|\boldsymbol{u}(t+h)\|^2_{\mathbf{V}_\sigma}\|\partial_t^h \boldsymbol{u}\|^2 + C\|\partial_t^h \boldsymbol{f}\|^2 \\ &\le C\|\partial_t^h \boldsymbol{u}\|^2 + C\|\partial_t^h \boldsymbol{f}\|^2,\end{aligned}$$

where we have used Lemma 7.5.3. On account of the inequality

$$\|\partial_t^h v\|_{L^2(t,t+1+h;H)} \le \|v_t\|_{L^2(t,t+1;H)},$$

applying the uniform Gronwall lemma, we infer that

$$\|\partial_t^h \boldsymbol{u}\|_{L^\infty(4\sigma,t;\mathbf{H}} + \|\nabla \partial_t^h \boldsymbol{u}\|_{L^2(t,t+1;\mathbf{H})} \le C, \quad \forall\, t \ge 4\sigma.$$

Hence, we conclude that

$$\|\partial_t \boldsymbol{u}\|_{L^\infty(4\sigma,t,\mathbf{H})} + \|\nabla \partial_t \boldsymbol{u}\|_{L^2(t,t+1;\mathbf{H})} \le C, \quad \forall\, t \ge 4\sigma. \tag{7.5.11}$$

Therefore, an application of [9, Theorem 1.1] yields

$$\|\boldsymbol{u}(t)\|_{\mathbf{W}^{1,4}(\Omega)} \le C, \quad \forall\, t \ge 4\sigma,$$

and we conclude that (7.5.9) holds. Finally, we can apply [51, Corollary 4.2] to the nonlocal Cahn–Hilliard equation with convective term and infer (7.5.10). □

Remark 7.5.9. As we observed in Section 7.3 we can still identify the weak solutions to problem (7.0.1)-(7.0.2) with the weak solutions to a similar problem with a regular potential. Then, we can generalize the results on the longtime behavior contained in Section 7.3. More precisely, we know from [62] that (7.0.1)-(7.0.2) generates a dissipative dynamical systems which possesses a global attractor. Then, the regularity of the global attractor as well as the convergence of any weak solution to a single equilibrium proved in [65] for a regular potential can be extended to the present case. The same can be told for results on the existence of an exponential attractor proven in [62, Section 5].

CHAPTER 8

The nonlocal Hele-Shaw-Cahn-Hilliard system

This chapter is devoted to the mathematical analysis of the nonlocal version of the Hele–Shaw-Cahn–Hilliard system. First, we prove the well-posedness of weak solutions in both two and three dimensions. Then, we show the existence of global in time strong solutions. Consequently, due to the parabolic nature of the system, we deduce that weak solutions become instantaneously strong solutions. In addition, by using the method introduced in Chapter 7, we obtain the validity of the separation property in dimension two. Finally, we discuss the longtime behavior.

In a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the nonlocal Hele–Shaw–Cahn–Hilliard system with matched viscosities ($\nu = 1$) reads as follows

$$\begin{cases} \boldsymbol{u} = -\nabla \pi + \mu \nabla \varphi, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi = \Delta \mu, \\ \mu = \Psi'(\varphi) - J * \varphi, \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{8.0.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = \partial_{\boldsymbol{n}} \mu = 0, & \text{on } \partial\Omega \times (0,T), \\ \varphi(\cdot, 0) = \varphi_0(\cdot), & \text{in } \Omega. \end{cases} \tag{8.0.2}$$

General agreement. Throughout this chapter, if it is not otherwise stated, we indicate by C a generic positive constant depending only on the domain and on structural quantities. The constant C may vary from line to line and even within the same line. Any further dependence will be explicitly pointed out if necessary.

8.1 Well-posedness

We prove that problem (8.0.1)-(8.0.2) is well posed with respect to the notion of weak solutions in $\Omega \subset \mathbb{R}^3$. To this purpose, we remind that the singular potential and the interaction kernel satisfy the basic assumptions (H) and (K).

By weak solution we mean the following

Definition 8.1.1. *Let φ_0 be a measurable function with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. A triple $(\boldsymbol{u}, \pi, \varphi)$ is a weak solution to problem* (8.0.1)-(8.0.2) *on $[0,T]$ if*

$$\begin{aligned}
&\boldsymbol{u} \in L^2(0,T;\mathbf{H}_\sigma), \quad \pi \in L^1(0,T;W^{1,\frac{3}{2}}(\Omega)),\\
&\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V'),\\
&\varphi \in L^\infty(\Omega \times (0,T)) \text{ with } |\varphi(x,t)| < 1 \text{ a.e. } (x,t) \in \Omega \times (0,T),\\
&\Psi'(\varphi) \in L^2(0,T;V),\\
&\mu \in L^2(0,T;V),
\end{aligned}$$

such that

$$\langle \partial_t \varphi, v \rangle - (\boldsymbol{u}\varphi, \nabla v) + (\nabla \mu, \nabla v) = 0, \quad \forall\, v \in V, \tag{8.1.1}$$

for almost every $t \in (0,T)$, where

$$\boldsymbol{u} = -\nabla \pi + \mu \nabla \varphi, \quad \mu = \Psi'(\varphi) - J * \varphi,$$

for almost every $(x,t) \in \Omega \times (0,T)$ and satisfies $\varphi(0,\cdot) = \varphi_0$ a.e. in Ω.

Remark 8.1.2. We deduce from the assumptions $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$ that the class of admissible initial conditions consist of $\varphi_0 \in L^\infty(\Omega)$ such that $|\varphi_0(x)| \leq 1$, for almost every $x \in \Omega$. However, they cannot be pure phases, namely $\varphi_0 \not\equiv \pm 1$, due to the restriction on the total mass. In addition, concerning the initial condition, note that $\varphi \in \mathcal{C}([0,T],H)$. Also, any weak solution satisfies the mass conservation property, namely

$$\overline{\varphi}(t) = \overline{\varphi}_0, \quad \forall\, t \geq 0.$$

Remark 8.1.3. On account of the regularity satisfied by φ, any weak solution fulfils the identity

$$\mu \nabla \varphi = \nabla \Big(\Psi(\varphi) - (J * \varphi)\varphi \Big) + (\nabla J * \varphi)\varphi,$$

for almost every $(x,t) \in \Omega \times (0,T)$. Thus, the Darcy's law is equivalent to

$$\boldsymbol{u} = -\nabla \pi^* + (\nabla J * \varphi)\varphi, \quad \pi^* = \pi - \Psi(\varphi) + (J * \varphi)\varphi, \qquad \text{a.e. in } \Omega \times (0,T).$$

Its weak formulation becomes

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}x = \int_\Omega (\nabla J * \varphi)\varphi \cdot \boldsymbol{v} \,\mathrm{d}x, \quad \text{a.e. in } (0,T), \tag{8.1.2}$$

for any $\boldsymbol{v} \in \mathbf{H}_\sigma$. In particular, the modified pressure $\pi^* \in L^2(0,T;V_0)$. However, since the product $\mu \nabla \varphi$ is bounded in $L^1(0,T;\mathbf{W}^{1,\frac{3}{2}}(\Omega))$, the original pressure π belongs to $L^1(0,T;\mathbf{W}^{1,\frac{3}{2}}(\Omega))$.

Our first result regarding the existence of a weak solution is given by

Theorem 8.1.4. *Let φ_0 be a measurable function with $\psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Then, for any $T > 0$, there exists at least a weak solution $(\boldsymbol{u}, \pi, \varphi)$ to problem* (8.0.1)-(8.0.2) *on $[0, T]$ satisfying*

$$\sup_{t \geq 0} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1. \tag{8.1.3}$$

In addition, assuming that $|\overline{\varphi}_0| \leq m$ for some $m \in [0, 1)$, there exists $C = C(m)$, independent of the initial datum, such that

$$\|\boldsymbol{u}\|_{L^2(t,t+1;\mathbf{H}_\sigma)} + \|\mu\|_{L^2(t,t+1;V)} + \|\varphi\|_{L^2(t,t+1;V)} \leq C, \quad \forall\, t \geq 0. \tag{8.1.4}$$

A solution in the sense of Definition 8.1.1 will be constructed through several approximating problems. We introduce an artificial viscosity $-\varepsilon\Delta\boldsymbol{u}$, with $\varepsilon > 0$, in the Darcy's equation and we replace the singular potential with the family of regular ones Ψ_λ, defined in Section 3.2. In this framework, the existence of a pair $(\boldsymbol{u}^{\varepsilon,\lambda}, \varphi^{\varepsilon,\lambda})$ is carried out via the standard Galerkin scheme (the pressure will be recovered at the end of this argument). Then we aim to derive some uniform estimates with respect to the approximation parameters λ and ε. After that, it will be convenient to pass to the limit as λ which goes to 0, with $\varepsilon > 0$ fixed. In such a way, the velocity field is bounded in $L^2(0, T; \mathbf{V}_\sigma)$ at this stage. This facilitates the goal of finding an estimate for $\partial_t\varphi^{\varepsilon,\lambda}$. Next, taking advantage of the uniform bound in $L^\infty(\Omega \times (0, T))$ of φ^ε, we recover the limit system via compactness letting ε go to 0.

Proof. The proof will be divided into five steps.

1. A two levels approximation problem. For any given $\varepsilon > 0$ and $\lambda > 0$, we consider the nonlocal Brinkman–Cahn–Hilliard system (see [20] and [47] and references therein)

$$\begin{cases} -\varepsilon\Delta\boldsymbol{u} + \boldsymbol{u} = -\nabla\pi + \mu\nabla\varphi, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi - \Delta\mu = 0, \\ \mu = \Psi_\lambda'(\varphi) - J * \varphi, \end{cases} \qquad \text{in } \Omega \times (0, T), \tag{8.1.5}$$

subject to the following boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = \frac{\partial\mu}{\partial\boldsymbol{n}} = 0, & \text{on } \partial\Omega \times (0, T), \\ \varphi(\cdot, 0) = \varphi_0, & \text{in } \Omega. \end{cases} \tag{8.1.6}$$

The family of regular functions $\Psi_\lambda : \mathbb{R} \to \mathbb{R}$ satisfies (cf. Section 3.2)

(i) for any $0 < \overline{\lambda} \leq 1$, there exists $C > 0$ such that

$$\Psi_\lambda(s) \geq \frac{1}{4\overline{\lambda}}s^2 - C, \quad \forall\, s \in \mathbb{R}, \forall\, \lambda \in (0, \overline{\lambda}]; \tag{8.1.7}$$

(ii) Ψ_λ is convex with

$$\Psi_\lambda''(s) \geq \frac{\alpha}{1 + \alpha}, \quad \forall\, s \in \mathbb{R}; \tag{8.1.8}$$

(iii) Ψ_λ' is Lipschitz on $\mathbb{R}$ with constant $\frac{1}{\lambda}$;

(iv) $\Psi_\lambda(s) \nearrow \Psi(s)$ and $|\Psi'_\lambda(s)| \nearrow |\Psi'(s)|$ for every $s \in \mathbb{R}$ as $\lambda \to 0$ and, in addition, Ψ'_λ converges uniformly to Ψ' on any interval $[a,b] \subset (-1,1)$;

(v) $\Psi_\lambda(0) = \Psi'_\lambda(0) = 0$, for all $\lambda > 0$.

As previously anticipated, arguing as in [47] and Chapter 7, one can prove the existence of a global weak solution to problem (8.1.5)-(8.1.6) via a Galerkin scheme, by exploiting standard energy estimates and then passing to the limit in the usual way. More precisely, given an initial datum $\varphi_0 \in H$, for any $T > 0$, there exists a pair $(\boldsymbol{u}, \varphi)$ such that

$$\begin{aligned}
&\boldsymbol{u} \in L^2(0,T;\mathbf{V}_\sigma),\\
&\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V), \quad \partial_t\varphi \in L^{\frac{4}{3}}(0,T;V'),\\
&\mu = \Psi'_\lambda(\varphi) - J * \varphi \in L^2(0,T;V),
\end{aligned}$$

and satisfies (see Remark 8.1.3)

$$\varepsilon(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}) = ((\nabla J * \varphi)\varphi, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma, \tag{8.1.9}$$

$$\langle \partial_t \varphi, v\rangle - (\boldsymbol{u}\varphi, \nabla v) + (\nabla\mu, \nabla v) = 0, \quad \forall\, v \in V, \tag{8.1.10}$$

for almost every $t \in (0,T)$. Furthermore, introducing the regularized free energy

$$\mathcal{E}_H^\lambda(\psi) = \int_\Omega \Psi_\lambda(\psi(x))\,\mathrm{d}x - \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\psi(x)\psi(y)\,\mathrm{d}x\mathrm{d}y,$$

the energy inequality holds

$$\mathcal{E}_H^\lambda(\varphi(t)) + \int_0^t \varepsilon\|\nabla \boldsymbol{u}(\tau)\|^2 + \|\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2\,\mathrm{d}\tau \le \mathcal{E}_H^\lambda(\varphi_0), \tag{8.1.11}$$

for almost every $t \ge 0$.

2. A priori bounds based on the energy estimate. We consider an admissible initial datum such that φ_0 is measurable with $\Psi(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. According to the previous stage, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a pair $(\boldsymbol{u}^{\varepsilon,\lambda}, \varphi^{\varepsilon,\lambda})$ satisfying the weak formulation (8.1.9)-(8.1.10) and the energy inequality (8.1.11). Our next goal is to show some a priori uniform bounds with respect to ε and λ.
By virtue of the properties of Ψ_λ, we deduce that $\mathcal{E}_H^\lambda(\varphi_0) \le \mathcal{E}_H(\varphi_0)$. In light of (8.1.11), we find

$$\mathcal{E}_H^\lambda(\varphi^{\varepsilon,\lambda}(t)) + \int_0^t \varepsilon\|\boldsymbol{u}^{\varepsilon,\lambda}(\tau)\|_{\mathbf{V}_\sigma}^2 + \|\boldsymbol{u}^{\varepsilon,\lambda}(\tau)\|^2 + \|\nabla\mu^{\varepsilon,\lambda}(\tau)\|^2\,\mathrm{d}\tau \le \mathcal{E}_H(\varphi_0) \tag{8.1.12}$$

for almost every $t \in [0,T]$. We notice that, as a consequence of property (i), we have in particular

$$\begin{aligned}
\|\varphi^{\varepsilon,\lambda}\|_{L^\infty(0,T;H)}^2 + \int_0^T \varepsilon\|\boldsymbol{u}^{\varepsilon,\lambda}(\tau)\|_{\mathbf{V}_\sigma}^2 + \|\boldsymbol{u}^{\varepsilon,\lambda}(\tau)\|^2 + \|\nabla\mu^{\varepsilon,\lambda}(\tau)\|^2\,\mathrm{d}\tau&\\
\le \mathcal{E}_H(\varphi_0) + C,&
\end{aligned} \tag{8.1.13}$$

where C is independent of ε and λ. Taking the gradient of $\mu^{\varepsilon,\lambda}$ and testing by $\nabla\varphi^{\varepsilon,\lambda}$, we also obtain

$$(\Psi''_\lambda(\varphi^{\varepsilon,\lambda})\nabla\varphi^{\varepsilon,\lambda}, \nabla\varphi^{\varepsilon,\lambda}) = (\nabla\mu^{\varepsilon,\lambda}, \nabla\varphi^{\varepsilon,\lambda}) + (\nabla J * \varphi^{\varepsilon,\lambda}, \nabla\varphi^{\varepsilon,\lambda}).$$

By property (ii) and the Young inequality for convolution product, we reach

$$\|\nabla\varphi^{\varepsilon,\lambda}\|^2 \le C\|\nabla\mu^{\varepsilon,\lambda}\|^2 + C\|\varphi^{\varepsilon,\lambda}\|^2.$$

Thus, integrating in time and using (8.1.12)-(8.1.13), we arrive at

$$\int_0^T \|\nabla\varphi^{\varepsilon,\lambda}(\tau)\|^2 \,\mathrm{d}\tau \le C(1+T)\big(1+\mathcal{E}_H(\varphi_0)\big). \tag{8.1.14}$$

We now prove a uniform estimate of $\mu^{\varepsilon,\lambda}$ in V. It is sufficient to find a control of the total mass $\overline{\mu}^{\varepsilon,\lambda}$, that is

$$\overline{\mu}^{\varepsilon,\lambda} = \frac{1}{|\Omega|}\int_\Omega \Psi'_\lambda(\varphi^{\varepsilon,\lambda})\,\mathrm{d}x - \frac{1}{|\Omega|}\int_\Omega J * \varphi^{\varepsilon,\lambda}\,\mathrm{d}x.$$

Thanks to the monotonicity of Ψ'_λ, it is possible to show (see [63] and Chapter 7) that

$$\|\Psi'_\lambda(\varphi^{\varepsilon,\lambda})\|_{L^1(\Omega)} \le C\int_\Omega (\varphi^{\varepsilon,\lambda} - \overline{\varphi}_0)\Psi'_\lambda(\varphi^{\varepsilon,\lambda})\,\mathrm{d}x + C, \tag{8.1.15}$$

where C depends on $\overline{\varphi}_0$ and Ψ but is independent of ε and λ. Then, testing $\mu^{\varepsilon,\lambda}$ by $\varphi^{\varepsilon,\lambda} - \overline{\varphi}_0$ yields

$$\int_\Omega (\varphi^{\varepsilon,\lambda} - \overline{\varphi}_0)\Psi'_\lambda(\varphi^{\varepsilon,\lambda})\,\mathrm{d}x \le C\|\nabla\mu^{\varepsilon,\lambda}\|\|\varphi^{\varepsilon,\lambda} - \overline{\varphi}_0\| + C\|\varphi^{\varepsilon,\lambda}\|^2 + C$$

Collecting the above estimates and using (8.1.13), we find

$$\int_0^T \|\Psi'_\lambda(\varphi^{\varepsilon,\lambda})(\tau)\|^2_{L^1(\Omega)}\,\mathrm{d}\tau \le C(1+T)\big(1+\mathcal{E}_H(\varphi_0)\big)^2, \tag{8.1.16}$$

which, in turn, implies

$$\int_0^T \|\mu^{\varepsilon,\lambda}(\tau)\|^2_V\,\mathrm{d}\tau \le C(1+T)\big(1+\mathcal{E}_H(\varphi_0)\big)^2. \tag{8.1.17}$$

Observe that all the above estimates are independent of ε and λ.

3. The BCH system with singular potential: the limit $\lambda \to 0^+$. Our goal is to perform the limit $\lambda \to 0^+$. To this end, we need to derive a uniform control of $\partial_t\varphi^{\varepsilon,\lambda}$. By comparison, using Sobolev embedding and (3.1.11) with $p = 3$, we have

$$\begin{aligned}\|\partial_t\varphi^{\varepsilon,\lambda}\|_{V'} &\le \|\nabla\mu^{\varepsilon,\lambda}\| + \|\boldsymbol{u}^{\varepsilon,\lambda}\|_{\mathbf{L}^6(\Omega)}\|\varphi^{\varepsilon,\lambda}\|_{L^3(\Omega)}\\ &\le \|\nabla\mu^{\varepsilon,\lambda}\| + C\|\boldsymbol{u}^{\varepsilon,\lambda}\|_{\mathbf{V}_\sigma}\|\varphi^{\varepsilon,\lambda}\|^{\frac12}\|\varphi^{\varepsilon,\lambda}\|^{\frac12}_V.\end{aligned}$$

Hence, we get

$$
\begin{aligned}
&\int_0^T \|\partial_t \varphi^{\varepsilon,\lambda}(\tau)\|_{V'}^{\frac{4}{3}} \,\mathrm{d}\tau \\
&\leq C \int_0^T (1 + \|\nabla \mu^{\varepsilon,\lambda}\|^2) \,\mathrm{d}\tau \\
&\quad + C\|\varphi^{\varepsilon.\lambda}\|_{L^\infty(0,T;H)}^{\frac{2}{3}} \Big(\int_0^T \|\boldsymbol{u}^{\varepsilon,\lambda}(\tau)\|_{\mathbf{V}_\sigma}^2 \,\mathrm{d}\tau \Big)^{\frac{2}{3}} \Big(\int_0^T \|\varphi^{\varepsilon,\lambda}(\tau)\|_V^2 \,\mathrm{d}\tau \Big)^{\frac{1}{3}} \\
&\leq C(1+T)\big(1 + \mathcal{E}_H(\varphi_0)\big)^{\frac{4}{3}}.
\end{aligned}
$$

Here C is independent of λ but depends on ε. Collecting the above estimates, we deduce that

$$\|\boldsymbol{u}^{\varepsilon,\lambda}\|_{L^2(0,T;\mathbf{V}_\sigma)} \leq C, \tag{8.1.18}$$

$$\|\varphi^{\varepsilon,\lambda}\|_{L^\infty(0,T;H)} \leq C, \tag{8.1.19}$$

$$\|\varphi^{\varepsilon,\lambda}\|_{L^2(0,T;V)} \leq C, \tag{8.1.20}$$

$$\|\partial_t \varphi^{\varepsilon,\lambda}\|_{L^{\frac{4}{3}}(0,T;V')} \leq C, \tag{8.1.21}$$

$$\|\mu^{\varepsilon,\lambda}\|_{L^2(0,T;V)} \leq C, \tag{8.1.22}$$

where the constant C depends on the initial energy $\mathcal{E}_H(\varphi_0)$, $\overline{\varphi}_0$, the form of Ψ and ε, but is independent of λ. Thank to the uniform controls (8.1.18)-(8.1.22), letting $\lambda \to 0$, we deduce the following weak convergence results (up to subsequences)

$$\boldsymbol{u}^{\varepsilon,\lambda} \rightharpoonup \boldsymbol{u}^{\varepsilon}, \qquad \text{weakly in } L^2(0,T;\mathbf{V}_\sigma), \tag{8.1.23}$$

$$\varphi^{\varepsilon,\lambda} \rightharpoonup \varphi^{\varepsilon}, \qquad \text{weakly star in } L^\infty(0,T;H), \tag{8.1.24}$$

$$\varphi^{\varepsilon,\lambda} \rightharpoonup \varphi^{\varepsilon}, \qquad \text{weakly in } L^2(0,T;V), \tag{8.1.25}$$

$$\partial_t\varphi^{\varepsilon,\lambda} \rightharpoonup \partial_t\varphi^{\varepsilon}, \qquad \text{weakly in } L^{\frac{4}{3}}(0,T;V'), \tag{8.1.26}$$

$$\mu^{\varepsilon,\lambda} \rightharpoonup \mu^{\varepsilon}, \qquad \text{weakly in } L^2(0,T;V). \tag{8.1.27}$$

Besides, according to (8.1.20) and (8.1.21), an application of the Aubin–Lions compactness lemma entails

$$\varphi^{\varepsilon,\lambda} \to \varphi^{\varepsilon}, \qquad \text{strongly in } L^2(0,T;L^p(\Omega)), \tag{8.1.28}$$

for $p \in [2,6)$. In turn, this gives the pointwise convergence

$$\varphi^{\varepsilon,\lambda} \to \varphi^{\varepsilon}, \qquad \text{a.e. in } \Omega \times (0,T). \tag{8.1.29}$$

On account of the monotonicity of Ψ' and the uniform bound (8.1.16), it is possible to show (see, e.g., Chapters 5 and 6) that the limit function φ^ε fulfils

$$\varphi^\varepsilon \in L^\infty(\Omega \times (0,T)) \ \text{ such that } |\varphi^\varepsilon(x,t)| < 1 \ \text{ a.e. in } \Omega \times (0,T). \tag{8.1.30}$$

As a consequence, we deduce from property (iv) and the pointwise convergence (8.1.29) that

$$\Psi'_\lambda(\varphi^{\varepsilon,\lambda}) \to \Psi'(\varphi^\varepsilon), \qquad \text{a.e. in } \Omega \times (0,T).$$

By comparison, we also get

$$\|\Psi'_\lambda(\varphi^{\varepsilon,\lambda})\|_{L^2(0,T;V)} \le C,$$

with C independent of λ. Then, we infer from a standard argument that

$$\Psi'_\lambda(\varphi^{\varepsilon,\lambda}) \rightharpoonup \Psi'(\varphi^\varepsilon), \qquad \text{weakly in } L^2(0,T;V). \tag{8.1.31}$$

Regarding the product terms, it follows from (K), (8.1.24), (8.1.28) that

$$(\nabla J * \varphi^{\varepsilon,\lambda})\varphi^{\varepsilon,\lambda} \rightharpoonup (\nabla J * \varphi^\varepsilon)\varphi^\varepsilon, \qquad \text{weakly in } L^{\frac{4}{3}}(0,T;H),$$

and

$$\boldsymbol{u}^{\varepsilon,\lambda}\varphi^{\varepsilon,\lambda} \rightharpoonup \boldsymbol{u}^\varepsilon\varphi^\varepsilon, \qquad \text{weakly in } L^{\frac{4}{3}}(0,T;H).$$

Therefore, a passage to the limit in the weak formulation (8.1.32)-(8.1.33) yields

$$\varepsilon(\nabla \boldsymbol{u}^\varepsilon, \nabla \boldsymbol{v}) + (\boldsymbol{u}^\varepsilon, \boldsymbol{v}) = ((\nabla J * \varphi^\varepsilon)\varphi^\varepsilon, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbf{V}_\sigma, \tag{8.1.32}$$

$$\langle \partial_t \varphi^\varepsilon, v\rangle - (\boldsymbol{u}^\varepsilon\varphi^\varepsilon, \nabla v) + (\nabla \mu^\varepsilon, \nabla v) = 0, \quad \forall\, v \in V, \tag{8.1.33}$$

for almost every $t \in (0,T)$, where

$$\mu^\varepsilon = \Psi'(\varphi^\varepsilon) - J * \varphi^\varepsilon, \qquad \text{a.e. in } \Omega \times (0,T).$$

Furthermore, owing to the above convergences, we can pass to the limit into the energy inequality (8.1.12) obtaining

$$\mathcal{E}_H(\varphi^\varepsilon(t)) + \int_0^t \varepsilon\|\nabla \boldsymbol{u}^\varepsilon(\tau)\|^2 + \|\boldsymbol{u}^\varepsilon(\tau)\|^2 + \|\nabla\mu^\varepsilon(\tau)\|^2 \,\mathrm{d}\tau \le \mathcal{E}_H(\varphi_0), \tag{8.1.34}$$

for almost every $t \in [0,T]$, for any given $T > 0$.

4. The vanishing viscosity limit $\varepsilon \to 0^+$. First, according to (8.1.30) and (8.1.34), it follows immediately that

$$\|\varphi^\varepsilon\|_{L^\infty(\Omega\times(0,T))} \le 1,$$
$$\|\boldsymbol{u}^\varepsilon\|_{L^2(0,T;\mathbf{H}_\sigma)} \le C$$
$$\|\nabla\mu^\varepsilon\|_{L^2(0,T;H)} \le C.$$

Repeating line by line all the estimates performed in Step 2, we arrive at

$$\|\varphi^\varepsilon\|_{L^2(0,T;V)} \le C,$$
$$\|\mu^\varepsilon\|_{L^2(0,T;V)} \le C,$$
$$\|\Psi'(\varphi^\varepsilon)\|_{L^2(0,T;V)} \le C.$$

We need to find a control for $\partial_t\varphi$ that is independent of ε. On the other hand, thanks to the uniform L^∞-bound of φ^ε (cf. Remark 7.1.2), we have by comparison

$$\|\partial_t\varphi^\varepsilon\|_{V'} \le \|\nabla\mu^\varepsilon\| + \|\boldsymbol{u}^\varepsilon\|.$$

Hence, this leads to

$$\|\partial_t\varphi^\varepsilon\|_{L^2(0,T;V')} \le C.$$

Being all the above bounds independent of λ, the following weak convergence results hold (up to a subsequence)

$$\boldsymbol{u}^\varepsilon \rightharpoonup \boldsymbol{u}, \qquad \text{weakly in } L^2(0,T;\mathbf{H}_\sigma), \tag{8.1.35}$$

$$\varphi^\varepsilon \rightharpoonup \varphi, \qquad \text{weakly star in } L^\infty(\Omega\times(0,T)), \tag{8.1.36}$$

$$\varphi^\varepsilon \rightharpoonup \varphi, \qquad \text{weakly in } L^2(0,T;V), \tag{8.1.37}$$

$$\partial_t\varphi^\varepsilon \rightharpoonup \partial_t\varphi, \qquad \text{weakly in } L^2(0,T;V'), \tag{8.1.38}$$

$$\mu^\varepsilon \rightharpoonup \mu, \qquad \text{weakly in } L^2(0,T;V). \tag{8.1.39}$$

Thanks to (8.1.35)-(8.1.39), and applying a similar argument to the one employed in Step 3, we are able to pass to the limit as $\varepsilon \to 0$ in the weak formulation (8.1.32)-(8.1.33) and the limit pair $(\boldsymbol{u},\varphi)$ satisfies

$$(\boldsymbol{u},\boldsymbol{v}) = ((\nabla J*\varphi)\varphi,\boldsymbol{v}), \quad \forall\,\boldsymbol{v}\in\mathbf{V}_\sigma, \tag{8.1.40}$$

$$\langle\partial_t\varphi, v\rangle - (\boldsymbol{u}\varphi,\nabla v) + (\nabla\mu,\nabla v) = 0, \quad \forall\, v\in V, \tag{8.1.41}$$

for almost every $t\in(0,T)$, where

$$\mu = \Psi'(\varphi) - J*\varphi.$$

To conclude the proof, we need to comply with the weak formulation stated in Definition 8.1.4 by recovering the pressure π. In this regard, making use of a density argument, we observe that $\boldsymbol{u} = \Pi((\nabla J*\varphi)\varphi)$. Thus, in accordance with (3.5.2), there exists $\pi^*\in L^2(0,T;V_0)$ such that $\boldsymbol{u} = -\nabla\pi^* + (\nabla J*\varphi)\varphi$. Owing to Remark 8.1.3, we conclude that $\pi = \pi^* + \Psi(\varphi) - (J*\varphi)\varphi \in L^1(0,T;W^{1,\frac{3}{2}}(\Omega))$ and $\boldsymbol{u} = -\nabla\pi + \mu\nabla\varphi$. All identities here hold almost everywhere in $\Omega\times(0,T)$.

5. Uniform dissipative estimates. On account of the regularity (8.1.35)-(8.1.39), passing to the limit in (8.1.34) as ε goes to 0, we get

$$\mathcal{E}_H(\varphi(t)) + \int_0^t \|\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2\,\mathrm{d}\tau \le \mathcal{E}_H(\varphi_0),$$

for almost any $t\ge 0$. Since $\varphi\in\mathcal{C}([0,T],H)$, we deduce that $\int_\Omega F(\varphi(t))\,\mathrm{d}x$ is bounded for any $t\ge 0$. In turn, it easily follows

$$\sup_{t\ge 0}\|\varphi(t)\|_{L^\infty(\Omega)} \le 1.$$

Moreover, due to the assumptions on the admissible initial datum, it is clear that

$$\mathcal{E}_H(\varphi_0) \le C,$$

where C is a positive constant depending on Ψ, but not on φ_0. Thus, we deduce that

$$\int_0^t \|\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2\,\mathrm{d}\tau \le C, \quad \forall\, t\ge 0, \tag{8.1.42}$$

Arguing as in Step 2, we take the gradient of μ and multiply by $\nabla\varphi$. After standard computations, we find

$$\|\nabla\varphi\| \le C(1+\|\nabla\mu\|). \tag{8.1.43}$$

Then, aiming to estimate $\overline{\mu}$, we recall the useful inequality (see [118] and Chapter 7)

$$\|\Psi'(\varphi)\|_{L^1(\Omega)} \leq C \int_\Omega (\varphi - \overline{\varphi}_0)\Psi'(\varphi)\,\mathrm{d}x + C,$$

where C depends on m. Thus, testing μ by $\varphi - \overline{\varphi}$, we easily obtain

$$\|\Psi'(\varphi)\|_{L^1(\Omega)} \leq C(1 + \|\nabla\mu\|),$$

which, in turn, entails

$$\|\mu\|_V \leq C(1 + \|\nabla\mu\|). \tag{8.1.44}$$

Therefore, (8.1.4) follows from an integration in time of (8.1.42)-(8.1.44) in time on $(t, t+1)$. The proof is complete. □

Remark 8.1.5. A consequence of the proof of Theorem 8.1.4 is the existence of a global weak solution to the nonlocal Brinkman–Cahn–Hilliard system with singular potential.

Our next aim is to show the uniqueness of weak solutions. We remind that, in the local case, this is known only in two dimensions (see Chapter 5). To this purpose, we observe that the regularity of the velocity field $\boldsymbol{u} \in L^2(0,T;\mathbf{H}_\sigma)$ is not sufficient. Nonetheless, taking advantage of the equivalent formulation of the Darcy's law (see Remark 8.1.3) and using the global L^∞-bound of φ, we prove that the velocity field of any weak solution is indeed more regular.

Lemma 8.1.6. *Let $(\boldsymbol{u}, \pi, \varphi)$ be a weak solution in the sense of Definition 8.1.1. For any $p \in (1,\infty)$, there exists $C = C(p) > 0$, independent of the initial datum, such that*

$$\|\boldsymbol{u}\|_{L^\infty(0,\infty;\mathbf{L}^p(\Omega))} \leq C. \tag{8.1.45}$$

Futhermore, there exists $C > 0$, independent of the initial datum, such that

$$\|\boldsymbol{u}\|_{L^2(t,t+1;\mathbf{V})} \leq C, \quad \forall\, t \geq 0. \tag{8.1.46}$$

Proof. Observe first that, by Remark 8.1.3, up to a redefinition of the pressure, $\boldsymbol{u}$ solves the equation $\boldsymbol{u} = -\nabla\pi^* + (\nabla J * \varphi)\varphi$. Being φ essentially bounded, it is easily seen that

$$\|(\nabla J * \varphi)\varphi\|_{\mathbf{L}^\infty(\Omega\times(0,T))} \leq C, \quad \forall\, T > 0.$$

Thus, (8.1.45) follows from (3.5.1). Let us now apply the curl operator to the Darcy's law. We find (in a distributional sense)

$$\nabla \times \boldsymbol{u} = -(\nabla J * \varphi) \times \nabla\varphi. \tag{8.1.47}$$

Exploiting once more the L^∞-bound of φ, we reach

$$\|(\nabla J * \varphi) \times \nabla\varphi\| \leq C\|\nabla\varphi\|.$$

Then, owing to (8.1.4), we end up with

$$\int_t^{t+1} \|(\nabla \times \boldsymbol{u})(\tau)\|^2\,\mathrm{d}\tau \leq C.$$

By virtue of (3.5.3), the above inequality entails (8.1.46) and this yields the proof. □

We are now in a position to prove the following continuous dependence estimate.

Theorem 8.1.7. *Let* φ_{01} *and* φ_{02} *be two measurable initial data with* $\Psi(\varphi_{0i}) \in L^1(\Omega)$ *and* $|\overline{\varphi}_{0i}| < 1$*,* $i = 1, 2$*. Then, there exists* $C = C(T) > 0$ *such that*

$$\|\varphi_1(t) - \varphi_2(t)\|_{V'}^2 \le C\|\varphi_{01} - \varphi_{02}\|_{V'}^2 + C|\overline{\varphi}_{01} - \overline{\varphi}_{02}|, \tag{8.1.48}$$

for all $0 \le t \le T$*. In particular, the weak solution to* (8.0.1)-(8.0.2) *is unique.*

Proof. Let $(\boldsymbol{u}_1, \pi_1, \varphi_1)$ and $(\boldsymbol{u}_2, \pi_2, \varphi_2)$ be two weak solutions to problem (8.0.1)-(8.0.2) corresponding to φ_{01} and φ_{02}, respectively. Setting $\varphi = \varphi_1 - \varphi_2$, $\mu = \mu_1 - \mu_2$ and $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, we have (cf. Remark 8.1.3)

$$(\boldsymbol{u}, \boldsymbol{v}) = ((\nabla J * \varphi)\varphi_1, \boldsymbol{v}) + ((\nabla J * \varphi_2)\varphi, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbf{H}_\sigma, \tag{8.1.49}$$
$$\langle \partial_t \varphi, v\rangle + (\nabla \mu, \nabla v) = (\mathbf{u}\varphi_1, \nabla v) + (\mathbf{u}_2 \varphi, \nabla v), \quad \forall v \in V, \tag{8.1.50}$$

for almost every $t \in (0, T)$, where

$$\mu = \Psi(\varphi_1) - \Psi(\varphi_2) - J * \varphi.$$

Taking $v = 1$ in (8.1.50) we readily obtain that $\overline{\varphi}(t) = \overline{\varphi}(0)$ for all $t \in [0, T]$. Let us now take $v = \mathcal{N}(\varphi - \overline{\varphi})$ in (8.1.50). By definition of $\mathcal{N}$ and using the chain rule (3.3.5), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \overline{\varphi}\|_{V_0'}^2 + (\mu, \varphi - \overline{\varphi}) = \mathcal{I}_1 + \mathcal{I}_2, \tag{8.1.51}$$

where

$$\mathcal{I}_1 = (\boldsymbol{u}\varphi_1, \nabla \mathcal{N}(\varphi - \overline{\varphi})), \quad \mathcal{I}_2 = (\boldsymbol{u}_2\varphi, \nabla \mathcal{N}(\varphi - \overline{\varphi})).$$

Thanks to (H), we have

$$(\Psi'(\varphi_1) - \Psi'(\varphi_2), \varphi) \ge \alpha\|\varphi\|^2, \tag{8.1.52}$$

On the other hand, recalling the conservation of the total mass, note that

$$(\Psi'(\varphi_1) - \Psi'(\varphi_2), \overline{\varphi}) \le |\overline{\varphi}(0)|\Big(\|\Psi'(\varphi_1)\|_{L^1(\Omega)} + \|\Psi'(\varphi_2)\|_{L^1(\Omega)}\Big). \tag{8.1.53}$$

Moreover, recalling once more the definition of $\mathcal{N}$, we deduce that

$$\begin{aligned}(J * \varphi,\, \varphi - \overline{\varphi}) &= (\nabla J * \varphi,\, \nabla \mathcal{N}(\varphi - \overline{\varphi}))\\ &\le \frac{\alpha}{2}\|\varphi\|^2 + C\|\varphi\|_{V_0'}^2. \end{aligned} \tag{8.1.54}$$

Therefore, we infer

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2 + \frac{\alpha}{2}\|\varphi\|^2 \le \mathcal{I}_1 + \mathcal{I}_2 + C\|\varphi\|_*^2 + |\overline{\varphi}(0)|\mathcal{W}, \tag{8.1.55}$$

where

$$\mathcal{W} = C\Big(\|\Psi'(\varphi_1)\|_{L^1(\Omega)} + \|\Psi'(\varphi_2)\|_{L^1(\Omega)}\Big).$$

Let us proceed to estimate the terms $\mathcal{I}_i$, $i = 1, 2$. We have

$$\mathcal{I}_1 \le \|\boldsymbol{u}\|\|\varphi - \overline{\varphi}\|_{V_0'}$$

In order to find a control of $\boldsymbol{u}$ in term of φ, we take $\boldsymbol{v}=\boldsymbol{u}$ in (8.1.49) getting

$$\|\boldsymbol{u}\|^2 = ((\nabla J*\varphi)\varphi_1,\boldsymbol{u}) + ((\nabla J*\varphi_2)\varphi,\boldsymbol{u}).$$

After standard computations, we obtain

$$\|\boldsymbol{u}\|^2 \leq \Big(\|\varphi_1\|_{L^\infty(\Omega)} + \|\varphi_2\|_{L^\infty(\Omega)}\Big)\|\nabla J\|_{\mathbf{L}^1(\Omega)}\|\boldsymbol{u}\|\|\varphi\|,$$

Hence, we arrive at

$$\|\boldsymbol{u}\| \leq C\|\varphi\|,$$

which, in turn, gives

$$\mathcal{I}_1 \leq \frac{\alpha}{8}\|\varphi\|^2 + C\|\varphi\|_*^2.$$

Regarding $\mathcal{I}_2$, by using (3.1.11) with $p=3$, (3.3.6) and (8.1.45), we find the control

$$\begin{aligned}\mathcal{I}_2 &\leq \|\boldsymbol{u}_2\|_{\mathbf{L}^6(\Omega)}\|\varphi\|\|\nabla\mathcal{N}(\varphi-\overline{\varphi})\|_{\mathbf{L}^3(\Omega)}\\ &\leq C\|\boldsymbol{u}_2\|_{\mathbf{L}^6(\Omega)}\|\varphi\|\|\varphi-\overline{\varphi}\|^{\frac12}\|\varphi-\overline{\varphi}\|_{V_0'}^{\frac12}\\ &\leq \frac{\alpha}{8}\|\varphi\|^2 + C\|\varphi\|_*^2.\end{aligned}$$

Combining all the previous estimates, we end up with the differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2 + \frac{\alpha}{4}\|\varphi\|^2 \leq C\|\varphi\|_*^2 + |\overline{\varphi}(0)|\mathcal{W}.$$

Therefore, taking in account that $\mathcal{W}\in L^1(0,T)$, an application of the Gronwall lemma yields

$$\|\varphi_1(t)-\varphi_2(t)\|_*^2 \leq C\Big(\|\varphi_{01}-\varphi_{02}\|_*^2 + |\overline{\varphi}_{01}-\overline{\varphi}_{02}|\Big), \quad \forall\, t\in[0,T].$$

By the equivalence of the norms, (8.1.48) immediately follows. The proof is complete. $\square$

We can also deduce the validity of the energy identity from Lemma 8.1.6. This identity will play a crucial role to study the longtime behavior (see Section 8.4).

Proposition 8.1.8. *Let φ_0 be a measurable function with $\Psi(\varphi_0)\in L^1(\Omega)$ and $|\overline{\varphi}_0|<1$. Then, the unique weak solution $(\boldsymbol{u},\pi,\varphi)$ to problem* (8.0.1)-(8.0.2) *satisfies the energy identity*

$$\mathcal{E}_H(\varphi(t)) + \int_s^t \|\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2\,\mathrm{d}\tau = \mathcal{E}(\varphi(s)), \quad \forall\, 0\leq s\leq t<\infty. \quad (8.1.56)$$

Proof. Let us introduce the modified energy functional $\mathcal{L}: H\to\mathbb{R}$ given by

$$\mathcal{L}(\psi) := \int_\Omega \Psi(\psi)\,\mathrm{d}x - \frac{1}{2}(J*\psi,\psi) + \frac{\kappa}{2}\|\psi\|^2.$$

By virtue of (K), taking $\kappa>0$ large enough, it is easily seen that $\mathcal{L}$ is proper, convex and lower semicontinuous. On account of the regularity $\varphi\in L^2(0,T;V)$, $\varphi_t\in L^2(0,T;V')$

and $\Psi'(\varphi) \in L^2(0,T;V)$, we learn from [40, Proposition 4.2] that $\mathcal{L}(\varphi(\cdot))$ is absolutely continuous on $[0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\varphi) = \langle \Psi'(\varphi) - J * \varphi + \kappa\varphi, \varphi_t\rangle, = \langle \mu, \varphi_t\rangle + \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2,$$

for almost any $t \in (0,T)$. Taking $v = \mu$ in (8.1.1) and summing up to the above equality, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_H(\varphi) - (\boldsymbol{u}\varphi, \nabla\mu) + \|\nabla\mu\|^2 = 0.$$

Then, testing the Darcy's law by $\boldsymbol{u}$, we have

$$\|\boldsymbol{u}\|^2 = (\mu\nabla\varphi, \boldsymbol{u}). \tag{8.1.57}$$

Note that the above test by $\boldsymbol{u}$ is well defined due to Lemma 8.1.6 (see, in particular, (8.1.45)). By the classical result on the product rule in Sobolev spaces, we can rewrite (8.1.57) as

$$\|\boldsymbol{u}\|^2 = -(\varphi\nabla\mu, \boldsymbol{u}). \tag{8.1.58}$$

Thus, collecting the above equalities, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_H(\varphi) + \|\boldsymbol{u}\|^2 + \|\nabla\mu\|^2 = 0. \tag{8.1.59}$$

An integration on (s,t), $0 \le s < t$, entails the energy identity (8.1.56). □

8.2 Strong Solutions and Regularity Properties

In this section we prove the existence of the (unique) strong solution to problem (8.0.1)-(8.0.2) under a natural assumption on the initial datum. This will be achieved via a priori higher order estimates. This result (together with the uniqueness of weak solutions) in the three dimensional case points out the gap between the local and the nonlocal versions of the Hele–Shaw–Cahn–Hilliard model with singular potential. Indeed, in the former case, we remind that the existence of a global in time strong solution has been proven in Chapter 5 only for initial data close to local minimizers of the Ginzburg-Landau free-energy. On account of the parabolic dissipative nature of the system, we also show the time regularization of weak solutions.

Theorem 8.2.1. *Let φ_0 be a measurable function with $\Psi(\varphi_0) \in L^1(\Omega)$, $|\overline{\varphi}_0| < 1$ and $\nabla\Psi'(\varphi_0) \in H$. Then, the weak solution is a strong solution to problem* (8.0.1)-(8.0.2) *on $[0,T]$ such that*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;\mathbf{V}) \cap L^{\frac{8}{d}}(0,T;\mathbf{W}^{1,4}(\Omega)) \cap L^2(0,T;\mathbf{W}^{1,p}(\Omega)),\\
&\pi \in L^\infty(0,T;W^{1,\frac{3}{2}}(\Omega)) \cap L^2(0,T;V_0),\\
&\varphi \in L^\infty(0,T;V) \cap L^{\frac{8}{d}}(0,T;W^{1,4}(\Omega)) \cap L^2(0,T;W^{1,p}(\Omega)),\\
&\partial_t\varphi \in L^\infty(0,T;V') \cap L^2(0,T;H),\\
&\Psi'(\varphi) \in L^\infty(0,T;V),
\end{aligned}$$

for any $4 < p < \infty$ if $d = 2$ or $4 < p \le 6$ if $d = 3$.

Proof. Let $(\boldsymbol{u}, \pi, \varphi)$ be the unique global in time weak solution to problem (8.0.1)-(8.0.2). Thanks to the additional assumption on the initial datum, we aim to show higher order regularity properties. We divide the proof into three steps.

1. Smoothing effect on time derivatives. For any $h > 0$, let us introduce the difference quotient of a function v by

$$\partial_t^h v = \frac{1}{h}\Big(v(t+h) - v(t)\Big).$$

We first consider the Darcy's equation at two different times t and $t+h$. By subtracting them, it is evident that (cf. Remark 8.1.3)

$$(\partial_t^h \boldsymbol{u}, \boldsymbol{v}) = ((\nabla J * \partial_t^h \varphi)\varphi(t+h), \boldsymbol{v}) + ((\nabla J * \varphi)\partial_t^h \varphi, \boldsymbol{v}),$$

for all $\boldsymbol{v} \in \mathbf{H}_\sigma$ and for almost any $t > 0$. Choosing $\boldsymbol{v} = \partial_t^h \boldsymbol{u}$, we deduce that

$$\|\partial_t^h \boldsymbol{u}\|^2 = ((\nabla J * \partial_t^h \varphi)\varphi(t+h), \partial_t^h \boldsymbol{u}) + ((\nabla J * \varphi)\partial_t^h \varphi, \partial_t^h \boldsymbol{u}),$$

By (K) and (8.1.3), we have

$$\begin{aligned}
((\nabla J * \partial_t^h \varphi)\varphi(t+h), \partial_t^h \boldsymbol{u}) &+ ((\nabla J * \varphi)\partial_t^h \varphi, \partial_t^h \boldsymbol{u}) \\
&\leq \Big(\|\varphi(t+h)\|_{L^\infty(\Omega)}\|\nabla J\|_{\mathbf{L}^1(\Omega)} + \|\nabla J * \varphi\|_{\mathbf{L}^\infty(\Omega)}\Big)\|\partial_t^h \varphi\|\|\partial_t^h \boldsymbol{u}\| \\
&\leq C\|\partial_t^h \varphi\|\|\partial_t^h \boldsymbol{u}\|,
\end{aligned}$$

which, in turn, gives

$$\|\partial_t^h \boldsymbol{u}\| \leq C\|\partial_t^h \varphi\|. \tag{8.2.1}$$

Subtracting now the weak formulation (8.1.1) evaluated at time t from the one at time $t+h$ we get

$$\langle \partial_t \partial_t^h \varphi, v\rangle + (\nabla \partial_t^h \mu, \nabla v) = (\varphi(t+h)\partial_t^h \boldsymbol{u}, \nabla v) + (\boldsymbol{u}\partial_t^h \varphi, \nabla v), \tag{8.2.2}$$

for every $v \in V$ and almost any $t \in (0,T)$. Taking $v = \mathcal{N}(\partial_t^h \varphi)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h \varphi\|_{V_0'}^2 + (\partial_t^h \mu, \partial_t^h \varphi) = \mathcal{J}_1 + \mathcal{J}_2, \tag{8.2.3}$$

where

$$\mathcal{J}_1 = (\varphi(t+h)\partial_t^h \boldsymbol{u}, \nabla \mathcal{N}(\partial_t^h \varphi)), \quad \mathcal{J}_2 = (\boldsymbol{u}\partial_t^h \varphi, \nabla \mathcal{N}(\partial_t^h \varphi)).$$

By virtue of (H) and (K), we have

$$\frac{1}{h}(\Psi'(\varphi(t+h)) - \Psi'(\varphi(t)), \partial_t^h \varphi) \geq \alpha\|\partial_t^h \varphi\|^2,$$

and

$$\begin{aligned}
(J * \partial_t^h \varphi, \partial_t^h \varphi) &= (\nabla J * \partial_t^h \varphi, \nabla \mathcal{N}\partial_t^h \varphi) \\
&\leq \frac{\alpha}{6}\|D^h \varphi\|^2 + C\|D^h \varphi\|_{V_0'}^2.
\end{aligned}$$

Let us control the terms $\mathcal{J}_i$, $i = 1, 2$. By (3.3.6), (3.1.11), (8.1.45) and (8.2.1), we find

$$\begin{aligned}\mathcal{J}_1 &\le C\|\partial_t^h \boldsymbol{u}\|\|\partial_t^h \varphi\|_{V_0'}\\ &\le \frac{\alpha}{6}\|\partial_t^h\varphi\|^2 + C\|\partial_t^h\varphi\|_*^2,\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_2 &\le \|\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)}\|\partial_t^h\varphi\|\|\nabla\mathcal{N}\partial_t^h\varphi\|_{\mathbf{L}^3(\Omega)}\\ &\le C\|\partial_t^h\varphi\|^{3/2}\|\partial_t^h\varphi\|_{V_0'}^{1/2}\\ &\le \frac{\alpha}{6}\|\partial_t^h\varphi\|^2 + C\|\partial_t^h\varphi\|_*^2.\end{aligned}$$

Therefore, collecting the above estimates, we arrive at the differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\varphi\|_*^2 + \frac{\alpha}{2}\|\partial_t^h\varphi\|^2 \le C\|\partial_t^h\varphi\|_*^2. \tag{8.2.4}$$

Thus, an application of the Gronwall lemma yields

$$\|\partial_t^h\varphi(t)\|_*^2 \le \|\partial_t^h\varphi(0)\|_*^2 e^{CT}, \quad \forall\, t \in [0,T]. \tag{8.2.5}$$

At this point, to deduce a global in time estimate we need to find a bound for $\|\partial_t^h\varphi(0)\|_*$. Taking advantage of (H, we observe that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi-\varphi_0\|_{V_0'}^2 &= \langle\partial_t\varphi, \mathcal{N}(\varphi-\varphi_0)\rangle\\ &= (\mu, \varphi-\varphi_0) + (\boldsymbol{u}\varphi, \nabla\mathcal{N}(\varphi-\varphi_0))\\ &= -(\Psi'(\varphi), \varphi-\varphi_0) + (J*\varphi, \varphi-\varphi_0) + (\boldsymbol{u}\varphi, \nabla\mathcal{N}(\varphi-\varphi_0))\\ &\le -(\nabla\Psi'(\varphi_0), \nabla\mathcal{N}(\varphi-\varphi_0)) + (\nabla J*\varphi, \nabla\mathcal{N}(\varphi-\varphi_0))\\ &\quad + (\boldsymbol{u}\varphi, \nabla\mathcal{N}(\varphi-\varphi_0))\\ &\le \big(C + \|\nabla\Psi'(\varphi_0)\|\big)\|\varphi-\varphi_0\|_{V_0'},\end{aligned}$$

for almost every $t \ge 0$. An integration in time leads to

$$\frac{1}{2}\|\varphi(t)-\varphi_0\|_{V_0'}^2 \le \big(C + \|\nabla\Psi'(\varphi_0)\|\big)\int_0^t \|\varphi(\tau)-\varphi_0\|_{V_0'}\,\mathrm{d}\tau, \quad \forall\, t \ge 0.$$

Accordingly, an application of the integral Gronwall lemma (see section 3.7) implies

$$\|\varphi(t)-\varphi_0\|_{V_0'} \le \big(C + \|\nabla\Psi'(\varphi_0)\|\big)t, \quad \forall\, t \ge 0.$$

Taking $t = h$, it follows that

$$\|\partial_t^h\varphi(0)\|_{V'} \le \big(C + \|\nabla\Psi'(\varphi_0)\|\big), \quad \forall\, h > 0. \tag{8.2.6}$$

Combining (8.2.5) with (8.2.6), we end up with

$$\|\partial_t^h\varphi(t)\|_{V'} \le C, \quad \forall\, t \in [0,T],$$

where C depends on φ_0 and T. Owing to (8.2.4), we also infer that

$$\|D^h\varphi\|_{L^2(0,T;H)} \le C. \tag{8.2.7}$$

Recalling that $\partial_t^h\varphi$ converges to φ_t weakly in $L^\infty(0,\infty;V')$ as $h\to 0$, we end up with

$$\|\partial_t\varphi\|_{L^\infty(0,T;V')}+\|\partial_t\varphi\|_{L^2(0,T;H)}\le C. \tag{8.2.8}$$

Besides, by (8.2.1), (8.2.7), we deduce that

$$\|\partial_t\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}_\sigma)}\le C. \tag{8.2.9}$$

2. $L^\infty-$**in time uniform estimates.** Our next concern is to establish global in time bounds on $\boldsymbol{u}$, p and φ. By (8.1.3), (8.1.44), (8.1.45), and (8.2.8), it is easily seen that

$$\|\mu\|_{L^\infty(0,T;V)}\le C. \tag{8.2.10}$$

Then, by virtue of (8.1.43), we get

$$\|\varphi\|_{L^\infty(0,T;V)}\le C. \tag{8.2.11}$$

In light of the latter bound, by using (3.5.3) and (8.1.47), we have

$$\|\boldsymbol{u}\|_{\mathbf{V}}\le C\big(1+\|\varphi\|_{L^\infty(\Omega)}\|\nabla\varphi\|\big), \tag{8.2.12}$$

which, in turn, entails

$$\|\boldsymbol{u}\|_{L^\infty(0,T;\mathbf{V})}\le C. \tag{8.2.13}$$

As a consequence, we obtain

$$\|\pi\|_{L^\infty(0,T;W^{1,\frac{3}{2}}(\Omega))}\le C.$$

3. Higher order estimates in space. We proceed by proving higher order regularity with respect to the space variables. In this regard, by (8.1.3) and (8.1.45), we observe that

$$\|\boldsymbol{u}\varphi\|_{L^\infty(0,T;\mathbf{L}^p(\Omega))}\le C,\quad \forall\, p\ge 2. \tag{8.2.14}$$

Then, interpreting (8.1.1) as the Neumann problem for μ, the regularity theory in $(W^{1,p})'$ (see [109] and Section 3.3) entails that, for any $p>1$, there exists $C=C(p)>0$ such that

$$\|\mu\|_{W^{1,p}(\Omega)}\le C\big(\|\varphi_t\|_{(W^{1,p'}(\Omega))'}+\|\boldsymbol{u}\cdot\nabla\varphi\|_{(W^{1,p'}(\Omega))'}+\|\mu\|_{(W^{1,p'}(\Omega))'}\big), \tag{8.2.15}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. In light of (8.2.10) and (8.2.14), we have

$$\begin{aligned}\|\mu\|_{W^{1,p}(\Omega)}&\le C\big(\|\varphi_t\|+\|\boldsymbol{u}\varphi\|_{\mathbf{L}^p(\Omega)}+\|\mu\|\big)\\&\le C\big(1+\|\varphi_t\|\big),\end{aligned}$$

for any $1<p<\infty$ if $d=2$ and $1<p\le 6$ if $d=3$. This gives

$$\|\mu\|_{L^2(0,T;W^{1,p}(\Omega))}\le C.$$

To recover further integrability on φ, we argue as in Chapter 7 (see also [65]). We first deduce by comparison that $F''(\varphi)\nabla\varphi\in L^2(0,T;\mathbf{L}^p(\Omega))$, for any $p\ge 1$. Then, exploiting (H), we reach

$$\|\nabla\varphi\|_{\mathbf{L}^p(\Omega)}\le C\big(1+\|\nabla\mu\|_{\mathbf{L}^p(\Omega)}\big), \tag{8.2.16}$$

which, in turn, entails

$$\|\varphi\|_{L^2(0,T;W^{1,p}(\Omega))} \le C. \tag{8.2.17}$$

Accordingly, we can improve the integrability of the convective term $\boldsymbol{u}\cdot\nabla\varphi$. By (8.1.45) and (8.2.17), we find

$$\|\boldsymbol{u}\cdot\nabla\varphi\|_{L^2(0,T;H)} \le C.$$

Due to this, the regularity theory of the Neumann problem in H yields

$$\|\mu\|_{L^2(0,T;H^2(\Omega))} \le C.$$

By (3.1.10) and (3.1.11) with $p = 4$ and the above estimates, we also learn that

$$\|\nabla\mu\|_{L^{\frac{8}{d}}(0,T;\mathbf{L}^4(\Omega))} + \|\nabla\varphi\|_{L^{\frac{8}{d}}(0,T;\mathbf{L}^4(\Omega))} \le C.$$

Finally, we consider again (8.1.47). It follows that, for any $1 < p < \infty$ if $d = 2$ or $1 < p \le 6$ if $d = 3$, there exists $C = C(p) > 0$ such that

$$\|\nabla\times\boldsymbol{u}\|_{L^{\frac{8}{d}}(0,T;\mathbf{L}^4(\Omega))} + \|\nabla\times\boldsymbol{u}\|_{L^2(0,T;\mathbf{L}^p(\Omega))} \le C$$

On account of [84, Theorem 3.5], there exists g such that $\nabla\times\mathrm{g} = \boldsymbol{u}$ and $-\Delta\mathrm{g} = \nabla\times\boldsymbol{u}$ in Ω, with $\mathrm{g}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. By the regularity theory for the Neumann problem, we obtain

$$\|\mathrm{g}\|_{L^{\frac{8}{d}}(0,T;\mathbf{W}^{2,4}(\Omega))} + \|\mathrm{g}\|_{L^2(0,T;\mathbf{W}^{2,p}(\Omega))} \le C.$$

Thus, we infer that

$$\|\boldsymbol{u}\|_{L^{\frac{8}{d}}(0,T;\mathbf{W}^{1,4}(\Omega))} + \|\boldsymbol{u}\|_{L^2(0,T;\mathbf{W}^{1,p}(\Omega))} \le C.$$

The further regularity of the pressure easily follows from (3.5.2) and the above estimates. This concludes the proof. □

We are now in a position to state that any weak solution is indeed a strong one on (σ,∞), for any $\sigma > 0$. To this aim, we consider a generic weak solution $(\boldsymbol{u},\pi,\varphi)$ departing from a measurable initial datum φ_0 such that $|\overline{\varphi}_0| \le m$, for a fixed $m \in [0,1)$. Accordingly, throughout this section, the generic positive constant C may depend on m, but will be independent of the initial datum.

Theorem 8.2.2. *Let* $(\boldsymbol{u},\pi,\varphi)$ *be a weak solution to problem* (8.0.1)-(8.0.2). *For any* $\sigma > 0$*, there exists* $C = C(\sigma) > 0$ *such that*

$$\|\partial_t\varphi\|_{L^\infty(\sigma,\infty;V')} + \|\mu\|_{L^\infty(\sigma,\infty;V)} \le C,$$

$$\sup_{t\ge\sigma}\|\boldsymbol{u}(t)\|_{\mathbf{V}} + \sup_{t\ge\sigma}\|\varphi(t)\|_V \le C,$$

$$\|\partial_t\boldsymbol{u}\|_{L^2(t,t+1;\mathbf{H}_\sigma)} + \|\partial_t\varphi\|_{L^2(t,t+1;H)} + \|\mu\|_{L^2(t,t+1;H^2(\Omega))} + \|\nabla\varphi\|_{L^{\frac{8}{d}}(0,T;\mathbf{L}^4(\Omega))} \le C,$$

for every $t \ge \sigma$*. Moreover, for any* $4 < p < \infty$ *if* $d = 2$ *and* $4 < p \le 6$ *if* $d = 3$ *and* $\sigma > 0$*, there exists* $C = C(\sigma,p)$ *such that*

$$\|\boldsymbol{u}\|_{L^2(t,t+1;\mathbf{W}^{1,p}(\Omega))} + \|\varphi\|_{L^2(t,t+1;W^{1,p}(\Omega))} + \|\mu\|_{L^2(t,t+1;W^{1,p}(\Omega))} \le C, \quad \forall\, t \ge \sigma.$$

The proof of Theorem 8.2.2 can be obtained by arguing as in Theorem 8.2.1. Indeed, recalling the useful inequality

$$\|D^h\varphi\|_{L^2(t,t+1;V')} \leq \|\varphi_t\|_{L^2(t,t+1+h;V')}, \quad \forall\, t \geq 0,$$

the only difference consists in applying the uniform Gronwall lemma (see Section 3.7) to the differential inequality (8.2.4). Then, all the desired estimates follow by repeating line by line the arguments employed in the above Steps 2 and 3. In particular, integrating on the time interval $(t, t+1)$, all the constants turn out to be independent of the time variable.

8.3 The Separation Property in Two Dimensions

In this section we address the quantitative property concerning the instantaneous and uniform in time separation from the pure phases. In other words, in the two dimensional case we can prove that the concentration parameter stays away from the singular values of the potential. Consequently, being the potential and its derivative globally bounded in L^∞-norm, we are also able to show a further regularization property.

Let us fix $m \in [0,1)$ and consider a weak solution $(\boldsymbol{u}, \pi, \varphi)$ such that $|\overline{\varphi}_0| \leq m$. In particular, we remind once more that the generic positive constant C may depend on m, but will be independent of the initial datum. Thanks to the regularity of $\boldsymbol{u}$ we can exploit the result proved in Chapter 7 in the present case. More precisely, we have

Theorem 8.3.1. *Let $d = 2$. Assume that $\Psi \in^3 (-1,1)$ satisfies (H.3) and (H.4). Then, for any $\sigma > 0$, there exists $\delta = \delta(\sigma, m) > 0$ such that*

$$\sup_{t \geq 3\sigma} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta.$$

Proof. The result follows by arguing as in the proof of Theorem 7.5.7. Indeed, the regularity for $\boldsymbol{u}$ given by (8.1.45) is enough for the estimates performed therein. □

Taking advantage of the above result we can prove further regularization properties, namely,

Theorem 8.3.2. *Let $d = 2$ and let the assumptions of Theorem 8.3.1 hold. Then, for any $\sigma > 0$ and $p \geq 1$, there exists $C = C(\sigma, p) > 0$ such that*

$$\|\partial_t \boldsymbol{u}\|_{L^\infty(4\sigma,\infty;\mathbf{H}_\sigma)} + \|\partial_t \varphi\|_{L^\infty(4\sigma,\infty;H)} \leq C,$$

and

$$\sup_{t \geq 4\sigma} \|\mu(t)\|_{H^2(\Omega)} + \sup_{t \geq 4\sigma} \|\varphi(t)\|_{W^{1,p}(\Omega)} \leq C.$$

In addition, assume that the assumption (K.1) holds. Then, there exists $C = C(\sigma)$ such that

$$\sup_{t \geq 4\sigma} \|\boldsymbol{u}(t)\|_{\mathbf{H}^2(\Omega)} + \sup_{t \geq 4\sigma} \|\varphi(t)\|_{H^2(\Omega)} \leq C.$$

Proof. Let us test (8.2.2) by $\partial_t^h \varphi$. We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h \varphi\|^2 + (\nabla \partial_t^h \mu, \nabla \partial_t^h \varphi) = \mathcal{H}_1 + \mathcal{H}_2,$$

where
$$\mathcal{H}_1 = (\varphi(t+h)\partial_t^h \boldsymbol{u}, \nabla \partial_t^h \varphi), \quad \mathcal{H}_2 = (\boldsymbol{u}\partial_t^h \varphi, \nabla \partial_t^h \varphi).$$
By (H), we first notice that
$$(\nabla \partial_t^h \mu, \nabla \partial_t^h \varphi) \geq \alpha \|\nabla \partial_t^h \varphi\|^2 - |(\nabla J * \partial_t^h \varphi, \nabla \partial_t^h \varphi)| - |(\nabla \varphi \partial_t^h \Psi''(\varphi), \nabla \partial_t^h \varphi)|.$$
It is evident that
$$|(\nabla J * \partial_t^h \varphi, \nabla \partial_t^h \varphi)| \leq \frac{\alpha}{8}\|\nabla \partial_t^h \varphi\|^2 + C\|\partial_t^h \varphi\|^2.$$
On the other hand, recalling that
$$D^h \Psi''(\varphi) = D^h \varphi \int_0^1 \Psi'''(s\varphi(t+h) + (1-s)\varphi(t))\,\mathrm{d}s,$$
from the assumptions on Ψ and exploiting (3.1.7) and Theorem 8.3.1, we infer
$$\begin{aligned}|(\nabla \varphi \partial_t^h \Psi''(\varphi), \nabla \partial_t^h \varphi)| &\leq C\|\nabla \varphi\|_{\mathbf{L}^4(\Omega)} \|\partial_t^h \varphi\|_{L^4(\Omega)} \|\nabla \partial_t^h \varphi\| \\ &\leq \frac{\alpha}{8}\|\nabla \partial_t^h \varphi\|^2 + C\|\nabla \varphi\|_{\mathbf{L}^4(\Omega)}^4 \|\partial_t^h \varphi\|^2.\end{aligned}$$
We proceed by estimating the right-hand side $\mathcal{H}_i$, $i=1,2$. By (8.1.3), (8.1.45) and (8.2.1), we obtain
$$\begin{aligned}\mathcal{H}_1 &\leq C\|\partial_t^h \varphi\| \|\nabla \partial_t^h \varphi\| \\ &\leq C\|\partial_t^h \varphi\|^2 + \frac{\alpha}{8}\|\nabla \partial_t^h \varphi\|^2,\end{aligned}$$
and
$$\begin{aligned}\mathcal{H}_2 &\leq C\|\boldsymbol{u}\|_{\mathbf{L}^4(\Omega)} \|\partial_t^h \varphi\|_{L^4(\Omega)} \|\nabla \partial_t^h \varphi\| \\ &\leq \frac{\alpha}{8}\|\nabla \partial_t^h \varphi\|^2 + C\|\partial_t^h \varphi\|^2.\end{aligned}$$
Combining all the previous estimates, we find the differential inequality
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h \varphi\|^2 + \frac{\alpha}{2}\|\nabla \partial_t^h \varphi\|^2 \leq C\big(1 + \|\nabla \varphi\|_{\mathbf{L}^4(\Omega)}^4\big)\|\partial_t^h \varphi\|^2. \tag{8.3.1}$$
Thanks to Theorem 8.2.2, an application of the uniform Gronwall lemma leads to
$$\|\partial_t^h \varphi(t)\| + \|\nabla \partial_t^h \varphi\|_{L^2(t,t+1;\mathbf{H})} \leq C, \quad \forall\, t \geq 4\sigma.$$
Here C is a positive constant depending on σ. A final passage to the limit as $h \to 0$, together with (8.2.1), entails
$$\|\partial_t \varphi\|_{L^\infty(4\sigma,\infty;H)} + \|\partial_t \boldsymbol{u}\|_{L^\infty(4\sigma,\infty;\mathbf{H}_\sigma)} \leq C \tag{8.3.2}$$
and
$$\|\partial_t \varphi\|_{L^2(t,t+1;V)} \leq C, \quad \forall\, t \geq 4\sigma.$$
Then, using (8.2.15) together with (8.2.14) and (8.3.2), we infer that, for any $p > 2$, there exists $C = C(p) > 0$ such that
$$\|\mu\|_{L^\infty(4\sigma,\infty;W^{1,p}(\Omega))} \leq C.$$

Thanks to (8.2.16), we end up with

$$\|\varphi\|_{L^\infty(4\sigma,\infty;W^{1,p}(\Omega))} \leq C. \tag{8.3.3}$$

Now, by (8.1.45) and (8.3.3), we get

$$\|\boldsymbol{u}\cdot\nabla\varphi\|_{L^\infty(4\sigma,\infty;H)} \leq C.$$

Thus, on account of (8.3.2), the regularity theory of the Neumann problem yields

$$\|\mu\|_{L^\infty(4\sigma,\infty;H^2(\Omega))} \leq C.$$

By comparison, we easily have

$$\|\Psi'(\varphi)\|_{L^\infty(4\sigma,\infty;H^2(\Omega))} \leq C.$$

Due to the validity of the separation property, by using the classical result on composition of functions in Sobolev spaces, we deduce that

$$\|\varphi\|_{L^\infty(4\sigma,\infty;H^2(\Omega))} \leq C.$$

Finally, by (8.1.47) and the above estimate, we reach

$$\|\nabla\times\boldsymbol{u}\|_{L^\infty(4\sigma,\infty;\mathbf{V})} \leq C.$$

which, in turn, gives

$$\|\boldsymbol{u}\|_{L^\infty(4\sigma,\infty;\mathbf{H}^2(\Omega))} \leq C.$$

In order to conclude the proof, we show that the estimates regarding $\boldsymbol{u}$, μ and φ holds for all $t \geq 4\sigma$. To this aim, it is enough to prove an $L^\infty(L^2)$-bound on $\partial_t\mu$. Indeed we have

$$\|\partial_t\mu\| \leq \|\Psi''(\varphi)\partial_t\varphi\| + \|J*\partial_t\varphi\| \leq \big(1+\|\Psi''(\varphi)\|_{L^\infty(\Omega)}\big)\|\partial_t\varphi\|.$$

Hence, by Theorem 8.3.1 and (8.3.2), we find

$$\|\partial_t\mu\|_{L^\infty(4\sigma,\infty;H)} \leq C.$$

As a consequence $\mu \in \mathcal{C}([4\sigma,T],H)$, for any $T \geq 4\sigma$. Thus, for any $\sigma > 0$, there exists $C = C(\sigma)$ such that

$$\sup_{t\geq 4\sigma}\|\mu(t)\|_{H^2(\Omega)} \leq C.$$

On account of (8.3.2), the same result holds for $\boldsymbol{u}$ and φ, thus concluding the proof. □

8.4 Longtime Behavior

In this section we provide a description on the asymptotic behavior of solutions as time goes to ∞. In the first part we define the semigroup map related to problem (8.0.1)-(8.0.2) and we show the existence of the global attractor. In the second part we prove that any weak solution does converge to a single equilibrium in dimension two.

The Infinite Dimensional Dynamical System

We define a semigroup on a suitable phase space as a consequence of Theorem 8.1.4. Indeed, for any $\varphi_0 \in L^\infty(\Omega)$ such that $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$ and $|\overline{\varphi}_0| < 1$, there exists a unique global in time weak solution $(\boldsymbol{u}, \pi, \varphi)$. Then, for any given $m \in [0,1)$, we define

$$\mathcal{V}_m = \Big\{ \varphi \in L^\infty(\Omega) : \|\varphi\|_{L^\infty(\Omega)} \leq 1 \text{ and } |\overline{\varphi}| = m \Big\},$$

and we equip it with the metric

$$\mathbf{d}(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|.$$

Then, for any $\varphi_0 \in \mathcal{V}_m$, we set

$$\varphi(t) := \mathcal{S}(t)\varphi_0,$$

φ being the unique global in time weak solution to (8.0.1)-(8.0.2). It is immediate to check that the one-parameter family of maps $\mathcal{S}(t)$ on $\mathcal{V}_m$ satisfies the semigroup properties (see [149]). Moreover, we also deduce that $t \mapsto S(t)\varphi_0 \in \mathcal{C}([0,\infty), \mathcal{V}_m)$, for every $\varphi_0 \in \mathcal{V}_m$. On account of Theorem 8.1.7, we have a continuous dependence estimate with respect to the initial data in a dual norm. Nevertheless, appealing to the instantaneous regularity, we are able to show the following property

Proposition 8.4.1. *For any $t \geq 0$, $\mathcal{S}(t) \in \mathcal{C}(\mathcal{V}_m, \mathcal{V}_m)$.*

Proof. The case $t = 0$ is trivial. We consider $t > 0$ and a sequence $\{\varphi_{0n}\} \subset \mathcal{V}_m$ such that $\mathbf{d}(\varphi_{0n}, \varphi_0) \to 0$, with $\varphi_0 \in \mathcal{V}_m$. Due to Theorem 8.2.2, we infer that

$$\|\varphi(t)\|_V + \|\varphi_n(t)\|_V \leq C.$$

Hence, by interpolation and using (8.1.48), we obtain

$$\begin{aligned} \mathbf{d}(\varphi_n(t), \varphi(t)) &\leq \|\varphi_n(t) - \varphi(t)\|_{V'}^{\frac{1}{2}} \|\varphi_n(t) - \varphi(t)\|_V^{\frac{1}{2}} \\ &\leq C\|\varphi_{0n} - \varphi_0\|_{V'}^{\frac{1}{2}} \\ &\leq C\mathbf{d}(\varphi_{0n}, \varphi_0)^{\frac{1}{2}}. \end{aligned}$$

The claim follows. □

The existence of the global attractor is given by

Theorem 8.4.2. *The dynamical system $(\mathcal{V}_m, \mathcal{S}(t))$ has a connected global attractor $\mathcal{A}$ that is bounded in V.*

Proof. Observe that, on account of Theorems 8.2.2, there exists a positive constant C, independent of the initial datum, such that

$$\sup_{t \geq 1} \|\varphi(t)\|_V \leq C.$$

This entails the existence of a compact absorbing set in $\mathcal{V}_m$. The existence of the global attractor is thus implied by [149, Theorem 1.1]. □

Remark 8.4.3. It is worth mentioning that the global attractor is more regular in the two dimensional case. Indeed, by Theorem 8.3.2, it follows that $\mathcal{A}$ is bounded in $H^2(\Omega)$. Furthermore, due to Theorem 8.3.1, the finite dimensionality of the global attractor and the existence of exponential attractors can also be proved. We refer the reader to Chapter 7 and references therein.

Convergence to Single Stationary State

We focus here on the longtime behavior of the single trajectory in two dimensions. Thanks to the validity of the separation property, Theorem 8.3.2 and the Sobolev compact embedding $H^2(\Omega) \hookrightarrow C^\beta$, for some $\beta > 0$, we are able to adapt the strategy devised for local Cahn–Hilliard type equations (see [1]) within the nonlocal framework.

Given $\varphi_0 \in \mathcal{V}_m$, let $(\boldsymbol{u}, \pi, \varphi)$ be the related unique global in time weak solution. We introduce the ω-limit set associated to φ_0 by

$$\omega(\varphi_0) = \big\{ \tilde{\varphi} \in \mathcal{V}_m : \exists\, t_n \to \infty \text{ such that } \varphi(t_n) \to \tilde{\varphi} \big\}.$$

On account of Theorems 8.2.2-8.3.2, there exists $C > 0$ such that

$$\|\varphi(t)\|_{V \cap C^\beta(\overline{\Omega})} \leq C, \quad \forall\, t \geq 1. \tag{8.4.1}$$

By the Sobolev compact embedding results, we deduce that the set $\omega(\varphi_0)$ is non-empty, compact, connected in $\mathcal{V}_m$ and, in particular,

$$\lim_{t \to \infty} \operatorname{dist}(\varphi(t), \omega(\varphi_0)) = 0, \quad \text{ in } L^\infty(\Omega). \tag{8.4.2}$$

We now proceed to characterize $\omega(\varphi_0)$. To this aim, we introduce the notion of equilibrium point (or stationary solution), that is, a function $\varphi_\infty \in V \cap \mathcal{V}_m$ satisfying that stationary nonlocal Cahn–Hilliard equation

$$\Psi'(\varphi_\infty) - J * \varphi_\infty = \mu_\infty, \quad \text{ in } \Omega, \tag{8.4.3}$$

where $\mu_\infty \in \mathbb{R}$ and $\overline{\varphi}_\infty = \overline{\varphi}_0 = m$. Moreover, $\boldsymbol{u}_\infty = 0$ and $\pi_\infty = \mu_\infty \varphi_\infty$. Besides, any stationary solution fulfils the separation property.

Lemma 8.4.4. *For any $\varphi_\infty \in V \cap \mathcal{H}_m$ satisfying* (8.4.3)*, there exists $\delta > 0$ such that*

$$\|\varphi_\infty\|_{L^\infty(\Omega)} \leq 1 - \delta.$$

Proof. Let us observe that

$$\|\Psi'(\varphi_\infty)\|_{L^\infty(\Omega)} \leq \mu_\infty + \|J\|_{L^1(\Omega)}.$$

Hence, $\Psi'(\varphi_\infty)$ is uniformly bounded. Due to the assumption (H), the conclusion follows. $\square$

Let us consider the set of all stationary points

$$\mathcal{L}_m = \Big\{ \varphi_\infty \in V \cap \mathcal{V}_m : \varphi_\infty \text{ satisfies (8.4.3)} \Big\}.$$

We remind that $\varphi \in \mathcal{C}([0,\infty), \mathcal{V}_m)$ and the energy $\mathcal{E}_H(\varphi)$ satisfies the energy equality (8.1.56). Due to this, we learn that $\mathcal{E}_H(\varphi)$ is a Lyapunov function. Thus, we infer that $\omega(\varphi_0)$ consists of stationary states, namely $\omega(\varphi_0) \subset \mathcal{L}_m$.

We conclude by applying the standard strategy in order to prove that $\omega(\varphi_0)$ is a singleton φ_∞. We report here the main tool to prove the convergence to equilibrium, that is the well-known Lojasiewicz–Simon inequality (see, e.g., [68])

Proposition 8.4.5. *Let $P_0 : H \to L_0^2$ be the projector operator. Assume that Ψ is real analytic in $(-1, 1)$, $\varphi \in V \cap L^\infty(\Omega)$ is such that $-1+\gamma \leq \varphi(x) \leq 1-\gamma$ for some $\gamma \in (0, 1)$ and for all $x \in \overline{\Omega}$ and $\varphi_\infty \in \omega(\varphi_0)$. Then, there exist $\theta \in (0, \frac{1}{2})$, $\eta > 0$ and a positive constant C such that*

$$|\mathcal{E}(\varphi) - \mathcal{E}(\varphi_\infty)|^{1-\theta} \leq C\|P_0(\Psi'(\varphi) - J * \varphi)\|_*, \tag{8.4.4}$$

whenever $\|\varphi - \varphi_\infty\| \leq \eta$.

The main result of this section is the following

Theorem 8.4.6. *Assume that Ψ is real analytic on $(-1, 1)$. Then, φ converges to an equilibrium φ_∞, namely,*

$$\lim_{t\to\infty} \|\varphi(t) - \varphi_\infty\|_{L^\infty(\Omega)} = 0, \tag{8.4.5}$$

where φ_∞ fulfills (8.4.3).

Proof. Thanks to the energy identity (8.1.56), it follows that $\mathcal{E}_H(\varphi(t))$ is non increasing, $\mathcal{E}_H(\varphi(t)) \geq \mathcal{E}_H(\varphi_\infty)$ and $\mathcal{E}_H(\varphi(t)) \to M$, where $M = \mathcal{E}_H(\varphi_\infty)$, for any $\varphi_\infty \in \omega(\varphi_0)$. Without loss of generality, we consider $\mathcal{E}_H(\varphi(t)) > \mathcal{E}_H(\varphi_\infty)$, for all $t \geq 0$. Otherwise, if there exists $\overline{t} > 0$ such that $\mathcal{E}_H(\varphi(t)) = \mathcal{E}_H(\varphi_\infty)$, it is evident that $\varphi(t) = \varphi(\overline{t})$, for all $t \geq \overline{t}$, and the claim follows. On the other hand, we fix $\theta \in (0, \frac{1}{2})$ and $\eta > 0$ given by Proposition 8.4.5. Via a contradiction argument, it is possible to show that there exists $t^* > 0$ such that $\|\varphi(t) - \varphi_\infty\| \leq \eta$, for all $t \geq t^*$ (see, e.g., [65]). Then, by Proposition 8.4.5, for any $t \geq t^*$, we have

$$\begin{aligned}\Big(\mathcal{E}_H(\varphi) - \mathcal{E}_H(\varphi_\infty)\Big)^{1-\theta} &\leq \Big(C\|P_0(\Psi'(\varphi) - J * \varphi)\|_*^{\frac{1}{1-\theta}}\Big)^{1-\theta} \\ &\leq C\|\nabla\mu\|.\end{aligned}$$

By using (8.1.56) together with the above inequalities, we deduce that

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{E}_H(\varphi) - \mathcal{E}_H(\varphi_\infty)\Big)^{\theta} &= -\theta\Big(\mathcal{E}_H(\varphi) - \mathcal{E}_H(\varphi_\infty)\Big)^{\theta-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_H(\varphi) \\ &\geq \frac{\theta\big(\|\boldsymbol{u}\|^2 + \|\nabla\mu\|^2\big)}{C\|\nabla\mu\|} \\ &\geq C\|\nabla\mu\|.\end{aligned}$$

An integration on the time interval (t^*, ∞), for t^* sufficiently large, leads to

$$\int_{t^*}^{\infty} \|\nabla\mu(\tau)\| \,\mathrm{d}\tau < \infty.$$

Also, in light of (8.1.3) and (8.1.58), we obtain

$$\int_{t^*}^{\infty} \|\boldsymbol{u}(\tau)\| \,\mathrm{d}\tau < \infty.$$

By comparison, we find

$$\int_{t^*}^{\infty} \|\partial_t\varphi(\tau)\|_{V'} \,\mathrm{d}\tau < \infty.$$

Thus, we conclude that $\varphi(t)$ converges in V' as t goes to ∞. Using the interpolation $\|u\|_{L^\infty} \leq C\|u\|_{V'}^{\beta}\|u\|_{C^\beta}^{1-\beta}$, for some $\beta \in (0, 1)$ and the uniform estimate (8.4.1), we deduce (8.4.5). □

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