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THE PROBLEM OF RANKING PLAYERS: FROM SEMIVALUES TO GAMES WITH ABSTENTION

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Preface

THE work presented in this thesis has been developed during my three years as a Ph.D student in Politecnico di Milano and during the months I spent, during my second year, in the Manresa School of the Universitat Politècnica de Catalunya.

The results included in chapter 2 are taken from G. Bernardi and R. Lucchetti (2015), “Generating Semivalues via Unanimity Games”, published in *Journal of Optimization Theory and Applications* and from G. Bernardi and J. Freixas (2017) “The Shapley value under the Felsenthal and Machover Bargaining Model” currently under revision.

Some of the results provided in chapter 3 are part of G. Bernardi (2017), “A New Axiomatization of the Banzhaf Index for Games with Abstention”, in *Group Decision and Negotiation* and of G. Bernardi and J. Freixas (2017) “An axiomatization for two power indices for (3,2)-simple games” currently under revision for a special issue after the SING13 meeting.

The work contained in chapter 5 is part of G. Bernardi, R. Lucchetti and S. Moretti (2017), “Ranking Objects from a Preference Relation over their Subsets”, in *Cahier du Lamsade* that is also submitted.

During these three years, I delivered several seminars within the GTAC group and the P.I.G.S. meetings in Milano to present the different results of my thesis. Moreover, I had the opportunity to present part of the work contained in chapter 3 and in chapter 4 at the 13^{th} European Meeting on Game Theory in Paris in the talk “Two power indices for games with abstention”.

Abstract

IN this thesis we analyse the problems of evaluating the power of players in different voting situations and of creating a ranking among them.

We start our analysis in the classical set of cooperative game theory. We present a theorem to characterize the family of semivalues, a class of solution concepts, by means of their behaviour on unanimity games and to establish a connection between semivalues and completely monotonic sequences. Then, we provide a new formula to compute the Shapley value due to a different interpretation of the value that is particularly meaningful in the voting context.

Secondly, we examine the model of games with abstention. We introduce some properties for the indices for games with abstention and generalize to this set some of the properties that have been provided in literature to characterize the corresponding power indices for simple games. We use these results to provide two different axiomatizations for the Banzhaf index for games with abstention and a characterization of the Shapley-Shubik index for games with abstention.

We then focus on multichoice cooperative games, an extension of the classical model of cooperative games, to describe situations in which players can have different levels of participation in the cooperation (or they have to vote among different alternatives). We analyse and compare the different models studied in literature and we define a new value for multichoice cooperative games in the spirit of the Shapley value. As a

consequence of our result, we provide an explicit formula to compute the Shapley-Shubik index for games with abstention.

In the last part of the thesis, we consider the problem of ranking players from a new perspective: we remove the structure of coalitional game and suppose that only an ordinal ranking among players is available. We present two functions that associate a ranking over the players, given a preference profile over the subsets formed by those players. We also provide an axiomatic characterization of these two functions by means of a set of axioms we introduce and discuss.

Riassunto

QUESTA tesi si occupa di studiare il problema di creare ordinamenti tra i giocatori, sia utilizzando gli indici di potere all'interno di un gioco cooperativo, sia affrontando il problema da un punto di vista più generale.

I primi risultati ottenuti sono nell'ambito classico della teoria dei giochi cooperativi. Viene presentato un teorema che caratterizza i *semivalues*, una famiglia di soluzioni per i giochi cooperativi, descrivendo il loro comportamento sui giochi di unanimità e stabilendo un collegamento tra i *semivalues* e le successioni completamente monotone. In seguito, viene presentata una nuova formula per calcolare l'indice di Shapley, che è dedotta da una diversa interpretazione di questo indice, particolarmente significativa nel contesto delle situazioni di voto.

Come secondo argomento, vengono esaminati i modelli di giochi con astensione. Vengono introdotti alcuni nuovi assiomi e generalizzate alcune delle proprietà presenti in letteratura per caratterizzare gli indici di potere. Questo permette di dimostrare due diverse assiomatizzazioni per l'indice di Banzhaf per i giochi con astensione e una caratterizzazione dell'indice di Shapley-Shubik per i giochi con astensione.

Successivamente, si sposta l'attenzione sul modello dei giochi *multichoice*, un ampliamento del modello classico di gioco cooperativo per permettere ai giocatori di cooperare a diversi livelli (o di votare tra diverse alternative). I diversi modelli presenti in letteratura vengono confrontati tra loro, dopodiché viene definito un nuovo valore, come con-

cetto di soluzione per i giochi *multichoice*, in linea con lo spirito della definizione indice di Shapley. Come conseguenza di questo risultato, si ottiene una formula per calcolare esplicitamente l'indice di Shapley-Shubik per i giochi con astensione.

Infine, nell'ultima parte della tesi, il problema di ordinare i giocatori è affrontato da una prospettiva diversa: la struttura di gioco cooperativo viene rimossa e si suppone che ci sia a disposizione solo un ordine qualitativo tra i giocatori. In questo contesto vengono definite due funzioni che creano un ordinamento dei giocatori a partire da un ordinamento dei sottoinsiemi dei giocatori stessi. Inoltre vengono introdotti e discussi alcune proprietà utilizzate successivamente per fornire una caratterizzazione assiomatica di queste due funzioni.

Contents

CHAPTER 1

Introduction

In many real-life situations that are studied in voting theory or, more generally, in social choice theory, it is necessary to rank individuals or objects. For instance this problem might arise when it is necessary to have a unifying preference profile over employees in a working group, or to evaluate the power of each voter in a decision making procedure or to give different marks to some students after a workshop...

A classical approach to this problem is provided by cooperative game theory; in particular, voting situations can be modelled as simple games, a particular family of games with transferable utility and power indices, that are values restricted to the family of simple games, provide a natural ranking among players.

From this classical approach many different questions may arise: is the use of a cooperative solution concept legitimate in the context of simple games? Are there any solution concepts more suitable than others? Why? Is it possible to use power indices also with different models of games, for instance in order to take into account the possibility of abstention in a voting procedure? Which indices can be generalized and what aspects of the decision process do they represent? Which different approach can be studied, besides cooperative game theory, to solve this problem?

In this thesis we analyse this topic and try to provide answers to some of these questions.

Cooperative theory with transferable utility was introduced in Von Neumann and Morgenstern (1945) to model situations in which players can cooperate, and reach different goals forming coalitions. A cooperative game with transferable utility is a function that assigns to each coalition, i.e. group of player, a number representing the worth of that coalition.

The Shapley value was introduced by Shapley (1953) as a solution concept for this class of games; the main idea of the value is to assign to each player a real number that represents his contribution to the game or what he is expecting from taking part to the cooperative game. The Shapley value was introduced as the unique function satisfying some reasonable axioms: linearity, anonymity, i.e. the value of a player does not depend on his name; null-player, i.e. the value for players that do not bring a contribution to any coalition should be zero; and efficiency, i.e. what players get all together is distributed among all of them.

Games in which the outcomes are only 0 or 1 are called *simple games* and are used as a model for binary voting situations: the idea is that players in favour of a proposal group together in coalitions, if a coalition is winning, i.e. it is formed by players that being in favour can pass the discussed bill, its value is 1, otherwise it is 0. Power indices are used to measure the power of each player in the decision-making procedure. Shapley and Shubik (1954) defined the Shapley-Shubik power index as a restriction to simple games of the Shapley value; the other popular power index in this context is the Banzhaf index, defined independently in Penrose (1946) and in Banzhaf (1964). The first axiomatic foundation of the Shapley-Shubik and the Banzhaf indices are due to Dubey (1975) and Dubey and Shapley (1979), respectively. These works introduce five axioms that are the correspondent for simple games of the classical axioms for cooperative games defined in Shapley (1953): null player, anonymity, transfer, efficiency for the Shapley-Shubik index and Banzhaf total power for the Banzhaf index. However, the interpretation of some of these axioms is not intuitively clear in the context of simple games; for instance: why should we impose efficiency, when we just want to compare the power of players? And, on the other hand, what is the theoretical relevance of the Banzhaf total power property, which is extremely tautological?

The discussion of the classical approach and the related axioms brought other ideas to define larger families of values, such as in Dubey and Weber (1977) and Weber (1988), or different indices more suitable for the voting context, that take into account other aspects of the voting procedure as in Deegan and Packel (1978), Johnston (1978), and Holler (1982). For a comparison among the different measures of power we refer to Felsenthal and Machover (2005), Laruelle and Valenciano (2008), and Bertini, Freixas, Gambarelli, and Stach (2013).

On the other hand, the classical approach was improved by other characterizations such as Laruelle and Valenciano (2001), Feltkamp (1995) and Felsenthal and Machover (1996) for the Shapley value,while Nowak (1997), Lehrer (1988), Albizuri (2001), and Barua, Chakravarty, and Roy (2005) refer to the Banzhaf value. Moreover, the Shapley value gained more credibility thanks to different analysis and applications, see for instance Roth (1988), Monderer and Samet (2002), and Moretti and Patrone (2008). Finally due to an increasing interest for cooperative game theory, other models of games were defined to capture different aspects of real-life decision making process, for instance if voters can abstain as in Bilbao, Fernández, Jiménez, and Lebrón (2000) and Felsenthal and Machover (1997). In many of these models, values were defined as solution concepts that generalize the ideas behind the classical ones, see for instance Hsiao and Raghavan (1993), Bolger (1993), Albizuri, Santos, and Zarzuelo (1999), Amer, Carreras, and Magaña (1998a), Freixas (2005a), Freixas (2005b), and Bilbao, Fernández, Jiménez, and López (2008a).

In this work we start our analysis in the classical cooperative theory context, where the Shapley and the Banzhaf values are two of the most common solution concepts, especially in the voting situations in which we want to evaluate the influence of players. Actually these two values have some common properties: they both satisfy the linearity, anonymity and null player properties; these three properties characterize a larger family of values, called *semivalues*, that are a subfamily of *probabilistic values*, as discussed in Weber (1988), Dubey and Weber (1977), and Dubey, Neyman, and Weber (1981).

Different works focus on semivalues, providing either a theoretical interpretation, see Carreras and Freixas (1999) and Carreras, Freixas, and Puente (2003), or some practical applications, as in Lucchetti, Moretti, and Patrone (2015) and Moretti, Patrone, and Bonassi (2007). In particular in Lucchetti, Radrizzani, and Munarini (2011) a new family of values were defined with the idea of finding some values with an intermediate behaviour between the Shapley and the Banzhaf values, at least for some classes of games. Values can be defined, for a fixed set N of players, on the class of unanimity games, and extended to all games by linearity. On the unanimity game u_S, where the unique minimal winning coalition is the coalition S, the Shapley value assigns the same power (thanks to the anonymity axiom) to the members of the winning coalition, which is inversely proportional to the size of the coalition; and it assigns zero to all other players (since it satisfies the null player property).

On the other hand, the Banzhaf value also fulfils the symmetry and null player properties, but it assigns a positive power to players in S, which is inversely exponential to the size of the coalition. The new values considered in Lucchetti, Radrizzani, and Munarini (2011) were defined assigning a power inversely proportional to some power of the size of the winning coalition to non-null players; moreover, it was proved that these values are actually regular semivalues. This approach raised the following question: suppose to define a value by assigning the positive value α_s to the players in S in the unanimity game u_S, where $s = |S|$, and zero to all other players, then we suppose to extend the value by linearity on the whole space of games; under which conditions the assignment of the positive number α_s actually defines a semivalue? In chapter 2 we provide an answer to this question.

In the same chapter we also provide a new formula for computing the Shapley value, deduced from a bargaining procedure different from the classical one. Actually, in the last section of Shapley (1953), there is a description of a *bargaining procedure* which produces the value of the game as an expected outcome. In the bargain process players agree to play the game and form the grand coalition N in the following way: (1) starting with a single player, the coalition adds one player at a time until everyone has been admitted. (2) The order in which players join the coalition is determined by chance, with all arrangements equally probable. (3) Each player, on his admission, demands and is promised the amount of adherence he contributes to the value of the coalition (as determined by the characteristic function). In the bargaining procedure underlying the model of simple games all players are assumed to vote "yes", and only the *order* in which they cast their votes is allowed to vary. Shapley and Shubik interpret the order of "voting" as an indication of the relative degrees of support by the different members, with the most enthusiastic members deciding first, etc. In their work Shapley and Shubik adapted the bargaining procedure proposed by Shapley (1953) to simple games. This means that players are willing to vote for some bill, they vote in a randomly chosen order and all $n!$ orderings are equally likely. As soon as the proposal is approved, the last voter is the *pivotal* player who takes the credit for having passed it. The Shapley-Shubik index is then the ratio of the number of times the player is pivotal to $n!$, the total number of orderings. This index is indeed the probability of each player of being pivotal and thus it is always a number between 0 and 1. Moreover, Shapley and Shubik noticed that their index is also a measure of the power of players in blocking a resolution: if we suppose players are queuing in all the possible orderings and vote against the proposal, then the Shapley-Shubik index is the ratio of the number of times the player is the last needed in order to block the bill to $n!$.

In simple games, considered as binary voting systems, coalitions are either winning or losing. Players are supposed to be in favour or against the bill. The characteristic function assigns the value of 0 to losing coalitions and, somewhat arbitrarily, the value of 1 to winning coalitions. In this voting context it does not seem natural to expect all voters to vote in the same way (either all of them "yes" or all of them "no"). According to Felsenthal and Machover (1996), the natural bargaining procedure in this context should allow voters to vote for any of the two options, and the idea of pivotal voter, as the one who clinches the outcome, should still be the same. They fixed this gap in 1996 and established the appropriate *bargaining procedure* for simple games regarded as binary voting systems. Although their approach is also valid for cooperative games, it is in simple games where the bargaining model has the most trustworthy interpretation. Thus, it seems more natural to let players vote in any ordering in favour or against the proposal, with equal probability, and then consider how many times their vote is pivotal, either in approving or in blocking the bill. This alternative characterization of the probability space in which players are voting is provided by the space of binary roll-calls, in which players are ordered in all the possible ways and can vote either yes or no. Roll-calls take into account all the $2^n n!$ possible orderings in which players can vote either in favour or against a bill. In Felsenthal and Machover (1996) a power index of a player is defined as his expected probability of being pivotal in the space of roll-calls, under the uniform discrete distribution.

Felsenthal and Machover proved the equivalence of their index with the Shapley-Shubik power index showing that their value for cooperative games satisfies the axioms of the Shapley value. As they wrote in their paper they avoided a "direct" proof by proving the equivalence of the two expressions derived from the two bargaining procedures for the Shapley-Shubik index. This direct approach, as noted by Felsenthal and Machover (1996), implies a *"combinatorial fact that is certainly non-trivial, and may be of some independent interest"*. In chapter 2 we provide two different direct proofs of this result and deduce the explicit formula to compute the Shapley value that can be derived from Felsenthal and Machover's bargaining model. The main interest of this result, however, is not the formula itself to compute the Shapley value for cooperative games, but the possibility to generalize the value to non-binary voting system, as we will discuss in chapter 4. Actually the model of roll-call can be easily generalized to situations in which players can vote among many different alternatives, while it is not clear how to extend the classical bargaining procedure described in Shapley (1953).

In chapter 3 we focus on one particular extension of simple games: games with abstention, that have been defined by Felsenthal and Machover (1997) to describe voting procedures in which abstention is a third, separated option, different from being in favour or against a bill. In literature there are different examples in which the model of games with abstention is more realistic and appropriate than the classical model with simple games to describe some voting procedures (see, for instance, Chapter 8 in Felsenthal and Machover (1998)). Our interest in this family of games is related to the evaluation of the power for each voter. So we focus on the extension of the Banzhaf and the Shapley-Shubik indices from simple games to games with abstention.

In particular, the Banzhaf index for games with abstention was defined in Felsenthal and Machover (1998) and extended to games with multiple levels in Freixas (2005a). Also the Shapley-Shubik index for games with abstention is discussed in Felsenthal and Machover (1997) and Felsenthal and Machover (1998) and extended to games with multiple levels in Freixas (2005b), following the Felsenthal and Machover bargaining procedure for simple games. More recently, Freixas and Lucchetti (2016) provided an axiomatization of the Banzhaf index for games with abstention as a two component power index. Their idea is to split the index in two parts in order to highlight the power of players in two different cases: when they are crucial in switching from voting in favour to abstaining and when they switch from abstaining to voting no.

In our work we want to provide a new axiomatization of the Banzhaf index for games with abstention characterizing the index and its two components, with the purpose of highlighting some of its properties, and of the Shapley-Shubik index for games with abstention, thus enforcing the idea of using these indices to evaluate the power of players in these situations.

We use two different approaches to characterize the two indices. The first one follows the classical axiomatization for simple games, see Dubey (1975) and Dubey and Shapley (1979). The axioms used to characterize the indices are anonymity, transfer, null player, efficiency for the Shapley-Shubik index, and Banzhaf total power for the Banzhaf index. We provide a similar characterizations for the Shapely-Shubik and the Banzhaf indices for games with abstention. However, these classical axioms, generalized in the context of (3,2)-simple games, are not sufficient to uniquely characterize the indices on the space of games with abstention. It is necessary to add another property. As we present in chapter 2, the behaviour on unanimity games is crucial in order to uniquely characterize an index on simple or coalitional games. For this reason the new axiom we are going to introduce, describe the behaviour of a power index on unanimity games.

Since in a game with abstention players can vote in three different ways, given a tripartition S, in the unanimity game u_S there are three different types of players: players in S_3 are null-players, while players in S_1 and in S_2 are not. Of course, the role of players voting yes and of those abstaining in a minimal winning tripartition is different, thus, also the power of these two types of non-null players should be different. The axioms we propose compare the power of a player in a unanimity game when he votes "yes" in a minimal winning tripartition and when he abstains in the same situation, i.e. all other players do not change their vote. We give different conditions for the differences of power in the two situations and use these conditions to deduce the Shapley-Shubik and the Banzhaf indices for games with abstention. We also prove that the five axioms used are independent, and thus all of them are necessary to characterize the indices.

In the second approach we focus only on the Banzhaf index for games with abstention. We follow and merge the works of Freixas and Lucchetti (2016), regarding games with abstention, and Laruelle and Valenciano (2001), that is about values for cooperative games. Actually in this second work, a new set of axioms is introduced and the crucial element is the *modified game*, resulting from dropping a minimal winning coalition from a given game. In their work they translate the four classical axioms using this modified game, to define new axioms that have a clear meaning and make sense one by one, independently from each other. Thus, we generalize Laurelle and Valenciano's axioms from simple games to the context of games with abstention; then we characterize the (raw) Banzhaf index for games with abstention and each of its two components using the modified game in which a minimal winning tripartition has been removed. Unfortunately this approach can not be used also to characterize the Shapley-Shubik power index, since there is not an explicit formula that can describe the change of power when comparing a game with its modified version.

In chapter 4 we look for such an explicit formula to at least compute the Shapley-Shubik power index for games with abstention. Actually, we present a more general result and adopt the bargaining procedure described by Felsenthal and Machover to define a value for multichoice games. In the roll-call model for simple games players are allowed to vote either yes or no, and this can be naturally generalized with players queueing in a random order and then voting any of the possible alternatives. In Hsiao and Raghavan (1993) a value for multichoice games is defined, this value is a matrix in which each element represents the effort of a player with respect of an action. Instead the value we are providing is a vector, in which each component represents the value associated to each player in taking part to the game, since we do not want to discriminate among different actions. Our idea is to give a unified measure of the power or influence that each player has in taking part to the decision procedure associated to the game. For this reason we avoid the axiomatic approach, that is used by Hsiao and Raghavan (1993) (and also by Bilbao, Fernández, Jiménez, and López (2008a) and Bolger (1993) to define extension of the Shapley value in context with more than one alternative in the input), but we rather to follow the bargaining approach described in Felsenthal and Machover (1996).

The result we are providing is then an explicit formula to compute the Shapley-Shubik power index for $(j, 2)$-simple games and, at the same time, a formula to define the Shapley value for cooperative games under the Felsenthal and Machover's roll-calls models. Our result is particularly interesting in the family of games with abstention since it allows to compute the Shapley-Shubik index explicitly without using the concept of pivotal player in a roll-call. For instance, one of the improvements the explicit formula provides is the possibility of using generating functions to compute the indices in a weighted majority game. We discuss these aspects in chapter 4 and provide also a comparison among our value and other values defined in analogous context.

In the last chapter, we examine the problem of ranking players from a completely different point of view. We remove the structure of coalitional game and suppose that only an ordinal ranking among players is available. A similar situation can not be represented by a coalitional games and power indices can not help, since, in general, changing power index can change the ranking among players, see for instance Carreras and Freixas (2008) and Freixas (2010). Moreover, in many situations it is possible to have an ordinal ranking over groups, without a specific characteristic function to have a cardinal comparisons among the groups. In these situations to select a random characteristic function to represent the ordinal relation would be a bad idea, since it is well known that the same ordering on the subsets, when described by different utility functions, can provide different ranking among players, as shown in Lucchetti, Moretti, and Patrone (2015). For all these reasons, we want to construct a procedure which is purely ordinal.

Thus the problem becomes to define a function on the set of the complete pre-orders on the non-empty subsets of a given finite set N and valued on the complete pre-orders on N. A similar problem has been investigated in Moretti (2015), where the author studies an alternative notion of power index for cooperative games that is invariant to the choice of the characteristic function representing the ranking over the coalitions. However, such invariant power index is properly defined for a limited class of total pre-orders (over the set of all the coalitions). Alternatively, in Moretti and Oztürk (2016) the authors analyse the ranking function problem using a property-driven approach, and they prove several impossibility theorems showing that no ranking function satisfies a given set of attractive properties.

Our approach is different, even if we use a classical tool in Social Choice and Game Theory: we propose a small set of properties that such a function should satisfy, and we prove that these properties are enough to identify a unique function.

Some of the properties we consider are, in some sense, classical properties in the context of social choice and game theory: they recall some form of anonymity and monotonicity. To these properties we add a new one that reflects the *philosophy* underlying our procedure and our idea of solution. It is clear that different criteria can affect the ranking, since the focus can be put on different aspects: for instance, one can decide that outstanding performance of a group should be very important, thus people in the first groups should also be ranked in the first positions; dually, it could be preferable to punish groups with bad performances and rank in the last positions those people. Also, according to the given context, it is conceivable that the size of the groups plays a role. For instance groups neither too small nor too large could be encouraged, and thus people participating in groups of a prescribed size and well ranked should get a good individual ranking. We want to stress this aspect, to underline that the function we propose is interesting, but for sure it is not the only one potentially available; rather, it is likely that our function is not the best one in every possible context. As for cooperative games we have many solution concepts, and each one of them can be of interest and preferable to others in some situations, in the same way in this context we believe that several useful functions can be defined and studied.

The thesis is organized as follows. At the beginning of each chapter one or two sections are devoted to introduce all the necessary preliminaries to understand our contribution. This means that every chapter is more or less self-contained from a notational and conceptual point of view. At the end of each chapter a brief conclusion shows the links between the different contributions. Moreover, a small abstract at the beginning of each chapter should help the reader to orientate in the setting and in the original works presented.

After this general introduction, chapter 2 is devoted to the study of values for coalitional games, the new formula for the Shapley value is presented and the characterization of the family of semivalues using unanimity games.

The central part of the thesis, chapter 3, deals with some properties for power indices for games with abstention, and provides two different axiomatizations for the Banzhaf index for games with absention and one characterization of the Shapley-Shubik index for games with abstention that allow to compute the index by means of a recursive formula.

In chapter 4 we present the new formula to compute a value for multi-choice cooperative games following the Felsenthal and Machover's bargaining model for simple games; we discuss the differences among our value and the others values defined in literature for games with several levels of approval.

Finally, in chapter 5 we discuss the problem of ranking players starting from an ordinal relations among the coalitions. We provide a characterization of two different solution ideas in this setting.

We present some general conclusion of our work and possible future developments in chapter 6.

CHAPTER 2

Values for coalitional games

AMONG the different solution concepts for coalitional games, the Shapley and the Banzhaf values are two of the most known and used, in particular as power indices for simple games. Both values are actually semivalues, a family of solution concepts that satisfy the linearity, anonymity and null player properties.

In this chapter we provide a new formula to compute the Shapley value due to a different interpretation of the value, proposed by Felsenthal and Machover that is particularly interesting in the voting context. Moreover, we present a theorem to characterize all semivalues by means of their behaviour on unanimity games.

2.1 Cooperative and simple games

Let us introduce some definitions and notations that we are going to use in the following. We refer to Maschler, Solan, and Zamir (2013) for a more accurate illustration of these concepts.

Definition 2.1. A *coalitional game with transferable utility (TU-game)* is given by the pair (N, v) where $v : 2^N \to \mathbb{R}$ and $v(\emptyset) = 0$.

The elements of the set N are the *players* of the game. We denote with n the cardinality of N, its subsets are called *coalitions* and 2^N is the set of all coalitions. The function v is the *characteristic function* of the game that assigns to any coalition of players S, a real value $v(S)$. The dual v^* of a game v is defined as $v^*(S) = v(N) - v(N \setminus S)$.

We denote with $\mathcal{G}^N$ the set of all TU-games on the finite set N.

Definition 2.2. For any coalition $S \neq \emptyset$ the *unanimity game* (N, u_S) is defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases}$$

The set $\mathcal{G}^N$ is isomorphic to the Euclidean space of dimension $2^n - 1$, thus $\mathcal{G}^N$ is a vector space. A basis for this space is given by the collection of the unanimity games $\{u_S : S \in 2^N, S \neq \emptyset\}$. Moreover, given two games $v, w \in \mathcal{G}^N$ the game $v + w$ is defined as

$$(v + w)(S) = v(S) + w(S) \text{ for any } S \in 2^N.$$

Given a game $v \in \mathcal{G}^N$ and a real number α the game αv is defined as

$$(\alpha v)(S) = \alpha v(S) \text{ for any } S \in 2^N.$$

Example 2.1. Three friends wants to share a taxi ride. This situation can be modelled with a coalitional game with set of players $N = \{a, b, c\}$ and where the value of each coalition is the cost of the ride for players belonging to it.

For instance, we can have $v(\{a\}) = v(\{b\}) = 10$; $v(\{c\}) = 14$ to represent the cost of the ride for each player by themselves; $v(\{a, b\}) = 12$ and $v(\{a, c\}) = v(\{b, c\}) = 18$ if two of them share the ride and $v(N) = 20$ if they go all together.

A subclass of the TU-games is the family of *simple games*.

Definition 2.3. A game (N, v) is *simple* if $v : 2^N \to \{0, 1\}$, $v(N) = 1$ and v is monotonic, i.e. if $S \subseteq T$ then $v(S) \leq v(T)$.

Simple games can be used as a model for voting situation, in which there is a group of people discussing to take a decision, for instance to approve or reject a bill. In this context, players are usually called *voters* and we say that a coalition S is *winning* if $v(S) = 1$ and it is losing otherwise. A simple game can be described only by the set of its winning coalitions:

$$\mathcal{W} = \{S \in 2^N : v(S) = 1\}$$

or by the set of its *minimal winning coalitions*:

$$\mathcal{W}^m(v) = \{S \in 2^N : v(S) = 1 \text{ and } v(T) = 0, \text{ for any } T \subset S\}.[1]$$

We denote with $\mathcal{SG}^N$ the set of all simple games on the finite set N.

It is possible to define operations on $\mathcal{SG}^N$ in the following way. Given two games $v, w \in \mathcal{SG}^N$, we define the *disjunction* $v \vee w$ as the game such that

$$(v \vee w)(S) = \max\{v(S), w(S)\} \text{ for any } S \in 2^N,$$

and the *conjunction* $v \wedge w$ as

$$(v \wedge w)(S) = \min\{v(S), w(S)\} \text{ for any } S \in 2^N.$$

Remark 2.1. Let $v, w \in \mathcal{SG}^N$, then $\mathcal{W}(v \vee w) = \mathcal{W}(v) \cup \mathcal{W}(w)$ and $\mathcal{W}(v \wedge w) = \mathcal{W}(v) \cap \mathcal{W}(w)$. This implies that if v is a simple game with $\mathcal{W}^m(v) = \{S_1, \ldots, S_k\}$ then $v = u_{S_1} \vee \cdots \vee u_{S_k}$.

Furthermore, given two unanimity games u_S and u_T, then their conjunction is still a unanimity game and in particular $u_S \wedge u_T = u_Z$ where $Z = S \cup T$.

Example 2.2 (UN Security Council)**.** A resolution is approved by the United Nation Security Council if the five permanent members (China, France, Russia, UK, and US) are not against it and there are at least nine members in favour.

This situation can be described by a simple game with 15 players, in which $P \subseteq N$ is the set of permanent members and a coalition S is winning if and only if $|S| \geq 9$ and $P \subseteq S$.

Actually this model does not take into account the possibility of a permanent member to abstain without explicitly express his veto against a proposal. If the UNSC is modelled as a simple game, the abstention of a permanent member is interpreted as a vote against the bill, however in the following chapter we will discuss the model of voting games with abstention that can give a more realistic representation of the voting procedure within the UNSC, as discussed for the first time in Felsenthal and Machover (1998).

[1] We use $S \subset T$ if $S \subseteq T$ and $S \neq T$.

A special class of simple games is the one of weighted majority game that are usually used to model voting situation in which there is a quota to reach in order to approve a bill.

Definition 2.4. A simple game (N, v) is a *weighted majority game* if there is a number q called *quota* and a vector $w = (w_1, \dots, w_n)$ of *weights* such that

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

A weighted majority game is denoted by $v = [q; w_1, \dots, w_n]$. Of course, every weighted majority game is a simple game, however not all simple games can be represented as weighted majority games.

Example. The UNSC can be represented as a weighted majority game:

$$v = [39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

The weight of a coalition is greater than 39 if and only if there are at least four players plus the five permanent members.

Example 2.3. A corporation has three different stockholders: a, b and c. Suppose the first one has 50% of the stocks, the second the 49% and the third the 1%; decision are taken by majority according to the number of shares each player has. This can be modelled as the weighted majority game

$$[51; 50, 49, 1].$$

The winning coalitions are $\{a, b\}, \{a, c\}, \{a, b, c\}$. Even if players b and c have different weights their role in the game is the same.

2.2 Values and power indices

Among the different solution concepts, we focus on values for coalitional games and power indices for simple games, since they both provide a natural ranking among players.

Definition 2.5. A *value* is a function $\varphi : \mathcal{G}^N \to \mathbb{R}^n$ that assigns to every coalitional game a vector $(\varphi_1, \dots, \varphi_n(v))$, where $\varphi_a(v)$ is the value of player a according to φ.

A *power index* is a function $\varphi : \mathcal{SG}^N \to \mathbb{R}^n$ that assigns to every simple game a vector $(\varphi_1, \dots, \varphi_n(v))$, where $\varphi_a(v)$ is interpreted as the a priori power of player a according to φ.

The most general approach to the problem of defining family of values as solution concepts for coalitional games can be found in Dubey and Weber (1977) and Weber (1979). In these works the family of *probabilistic values* is introduced.

Definition 2.6 (Probabilistic values)**.** Given a player a, let $\{p_S^a,\ S \subseteq N \smallsetminus a\}$ be a probability distribution over the set of coalitions not containing a, that is a family of constant such that $S \subseteq N \smallsetminus a$, $p_S^a \geq 0$ and $\sum_{S\subseteq N\smallsetminus a} p_S^a = 1$. Then a *probabilistic value* ψ on $\mathcal{G}^N$ is defined as

$$\psi_a(v) = \sum_{S\subseteq N\smallsetminus a} p_S^a[v(S\cup\{a\}) - v(S)].$$

for every game v.

Probabilistic values can be seen as the expected payoff of player a if we see his participation to a game as consisting of joining a coalition S with probability p_S^a and then receiving as a reward his marginal contribution $v(S\cup\{a\}) - v(S)$.

It is possible to give a characterization of the probabilistic values by means of some properties, that in this context are also called axioms, since they form a basis for the theory developed from them. First of all, a player a in a game v is called a *null player* if $v(S\cup\{a\}) = v(S)$ for any $S \in 2^{N\smallsetminus\{a\}}$. Then it is reasonable to assume that a null player will not get anything by playing a game.

Null player A value (or a power index) φ satisfies the *null player property* if given a null player a for the game v, then

$$\varphi_a(v) = 0.$$

Positivity Given a monotonic game v, i.e. a game such that $v(S) \leq v(T)$ for any $S \subseteq T$, a value φ satisfies *positivity* if

$$\varphi_a(v) \geq 0$$

for any $a \in N$.

The second axiom is quite natural from a mathematical point of view:

Linearity A value φ satisfies *linearity* if it is a linear operator on $\mathcal{G}^N$, i.e. $\forall v, w \in \mathcal{G}^N, \lambda \in \mathbb{R}$ it holds

$$\varphi(v+w) = \varphi(v) + \varphi(w) \text{ and } \varphi(\lambda v) = \lambda\varphi(v).$$

These axioms characterize the family of probabilistic values, as the following theorem shows.

Theorem 2.1 (Dubey and Weber (1977))**.** *Let $\psi : \mathcal{G}^N \to \mathbb{R}^n$ be a value on $\mathcal{G}^N$ that satisfies linearity, positivity and the null player properties. Then ψ is a probabilistic value and the following formula holds*

$$\psi_a(v) = \sum_{S \subseteq N \smallsetminus a} p_S^a [v(S \cup \{a\}) - v(S)]$$

for some p_S^a such that $p_S^a \geq 0$ for all $S \subseteq N \smallsetminus \{a\}$ and $\sum_{S \subseteq N \smallsetminus a} p_S^a = 1$. Moreover, every probabilistic value satisfies these properties.

Other properties that have been defined in order to characterize values and power indices are the following (see Shapley and Shubik (1954), Dubey (1975), and Dubey and Shapley (1979)).

Transfer A power index φ satisfies *transfer* if $\forall v, w \in \mathcal{SG}^N$, it holds

$$\varphi(v \vee w) = \varphi(v) + \varphi(w) - \varphi(w \wedge v).$$

Anonymity A value (or a power index) φ satisfies *anonymity* if for every permutation ϑ on N, every game v and every player a, it holds

$$\varphi_a(\vartheta v) = \varphi_{\vartheta(a)}(v)$$

where $(\vartheta v)(S) = v(\vartheta S)$ for any $S \in 2^N$.

Efficiency A value (or a power index) φ satisfies *efficiency* if

$$\sum_{i \in N} \varphi_i(v) = v(N)$$

for every game v.

The Banzhaf power index was defined independently by Penrose (1946) and by Banzhaf (1964). This index was considered as a measure of the power of players in voting committee, thus it was initially defined only for simple games.

Definition 2.7 (Banzhaf value)**.** The Banzhaf value β is defined as

$$\beta_a(v) = \sum_{S \subseteq N \smallsetminus \{a\}} \frac{1}{2^{n-1}} [v(S \cup \{a\}) - v(S)]$$

for any game $v \in \mathcal{G}^N$ and every $a \in N$.

The Banzhaf value can be seen as a probabilistic value for which a player a is equally likely to join any coalition in N:

$$p_S^a = \frac{1}{2^{n-1}}.$$

Moreover the Banzhaf value (index) satisfies linearity (transfer), anonymity and null player property.

2.3 The Shapley value and the Shapley-Shubik index

The Shapley value for cooperative games, introduced by Shapley (1953) and the Shapley-Shubik index for simple games, introduced by Shapley and Shubik (1954), are recognized as a very important solution concept for coalitional games and have attracted enormous attention by scholars, see for instance: Roth (1988), Monderer and Samet (2002), Winter (2002), Moretti and Patrone (2008).

In the original work, Shapley defined the value following a *deductive approach*, as the unique function satisfying three axioms that he called anonymity, efficiency and law of aggregation (equivalent to linearity). From these axioms, Shapley derived the explicit well-known formula for his value in terms of the marginal contributions of players.

Definition 2.8 (Shapley value)**.** The Shapley value ϕ is defined as

$$\phi_a(v) = \sum_{S \subseteq N \setminus \{a\}} \frac{s!(n-s-1)!}{n!}[v(S \cup \{a\}) - v(S)] \tag{2.1}$$

for any game $v \in \mathcal{G}^N$ and every $a \in N$[2].

Following the model of probabilistic values, the Shapley value arises from the belief that a player a will join a coalition that is equally likely to be of any size $s = 0, \dots, n-1$ and all coalitions of size s are equally likely:

$$p_S^a = \frac{1}{n}\frac{1}{\binom{n-1}{s}} = \frac{s!(n-1-s)!}{n!}.$$

The Shapley-Shubik index, introduced in Shapley and Shubik (1954), is the restriction of the Shapley value to simple games. In their work Shapley and Shubik adopted the bargaining procedure proposed by Shapley (1953) for cooperative games to simple games. This means that players are willing to vote for some bill, they vote in a randomly chosen

[2] We denote with s the cardinality of S.

order and all $n!$ orderings are equally likely. As soon as the proposal is approved, the last voter is the *pivotal* player who takes the credit for having passed it. The Shapley-Shubik index is the ratio of the number of times the player is pivotal under this scheme to $n!$ (the total number of orderings). However, it is not clear why, in order to model a voting situation, players are supposed to vote only in favour of the proposal and then they receive credits when the proposal is approved thanks to their vote. It would seem more reasonable to count the number of time a player is pivotal either in approving or blocking a resolution. Felsenthal and Machover (1996) proposed an alternative bargaining model to capture these ideas.

2.3.1 The Shapley value under the Felsenthal and Machover bargaining model

Let us present the bargaining model described by Shapley (1953) in the last section of his seminal work and the bargaining model introduced by Felsenthal and Machover (1996). To do so, we are going to define the space of binary roll-calls and the idea of pivotal player in a roll-call. We mainly use the notation from Felsenthal and Machover (1996). Let s and d (short for *sinister* and *dexter*) denote the left-hand and right-hand projection from the Cartesian product of two finite sets $A \times B$, that is $s(a, b) = a$ and $d(a, b) = b$.

A *roll-call* is a map $R : N \to \{1, 2, \dots, n\} \times \{-1, 1\}$, such that sR is a bijection from N to $\{1, 2, \dots, n\}$. Thus sR induces a total order on N, we refer to it as the *queue* of players in R. If $dR(a) = 1$ we say that a is *positive in* R and it means that a votes “yes" in R; if $dR(a) = -1$ we say that a is *negative in* R and it means that a votes “no". We interpret the roll-call R as the players ordering according to sR and voting “yes" or “no" according to their left-hand projection: $sR(a) = i$ means that a is the i^{th} to vote and $dR(a) = 1$ (or $dR(a) = -1$) means that a votes “yes" (or “no").

Let $\mathcal{R}$ be the set of all roll-calls, $\mathcal{R}^+$ be the set of roll-calls for which all players are positive, and $\mathcal{R}^-$ be the set of roll-calls for which all players are negative. It holds $|\mathcal{R}| = 2^n n!$ and $|\mathcal{R}^+| = |\mathcal{R}^-| = n!$. Note that in the bargaining model introduced by Shapley both sets $\mathcal{R}^+$ and $\mathcal{R}^-$ are considered with a uniform probability distribution over their elements.

Given a roll-call R the sets of positive and negative players in R are the following:

$$\mathcal{Y}(R) = \{x \in N : dR(x) = 1\} \qquad \mathcal{N}(R) = \{x \in N : dR(x) = -1\}.$$

For any player a, we will also use the sets of positive and negative players who do not vote after a:

$$\mathcal{Y}(R, a) = \{x \in N : dR(x) = 1 \wedge sR(x) \leq sR(a)\}$$
$$\mathcal{N}(R, a) = \{x \in N : dR(x) = -1 \wedge sR(x) \leq sR(a)\}.$$

Given a roll-call R and a player a, the *marginal contribution* of a to $v(\mathcal{Y}(R))$ (or to $v^*(\mathcal{N}(R))$) is defined as

$$M(v, R, a) = \begin{cases} v(\mathcal{Y}(R,a)) - v(\mathcal{Y}(R,a) \smallsetminus \{a\}) & \text{if } dR(a) = 1 \\ v^*(\mathcal{N}(R,a)) - v^*(\mathcal{N}(R,a) \smallsetminus \{a\}) & \text{if } dR(a) = -1. \end{cases}$$

In the last section of Shapley (1953), it is showed that the Shapley value $\phi_a(v)$ is the expected value of $M(v, R, a)$ in the space of positive roll-calls, $\mathcal{R}^+$.

Theorem 2.2 (Shapley (1953))**.** *The Shapley value of player $a \in N$ in the game $v \in \mathcal{G}^N$ is*

$$\phi_a(v) = \sum_{R \in \mathcal{R}^+} \frac{M(v, R, a)}{n!}.$$

On the other hand, Felsenthal and Machover (1996) proved that $\phi_a(v)$ is the expected value of $M(v, R, a)$ in the probability space of roll-calls, $\mathcal{R}$.

Theorem 2.3 (Felsenthal and Machover (1996))**.** *The Shapley value of player $a \in N$ in the game $v \in \mathcal{G}^N$ is*

$$\phi_a(v) = \sum_{R \in \mathcal{R}} \frac{M(v, R, a)}{2^n n!}. \tag{2.2}$$

The new interpretation given by Felsenthal and Machover is particularly interesting in the context of simple games modeling a voting situation.

Observe that for each $R \in \mathcal{R}$ there is a unique voter a such that $M(v, R, a) = 1$, while $M(v, R, x) = 0$ for all $x \neq a$. We say that a is the *pivotal* player in R for the game v and we write $a = piv(v, R)$.

The pivotal player can be characterized also as the first player a such that if $\mathcal{Y}(R)$ wins in v then $\mathcal{Y}(R,a)$ wins in v, and if $\mathcal{Y}(R)$ loses in v then $N \smallsetminus \mathcal{N}(R,a)$ loses in v. In other words, $piv(v,R)$ is the first player a in the queue of R such that for any roll-call R' with $sR' = sR$ and $dR'(x) = dR(x)$ for any x that precedes a, then $\mathcal{Y}(R')$ wins in v *if and only if* $\mathcal{Y}(R)$ wins in v.

Shapley and Shubik (1954) defined the *Shapley-Shubik index* as the restriction of the Shapley value to the class of simple games, in order to measure the power of a player in a simple game. From Theorem 2.2 and the definition of pivotal player it follows:

Corollary 2.1. *The Shapley-Shubik index of player $a \in N$ in the game $v \in \mathcal{SG}^N$ is*

$$\phi_a(v) = \frac{|\{R \in \mathcal{R}^+ : a = piv(v,R)\}|}{n!}.$$

On the other hand, as a consequence of Theorem 2.3, Felsenthal and Machover (1996) proved the following.

Corollary 2.2. *The Shapley-Shubik index of player $a \in N$ in the game $v \in \mathcal{SG}^N$ is*

$$\phi_a(v) = \frac{|\{R \in \mathcal{R} : a = piv(v,R)\}|}{2^n n!}. \tag{2.3}$$

Felsenthal and Machover (1996) remarked that equations (2.2) and (2.3) are not useful from a practical computational point of view, while their value is conceptual. The only interest they have in (2.2) is to deduce (2.3), which they do regard as conceptually more attractive than the result in Corollary 2.1.

As (2.1) is the explicit formula to compute the Shapley value (and the Shapley-Shubik index) associated to the bargaining procedure described by Shapley (1953), in Felsenthal and Machover (1996) there is not such an equivalent expression to explicitly compute the Shapley value.

We fill this gap providing the the explicit formula associated to the Felsenthal and Machover bargaining model. In order to do so, we define a value Φ for any $a \in N$ as follows.

$$\Phi_a(v) = \sum_{S \subseteq N \smallsetminus \{a\}} \Gamma^n(s)[v(S \cup \{a\}) - v(S)] \tag{2.4}$$

where $s = |S|$ and

$$\Gamma^n(s) = \frac{s!}{2^n n!}\sum_{k=0}^{s}\frac{(n-k-1)!}{(s-k)!}2^k + \frac{(n-s-1)!}{2^n n!}\sum_{k=0}^{n-s-1}\frac{(n-k-1)!}{(n-s-1-k)!}2^k \tag{2.5}$$

for any $s = 0, \dots, n-1$.

Then, the main result is the following.

Theorem 2.4. *The explicit formula* (2.4)*, which describes a value* Φ *in terms of marginal contribution, is associated to the bargaining model described by Felsenthal and Machover. In particular, formula* (2.4) *can be deduced from Theorem 2.3 for cooperative games and from Corollary 2.2 for simple games.*

Proof. In the following proof, we focus on simple games, but the same argument can be easily generalized to describe any coalitional game since in our reasoning it does not matter which are the values of the marginal contributions.

We start considering two different situations, according to player a voting either yes or no.

If a is a positive player in a roll-call R, then $dR(a) = 1$ and

$$M(v, R, a) = v(\mathcal{Y}(R, a)) - v(\mathcal{Y}(R, a) \setminus \{a\}) = 1.$$

If we take $S = \mathcal{Y}(R, a)$, then there is a correspondence between the roll-calls for which a is a positive pivotal player and the coalitions S such that $a \in S$ and $v(S) - v(S \setminus \{a\}) = 1$. In particular, any S of this type is associated to all the roll-calls in which all the players in S vote yes before a, while the players not in S vote no if they are before a or vote either yes or no if they are after a in the queue of R. This means that for any S such that $a \in S$ and $v(S) - v(S \setminus \{a\}) = 1$ the number of associated roll-calls is

$$\sum_{j=0}^{n-s} \overbrace{\binom{n-s}{j}}^{\substack{\text{combinations}\\ \text{of players} \notin S\\ \text{before } a}} \underbrace{(s-1+j)!}_{\substack{\text{orderings of voters}\\ \text{before } a}} \overbrace{(n-s-j)!}^{\substack{\text{orderings}\\ \text{of voters}\\ \text{after } a}} \underbrace{2^{n-s-j}}_{\substack{\text{ways to vote}\\ \text{for players}\\ \text{after } a}} \stackrel{\text{def}}{=} \gamma^n(n-s).$$

Thus, if a is a positive player, the number of roll-calls for which a is pivotal is given by

$$\phi_a^+(v) = \sum_{S \subseteq N : a \in S} \gamma^n(n-s)[v(S) - v(S \setminus \{a\})]$$

or equivalently

$$\phi_a^+(v) = \sum_{S \subseteq N \setminus \{a\}} \gamma^n(n-s-1)[v(S \cup \{a\}) - v(S)]. \tag{2.6}$$

Consider now the other situation in which player a votes no. If a is a negative pivotal player in a roll-call R, then $dR(a) = -1$ and

$$M(v, R, a) = v^*(\mathcal{N}(R,a)) - v^*(\mathcal{N}(R,a) \setminus \{a\}) = 1.$$

Using the definition of dual game v^*, we have

$$M(v, R, a) = v((N \setminus \mathcal{N}(R,a)) \cup \{a\}) - v(N \setminus \mathcal{N}(R,a)).$$

If we take $S = N \setminus \mathcal{N}(R,a) \cup \{a\}$, then there is a correspondence between the roll-calls for which a is a negative pivotal player and the coalitions S such that $a \in S$ and $v(S) - v(S \setminus \{a\}) = 1$. In particular, any S of this type is associated to all roll-calls in which all players not in S (i.e. players in $\mathcal{N}(R,a) \setminus \{a\}$) vote no before a, while players in S vote yes if they are before a and vote either yes or no if they are after a in the queue of R. This means that for any S such that $a \in S$ and $v(S) - v(S \setminus \{a\}) = 1$ the number of associated roll-calls is

$$\sum_{j=0}^{s-1} \overbrace{\binom{s-1}{j}}^{\substack{\text{combinations} \\ \text{of players} \in S \\ \text{before } a}} \underbrace{(n-s+j)!}_{\substack{\text{orderings of voters} \\ \text{before } a}} \overbrace{(s-1-j)!}^{\substack{\text{orderings} \\ \text{of voters} \\ \text{after } a}} \underbrace{2^{s-1-j}}_{\substack{\text{ways to vote} \\ \text{for players} \\ \text{after } a}} \stackrel{\text{def}}{=} \gamma^n(s-1).$$

Thus, if a is a negative player, the number of roll-calls for which a is pivotal is given by

$$\phi_a^-(v) = \sum_{S \subseteq N : a \in S} \gamma^n(s-1)[v(S) - v(S \setminus \{a\})]$$

or equivalently

$$\phi_a^-(v) = \sum_{S \subseteq N \setminus \{a\}} \gamma^n(s)[v(S \cup \{a\}) - v(S)]. \tag{2.7}$$

According to (2.3) the Shapley-Shubik index of a player a is given by the number of roll-calls for which a is a pivotal player divided by $2^n n!$. Thus,

$$\phi_a(v) = \frac{\phi_a^+(v) + \phi_a^-(v)}{2^n n!}.$$

Using equations (2.6) and (2.7) we have

$$\phi_a(v) = \sum_{S \subseteq N : a \in S} \frac{\gamma^n(s) + \gamma^n(n-s-1)}{2^n n!}[v(S \cup \{a\}) - v(S)]$$

that is equivalent to (2.4) thanks to definition of $\Gamma^n(s)$ given in (2.5).

□

The next result establishes the expected coherence between the two bargaining models for the Shapley value, i.e., the two formulas to compute the value in terms of marginal contributions under the two associated bargaining models are equivalent.

Theorem 2.5. *The values Φ and ϕ for cooperative games coincide.*

We want to directly prove the equivalence of formulas (2.1) and (2.4) this will be done with two different proofs: the first one by induction and the second one using generating functions. Note that the two equations have the same structure and the only difference is the coefficient multiplying the marginal contribution to each coalition S such that $a \notin S$. Thus, to prove Theorem 2.5 we just need the following lemma regarding the combinatorial identity between the two coefficients.

Lemma 2.1. *For any n and any $s = 0, \ldots, n-1$, it holds*

$$\frac{s!(n-s-1)!}{n!} = \frac{s!}{2^n n!} \sum_{k=0}^{s} \frac{(n-k-1)!}{(s-k)!} 2^k + \frac{(n-s-1)!}{2^n n!} \sum_{k=0}^{n-s-1} \frac{(n-k-1)!}{(n-s-1-k)!} 2^k.$$

Proof. First of all, simplifying the identity, the thesis is equivalent to the following equation

$$s!(n-s-1)!2^n = s! \sum_{k=0}^{s} 2^k \frac{(n-k-1)!}{(s-k)!} + (n-s-1)! \sum_{k=0}^{n-s-1} 2^k \frac{(n-k-1)!}{(n-s-k-1)!} \tag{2.8}$$

for all n and $0 \leq s \leq n-1$.

Observe that if $n = 1$, then $s = 0$, then this equality reduces to $2 = 1 + 1$, that is trivially true. Since we are dealing with voting games, we assume that there is not only one player and so $n \geq 2$.
We proceed in proving (2.8) using induction on n.

If $n = 2$ and $s = 0$ we have $2^2 = 1 + 1 + 2$. If $n = 2$ and $s = 1$ we have $2^2 = 1 + 2 + 1$. So the thesis is true for $n = 2$.

Now, we assume that (2.8) is true for n and all $0 \leq s \leq n-1$ and we prove it for $n+1$ and $0 \leq s \leq n$.
We first consider the extreme cases $s = 0$ and $s = n$ and prove them directly. Secondly, we prove the statement for each s with $0 < s < n$, using the induction hypothesis for n with s and $s-1$.

First step: For $n+1$ and $s = 0$ (or $s = n$), equality (2.8) becomes

$$2^{n+1} n! = n! + n! \sum_{k=0}^{n} 2^k.$$

Thanks to the induction hypothesis (for n and $s = 0$) we have

$$2^n (n-1)! = (n-1)! + (n-1)! \sum_{k=0}^{n-1} 2^k.$$

Then we can write the right side of our claim as

$$\begin{aligned} n! + n! \sum_{k=0}^{n} 2^k &= n! + 2^n n! + n! \sum_{k=0}^{n-1} 2^k \\ &= n! + 2^n n! + n[2^n (n-1)! - (n-1)!] \\ &= n! + 2^n n! + 2^n n! - n! = 2^{n+1} n! \end{aligned}$$

and this proves the first part.

Second step: We now want to prove the thesis for $n+1$, thus, we have to show that the following is true

$$2^{n+1} s!(n-s)! \stackrel{?}{=} s! \sum_{k=0}^{s} \frac{(n-k)!}{(s-k)!} 2^k + (n-s)! \sum_{k=0}^{n-1} \frac{(n-k)!}{(n-s-k)!} 2^k. \quad (2.9)$$

Thanks to induction, if we take n and s we have

$$s!(n-s-1)! 2^n = s! \sum_{k=0}^{s} 2^k \frac{(n-k-1)!}{(s-k)!} + (n-s-1)! \sum_{k=0}^{n-s-1} 2^k \frac{(n-k-1)!}{(n-s-k-1)!} \quad (2.10)$$

and if we take $s-1$

$$(s-1)!(n-s)!2^n = (s-1)!\sum_{k=0}^{s-1} 2^k \frac{(n-k-1)!}{(s-1-k)!} + (n-s)!\sum_{k=0}^{n-s} 2^k \frac{(n-k-1)!}{(n-s-k)!}. \tag{2.11}$$

We work on the right-hand side of equation (2.9) and rewrite each of the two addends in the following way

$$\begin{aligned} s!\sum_{k=0}^{s} \frac{(n-k)!}{(s-k)!}2^k &= s!(n-s)!2^s + s!\sum_{k=0}^{s-1} \frac{(n-k)!}{(s-k)!}2^k \\ &= s!(n-s)!2^s + s!\sum_{k=0}^{s-1} \frac{(n-k-1)!}{(s-k-1)!}2^k \frac{n-k}{s-k}, \end{aligned}$$

writing $\frac{n-k}{s-k}$ as $\frac{n-s}{s-k}+1$,

$$\begin{aligned} &= s!(n-s)!2^s + s!\sum_{k=0}^{s-1} \frac{(n-k-1)!}{(s-k-1)!}2^k \big(\frac{n-s}{s-k}+1\big) \\ &= s!(n-s)!2^s + s!(n-s)\sum_{k=0}^{s-1} \frac{(n-k-1)!}{(s-k)!}2^k + \\ &\quad + s!\sum_{k=0}^{s-1} \frac{(n-k-1)!}{(s-k-1)!}2^k, \end{aligned}$$

the first term can be moved inside the sum, to get

$$= s!(n-s)\sum_{k=0}^{s} \frac{(n-k-1)!}{(s-k)!}2^k + s!\sum_{k=0}^{s-1} \frac{(n-k-1)!}{(s-k-1)!}2^k.$$

Analogously the second term in (2.9) can be written as

$$\begin{aligned} (n-s)!\sum_{k=0}^{n-s} \frac{(n-k)!}{(n-s-k)!}2^k &= s(n-s)!\sum_{k=0}^{n-s} \frac{(n-k-1)!}{(n-s-k)!}2^k + \\ &\quad + (n-s)!\sum_{k=0}^{n-s-1} \frac{(n-k-1)!}{(n-s-k-1)!}2^k. \end{aligned}$$

If we now sum these expression the right-hand side of (2.9) becomes

$$s\left[(s-1)!\sum_{k=0}^{s-1}\frac{(n-k-1)!}{(s-k-1)!}2^k+(n-s)!\sum_{k=0}^{n-s}\frac{(n-k-1)!}{(n-s-k)!}2^k\right]+$$
$$(n-s)\left[s!\sum_{k=0}^{s}\frac{(n-k-1)!}{(s-k)!}2^k+(n-s-1)!\sum_{k=0}^{n-s-1}\frac{(n-k-1)!}{(n-s-k-1)!}2^k\right].$$

Using the induction hypothesis and in particular (2.10) and (2.11) and replacing everything in the right-hand side of (2.9), we finally get

$$\begin{aligned}2^{n+1}s!(n-s)! &= s[2^n(s-1)!(n-s)!]+(n-s)[2^n s!(n-s-1)!]\\ &= 2^n s!(n-s)!+2^n s!(n-s)!\\ &= 2^{n+1}s!(n-s)!.\end{aligned}$$

□

We are going to provide another proof of Theorem 2.5 using the tool of formal series and generating function. Let us recall the results we are going to use. Given a sequence $\{u_n\}_n$ the *formal serie* of $\{u_n\}_n$ is

$$U(t)=\sum_{n\geq 0}u_nt^n=u_0+u_1t+u_2t^2+\dots$$

The *multiplication* of two formal series $U(t)$ and $V(t)$ is given by

$$U(t)\cdot V(t)=\big(\sum_{n\geq 0}u_nt^n\big)\cdot\big(\sum_{n\geq 0}v_nt^n\big)=\sum_{n\geq 0}\big(\sum_{k=0}^{n}u_kv_{n-k}\big)t^n.$$

Moreover, differentiating k times the geometric serie $\frac{1}{1-t}=\sum_{n\geq 0}t^n$, we get the following identity:

$$\frac{1}{(1-t)^{k+1}}=\sum_{n\geq 0}\binom{n+k}{k}t^n.$$

Thanks to the previous identities we can prove the following.

Lemma 2.2. *Let m and s be non-negative integers, then*

$$\sum_{k=0}^{s}\binom{m+s-k}{m}2^k+\sum_{k=0}^{m}\binom{m+s-k}{s}2^k=2^{m+s+1}. \tag{2.12}$$

Proof. Let us consider the sequence $a_{m,s} = \sum_{k=0}^{s} \binom{m+s-k}{m} 2^k$, the formal series of $a_{m,s}$ is

$$\begin{aligned}
\sum_{m\geq 0}\left\{\sum_{s\geq 0}\left[\sum_{k=0}^{s}\binom{m+s-k}{m}2^k t^s\right]u^m\right\} &= \sum_{m\geq 0}\left[\sum_{s\geq 0}2^s t^s \sum_{s\geq 0}\binom{m+s}{m}t^s\right]u^m \\
&= \sum_{m\geq 0}\left[\frac{1}{1-2t}\cdot\frac{1}{(1-t)^{m+1}}\right]u^m \\
&= \frac{1}{1-2t}\sum_{m\geq 0}\frac{1}{(1-t)^{m+1}}u^m \\
&= \frac{1}{1-2t}\cdot\frac{1}{1-t-u}.
\end{aligned}$$

The two addends on the left-hand side of Equation (2.12) are symmetrical, thus we get

$$\frac{1}{1-2t}\cdot\frac{1}{1-t-u}+\frac{1}{1-2u}\cdot\frac{1}{1-t-u}=\frac{2}{(1-2t)(1-2u)}.$$

On the other hand, the formal series of the sequence $b_{m,s} = 2^{m+s+1}$ is

$$\begin{aligned}
\sum_{m\geq 0}\left\{\sum_{s\geq 0}\left[2^{m+s+1}t^s\right]u^m\right\} &= 2\sum_{m\geq 0}[\sum_{s\geq 0}2^s t^s 2^m u^m] \\
&= 2\frac{1}{1-2t}\cdot\sum_{m\geq 0}2^m u^m \\
&= 2\frac{1}{1-2t}\frac{1}{1-2u}
\end{aligned}$$

and this complete the proof. □

Remark 2.2. Lemma 2.2 and Lemma 2.1 are equivalent: the identity in Lemma 2.1 can be transformed in equation (2.12) by taking $m = n - s - 1$ and rearranging the terms.

2.4 Semivalues

Let us now consider a subfamily of the probabilistic values: *semivalues*. We are going to give a way to generate new family of semivalues, but for a complete overview over this topic refer to Dubey, Neyman, and Weber (1981), Dubey and Weber (1977), Monderer and Samet (2002), and Carreras, Freixas, and Puente (2003) for the restriction of semivalues to simple games.

Definition 2.9 (Semivalue)**.** A *semivalue* on $\mathcal{G}^N$ is a value ψ defined as

$$\psi_a^N(v) = \sum_{S \subset N \setminus \{a\}} p_s[v(S \cup \{a\}) - v(S)]$$

for any $a \in N, v \in \mathcal{G}^N$ with $p_s \geq 0$ for every $s = 0, \dots, n-1$ and $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$.

A semivalue with weighting coefficients p_k is *regular* if $p_k > 0 \, \forall k = 0, \dots, n-1$.

Thus a semivalue π^N can be identified by an element of the simplex

$$\Sigma := \{x \in \mathbb{R}^{n-1} : x_i \geq 0 \quad \wedge \quad \sum_{s=0}^{n-1} \binom{n-1}{s} x_s = 1\}.$$

In the sequel, we shall identify the semivalue π^N by means of the n-dimensional vector $(p_0, \dots, p_{n-1})$.

Theorem 2.6. *A semivalue is a probabilistic index that satisfies anonymity and, vice-versa, any probabilistic index that satisfies anonymity is a semivalue.*

Clearly both the Shapley value and the Banzhaf value are regular semivalues. In particular, they have the following features: the Banzhaf value is the only one for which $p_s = p_t$ for all s, t and the Shapley value is the only one fulfilling efficiency. Another family of regular semivalues on $\mathcal{G}^N$, defined in Carreras and Freixas (1999), is the family of the so called *binomial semivalues*, where $p_s = q^s(1-q)^{n-s-1}$, and $0 < q < 1$.

In Lucchetti, Radrizzani, and Munarini (2011) a new family of values on $\mathcal{G}^N$ is introduced: the *c-values*. Every value $\sigma^{N,c}$ is defined first of all on the base of the unanimity games

$$\sigma_a^{N,c}(u_S) = \begin{cases} \frac{1}{s^c} & \text{if } a \in S \\ 0 & \text{otherwise.} \end{cases}$$

and then extended by linearity on the whole space $\mathcal{G}^N$.

In the paper it is proved that, when the parameter a ranges over the non-negative real numbers, the family $\sigma_i^{N,c}$ describes a curve in the interior of the simplex Σ, containing the Shapley value (obtained by $c = 1$). Moreover a similar relation holds also for the Banzhaf value:

$$\beta_a(u_S) = \begin{cases} \frac{1}{2^{s-1}} & \text{if } a \in S \\ 0 & \text{otherwise,} \end{cases}$$

and, in general, for any q-binomial value

$$\phi_a(u_S) = \begin{cases} q^{s-1} & \text{if } a \in S \\ 0 & \text{otherwise.} \end{cases}$$

The approach used in Lucchetti, Radrizzani, and Munarini (2011), motivated by a specific application in molecular biology, raises the following natural question: let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ be a given sequence, and define a linear value $\pi^{N,\alpha}$ on $\mathcal{G}^N$, acting in the following way on the class of the unanimity games:

$$\pi_a^{N,\alpha}(u_S) = \begin{cases} \alpha_s & \text{if } a \in S \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Then the question is: under which conditions on the coefficients α_s the value $\pi^{N,\alpha}$ is a (regular) semivalue? A discussion of this topic is Carreras, Freixas, and Puente (2003). In next section we provide a sufficient condition, that, as we shall see, allows creating new specific families of semivalues. Moreover, it is possible to extend to $\mathcal{G}$ the concept of semivalue just by requiring that an operator π on $\mathcal{G}$ is a semivalue provided $\pi^N = \pi|_{\mathcal{G}}^N$ is a semivalue for all n. In this case we can characterize the conditions under which the sequence $\{\alpha_s\}_{s\in\mathbb{N}}$ generates a regular semivalue.

2.4.1 Generating semivalues

In order to prove our results we need the following result (see Carreras, Freixas, and Puente (2003) and Lucchetti, Radrizzani, and Munarini (2011).

Proposition 2.1. *Suppose, for each $t = 1, \ldots, n$, positive numbers α_t are given and suppose $\pi^N : \mathcal{G}^N \to \mathbb{R}^n$ is a linear value assigning α_t to all players of the coalition T in the unanimity game u_T, and zero to all players not in T, for all coalitions T such that $|T| = t$. Then $\pi^{N,\alpha}$ verifies the following formula:*

$$\pi_a^{N,\alpha}(v) = \sum_{S\subseteq N\setminus\{a\}} \left(\sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} \right) [v(S\cup\{a\}) - v(S)].$$

Thus $\pi^{N,\alpha}$ is characterized by the fact that it is of the form

$$\pi_a^{N,\alpha}(v) = \sum_{S \subseteq N \setminus \{a\}} p_s^n [v(S \cup \{a\}) - v(S)],$$

where

$$p_s^n = \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1}.$$

It follows that $\pi^{N,\alpha}$ is a semivalue provided the coefficient α_s fulfill:

1. for any $s = 0, \ldots, n-1$:
$$\sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} \geq 0;$$

2. $$\sum_{s=0}^{n-1} \binom{n-1}{s} \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} = 1.$$

The second item follows immediately from the following result, proven in Lucchetti, Radrizzani, and Munarini (2011).

Proposition 2.2. *Take, for every $n \in \mathbb{N}$, real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then*

$$\sum_{s=0}^{n-1} \binom{n-1}{s} \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} = \alpha_1.$$

Thus the second condition is fulfilled by requiring $\alpha_1 = 1$ and the problem becomes to provide sufficient conditions under which the first issue is verified. To analyze this, we need some preliminaries. We start with the following definition that can be found in Widder (1946), (p. 108).

Definition 2.10. Let μ_n be a sequence of real numbers, then the backward difference operator Δ^k is defined by

$$\Delta^0 \mu_n = \mu_n \qquad \Delta^k \mu_n = \Delta^{k-1} \mu_{n+1} - \Delta^{k-1} \mu_n$$

for $n = 0, 1, 2, \ldots,$ and $k = 1, 2, \ldots$ The sequence $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic if its elements are non-negative and

$$(-1)^k \Delta^k \mu_n \geq 0$$

for every $k, n = 0, 1, 2, \ldots$

The backward difference operator can be written also as

$$\Delta^k \mu_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu_{n+k-j}.$$

The following result, whose proof can be found for instance in Lorch and Moser (1963), (p. 171), will be used in detecting regularity of semivalues.

Proposition 2.3. *Let $\{\mu_k\}_{k=0}^{+\infty}$ be a completely monotonic sequence. Then*

$$(-1)^k \Delta^k \mu_n > 0$$

for every $n, k = 0, 1, \dots$ unless $\mu_1 = \mu_2 = \cdots = \mu_n = \dots$, that is the sequence is constant except at most for the first term.

Now we prove the following Lemma, the key ingredient for the main result.

Lemma 2.3. *Given the value $\pi^{N,\alpha}$, with associated vector $(p_0^n, \dots, p_{n-1}^n)$ the following formula holds, for all $s = 0, \dots, n-1$:*

$$(-1)^{n-s-1} \Delta^{n-s-1} \alpha_{s+1} = p_s^n.$$

Proof. Observe that, for all l

$$(-1)^l \Delta^l \alpha_m = (-1)^l \sum_{j=0}^{l} (-1)^j \binom{l}{j} \alpha_{m+l-j} = (-1)^l \sum_{k=l}^{0} (-1)^{l-k} \binom{l}{l-k} \alpha_{m+k}$$

$$= \sum_{k=0}^{l} (-1)^k \binom{l}{k} \alpha_{m+k}.$$

Since this equation holds for every $l \geq 0$ and every $m \geq 1$ we can set $l = n - s - 1$ and $m = s + 1$ to get

$$(-1)^{n-s-1} \Delta^{n-s-1} \alpha_{s+1} = \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} = p_s^n.$$

□

From the above results we get the following Theorem.

Theorem 2.7. *Let $\{\alpha_s\}_{s \in \mathbb{N}}$ be a completely monotonic sequence such that $\alpha_1 = 1$ and let $\pi^{N,\alpha}$ be the value defined on $\mathcal{G}^N$ as in Equation (2.13). Then $\pi^{N,\alpha}$ is a semivalue.*

2.4.2 Semivalues on $\mathcal{G}$

We now consider the space $\mathcal{G}$ of all finite games. This means that $\mathcal{G} = \cup_N \mathcal{G}^N$ where N is a finite set of players.

Definition 2.11. A semivalue on $\mathcal{G}$ is an operator π on $\mathcal{G}$ such that its restriction to $\mathcal{G}^N$ is a semivalue for all N.

Let moreover $\mathcal{S}$ be the space of the real valued sequences:

$$\mathcal{S} = \{(\alpha_1, \alpha_2, \dots, \alpha_s, \dots) : \alpha_s \in \mathbb{R}\ \forall s \geq 1, \alpha_1 = 1, \alpha_s \geq 0\ \forall s\}.$$

Finally, given a sequence $\alpha \in \mathcal{S}$ define π^α on $\mathcal{G}$ in the following way:

$$\pi^\alpha(v) = \pi^{N,\alpha}(v)$$

for every v such that $v \in \mathcal{G}^N$. Thus, to player i in the game $v \in \mathcal{G}^N$, the operator π^α assigns:

$$\pi_a^\alpha(v) = \sum_{S \subseteq N \setminus \{a\}} \left(\sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} \right) [v(S \cup \{a\}) - v(S)].$$

Putting together the above results we get the following.

Theorem 2.8. *π^α is a semivalue on $\mathcal{G}$ if and only if $\alpha \in \mathcal{S}$ is completely monotonic.*

We explicitly notice that if π^α is a semivalue on $\mathcal{G}$, then it is defined for every $n \in \mathbb{N}$ $\mathbf{p^n} = \{p_k^n\}_{k=0}^{n-1}$ such that $\mathbf{p^n}$ is the vector of weighting coefficients associated to $\pi|_{\mathcal{G}}^N$. The coefficients fulfill the formula

$$p_s^n = \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1}.$$

With the choice of $s = n-1$ the formula above shows that

$$\alpha_n = p_{n-1}^n$$

for all $n = 1, 2, \dots$. So thanks to the previous theorem we have the following

Corollary 2.3. *π^α is a semivalue on $\mathcal{G}$ if and only if the sequence $\{p_n^{n+1}\}_{n=0,1\dots}$ is completely monotonic.*

The next result we want to prove deals with semivalues which are not regular. To prove it, we need a preliminary result.

Proposition 2.4. *For all s, n such that $s \leq n-1$, the following formula hods:*

$$p_s^n = p_s^{n+1} + p_{s+1}^{n+1}. \tag{2.14}$$

Proof. Remembering the formula

$$p_s^n = \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1}$$

we have that

$$\begin{aligned}
p_s^{n+1} + p_{s+1}^{n+1} &= \sum_{k=0}^{n-s} (-1)^k \binom{n-s}{k} \alpha_{s+k+1} + \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+2} \\
&= \alpha_{s+1} + \sum_{k=1}^{n-s-1} (-1)^k \left[\binom{n-s}{k} - \binom{n-s-1}{k-1} \right] \alpha_{s+k+1} \\
&= \sum_{k=1}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} + \alpha_{s+1} \\
&= \sum_{k=0}^{n-s-1} (-1)^k \binom{n-s-1}{k} \alpha_{s+k+1} = p_s^n.
\end{aligned}$$

□

We now provide the formal definition of regular and irregular semivalues on the space $\mathcal{G}$. Remember that on $\mathcal{G}^N$ regularity of π^α means that $p_s^n > 0$ for all s.

Definition 2.12. A semivalue on $\mathcal{G}$ is regular iff its restriction to $\mathcal{G}^N$ is a regular semivalue for all n. A semivalue which is not regular is called irregular.

In the next definition we introduce two irregular semivalues, extending the definitions of marginal and dictatorial values given in Carreras, Freixas, and Puente (2003).

Definition 2.13. The marginal value μ on $\mathcal{G}$ is the value such that its restriction to $\mathcal{G}^N$ is described by the vector $(0, 0, \ldots, 1)$. The dictatorial value δ on $\mathcal{G}$ is the semivalue such that its restriction to $\mathcal{G}^N$ is described by the vector $(1, 0, \ldots, 0)$.

Theorem 2.9. *The values μ and δ are semivalues on $\mathcal{G}$. Moreover, let π^α be a irregular semivalue on $\mathcal{G}$. Then, there is $q \in [0,1]$ such that $\mathbf{p^n}$ is of the form $(1-q, 0, \ldots, 0, q)$, for every n.*

Proof. The dictatorial value is generated by the sequence $\alpha_1 = 1$, $\alpha_n = 0$ for all $n > 1$, the marginal value by the sequence $\alpha_n = 1$, for all $n \geq 1$. Both are completely monotonic sequences. Moreover, by definition a semivalue on $\mathcal{G}$ is irregular if and only if its restriction to some set N has a vanishing element p_s^n. From Theorem 2.8 we know that α must be a completely monotonic sequence. On the other hand, as Proposition 2.3 shows, the only case when a monotonic sequence allows for a vanishing coefficient p_s^n, is when the sequence is constant, with the only possible exception of the first term (always equal to one in our contest). Thus $\alpha = (1, q, \ldots, q, \ldots)$, for some $q \in [0,1]$. We claim that $\mathbf{p^n}$ is of the form $(1-q, 0, \ldots, 0, q)$ for all n. First of all, remember that for all n it holds that $p_{n-1}^n = \alpha_n$; thus necessarily the last component of the vector $\mathbf{p^n}$ is q. So the result immediately follows for $n = 2$, remembering that $p_0^2 + p_1^2 = 1$. Now it is easy to show the claim by induction, just using formula (2.14). This ends the proof. $\square$

Thus a irregular semivalue assigns, for all n a fixed probability to the fact that the players act alone and the complement to the fact that they act all together. As a conclusion, we underline that other semivalues, like the basic semi indices defined in Carreras, Freixas, and Puente (2003), do not extend to semivalues on the whole $\mathcal{G}$.

Remark 2.3. The following is a natural question to address: since a semivalue on $\mathcal{G}$ automatically (by definition) generates a semivalue on $\mathcal{G}^N$ for all N, then conversely given a semivalue on some fixed set N of players, can it be extended to a semivalue on $\mathcal{G}$? First of all, as it is easy to see, irregular semivalues can not be extended, with the exception of those described by Theorem 2.9. Instead given a regular semivalue on N, it is possible to extend it on all T such that $|T| < |N|$; but it is not always possible to extend it on bigger sets. For instance it is not possible to extend the semivalue given by the vector $\mathbf{p^3} = (\varepsilon, \frac{1}{2} - \varepsilon, \varepsilon)$ if $\varepsilon < \frac{1}{6}$. Moreover when a semivalue associate to the vector $\mathbf{p^n}$ can be extended, the extension is not unique and there is a family of candidates $\mathbf{p^{n+1}}$, depending from a parameter. For instance consider the Shapley value for $n = 4$, that is $\mathbf{p^4} = (\frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{4})$. Then we can choose $p_1^5 = p_2^5 = p_3^5 = \frac{1}{24}$ and get $p_0^5 = p_4^5 = \frac{5}{24}$. In general we can choose $p_j^n = \frac{1}{3 \cdot 2^{n-2}}$ if

$j \neq 0, n-1$ and obtain a regular semivalue, with weighting coefficients different from the ones of the Shapley value.

2.5 Conclusion

There are two new main results provided in this chapter: the first one regarding the Shapley value and the Shapley-Shubik index and the second one regarding the possibility to define new family of semivalues. We were able to provide a generic description of the family of semivalues, using completely monotonic sequences, thus it could be of future interest to investigate some of these sequences in order to generate new semivalues with particular properties.

On the other hand, the main interest of the first result is summarized in equation (2.4), even if the interest is not the new formula itself, but its consequences. First of all, this equivalent expression to compute the Shapley value is useful to deduce (2.3) and constitutes an alternative proof of the Felsenthal and Machover (1996) result, not requiring the use of axioms. Secondly, the binary formula we propose provides the clue for obtaining an extension of the Shapley value to ternary games, $(j, 2)$-simple games and multi-choice cooperative games, as we will discuss in details in chapter 4.

CHAPTER 3

Power indices for games with abstention

SIMPLE games can be seen as a model for voting procedures where players in favour of a proposal form a coalition, while players outside the coalition are voting against it. This idea can be generalized in order to model other voting procedures in which players have the possibility of abstaining, too. Games with abstention have been defined with this purpose. The Banzhaf and the Shapley-Shubik power indices for games with abstention can be used as solution concepts and to evaluate the power of players in the decision making process.

In this chapter we focus our analysis on the properties of these two power indices for games with abstention and provide different characterizations of them. Our approach is to generalize to the set of games with abstention some of the properties that have been provided in literature to characterize the corresponding power indices for simple games.

3.1 Games with abstention

A game with abstention is a model of a voting situation in which players have three different possibilities: voting "yes", voting "no", and abstaining. This is a generalization of the standard model of simple games in which players can only vote in support of or against the status quo. Felsenthal and Machover (1997) introduced *ternary voting games* in order to generalize the model of simple games and include the possibility of abstention. In particular, chapter 8 in Felsenthal and Machover (1998) presents a complete discussion on the role of abstention in voting procedure. After some years, Freixas and Zwicker (2003) introduced (j,k) games to capture voting procedure in which players can choose among j levels of support and there are k possible outcomes. Felsenthal and Machover's model for games with abstention can actually be seen as a $(3,2)$-game. In our work, we refer to these two approaches to describe games with abstention, since they seem a natural extension of the classical cooperative approach.

Given a finite set of players N of cardinality n, in simple games the set 2^N represents the set of all coalitions. Actually any coalition $T \in 2^N$ can be seen as a bipartition (T_1, T_2) in which $T_1 = T$ and $T_2 = N \setminus T$. We can view a coalition T as the set of players supporting a decision and the coalition $N \setminus T$ as the set of players against it. Analogously, in the context of games with abstention we consider the set 3^N of all tripartitions. By tripartition we denote any element $S = (S_1, S_2, S_3)$, where S_1, S_2, S_3 are mutually disjoint subsets of N such that $S_1 \cup S_2 \cup S_3 = N$, and any S_k can be empty. An element $S = (S_1, S_2, S_3) \in 3^N$ describes a voting situation in which the players in S_k are voting at "level k" of approval. It is supposed that level 1 is the highest, level 2 is the intermediate, and level 3 is the lowest. Hereafter, with the idea of modelling a voting situation, we will say that players in S_1 are voting "yes", players in S_2 are abstaining, players in S_3 are voting "no".

A partial order $\subseteq$ on the set 3^N is defined as follows. If $S, T \in 3^N$, then $S \subseteq T$ means $S_1 \subseteq T_1$ and $S_2 \subseteq T_1 \cup T_2$. In other words, a tripartition S is contained in the tripartition T if players in T are either voting as in S or increasing their level of support. This means that S can be transformed into T by shifting one or more players to higher levels of approval. For instance[1] $(a, b, c) \subseteq (ab, c, \emptyset)$, since the second tripartition is obtained from the first one when player b changes from abstaining to voting "yes" and player c switches from voting "no" to abstaining. The tripartition $(\emptyset, \emptyset, N)$ is the minimum of the order $\subseteq$, while the maximum is the tripartition $(N, \emptyset, \emptyset)$.

[1] To simplify the notation we omit the braces to denote the sets in a tripartition, for instance the informal notation (a, b, c) stands for $(\{a\}, \{b\}, \{c\})$.

Definition 3.1. A *game with abstention* or *(3,2)-simple game* is a pair (N, v) in which N is the set of players (or voters) and $v : 3^N \to \{0, 1\}$ is a function that is monotonic, i.e. if $S \subseteq T$ then $v(S) \leq v(T)$, and such that $v(\emptyset, \emptyset, N) = 0$ and $v(N, \emptyset, \emptyset) = 1$.

We denote by $\mathfrak{T}^N$ the set of all games with abstention on the finite set N.

As for simple games, any game $v \in \mathfrak{T}^N$ can be described by the set of *winning tripartitions*

$$\mathcal{W}(v) = \{S \in 3^N : v(S) = 1\}$$

or by the set of *minimal winning tripartitions*

$$\mathcal{W}^m(v) = \{S \in 3^N : v(S) = 1 \text{ and } v(T) = 0, \text{ for any } T \subset S\}.$$[2]

Definition 3.2 (Unanimity game)**.** For any tripartition $S \neq (\emptyset, \emptyset, N)$, the *unanimity game* u_S is defined as

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases}$$

Given two games $v, w \in \mathfrak{T}^N$, we define from them the following games:

Disjunction: the game $v \vee w$ defined as $(v \vee w)(S) = \max\{v(S), w(S)\}$.

Conjunction: the game $v \wedge w$ defined as $(v \wedge w)(S) = \min\{v(S), w(S)\}$.

Let us make some remarks about these operations:

1. $\mathcal{W}(v \vee w) = \mathcal{W}(v) \cup \mathcal{W}(w)$ and $\mathcal{W}(v \wedge w) = \mathcal{W}(v) \cap \mathcal{W}(w)$;
2. if $\mathcal{W}^m(v) = \{S_1, \ldots, S_t\}$ then $v = u_{S_1} \vee \cdots \vee u_{S_t}$;
3. given two unanimity games u_S and u_T, then their conjunction is still a unanimity game and in particular $u_S \wedge u_T = u_Z$ with $Z_1 = S_1 \cup T_1$, $Z_2 = (S_2 \cup T_2) \setminus Z_1$ and $Z_3 = N \setminus (Z_1 \cup Z_2)$;
4. if we omit the conditions $v(\emptyset, \emptyset, N) = 0$ and $v(N, \emptyset, \emptyset) = 1$ in Definition 3.1, then $\mathfrak{T}^N$ is a distributive lattice with the operations $\wedge, \vee$. The game $v \equiv 1$ in which all tripartitions are winning is its supremum; the game $v \equiv 0$ in which all tripartitions are losing is its infimum.

[2]We use $S \subset T$ if $S \subseteq T$ and $S \neq T$.

3.2 Power indices

As for simple games, power indices for games with abstention can be defined to evaluate the influence of players in a voting procedure and generate a ranking among voters.

Definition 3.3. A *power index for games with abstention* is a function $\varphi : \mathfrak{T}^N \to \mathbb{R}^n$, that assigns to every game v a vector $\varphi(v)$.

We denote with $\varphi_a(v)$ its a^{th} component, representing the measure of the power of player a according to φ in the voting system described by v.

We focus our attention on the Banzhaf and the Shapley-Shubik indices that have been the first indices generalized from simple games to games with abstention (see Felsenthal and Machover (1997), Freixas (2005a), and Freixas (2005b)). Of course, other indices can be considered, for instance Freixas (2012) focused on probabilistic indices for games with abstention

3.2.1 The Banzhaf index for games with abstention

Let us now describe the Banzhaf index for games with abstention as defined in Felsenthal and Machover (1998) and in Freixas (2005a). Before the definition of the index, we introduce some notation.

Given a tripartition $S = (S_1, S_2, S_3)$ and a player $a \notin S_3$ we denote with $S_{\downarrow a}$ the tripartition in which player a decreases his support of one level

$$S_{\downarrow a} = \begin{cases} (S_1 \setminus \{a\}, S_2 \cup \{a\}, S_3) & \text{if } a \in S_1 \\ (S_1, S_2 \setminus \{a\}, S_3 \cup \{a\}) & \text{if } a \in S_2. \end{cases}$$

Of course, there is also the possibility that player $a \in S_1$ switches from supporting a decision to vote against it

$$S_{\downarrow\downarrow a} = (S_1 \setminus \{a\}, S_2, S_3 \cup \{a\}).$$

In an analogous way, given S and a player $a \notin S_1$, we define the tripartition in which a increases the support

$$S_{\uparrow a} = \begin{cases} (S_1 \cup \{a\}, S_2 \setminus \{a\}, S_3) & \text{if } a \in S_2 \\ (S_1, S_2 \cup \{a\}, S_3 \setminus \{a\}) & \text{if } a \in S_3 \end{cases}$$

and if $a \in S_3$

$$S_{\uparrow\uparrow a} = (S_1 \cup \{a\}, S_2, S_3 \setminus \{a\}).$$

Definition 3.4 (Banzhaf index for games with abstention)**.** For any game $v \in \mathfrak{T}^N$ and any player $a \in N$, define $\eta_a(v)$ as the number of yes-no swings for player a, that is

$$\eta_a(v) = |\{S : a \in S_1 \text{ and } v(S) - v(S_{\downarrow\downarrow a}) = 1\}|.$$

The function η is the *raw Banzhaf index for games with abstention*, while the *Banzhaf index for games with abstention* is the function $\beta : \mathfrak{T}^N \to \mathbb{R}^n$ defined as

$$\beta_a(v) = \frac{\eta_a(v)}{3^{n-1}} \qquad \text{for any } a \in N.$$

The previous definition generalizes the classical one from simple games to games with abstention, assuming that all tripartitions have the same probability to form. However, it can be observed that in this context a more explicit approach is to split the index in two components, as described by Freixas and Lucchetti (2016).

Definition 3.5 (Banzhaf two components index)**.** For any game $v \in \mathfrak{T}^N$ and any player $a \in N$, define $\eta_a^{YA}(v)$ and $\eta_a^{AN}(v)$ as the number of yes-abstain and abstain-no swings for player a, that is

$$\eta_a^{YA}(v) = |\{S : a \in S_1 \text{ and } v(S) - v(S_{\downarrow a}) = 1\}|,$$
$$\eta_a^{AN}(v) = |\{S : a \in S_2 \text{ and } v(S) - v(S_{\downarrow a}) = 1\}|.$$

Thus, the *YA-Banzhaf index* and the *AN-Banzhaf index* are defined as

$$\beta_a^{YA}(v) = \frac{\eta_a^{YA}(v)}{3^{n-1}} \qquad \beta_a^{AN}(v) = \frac{\eta_a^{AN}(v)}{3^{n-1}}.$$

Remark 3.1. From the two previous definitions it holds that

$$\eta(v) = \eta^{YA}(v) + \eta^{AN}(v)$$

and $\beta(v) = \beta^{YA}(v) + \beta^{AN}(v)$.

3.2.2 The Shapley-Shubik index for games with abstention

The Shapley-Shubik power index for games with abstention was introduced in Felsenthal and Machover (1997) using the idea of roll-calls, as an extension of the bargaining model introduced in Felsenthal and Machover (1996), that we discussed in subsection 2.3.1. As for simple games, players are supposed to queue in a random order and vote either yes or no, then for games with abstention players have also the possibility of abstaining. The Shapley-Shubik index is still the expected probability of a player being pivotal, in the sense of fixing the result of the voting, under this scheme.

Let us introduce the mathematical notation beyond this model. Let $\mathbf{Q}_N$ be the space of all permutations of N, and let 3^N be the set of all tripartitions of N. The *ternary roll-call space* $\mathbf{R}_N$ is defined as

$$\mathbf{R}_N = \mathbf{Q}_N \times 3^N.$$

Each roll-call R is given by a queue qR and a tripartition tR, that is $R = (qR, tR)$ where qR represents the order in which players are voting and tR represents how each one of them is voting. For instance, $qR(a) = i$ means that a is the i^{th} to vote and $a \in tR_1$ means that a is voting "yes". The number of the elements in $\mathbf{R}_N$ is $n!3^n$.
A player a is said to be *pivotal* in R for the game v (and we write $piv(R, v)$) if after a's vote the outcome is decided, no matter what the players after a in qR are going to vote. This means that a is the first player whose vote fixes the outcome of the voting procedure: either as winning or losing.

The Shapley-Shubik power index for games with abstention is defined as an analogous to the Shapley-Shubik index for simple games deduced in Corollary 2.2.

Definition 3.6 (Shapley-Shubik index for games with abstention)**.** For any $v \in \mathfrak{T}^N$ and any player $a \in N$, the *Shapley-Shubik index for (3,2)-simple games* is defined as

$$\phi_a(v) = \frac{|\{R \in \mathbf{R}_N : a = piv(R, v)\}|}{3^n n!}. \tag{3.1}$$

Example 3.1. In Table 3.1 the Shapley-Shubik and the Banzhaf indices are provided for all the games with abstention (up to isomorphism) on $N = \{1, 2\}$.

For the results we are going to discuss in next sections, we introduce some notation related to roll-calls. Consider the set of roll-calls $\mathbf{R}_N$; for any player $a \in N$, we define the following subsets which form a partition of $\mathbf{R}_N$:

$$\mathcal{R}_a^{yes} = \{\text{roll-calls in which player } a \text{ votes "yes"}\}$$
$$\mathcal{R}_a^{abs} = \{\text{roll-calls in which player } a \text{ abstains}\}$$
$$\mathcal{R}_a^{no} = \{\text{roll-calls in which player } a \text{ votes "no"}\}$$

	W^m	S-S index	Bz index
1	$(12, \emptyset, \emptyset)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{3})$
2	$(1, 2, \emptyset)$	$(\frac{2}{3}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{3})$
3	$(1, 2, \emptyset)$ and $(2, 1, \emptyset)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{2}{3}, \frac{2}{3})$
4	$(1, 2, \emptyset)$ and $(2, \emptyset, 1)$	$(\frac{1}{6}, \frac{5}{6})$	$(\frac{1}{3}, \frac{1}{3})$
5	$(1, \emptyset, 2)$	$(1, 0)$	$(1, 0)$
6	$(1, \emptyset, 2)$ and $(2, \emptyset, 1)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{2}{3}, \frac{2}{3})$
7	$(1, \emptyset, 2)$ and $(\emptyset, 12, \emptyset)$	$(\frac{5}{6}, \frac{1}{6})$	$(1, \frac{2}{3})$
8	$(1, \emptyset, 2)$ and $(\emptyset, 2, 1)$	$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{3}, \frac{2}{3})$
9	$(\emptyset, 12, \emptyset)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{2}{3}, \frac{2}{3})$
10	$(\emptyset, 1, 2)$	$(1, 0)$	$(1, 0)$
11	$(\emptyset, 1, 2)$ and $(\emptyset, 2, 1)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{2}{3}, \frac{2}{3})$
12	$(1, \emptyset, 2)$ and $(2, \emptyset, 1)$ and $(\emptyset, 12, \emptyset)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{2}{3}, \frac{2}{3})$

Table 3.1: *Shapley-Shubik and Banzhaf indices for* $(3, 2)$*-simple games with two players.*

Thus,

$$\mathbf{R}_N = \mathcal{R}_a^{yes} \cup \mathcal{R}_a^{abs} \cup \mathcal{R}_a^{no}$$

and $|\mathcal{R}_a^{yes}| = |\mathcal{R}_a^{abs}| = |\mathcal{R}_a^{no}| = n!3^{n-1}$

Given a player a and a roll-call $R = (qR, tR) \notin \mathcal{R}_a^{no}$, we define the roll-call $R_{\downarrow a}$ in which players are in the same order as in R, all players in $N \setminus \{a\}$ vote as in R, while a decreases the support of one level

$$R_{\downarrow a} = (qR, tR_{\downarrow a}).$$

Note that if $R \in \mathcal{R}_a^{yes}$, then $R_{\downarrow a} \in \mathcal{R}_a^{abs}$; if $R \in \mathcal{R}_a^{abs}$, then $R_{\downarrow a} \in \mathcal{R}_a^{no}$.

We also define the roll-call in which a decreases the support of two levels, changing the vote from "yes" to "no": if $R \in \mathcal{R}_a^{yes}$, then $R_{\downarrow\downarrow a} \in \mathcal{R}_a^{no}$ is

$$R_{\downarrow\downarrow a} = (qR, tR_{\downarrow\downarrow a})$$

For a roll-call $R \in \mathcal{R}_a^{yes}$ we analogously define

$$R_{\uparrow a} = (qR, tR_{\uparrow a})$$

and if $R \in \mathcal{R}_a^{no}$

$$R_{\uparrow\uparrow a} = (qR, tR_{\uparrow\uparrow a}).$$

Note that, for instance, if $R \in \mathcal{R}_a^{yes}$ we have:

$$(R_{\downarrow a})_{\uparrow a} = R$$

which shows that there is a one-to-one correspondence between $\mathcal{R}_a^{yes}$ and $\mathcal{R}_a^{abs}$ with these changes. In addition, from

$$(R_{\downarrow\downarrow a})_{\uparrow\uparrow a} = R$$

the one-to-one correspondence of the three sets $\mathcal{R}_a^{yes}, \mathcal{R}_a^{abs}, \mathcal{R}_a^{no}$ follows.

We also introduce the following sets, for any player $a \in N$ and any game v:

$$\begin{aligned} Y_{a,v} =& \{R \in \mathcal{R}_a^{yes} : a = piv(R, v)\ \} \\ A_{a,v} =& \{R \in \mathcal{R}_a^{abs} : a = piv(R, v)\ \} \\ N_{a,v} =& \{R \in \mathcal{R}_a^{no}\ : a = piv(R, v)\ \}. \end{aligned}$$

and the following subsets of $A_{a,v}$ and $N_{a,v}$:

$$\begin{aligned} AY_{a,v} =& \ \{R \in A_{a,v} : R_{\uparrow a} \in Y_{a,v}\} \quad & A\overline{Y}_{a,v} =& \ \{R \in A_{a,v} : R_{\uparrow a} \notin Y_{a,v}\} \\ NY_{a,v} =& \{R \in N_{a,v} : R_{\uparrow\uparrow a} \in Y_{a,v}\} \quad & N\overline{Y}_{a,v} =& \{R \in N_{a,v} : R_{\uparrow\uparrow a} \notin Y_{a,v}\}. \end{aligned}$$

Thanks to the previous notation, the Shapley-Shubik index as defined in (3.1) can be written as

$$\phi_a(v) = \frac{1}{3^n n!} \left[|Y_{a,v}| + |A_{a,v}| + |N_{a,v}|\right]$$

or as

$$\phi_a(v) = \frac{1}{3^n n!} \left[|Y_{a,v}| + |AY_{a,v}| + |A\overline{Y}_{a,v}| + |NY_{a,v}| + |N\overline{Y}_{a,v}|\right].$$

3.3 Axioms for power indices

Let us present some of the axioms for games with abstention, analogous to the classical axioms that we presented in section 2.2. These axioms are an extension of the axioms for simple games to the family of games with abstention and to (j, k) games, see Freixas (2005a), Freixas (2005b), and Freixas and Lucchetti (2016).

Axiom 3.1 (Transfer)**.** An index φ satisfies *transfer* if for any $v, w \in \mathfrak{T}^N$

$$\varphi(v) + \varphi(w) = \varphi(v \wedge w) + \varphi(v \vee w).$$

Axiom 3.2 (Anonymity)**.** An index φ satisfies *anonymity* if for any $v \in \mathfrak{T}^N$, any permutation π of N and any $a \in N$

$$\varphi_a(\pi v) = \varphi_{\pi(a)}(v)$$

where $(\pi v)(S) = v(\pi(S))$.

In a game with abstention players have three different ways of changing their vote: when they move from voting yes to abstainining or to voting no and when they vote changes from abstention to no. For this reason, we have different types of null players.

Definition 3.7 (Null players)**.** Let $v \in \mathfrak{T}^N$, then a voter $a \in N$ is called

- *YN-null* player if $v(S) = v(S_{\downarrow\downarrow a})$ for any tripartition S such that $a \in S_1$;
- *YA-null* player if $v(S) = v(S_{\downarrow a})$ for any tripartition S such that $a \in S_1$;
- *AN-null* player if $v(S) = v(S_{\downarrow a})$ for any tripartition S such that $a \in S_2$.

Note that a is a YN-null player if and only if a is both YA-null and AN-null, and if and only if $a \in S_3$ for any $S \in \mathcal{W}^m(v)$.

Axiom 3.3 (*X*-null player)**.** An index φ satisfies the *X-null player* property if a is a *X*-null player in a game v, then

$$\varphi_a(v) = 0,$$

with *X*=*YN, YA, AN* .

Axiom 3.4 (Efficiency)**.** An index φ satisfies *efficiency* if for any $v \in \mathfrak{T}^N$

$$\sum_{a \in N} \varphi_a(v) = 1.$$

Axiom 3.5 (Banzhaf total power)**.** An index φ satisfies the *Banzhaf total power property* if for any $v \in \mathfrak{T}^N$

$$\sum_{a \in N} \varphi_a(v) = \frac{1}{3^{n-1}} \sum_{i=1}^{n} \sum_{\substack{S \in 3^N \\ a \in S_1}} [v(S) - v(S_{\downarrow\downarrow a})].$$

In the following lemmas we show which of these properties are satisfied by the Shapley-Shubik index for games with abstention and by the Banzhaf index for games with abstention.

Lemma 3.1. *The Shapley-Shubik index for* $(3,2)$*-simple games satisfies the anonymity, the null player, the transfer and the efficiency axioms.*

Proof. **Anonymity** Let π be a permutation of N. Given a roll-call $R = (qR, tR)$, define $\pi R = (\pi(qR), \pi(tR))$. This means that if $\pi(a) = b$ then b votes in πR in the same position and in the same level of approval of a in R.
If a is pivotal in the game v for the roll-call R, then $\pi(a)$ is pivotal in the game πv for the roll-call $\pi(R)$. Then

$$\begin{aligned}\phi_a(v) &= \frac{|\{R : a = piv(R,v)\}|}{3^n n!} \\ &= \frac{|\{\pi R : \pi a = piv(\pi R, \pi v)\}|}{3^n n!} = \phi_{\pi(a)}(\pi v).\end{aligned}$$

So the Shapley-Shubik index for $(3,2)$-simple games satisfies the anonymity axiom.

Null player If a is a null player in a game v, there is not a roll-call R such that $a = piv(R,v)$. Then $\phi_a(v) = 0$.

Transfer Let v and w be two games with abstention, then consider the following sets of roll-calls:

$$\begin{aligned}A &= \{R : a \text{ is pivotal in } v \text{ and in } w\} \\ B &= \{R : a \text{ is pivotal in } v \text{ but not in } w\} \\ C &= \{R : a \text{ is pivotal in } w \text{ but not in } v\}.\end{aligned}$$

Note that A and B form a partition of the set of roll-calls for which a is pivotal in v, while A and C form a partition of the set of roll-calls for which a is pivotal in w. Note also that A is the set of roll-calls for which a is pivotal in the game $v \wedge w$, while A, B, C form a partition of the set of roll-calls in which a is pivotal in the game $v \vee w$. For any $a \in N$ we have

$$\begin{aligned}\phi_a(v) &= \frac{|A| + |B|}{3^n n!} \qquad \phi_a(w) = \frac{|A| + |C|}{3^n n!} \\ \phi_a(v \vee w) &= \frac{|A| + |B| + |C|}{3^n n!} \qquad \phi_a(v \wedge w) = \frac{|A|}{3^n n!}.\end{aligned}$$

Thus $\phi(v \vee w) + \phi(v \wedge w) = \phi(v) + \phi(w)$ and the Shapley-Shubik index satisfies the transfer axiom.

Efficiency In every roll-call there is one and only one player that is pivotal, from the definition of the Shapley-Shubik index in (3.1), we get that it satisfies efficiency.

□

Remark 3.2. As it is well-known, these axioms for simple games are independent and they fully characterize the Shapley-Shubik index for simple games. This is not true for $(3,2)$-simple games. For instance, let $\bar{\phi}$ be the standard Shapley-Shubik index for simple games. Then consider the index φ for $(3,2)$-simple games defined as $\varphi(v) = \bar{\phi}(V)$ where V is the simple game associated to the $(3,2)$-simple game v and defined as

$$V(S) = 1 \iff v(S, N \setminus S, \emptyset) = 1.$$

Then φ satisfies the anonymity, null player, transfer and efficiency axioms for $(3,2)$-simple games since the Shapley-Shubik index satisfies them on simple games. However, φ is different from ϕ, for instance

$$\varphi(u_{(a,b,\emptyset)}) = (1,0)$$

while

$$\phi(u_{(a,b,\emptyset)}) = \left(\frac{2}{3}, \frac{1}{3}\right).$$

Lemma 3.2. *The Banzhaf index for $(3,2)$-simple games satisfies the anonymity, null player, transfer and Banzhaf total power axioms.*

Proof. **Anonymity** The Banzhaf index for games with abstention satisfies anonymity: actually if $a \in N$ and a is yes-no swinger for tripartition S, then it is clear that πa is a yes-no swinger for πS.

Null player If a is a null player in the game v, then $v(S) = v(S_{\downarrow\downarrow a})$ for all $S \in 3^N$ such that $a \in S_1$. So $\beta_a(v) = 0$.

Transfer Let v and w be two games with abstention and V and W be the set of their winning tripartitions, then consider the following sets:

$$\begin{aligned}
A &= \{S \in 3^N : a \in S_1, S \in V \setminus W, S_{\downarrow\downarrow a} \notin V\} \\
B &= \{S \in 3^N : a \in S_1, S \in W \setminus V, S_{\downarrow\downarrow a} \notin W\} \\
C &= \{S \in 3^N : a \in S_1, S \in V \cap W, S_{\downarrow\downarrow a} \in W \setminus V\} \\
D &= \{S \in 3^N : a \in S_1, S \in V \cap W, S_{\downarrow\downarrow a} \in V \setminus W\} \\
E &= \{S \in 3^N : a \in S_1, S \in V \cap W, S_{\downarrow\downarrow a} \notin V \cap W\}.
\end{aligned}$$

Note that the sets A, C and E form a partition for the set of yes-no swings of a in v; B, D and E form a partition for the set of yes-no swings of a in w. All the five sets form the set of yes-no swings for player a in $v \vee w$, while E is the set of swings for player a in $v \wedge w$. So the Banzhaf index, which counts the number of yes-no swings, satisfies the transfer axiom.

Banzhaf total power This axiom is trivially satisfied from the definition of the Banzhaf index.

□

Remark 3.3. Again, anonymity, null player, transfer and Banzhaf total power are independent axioms on simple games, but the Banzhaf index for $(3, 2)$-games is not uniquely determined using only these four. For instance, let $\bar{\beta}$ be the standard Banzhaf index for simple games, consider the index φ for $(3, 2)$-simple games defined as $\varphi(v) = \bar{\beta}(V)$ where V is the simple game associated to the game v and defined as

$$V(S) = 1 \iff v(S, N \setminus S, \emptyset) = 1.$$

Then φ satisfies the anonymity, null player, transfer and Banzhaf total power axioms since the Banzhaf index satisfies them on simple games, but it is different from β, for instance

$$\varphi(u_{(a,b,\emptyset)}) = (1, 0)$$

while

$$\beta(u_{(a,b,\emptyset)}) = \left(\frac{2}{3}, \frac{1}{3}\right).$$

3.4 Classical axiomatizations

3.4.1 A new axiom for the Shapley-Shubik index for games with abstention

As we have seen in remark 3.2 transfer, anonymity, null player and efficiency are not enough to uniquely determine the Shapley-Shubik power index for unanimity games. As we shall see, the addition of a new axiom is sufficient for the characterization. As we discussed in section 2.4, the behaviour of a value on the family of unanimity games is significant in order to characterize it. Following this idea, we introduce a new axiom that describe the behaviour of a power index on unanimity games, and in particular evaluate how the power of a player changes when he moves from voting "yes" to abstaining.

This axiom, together with null player, anonymity and efficiency allows to obtain the value for players in every unanimity game with a recursive procedure. Once the value is defined over unanimity games, it can be extended to all games with the transfer axiom and thus, it would be uniquely determined for every player in every game.

Yes-abstain loss on unanimity game An index ψ satisfies the *new axiom* if for any tripartition $S \in 3^N$ such that $S_1 \neq \emptyset$ it holds:

$$\psi_a(u_S) - \psi_a(u_{S\downarrow a}) = \psi_a(u_{S\downarrow a}) - f(s_1, s_2) \qquad (3.2)$$

for any $a \in S_1$, where[3]

$$f(s_1, s_2) = \frac{2^{s_2}}{3^{s_1+s_2-1}} \frac{1}{(s_1 + s_2)}.$$

Note that $g(s_1, s_2) = \frac{2^{s_2}}{3^{s_1+s_2-1}}$ is the probability, under uniform distribution over tripartitions, of having a winning tripartition in the game u_S with a voting "yes". On the other hand $f(s_1, s_2) = g(s_1, s_2)\frac{1}{s_1+s_2}$ is the same probability divided by the number of active players (players that are not null) in the game u_S.

Thus, the meaning of the yes-abstain loss on unanimity game is that the gain player a has voting yes instead of abstaining is equal to the power of player a if he is in the second level of approval minus the probability of having an active player in a winning tripartition. Equation

[3] We use the notation $s_i = |S_i|$ for $i = 1, 2, 3$.

(3.2) can be stated also as

$$\psi_a(u_{S\downarrow a}) = \frac{\psi_a(u_S) + f(s_1, s_2)}{2}$$

showing that, in a unanimity game u_S, the power of player a in case he abstains is an average between the power of a when he votes "yes" and the probability of being an active player in a winning tripartition in the game u_S.

First of all, we have to prove that the Shapley-Shubik index for $(3,2)$-simple games satisfies the Yes-abstain loss on unanimity game. From now on, we fix a tripartition S with $a \in S_1$ and consider the game u_S. We want to compute $\phi_a(u_S)$ and then compare it with $\phi_a(u_{S\downarrow a})$.

Lemma 3.3. *Let u_S be a unanimity game, then player $a \in S_1$ is pivotal in the roll-call $R \in \mathcal{R}_a^{no}$ if and only if a is pivotal in the roll-call $R_{\uparrow a} \in \mathcal{R}_a^{abs}$.*

Proof. A roll-call is winning in the game u_S if and only if all players belonging to S_1 are voting "yes" and all players belonging to S_2 are not voting "no".
If $a \in S_1$ is pivotal by voting "no" in the roll-call R, then the outcome of R is negative. The roll-call $R_{\downarrow a}$ represents the same situation of R with the only difference that player a abstains instead of voting no. But a is still pivotal abstaining and fixing as negative the outcome of the roll-call. Analogously, if player a is pivotal by abstaining, in the same situation a is also pivotal by voting "no".

□

Remark. Lemma 3.3 implies that $|A_{a,u_S}| = |N_{a,u_S}|$, but also $|AY_{a,u_S}| = |NY_{a,u_S}|$ and $|A\overline{Y}_{a,u_S}| = |N\overline{Y}_{a,u_S}|$, because of the one-to-one correspondence among each pair of the sets $\mathcal{R}_a^{yes}$, $\mathcal{R}_a^{abs}$, and $\mathcal{R}_a^{no}$.

Lemma 3.4. *Let u_S be a unanimity game, if player $a \in S_1$ is pivotal in the roll-call $R \in \mathcal{R}_a^{yes}$, then a is pivotal in the roll-call $R_{\downarrow a} \in \mathcal{R}_a^{abs}$ and in the roll-call $R_{\downarrow\downarrow a} \in \mathcal{R}_a^{no}$*

Proof. A roll-call is winning in the game u_S if and only if all players belonging to S_1 are voting "yes" and all players belonging to S_2 are not voting "no".
If $a \in S_1$ is pivotal in the roll-call R by voting "yes", this means that after a's vote the outcome is positive and all the other players in S_1 and

S_2 voted before a. This also means that after a only some of the players belonging to S_3 are going to vote, but they are null players and can not be pivotal.
In the roll-call R, a is the last player who has the power to change the outcome of the game, thus a is pivotal also in $R_{\downarrow\downarrow a}$ voting "no" and in $R_{\downarrow a}$ abstaining. □

The converse is not true. For instance consider the tripartition $S = (a, b, c)$ and the game u_S. In any roll-call in which a is the first to vote, he is pivotal abstaining or voting "no". On the other hand if a votes "yes" as first player, then b is pivotal: if she votes "no" the outcome is negative, while if she abstains or votes "yes" the outcome is positive.

Remark. Note that by definition $|AY_{a,u_S}| \leq |Y_{a,u_S}|$ and $|NY_{a,u_S}| \leq |Y_{a,u_S}|$. Lemma 3.4 implies that $|Y_{a,u_S}| \leq |AY_{a,u_S}|$ and $|Y_{a,u_S}| \leq |NY_{a,u_S}|$. Thanks to these considerations and Lemma 3.3 we have $|Y_{a,u_S}| = |AY_{a,u_S}| = |NY_{a,u_S}|$.

Hence, from the previous remarks, the Shapley-Shubik index for games with abstention on the unanimity game u_S of player $a \in S_1$ is

$$\begin{aligned}
\phi_a(u_S) =& \frac{1}{3^n n!}\big(|Y_{a,u_S}| + |A_{a,u_S}| + |N_{a,u_S}|\big) \\
=& \frac{1}{3^n n!}\big(|Y_{a,u_S}| + |AY_{a,u_S}| + |A\overline{Y}_{a,u_S}| + |NY_{a,u_S}| + |N\overline{Y}_{a,u_S}|\big) \\
=& \frac{1}{3^n n!}\big(3|Y_{a,u_S}| + 2|N\overline{Y}_{a,u_S}|\big) \qquad (3.3)
\end{aligned}$$

It is possible to calculate the value $|Y_{a,u_S}|$ thanks to the following lemma.

Lemma 3.5. *A player $a \in S_1$ is pivotal in the game u_S for the roll-call $R \in \mathcal{R}_a^{yes}$ if and only if all players in $(S_1 \smallsetminus \{a\}) \cup S_2$ are before him in qR, they vote "yes" if they belong to S_1, and they vote "yes" or abstain if they belong to S_2.*
In particular

$$|Y_{a,u_S}| = 2^{s_2} 3^{s_3} \frac{(s_1 + s_2 + s_3)!}{s_1 + s_2}.$$

Proof. A roll-call is winning in the game u_S if and only if all players belonging to S_1 are voting "yes" and all players belonging to S_2 are not voting "no". Thus if a player $a \in S_1$ is pivotal in the roll-call $R \in \mathcal{R}_a^{yes}$, he is the last player of the set $S_1 \cup S_2$ to vote and he is the last player that has the possibility to influence the outcome and fix it as positive. All the players before him were not pivotal, so they voted "yes" if they belong to S_1, they abstained or voted "yes" if they belong to S_2.

$$\underbrace{\begin{array}{c}\text{players in } S_1 \text{ and in } S_2\\ \text{not pivotal}\\ \text{and } j \text{ players in } S_3\end{array}}_{\binom{s_3}{j}(s_1+s_2+j-1)!2^{s_2}3^j} \quad \begin{array}{c}(s_1+s_2+j)^{th}\\ \downarrow\\ a\end{array} \quad \underbrace{\begin{array}{c}s_3-j \text{ players in } S_3\\ \text{can vote anything}\end{array}}_{(s_3-j)!3^{s_3-j}}$$

Figure 3.1: *Roll-calls in which $a \in S_1$ is pivotal by voting "yes"*

To prove the second part of the thesis, we have to count the number of roll-calls $R \in Y_{a,u_S}$. Actually, if $j = 0, \dots, s_3$, player a can vote j positions after that all the players in $S_1 \smallsetminus \{a\} \cup S_2$ voted. This means that j players belonging to S_3 vote before a and $s_3 - j$ players belonging to S_3 vote after a. There are $\binom{s_3}{j}$ different ways to choose the j players, then $(s_1 + s_2 + j - 1)!2^{s_2}3^j$ possibilities for the players before a and $(s_3 - j)!3^{s_3-j}$ for the players after a. Hence,

$$\begin{aligned}|Y_{a,u_S}| =& \sum_{j=0}^{s_3} \binom{s_3}{j} (s_1 + s_2 + j - 1)!2^{s_2}3^j(s_3 - j)!3^{s_3-j}\\ =& 2^{s_2}3^{s_3}s_3! \sum_{j=0}^{s_3} \frac{(s_1 + s_2 + j - 1)!}{j!}\\ =& 2^{s_2}3^{s_3}\frac{(s_1 + s_2 + s_3)!}{s_1 + s_2}.\end{aligned}$$

□

We can now discuss how the power of a player changes in a unanimity game when he decreases the support of one level. We evaluate the Shapley-Shubik index for $(3, 2)$-simple games of a player $a \in S_1$ in the unanimity game $u_{S\downarrow a}$, generated by the tripartition $(S_1 \smallsetminus \{a\}, S_2 \cup \{a\}, S_3)$.
Firstly note that if a is pivotal by abstaining in $u_{S\downarrow a}$, then in the same situation a is pivotal also by voting "yes"; this means that $|A\overline{Y}_{a,u_{S\downarrow a}}| = 0$. Then observe that if $R \notin A\overline{Y}_{a,u_S}$ and a is pivotal in R for u_S, then a is

pivotal in R also for $u_{S\downarrow a}$. This means that

$$\begin{gathered} Y_{a,u_{S\downarrow a}} = Y_{a,u_S} \\ AY_{a,u_{S\downarrow a}} = AY_{a,u_S} \qquad A\overline{Y}_{a,u_{S\downarrow a}} = \emptyset \\ NY_{a,u_{S\downarrow a}} = NY_{a,u_S} \qquad N\overline{Y}_{a,u_{S\downarrow a}} = N\overline{Y}_{a,u_S}. \end{gathered}$$

Hence,

$$\phi_a(u_{S\downarrow a}) = \frac{1}{3^n n!}\big(3|Y_{a,u_S}| + |N\overline{Y}_{a,u_S}|\big). \tag{3.4}$$

Finally, to establish how the index changes when a player switches from voting "yes" to abstaining in a unanimity game, we compare equations (3.3) and (3.4) obtain the following:

$$\phi_a(u_S) - \phi_a(u_{S\downarrow a}) = \frac{1}{3^n n!}|N\overline{Y}_{a,u_S}|. \tag{3.5}$$

Unfortunately, in general, there is not a clear explicit formula to compute $|N\overline{Y}_{a,u_S}|$. However, we can make the following comparison with $|Y_{a,u_S}|$ that has been explicitly computed in Lemma 3.5:

$$2\phi_a(u_{S\downarrow a}) - \phi_a(u_S) = \frac{3}{3^n n!}|Y_{a,u_S}|,$$

that can also be written as

$$\phi_a(u_S) - \phi_a(u_{S\downarrow a}) = \phi_a(u_{S\downarrow a}) - \frac{3}{3^n n!}|Y_{a,u_S}|.$$

We can finally prove that the Yes-abstain loss on unanimity game is needed for the characterization of the Shapley-Shubik power index.

Proposition 3.1. *The Shapley-Shubik index satisfies the yes-abstain loss on unanimity game.*

Proof. We have seen that

$$\phi_a(u_S) - \phi_a(u_{S\downarrow a}) = \phi_a(u_{S\downarrow a}) - \frac{3}{3^n n!}|Y_{a,u_S}|$$

and, thanks to Lemma 3.5

$$|Y_{a,u_S}| = 2^{s_2}3^{s_3}\frac{(s_1+s_2+s_3)!}{s_1+s_2}.$$

Since $n = s_1 + s_2 + s_3$ we obtain the thesis.

□

As a consequence of the yes-abstain loss on unanimity game, it is possible to derive boundaries for the ratio between $\phi_a(u_S)$ and $\phi_a(u_{S\downarrow a})$ with $a \in S_1$.

Proposition 3.2. *Consider the unanimity game u_S with $S = (S_1, S_2, S_3) \neq (\emptyset, \emptyset, N)$ and $a \in S_1$. Then*

$$1 \leq \frac{\phi_a(u_S)}{\phi_a(u_{S\downarrow a})} < 2.$$

Moreover, $\phi_a(u_S) = \phi_a(u_{S\downarrow a})$ if and only if $S = (a, \emptyset, N \smallsetminus \{a\})$.

Proof. From the yes-abstain loss on unanimity game (3.2), we get that $2\phi_a(u_{S\downarrow a}) - \phi_a(u_S) > 0$ and thus

$$\frac{\phi_a(u_S)}{\phi_a(u_{S\downarrow a})} < 2.$$

Note that the inequality is strict because $f(s_1, s_2)$ is always positive.
On the other hand from equation (3.5) we have that

$$\phi_a(u_S) - \phi_a(u_{S\downarrow a}) = \frac{1}{3^n n!}|N\overline{Y}_{a,u_S}|.$$

Since $|N\overline{Y}_{a,u_S}| \geq 0$, this proves that

$$\frac{\phi_a(u_S)}{\phi_a(u_{S\downarrow a})} \geq 1.$$

To prove the second part of the thesis, note that $\phi_a(u_S) = \phi_a(u_{S\downarrow a})$ if and only if $|N\overline{Y}_{a,u_S}| = 0$, this means that every time a is pivotal voting "no", then he is also pivotal voting "yes". This happens if and only if a is the only not-null player in the game, so $S = (a, \emptyset, N \smallsetminus \{a\})$.
□

We can now state the main theorem to show the uniquely characterization of the Shapley-Shubik power index for games with abstention.

Theorem 3.1 (Shapley-Shubik index for $(3,2)$-simple games)**.** *Let $\psi : \mathfrak{T}^N \to \mathbb{R}^n$ be an index for $(3,2)$-simple games, then ψ satisfies anonymity, null player, transfer, efficiency and the yes-abstain loss on unanimity game if and only if ψ is the Shapley-Shubik index for $(3,2)$-simple games.*

Proof. In Lemma 3.1 and in Proposition 3.1 it is proved that the Shapley-Shubik index for $(3,2)$-simple games satisfies all the axioms, we just need to prove that only one index satisfies all of them.
So, let ψ be an index that satisfies the hypothesis. We will prove that it is uniquely determined on a game v, using induction on the number of minimal winning tripartitions of v.

First, suppose that $|\mathcal{W}^m(v)| = 1$. Then $v = u_S$ for some tripartition $S \in 3^N$ and $S \neq (\emptyset, \emptyset, N)$. We again use induction on the number of elements in S_2.

$|S_2| = 0$ Then $S = (S_1, \emptyset, N \setminus S_1)$ for some $S_1 \subseteq N$. All players in $S_3 = N \setminus S_1$ are null players, so if $c \in S_3$: $\psi_c(u_S) = 0$, on the other hand all players in S_1 have the same role, thus, thanks to the anonymity and efficiency axioms we have $\psi_a(u_S) = \frac{1}{s_1}$, for any $a \in S_1$.

$|S_2| = t + 1$ Suppose now that the thesis is true for any tripartition T such that $|T_2| \leq t$, we want to prove it for a tripartition S such that $|S_2| = t + 1$. Given the tripartition $S = (S_1, S_2, S_3)$, there exist a player $p \in S_2$ and a tripartition $T = (T_1, T_2, T_3)$ such that $T_{\downarrow p} = S$ and $|T_2| = t$. Since ψ satisfies the yes-abstain loss on unanimity game:

$$\psi_p(u_S) = \psi_p(u_{T\downarrow p}) = \frac{1}{2}[\psi_p(u_T) + f(t_1, t_2)]$$

then the induction hypothesis and anonymity imply that $\psi_b(u_S)$ is uniquely determined for all players $b \in S_2$.
Thanks to anonymity and efficiency:

$$s_1\psi_a(u_S) + s_2\psi_b(u_S) = 1,$$

so we can determine $\psi_a(u_S)$ for $a \in S_1$. All players in S_3 are null, so $\psi_c(u_S) = 0$ if $c \in S_3$.
Thus, ψ coincides with the Shapley-Shubik power index for $(3,2)$-simple games for any unanimity game u_S.

Now, suppose that the thesis holds for any game v such that $|\mathcal{W}^m(v)| \leq k - 1$; we need to prove it for v such that $|\mathcal{W}^m(v)| = k$.
If $\mathcal{W}^m(v) = \{S^1, \ldots, S^k\}$, then $v = u_{S^1} \vee u_{S^2} \vee \cdots \vee u_{S^k}$, since ψ satisfies the transfer axiom:

$$\psi(v) = \psi(u_{S^1}) + \psi(u_{S^2} \vee \cdots \vee u_{S^k}) - \psi(u_{S^1} \wedge u_{S^2} \wedge \cdots \wedge u_{S^k}).$$

The conjunction of unanimity games is still a unanimity game, so all games in the right-hand side of the previous equation have a number of minimal winning tripartitions smaller than $|W^m(v)|$. Using the induction hypothesis, ψ coincides with the Shapley-Shubik index for all of them and this ends the proof.

□

Now, that we proved that the Shapley-Shubik power index for games with abstention is uniquely characterized by the five axioms we can show how these axioms allow to compute the index for every unanimity game by means of a recursive procedure.

Consider the unanimity game u_S with the set S_3 of people voting no. Then consider the game u_{T^0} where $T^0 = (\emptyset, N \setminus S_3, S_3)$. For every player $a \in S_3$ thanks to null player we have $\phi_a(u_{T^0}) = 0$, for every $a \in N \setminus S_3$ thanks to anonymity and efficiency it holds

$$\phi_a(u_T) = \frac{1}{n - s_3}.$$

Then consider the game u_{T^1} where $T^1 = (p, N \setminus (S_3 \cup p), S_3))$ for some player $p \in S_1$. Thanks to (3.2) we can compute $\phi_p(u_{T^1})$, then for any $a \neq p \in N \setminus S_3$, $\phi_a(u_{T^1})$ can be computed using efficiency and players in S_3 are still null players.

It is clear that this process can be reiterated until we reach the games u_S and establish the value for all players in S_1 and S_2 using the yes-abstain loss on unanimity game and efficiency.

3.4.2 Independence of the axioms

We now prove the independence of the five axioms used in Theorem 3.1 and show that all of them are necessary to uniquely characterize the Shapley-Shubik power index for $(3, 2)$-simple games. We are going to give examples of power indices for games with abstention that satisfy only four of them, as summarized in Table 3.2.

Not anonymity Consider the index ψ^1 defined on unanimity games as follows.

- If $s_3 = n - 2$ then for any two players a and b such that $a < b$,
  - if $S = (ab, \emptyset, N \setminus \{a, b\})$, then

$$\psi^1_a(u_S) = \frac{1}{2} + \varepsilon, \quad \psi^1_b(u_S) = \frac{1}{2} - \varepsilon$$

	Anonymity	Null	Transfer	Efficiency	Yes-abstain loss on unanimity game
ψ^1	-	✓	✓	✓	✓
ψ^2	✓	-	✓	✓	✓
ψ^3	✓	✓	-	✓	✓
ψ^4	✓	✓	✓	-	✓
φ	✓	✓	✓	✓	-

Table 3.2: *Independence of the axioms that characterize the Shapley-Shubik index for games with abstention*

– if $S = (a, b, N \setminus \{a, b\})$, then

$$\psi^1_a(u_S) = \frac{2}{3} + \frac{\varepsilon}{2}, \quad \psi^1_b(u_S) = \frac{1}{3} - \frac{\varepsilon}{2}$$

– if $S = (b, a, N \setminus \{a, b\})$, then

$$\psi^1_a(u_S) = \frac{1}{3} + \frac{\varepsilon}{2}, \quad \psi^1_b(u_S) = \frac{2}{3} - \frac{\varepsilon}{2}$$

– if $S = (\emptyset, ab, N \setminus \{a, b\})$, then

$$\psi^1_a(u_S) = \frac{1}{2} + \frac{\varepsilon}{4}, \quad \psi^1_b(u_S) = \frac{1}{2} - \frac{\varepsilon}{4}$$

where $\varepsilon > 0$;

- if $s_3 \neq n - 2$, $\psi^1(u_S) = \phi(u_S)$ where ϕ is the Shapley-Shubik index for $(3, 2)$-simple games.

Then extend ψ^1 to $\mathfrak{T}^N$ using transfer.

It is clear that this index satisfies null player and efficiency. It also satisfies the yes-abstain loss on unanimity game, because it coincides with the Shapley-Shubik index for $(3, 2)$-simple games when $s_3 \neq n-2$ and if $s_3 = n - 2$ the yes-abstain loss on unanimity game is satisfied by the definition of ψ^1, as we can check:

$$2\psi^1_a(u_{(b,a,N\setminus\{a,b\})}) - \psi^1_a(u_{ab,\emptyset,N\setminus\{a,b\}}) = \frac{1}{6} = f(2, 0)$$
$$2\psi^1_b(u_{(a,b,N\setminus\{a,b\})}) - \psi^1_b(u_{ab,\emptyset,N\setminus\{a,b\}}) = \frac{1}{6} = f(2, 0)$$
$$2\psi^1_a(u_{(\emptyset,ab,N\setminus\{a,b\})}) - \psi^1_a(u_{a,b,N\setminus\{a,b\}}) = \frac{1}{3} = f(1, 1)$$
$$2\psi^1_b(u_{(\emptyset,ab,N\setminus\{a,b\})}) - \psi^1_b(u_{b,a,N\setminus\{a,b\}}) = \frac{1}{3} = f(1, 1)$$

However, ψ^1 does not satisfy anonymity, because for instance

$$\psi^1_a(u_{(ab,\emptyset,N\setminus\{a,b\})}) - \psi^1_b(u_{(ab,\emptyset,N\setminus\{a,b\})}) = 2\varepsilon \neq 0.$$

Not null player Consider the index ψ^2 defined on unanimity games as follows.

- If $S = (\emptyset, a, N \setminus \{a\})$ for some $a \in N$, then

  $$\psi^2_a(u_S) = 1 - \varepsilon \quad \psi^2_b(u_S) = \frac{\varepsilon}{n-1}$$

  for any $b \neq a$, with $\varepsilon > 0$;
- if $S = (a, \emptyset, N \setminus \{a\})$ for some $a \in N$, then

  $$\psi^2_a(u_S) = 1 - 2\varepsilon \quad \psi^2_b(u_S) = \frac{2\varepsilon}{n-1}$$

  for any $b \neq a$, with $\varepsilon > 0$;
- for any other $S \in 3^N$, $\psi^2(u_S) = \phi(u_S)$ where ϕ is the Shapley-Shubik index for $(3,2)$-simple games.

Then extend ψ^2 to $\mathfrak{T}^N$ using transfer.
It is clear that this index satisfies anonymity and efficiency. It also satisfies the yes-abstain loss on unanimity game since it coincides with the Shapley-Shubik index on unanimity games such that $s_3 \neq n-1$ and if $s_3 = n-1$ and $a \in S_1$

$$2\psi^2(u_{S\downarrow a}) - \psi^2(u_S) = 1 = f(1,0).$$

However, ψ^2 does not satisfy null player: any $b \neq a$ is a null player in the game $u_{(\emptyset,a,N\setminus\{a\})}$ but $\psi^2_b(u_{(\emptyset,a,N\setminus\{a\})}) = \frac{\varepsilon}{n-1} \neq 0$.

Not transfer Consider the index ψ^3 defined as $\psi^3(u_S) = \phi(u_S)$ for any unanimity game u_S and for any other game v as

$$\psi^3_a(v) = \begin{cases} 0 & \text{if } a \text{ is a null player} \\ \frac{1}{k} & \text{otherwise,} \end{cases}$$

where $k = |\{p \in N : p \text{ is not a null player in } v\}|$.
The index ψ^3 satisfies the null player, the anonymity, and the efficiency axioms; it also satisfies the yes-abstain loss on unanimity game, since it coincides with the Shapley-Shubik index on unanimity games. However,

from its definition, it is clear that $\psi^3(v)$ does not satisfy the transfer axiom.

Not efficiency Consider the index ψ^4 defined on unanimity game as

$$\psi^4_a(u_S) = \begin{cases} 1 & \text{if } a \in S_1 \\ \frac{1}{2} + \frac{2^{s_2-1}}{3^{s_1+s_2-1}(s_1+s_2)} & \text{if } a \in S_2 \\ 0 & \text{if } a \in S_3 \end{cases}$$

and extended to $\mathfrak{T}^N$ using transfer. Then ψ^4 satisfies anonymity, null player and transfer. It also satisfies yes-abstain loss on unanimity game since for any tripartition S with $a \in S_1$:

$$\begin{aligned} 2\psi^4_a(u_{S\downarrow a}) &= 2\Big[\frac{1}{2} + \frac{2^{s_2-1}}{3^{s_1+s_2-1}(s_1+s_2)}\Big] \\ &= 1 + \frac{2^{s_2}}{3^{s_1+s_2-1}(s_1+s_2)} \\ &= \psi^4_a(u_S) + f(s_1, s_2) \end{aligned}$$

However, ψ^4 does not satisfy efficiency, for instance $\psi^4(u_{(N,\emptyset,\emptyset)}) = (1, \ldots, 1)$ so that $\sum_{a\in N} \psi^4(u_{(N,\emptyset,\emptyset)}) = n \neq 1$.

Not yes-abstain loss on unanimity game As we explained in remark 3.2, the index φ, that is Shapley-Shubik index for simple games computed on the simple games associated to the game with abstention, satisfies null player, anonymity, transfer, and efficiency, but does not satisfy the yes-abstain loss on unanimity game.

3.4.3 A similar approach for the Banzhaf index for games with abstention

We want to give a characterization of the Banzhaf index for games with abstention analogous to the one of the Shapley-Shubik given in the previous section. In section 3.3 we proved that this index satisfies transfer, anonymity, null player and the Banzhaf total power; however in the context of games with abstention these four axioms are not enough to uniquely characterize the index.

Let us start with a preliminary lemma that shows how simple is to compute the Banzhaf index for $(3, 2)$-simple games on unanimity games.

Lemma 3.6. *Let $S \neq (\emptyset, \emptyset, N)$ be a tripartition of N, then the Banzhaf index for $(3,2)$-simple games on the unanimity games u_S is*

$$\beta_p(u_S) = \begin{cases} \frac{2^{s_2}}{3^{s_1+s_2-1}} & \text{if } p \in S_1 \\ \frac{2^{s_2-1}}{3^{s_1+s_2-1}} & \text{if } p \in S_2 \\ 0 & \text{if } p \in S_3. \end{cases}$$

Proof. We have to compute $\eta_a(u_S)$ for any player $a \in N$. Assume that $a \in S_1$. Remember $u_S(T) = 1$ if and only if $S_1 \subseteq T_1$ and $S_2 \subseteq T_1 \cup T_2$. Note that if $a \in S_1 \cap T_1$, the condition $u_S(T) = 1$ implies $u_S(T_{\downarrow\downarrow a}) = 0$. Moreover, if $a \in S_1$, the conditions $a \in T_1$ and $u_S(T) = 1$ are equivalent to $S \subseteq T$. Hence,

$$\begin{aligned} \eta_a(u_S) &= |\{T \in 3^N : a \in T_1, u_S(T) - u_S(T_{\downarrow\downarrow a}) = 1\}| \\ &= |\{T \in 3^N : a \in T_1, u_S(T) = 1\}| \\ &= |\{T \in 3^N : S \subseteq T\}| = 2^{s_2} 3^{s_3}. \end{aligned}$$

Assume now $a \in S_2$. Analogously, it holds

$$\eta_a(u_S) = |\{T \in 3^N : a \in T_1 \text{ and } S \subseteq T\}| = 2^{s_2-1} 3^{s_3}.$$

Finally, suppose $a \in S_3$. Players in S_3 are null, so $\beta_a(u_S) = 0$. Since the Banzhaf index for $(3,2)$-simple games is given by

$$\beta_a(u_S) = \frac{\eta_a(u_S)}{3^{n-1}}$$

and $n = s_1 + s_2 + s_3$, we have the thesis.

□

Remark. Note that, in particular if $a \in S_1$ and $b \in S_2$, it holds

$$\beta_a(u_S) = 2\beta_b(u_S).$$

In the Dubey and Shapley (1979) characterization of the Banzhaf power index for simple games, the Banzhaf total power axiom is introduced in order to replace efficiency, that is used for the Shapley-Shubik power index. However, the Banzhaf total power axiom is not a convincing axiom; some subsequent axiomatic characterization of the Banzhaf power index avoided this axiom, see for instance Laruelle and Valenciano (2001), Lehrer (1988), and Albizuri (2001).

We want to follow the classical approach and use the same set of axioms to characterize the Shapley-Shubik and the Banzhaf indices for $(3, 2)$-simple games. However, we will replace the Banzhaf total power with a weaker condition that refers only to unanimity games.

Total power on unanimity games For any tripartition $S \neq (\emptyset, \emptyset, N)$

$$\sum_{a \in N} \psi_a(u_S) = (2s_1 + s_2) \frac{2^{s_2 - 1}}{3^{s_1 + s_2 - 1}}$$

Lemma 3.7. *The Banzhaf index for $(3, 2)$-simple games satisfies the total power on unanimity games axiom.*

Proof. Consider a unanimity game u_S, then thanks to Lemma 3.6 and anonymity we have

$$\begin{aligned}
\sum_{p \in N} \beta_p(u_S) &= \sum_{a \in S_1} \beta_a(u_S) + \sum_{b \in S_2} \beta_b(u_S) \\
&= s_1 \beta_a(u_S) + s_2 \beta_b(u_S) \\
&= 2s_1 \beta_b(u_S) + s_2 \beta_b(u_S) \\
&= (2s_1 + s_2) \frac{2^{s_2 - 1}}{3^{s_1 + s_2 - 1}}
\end{aligned}$$

with $a \in S_1$ and $b \in S_2$. $\square$

As we have previously done for the Shapley-Shubik index on $(3, 2)$-simple games, it is necessary to add another axiom in order to uniquely characterize the Banzhaf index on $(3, 2)$-simple games. The new axiom defined in equation (3.2), describes what a player is losing when passing from voting "yes" to abstaining; we have an analogous of the yes-abstain loss on unanimity game for the Banzhaf value that is the following:

Yes-abstain null lost Let u_S be a unanimity game and $a \in S_1$ then

$$\psi_a(u_S) = \psi_a(u_{S \downarrow a}).$$

The following proposition is the analogous for the Banzhaf index for $(3, 2)$-simple games of Proposition 3.1 and Proposition 3.2 for the Shapley-Shubik index for $(3, 2)$-simple games.

Proposition 3.3. *Consider the unanimity game u_S with $S = (S_1, S_2, S_3) \neq (\emptyset, \emptyset, N)$ and $a \in S_1$. Then the Banzhaf index for $(3,2)$-simple games satisfies the yes-abstain null lost axiom and*

$$\beta_a(u_S) = \beta_a(u_{S\downarrow a}).$$

Proof. The thesis follows from applying Lemma 3.6 to the game u_S and $u_{S\downarrow a}$. □

We can now prove the characterization of the Banzhaf index for $(3,2)$-simple games following the spirit of Dubey and Shapley (1979).

Theorem 3.2. *Let $\psi : \mathfrak{T}^N \to \mathbb{R}^n$ be an index for $(3,2)$-simple games, then ψ satisfies anonymity, null player, transfer, total power on unanimity games and yes-abstain null lost if and only if ψ is the Banzhaf index for $(3,2)$-simple games.*

Proof. We already proved that the Banzhaf index for $(3,2)$-simple games satisfies all these properties, so we just need to prove that if ψ is a power index that satisfies the hypothesis, then it is uniquely determined. We use induction on the number of minimal winning tripartitions of the game v.

Suppose that $|\mathcal{W}^m(v)| = 1$, then $v = u_S$ for some tripartition S. So we start proving that ψ coincides with the Banzhaf index on unanimity games.

We again use induction, this time on the cardinality of S_2.

$|S_2| = 0$. Then $S = (S_1, \emptyset, N \setminus S_1)$ for some $S_1 \subseteq N$. Then players in $S_3 = N \setminus S_1$ are null, so $\psi_c(u_S) = 0$ for all $c \in S_3$. Players in S_1 are symmetric and thanks to anonymity and total power on unanimity game, if $a \in S_1$

$$\beta_a(u_{(S_1, \emptyset, N \setminus S_1)}) = \frac{1}{s_1} \frac{2 s_1 2^{-1}}{3^{s_1 - 1}} = \frac{1}{3^{s_1 - 1}}.$$

So, ψ is uniquely determined on unanimity games with $s_2 = 0$ and it coincides with the Banzhaf index for $(3,2)$-simple games.

$|S_2| = t + 1$. Suppose now that the thesis is true for any tripartition T such that $|T_2| \leq t$, we want to prove this for a tripartition S such that $|S_2| = t + 1$. Given a tripartition $S = (S_1, S_2, S_3)$ such that $|S_2| = t + 1$, there exist a player $p \in S_2$ and a tripartition $T = (T_1, T_2, T_3)$ such that $T_{\downarrow p} = S$ and $T_2 = t$. Since ψ satisfies yes-abstain null lost:

$$\psi_p(u_S) = \psi_p(u_{T\downarrow p}) = \psi_p(u_T),$$

then the induction hypothesis and anonymity imply that $\psi_b(u_S)$ is uniquely determined for all players $b \in S_2$.
Using again anonymity and the total power on unanimity game:

$$s_1\psi_a(u_s) + s_2\psi_b(u_S) = (2s_1 + s_2)\frac{2^{s_2-1}}{3^{s_1+s_2-1}}$$

with $a \in S_1$ and $b \in S_2$. So, we can determine $\psi_a(u_S)$ for $a \in S_1$. Thanks to null player we have that $\psi_c(u_S) = 0$ if $c \in S_3$. Thus, ψ coincides with the Banzhaf index for $(3, 2)$-simple games for any unanimity game u_S.

We suppose that the thesis holds for any game v such that $|\mathcal{W}^m(v)| \leq k - 1$, and prove it for v such that $|\mathcal{W}^m(v)| = k$.
If $\mathcal{W}^m(v) = \{S^1, \dots, S^k\}$, then $v = u_{S^1} \vee u_{S^2} \vee \cdots \vee u_{S^k}$, since ψ satisfies the transfer axiom:

$$\psi(v) = \psi(u_{S^1}) + \psi(u_{s^2} \vee \cdots \vee u_{S^k}) - \psi(u_{S^1} \wedge u_{S^2} \wedge \cdots \wedge u_{S^k})$$

The conjunction of unanimity games is still a unanimity game, so all games on the right-hand side of the previous equation have a number of minimal winning tripartitions smaller than $|W^m(v)|$. Using the induction hypothesis, ψ coincides with the Banzhaf index for $(3, 2)$-simple games for all of them and this ends the proof.

□

Remark. In the characterization given by the previous theorem the yes-abstain null lost can be replaced by the following property:

$$\psi_a(u_S) = 2\psi_b(u_S)$$

for any $a \in S_1$ and any $b \in S_2$.

3.4.4 Independence of the axioms

The five axioms for $(3, 2)$-simple games used in Theorem 3.2 are independent. We are going to give examples of power indices on games with abstention that satisfy only four of them, as summarized in Table 3.3.

Not anonymity Consider the index γ^1 defined on unanimity games as follows.

- If $s_3 = n - 2$ then for any two players a and b such that $a < b$,

	Anonymity	Null	Transfer	Total power	Yes-abstain
γ^1	-	✓	✓	✓	✓
γ^2	✓	-	✓	✓	✓
γ^3	✓	✓	-	✓	✓
γ^4	✓	✓	✓	-	✓
γ^5	✓	✓	✓	✓	-

Table 3.3: *Independence of the axioms to characterize the Banzhaf index for* $(3,2)$*-simple games*

– if $S = (ab, \emptyset, N \setminus \{a,b\})$, then

$$\gamma_a^1(u_S) = \frac{1}{3} + \varepsilon, \quad \gamma_b^1(u_S) = \frac{1}{3} - \varepsilon;$$

– if $S = (a, b, N \setminus \{a,b\})$, then

$$\gamma_a^1(u_S) = \frac{2}{3} + \varepsilon, \quad \gamma_b^1(u_S) = \frac{1}{3} - \varepsilon;$$

– if $S = (b, a, N \setminus \{a,b\})$, then

$$\gamma_a^1(u_S) = \frac{1}{3} + \varepsilon, \quad \gamma_b^1(u_S) = \frac{2}{3} - \varepsilon;$$

– if $S = (\emptyset, ab, N \setminus \{a,b\})$, then

$$\gamma_a^1(u_S) = \frac{2}{3} + \varepsilon, \quad \gamma_b^1(u_S) = \frac{2}{3} - \varepsilon;$$

where $\varepsilon > 0$.

- If $s_3 \neq n-2$, $\gamma^1(u_S) = \beta(u_S)$ where β is the Banzhaf index for $(3,2)$-simple games.

Then extend γ^1 to $\mathfrak{T}^N$ using transfer.
This index satisfies null player, total power on unanimity games and yes-abstain null lost. However, γ^1 does not satisfy anonymity, because for instance

$$\gamma_a^1(u_{(ab,\emptyset,N\setminus\{a,b\})}) \neq \gamma_b^1(u_{(ab,\emptyset,N\setminus\{a,b\})}).$$

Not null player Consider the index γ^2 defined on unanimity games as follows.

- If $S = (\emptyset, a, N \setminus \{a\})$ or $S = (a, \emptyset, N \setminus \{a\})$ for some $a \in N$, then
$$\gamma_a^2(u_S) = 1 - \varepsilon \quad \gamma_b^2(u_S) = \frac{\varepsilon}{n-1}$$
for any $b \neq a$ and with $\varepsilon > 0$;
- for any other $S \in 3^N$, $\gamma^2(u_S) = \beta(u_S)$ where β is the Banzhaf index for $(3,2)$-simple games.

Then extend γ^2 to $\mathfrak{T}^N$ using transfer.
This index satisfies anonymity, total power on unanimity games and yes-abstain null lost. However, γ^2 does not satisfy null player: any $b \neq a$ is a null player in the game $u_{(\emptyset,a,N\setminus\{a\})}$ but $\gamma_b^2(u_{(\emptyset,a,N\setminus\{a\})}) = \frac{\varepsilon}{n-1} \neq 0$.

Not transfer Consider the index γ^3 defined as $\gamma^3(u_S) = \beta(u_S)$ for any unanimity game u_S and for any other game v

$$\gamma_a^3(v) = \begin{cases} 0 & \text{if } a \text{ is a null player} \\ \frac{1}{k} & \text{otherwise} \end{cases}$$

where $k = |\{p \in N : p \text{ is not a null player in } v\}|$.
The index γ^3 satisfies the null player and the anonymity axioms. It also satisfies the total power on unanimity games and the yes-abstain null power, since it coincides with the Banzhaf index on unanimity games. From the definition of $\gamma^3(v)$ it is clear that this index does not satisfy the transfer axiom.

Not total power on unanimity games Consider the index γ^4 defined on unanimity game as

$$\gamma_a^4(u_S) = \begin{cases} \frac{1}{s_1+s_2} & \text{if } a \in S_1 \cup S_2 \\ 0 & \text{if } a \in S_3 \end{cases}$$

and extended to $\mathfrak{T}^N$ using transfer.
The index γ^4 satisfies anonymity, null player, and yes-abstain null lost. However, γ^4 satisfies efficiency instead of total power on unanimity game.

Not yes-abstain null lost Consider the index γ^5 defined on unanimity game as
$$\gamma_a^5(u_S) = \begin{cases} \frac{2s_1+s_2}{s_1+s_2} \frac{2^{s_2-1}}{3^{s_1+s_2-1}} & \text{if } a \in S_1 \cup S_2 \\ 0 & \text{if } a \in S_3 \end{cases}$$

and extended $\mathfrak{T}^N$ using transfer.
The index γ^4 satisfies anonymity, null player, and total power on unanimity games. However, it does not satisfy yes-abstain null lost:

$$\gamma_a^5(u_S) = \frac{2s_1 + s_2}{s_1 + s_2} \frac{2^{s_2-1}}{3^{s_1+s_2-1}} \neq \frac{2s_1 + s_2 - 1}{s_1 + s_2} \frac{2^{s_2}}{3^{s_1+s_2-1}} = \gamma_a^5(u_{S\downarrow a}).$$

3.5 Another approach for the Banzhaf index for games with abstention

The previous axiomatization for the Banzhaf index is unsatisfactory from different reasons. Even if it is nice to have similar results for both the Banzhaf and the Shapley-Shubik power indices, it is clear that the Banzhaf total power, or the Total power on unanimity games, are tautological axioms not so many interesting from an a priori analysis of the power of players. Moreover, in the context of games with abstention the Banzhaf index can be decomposed in two different indices that can better capture the influence of players in the game.

For these reasons we follow the approach of Laruelle and Valenciano (2001) for simple games and merge it with the work of Freixas and Lucchetti (2016) for games with abstention to provide a new characterization of the Banzhaf index for games with abstention.

3.5.1 A new set of axioms

In this section we introduce the new axioms to characterize the Banzhaf index for games with abstention and its two components. We also discuss their connections with the generalization to the context of games with abstention of the classical axioms introduced by Dubey and Shapley (1979).

In our axiomatization we describe how a power index changes when we remove a minimal winning tripartition from a game, the crucial element will be the game v_S^* defined in the following way.

Definition 3.8. Given a game $v \neq u_{(N,\emptyset,\emptyset)} \in \mathfrak{T}^N$ and $S \in \mathcal{W}^m(v)$, the game v_S^* is such that $\mathcal{W}(v_S^*) = \mathcal{W}(v) \setminus S$.

From the previous Definition we have

$$v_S^*(T) = \begin{cases} v(T) & \text{if } T \neq S \\ 0 & \text{if } T = S. \end{cases}$$

Some of the axioms we are going to use will propose comparisons among the value of a power index on the game v and on the game v_S^*.

We are going to use Axiom 3.2 and Axiom 3.3, but we replace transfer and the tautological Banzhaf total power with two other axioms that are a generalization to the family of games with abstention of the two axioms introduced by Laruelle and Valenciano (2001) for simple games: transfer* and average balance.

Axiom 3.6 (Transfer*)**.** For any $v, w \neq u_{(N,\emptyset,\emptyset)} \in \mathfrak{T}^N$ and any minimal winning tripartition S for both games (i.e. $S \in \mathcal{W}^m(v) \cap \mathcal{W}^m(w)$) then

$$\varphi_i(v) - \varphi_i(v_S^*) = \varphi_i(w) - \varphi_i(w_S^*)$$

for all $i \in N$.

The transfer axiom is a way to translate linearity from the set of cooperative games to the set of simple games, but it is not intuitive to see its interpretation as the exchange of power from the games v and w and the disjunction and conjunction games. On the other hand, the previous axiom is very clear: when we remove a minimal winning tripartition from two different games, players' power should change in the same way. Actually, Laruelle and Valenciano (2001) proved that transfer and transfer* are equivalent axioms and this result holds also in our context.

Proposition 3.4. *Transfer and transfer* are equivalent.*

Proof. Let φ be a power index that satisfies transfer. Take $v, w \in \mathfrak{T}^N$ and $S \in \mathcal{W}^m(v) \cap \mathcal{W}^m(w)$. Then $v = v_S^* \vee u_S$ and $w = w_S^* \vee u_S$, where u_S is the unanimity game generated by the tripartition S. Using transfer it holds that:

$$\begin{aligned} \varphi(v) &= \varphi(v_S^* \vee u_S) = \varphi(v_S^*) + \varphi(u_S) - \varphi(v_S^* \wedge u_S) \\ \varphi(w) &= \varphi(w_S^* \vee u_S) = \varphi(w_S^*) + \varphi(u_S) - \varphi(w_S^* \wedge u_S). \end{aligned}$$

Since $v_S^* \wedge u_S = (u_S)_S^*$ and $w *_S \wedge u_S = (u_S)_S^*$, φ satisfies transfer*.

On the other hand, suppose φ satisfies transfer*. Then $\mathcal{W}(v \wedge w) = \mathcal{W}(v) \cap \mathcal{W}(w)$ and $\mathcal{W}(v \vee w) = \mathcal{W}(v) \cup \mathcal{W}(w)$, so it holds that

$$\mathcal{W}(v) \setminus \mathcal{W}(v \vee w) = \mathcal{W}(v \wedge w) \setminus \mathcal{W}(w).$$

This means that reaching w from $v \wedge w$ takes dropping one-by-one exactly the same winning coalitions as reaching $v \vee w$ from v. By transfer* the effect on the index of deleting a minimal winning coalition is the same in any game; consequently transfer holds:

$$\varphi(v) - \varphi(v \vee w) = \varphi(v \wedge w) - \varphi(w).$$

□

Now we introduce some conditions to replace the Banzhaf total power axiom: the average X-balance (where X stands for yes-no, yes-abstain and abstain-no). These axioms translate in the context of games with abstention the average gain-loss balance defined by Laruelle and Valenciano (2001). The average gain-loss balance axiom is the one differentiating the Banzhaf index from the Shapley-Shubik index and from any other semivalue (for an overview on semivalues we refer to Monderer and Samet (2002)). In the context of games with abstention, this axiom is formed by two different conditions. For instance, if we consider the average yes-no balance there is one condition for players voting "yes" or "no" and one for players abstaining; the idea is that when we compare the games v and v_S^*, players in S_1 lose power as players in S_3 gain power. Instead players in S_2 gain and lose the same power in changing game from v to v_S^* because they are crucial in both games but in two opposite ways: moving from "abstain" to "yes" and moving from "abstain" to "no".

Axiom 3.7 (Average yes-no balance)**.** For any $v \neq u_{(N,\emptyset,\emptyset)} \in \mathfrak{T}^N$ and any $S = (S_1, S_2, S_3) \in \mathcal{W}^m(v)$

$$s_3 \sum_{i \in S_1} \Big[\varphi_i(v) - \varphi_i(v_S^*)\Big] = s_1 \sum_{i \in S_3} \Big[\varphi_i(v_S^*) - \varphi_i(v)\Big]$$

and

$$\varphi_i(v) = \varphi_i(v_S^*) \qquad \forall i \in S_2.$$

In order to characterize the two components of the Banzhaf index, we also need the following specific axioms.

Axiom 3.8 (Average yes-abstain balance)**.** For any $v \neq u_{(N,\emptyset,\emptyset)} \in \mathfrak{T}^N$ and any $S = (S_1, S_2, S_3) \in \mathcal{W}^m(v)$

$$s_2 \sum_{i \in S_1} \Big[\varphi_i(v) - \varphi_i(v_S^*)\Big] = s_1 \sum_{i \in S_2} \Big[\varphi_i(v_S^*) - \varphi_i(v)\Big]$$

and

$$\varphi_i(v) = \varphi_i(v_S^*) \qquad \forall i \in S_3.$$

Axiom 3.9 (Average abstain-no balance)**.** For any $v \neq u_{(N,\emptyset,\emptyset)} \in \mathfrak{T}^N$ and any $S = (S_1, S_2, S_3) \in \mathcal{W}^m(v)$

$$s_3 \sum_{i \in S_2} \Big[\varphi_i(v) - \varphi_i(v_S^*)\Big] = s_2 \sum_{i \in S_3} \Big[\varphi_i(v_S^*) - \varphi_i(v)\Big]$$

and

$$\varphi_i(v) = \varphi_i(v_S^*) \qquad \forall i \in S_1.$$

The last axiom we introduce states that all players voting in the same level in S lose or gain the same power when moving from v to v_S^*.

Axiom 3.10 (Symmetric gain-loss)**.** For any $v \in \mathfrak{T}^N$ and any $S \in \mathcal{W}^m(v)$, for any $i, j \in S_1$ (or $i, j \in S_2$, or $i, j \in S_3$)

$$\varphi_i(v) - \varphi_i(v_S^*) = \varphi_j(v) - \varphi_j(v_S^*).$$

The previous axiom describes a symmetry among players, but in the characterization provided by Laruelle and Valenciano (2001) it is used as a substitute for transfer. Actually symmetric gain-loss and transfer are independent, but if anonymity is assumed then there is a correlation with symmetric gain-loss and transfer*, as the next proposition and the following counter-example show.

Proposition 3.5. *Anonymity and transfer* imply symmetric gain-loss.*

Proof. Let $v \in \mathfrak{T}^N$, $S \in \mathcal{W}^m(v)$, $i, j \in S_1$ and π be a transposition between i and j. Then $\pi(S) = S$ and $S \in \mathcal{W}^m(v) \cap \mathcal{W}^m(\pi v)$.
Note that $(\pi v)_S^* = \pi(v_{\pi(S)}^*) = \pi(v_S^*)$. Using anonymity and transfer*

$$\begin{aligned}\varphi_i(v) - \varphi_i(v_S^*) &= \varphi_i(\pi v) - \varphi_i((\pi v)_S^*) = \varphi_i(\pi v) - \varphi_i(\pi(v_S^*)) \\ &= \varphi_{\pi(i)}(v) - \varphi_{\pi(i)}(v_S^*) = \varphi_j(v) - \varphi_j(v_S^*).\end{aligned}$$

If $i, j \in S_2$ or $i, j \in S_3$, the proofs are analogous.

$\square$

It is possible to find an index that satisfies anonymity and symmetric gain-loss but does not satisfy transfer*.

Example 3.2. Let $N = \{a, b\}$ and the games v, w defined by

$$\begin{aligned}\mathcal{W}^m(v) &= \{(\emptyset, ab, \emptyset), (a, \emptyset, b)\} \\ \mathcal{W}^m(w) &= \{(\emptyset, ab, \emptyset)\}\end{aligned}$$

If we take $S = (\emptyset, ab, \emptyset)$ then we have $\mathcal{W}^m(v_S^*) = \{(a, \emptyset, b), (b, a, \emptyset)\}$; $\mathcal{W}^m(w_S^*) = \{(a, b, \emptyset), (b, a, \emptyset)\}$.
Let φ be the index defined as

$$\begin{aligned} \varphi(v) = &\ (1, 0.6) \qquad & \varphi(v_S^*) = (0.7, 0.3) \\ \varphi(w) = &\ (0.5, 0.5) \qquad & \varphi(w_S^*) = (0.3, 0.3) \end{aligned}$$

extended by anonymity to all games that are symmetric to these four and defined as zero for all the other games.
Then φ satisfies symmetric gain-loss (S is the only tripartition for which there are two players belonging to the same subset). But φ does not satisfy transfer*: if we consider $S \in \mathcal{W}^m(v) \cap \mathcal{W}^m(w)$ then $\varphi_a(v) - \varphi_a(v_S^*) = 0.3 \neq \varphi_a(w) - \varphi_a(w_S^*) = 0.2$.

3.5.2 Characterization of the Banzhaf index

We can now state and prove the main result with the characterization of the (raw) Banzhaf index for games with abstention and its two components. In the characterization of the index there are two different steps: the existence of a function satisfying the axioms and then the uniqueness of that function, up to a scalar factor. The following lemma and its corollary will be used in both part of the proof, they provide a measure of the difference between the power of players in v and in v_S^*, when we consider the raw Banzhaf index for games with abstention or one of its two components.

Lemma 3.8. *Let $v \neq u_{(N,\emptyset,\emptyset)} \in \mathfrak{T}^N$ and $S = (S_1, S_2, S_3) \in \mathcal{W}^m(v)$, then*

$$\eta_i^{YA}(v) - \eta_i^{YA}(v_S^*) = \begin{cases} +1 & \textit{if } i \in S_1 \\ -1 & \textit{if } i \in S_2 \\ 0 & \textit{if } i \in S_3 \end{cases}$$

and

$$\eta_i^{AN}(v) - \eta_i^{AN}(v_S^*) = \begin{cases} 0 & \textit{if } i \in S_1 \\ +1 & \textit{if } i \in S_2 \\ -1 & \textit{if } i \in S_3. \end{cases}$$

Proof. The games v and v_S^* are identical, except for the value of the tripartition S; so all players will have the same number of swings except the one concerning S. In particular

- if $i \in S_1$, then i is crucial in v when moving from S to $S_{\downarrow i}$ but he is not in v^*_S because both S and $S_{\downarrow i}$ are losing tripartitions, thus $\eta_i^{YA}(v) = \eta_i^{YA}(v^*_S) + 1$ and $\eta_i^{AN}(v) = \eta_i^{AN}(v^*_S)$;
- if $i \in S_3$, then i is crucial in v^*_S when moving from S to $S_{\uparrow i}$, but he is not in v, thus $\eta_i^{AN}(v) = \eta_i^{AN}(v^*_S) - 1$ and $\eta_i^{YA}(v) = \eta_i^{YA}(v^*_S)$;
- if $i \in S_2$, then he is a YA crucial player in v^*_S when moving from S to $S_{\uparrow i}$, but he is not crucial in v; conversely if i moves down from S to $S_{\downarrow i}$ he is a crucial player in v and not in v^*_S. So $\eta_i^{YA}(v) = \eta_i^{YA}(v^*_S) - 1$ and $\eta_i^{AN}(v) = \eta_i^{AN}(v^*_S) + 1$.

□

From the previous lemma and the definition of the raw Banzhaf index for games with abstention, we immediately have the following.

Corollary 3.1. *Let $v \neq u_{(N,\emptyset,\emptyset)} \in \mathfrak{T}^N$ and $S = (S_1, S_2, S_3) \in \mathcal{W}^m(v)$, then*

$$\eta_i(v) - \eta_i(v^*_S) = \begin{cases} +1 & \textit{if } i \in S_1 \\ 0 & \textit{if } i \in S_2 \\ -1 & \textit{if } i \in S_3. \end{cases}$$

We can now state and prove our main result regarding the characterization of the (raw) Banzhaf index for games with abstention.

Theorem 3.3. *Let φ be a value on the class of $(3, 2)$-games. Then φ satisfies anonymity, YN-null player, transfer* and average yes-no balance if and only if $\varphi = \alpha\eta$ for some real number $\alpha > 0$.*

Proof. ($\Leftarrow$) If $\varphi = \alpha\eta$, then using Corollary 3.1 and the definition of η, it is trivial to check that φ satisfies all the listed properties.

($\Rightarrow$) Let φ be a value that satisfies the four properties. Note that thanks to Proposition 3.5, φ satisfies also symmetric gain-loss.
Define $\alpha := \varphi_i(u_{(N,\emptyset,\emptyset)})$ for any player i. Thanks to anonymity α is well-defined. We will prove that for any $v \in \mathfrak{T}^N$, $\varphi(v) = \alpha\eta(v)$, proceeding by induction on the number of winning tripartitions in v.

If $|\mathcal{W}(v)| = 1$ then $v = u_{(N,\emptyset,\emptyset)}$. Anonymity implies $\varphi_i(u_{(N,\emptyset,\emptyset)}) = \varphi_j(u_{(N,\emptyset,\emptyset)})$ for any $i, j \in N$, moreover $\eta_i(u_{(N,\emptyset,\emptyset)}) = 1$ and the thesis follows from the choice of α.

Suppose now that the thesis is true for any game w such that $|\mathcal{W}(w)| < t$ and let us prove it for a game v with $|\mathcal{W}(v)| = t$. We consider two different situations depending on the number of minimal winning tripartitions in v.

If $|\mathcal{W}^m(v)| = 1$ then $v = u_S$ for some tripartition S. Since φ satisfies average yes-no balance and symmetric gain-loss we can write

$$\varphi_i(u_S) - \varphi_i((u_S)^*_S) = \varphi_j((u_S)^*_S) - \varphi_j(u_S) \qquad \forall i \in S_1, j \in S_3 \quad (3.6)$$

but $\varphi_j(u_S) = 0$ because $j \in S_3$ and φ satisfies *YN*-null player. Thanks to the induction hypothesis and Corollary 3.1, for any $i \in S_1$ and $j \in S_3$:

$$\begin{aligned}\varphi_i(u_S) &= \varphi_j((u_S)^*_S) + \varphi_i((u_S)^*_S) \\ &= \alpha\eta_j((u_S)^*_S) + \alpha\eta_i((u_S)^*_S) \\ &= \alpha[\eta_j(u_S) + 1 + \eta_i(u_S) - 1] \\ &= \alpha\eta_i(u_S)\end{aligned}$$

since $\eta_j(u_S) = 0$.

If $i \in S_2$ thanks to average yes-no balance, the induction hypothesis and Corollary 3.1 we have $\varphi_i(u_S) = \varphi_i((u_S)^*_S) = \alpha\eta_i((u_S)^*_S) = \alpha\eta_i(u_S)$. If $i \in S_3$ then the thesis is trivial thanks to the *YN*-null player property.

So far, we proved that $\varphi(v) = \alpha\eta(v)$ for any v such that $\mathcal{W}^m(v) = 1$. Now, if $|\mathcal{W}^m(v)| > 1$ there are in v at least two minimal winning tripartitions S and T. Note that if we remove two minimal winning tripartitions from a game the result does not depend on the order in which we remove them, i.e. $(v^*_S)^*_T = (v^*_T)^*_S$. To simplify the notation, we use $v^*_{S,T}$ to denote this game. We also have that $S \in \mathcal{W}^m(v) \cap \mathcal{W}^m(v^*_T)$ and thanks to transfer* it holds

$$\varphi_i(v) - \varphi_i(v^*_S) = \varphi_i(v^*_T) - \varphi_i((v^*_{T,S})$$

for any $i \in N$.

Then using the induction hypothesis and Corollary Corollary 3.1, if $i \in S_1$ we have

$$\begin{aligned}\varphi_i(v) &= \varphi_i(v^*_S) + \varphi_i(v^*_T) - \varphi_i(v^*_{T,S}) \\ &= \alpha[\eta_i(v^*_S) + \eta_i(v^*_T) - \eta_i(v^*_{T,S})] \\ &= \alpha[\eta_i(v^*_S) + 1] \\ &= \alpha\eta_i(v).\end{aligned}$$

Analogously if $i \in S_2$ and $i \in S_3$ we can prove that $\varphi_i(v) = \alpha\eta_i(v)$ for any $v \in \mathfrak{T}^N$.

□

In the previous theorem we proved that, up to a scalar factor α, there is a unique function that satisfies anonymity, transfer*, YN-null player and yes-no average balance. For instance, if $\alpha = 1$ the value is the raw Banzhaf index η, while if $\alpha = \frac{1}{3^{n-1}}$ we get the Banzhaf index for games with abstention.

In the following theorem we show that the two components of the Banzhaf index for games with abstention can be analogously characterized.

Theorem 3.4. *Let φ be a value on the class of $(3,2)-$games. Then φ satisfies*

1. *anonymity, YA-null player, transfer* and average yes-abstain balance if and only if $\varphi = \alpha\eta^{YA}$ for some real number $\alpha > 0$;*
2. *anonymity, AN-null player, transfer* and average abstain-no balance if and only if $\varphi = \alpha\eta^{AN}$ for some real number $\alpha > 0$.*

Proof. 1. ($\Rightarrow$) If $\varphi = \alpha\eta^{YA}$, then using Corollary 3.1 and the definition of η^{YA}, it is trivial to check that φ satisfies all the listed properties.

($\Leftarrow$) The second part of the proof is very similar to the proof of Theorem 3.3: we choose $\alpha = \varphi_i(u_{(N,\emptyset,\emptyset)})$ then proceed by induction on the number of winning tripartitions. YN-null player and yes-no average balance are replaced by YA-null player and yes-abstain average balance. The correspondent of Equation (3.6) is

$$\varphi_i(u_S) - \varphi_i((u_S)^*_S) = \varphi_j((u_S)^*_S) - \varphi_j(u_S) \qquad \forall i \in S_1, j \in S_2.$$

From that, following the same steps as in the proof of Theorem 3.3 and using Lemma 3.8 we get the thesis.

2. ($\Rightarrow$) If $\varphi = \alpha\eta^{AN}$, then using Corollary 3.1 and the definition of η^{AN}, it is trivial to check that φ satisfies all the listed properties.

($\Leftarrow$) This proof is also very similar to the proof of Theorem 3.3: we we choose $\alpha = \varphi_i(u_{(N,\emptyset,\emptyset)})$, proceed by induction and replace YN-null player and yes-no average balance with AN-null player and abstain-no average balance. In this case, the correspondent of Equation (3.6) is

$$\varphi_i(u_S) - \varphi_i((u_S)^*_S) = \varphi_j((u_S)^*_S) - \varphi_j(u_S) \qquad \forall i \in S_2, j \in S_3.$$

From that, following the same steps as in the proof of Theorem 3.3 and using Lemma 3.8 we get the thesis.

□

Remark 3.4. The previous theorems and Definition 3.4 imply that a value φ that satisfies anonymity, YN-null player, transfer* and yes-no average balance can be decomposed as the sum of two function φ_1, which satisfies, anonymity, transfer*, YA-null player, yes-abstain balance and φ_2, which satisfies, anonymity, transfer*, AN-null player and abstain-no balance.

Note that, in general, the opposite is not true: the sum of two functions φ_1 and φ_2, as before, satisfies anonymity, transfer* and YN-null player but does not satisfy yes-no balance. For instance, take $\varphi_1 = \eta^{YA}$ and $\varphi_2 = \beta^{AN}$, then using Lemma 3.8 it is easy to see that $\varphi_1 + \varphi_2$ does not satisfy yes-no balance, actually there is not $\alpha > 0$ such that $\alpha\eta$ is equal to the sum $\eta^{YA} + \beta^{AN}$.

Let us make some concluding remarks and comparison with our work and the work of Laruelle and Valenciano (2001) for simple games to stress the difference among the classical model and games with abstention. Laurelle and Valenciano characterized the Banzhaf index on the space of super additive games, using the equivalent for simple games of the axioms we defined in this chapter. However, in our main theorem we use transfer*, the restatement of the transfer axiom, instead of symmetric gain-loss, which is used by Laruelle and Valenciano (2001). This is because we were looking for a more general result and wanted to characterize the index on the whole family of (3,2)-simple games not only on super-additive games. Super-additivity for simple games means that it is not possible to have a winning coalition S such that also $N \setminus S$ is winning. This condition in particular implies that any two winning coalitions have at least one player in common. It is not clear how to translate such a condition in the context of tripartition games: what is the complement set of a tripartition? What is the interpretation of super-additivity for a game with abstention? Is it meaningful to study just that class of games? All these questions are still open and could be further investigated.

3.6 Conclusion

In this chapter we gave a characterization of the two main power indices extended from simple games to simple games with abstention: the Banzhaf and the Shapley-Shubik power indices. Firstly, we followed the classical approach using a generalized version of the transfer, null player, symmetric, efficiency and total power axioms. Since these axioms are not sufficient, we added other axioms regarding the behaviour of a power index on unanimity games in order to uniquely characterize the indices. However, the interpretation of these new axioms is not clear and their meaning is not transparent as one would like them to be. It may be interesting for future works to develop new studies about the properties of these indices in order to increase their use.

We started this new development following the work in Laruelle and Valenciano (2001) to give a characterization of the Banzhaf index for games with abstention. Of course, it would be interesting to provide another axiomatization also for the Shapley-Shubik index for games with abstention. However, apparently, there is not e a theorem analogous to Theorem 3.3 for the Shapley-Shubik index for games with abstention because it was not possible to use the same set of axioms used for the Banzhaf index. In particular it was not possible to find a relation for the Shapley-Shubik index for games with abstention analogous of the ones stated in Corollary 3.1 and in Lemma 3.8 for the Banzhaf index for games with abstention. The main difficulty is that there is not an explicit formula to compute the Shapley-Shubik index for games with abstention that can be used to compare the power of players in a game v and in the modified game v_S^*.

Note that the yes-abstain loss on unanimity game we introduced, together with the null-player axiom, anonymity and efficiency, allows to compute the Shapley-Shubik index for (3,2)-simple games on unanimity games using a recursive formula. This is an improvement with respect to using the definition of roll-calls and pivotal players. However, in literature there is still not an explicit formula, even for unanimity games, to directly compute the Shapley-Shubik index for (3,2)-simple games. We provide such formula in the following chapter following the ideas developed in chapter 2 for simple games. This new formula will be deduced as a special case of a more general result for multichoice cooperative games and it will provide a somehow explicit way to compute the Shapley-Shubik index for games with abstention. However it will not solve the problem of having a result equivalent to Lemma 3.8 that provides an effective comparison of the power of players in a game v and in the modified game v_S^*.

CHAPTER 4

A value for multichoice cooperative games

COALITIONAL games with transferable utility describe a situation in which players are supposed to form coalitions to cooperate and reach a common goal. This classical model has been generalized by several authors to describe situations in which players can have different levels of participation in the cooperation (or vote among different alternatives).

In this chapter we discuss and compare different approaches to this problem, and then define a value for multichoice games. The value we are proposing is deduced from the Shapley-Shubik index for $(j, 2)$-games. We provide an explicit formula to compute this index and show that such formula can be generalized to define a value on the family of multichoice cooperative games.

4.1 Multichoice games

In chapter 3 we presented the model of games with abstention to model voting situation in which players can vote yes, no or abstain. Games with abstention can be seen as $(3, 2)$-games, since players have three alternatives and there are two possible outputs (winning or losing). This model can be generalized to (j, k) games, introduced by Freixas and Zwicker (2003), in which players vote choosing among j (ordered) different alternatives and there are k possible outcomes. If the outcomes are not a fixed number of elements, but the result is a real number reflecting the value that a j-partition can obtain, we have a multichoice game, as defined in Hsiao and Raghavan (1993). In this section we present these two models.

The following notation is similar to the one introduced in section 3.1 for games with abstention. N is the finite set of *players* or *voters* with cardinality n. A *j-partition* of the set N is a collection of j mutually disjoint subsets of N, $S_1, \ldots, S_j$, such that $S_1 \cup \cdots \cup S_j = N$, note that any S_l can be empty. We also use s_l or $|S_l|$to denote the cardinality of a subset S_l. We denote with J^N the set of all j-partitions on N.

A j-partition describes a division of players among the j different alternatives, or levels of support to a specific activity, or levels of voting support to a given proposal. Thus, players in S_l are taking the action of working at the l^{th} level and players in S_j are not doing anything. In a voting context, players in S_1 are assumed to vote for the highest level of approval, players in S_2 are voting for the second highest level of approval, and so on until players in S_j who are assumed to vote for the lowest level of approval.

It is possible to define a partial order $\subseteq$ on the set J^N. If $S, T \in J^N$, then $S \subseteq T$ means $S_l \subseteq \cup_{i=1}^{l} T_i$ for any $l = 1, \ldots, j$. In other words, a j-partition S is contained in a j-partition T if players in T are working or voting in the same or in a higher level than in S. We use $S \subset T$ if $S \subseteq T$ and $S \neq T$. The j-partition $(\emptyset, \ldots, \emptyset, N)$ is the minimum of the order $\subseteq$, while the maximum is the j-partition $(N, \emptyset, \ldots, \emptyset)$.

Voting situation in which players have more than two input alternatives can be described as a $(j, 2)$-simple game: voters can vote for j different levels to approve or reject a resolution.

Definition 4.1. Let N be a finite set and J^N be the set of all j-partitions on N; a $(j, 2)$-*simple game* is a function $v : J^N \to \{0, 1\}$ such that

- it is monotonic: if $S \subseteq T$ then $v(S) \leq v(T)$;
- $v(\emptyset, \ldots, \emptyset, N) = 0$ and $v(N, \emptyset, \ldots, \emptyset) = 1$.

We denote with $\mathcal{SJ}^N$ the space of all $(j, 2)$-simple games on the finite set N.

Note that simple games are (2,2)-simple games, while when voters have three ordered alternatives we have $(3, 2)$-simple games or *ternary voting games* as defined by Felsenthal and Machover (1997), when the middle level is abstention, as we discussed in chapter 3. In a $(j, 2)$-simple game, a j-partition S is *winning* if $v(S) = 1$, and it is *losing* otherwise.

The family of $(j, 2)$-simple games is a subclass of the family of (j, k) games introduced by Freixas and Zwicker (2003).

Definition 4.2. Let N be a finite set and J^N be the set of all j-partitions on N; let $w_1, w_2, \ldots, w_k$ be k objects with a strict linear ordering $w_1 \succ w_2 \succ \cdots \succ w_k$. Then a (j, k)*-simple game* is a function $v : J^N \to \{w_1, w_2, \ldots, w_k\}$ that is monotonic, i.e., for any $S, T \in J^N$, such that $S \subseteq T$, then $v(S) \leq v(T)$.

Given a (j, k)-simple game, it is possible to add a *numeric evaluation* $\alpha : \{w_1, w_2, \ldots, w_k\} \to \mathbb{R}^k$ that assigns to each output a real number, we write $\alpha(w_i) = \alpha_i$, and that preserves the order, that is $\alpha_i > \alpha_{i+1}$. It is standard to normalize by taking $\alpha_k = 0$

A (j, k)-simple game with a numeric evaluation α can be seen as a special subclass of multichoice games.

Definition 4.3. Let N be a finite set and J^N be the set of all j-partitions on N; a *multichoice cooperative game* is a function $v : J^N \to \mathbb{R}$ such that $v(\emptyset, \ldots, \emptyset, N) = 0$.

A multichoice game is *nondrecreasing* or monotonic, if for any j-partitions S, T, such that $S \subseteq T$ then $v(S) \leq v(T)$.

We denote with $\mathcal{J}^N$ the space of all multichoice cooperative games with j ordered alternatives on the finite set N

Ternary games that we described in chapter 3 were defined by Felsenthal and Machover (1997). The more general family of multichoice games was introduced by Hsiao and Raghavan (1993) to describe cooperative games in which players have more than one way of acting within a coalition. Actually Hsiao and Raghavan used the vector notation instead of the notation with partitions that we are using here. It is easy to see that both languages describe the same object.

A classical cooperative game is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. On the other hand, the power set 2^N can be identified with $\{0,1\}^n$ and any coalition S is associated to the vector $\mathbf{x}(S) = (x_1, \ldots, x_n)$ where

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Thus, a cooperative game can be seen also as a function $v : \{0,1\}^n \to \mathbb{R}$ and any coalition as a binary vector in $\mathbb{R}^n$. In an analogous way we can associate to a finite set of j actions a vector in $\{0, 1, \ldots, j-1\}^n$. In particular, Hsiao and Raghavan (1993) use the following elements:

- $\sigma_0, \sigma_1, \ldots, \sigma_{j-1}$ denote j actions, where σ_0 is the action of doing nothing;
- $\beta = \{0, 1, \ldots, j-1\}$ is a vector representing the actions;
- $x \in \beta^n$ is a vector in which $x_a = l$ if and only if player a takes action σ_l;
- then a *mult-ichoice cooperative game* is given by the set of actions and by the characteristic function $v : \beta^n \to \mathbb{R}$ such that $v(\mathbf{0}) = 0$.

Of course, there is a correspondence among the elements in β^n and in a j-partition of N. In our notation, given a vector $\mathbf{x} \in \beta^n$, the associated j-partition S is such that

$$a \in S_l \iff x_a = j - l.$$

Another difference with the seminal work of Hsiao and Raghavan (1993) is that they also consider a function that associates to every action a *weight* in order to discriminate among actions. In our model, we assume that the effort and the consequences of each action taken by players is already captured by the value function v, so weights are not necessary and all the information they would provide are unified by the value of the function for any j-partition.

4.2 The roll-call model

In this section we recall the model of roll-calls in order to define the Shapley-Shubik index. This model was introduced by Felsenthal and Machover (1996) for simple games and then extended in Felsenthal and

Machover (1997) to voting games with abstention and in Freixas (2005b) to (j, k) games.

Given the finite set N, let $\mathcal{Q}$ be the space of all permutations of N. Then the *j-roll-call space* $\mathbf{R}^J$ is defined as

$$\mathbf{R}^J = \mathcal{Q} \times J^N.$$

Each roll-call $R \in \mathbf{R}^J$ is given by a pair (qR, jR) where qR is a bijection from N to $\{1, 2, \ldots, n\}$ and jR is a j-partition of N, i.e. $jR = (jR_1, jR_2, \ldots, jR_j)$. Thus, qR induces a total order on N that we call the *queue* of players in the roll-call R, $qR(a) = i$ means that player a is voting in the i^{th} position. On the other hand, jR represents how each player is voting in the roll-call R: if $a \in jR_l$, then player a votes in the l^{th} level of approval. In other words, a roll-call describes a permutation of players in which every player can vote in any of the j possible levels of support. It is easy to see that the number of elements in $\mathbf{R}^J$ is $n!j^n$.

Before dealing with the general case, let us focus again on $(j, 2)$-simple games, i.e. games in which the image of the function v is $\{0, 1\}$. These games can be seen as a generalized model for voting situations in which there are winning and losing j-partitions: according to how each voter votes and according to the role of each voter in the game. Then a proposal can be approved or rejected. Voting games with abstention are a subclass of this family of games.

First of all, we say that a roll-call R is *positive* (or that R has a positive outcome) if $v(jR) = 1$, and that R is *negative* if $v(jR) = 0$. The interpretation is straightforward, a roll-call is positive if the result of the voting process is 1 and the bill in discussion is approved, otherwise it is negative and the bill is rejected.

A player a is said to be *pivotal* in R for the game v, we write $a = piv(R, v)$, if after a's vote the outcome is decided, no matter what the players after a are going to vote. This means that a is the first player in the queue of R for which the following holds. For any other roll-call R' such that $qR' = qR$ and $jR'(x) = jR(x)$ for any x such that $qR(x) \leq qR(a)$, then R is positive if and only if R' is positive. We say that a is *positively* pivotal if the roll-call has a positive outcome and *negatively* pivotal if the outcome of the roll-call is negative. This means that if a is positively pivotal, then after a's vote the outcome is fixed and it will be positive even in the extreme case in which all the players after a are voting in the last level of approval (i.e. all players after a in the queue qR belongs to jR_j). Analogously, if a is negatively pivotal, then the outcome is fixed and it will be negative even in the extreme case that all voters after a are voting in the first level (i.e. they belong to jR_1).

The Shapley-Shubik index for $(j, 2)$-simple games is defined as the probability of each player of being pivotal in the space of roll-calls under the discrete uniform distribution, see Felsenthal and Machover (1997) and Freixas (2005b).

Definition 4.4. For any $v \in \mathcal{SJ}^N$ and any player $a \in N$, the *Shapley-Shubik index for (j,2)-simple games* is defined as

$$\phi_a(v) = \frac{|\{R \in \mathbf{R}^J : a = piv(R, v)\}|}{j^n n!}. \tag{4.1}$$

Unfortunately, the previous definition it is not useful from a practical computational point of view. Note also that this definition can not be easily extended to define the Shapley value for multichoice cooperative games, since the idea of pivotal player can not be extended when the outcomes are more than two real numbers.

Let v be a (j, k) game on N as in Definition 4.2. To define a power index, with the flavor of the Shapley-Shubik index, for this class of games we need to extend the concept of pivotal players. We say that a player $a \in N$ is a $i-$pivot in the roll-call R for the game v, and we write $a = i - piv(R, v)$, if a is the player whose vote clinches the outcome to a result that is no less than v_i or no more than v_{i+1}. In other words, independently of how players after a are going to vote if a is the i-pivot, a is the player who clinches the outcome to v_h with $v_h \geq v_i$ or $v_h < v_{i+1}$. We say that a is positively i-pivotal if a's vote fixes the outcome to be at least v_i, this means that even if all player after a are going to vote for the last level of support the result is v_i or even better than that. On the other hand, we say that a is negatively i-pivotal if after a's vote the result can not be better than v_{i+1}, this means that even if all the players voting after a are giving the maximum support, the outcome will be v_i or worse than it.

The Shapley-Shubik power index for (j, k)-simple games has been defined in Freixas (2005b) in the following way.

Definition 4.5. For any (j, k)-simple game v with numeric evaluation α and any player $a \in N$, the *Shapley-Shubik power index for a* (j, k)-

simple game is defined as

$$\phi_a(v) = \frac{1}{n!j^n\alpha_1} \sum_{i=1}^{k-1} (\alpha_i - \alpha_{i+1}) |\{R \in \mathcal{R} : a = i - piv(R, v)\}|. \quad (4.2)$$

It is easy to see that if $k = 2$ and we choose a uniform numeric evaluation. i.e., $\alpha = (1, 0)$, then the previous formula reduces to Equation (4.1).

Remark 4.1. In the previous definition there is a factor $\frac{1}{\alpha_1}$ in order to define a power index and have a function which take values only in the interval $[0, 1]$. In general, we can remove the factor α_1 in the denominator if we want to avoid the normalization of the index and define a general value for (j, k)-games.

4.3 The Shapley value for multichoice games

Let us introduce the following notation, given a j-partition S, we define the j-partitions $S_{a\downarrow_l}$ and $S_{a\uparrow_l}$ in which player a has moved from the highest (or the lowest) level to the l^{th} level. In particular, if $a \in S_1$

$$S_{a\downarrow_l} = (S_1 \setminus \{a\}, \ldots, S_l \cup \{a\}, \ldots, S_j)$$

for any $l = 2, \cdots, j$; and if $a \in S_j$

$$S_{a\uparrow_l} = (S_1, \ldots, S_l \cup \{a\}, \ldots, S_j \setminus \{a\})$$

for any $l = 1, \cdots, j - 1$.

We can now introduce the definition of a value with the same flavour of the Shapley value for cooperative games that extends both: the Shapley value itself and the Shapley-Shubik index for $(j, 2)$-simple games as defined in Equation 4.1. The formula we provide depends only on the characteristic function v that describes the game and has a structure similar to the classical formula used for the Shapley value in cooperative theory (as defined in Shapley (1953)) in terms of some "marginal contributions". The idea of the following value is analogous to the idea behind the Shapley value in the cooperative context; the value proposes a division among players of the total benefit $v(N, \emptyset, \ldots, \emptyset)$ they can get if they cooperate at the highest action.

Definition 4.6 (Shapley value for multichoice cooperative games)**.** For any $v \in \mathcal{J}^N$ and any player $a \in N$, the *Shapley value for multichoice cooperative game* is defined as

$$\Phi_a(v) = \sum_{\substack{S\in 3^N:\\ a\in S_1}} \left\{ \sum_{l=2}^{j} \frac{\gamma^n(s_1-1)}{j^n n!} \big[v(S) - v(S_{a\downarrow_l})\big] \right\} + \\ + \sum_{\substack{S\in 3^N:\\ a\in S_j}} \left\{ \sum_{l=1}^{j-1} \frac{\gamma^n(s_j-1)}{j^n n!} \big[v(S_{a\uparrow_l}) - v(S)\big] \right\} \tag{4.3}$$

where

$$\gamma^n(t) = t! j^t \sum_{i=0}^{t} \frac{(n-t-1+i)!}{j^i i!} \tag{4.4}$$

for any $t = 0, \ldots, n-1$.

Let us explain the previous formula starting with the meaning of Equation (4.4) which defines the coefficients $\gamma^n(t)$. Given a subset T of N and a player $a \in N \setminus T$, if $t = |T|$, the aim of this coefficient is to count the number of roll-calls such that

- all players in $N \setminus T$ vote before a in a fixed unique way;
- players in T can vote either before or after a;
- if players in T vote before a, they vote in a unique way, while if the vote after a they can vote any of the j-possible alternatives.

We can state these considerations in the following general lemma:

Lemma 4.1. *Let T be a subset of N of cardinality t and $a \in N \setminus T$, then $\gamma^n(t)$ as defined in Equation (4.4) counts the number of permutation of elements such that the elements in $N \setminus T$ precede a, while the elements in T can be either before or after a and, in the second case, they can be of j different types.*

Proof. In the ordering the element a appears after all players in $N \setminus T$ and some of the players in T, this means that for any $i = 1, \ldots, t$, player a is the $n - 1 - (t - i)^{th}$ player in the ordering. Note that

- there are $\binom{t}{i}$ ways for choosing the elements in T that are precede a;
- the elements before a are $n - t - 1 + i$ since there are all players in $N \setminus T$, minus a, plus the i players chosen among those in T;

- there are $t-i$ elements coming after a;
- there are $(n-t-1+i)!$ possible permutations for the elements before a ;
- there are $(t-i)!$ possible permutations for the elements after a than can be of j^{t-i} different types.

Thus,

$$\begin{aligned}\gamma^n(t) &= \sum_{i=0}^{t} \binom{t}{i} (n-t-i+1)!(t-i)!j^{t-i} \\ &= t!j^t \sum_{i=0}^{t} \frac{(n-t-1+i)!}{j^i i!}.\end{aligned}$$

□

Now that it is clear what the coefficient $\gamma^n(t)$ is counting we can explain why Equation (4.3) describes the Shapley value for j-cooperative games and it is a generalization of the Shapley-Shubik index we discussed in the previous section.

Theorem 4.1. *The explicit formula* (4.3)*, which describes a value* Φ *for a multichoice cooperative game in terms of marginal contributions, is associated to the bargaining model described by Felsenthal and Machover extended to multichoice cooperative games.*

In particular, formula (4.3) *generalizes formula* (4.1) *for* $(j,2)$*-simple games.*

Proof. In the roll-call model introduced by Felsenthal and Machover players are queuing in a random order and each one of them is voting (or acting) in one of the j possible levels. The Shapley value, under this scheme, is seen as the expected marginal contribution of each player when taking part in this process.

In order to compute the value, let us recall that in a $(j,2)$-simple game a player a can be pivotal in two different ways: either fixing the outcome to be 1 or blocking the outcome to the value 0. Analogously in a multichoice cooperative game a player has two different pivotal abilities: block the situation avoiding the j-partition to reach a bigger result or, on the other hand, guarantee at least a minimum outcome. Thus, in order to compute the Shapley-Shubik index we have two different situations, which correspond to the two different addends in equation (4.3).

Blocking capacity Let us consider the situation in which a player is pivotal blocking the j-partition to reach a better result. This means that there is a j-partition in which player a is voting at level l (for some $l > 1$) and he is blocking the part of gain the others would get if he votes in the maximum level. In other words, there is a j-partition S such that $a \in S_1$ and $v(S) - v(S_{a\downarrow_l}) \neq 0$, the number $v(S) - v(S_{a\downarrow_l})$ measures in some sense the capacity of blocking of player a in the j-partition S.

Moreover, for any j-partition S such that the $v(S) - v(S_{a\downarrow_l}) \neq 0$, there are many other associated roll-calls such that player a is pivotal in the same way. Actually a blocking capacity is $v(S) - v(S_{a\downarrow_l})$ every time

- a votes after players in $N \setminus S_1$;
- players in S_1 can vote either before or after a;
- players before a in the queue vote according to the level they are in S;
- players after a can vote anything.

This means that for any j-partition S such that $a \in S_1$ and $v(S) - v(S_{a\downarrow_l}) \neq 0$ the number of associated roll-calls is $\gamma^n(s_1 - 1)$.

Since this can happen when player a votes at any level $l = 2, \ldots, j$, if we sum over all the j-coalition and over all the levels of voting we have that

$$\sum_{S:a\in S_1} \sum_{l=2}^{j} \gamma^n(s_1 - 1)\big[v(S) - v(S_{a\downarrow_l})\big] \tag{4.5}$$

is the expected blocking capacity of player a in the game v. If v is a $(j, 2)$-simple game, then (4.5) is the number of roll-calls for which a is negatively pivotal.

Approval capacity Let us now consider when a player is pivotal in guaranteeing at least a minimum result to a j-partition. This means that there is a j-partition in which player a is voting at a level l (with $l < j$) guaranteeing a minimum level that the other would not get if a vote for the lower level. In other words, there is a j-partition S such that $a \in S_j$ and $v(S_{a\uparrow_l}) - v(S) \neq 0$, the number $v(S_{a\uparrow_l}) - v(S)$ measures, in some sense, the capacity of player a of approval in the j-partition S.

Actually a approval capacity is $v(S_{a\uparrow_l}) - v(S)$ every time that:

- a votes after players in $N \setminus S_j$;

- players in S_j can vote either before or after a;
- players before a in the queue vote according to the level they are in S;
- players after a can vote anything.

This means that for any j-partition S such that $a \in S_j$ and $v(S_{a\uparrow_l}) - v(S) \neq 0$ the number of associated roll-calls is $\gamma^n(s_j - 1)$.

Since player a can be positively pivotal voting at any level $l = 1, \ldots, j-1$, if we sum over all the j-coalition and over all the levels of voting we have that

$$\sum_{S: a \in S_j} \sum_{l=1}^{j-1} \gamma^n(s_j - 1)\big[v(S_{a\uparrow_l}) - v(S)\big] \tag{4.6}$$

is the expected approval capacity of player a in the game v. Note that this is also the number of roll-calls for which a is positively pivotal if v is a $(j, 2)$-simple games.

In order to have the Shapley value under the Felsenthal and Machover bargaining model, we sum (4.5) and (4.6), then divide the result by the total number of roll-calls $j^n n!$ and get equation (4.3).

□

Formula (4.3) is useful to obtain an explicit formula to compute the Shapley-Shubik index for $(j, 2)$-simple games avoiding the count of all roll-calls. On the other hand, since Φ is expressed in terms of marginal contribution the formula can also give a hint about how to generalize other values, such as the Banzhaf value, from the classic cooperative approach to the family of multichoice games.

Remark 4.2. If we consider simple games as $(2, 2)$-simple games, then equation (4.3) reduces to

$$\begin{aligned}
\phi_a(v) =& \frac{1}{2^n n!} \sum_{S: a \in S_1} \gamma^n(s_2 - 1)\big[v(S) - v(S_{a\downarrow_2})\big] + \\
& + \frac{1}{2^n n!} \sum_{S: a \in S_2} \gamma^n(s_1 - 1)\big[v(S_{a\uparrow_1}) - v(S)\big] \\
=& \sum_{T \subseteq N \setminus \{a\}} \frac{\gamma^n(n - t - 1) + \gamma^n(t)}{2^n n!} [v(T \cup \{a\}) - v(T)].
\end{aligned}$$

The equivalence with this formula with the standard formula to compute the Shapley value was stated in Felsenthal and Machover (1996) and a direct proof of the equivalence is discussed in chapter 2 and proved in Lemma 2.1.

Remark 4.3. Let v be a (j,k)-game with numeric evaluation α, then $\frac{1}{\alpha_1}\Phi(v)$ is equivalent to $\phi(v)$ as defined in (4.2).

4.4 Explicit formulas for games with abstention

In this section we focus on games with abstention, i.e. games in which players can vote yes, no or abstain. In chapter 3 we discussed this model and the properties of two power indices for this class of games: the Banzhaf and the Shapley-Shubik power index for games with abstention. However in this chapter we provided a formula to compute a value for multichoice games, the restriction of this formula to ternary games (i.e. multichoice games in which there are three levels in input) provide an explicit formula to compute also the Shapley-Shubik index for $(3,2)$-simple games.

Formula (4.3) for ternary game v can be written as

$$\begin{aligned}\phi_a(v) = &\sum_{S:a\in S_1}\left[\frac{\gamma^n(s_3)+\gamma^n(s_1-1)}{3^n n!}\right][v(S)-v(S_{\downarrow\downarrow a})]+\\ &+\sum_{S:a\in S_1}\frac{\gamma^n(s_1-1)}{3^n n!}[v(S)-v(S_{\downarrow a})]+\sum_{S:a\in S_2}\frac{\gamma^n(s_3)}{3^n n!}[v(S)-v(S_{\downarrow a})]\end{aligned}\tag{4.7}$$

for any player $a\in N$, where the coefficients $\gamma^n(t)$ are

$$\gamma^n(t) = 3^t t!\sum_{j=0}^{s}\frac{(n-t-1+j)!}{3^j j!}$$

for any $t=0,\dots,n-1$.

To compute the Shapley-Shubik index using the definition of pivotal player it is necessary to check who is the pivotal player in all the roll-calls, this means that there are $3^n n!$ elements to check. On the other hand, using the formula we provide it is only necessary to evaluate the marginal contribution for any tripartition S, that means that there are 3^n elements to consider. Even if this is a huge number, there is a significant improvement with respect to the factorial numbers.

	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=\dots$
$n=1$	1						
$n=2$	1	4					
$n=3$	2	5	26				
$n=4$	6	12	36	240			
$n=5$	24	42	96	348	2904		
$n=6$	120	192	372	984	4296	43680	
$\dots$							

Table 4.1: *Numerical coefficients to compute $\gamma^n(s)$ for ternary games for $n \leq 6$.*

The following example is taken from Felsenthal and Machover (1998). We use it to explicitly compute the Shapley-Shubik index for games with abstention using formula (4.7) and comparing this result with the computation deduced using Definition 3.6.

Example 4.1. Felsenthal and Machover (1998) Let $N = \{a, b, c\}$, suppose that players can vote yes, no or abstain and that a resolution is approved if and only if a is in favour and at least one of the other two players is not against it. This voting process is described by the $(3, 2)$-simple game v with minimal winning tripartitions $\{(a, b, c), (a, c, b)\}$.

Let us compute the Shapley-Shubik power index for player b using formula (4.7) and Table 4.1 for the value of the coefficient.

$$\begin{aligned}\phi_b &= (\gamma^3(1) + \gamma^3(1))[v((ab, \emptyset, c)) - v((a, \emptyset, bc))] + \gamma^3(1)[v((a, b, c)) - v((a, \emptyset, bc))] \\ &= 3\gamma^3(1) = \frac{5}{54}\end{aligned}$$

We can directly check this result counting the number of roll-calls for which player b is pivotal.
Player b is pivotal

- voting **yes** in the roll-calls in which she votes after a, who votes yes and c votes no if he is before b:

$$\begin{array}{lll} abc \times (ab, \emptyset, c) & abc \times (ab, c, \emptyset) & abc \times (abc, \emptyset, \emptyset) \\ acb \times (ab, \emptyset, c) & & \\ cab \times (ab, \emptyset, c) & & \end{array}$$

- **abstaining** in the roll-calls in which she votes after a, who votes yes and c votes no if he is before b:

$$\begin{array}{lll} abc \times (a,b,c) & abc \times (a,bc,\emptyset) & abc \times (ac,b,\emptyset) \\ acb \times (a,b,c) & & \\ cab \times (a,b,c) & & \end{array}$$

- voting **no** in the roll-calls in which she votes after c who votes no, and a votes yes if he is before b:

$$\begin{array}{lll} cba \times (a,\emptyset,bc) & cba \times (\emptyset,a,bc) & cba \times (\emptyset,\emptyset,abc) \\ acb \times (a,\emptyset,bc) & & \\ cab \times (a,\emptyset,bc) & & \end{array}$$

Thus, the number of roll-calls for which a is pivotal is 15 and the Shapley-Shubik index is $\frac{5}{54}$. Note that all these roll-calls are associated to the tripartition $S = (ab,\emptyset,c)$ that is the only one such that $b \in S_1$ and $v(S) - v(S_{\downarrow\downarrow b}) = 1$.

In the case of $(3,2)$-simple games the formula to compute the Shapley-Shubik index can be written in a more compact way. First of all, note that if v is a $(3,2)$-simple game, it takes values only in $\{0,1\}$. This means that all the marginal contributions in formula (4.7) are either 0 or 1, of course we are interested only in tripartitions such that the marginal contributions are equal to 1. Let S be a tripartition such that $a \in S_1$ and $v(S) - v(S_{\downarrow\downarrow a}) = 1$. Then thanks to monotonicity one and only one of the following holds: $v(S) - v(S_{\downarrow a}) = 1$ or $v(S_{\downarrow a}) - v(S_{\downarrow\downarrow a}) = 1$. Define the following sets

$$\begin{aligned} \mathcal{C}_a^{YA} &= \{S \in 3^N :\ a \in S_1, S \in W, S_{\downarrow a} \notin W\} \\ &= \{S \in 3^N :\ a \in S_1, v(S) - v(S_{\downarrow a}) = 1\} \\ \mathcal{C}_a^{AN} &= \{S \in 3^N :\ a \in S_1, S_{\downarrow a} \in W, S_{\downarrow\downarrow a} \notin W\ \} \\ &= \{S \in 3^N :\ a \in S_1, v(S_{\downarrow a}) - v(S_{\downarrow\downarrow a}) = 1\}. \end{aligned}$$

We say that $\mathcal{C}_a^{YA}$ is the set of all tripartitions such that player a is a yes-abstention swinger; while $\mathcal{C}_a^{AN}$ is the set of all tripartitions for which player a is an abstention-no swinger. Of course we can also define the set of tripartition for which a is a yes-no swinger

$$\begin{aligned}\mathcal{C}_a^{YN} &= \{S \in 3^N :\ a \in S_1, S \in W, S_{\downarrow\downarrow a} \notin W\ \} \\ &= \{S \in 3^N :\ a \in S_1, v(S) - v(S_{\downarrow\downarrow a}) = 1\} \\ &= \mathcal{C}_a^{YA} \cup \mathcal{C}_a^{AN}\end{aligned}$$

Using this notation, we can write formula (4.7) just for simple games in a more compact way:

$$\phi_a(v) = \sum_{S \in \mathcal{C}_a^{YA}} \left[\frac{\gamma^n(s_3) + 2\gamma^n(s_1 - 1)}{3^n n!}\right] + \sum_{S \in \mathcal{C}_a^{AN}} \left[\frac{2\gamma^n(s_3) + \gamma^n(s_1 - 1)}{3^n n!}\right] \tag{4.8}$$

As we discussed in chapter 3, the Banzhaf index for games with abstention has been defined as a two component power index in order to stress the difference of the power of a player when passing from voting yes to abstaining and from abstaining to voting no. If $\beta_a(v) = \beta_a^{YA}(v) + \beta_a^{AN}(v)$, using the previous notation we can write the two components as

$$\beta_a^{YA}(v) = \frac{|\mathcal{C}_a^{YA}|}{3^{n-1}} \qquad \beta_a^{AN}(v) = \frac{|\mathcal{C}_a^{AN}|}{3^{n-1}}.$$

Thanks to formula (4.8) we can define the Shapley-Shubik index as a two component index, too. ϕ^{YA} measures the power when switching from yes to abstention and ϕ^{AN} measures the power when switching from abstention to no:

$$\begin{aligned}\phi_a^{YA}(v) &= \sum_{S \in \mathcal{C}_a^{YA}} \frac{\gamma^n(s_3) + 2\gamma^n(s_1 - 1)}{3^n n!} \\ \phi_a^{AN}(v) &= \sum_{S \in \mathcal{C}_a^{AN}} \frac{2\gamma^n(s_3) + \gamma^n(s_1 - 1)}{3^n n!}\end{aligned}$$

for any player $a \in N$.
Of course, the Shapley-Shubik index for $(3, 2)$-simple games is then given by the sum of these two components.

Example 4.2 (UNSC)**.** As noted by Felsenthal and Machover (1998) the UN Security Council can be modeled as a $(3, 2)$-simple game: a resolution is approved if there are at least nine members in favor and permanent members are not against it. Thus, even if some of the permanent members abstain, without the veto of some other permanent member, a resolution can be carried on. The resulting game v has 15 players, with the subset P of the five permanent member, and a tripartition S is winning iff

$$|S_1| \geq 9 \qquad \wedge \qquad S_3 \cap P = \emptyset.$$

We can compute the Shapley-Shubik power index using equation (4.8). For a permanent member p we have

$$\mathcal{C}_p^{YA}(v) = \{S : p \in S_1, |S_1| = 9 \wedge S_3 \cap P = \emptyset\}$$

and

$$\mathcal{C}_p^{AN}(v) = \{S : p \in S_1, |S_1| > 9 \wedge S_3 \cap P = \emptyset\}.$$

So

$$\phi_p(v) = \sum_{s_3=0}^{6} \left\{ \frac{2\gamma^{15}(8) + \gamma^{15}(s_3)}{3^{15}15!} \sum_{\substack{j=\max\{0,\\ s_3-2\}}}^{4} \binom{4}{j}\binom{10}{8-j}\binom{j+2}{s_3} \right\} +$$

$$+ \sum_{s_1=10}^{15} \sum_{s_3=0}^{15-s_1} \left\{ \frac{\gamma^{15}(s_1-1) + 2\gamma^{15}(s_3)}{3^{15}15!} \sum_{\substack{j=\max\{0,\\ s_1+s_3-11\}}}^{4} \binom{4}{j}\binom{10}{s_1-1-j}\binom{11-s_1+j}{s_3} \right\}$$

On the other hand, for a non-permament q we have

$$\mathcal{C}_q^{YA}(v) = \{S : q \in S_1, |S_1| = 9 \wedge |S_3 \cap P| = \emptyset\}$$

and $\mathcal{C}_q^{AN}(v) = \emptyset$. Thus,

$$\phi_q(v) = \sum_{s_3=0}^{6} \left\{ \frac{2\gamma^{15}(8) + \gamma^{15}(s_3)}{3^{15}15!} \sum_{j=\max\{0,s_3-1\}}^{5} \binom{5}{j}\binom{9}{8-j}\binom{j+1}{s_3} \right\}$$

Using these formulas we get

$$\phi_p(v) = 0.16338987329859317 \qquad \phi_q(v) = 0.01830506335070341.$$

Observe that

$$\frac{\phi_p(v)}{\phi_q(v)} \approx 8.93$$

4.4.1 Banzhaf value for ternary games

As we discussed in the previous section the Banzhaf index for $(3,2)$-simple games can also be written as

$$\beta_a(v) = \frac{|\mathcal{C}_a^{YN}|}{3^{n-1}} = \frac{|\mathcal{C}_a^{YA}| + |\mathcal{C}_a^{AN}|}{3^{n-1}}$$

If we use the same structure of the formula for the Shapley-Shubik index (4.8) we can write

$$\beta_a(v) = \sum_{S \in \mathcal{C}_a^{YA}} [\delta^n(s_3) + 2\delta^n(s_1 - 1)] + \sum_{S \in \mathcal{C}_a^{AN}} [2\delta^n(s_3) + \delta^n(s_1 - 1)]$$

where $\delta^n(s) = \frac{1}{3^{n-2}}$.

Thus, as we already noted for the Shapley value, we can then extend the Banzhaf index and define the Banzhaf value for ternary cooperative games.

Definition 4.7. Let v be a ternary cooperative game, then the *Banzhaf value for ternary games* is given by

$$\begin{aligned}\beta_a(v) =& \frac{2}{3^{n-2}} \sum_{S: a \in S_1} [v(S) - v(S_{\downarrow\downarrow a})] + \frac{1}{3^{n-2}} \sum_{S: a \in S_1} [v(S) - v(S_{\downarrow a})] + \\ &+ \frac{1}{3^{n-2}} \sum_{S: a \in S_2} [v(S) - v(S_{\downarrow a})]\end{aligned}$$

for any player $a \in N$.

4.4.2 Weighted games and generating function

In Freixas and Zwicker (2003) weighted (j,k)-games are described to model some voting situations. In particular a weighted $(3,2)$-simple weighted majority game consists of a real number, quota, q and a vector of weights for each voter a, (ω_a^Y, ω_a^N), where $\omega_a^Y \geq 0$ is the weight when a votes yes and $\omega_a^N \leq 0$ is the weight when a votes no. A weighted representation for the game is then $[q; (\omega_1^Y, \omega_1^N), \cdots, (\omega_n^Y, \omega_n^N)]$[1] The game is then defined as

$$v(S) = 1 \iff \omega(S) \stackrel{def}{=} \sum_{i \in S_1} \omega_i^Y + \sum_{i \in S_3} \omega_i^N \geq q$$

[1] Without loss of generality, we assume that when players abstain their weight is 0.

for all $S \in J^N$ Since we have the explicit formula (4.8), in case of a weighted $(3,2)$-game we can compute the Shapley-Shubik power index using the model of generating functions as done by Brams and Affuso (1976) for simple games and Freixas (2012) for $(3,2)$-simple games.

Definition 4.8. Let $v = [q; (\omega_1^Y, \omega_1^N), \cdots, (\omega_n^Y, \omega_n^N)]$ be a $(3,2)$-simple weighted game with abstention. For any $a \in N$, the generating function is defined as

$$F_a(x) = \prod_{p \in N, p \neq a} \left(y x^{\omega_p^Y} + 1 + t x^{\omega_p^N} \right)$$

If we do the computations, the function $F_a(x)$ becomes

$$F_a(x) = \sum_{k=\underline{\omega}}^{\overline{\omega}} \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} b_{k,i,j} y^i t^j x^k$$

where $\underline{\omega} = \sum_{i \in N} \omega_i^N$ and $\overline{\omega} = \sum_{i \in N} \omega_i^Y$.

In the previous formula the coefficient $b_{k,i,j}$ counts the number of tripartitions S of total weight k such that there are i players in S_1 and j players in S_3. Using these coefficients Equation (4.8) becomes

$$\Phi_a(v) = \sum_{k=q-\omega_a^Y}^{q-1} b_{k,i,j} \frac{2\gamma^n(i) + \gamma^n(j)}{3^n n!} + \sum_{k=q}^{q-\omega_a^N-1} b_{k,i,j} \frac{\gamma^n(i) + 2\gamma^n(j)}{3^n n!}$$

for any player a such that $\omega_a^Y \neq 0$ and $\omega_a^N \neq 0$.

Example 4.3. The UNSC can be represented as a weighted $(3,2)$-simple games v where the quota is $q = 9$, the weights for a permanent member p are $(1, -6)$ and for a non permanent member q are $(1, 0)$. Using generating functions the Shapley-Shubik index that we get is the same as in Example 4.2.

$$\phi_p(v) = 0.16338987329859317 \qquad \phi_q(v) = 0.01830506335070341.$$

Example 4.4. A modified version of the United Security Council game is to add the possibility of approval of a resolution if one permanent members is against it but all the other members are in favour. This means that for any permanent member p, we should add $(N \setminus p, \emptyset, p)$ to the winning tripartitions.

This new game can be represented as a weighted game with quota $q = 9$ and vector of weights for the permanent members $(1, -5)$ and $(1, 0)$ for non permanent members.

Using the generating function method, the Shapley-Shubik index for a permanent member p and a non permanent member r are:

$$\phi_p = 0.13958034451108942 \qquad \phi_q = 0.030209827744455294.$$

Note that

$$\frac{\phi_p(v)}{\phi_q(v)} \approx 4.62$$

This considerable lower proportion for $\frac{\phi_p(v)}{\phi_q(v)}$ with respect to the the results in Example 4.2 would make the UNSC voting system more egalitarian.

4.5 A comparison among different values and models

We conclude this chapter with an overview of other models that generalize cooperative games to games in which players can choose among different alternatives or levels of effort.

4.5.1 Bi-cooperative games

Bilbao, Fernández, Jiménez, and Lebrón (2000) introduced the model of bi-cooperative games to model voting situation in which players explicitly vote in favour or against a proposal, but they also have the third separated option to abstain. Thus, the original idea is very similar to the model of games with abstention introduced by Felsenthal and Machover (1997) that we discussed in chapter 3.

As $(3, 2)$-simple games, also bi-cooperative games are defined on the set 3^N of tripartitions, but in Bilbao's work a tripartition is denoted just with two sets (S, T), players in S are supposed to vote in favour, players in T are supposed to vote against and all players in $N \setminus (S \cup T)$ are abstaining. A bi-cooperative game is then a function $b : 3^N \to \mathbb{R}$ such that $b(\emptyset, \emptyset) = 0$. This last normalization condition is different to the one used for games with abstention, in which a tripartition gets zero if all players are voting no. Behind the bi-cooperative model there is the idea that the default option for players is to abstain and they should explicitly vote in favour or against a proposal forming the two disjoint sets.

Several solution concepts for bi-cooperative games have been studied: Bilbao, Fernández, Jiménez, and López (2007) present the core and the Weber set, while Bilbao, Fernández, Jiménez, and López (2008b), Labreuche and Grabisch (2008), and Bilbao, Fernández, Jiménez, and López (2010) discuss different kind of values. In particular, in Bilbao, Fernández, Jiménez, and López (2008a) a Shapley value for a bi-cooperative game b is defined as

$$\phi_a(b) = \sum_{(S,T)\in 3^{N\setminus\{a\}}} \Big\{\overline{p}_{s,t}[b(S\cup a,T) - b(S,T)] + \underline{p}_{s,t}[b(S,T) - b(S,T\cup a)]\Big\}$$

where

$$\overline{p}_{s,t} = \frac{(n+s-t)!(n+t-s-1)!}{(2n)!}2^{n-s-t}$$

and

$$\underline{p}_{s,t} = \frac{(n+t-s)!(n+s-t-1)!}{(2n)!}2^{n-s-t}$$

Using the notation of tripartition that we introduced in chapter 3 the Bilbao's value for bi-cooperative game can be written as

$$\phi_a(b) = \sum_{\substack{S\in 3^N\\ a\in S_1}} p(s_1,s_3)[b(S) - b(S_{\downarrow a})] + \sum_{\substack{S\in 3^N\\ a\in S_3}} p(s_3,s_1)[b(S_{\uparrow a}) - b(S)] \tag{4.9}$$

where

$$p(s_i,s_j) = \frac{(2s_i+s_2-1)!(2s_j+s_2)!}{(2n)!}2^{s_2+1}.$$

The main difference with the value defined in Equation (4.7) is that the marginal contributions considered by Bilbao are only when a player moves from abstention to voting in favour or against, while we also consider the contribution from being in favour to being against. In Bilbao, Fernández, Jiménez, and López (2008a) starting with the hypothesis that the value will depend on the these marginal contributions, the coefficient $\overline{p}_{s,t}$ and $\underline{p}_{s,t}$ are deduced using linearity, efficiency, the dummy player property, symmetry and another axiom they called structural axiom. Of course, the resulting coefficients are different from ours and so are the the indices defined by (4.7) and by (4.9).

To conclude this brief comparison between $(3,2)$-simple games and bi-cooperative games, note that there are no other works following the ideas of bi-cooperative games to generalize the model to games in which players have more than three alternatives in input, as it has been done with (j,k)-games and multichoice games.

4.5.2 Bolger's model: games with r alternatives

The model of games with r alternatives was introduced in Bolger (1993): players must choose one, and only one, of the r available alternatives. The main tools in this model are *arrangements* and *embedded coalitions*. Given the set of players N and the r alternatives, an arrangement $\Gamma = (C(1), \ldots, C(r))$ is the equivalent of a r-partition of the set: players in $C(i)$ are choosing alternative i, for any $i = 0, \ldots, r$. Given an arrangement Γ, an embedded coalition is a pair $(C(i), \Gamma)$, where $C(i) \in \Gamma$.

To any arrangement Γ is associated a r-dimensional vector of real numbers, in which the i-th components is interpreted as the value of the coalition $C(i)$ with respect to the arrangement Γ. Thus the game is a function $v : 2^N \times R^N \to \mathbb{R}^n$ which assigns a real value $v(C(i), \Gamma)$ to any embedded coalition.

It is clear that this model is different from the one of multichoice cooperative games, even if the ideas behind the two models are similar: the second one assigns a unique value to any r-partition, while the first one assigns a different value to any set of the r-partition with respect to the partition itself. However, multichoice games can be seen as a special subclass of games with r alternatives if $v(C(i), \Gamma)$ depends only on the arrangement Γ, that is $v(C(i), \Gamma) = v(C(j), \Gamma)$ for any $i, j = 1, \ldots, r$.

In Bolger (1983) a Banzhaf-type value for games with r alternatives was used; this power index was studied also in subsequent works from the same author: Bolger (1986) and Bolger (2000). On the other hand, in Bolger (1993) a value in the spirit of the Shapley value is defined for games with r alternatives, this value is deduced following an axiomatic approach and imposing some properties such as efficiency, symmetry and linearly dependence from the marginal contribution. In Bolger (2000) the value is revised with the discussion of other properties, other solution concepts for this class of games are presented in Albizuri, Santos, and Zarzuelo (1999), Albizuri and Zarzuelo (2000), Amer, Carreras, and Magaña (1998b), and Amer, Carreras, and Magaña (1998a).

Let us focus on the value defined in Bolger (1993). Bolger's value is a matrix which assigns a real number to every player and every action. In particular for any game with r alternatives v and any player $a \in N$ which is taking action j the value is defined as

$$\vartheta_a^j(v) = \sum_{S : a \in S_j} \sum_{\substack{k=0 \\ k \neq j}}^{r} f(s_j, n) \left[v(S_j, S) - v(S_j \setminus \{a\}, S_{a \downarrow k}) \right]$$

where

$$f(s_j, n) = \frac{(s_j - 1)!(n - s_j)!}{n!(r-1)^{n-s_j-+1}}.$$

Besides the fact that Bolger's value is a matrix instead of a vector, the main difference with the value we are proposing is that here the marginal contribution of each player is considered only when he moves from the considered action to another one. Moreover, the coefficient that Bolger deduced from a set of axioms are different from the one we defined in equation (4.4).

It is possible to model a multichoice cooperative game as a Bolger's game with the same value for any embedded coalition with respect to the same arrangement, then we can compute Bolger's value for the maximum level of action to get a similar value to the one we are discussing in this chapter. This approach is followed in Bolger (1993), where the UN Security Council game is presented and the value obtained are slightly different from the one we computed in Example 4.2. This may seem just an approximation problem, but it is not as we will show in Example 4.5: Bolger's value and the Shapley value defined in (4.3) give different numbers.

4.5.3 Hsiao and Raghavan's value for multichoice games

Multichoice games were defined in Hsiao and Raghavan (1993) and we already presented the model in section 4.1. However, in the original work there is also an axiomatic approach to define a value as matrix ϕ such that $\phi_a^l(v)$ is the value of player a when he takes the action at the l^{th} level in the game v.

Let us recall the elements in Hsiao and Raghavan's work:

- let $\sigma_0, \sigma_1, \ldots, \sigma_{j-1}$ be j actions, where σ_0 is the action of doing nothing;

- let $\beta = \{0, 1, \ldots, j-1\}$ be a vector representing the actions;

- then $\mathbf{x} \in \beta^n$ is a vector in which $x_a = l$ if and only if player a is takes action σ_l;

- finally, the characteristic function is $v : \beta^n \to \mathbb{R}$ such that $v(\mathbf{0}) = 0$.

To define the value, there is also a function w of *weight* associated to the vector of actions β: $w : \beta \to \mathbb{R}^+$ such that $w(0) = 0 \leq w(1) \leq \cdots \leq$

$w(m)$. The value is then defined as

$$\phi_a^l(v) = \sum_{k=1}^{l} \left\{ \sum_{\substack{\mathbf{x} \in \beta^{\mathbf{n}} \setminus \mathbf{0}: \\ x_a = k}} g(\mathbf{x}, l, a) \cdot [v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e_a})] \right\} \tag{4.10}$$

where

$$g(\mathbf{x}, l, a) = \sum_{T \subseteq M_a(\mathbf{x})} (-1)^t \frac{w(x_a)}{||\mathbf{x}|| + \sum_{r \in T} [w(x_r + 1) - w(x_r)]}$$

and

$$M_a(\mathbf{x}) = \{p : x_p \neq m, p \neq a\}, \qquad ||\mathbf{x}|| = \sum_{r=1}^{n} w(x_r),$$

while $\mathbf{e_a}$ is a vector with all components equal to 0, except for the a^{th} position in which there is a 1.

There are many difference with the value defined in formula (4.3). First of all, in our model we assume that the value function v captures all the efforts done by player when choosing an action, thus we do not consider weights and the value is obtained from the characteristic function v of the game. Secondly, we consider a unique value and do not make differences according to the action that is chosen, thus our value is a vector assigning a number to each player and not a matrix assigning a number to each player and each action. Moreover, the value defined by Hsiao and Raghavan takes into consideration the marginal contribution of each player when he changes from one level to the immediate higher level; our value instead consider the marginal contribution when moving from any level to the best or to the worst one. Finally, we deduced our value considering a bargaining procedure of forming j partitions, while in Hsiao and Raghavan (1993) there is not a probabilistic interpretation for the value.

In the particular case of $(3, 2)$-simple games, if we use a linear and uniform vector of weights, i.e. $w = (0, 1, 2)$, then with the notation of tripartitions formula (4.10) when player are voting yes (that is taking the highest level of action) becomes

$$\phi_a^2(v) = \sum_{S:a\in S_1} \left\{ \sum_{t=0}^{s_2+s_3} \binom{s_2+s_3}{t} (-1)^t \frac{2}{2s_1+s_2+t} [v(S) - v(S_{\downarrow a})] \right\} +$$
$$+ \sum_{S:a\in S_2} \left\{ \sum_{t=0}^{s_2+s_3-1} \binom{s_2+s_3-1}{t} (-1)^t \frac{2}{2s_1+s_2+t} [v(S) - v(S_{\downarrow a})] \right\} \tag{4.11}$$

The previous formula is clearly different from equation (4.7).

Let us conclude this section with an explicit example to show that the values of the three models discussed here are different from the value we are introducing following the Felsenthal and Machover approach for multichoice games.

Example 4.5. Let us consider again the $(3,2)$-simple game introduced by Felsenthal and Machover (1997) that we discussed in Example 4.1. The set of players is $N = \{a,b,c\}$ and the set of minimal winning tripartitions is $\{(a,b,c),(a,c,b)\}$. We can see this game as a game with abstention, but also as a special game of the other classes we discussed so far.

Let us start by considering v as a bi-cooperative game and compute the value for player b. The marginal contributions in equation (4.9) are always equal to zero, except for the tripartition $(a,\emptyset,bc)$ when b moves to the second level from the third one. Thus

$$\phi_b(v) = p(2,1) = \frac{3!2!}{6!}2 = \frac{1}{30}.$$

If we consider this game as a Bolger game with three alternatives, we have to compute the value when players take the best action, i.e. vote yes (this is analogous to what is done in Bolger (1993) to compute the index for the UNSC). Again, for player b the only significant marginal contribution is when player b moves from abstaining to voting no in tripartition (a,b,c),

$$\theta_b^2(v) = f(1,2) = \frac{1}{24}.$$

Finally to compute the Hsiao and Raghavan value we suppose that the vector of weights is linear and uniform, and consider the action of

Model of the game	Value
Game with abstention	$\left(\frac{22}{27}, \frac{5}{54}, \frac{5}{54}\right)$
Multi-choice game	$\left(\frac{10}{12}, \frac{1}{12}, \frac{1}{12}\right)$
Bi-cooperative game	$\left(\frac{14}{15}, \frac{1}{30}, \frac{1}{30}\right)$
Game with r-alternatives	$\left(\frac{22}{24}, \frac{1}{24}, \frac{1}{24}\right)$

Table 4.2: *Comparison among the different values for a game with abstention.*

voting yes. Using equation (4.11) we have that

$$\phi_b^2(v) = \sum_{t=0}^{1} \binom{1}{t} (-1)^t \frac{1}{3+t} = \frac{1}{12}.$$

All the values we discussed so far satisfy efficiency and symmetry, thus it is possible to compute the values also for players a and c. All the different values are summarized in Table 4.2. For this example the value we obtain is more egalitarian than the others, which assign almost all power to voter a.

4.6 Conclusion

In this chapter we presented a new value for multichoice cooperative games. The interest we had at the beginning was to find an explicit formula to compute the Shapley-Shubik index for games with abstention analogous to the formula for simple games we proposed in section 2.3. Actually we found a more general result that can be used not only for $(3, 2)$-simple games but also for ternary cooperative games and in general for multichoice games in which players can choose among different alternatives.

The new value we provided in this chapter is a generalization of the Shapley value following the Felsenthal and Machover bargaining approach extended to multichoice games. The restriction of this value to $(3, 2)$-simple games provides an explicit formula to compute the Shapley-Shubik index for games with abstention by means of marginal contribution and without having to check all roll-calls and the pivotal players.

We presented also a brief comparison with other models of games in which players can choose among different alternatives and show that the value we define is different from the other values presented in literature so far. This work could be further developed giving an extension of other values, or all the family of semivalues, to the family of multichoice cooperative games.

CHAPTER 5

Ranking objects

VALUES and power indices naturally provide a ranking over players, starting from a value over coalitions. However, in many real-life situation it can happen that one wants to rank individuals starting only from an ordinal ranking over coalitions, without having a characteristic function describing a game.

In this chapter we discuss this problem, we introduce some properties that a solution to this problem should satisfy and then we provide a characterization of a social ranking function.

5.1 Notation and preliminaries

Let us introduce the main definitions for the development of this chapter. A *binary relation* $\succcurlyeq$ on a finite set X is a subset of the cartesian product $X \times X$. For each $x, y \in X$, the notation $x \succcurlyeq y$ will be preferably used instead of the more formal $(x, y) \in \succcurlyeq$. The following are some standard properties for a binary relation $\succcurlyeq$:

- *reflexivity*: for each $x \in X$, $x \succcurlyeq x$;

- *transitivity*: for each $x, y, z \in X$, $x \succcurlyeq y$ and $y \succcurlyeq z \Rightarrow x \succcurlyeq z$;
- *completeness*: for each $x, y \in X$, $x \neq y \Rightarrow$ either $x \succcurlyeq y$ or $x \succcurlyeq y$;
- *antisymmetry*: for each $x, y \in X$, $x \succcurlyeq y$ and $y \succcurlyeq x \Rightarrow x = y$.

A reflexive, transitive and complete binary relation on X is called a *complete pre-order* or also, indifferently, a *preference relation* or a *ranking* over X.

A reflexive, transitive, complete and antisymmetric binary relation on X is called a *complete order* on X or also a *strict ranking* over X. $\boldsymbol{\mathcal{R}}(X)$ denotes the set of rankings (or complete pre-orders) on a given set X. Given a ranking $\succcurlyeq \in \boldsymbol{\mathcal{R}}(X)$, in general, there are elements $x \neq y$ such that both $x \succcurlyeq y$ and $y \succcurlyeq x$ hold: in this case we say that x and y are indifferent, and we write $x \sim y$.

In the following, we consider a *finite* set N of n elements that should be ranked and the set of its non-empty subsets is denoted by $\mathcal{P}(N)$.

Definition 5.1. A *social ranking function under coalition information*, or briefly a *ranking function*, is a function

$$r : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \boldsymbol{\mathcal{R}}(N).$$

In other words, r is a function providing a ranking of the objects of N, starting from any possible ranking on subsets of N.

In the sequel, in order to facilitate the reading, with a little abuse of notation, given a ranking $\succcurlyeq$ and a ranking function r we shall write

$$r(x) \succcurlyeq r(y)$$

instead of $x\ r(\succcurlyeq)\ y$ to indicate that, starting from the given ranking $\succcurlyeq$ on $\mathcal{P}(N)$, x is in relation with y according to the ranking function r. Analogously we write

$$r(x) \succ r(y)$$

to denote that r ranks x strictly better than y.

A trivial example of a ranking function is the function p that ranks the elements in N according to the ranking of the singletons in $\mathcal{P}(N)$. Given a ranking $\succcurlyeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ we have that

$$p(x) \succcurlyeq p(y) \iff \{x\} \succcurlyeq \{y\}.$$

However, this solution is extremely unsophisticated since all the information provided by the ranking over coalitions is not taken into account at all. The following example shows that, according to the situation we are modelling, there can be different criteria useful to select a ranking among players.

Example 5.1. Consider the set of players $\{a, b, c\}$ and the following ranking $\succcurlyeq$

$$\{a, b\} \sim \{a, c\} \succ \{b\} \succ \{c\} \sim \{a, b, c\} \succ \{a\} \sim \{b, c\}.$$

We can suppose that this ranking reflects the scientific works of three professor. If we consider the ranking p provided by the ranking among singletons we have $p(b) \succ p(c) \succ p(a)$.

However, player a is in the first two position when working with her colleagues, thus she seems a more important resource if we are interested, for instance, in a team work project.

We introduce another notation we will use in the following. Suppose we have a ranking $\succcurlyeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ of the form

$$S_1 \succcurlyeq S_2 \succcurlyeq S_3 \succcurlyeq \cdots \succcurlyeq S_{2^n-1}.$$

Unless the ranking is a complete order, some indifferences are present. We associate to this ranking the following notation

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

in which the subsets S_j have been grouped in the equivalence classes Σ_k generated by $\sim$. We use this notations to stress the strict ranking among some groups and the indifferences among the others. This means that all the sets in Σ_1 are indifferent to S_1 and are strictly better then the sets in Σ_2 and so on. Thus, for every j all coalitions in Σ_j are ranked at the same level, and are strictly better than any coalition in Σ_{j+1}. It is clear that $\succcurlyeq$ offers, for some $l = 1, \ldots, 2^n - 1$, l different levels of satisfaction with respect to the coalitions, with the first level, i.e. coalitions in Σ_1 are the best, and so on. For example, such equivalence classes could represent the levels of scientific productivity reached by different groups of researchers (for instance given by the best ranked journal on which a group has published articles). Note that if $\succcurlyeq$ is a complete order, then the two relations are the same and $\Sigma_i = S_i$ for any $i = 1, \ldots, 2^n - 1$.

5.2 Properties for a ranking function

In this section we introduce some properties that in our opinion a ranking function should reasonably satisfy, and we discuss their importance and interpretation.

Let us consider a strict preference relation $\succ$:

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

and an element $x \in N$. We denote by x_k the number of sets in Σ_k containing x:

$$x_k = |\{S \in \Sigma_k : x \in S\}|.$$

The first property we introduce is the following

Axiom 5.1 (Equal Treatment of Groups (ETG))**.** We say that a ranking function r satisfies the Equal Treatment of Groups property if, for any strict ranking $\succ$ and any $x, y \in N$ such that $x_k = y_k$ for any $k = 1, \ldots, l$, then

$$r(x) \sim r(y).$$

The *Equal Treatment of Groups* property requires that all groups count the same. To be more specific, what the above property wants to stress is the following. The ranking $\succcurlyeq$ provides a partition of the groups on l different levels of satisfaction and at each of these levels the objects x and y are present in the same number of groups. Thus, since we do not assume that some groups are more desirable than others (for instance according to some rule related to their cardinality or related to which players are inside them), the property requires that there cannot be a strict preference between x and y. For the sake of the example about professors' evaluation, let's say that if two professors belong to an equal number of equivalent groups for each level of scientific productivity, then the ETG property imposes to them the same ranking.

The next property is a kind of monotonicity property, and it serves, as it is clear from its definition, to break ties in a consistent way.

Axiom 5.2 (Monotonicity)**.** We say that a ranking function r is *monotone* if for any strict ranking $\succ$:

$$\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_u \succ \Sigma_{u+1} \succ \cdots \succ \Sigma_l$$

and any $x, y \in N$ such that $r(x) \sim r(y)$, then, for any subset $\Sigma \in \Sigma_{u+1}$ such that $\{x, y\} \cap \Sigma = \{x\}$ if we consider the strict ranking $\sqsupset$:

$$\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \quad \Sigma_u \cup \Sigma \quad \sqsupset \quad \Sigma_{u+1} \setminus \Sigma \quad \sqsupset \cdots \sqsupset \Sigma_l$$

it holds $r(x) \sqsupset r(y)$ i.e. if x and y were indifferent in the ranking provided by $\succ$, then in the second situation x is ranked strictly better than y.

The requirement is clear. Suppose that, for a given preference order, the ranking function ranks the two objects x and y at the same level of satisfaction (e.g., the two professors based on their scientific performance over different teams). Then for every pre-order obtained by the given one just strictly improving the ranking of some coalition containing x and not y (e.g., due to the fact that the educational quality is considered as a secondary criterion for the groups' evaluation, and the educational offer provided by the group containing professor x and not y is much more differentiated than the one of other groups with an equivalent scientific productivity), the ranking function now ranks x strictly better than y. In other words, since x in the new preference system has an improvement (because it belongs to a coalition that now is strictly better ranked than before), while y is exactly in the same situation as before, the tie present before between x and y can now be broken. We observe that this looks like a very reasonable criterion to break ties.

The two previous properties are general request to a ranking function and do not refer to any specific idea behind the way how to rank people looking at their performances when working in different groups. As far as the next property is concerned, on the contrary, a precise underlying idea on how to drive the ranking pops up: to consider the good performances more important than the bad ones.

Axiom 5.3 (Independence from the worst set)**.** We say that a ranking function r is *independent from the worst set* if for any strict ranking

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

with $l \geq 2$, $x, y \in N$ such that

$$r(x) \succ r(y)$$

then, it holds

$$r(x) \sqsupset r(y)$$

for any partition $T_1, \dots T_m$ of Σ_l and for any order $\sqsupset$ such that

$$\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset T_1 \sqsupset \cdots \sqsupset T_m.$$

Let us comment on the property. It claims the following. Suppose a strict ranking is already reached on a pair of objects, starting from a given preference relation

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l.$$

Suppose now to consider a new preference relation $\sqsupset$ on the subsets with the property that it is built from the previous one just refining *only* the information on the previous worst set Σ_l. Then the property simply requires that the ranking of the two objects does not change. Thus the property proposes to consider as irrelevant a further refinement of the preferences on the worst group and thus to ignore this further information, when a strict ranking among two elements is already defined. In our example, once a strict ranking between two professors is established on the basis of their scientific productivity over all groups, the possible use of a secondary criterion for groups' evaluation (e.g., the educational offer of a team) affecting only coalitions with the lowest scientific productivity, may not impact a strict ranking defined according to the most important evaluation's criteria.

We again stress the fact that, differently from the two previous properties, that should be fulfilled by any reasonable ranking function, this last property privileges an idea that, though very reasonable, is clearly not the unique reasonable criterion. In a subsequent section we shall see how it is possible to *dualize* the above property to get another, different but equally reasonable criterion, characterizing a different ranking function.

5.3 Two solutions to the ranking problem

5.3.1 The excellence ranking function

We can now introduce and characterize the ranking function we propose. First of all, let us recall that given a ranking $\succcurlyeq$ and any element $x \in N$, x_k is the number of sets containing x in Σ_k, that is

$$x_k = |\{S \in \Sigma_k : x \in S\}|$$

for $k = 1, \ldots, l$. Now, let $\theta_{\succcurlyeq}(x)$ be the l-dimensional vector $\theta_{\succcurlyeq}(x) = (x_1, \ldots, x_l)$ associated to $\succcurlyeq$. Consider the lexicographic order among vectors:

$$\mathbf{x} \geq_L \mathbf{y} \quad \text{if either } \mathbf{x} = \mathbf{y} \quad \text{or} \quad \exists j : x_i = y_i, i = 1, \ldots, j-1 \ \wedge x_j > y_j.$$

As it is well known, and easy to see, $\geq_L$ defines a complete order on $\mathbb{R}^l$. Now, we are ready for the main definition.

Definition 5.2. The *excellence ranking function* is the function

$$\boldsymbol{e} : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \boldsymbol{\mathcal{R}}(N)$$

defined for any $\succcurlyeq$ as

$$\boldsymbol{e}(x) \succcurlyeq \boldsymbol{e}(y) \qquad \text{if} \qquad \theta_{\succcurlyeq}(x) \ \geq_L \ \theta_{\succcurlyeq}(y).$$

Note that, in general, $\boldsymbol{e}$ provides a pre-order and not an order, even if $\geq_L$ is an order; this is due to the fact that $\theta_{\succcurlyeq}$ in general is not one-to-one; however, if $\succcurlyeq$ is a complete order, actually $\boldsymbol{e}(\succcurlyeq)$ provides an order and the $(2^n - 1)$-dimensional vector $\theta_{\succcurlyeq}(x)$ is boolean, i.e. made by only zeros and ones.

Example 5.2. Consider the set of elements $\{a, b, c\}$ and the following ranking $\succcurlyeq$

$$\{a, b\} \succ \{c\} \succ \{b\} \succ \{a, c\} \sim \{a, b, c\} \succ \{a\} \sim \{b, c\}.$$

Then we have

$$\begin{aligned} \theta_{\succcurlyeq}(a) &= (1, 0, 0, 2, 1) \\ \theta_{\succcurlyeq}(b) &= (1, 0, 1, 1, 1) \\ \theta_{\succcurlyeq}(c) &= (0, 1, 0, 2, 1) \end{aligned}$$

then the excellence ranking function ranks $\boldsymbol{e}(b) \succcurlyeq \boldsymbol{e}(a) \succcurlyeq \boldsymbol{e}(c)$.

In the remaining of this section we characterize the excellence function as the unique one fulfilling the three properties: ETG, monotonicity and independence from the worst set.

Theorem 5.1. *The excellence ranking function $\boldsymbol{e}$ satisfies axioms 5.1, 5.2 and 5.3.*

Proof. The ETG condition requires that for any ranking $\succcurlyeq$, if $\theta_{\succcurlyeq}(x) = \theta_{\succcurlyeq}(y)$ then $x \sim y$. This comes immediately from the definition of excellence function.
Let us consider the monotonicity property. Given a ranking $\succcurlyeq$ if $\boldsymbol{e}(x) \sim \boldsymbol{e}(y)$, this means that $\theta_{\succcurlyeq}(x) = \theta_{\succcurlyeq}(y)$. Now consider a new ranking $\sqsupset$ as in Axiom 5.2, we have that $\theta_{\sqsupset}(x) >_L \theta_{\succcurlyeq}(x)$, while $\theta_{\sqsupset}(y) =_L \theta_{\succcurlyeq}(y)$. So $\theta_{\sqsupset}(x) >_L \theta_{\sqsupset}(y)$ and $\boldsymbol{e}(x) \sqsupset \boldsymbol{e}(y)$.
Finally the third property is obvious from the definition of $\boldsymbol{e}$. □

Theorem 5.2. *Let φ be a ranking function that satisfies axioms 5.1, 5.2 and 5.3. Then for any $\succcurlyeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ and $x, y \in N$*

$$\boldsymbol{e}(x) \succ \boldsymbol{e}(y) \iff \varphi(x) \succ \varphi(y).$$

Proof. ($\Rightarrow$) Let $\succcurlyeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ such that $\boldsymbol{e}(x) \succ \boldsymbol{e}(y)$. This means that $\theta_{\succcurlyeq}(x) >_L \theta_{\succcurlyeq}(y)$. Let k be the first index such that $x_k > y_k$.

Define $\Sigma^* = \{A \in \Sigma_k : x \in A, y \notin A\}$ and let Σ be any subset of Σ^* such that $|\Sigma| = x_k - y_k$. This means that in Σ there are only sets containing x and not containing y and if we remove Σ from Σ_k then the number of elements in Σ_k containing x is the same of the number of elements containing y. Note that Σ is not empty and well-defined due to the hypothesis $\boldsymbol{e}(x) \succ \boldsymbol{e}(y)$.

Define the strict order $\sqsupset$ such that

$$\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \quad \Sigma_k \setminus \Sigma \quad \sqsupset \Sigma \cup \Sigma_{k+1} \cup \cdots \cup \Sigma_l,$$

then since φ satisfies axiom 5.1 we have that x and y are indifferent according to φ: $\varphi(x) - \varphi(y)$.

Consider now the relation $>$ in which Σ is moved up of one level:

$$\Sigma_1 > \Sigma_2 > \cdots > \quad \Sigma_k \quad > \Sigma_{k+1} \cup \cdots \cup \Sigma_l,$$

then since φ satisfies axiom 5.2 , if we compare $\sqsupset$ and $>$ we have $\varphi(x) > \varphi(y)$. Finally, thanks to axiom 5.3, when we consider the original ranking $\succcurlyeq$

$$\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \quad \Sigma_k \quad \succ \cdots \succ \Sigma_l,$$

we have $\varphi(x) \succ \varphi(y)$.

($\Leftarrow$) Let $\varphi(x) \succ \varphi(y)$. Suppose that $\boldsymbol{e}(x) \sim \boldsymbol{e}(y)$, then $\theta_{\succeq}(x) = \theta_{\succeq}(y)$ but this is impossible since φ satisfies axiom 5.1.

Suppose then that $\boldsymbol{e}(x) \prec \boldsymbol{e}(y)$, then we already proved that this would imply $\varphi(x) \prec \varphi(y)$. Thus the only possibility is that $\boldsymbol{e}(x) \succ \boldsymbol{e}(y)$. □

Corollary 5.1. *There is one and only one ranking function satisfying the properties of ETG, monotonicity and independence from the worst set. This is the excellence ranking function* $\boldsymbol{e}$.

Proof. The excellence ranking function fulfils the properties, according to Theorem 5.1. Theorem 5.2 takes care of the uniqueness argument, since it shows that a preference relation fulfilling the three properties denotes the same subset of $N \times N$ as the excellence ranking function. □

5.3.2 The dual ranking function

It is possible to define a ranking function dual to the excellence function, in the following way. Given two vectors $\mathbf{x}, \mathbf{y}$ we define the lexicographic* order $\geq_{L^*}$ as

$$\mathbf{x} \geq_{L^*} \mathbf{y} \quad \text{if either } \mathbf{x} = \mathbf{y} \quad \text{or} \ \exists j : x_i = y_i, \forall i > j \ \wedge x_j < y_j.$$

Definition 5.3. The *dual ranking function* is the function $\boldsymbol{w} : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \boldsymbol{\mathcal{R}}(N)$ defined for any $\succcurlyeq$ as

$$\boldsymbol{w}(x) \succcurlyeq \boldsymbol{w}(y) \qquad \text{if} \qquad \theta_{\succeq}(x) \quad \geq_{L^*} \quad \theta_{\succeq}(y).$$

It immediately appears that this ranking function acts, in a sense, dually to the excellence function, since it ranks at the last places objects present in the worst ranked groups. Here mediocrity is punished, before excellence was rewarded. Let us refer to our usual example: how the chairman of the department should rank her professors? Which property should choose? The answer can heavily depend on the way financial support of the research is distributed in the country. If the department is rewarded provided there are outstanding research teams, the independence from the worst set is a very reasonable assumption to make, thus the excellence ranking function would be a good choice. On the contrary, if a department is punished if there are groups with a very low scientific production, a competition at the lowest level is natural, as the dual function proposes, in order to enhance the level of quality of the worst ranked groups.

It is clear that in general the two ranking functions give different ranking among the elements, as the following example shows.

Example 5.3. Consider the ranking $\succcurlyeq$ defined on the power set of $N = \{a, b, c\}$ as

$$\{a, b\} \sim \{a, c\} \succ \{b\} \succ \{b, c\} \succ \{c\} \sim \{a, b, c\} \succ \{a\}.$$

We have that $\theta_{\succcurlyeq}(a) = (2, 0, 0, 1, 1)$, $\theta_{\succcurlyeq}(b) = (1, 1, 1, 1, 0)$ and $\theta_{\succcurlyeq}(c) = (1, 0, 1, 2, 0)$. Then the excellence function ranks $\boldsymbol{e}(a) \succ \boldsymbol{e}(b) \succ \boldsymbol{e}(c)$, while the dual excellence function ranks $\boldsymbol{w}(b) \succ \boldsymbol{w}(c) \succ \boldsymbol{w}(a)$.

The function $\boldsymbol{w}$ satisfies ETG and monotonicity. In order to characterize it we need to define a axiom in some sense symmetrical to axiom 5.3.

Axiom 5.4 (Independence from the best set)**.** We say that a ranking function is *independent from the best set* if for any strict order $\succ$

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

and $x, y \in N$ such that

$$r(x) \succ r(y),$$

then it holds

$$r(x) \sqsupset r(y)$$

for any partition $T_1, \dots T_m$ of Σ_1 and for any order $\sqsupset$

$$T_1 \sqsupset \cdots \sqsupset T_m \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset \Sigma_l.$$

The requirement of the above property is that if a strict ranking is already reached on a pair of objects, then it cannot change by refining the best preferred equivalence class: the ranking remains the same even if Σ_1 is partitioned according to any possible partition. Clearly the requirement proposed by this property is to ignore the information provided from the best ranked coalitions.

Theorem 5.3. *The dual ranking function* $\boldsymbol{w}$ *satisfies axioms 5.1, 5.2 and 5.4.*

Proof. From the definition of the $\boldsymbol{w}$ function it is trivial to check that it satisfies the three axioms. □

Theorem 5.4. *Let* φ *be a ranking function that satisfies axioms 5.1, 5.2 and 5.4. Then for any* $\succcurlyeq$ *and* $x, y \in N$

$$\boldsymbol{w}(x) \succ \boldsymbol{w}(y) \iff \varphi(x) \succ \varphi(y).$$

Proof. ($\Rightarrow$) Let $\succcurlyeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ such that $\boldsymbol{w}(x) \succ \boldsymbol{w}(y)$. This means that $\theta_{\succcurlyeq}(x) \geq_{L^*} \theta_{\succcurlyeq}(y)$. Let k be the index such that $x_k < y_k$ and $x_i = y_i$ for all $i > k$.

This means that in Σ_k there are some elements containing y and not x. Let $\Sigma^* = \{A \in \Sigma_i : i < k, x \in A, y \notin A\}$ and let Σ be any subset of Σ^* such that $|\Sigma| = y_k - x_k$. Σ is not empty, since the number of subsets is fixed and it holds $x_k < y_k$ and $x_i = y_i$ for all $i > k$. This means that in Σ there are sets containing x and not containing y and if we add Σ to Σ_k then the number of elements in Σ_k containing x is the same of the number of elements containing y.

Define the order $\sqsupset$ such that

$$(\Sigma_1 \cup \cdots \cup \Sigma_{k-1}) \setminus \Sigma \sqsupset \quad \Sigma_k \cup \Sigma \quad \sqsupset \Sigma_{k+1} \sqsupset \Sigma_{k+2} \sqsupset \cdots \sqsupset \Sigma_l,$$

then since φ satisfies axiom 5.1 we have $\varphi(x) - \varphi(y)$.

Consider now the relation $\geq$ in which Σ is moved up of one level:

$$\Sigma_1 \cup \Sigma_2 \cup \cdots \geq \quad \Sigma_k \quad \geq \Sigma_{k+1} \geq \cdots \geq \Sigma_l,$$

then since φ satisfies axiom 5.2 we have $\varphi(x) > \varphi(y)$. Finally, thanks to axiom 5.4, when we consider the original ranking $\succcurlyeq$

$$\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \quad \Sigma_k \quad \succ \cdots \succ \Sigma_l,$$

we have $\varphi(x) \succ \varphi(y)$.

($\Leftarrow$) Let $\varphi(x) \succ \varphi(y)$. Suppose that $\boldsymbol{w}(x) \sim \boldsymbol{w}(y)$, then $\theta_{\succcurlyeq}(x) = \theta_{\succcurlyeq}(y)$ but this is impossible since φ satisfies axiom 5.1.

Suppose then that $\boldsymbol{w}(x) \prec \boldsymbol{w}(y)$, then we already proved that $\varphi(x) \prec \varphi(y)$. Thus the only possibility is that $\boldsymbol{w}(x) \succ \boldsymbol{w}(y)$. $\square$

Corollary 5.2. *There is one and only one ranking function satisfying the properties of ETG, monotonicity and independence from the best set. This is the dual ranking function $\boldsymbol{w}$.*

Proof. The dual ranking function fulfils the properties, according to Theorem 5.3. Theorem 5.4 takes care of the uniqueness argument, since it shows that a preference relation fulfilling the three properties denotes the same subset of $N \times N$ as the dual ranking function. $\square$

5.4 Independence of the axioms

Let us prove that axioms 5.1, 5.2, 5.3 (and 5.4) are independent, thus they all are necessary in order to uniquely characterize the excellence ranking function and its dual ranking function.

ETG is not satisfied Given a finite set N, take any arbitrary order $<$ on it. Let $\boldsymbol{e}^*$ be the ranking function defined as $\boldsymbol{e}$ but if there is a ranking $\succcurlyeq$ and there are two elements x, y such that $\theta_{\succcurlyeq}(x) = \theta_{\succcurlyeq}(y) = (2^{n-1}, 0, \ldots, 0)$ (i.e. all sets with the elements x and y are in Σ_1) then $\boldsymbol{e}^*$ breaks the tie with $\boldsymbol{e}^*(x) \succ \boldsymbol{e}^*(y)$ if $x < y$.
For instance if we take $N = \{1, 2, \ldots, n\}$ and $>$ as the majority relation among integer numbers, then in case of indifference among all the subsets, $\boldsymbol{e}^*$ ranks the elements in increasing order.

This function satisfies Axioms 5.2 and 5.3, but does not satisfy ETG.

Monotonicity is not satisfied Let f be a ranking function defined as

$$f(x) \succcurlyeq f(y) \text{ if } x_j \geq y_j$$

where $j = \min\{k : \Sigma_k \cap \{x, y\} \neq \emptyset\}$.

The function f ranks players only counting the number of sets in the first level containing those players.
It is clear that f satisfies Axioms 5.1 and 5.3, since the ranking only depends on the number of sets in Σ_j. However, this function does not satisfy monotonicity: if we improve the situation of one player but do not change Σ_j, then the ranking does not change. For instance consider the following ranking $\succcurlyeq$ on $N = \{a, b\}$

$$\{a, b\} \succ \{b\} \sim \{a\}$$

then $f(a) \sim f(b)$. Consider now the ranking

$$\{a, b\} \sqsupset \{b\} \sqsupset \{a\},$$

then it still holds $f(a) - f(b)$, while monotonicity would require $f(a) \sqsupset f(b)$.

Independence from the worst set is not satisfied Consider $\boldsymbol{w}$ the dual function of $\boldsymbol{e}$ as defined in the previous paragraphs. Then it satisfies ETG since given a ranking $\succcurlyeq$ we have that $\boldsymbol{w}(x) \sim \boldsymbol{w}(y)$ if and only if $x_k = y_k$ for any k. It satisfies monotonicity since if $\boldsymbol{w}(x) \sim \boldsymbol{w}(y)$ and we take $\Sigma \subseteq \Sigma_{u+1}$ for some u such that $\Sigma \cap \{x, y\} = \{x\}$, then in the new ranking we have that $x_{u+1} < y_{u+1}$ and thus x will be ranked better than y.
Of course, $\boldsymbol{w}$ does not satisfy independence from the worst set. For instance given the ranking

$$\cdots \succ \{a, b\} \sim \{a\} \succ \{b\} \sim \{b, c\} \sim \{a, c\}$$

we have $\boldsymbol{w}(a) \succ \boldsymbol{w}(c) \succ \boldsymbol{w}(b)$. But if we change the last set in the following way

$$\cdots \sqsupset \{a, b\} - \{a\} \sqsupset \{b\} - \{b, c\} \sqsupset \{a, c\}$$

we get a new ranking: $\boldsymbol{w}(b) \sqsupset \boldsymbol{w}(a) \sqsupset \boldsymbol{w}(c)$.

Independence from the best set is not satisfied The excellence function $\boldsymbol{e}$ satisfies ETG and monotonicity but of course, it is not independent from the best set. Thus, it is also clear that the three axioms characterizing the dual ranking function are independent.

5.5 Conclusion

In this chapter we have proposed a general way to rank objects whenever it is available a ranking between the subsets of these objects. Our approach was classical, in the sense that we identified a rule by requiring the fulfilment of some general properties. Having this characterization in mind, there are other interesting issues to consider: first of all, it is clear that in practical situations it can happen that having a ranking on *all* subsets of N is an unrealistic requirement: in our paradigmatic example of the ranking of the professors in a department, it is unreasonable to expect that all members collaborate forming every possible research team. Moreover, there may be some specific reason to reward coalitions of a fixed size for instance or the collaborations with a specific element or group of elements. In all these situations, there may be a ranking over only a subset of $\mathcal{P}(N)$.

Of course, the excellence ranking can be defined, without any changes, on subsets of $\mathcal{P}(N)$. And clearly the properties that characterize the function hold if we restrict the domain. However uniqueness is not clear in this case. For instance, the equal treatment of groups property in a subset could be totally uninformative, since it is possible, for instance, that no pair of objects is present in the same number of coalitions. Thus a characterization in this case must depend from the type of coalitions that are actually ranked.

Secondly, as usual the main theorem we provide requires the ranking function to be defined on a very large set, namely the set of all pre-orders over the subsets of N. But it would be interesting to analyse the same problem of characterizing a ranking function defined only on (meaningful) subsets of $\boldsymbol{\mathcal{R}}(\mathcal{P}(N))$. For instance, there could be only k different levels to judge the collaboration between the elements, thus we can consider the elements in $\boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ for which the number of indifference sets is fixed. Another interesting point for future developments would be to consider the restriction to orders over $\mathcal{P}(N)$ and provide a characterization of the excellence ranking function in that framework.

CHAPTER 6

Concluding remarks

THE main work presented in this thesis is about the evaluation of power in different voting situations and the procedure to rank players in different contexts.

At the beginning of our work we analysed new contributions to the classical cooperative game theory. We proposed a procedure to generate other semivalues as solution concepts and examined the well-known Shapley value from a different point of view.

In the central part of the thesis we examined the model of games with abstention and of multichoice games. In particular, we provided some axiomatizations for two power indices for games with abstention and defined a new value for multichoice cooperative games with in the spirit of the Shapley value.

In the last part of the thesis, we considered the problem of ranking players from a new perspective. We presented two functions that associate a ranking over players, given a preference profile over the subsets formed by those players. We also provided an axiomatic characterization of these two functions.

Our work can be further developed in different directions. First of all, it would be interesting to find real life examples and applications for all the ideas presented in this thesis. An interesting line of application would be also to establish the connections between a ranking with power indices and with the lexicographic functions we introduced.

Secondly, a natural extension of our work is to provide other characterizations for power indices for games with abstention or define other values in the family of multichoice games. To reinforce the use of these indices as solution concepts, more properties can be studied and other mathematical aspects can be analysed. The main difficulties to develop this work is to identify which axioms can be properly generalized from simple and cooperative games to games with abstention and multichoice games.

Lastly, the more general approach to the ranking problem given an ordinal relation among coalitions seems to be quite innovative and promising. It is clear that other properties can be defined to capture different ideas and to extend the range of application of the results presented in our last chapter. Since there are different approaches and philosophies behind the problem of ranking players, it is possible to define new ranking functions and new axioms. Moreover, it is desirable to find other properties, that can also characterize the proposed ranking functions ands that are more self-explanatory and independent from the functions presented here.

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