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LOWER AND UPPER BOUNDS FOR BASKET OPTIONS

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Introduction

Multidimensional problems represent a great issue in the financial industry and an interesting research area in academy. In particular the mathematical and numerical difficulties issued by this class of problems are relevant even for the "simple" reference models, where the underlying processes are correlated Geometric Brownian motions with parameters eventually deterministic functions of time, such as the Black model [6] for Equity derivatives and Hull and White model [21] for interest rates where Zero Coupon Bonds' terminal distributions are described by multivariate log-normals. In this thesis we tackle two different multidimensional problems related to Basket Options: the first is the pricing and hedging European Basket Options in Equity markets and the second is related to the pricing of illiquid corporate Bonds. In the latter case we show that the problem can be modeled as a Lookback Basket Option. Both Basket Option problems can be dealt with Lower and Upper Bounds that in some cases can be very close to the exact solution but present several analytical and numerical disadvantages. The thesis is divided in two parts where each one is related to one of these two applications of Lower and Upper bounds to Basket Options.

European Basket Options considered in the first part, hereinafter also standard Basket Options, are a particular class of derivatives that can be viewed as Plain Vanilla European options on the weighted sum of a certain number of assets. In a standard Basket Option, as an index Option like Eurostoxx50 or SP500, tens or hundreds of underlyings can be involved and thus the computation of prices and Greeks can be extremely time consuming, inaccurate and numerically unstable with standard Monte Carlo techniques. For this reason it is common among practitioners to rely on closed formula Lower and Upper Bounds which admit also simple closed formulas for all sensitivities. As already underlined in the literature [9], even if is well known that constant volatility is not sufficient to describe real market data, the Black formula is still considered a landmark and hence studying its generalization on a multidimensional framework can be considered an interesting research area by itself. In this thesis we show that is already challenging the problem where one models the underlyings as Geometric Brownian Motions. In particular we focus our research on Lower and Upper Bounds that present a simple analytic form and show sensitivities similar to the Black ones. This problem has been first tackled by the seminal paper of Curran [10] and nowadays the best performing Lower Bound can be constructed starting from some intuitions of Carmona and Durrleman [9] and Lord [30].

Carmona and Durrleman derive a set of Lower Bounds with an analytic form equal to weighted sums of Black call like formulas. They also consider the problem of picking the best Lower Bound in the set introduced. This is a constrained optimization problem that is numerically solved by the authors using a gradient based method.

In this thesis we follow a similar approach with a different Lower Bound that admits Carmona and Durrleman one as a subcase.

This functional form comes from the idea developed by Lord in [30]. According to Lord the Lower Bound functional form is a weighted sum of Black like calls and puts. He considers the Lower Bound for a selected set of parameters. In this thesis we show that while the two functional form are both very simple, in general considering the second approach gives a significantly better Lower Bound in several cases and it can result in a more accurate approximation.We also show the equivalence of the two approaches for some particular cases.

In this thesis also the set of Upper Bounds introduced by Deelstra et al. in [12] is considered. Likewise the Lower Bound this set result in a very simple analytical form that reminds a linear combination of Black like calls. This set is then interesting for two reasons: first because of its simple analytic form and then because numerical comparisons with other set of Upper Bounds (Chapter 2) show that it is often the best performing.

Two are the innovative contributions of this first part of the thesis to the existing literature.

First we introduce this optimization problem for the Lower Bound and for the Upper Bound. We are able to characterize the problem in terms of existence and uniqueness for some financially relevant situations.

Second we we show that in general the optimization problem for the Lower (Upper) Bound presents several maxima (minima). Unfortunately the standard global optimization tools (e.g. GA in Matlab) sometimes do not reach the global maxima (minima) and it is always extremely slow for problems of financial interest. For this reason an ad-hoc optimization algorithm has been developed that is fast and accurate.

This first part of the thesis is divided as follows. In Chapter 1, after introducing the notation, we define the Lower and Upper Bound and related optimization problems. We also discuss the relations of such bounds with the literature. Chapter 2 is devoted to an analytical study of the two optimization problems. We show that both Upper and Lower

Bounds global and local minima are in relation with the matrix that describes the assets' correlation which characterizes the constraint geometry. In particular we show that the columns of such matrix play a relevant role. We also compare the performance of the optimal bounds with the literature via numerical examples. In Chapter 3 we develop a new algorithm for the numerical computation of the optimal Upper and Lower Bounds. We discuss the numerical issues presented by the optimization problems and we describe in detail the implementation of the algorithm proposed. The performance of such algorithm is then compared with GA of Matlab with numerical examples.

The second part of the thesis is devoted to study the problem of pricing a corporate illiquid coupon bond in a defaultable setting assuming an Hull and White dynamic for the Zero Rates. Following the seminal work of Longstaff [28] the estimation of the illiquidity premium is related to the running maximum (like a Lookback option) of the Bond during a time window, the time-to-liquidate the position, that estimates the time when the trader is able to liquidate his position. This problem has been extended to the Zero Coupon (hereinafter ZC) case of a default free Bond. The main contribution of this thesis is to propose a simple closed formula for a defaultable Coupon Bond in presence of illiquidity. In this case the approach proposed in [25] cannot be extended and we show that this approach is related to a Lookback Basket Option. Also in this case, as in the one of the first part, a closed formula solution would be desirable since the problem is strongly path dependent and Monte Carlo simulations can be inaccurate and time consuming. In this thesis we show that the price of such an illiquid Bond can be evaluated via a Lower Bound and an Upper Bound technique. We also show that, for all practical purposes, these two Bounds coincide and show two examples in the European market.

Contents

Part I

Basket options: a pricing and hedging problem

CHAPTER 1

Upper and lower bounds for basket options

1.1 Introduction

Pricing an European call option written on a linear combination of different underlyings is not a trivial task even in the basic log-normal framework, where the assets risk neutral dynamics are described by a multidimensional geometric Brownian motion. Within this framework when the underlying is one single asset the price of a European call is given by the celebrated Black formula, while when the number of underlyings increase a closed formula cannot be derived anymore because the distribution of the sum of log-normal variables is unknown. This task can be performed only via Monte Carlo techniques but the computation of the price and the sensitivities can be extremely demanding in terms of computational time. For this reason closed formula Lower and Upper Bounds similar to the Black formula have been developed. The literature on Basket Option bounds is quite extensive. Let us mention some of the relevant results to the present thesis. After the seminal work of Curran [10], sets of Lower Bounds obtained making use of the technique of conditioning with respect to a generic Gaussian variable and the Jensen inequality has been introduced in [9, 33, 34]. At the same time similar Lower Bounds were obtained first in [23] and then in [11–13,39] using the so called technique of conditional comonotonicity. To the best of our knowledge the best approximations, in the sense that always outperforms all the others approximations from below, known so far can be found in [30] by Lord where the author derive a set of Bounds which reminds a linear combinations of Black call and put Plain Vanillas and in [9] by Carmona and Durrleman where they consider a Lower Bound obtained by picking the best element in a set of Lower Bounds which reminds a weighted sum of only Black Plain Vanilla calls. Also a several approximations from above have been studied. In particular in [12] the Improved Comonotonic Upper Bound and the Partially Exact Comonotonic Upper Bound that we consider in this thesis are introduced while other upper bounds can be found in [12,30,34,40]. Closed formula bounds for basket and spread options are also studied in non Gaussian models in [8], where the indicated bounds depend on the multivariate characteristic function of the process; when considering Gaussian models the lower bound they propose is the same of [34]. It is also worthwhile to mention the model free approach for the upper bound, where assets' dynamics are not specified, first proposed in [26].

In this Chapter we introduce the notation and we define the two sets of Lower and Upper Bounds we recall the main results in the literature on the subject. For the Lower Bound we introduce a new approximation combining the idea of Lord [30] of considering sets of formulas with Black Plain Vanilla calls and puts and the idea of Carmona and Durrleman [9] of the taking the best element within the set. For the Upper Bound we recall the Improved Comonotonic Upper Bound of [12] giving a slightly more precise definition where the difference is that in our formulation a set of irregular points are well defined. Moreover at the end of Chapter we discuss the cases in which the two approximations default to the true value and the Greeks.

1.2 Basket call in a log-normal framework

Similarly to Carmona and Durrleman we consider a multidimensional Geometric Brownian Motion dynamics for n stocks $S_1, \ldots, S_n$ whose risk neutral dynamics is given by

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + v_i dW_i(t) \tag{1.1}$$

with some initial values $S_i(0), \ldots, S_n(0)$, where $W_i(t)$ are some correlated Brownian motions such that

$$dW_i(t)dW_j(t) = \rho_{ij}dt \tag{1.2}$$

and ρ is a positive definite correlation matrix. Under the assumption of deterministic interest rates, the risk neutral price of a standard Basket Option is

$$\mathcal{C} = B(0,T)\mathbb{E}\left[\left(\sum_{i=1}^{n} S_i(T) - \mathcal{K}\right)_+\right] \tag{1.3}$$

with $B(0,T)$ the discount factor up to T, $B(0,T) = e^{-rT}$. Let us introduce a simplified notation that puts in evidence the relevant free parameters in this pricing problem. We can consider the set of forwards with expiry T associated to the n stocks whose value in $t \in [0,T]$ is

$$F_i^T(t) = e^{(r-q_i)(T-t)} S_i(t) \qquad i = 1, \ldots, n; \quad t \in [0,T]. \tag{1.4}$$

Their martingale dynamics is

$$dF_i^T(t) = F_i^T(t) v_i dW_i(t) \qquad i = 1, \ldots, n; \quad t \in [0,T], \tag{1.5}$$

whose solution in $t = T$ with initial condition $F_i^T(0)$ is

$$F_i^T(T) = F_i^T(0) e^{-\frac{1}{2}v_i^2 t - v_i W_i(T)} \tag{1.6}$$

that is equivalent in law to

$$F_i^T(T) \stackrel{\text{"law"}}{=} F_i^T(0) e^{-\frac{1}{2}v_i^2 T - v_i \sqrt{T} g_i} \tag{1.7}$$

with $\{g_i\}_{i=1,\ldots,n}$ a set of correlated standard normal random variables such that

$$\mathbb{E}\left[g_i g_j\right] = \rho_{ij}. \tag{1.8}$$

Call Option (1.3) is then equivalent to

$$\mathcal{C} = B(0,T) F^T(0) C \tag{1.9}$$

with $F^T(0) = \sum_{i=1}^n F_i^T(0)$ the basket forward at value date and

$$C := \mathbb{E}[(X - K)_+] \tag{1.10}$$

and

$$X = \sum_{i=1}^n \omega_i e^{-\frac{1}{2}\sigma_i^2 - \sigma_i g_i}, \tag{1.11}$$

where we have defined the weights ω_i and the cumulated volatilities such that

$$\begin{cases} \omega_i := \dfrac{a_i F_i^T(0)}{F^T(0)} \\ \sigma_i := v_i \sqrt{T}. \end{cases} \tag{1.12}$$

According to this notation C (1.10) is, a part a set of constants, the quantity that we price that depends via (1.11) only on the $2n$ parameters $\{\omega_i\}_{i=1,\ldots,n}$ and $\{\sigma_i\}_{i=1,\ldots,n}$.

Remark. It is straightforward to show, as in Merton [31] that the problem is equivalent when both interest rates r, dividend yields $\{q_i\}_{i=1,\dots,n}$ and volatilities $\{v_i\}_{i=1,\dots,n}$ are deterministic functions of time. The European Basket Option depends only on the two sets $\{\omega_i\}_{i=1,\dots,n}$, $\{\sigma_i\}_{i=1,\dots,n}$.

Hereinafter we will always refer to the adimensional pricing problem (1.10). When $n = 1$ (1.10) becomes the celebrated Black formula. Even if it is probably the most popular formula in mathematical finance, in the following we want to write and derive it in a slightly different analytic form involving an optimization problem. The reason is that the derivation of the Lower Bound in the next section is conceptually similar to the following derivation of the Black formula. We start with a technical lemma (see [9], Proposition 2 pag.4).

Lemma 1.2.1. *For any integrable random variable V and given the set $\mathcal{A}$ of random variables taking values in $[0,1]$*

$$\mathbb{E}[V_+] = \sup_{Y\in\mathcal{A}} \mathbb{E}[YV]. \tag{1.13}$$

Proof. Being $Y \in [0,1]$ then

$$\mathbb{E}[YV] = \mathbb{E}[YV_+] - \mathbb{E}[YV_-] \leq \mathbb{E}[V_+].$$

and, denoting by $\mathbf{1}_{\{\cdot\}}$ the characteristic function, the equality is attained for $Y = \mathbf{1}_{\{V>0\}}$. □

We can now write the Black formula, normalized for the forward and the discount, in a revisited analytic form. Defining the density function of the standard normal variable with $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$ and the cumulative function $\phi(x) = \int_{-\infty}^{x} \varphi(t)dt$, the following proposition holds

Proposition 1.2.1 (Black76 adimensional formula)**.** *For $n = 1$ it holds that*

$$C = \sup_{y\in\mathbb{R}} \left[\phi(y+\sigma_1) - K\phi(y)\right]. \tag{1.14}$$

Proof. Defining $V = \left(e^{-\frac{1}{2}\sigma_1^2 - \sigma_1 g_1} - K\right)$, we have

$$C = \mathbb{E}\left[V_+\right].$$

Then defining the subset $\mathcal{G}$ of $\mathcal{A}$, which contains the random variables $Y(y)$ defined as $Y(y) := \mathbf{1}_{\{g_1<y\}}$ with $y \in \mathbb{R}$, we have that, being V monotone with respect to g_1, the element $\mathbf{1}_{\{V>0\}}$ belongs to $\mathcal{G}$. Then we can exploit Lemma 1.2.1 restricting to the set $\mathcal{G}$

and we obtain

$$
\begin{aligned}
C = & \sup_{Y(y)\in\mathcal{G}} \mathbb{E}\left[YV\right] \\
= & \sup_{y\in\mathbb{R}} \int_{-\infty}^{\infty} dt\varphi(t)\left(e^{-\frac{1}{2}\sigma_1^2-\sigma_1 t} - K\right)\mathbf{1}_{\{t<y\}} \\
= & \sup_{y\in\mathbb{R}} \left[\phi(y+\sigma_1) - K\phi(y)\right]
\end{aligned}
$$

which is the result. □

Explicitly computing the supremum over y the usual analytic form of the Black formula is obtained.

Proposition 1.2.2 (Black76 adimensional standard formula)**.** *For n=1 it holds that*

$$
C = \phi(y^\star + \sigma_1) - K\phi(y^\star), \tag{1.15}
$$

where $y^\star = -\frac{\log K}{\sigma_1^2} - \frac{1}{2}\sigma_1$ *is the solution of the equation*

$$
e^{-\frac{1}{2}\sigma_1^2-\sigma_1 y} - K = 0. \tag{1.16}
$$

Proof. Imposing the derivative w.r.t y of eq.(1.14) equal to zero, eq.(1.16) is obtained. This equation has only one zero which corresponds to a maximum. □

The value of the adimensional put option with strike K can be obtained via the adimensional put call parity

$$
P = C - 1 + K \tag{1.17}
$$

Let us stress stress that the key element in obtaining a closed form exact solution to the trivial one dimensional pricing problem is to define a simple subset of $\mathcal{A}$, which contains only elements with Gaussian measure and which also contains the element $\mathbf{1}_{\{(X-K)>0\}}$. When $n > 1$, it is impossible to define such a subset and a Lower Bound is obtained. This is indeed the technique used in [9] for obtaining their Lower Bound and that we also use in the next section. In addition all the others Lower Bound can be obtained by considering different sets $\mathcal{G}$ over which the supremum is computed. Let us also remark that, as shown by Caldana et al. in [8], this technique can be also generalized to non Gaussian models to obtain closed formula Lower Bounds to Basket options in terms of the characteristic function.

1.3 Lower Bound

In this section we introduce a Lower Bound for the Basket Options 1.10 relating with the existing literature on the subject. As discussed in the above section, the idea of [9] for obtaining a Lower Bound to the price of a basket call is to restrict the set over which the supremum of Lemma 1.2.1 is computed. In particular, instead of the set $\mathcal{A}$ containing all the random variables taking values in $[0,1]$, we restrict to the subset $\mathbf{1}_{\{\mathcal{G}\}}$ where $\mathcal{G}$ contains only variables with Gaussian measure. Here we exploit the same technique but we consider a wider set. Picking the best element within $\mathcal{G}$ results in a nonlinear constrained programming problem, hence we start defining the objective function of the optimization problem, the constraint and then we show the solution of such optimization problem is actually a Lower Bound.

Definition 1.3.1 (Lower Bound objective function). *Let us consider $x \in \mathbb{R}^n$ and $y = (y_1, y_2) \in \mathbb{R}^2$, then we define*

$$LB(x,y) := \begin{cases} \sum_{i=1}^n w_i \left[\phi\left(y_1 + \sigma_i x_i\right) + \phi\left(-y_2 - \sigma_i x_i\right)\right] - K\left[\phi\left(y_1\right) + \phi\left(-y_2\right)\right] & y_1 \leq y_2 \\ (1-K)_+ & y_1 > y_2 \end{cases} \tag{1.18}$$

We also introduce the domain where the supremum over x is computed.

Definition 1.3.2 (Ellipsoid $\mathcal{Q}$). *Given $x \in \mathbb{R}^n$ and $\rho \in \mathbb{R}^{n\times n}$ positive definite we define*

$$\mathcal{Q} := \left\{x \in \mathbb{R}^n \ : \ x \cdot \left(\rho^{-1}x\right) = 1\right\} \tag{1.19}$$

The domain $\mathcal{Q}$ is clearly an hyper-ellipsoid in $\mathbb{R}^n$ (hereinafter ellipsoid) because ρ^{-1} is a positive definite squared matrix in $\mathbb{R}^{n\times n}$. The Optimal Lower Bound is defined as the solution of the following problem

Definition 1.3.3 (Optimal Lower Bound). *Given the basket call* (1.10) *we define the Optimal Lower Bound through the following optimization problem:*

$$\mathrm{LB}_{opt} := \sup_{x\in\mathcal{Q}} \sup_{y\in\mathbb{R}^2} LB(x,y). \tag{1.20}$$

In the next proposition we show that formula (1.20) is a Lower Bound.

Theorem 1.3.1. *Given the call C in* (1.10)*, then LB_{otp} is a Lower Bound for C, i.e.*

$$\mathrm{LB}_{opt} \leq C. \tag{1.21}$$

Proof. Lemma 1.2.1 implies that for every subset $\mathcal{G}$ of the set $\mathcal{A}$ of of all the random variables taking values in $[0,1]$ and any integrable variable V

$$\left(\sup_{Y\in\mathcal{G}} \mathbb{E}\left[VY\right]\right)_+ \leq \mathbb{E}\left[V_+\right].$$

We then consider the subset $\mathcal{G}$ containing all the elements of the type $Y(\beta, y) = \mathbf{1}_{\{\beta \cdot g \leq y_1 \cup \beta \cdot g \geq y_2\}}$ parametrized by β and y where $\beta \in \mathbb{R}^n/\{0\}$ and $y \in \mathbb{R}^2$. When $y_1 \geq y_2$, $Y(\beta, y)$ is the variable constant equal to 1 and then

$$(\mathbb{E}\left[VY(\beta, y)\right])_+ = (\mathbb{E}\left[V\right])_+ .$$

Then taking $V = X - K$ we obtain

$$(\mathbb{E}\left[X - K\right])_+ = (1 - K)_+$$

The non trivial cases are obtained using the total probability formula. Defining $x \in \mathbb{R}^n$ the vector whose components x_i are the correlations between $\beta \cdot g$ and g_i,

$$x_i = \frac{\mathbb{E}[g_i \beta \cdot g]}{\mathbb{E}[g_i^2]\mathbb{E}[(\beta \cdot g)^2]} = \frac{(\rho\beta)_i}{\sqrt{\beta \cdot \rho\beta}},$$

we have

$$\begin{aligned}
\sup_{Y \in \mathcal{G}} \mathbb{E}\left[(S-K)Y\right] &= \sup_{\beta \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^2} \mathbb{E}\left[\mathbb{E}\left[S - K | \beta \cdot g\right] \mathbf{1}_{\{\beta \cdot g < y_1\}} \mathbf{1}_{\{\beta \cdot g > y_2\}}\right] \\
&= \sup_{x \in \mathcal{Q}} \sup_{y \in \mathbb{R}^2} \mathbb{E}\left[\left(\sum_{i=1}^n \omega_i e^{-\frac{1}{2}\sigma_i^2 x_i^2 - \sigma_i x_i \beta \cdot g} - K\right) \mathbf{1}_{\{\beta \cdot g < y_1\}} \mathbf{1}_{\{\beta \cdot g > y_2\}}\right] \\
&= \sup_{x \in \mathcal{Q}} \sup_{y \in \mathbb{R}^2} LB(x, y).
\end{aligned}$$

Now we only need to show that actually $x \in \mathcal{Q}$. Via direct computation we have

$$x \cdot \left(\rho^{-1} x\right) = \frac{(\rho\beta)}{\sqrt{\beta \cdot \rho\beta}} \cdot \left(\rho^{-1} \frac{(\rho\beta)}{\sqrt{\beta \cdot \rho\beta}} = 1\right).$$

$\square$

We can observe that LB_{opt} is a multidimensional extension of the Black formula: it is a linear combination of Black call and put formulas (1.14) where the volatilities σ_i are scaled by the factors x_i. Being the main goal of this part of the thesis the computation of the optimal Lower Bound LB_{opt} defined as the solution of the problem (1.20), it is interesting in first instance to look at the first order conditions. Defining the Lagrangian function

$$L(x, y, \mu) = LB(x, y) - \frac{\mu}{2}\left(x\left(\cdot \rho^{-1} x - 1\right)\right), \tag{1.22}$$

where μ is a Lagrange multiplier, the first order conditions

$$\begin{cases}
\frac{\partial L(x,y,\mu)}{\partial y_1} = 0 \\
\frac{\partial L(x,y,\mu)}{\partial y_2} = 0 \\
\frac{\partial L(x,y,\mu)}{\partial x_i} = 0 \qquad i = 1, \ldots, n \\
\frac{\partial L(x,y,\mu)}{\partial \mu} = 0
\end{cases} \tag{1.23}$$

reads as

$$\begin{cases} \sum_{i=1}^n \omega_i e^{-\frac{1}{2}\sigma_i^2 x_i^2 - \sigma_i x_i y_1} = K \\ \sum_{i=1}^n \omega_i e^{-\frac{1}{2}\sigma_i^2 x_i^2 - \sigma_i x_i y_2} = K \\ \omega_i \sigma_i \left[\varphi(y_1 + \sigma_i x_i) - \varphi(-y_2 - \sigma_i x_i)\right] = \left(\rho^{-1} x\right)_i \\ x \cdot (\rho^{-1} x) = 1 \end{cases} \tag{1.24}$$

This set of equations will be used in Chapter 3 for the numerical computation of the supremum and also in the next section when obtaining the Lord lower bound [30] as a non optimized formulation of our Lower Bound.

1.4 Connections with existing Lower Bounds

In this section we show the relation between the Lower Bounds defined in [9] and in [30] with the one derived in the above section. We also discuss the relation with the extension of the Lower Bound for non Gaussian models derived in [8].

Carmona and Durrleman

In the derivation of the Lower Bound of [9] the authors consider the subset $\mathcal{G}_{CD}$ of the random variables $\mathbf{1}_{\{\beta \cdot g \leq y_1\}}$ with $\beta \in \mathbb{R}^n$ and $y \in \mathbb{R}$ which is clearly a subset of $\mathcal{G}$ containing $\mathbf{1}_{\{\beta \cdot g \leq y_1 \cup \beta \cdot g \geq y_2\}}$ with $\beta \in \mathbb{R}^n$ and $y \in \mathbb{R}^2$ that we consider. Moreover they also define the optimal Lower Bound picking the best element within the set introduced. Roughly speaking their subset can be retrieved from ours fixing the value of y_2 (i.e. not taking the supremum over it) and then taking the limit for $y_2 \to \infty$. Thus considering such limit in our definition of the objective function 1.3.1 we obtain their definition:

Definition 1.4.1 (Carmona and Durrleman objective function)**.** *Given $x \in \mathcal{Q}$ and $y_1 \in \mathbb{R}$ we define*

$$\mathrm{CDLB}(x, y) := \sum_{i=1}^n w_i \phi\left(y_1 + \sigma_i x_i\right) - K \phi\left(y_1\right). \tag{1.25}$$

From this definition they define their Lower Bound as the optimal one:

Definition 1.4.2 (Carmona and Durrleman Lower Bound)**.**

$$\begin{aligned} \mathrm{CDLB}_{opt} :=& \sup_{x \in \mathcal{Q}} \sup_{y_1 \in \mathbb{R}} \lim_{y_2 \to \infty} LB(x, y) \\ =& \sup_{x \in \mathcal{Q}} \sup_{y_1 \in \mathbb{R}} \sum_{i=1}^n w_i \phi\left(y_1 + \sigma_i x_i\right) - K \phi\left(y_1\right) \end{aligned} \tag{1.26}$$

The advantage of their approach is twofold: first their lower bound has a simpler analytic form and second it can be generalized for negative weights ω_i (negative weights correspond to spread options). The drawback is that their formula is clearly less accurate and sometimes it results in a not accurate approximation to the true price. In particular in section 2.4 we are able to show that the contribution of the additional terms that appears in the formula we consider (which are similar to put Black formulas) have a dramatic impact when negative correlations appears.

Lord

Lord in [30] considers continuously sampled Asian options on an underlying that follows a Geometric Brownian Motion. This case is qualitatively similar to the case we consider in this thesis, because a discretely sampled Asian Option is equivalent to a standard Basket Option. Let us introduce the Lower Bound in [30] for a standard Basket Option, showing that is equivalent to our objective function (1.18), where it has been specified the $\sup$ w.r.t. $y \in \mathbb{R}^2$, i.e.

Definition 1.4.3 (Lord Lower Bound)**.**

$$LBL(x) := \sup_{y \in \mathbb{R}^2} LB(x, y). \tag{1.27}$$

In order to obtain his formula we need to compute the supremum over y and thus we must solve the first two equations of (1.24). For this purpose we consider the function $\ell(\lambda) := \sum_{i=1}^{n} \omega_i e^{-\frac{1}{2}(\sigma_i x_i)^2 - \sigma_i x_i \lambda}$ and, for any given vector $x \in \mathbb{R}^n/\{0\}$, the equation in λ

$$\ell(\lambda) = K. \tag{1.28}$$

The analytic form of the Lord lower bound is then strictly related to the following lemma.

Lemma 1.4.1. *For any given a vector $x \in \mathbb{R}^n/\{0\}$, eq.* (1.28) *could have two, one or no solutions. There are two relevant sets of cases.*

1) *All components in x have the same sign and :*
   - a. $x_i \geq 0\ \forall i$ *and* $\inf_\lambda \ell(\lambda) < K$. *Eq.*(1.28) *has one solution;*
   - b. $x_i \geq 0\ \forall i$ *and* $\inf_\lambda \ell(\lambda) \geq K$. *Eq.*(1.28) *has no solution;*
   - c. $x_i \leq 0\ \forall i$ *and* $\inf_\lambda \ell(\lambda) < K$. *Eq.(1.28) has one solution;*
   - d. $x_i \leq 0\ \forall i$ *and* $\inf_\lambda \ell(\lambda) \geq K$. *Eq.*(1.28) *has no solution.*

2) *At least one component of x is greater than zero and one is lower than zero. In this case there exists and it's unique $\hat{\lambda}$ s.t. $\hat{\lambda} = \min_\lambda \ell(\lambda)$ and*
   - a. $\hat{\lambda} < K$. *Eq.*(1.28) *has two solutions;*

b. $\hat{\lambda} = K$*. Eq.*(1.28) *has one solution;*
c. $\hat{\lambda} < K$*. Eq.*(1.28) *has no solution.*

Proof. A straightforward generalization of Lemma 1 and Lemma 2 in [30, sect. 4.3] □

In the following statement the function $LBL(x)$ is defined.

Definition 1.4.4 ($LBL(x)$)**.** *For every* $x \in \mathbb{R}^n$ *let us define the function* $LBL(x)$*, depending on the different solutions in the cases of the above Lemma*

$$LBL(x) := \begin{cases} c_1(\lambda^\star) & case: \; 1a \\ c_2(\lambda^\star) & case: \; 1b \\ c_1(\lambda^-) + c_2(\lambda^+) & case: \; 2a \\ 1 - K & cases: \; 1b, 1d, 2b, 2c \\ 0 & \text{for } x = 0 \end{cases} \tag{1.29}$$

where $\lambda^\star$ *is the solution of eq.*(1.28) *in the unique solution cases, while* λ^- *and* λ^+ *are respectively the smallest and the largest solutions in the two solutions case and*

$$\begin{aligned} c_1(\lambda) &\equiv \sum_{i=1}^{n} \omega_i \phi(\lambda + \tilde{\sigma}_i) - K\phi(\lambda) \\ c_2(\lambda) &\equiv \sum_{i=1}^{n} \omega_i \phi(-\lambda - \tilde{\sigma}_i) - K\phi(-\lambda) \end{aligned} \qquad \lambda \in \mathbb{R} \tag{1.30}$$

where $\phi(\cdot)$ *is the standard normal cumulate function.*

Theorem 1.4.1. *Given the call* C *in* (1.10) *and* $x \in \mathcal{Q}$*,* $LBL(x)$ *is a lower bound for* C*, i.e.* $LBL(x) \leq C$*.*

Proof. A straightforward generalization of the proof in [30, sect. 4.3] or by observing that

$$LBL(x) \leq \sup_{x \in \mathcal{Q}} LBL(x) = \mathrm{LB}_{opt} \leq C \tag{1.31}$$

where the last inequality is from theorem 1.4.1. □

Let us state some remark on the formula obtained. The equality

$$\mathrm{LB}_{opt} = \sup_{x \in \mathcal{Q}} LBL(x) \tag{1.32}$$

tells that we could either retrieve the optimal Lower Bound LB_{opt} we defined in (1.20) by optimizing $LBL(x)$ in the variable $x \in \mathcal{Q}$. Despite the optimal value are equal, our formulation is clearly favorable in terms of computational effort required for numerical optimization. In fact, the computation of $LBL(x)$ for some $x \in \mathcal{Q}$ requires an unconstrained two dimensional optimization. This means that if we want to compute the supremum in (1.32) we need to perform such optimization for each value of $x \in \mathcal{Q}$ explored. On the contrary with our definition of the objective function (1.18) we can compute LB_{opt} by a joint optimization of $x \in \mathcal{Q}$ and $y \in \mathbb{R}^2$.

Default of LB_{opt} to CDLB_{opt}

We conclude this section showing a relevant relation between the Lower Bound we introduced and the Lower Bound of Carmona and Durrleman.

Proposition 1.4.1. *Suppose that* $x^\star := \arg\max_{x\in\mathcal{Q}} LBL(x)$ *has all positive components, i.e.* $x_i^\star > 0$ *for* $i = 1, \ldots, n$*, then* $\mathrm{LB}_{opt} = \mathrm{CDLB}_{opt}$.

Proof. We observe from Lemma 1.4.1 that the optimal y_1 exist and it's finite while the optimal $y_2 \to \infty$, which is exactly the definition 1.4.2. □

This proposition states that if the optimal x is in the first hyperoctant, then we can consider only the simpler formula of Carmona and Durrleman and compute CDLB_{opt}. However at this point we cannot know a priori if this hypothesis is met. In Chapter 2 we tackle this problem and give some conditions under which $x^\star$ is in the first hyperoctant.

1.5 Upper bounds: ICUB and PECUB

After the introduction of a simple Lower Bound, this subsection is devoted to recall two upper bounds that are common in the literature and can be written as linear combinations of Black formulas (see e.g. [11–13, 23, 39, 40]): the improved comonotonic upper bound $ICUB(x)$ first introduced in [23], and the partially exact comonotonic upper bound $PECUB(x, d)$ introduced in [12] as an improvement of $ICUB(x)$ in some particular cases. The formal definition of the objective function $ICUB(x)$ presented in this subsection allows to highlight a difference with the original definition in [23]: as discussed in section 2.4 this difference is significant in some relevant cases. Similarly to the Lower Bound of Lord previous subsection we consider the function $u(\lambda, \vartheta) := \sum_{i=1}^n \omega_i e^{-\frac{1}{2}\sigma_i^2 - \sigma_i x_i \lambda - \sigma_i \sqrt{1-x_i^2}\theta(\lambda)}$ where $\theta(\lambda)$ is defined as the equation in θ

$$u(\lambda, \theta) = K. \tag{1.33}$$

The definition of $ICUB(x)$ is limited to a region of $\mathbb{R}^n$ delimited by the unitary cube. We first define this region and then, as in the lower bound case, state a lemma strictly related to the analytic form of $ICUB(x)$.

Definition 1.5.1 (Unitary cube)**.** *Let us define the unitary cube as*

$$\mathcal{B} := \{x \in \mathbb{R}^n : -1 < x_i < 1 \ \ \forall i\}$$

and denote its boundary with $\partial\mathcal{B}$.

Lemma 1.5.1. *Given $x \in \mathcal{B} \cup \partial\mathcal{B}$ and $\lambda \in \mathbb{R}$ eq.* (1.33) *could have one or no solutions. More precisely*

1) if $x \in \mathcal{B}$ eq.(1.33) *has one solution for each $\lambda \in \mathbb{R}$;*

2) if $x \in \partial\mathcal{B}$, then:

a. if $\inf_\vartheta u(\lambda, \vartheta) < K$, then eq.(1.33) *has one solution;*

b. if $\inf_\vartheta u(\lambda, \vartheta) \geq K$, then eq.(1.33) *has no solution.*

Proof. First notice that $u(\lambda, \vartheta)$ is decreasing in ϑ for each $\lambda \in \mathbb{R}$. Case 1) follows from $\inf_\vartheta u(\lambda, \vartheta) = 0$ for each $\lambda \in \mathbb{R}$. Case 2) from $\inf_\vartheta u(\lambda, \vartheta) > 0$ for each $\lambda \in \mathbb{R}$ □

The improved comonotonic upper bound ($ICUB(x)$) is defined as follows.

Definition 1.5.2 ($ICUB(x)$)**.** *For $x \in \mathcal{B} \cup \partial\mathcal{B}$ let us define the function $ICUB(x)$, depending on the different solutions in the cases of the above Lemma*

$$ICUB(x) := \int_{-\infty}^{\infty} d\lambda \, \phi(\lambda) \, c(\lambda) \tag{1.34}$$

where $\phi(\cdot)$ is the standard normal density function and

$$c(\lambda) := \begin{cases} \displaystyle\sum_{i=1}^{n} \omega_i e^{-\frac{1}{2}\tilde{\sigma}_i^2 - \tilde{\sigma}_i \lambda} \, \phi(\theta(\lambda) + \hat{\sigma}_i) - K \, \phi(\hat{\sigma}_i) & cases : 1, 2a \\ \displaystyle\sum_{i=1}^{n} \omega_i e^{-\frac{1}{2}\tilde{\sigma}_i^2 - \tilde{\sigma}_i \lambda} - K & case : 2b \end{cases}$$

being $\theta(\lambda)$ the solution of eq.(1.33)*.*

Theorem 1.5.1. *Given the stop-loss premium C in* (1.10) *and $x \in \mathcal{Q}$, $ICUB(x)$ is an upper bound for C, i.e. $ICUB(x) \geq C$.*

Proof. Straightforward applying Lemma 1.5.1. □

Let us underline that Definition 1.5.2 differs slightly from the one in [23]. This definition is more precise than the one in [23] for x on the boundary of the cube $\partial\mathcal{B}$, because for $x \in \partial\mathcal{B}$ eq. (1.33) can have no solution. We show in the next section that these correlation values are extremely relevant for the $ICUB(x)$. The set of upper bounds $ICUB(x)$ can be written in a revisited analytic form which is more suitable for the purpose of numerical optimization.

Proposition 1.5.1. *Given $x \in \mathcal{Q}$, $ICUB(x)$ can be written as*

$$ICUB(x) = \int_{-\infty}^{\infty} d\lambda \sup_{\theta(\lambda)\in\mathbb{R}} \left[\sum_{i=1}^{n} \omega_i \phi(\lambda + \sigma_i x_i) \phi \left(\sigma_i \sqrt{1 - x_i^2} + \theta(\lambda) \right) - K\phi(\lambda)\Phi(\theta(\lambda)) \right] \tag{1.35}$$

Proof. Observe that $c(\lambda)$ of (1.34) can be written in terms of an optimization problem exploiting Lemma 1.2.1 □

Thanks to theorem 1.5.1 we can now define the upper bound as the of the solution of the following constrained optimization problem

Definition 1.5.3 (Upper Bound)**.** *Given a basket call of the type* (1.10) *we define the upper bound through the following optimization problem:*

$$\mathrm{UB}_{opt} := \inf_{x\in\mathcal{Q}} ICUB(x) \tag{1.36}$$

Let us state an important remark on the formula above. The numerical computation of the infimum would be extremely time consuming because the computation of the objective function for each value of $x \in \mathcal{Q}$ requires the solution of a control problem. This issue will be overcome in Chapter 3 where we are able to deal with the optimization problem as it was a joint optimization of $(x, \theta(\lambda))$.

Another upper bound that, in some particular cases, plays a crucial role is the $PECUB(x, d)$ introduced in [12].

Definition 1.5.4 (partially exact)**.** *Given $x \in \mathbb{R}^n$ and defining the $\beta \in \mathbb{R}^n$ as*

$$\beta = \rho^{-1} x \tag{1.37}$$

and the Gaussian variable

$$\Lambda = \beta \cdot x \tag{1.38}$$

then x is partially exact if there exist $d \in \mathbb{R}$ s.t. $\Lambda < d$ implies $S > K$.

Definition 1.5.5 ($PECUB(x, d)$)**.** *Given $x \in \mathcal{B} \cup \partial\mathcal{B}$ partially exact we define*

$$PECUB(x, d) := \int_{-\infty}^{d} d\lambda\, \phi(\lambda)\, (\ell(\lambda) - K) + \int_{d}^{\infty} d\lambda\, \phi(\lambda)\, c(\lambda) \tag{1.39}$$

Theorem 1.5.2. *Given $x \in \mathcal{Q}$*

$$PECUB(x, d) \leq ICUB(x).$$

Proof. A straightforward generalization of the proof in [12, sect. 4.2.1] □

The above theorem, states that, whenever $PECUB(x, d)$ exists for a given x and d, this upper bound is always better than the corresponding $ICUB(x)$. However, let us stress that in literature $PECUB(x, d)$ can be computed only for three discrete values of the vector x [12], while, likewise the Lower Bound of the previous section, $ICUB(x)$ can be optimized for $x \in \mathcal{Q}$. In section 2.4 we show via numerical examples that UB should be preferred not only due to its simplicity, but also because performs even better than the three $PECUB(x, d)$.

1.6 Equality cases

In this section we discuss the cases in which the approximations introduced equal the true price. A first simple case is when $n = 1$ where both the Upper and Lower Bounds share the property of defaulting to the Black formula (1.14). This result can be trivially obtained by observing that the constraint in dimension 1 is $x^2 = 1$ and then substituting one of the two solutions (e.g $x = 1$) in Definitions 1.4.2 and 1.5.2. It is possible to show that also in dimension 2 the Upper Bound equals the true price. The derivation of this original result requires some preliminary results of Chapter 2, hence we refer to section 2.3 for it. A third case is when assets are perfectly correlated (positively or negatively). In [9] Carmona and Durrleman examined this point, but, in the context of basket options, their result holds only for positive correlated assets. Here we extend this result allowing negative entries of the correlations matrix. In the following proposition we indicate with η a vector whose first component is $\eta_1 = 1$, other components $\eta_{i\neq 1} = \pm 1$ correspond to the correlation of asset i with the first asset. Moreover with $\eta\eta^T$ we mean the row-column product, i.e. $(\eta\eta^T)_{ij} = \eta_i\eta_j$.

Proposition 1.6.1. *If $\rho = \eta\eta^T$ then the price C has a closed formula formulation. In particular $C = \mathrm{LB} = \mathrm{UB}$.*

Proof. The proof consists in showing that the lower bound LB and UB coincide and then both coincide with the exact price C. We start observing that the matrix ρ is singular and hence $\mathcal{Q}$ cannot be defined and we shall use the definition of the optimization problems in terms of $\beta = \rho x$. Choosing $\beta = \eta$ as guess solution we obtain $x = \eta$ and hence

$$\begin{aligned}\mathrm{LB} &= \sup_{y\in\mathbb{R}^2} \mathbb{E}\left[\sum_{i=1}^{n} \omega_i e^{-\frac{1}{2}\sigma_i^2 - \sigma_i\eta_i\beta\cdot g} \mathbf{1}_{\{\beta\cdot g<y_1\}} \mathbf{1}_{\{\beta\cdot g>y_2\}}\right] \\ &= \int_{-\infty}^{\infty} d\lambda\phi(\lambda)\left(\sum_{i=1}^{n} \omega_i e^{-\frac{1}{2}\sigma_i^2 - \sigma_i\eta_i\lambda} - K\right)_+\end{aligned} \tag{1.40}$$

On the other hand, always choosing $x = \eta$, for the UB we have

$$\begin{aligned}
&\sup_{\theta(\lambda)\in\mathbb{R}} \sum_{i=1}^{n} \omega_i e^{-\frac{1}{2}\sigma_i^2-\sigma_i\eta_i\lambda}\Phi\left(\theta(\lambda)\right) - K\Phi(\theta(\lambda)) \\
&= \sup_{s\in[0,1]} \sum_{i=1}^{n} \omega_i e^{-\frac{1}{2}\sigma_i^2-\sigma_i\eta_i\lambda} s - Ks \qquad (1.41)\\
&= \left(\sum_{i=1}^{n} \omega_i e^{-\frac{1}{2}\sigma_i^2-\sigma_i\eta_i\lambda} - K\right)_+
\end{aligned}$$

Hence by definition we have

$$\mathrm{UB} = \int_{-\infty}^{\infty} d\lambda\phi(\lambda) \left(\sum_{i=1}^{n} \omega_i e^{-\frac{1}{2}\sigma_i^2-\sigma_i\eta_i\lambda} - K\right)_+ \qquad (1.42)$$

which is equal to (1.40). □

Let us state a remark on the result obtained. In [9] the authors approximate the price of a basket call with a combination of only Black call style terms, the equality of LB with the true price holds only when all the correlations are positive. This result hence shows that adding Black put style terms is necessary when some assets are negative correlated with others.

1.7 Greeks

It is quite common in literature to deal with approximated analytic solutions to the problems (1.3.3) and (1.5.2). A drawback of this approach is that Greeks of both closed formulas could strongly depend on the variation of the optimization parameters. This point has been already highlighted in [9], hence in the following we will show the computation only for the Δ of LB. Extensions to other Greeks and to UB are analogous. In order to compute the variation of the approximated price in terms of the variation of one the underlying forwards $F_i^T(0)$ it is convenient to explicitly write the non adimensional lower bound

$$\begin{aligned}
\mathcal{LB} =& B(0,T)F^T(0)\mathrm{LB} \\
=& B(0,T)F^T(0) \sup_{x\in\mathcal{Q}} \sup_{y\in\mathbb{R}^2} \sum_{i=1}^{n} w_i \left[\Phi\left(y_1+\sigma_i x_i\right) + \Phi\left(-y_2-\sigma_i x_i\right)\right] - \mathcal{K}\left[\Phi\left(y_1\right)+\Phi\left(-y_2\right)\right] \\
=& B(0,T) \sup_{x\in\mathcal{Q}} \sup_{y\in\mathbb{R}^2} \sum_{i=1}^{n} a_i F_i(0) \left[\Phi\left(y_1+\sigma_i x_i\right) + \Phi\left(-y_2-\sigma_i x_i\right)\right] - \mathcal{K}\left[\Phi\left(y_1\right)+\Phi\left(-y_2\right)\right] \\
=& B(0,T) \sup_{x\in\mathcal{Q}} \sup_{y\in\mathbb{R}^2} \mathcal{LB}(x,y)
\end{aligned}$$

(1.43)

having defined the cost function $\mathcal{LB}(x, y)$ in the last equality. Then the first order conditions for this optimization problem reads as

$$\begin{cases} \sum_{i=1}^n a_i F_i(0) e^{-\frac{1}{2}\sigma_i^2 x_i^2 - \sigma_i x_i y_1} = \mathcal{K} \\ \sum_{i=1}^n a_i F_i(0) e^{-\frac{1}{2}\sigma_i^2 x_i^2 - \sigma_i x_i y_2} = \mathcal{K} \\ a_i F_i(0)\sigma_i \left[\varphi(y_1 + \sigma_i x_i) - \varphi(-y_2 - \sigma_i x_i)\right] = \left(\rho^{-1}x\right)_i \\ x \cdot \rho^{-1} x = 1 \end{cases} \tag{1.44}$$

The formula for the sensitivity is given by the following proposition

Proposition 1.7.1. *Denoting by $x^\star$ and $y^\star$ the global solution of the problem* (1.3.3)*, then*

$$\Delta_i = \frac{d\mathcal{LB}}{dF_i^T(0)} = a_i B(0, T) \left(\Phi(y_1^\star + \sigma_i x_i^\star) + \Phi(-y_2^\star - \sigma_i x_i^\star)\right) \tag{1.45}$$

Proof. From the first equality of (1.43) we observe that $x^\star$ and $y^\star$ are optimal for both $\mathcal{LB}$ and LB. Then from the last equality of (1.43) we have

$$\begin{aligned} \frac{d\mathcal{LB}}{dF_i^T(0)} =& \frac{\partial\mathcal{LB}}{\partial F_i^T(0)} + \frac{\partial\mathcal{LB}}{\partial y^\star} \cdot \frac{\partial y^\star}{\partial F_i^T(0)} + \frac{\partial\mathcal{LB}}{\partial x^\star} \cdot \frac{\partial x^\star}{\partial F_i^T(0)} \\ =& \frac{\partial\mathcal{LB}}{\partial F_i^T(0)} + \alpha\rho^{-1}x^\star \cdot \frac{\partial x^\star}{\partial F_i^T(0)} \\ =& \frac{\partial\mathcal{LB}}{\partial F_i^T(0)} + \frac{\alpha}{2}\frac{\partial}{\partial F_i^T(0)}\left(x^\star \cdot \rho^{-1}x^\star\right) \\ =& \frac{\partial\mathcal{LB}}{\partial F_i^T(0)} \end{aligned}$$

where α is a Lagrange multiplier and the second equality is obtained thanks to the first order conditions $\frac{\partial\mathcal{LB}}{\partial y^\star} = 0$ and $\frac{\partial\mathcal{LB}}{\partial x^\star} = \alpha\rho^{-1}x^\star$. Then explicitly computing $\frac{\partial\mathcal{LB}}{\partial F_i^T(0)}$ via simple differentiation, the result is obtained. $\square$

The former proposition shows that the lower bound sensitivity to the movement of one of the underlying forward prices is completely analogous to the Δ of the Black model. Moreover, being stationary (i.e not depending on the variation of the optimization parameters x and y) we expect it to give a very good approximation of the real value of the sensitivity. We stress also that this property holds only if we define the lower bound as a solution of an optimization problem. On the contrary if we define the lower as the value of the cost function in some point (x, y) different from $(x^\star, y^\star)$, the value of the sensitivities could significantly diverge from the real one even if the price approximation performs well. For a detailed numerical study of the convergence of the Greeks we refer to [9].

CHAPTER 2

Optimal bounds: analytical results

2.1 Introduction

This chapter is devoted to obtain some analytic results on the two optimization problems related to the computation of the Upper and Lower Bounds introduced in the previous Chapter. The aim of the Chapter is showing that the two optimization problems are non trivial because they can show multiple local optimal points and, for the Upper Bound, also irregular points. We also guess that for some particular market conditions the problem is "easy", i.e. admits a unique solution which can be then computed using local optimization tools like gradient based methods. Even if we are not able to find an exact condition, we focus on our guess of non-negative ρ, i.e. the case in which the correlation matrix has non negative entries, a financially relevant case that characterizes most baskets with equity stocks, one of the derivative markets where Basket Options are mostly used.

We show how optimal bounds are related to some characteristics in the geometry of the problem and in particular to $\{\rho_i\}_{i=1,..n}$, the column vectors of ρ. In a nutshell the main results are: i) for the optimal Lower Bound LB_{opt} we prove the existence of an optimal solution on the part of the ellipsoid delimited by the positive linear span of $\{\rho_i\}$; we also show some sufficient conditions for uniqueness of the global maximum ii) for the $ICUB(x)$ upper bound, $\{\rho_i\}$ are the only points where the bound has an angular point. In n dimensions for a ρ with a simple shape we prove that these points are local minima and this result looks to hold for a more general ρ.

2.2 Optimal upper and lower bounds: Preliminary results

This section is devoted in showing some preliminary results on the two optimization problems related to LB_{opt} and UB_{opt} some geometric properties of the constraint that will be used in the next section. Hereinafter we will always refer to the Lord formulation of the problem because it is more suitable for an analytical study.

2.2.1 Regularity

The following two lemmas on the regularity of the bounds hold.

Lemma 2.2.1. *$LBL(x)$ defined in Definition 1.4.4 is a continuous and differentiable function for $x \in \mathbb{R}^n$ except in the origin when $K = 1$.*

Proof. See Appendix A □

Lemma 2.2.2. *$ICUB(x)$ defined in 1.5.2 is a continuous and differentiable function in the cube $\mathcal{B}$ and continuous but not differentiable on the boundary $\partial\mathcal{B}$.*

Proof. See Appendix A □

A direct consequence of these two lemmas is that the optimization problems (1.3.3) and (1.5.3) are well defined because the bounds are continuous in a neighborhood of $\mathcal{Q}$, which is a compact set. Another property shared by both $LBL(x)$ and $ICUB(x)$ is given by the following lemma.

Lemma 2.2.3. *$LBL(x)$ and $ICUB(x)$ are symmetric functions of their argument. Both bounds share the property $LBL(x) = LBL(-x)$ and $ICUB(x) = ICUB(-x)$.*

Proof. Straightforward via direct computation □

2.2.2 Geometric properties of the constraint

The results in this thesis on the optimization problems (1.3.3) and (1.5.3) are strictly related to some geometric features of $\mathcal{Q}$. In this subsection we point out the main ones. Denoting with ρ_i the i^{th} column of ρ, let us define some useful quantities.

Definition 2.2.1 (Relevant geometrical sets)**.**

$$\pi_i \equiv \left\{ x \in \mathbb{R}^n : x = \sum_{j \neq i}^{n} \alpha_j \rho_j \right\}$$
$$\pi_+ \equiv \left\{ x \in \mathbb{R}^n : x = \sum_{j=1}^{n} \alpha_j \rho_j, \ \alpha_i > 0 \ \ \forall i \right\}$$
$$\pi_- \equiv \{ x \in \mathbb{R}^n : -x \in \pi_+ \}$$

and the relative intersections with the quadric

$$\mathcal{Q}_i \equiv \pi_i \cap \mathcal{Q} \qquad \mathcal{Q}_+ \equiv \pi_+ \cap \mathcal{Q} \qquad \mathcal{Q}_- \equiv \pi_- \cap \mathcal{Q} \qquad .$$

Given the above definitions it is possible to state two lemmas which characterize the ellipsoid $\mathcal{Q}$.

Lemma 2.2.4. *For a positive definite matrix ρ the quadric $\mathcal{Q}$ is entirely contained in the cube $\mathcal{B} \cup \partial\mathcal{B}$ and it is tangent to $\partial\mathcal{B}$ only in the points $\{\pm\rho_i\}_{i=1,\dots,n}$.*

Proof. See Appendix A □

Lemma 2.2.5. *Defining $\mathcal{D}$ as the elliptical cylinder having its axis on the vector e_i and tangent to $\mathcal{Q}$, $\mathcal{Q}_i$ is the set of points in which $\mathcal{D}$ is tangent to $\mathcal{Q}$.*

Proof. See Appendix A □

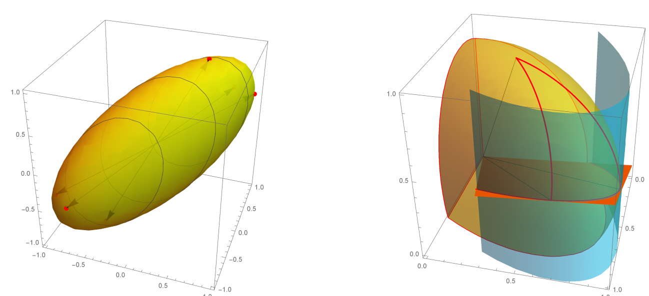

Figure 2.1: *The figure on the left shows that the ellipsoid $\mathcal{Q}$ is included in the unitary cube $\mathcal{B}$ (see Lemma 2.2.4); the red dots are the points $\pm\rho_i$, the only points lying on $\partial\mathcal{B}$. The figure on the right shows in red the boundary of π_+, in cyan we plot the tangent cylinder $\mathcal{D}$ having e_3 as axis and in orange π_3, the set of points where $\mathcal{D}$ is tangent to $\mathcal{Q}$ (see Lemma 2.2.5).*

2.3 Lower Bound: Domain of existence for the maximum and uniqueness

In this section we consider a non-negative correlation matrix ρ, i.e. the matrix ρ has all positive coefficients. In this case, the main result on the maximization problem (1.20) is provided by the following proposition.

Proposition 2.3.1. *Let ρ be a non-negative matrix then:*

i) there exists one point $x_M \in \mathcal{Q}_+$ solution of eq. (1.24) *which corresponds to a local maximum point.*

ii) there exists one point $x_m \in \mathcal{Q}/(\mathcal{Q}_+ \cup \partial\mathcal{Q}_+ \cup \mathcal{Q}_- \cup \partial\mathcal{Q}_-)$ solution of eq. (1.24) *which corresponds to a local minimum point.*

Proof. See Appendix A $\square$

A consequence of the symmetry property $LBL(x) = LBL(-x)$ (see Lemma 2.2.3) implies that there exists a local maximum point even in $\mathcal{Q}_-$. Let us state some remarks on

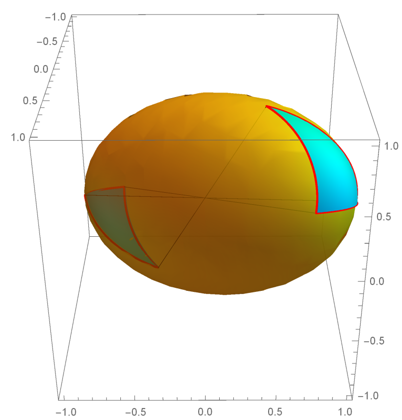

Figure 2.2: *We show in cyan $\mathcal{Q}_+$ (and $\mathcal{Q}_-$ on the other side) while in red $\partial\mathcal{Q}_+$ (and $\partial\mathcal{Q}_-$). Proposition 2.3.1 states that there exists a maximum point in each one of the two cyan zones while two minima are in the yellow zone.*

the above proposition. We have shown that, when the correlation matrix is non negative a maximum exists in the first hyperoctant. If we could state that this maximum is also global we could state, thanks to Proposition 1.4.1, that when ρ is non negative it is possible to consider only the simple formula of Carmona and Durrleman. Unfortunately this is not true in general. However analyzing numerically $LBL(x)$, one observes that the maximum is unique for $x \in \mathcal{Q}_+$ when total volatilities are not larger than 1 and for options not deeply ITM or OTM. This uniqueness can be understood qualitatively considering, for

$K = 1$, the first order Taylor expansion in $\{\sigma_i\}_{i=1,\cdots,n}$ for $LBL(x)$. In this case we get [1] $LBL(x) = (2\pi)^{-1/2}\underline{\sigma\omega}\cdot x + o(\sigma)$. When considering this linear approximation, uniqueness is guaranteed being $\mathcal{Q}$ concave. The solution for x is $\rho\underline{\sigma\omega}/\sqrt{\underline{\sigma\omega}\cdot\rho\underline{\sigma\omega}}$ which corresponds to the point x_{FA2} proposed in [12] and justified in [39]. This explains why often x_{FA2} is a good first guess for the optimal solution for positive ρ and it's expected to perform reasonably well for strikes “around" the ATM. For a generic ρ uniqueness is not granted as we show in a numerical example in the next section. For a non-negative ρ in most cases of financial interest one observes numerically a unique maximum. In proposition 2.3.2 of this section we are able to state a condition for the uniqueness of the lower bound focusing on $n = 2$. We start defining the function $\psi : \mathbb{R}^n \to \mathcal{Q}$, then after two technical lemmas we state the proposition.

Definition 2.1. *The function $\psi : \mathbb{R}^n \to \mathcal{Q}$ is defined as*

$$\psi(x) \equiv \frac{\rho\nabla LBL(x)}{\sqrt{\nabla LBL(x)\cdot\rho\nabla LBL(x)}} \qquad .$$

Lemma 2.3.1. *A point x is a solution of the Lagrange eq.* (1.24) *iff it is a fixed point for the function $\psi(x)$.*

Proof. See Appendix A □

Remark. In the above definition we should have first shown that the codomain of ψ is $\mathcal{Q}$, however by direct inspection it is esay to verify that $\psi(x)\cdot(\rho^{-1}\psi(x)) = 1$ for all $x \in \mathbb{R}^n$.

In the above lemma we have shown that the eq. $x = \psi(x)$ is equivalent to eq. (**??**). In the next chapter is pointed out that the (discrete-time) map $x^{i+1} = \psi(x^i)$ plays a crucial role in a fast selection of local maxima for the lower bound. In particular looking for a fixed point of this map and applying the contraction principle, the following lemma holds.

Lemma 2.3.2. *Given ρ a non-negative matrix, the Jacobian matrix of $\psi(x)$ is singular. For $n = 2$ the non-null eigenvalue is*

$$l(x) = -(1-\rho^2)\frac{\sigma_1^2\sigma_2^2\hat{\omega}_1(x)\hat{\omega}_2(x)}{\hat{\alpha}(x)^{3/2}}\left[\frac{\tilde{\sigma}_1\tilde{\sigma}_2}{\tilde{\sigma}_1\hat{\omega}_1(x)+\tilde{\sigma}_2\hat{\omega}_2(x)} + \lambda^\star\right]$$

where

$$\hat{\omega}_i(x) \equiv \omega_i\frac{e^{-\frac{1}{2}\tilde{\sigma}_i^2-\tilde{\sigma}_i\lambda^\star}}{K}$$

with $\hat{\alpha}(x) \equiv \underline{\sigma\hat{\omega}(x)}\cdot\rho\,\underline{\sigma\hat{\omega}(x)}$ and $\lambda^\star$ the solution of eq. (1.28).

[1] The notation $\underline{\sigma\omega}$ with σ and ω both in $\mathbb{R}^n$ indicates a vector $v \in \mathbb{R}^n$ with components $v_i = \sigma_i\omega_i$

Proof. See Appendix A $\square$

Finally bounding $|l(x)|$ from above $\forall x \in \mathcal{Q}$ we can state the desired proposition.

Proposition 2.3.2. *Given a non-negative matrix ρ and supposing, with no loss of generality, that $\sigma_1 \leq \sigma_2$, a sufficient condition under which $\mathrm{LB}(x)$ has a unique maximum point in $\mathcal{Q}_+$ is $l^\star < 1$. Furthermore if $K \geq 1$ the condition can be improved by $l_I^\star < \ell^\star < 1$, with*

$$\begin{aligned} l^\star &= (1-\rho^2)\sqrt{\sigma_1^2+\sigma_2^2}\left[\sigma_2 + \frac{1}{\rho\sigma_1}\max\left\{-\log(K),\ \left|\log\left(\frac{\sum_{i=1}^2 \omega_i e^{-\frac{1}{2}\sigma_i^2}}{K}\right)\right|\right\}\right] \\ l_I^\star &= (1-\rho^2)\sqrt{\sigma_1^2+\sigma_2^2}\max\left\{\sigma_2,\ \frac{\sigma_2^2+\log(K)}{\bar{\sigma}}\right\} \end{aligned}$$

where $\bar{\sigma} = \min\{\underline{\sigma\omega}\cdot\rho_1,\ \underline{\sigma\omega}\cdot\rho_2\}$.

Proof. See Appendix A $\square$

We want to stress that the condition for option parameters indicated by proposition 2.3.2 are rather intuitive: they are satisfied for a wide set of interesting financial parameters which are, as expected, strikes not so far from ATM (but even significantly OTM and ITM) and when each asset total volatility is small (and in particular it holds for small time-to-maturity). Furthermore for assets highly correlated ($\rho \simeq \pm 1$) the condition is always satisfied.

2.3.1 Upper bound

Even for the minimization problem (1.5.3) the columns of the correlation matrix play an important role. The main object here is not the positive linear span $\mathcal{Q}_+$ generated, but the columns themselves. We claim that these points are local minima points and often one of them is the global one.

We start showing some interesting features of these points.

Lemma 2.3.3. $\{\pm\rho_i\}_{i=1,..,n}$ *are the only points in $\mathcal{Q}$ where equation* (1.33) *has no solution for some λ. In particular there is no solution for*

$$\lambda \leq \lambda_i^* \equiv -\frac{\ln(K/\omega_i)}{\sigma_i} - \frac{\sigma_i}{2}$$

and one solution for $\lambda > \lambda_i^*$.

Proof. Lemma 1.5.1 states that no solution appear iff some component of x equals 1. As a direct consequence of Lemma 2.2.4 this happen only in $\pm\{\rho_i\}_{i=1,..,n}$, the only points that belong to $\partial\mathcal{B}$. The value of λ_i^* follows from direct calculation $\square$

The fact that $x = \pm\rho_i$ allows to prove a relevant property that holds for $PECUB(x, d)$ on the columns of ρ.

Proposition 2.3.3. *Considering $x = \rho_i$ and $i = 1, \cdots, n$, it always exists a $d = \lambda_i^*$ s.t. $PECUB(\rho_i, \lambda_i^*)$ is well defined and*

$$PECUB(\rho_i, \lambda_i^*) = ICUB(\rho_i) \quad .$$

Proof. See Appendix A □

Now we deal with the optimization problem for the $ICUB(x)$. We start with the two dimensional case which is fully characterized by the following proposition.

Proposition 2.3.4. *For $n = 2$, considering $ICUB(x)$ with $x \in \mathcal{Q}$, the true price C is equal $ICUB(x)$ when $x \notin \mathcal{Q}_+ \cup \mathcal{Q}_-$.*

Proof. See Appendix A □

It can be useful to show in a plot the above result. For $n = 2$, $\mathcal{Q}$ can be parameterized via the angle ϑ in the following way

$$\begin{cases} x_1(\vartheta) = \dfrac{\cos(\vartheta)}{r(\vartheta)} \\ x_2(\vartheta) = \dfrac{\sin(\vartheta)}{r(\vartheta)} \end{cases}$$

where $r(\vartheta) = \sqrt{[\cos(\vartheta), \sin(\vartheta)] \cdot \rho^{-1}[\cos(\vartheta), \sin(\vartheta)]}$. In figure 2.3 we show that outside $\mathcal{Q}^+$ $ICUB(x)$ is constant and equals the true price.

Finally we deal with the $n-$dimensional case. First for a non-negative ρ we show that ρ columns are the only irregular points for the $ICUB(x)$. Furthermore, in a particular but relevant case from a financial point of view, we show that this points are local minima. These results are stated in the two following propositions.

Proposition 2.3.5. *The points $\{\rho_i\}_{i=1,\cdots,n}$ are angular points for $ICUB(x)$.*

Proof. See Appendix A □

Proposition 2.3.6. *Let all the out of diagonal elements of ρ be equal, i.e. $\rho_{ij} = \hat{\rho} \ \forall i \neq j$, then ρ columns ρ_i are local minima.*

Proof. See Appendix A □

In the above proposition we have shown, for a class of ρ that is often chosen by market makers for equity index baskets, that ρ columns are local minima, so they are good candidates for being the solution of the optimization problem. Our claim is that, for a generic non-negative ρ, the global minimum is one of these local minima, in most cases of financial interest.

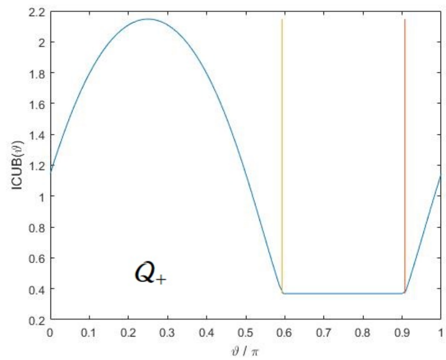

Figure 2.3: *All possible values for* $\mathrm{ICUB}(x)$ *in a 2 assets basket with* x *in half of the ellipse* $\mathcal{Q}$. *We consider the case with* $\omega = [0.5, 0.5]$ $\sigma = [0.2, 0.2]$, $K = 1.2$, $\rho = -0.3$. *The yellow line and the red line represent the two boundary points of* $\mathcal{Q}_+$, *i.e. the values that correspond to* ρ *columns. The function is constant outside* $\mathcal{Q}_+$ *(defined in Definition 2.2.1) as stated by proposition 2.3.4.*

2.4 Numerical examples

In this section we show some numerical examples. We report the exact prices computed via Monte Carlo (MC) simulations with 10^6 paths using the lower bound as control variate and the antithetic variable technique. The error reported is the standard error.

In Table 2.1 we consider a basket of 2 assets positive correlated. One observes numerically that the maximum in $\mathcal{Q}_+$ is always unique. The improvement with the optimal solution is less pronounced w.r.t. considering the conditioning variable $FA2$ near the ATM as expected, for other strikes the optimization brings a substantial improvement, in some cases of several basis points[2].

In Table 2.2 an example of a basket of two assets negatively correlated is shown. In the negative correlation cases we observe that the difference between the optimal point and x_{FA2} is higher. When ρ has some negative values, for a generic n, it can be noticed numerically that, even if the performance of the lower bound LB is not as good as in the positive correlation case, a very good performance for the UB is observed.

The example shown in figure 2.4 presents a non-unique local maximum for $LBL(x)$: this problem arises frequently when the assets are negatively correlated.

The fourth example in Table 2.3 is a three dimensional basket with a non-negative correlation matrix. $ICUB(\rho_b)$ is the lowest among $ICUB(\rho_i)$: as previously mentioned, it always corresponds to the heaviest log-normal. $ICUB(\rho_b)$ is the global minimum. Best PECUB indicates the best value possible when considering x one of the three values (called FA1, FA2, GA) for which a $PECUB(x, d)$ solution has been found in [12]. We also compare the bounds with PERSUB which is a partially exact upper bound based on

[2] A basis point is equal to 0.01%.

K	$\mathcal{C}$ $(s.e.)$	LB	$LBL(x_{FA2})$
0.6	40.096 (0.009)	40.095	40.023
0.7	30.685 (0.009)	30.627	30.383
0.8	22.462 (0.009)	22.288	21.979
0.9	15.947 (0.009)	15.682	15.511
1.0	11.238 (0.009)	10.944	10.918
1.1	8.031 (0.008)	7.776	7.766
1.2	5.905 (0.008)	5.713	5.612
1.3	4.449 (0.008)	4.348	4.127
1.4	3.464 (0.007)	3.410	3.088

Table 2.1: *Basket call option prices C (1.10) in $\%$ for $\omega = [0.8, 0.2]$, $\sigma = [0.2, 1.1]$, $\rho = 0.1$. We show MC prices (with standard errors in brackets), the lower bound in the optimal maximum and the one computed in x_{FA2}: we observe a significant improvement for all moneynesses and in some cases even of several basis points.*

K	$\mathcal{C}$ $(s.e.)$	LB	$LBL(x_{FA2})$
0.6	40.026 (0.016)	40.006	40.000
0.7	30.238 (0.015)	30.162	30.000
0.8	21.559 (0.015)	21.285	20.843
0.9	15.007 (0.014)	14.679	14.592
1.0	10.623 (0.013)	10.453	10.449
1.1	7.770 (0.013)	7.725	7.648
1.2	5.853 (0.012)	5.870	5.707
1.3	4.560 (0.012)	4.556	4.332
1.4	3.588 (0.011)	3.597	3.337

Table 2.2: *Basket call option prices C (1.10) in $\%$ for $\omega = [0.6, 0.4]$, $\sigma = [0.2, 8]$, $\rho = -0.6$. We show MC prices (with standard errors in brackets), the lower bound in the optimal maximum and the one computed in x_{FA2}. Even for negative correlations we observe a significant improvement for all moneynesses and in some cases of several basis points.*

lower bound of [12]; even in this case, best PERSUB indicates the lowest between the same three choices for x.

We can mention a counterexample in which the global minimum is not one of the ρ columns. This is the Asian option example of Table 2 in [40]. However the bound calculated conditioning on the heaviest log-return, the log-asset value at the last sampling time indicated with B_T, is very close to the optimal one. This behavior appears for assets highly correlated, as also other numerical examples suggest: in this case $ICUB(x)$ is substantially constant for $x \in \mathcal{Q}$.

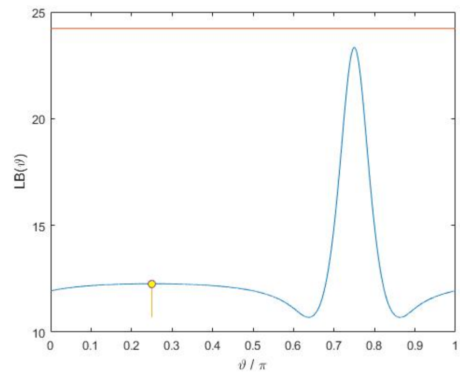

Figure 2.4: *All possible lower bounds for a two dimensional basket. $\mathcal{Q}$ is parameterized via the angle ϑ. $t = 1$, $\omega = [0.5, 0.5]$, $\sigma = [1, 1]$, $\rho = -0.95$, $K = 0.9$, $\mathrm{LB}(x^\star_\ell) = 23.337$, $\mathrm{LB}(x_{FA2}) = 12.264$, $C = 23.892(0.014)$. The red line is the true price value, with the yellow line and the spot we indicate the point which corresponds to $FA2$.*

K	$\mathcal{C}$ $(s.e.)$	ICUB(ρ_b)	$best$ PECUB	$best$ PERSUB	LB	$LBL(x_{FA2})$
0.6	43.936 (0.010)	44.125	44.682	44.439	43.886	43.797
0.7	37.258 (0.010)	37.498	38.243	37.938	37.193	37.125
0.8	31.616 (0.010)	31.890	32.828	32.520	31.542	31.499
0.9	26.924 (0.010)	27.183	28.289	28.084	26.814	26.792
1.0	22.978 (0.010)	23.247	24.481	24.516	22.873	22.865
1.1	19.689 (0.010)	19.956	21.290	21.542	19.589	19.588
1.2	16.931 (0.010)	17.200	18.606	18.982	16.849	16.849
1.3	14.647 (0.010)	14.887	16.341	16.861	14.556	14.551
1.4	12.728 (0.010)	12.939	14.419	15.100	12.630	12.618

Table 2.3: *Basket call option prices C (1.10) in % for $\omega = [0.33, 0.33, 0.33]$, $\sigma = [0.7, 0.6, 1]$, $\rho_{12} = 0.8$, $\rho_{13} = 0.7$, $\rho_{23} = 0.4$.*

2.5 Conclusions

In this chapter we have considered the problem of finding the optimal values for the bounds $LBL(x)$ and $ICUB(x)$ of a basket call when assets are lognormally distributed. In the financially relevant case where we consider a positive correlation matrix ρ we obtain some interesting results. For the lower bound we show the existence of a maximum on the part of the hyper-ellipsoid $\mathcal{Q}$ delimited by the positive linear span of ρ columns $\{\rho_i\}$; this region can be very limited in some cases as in presence of high correlations. We also present some sufficient conditions for the uniqueness of the maximum; as shown via a counterexample, this result does not hold true anymore in the negative correlation case. For the $ICUB(x)$ we show some relevant results for x equal to ρ columns. In these points we prove for

$n = 2$ and for some simple correlation matrices if a generic n is considered, that we reach a local minimum. The global minimum is generally reached choosing x among one of these points. We claim that this result holds for a generic positive correlation ρ.

APPENDIX $\mathcal{A}$

Proofs

Proof of Lemma 2.2.1. Let A be first hyperoctant $A = \{x : x_i > 0\}$ and B the opposite hyperoctant $B = \{x : x_i < 0\}$ and let C be the open set $C = \mathbb{R}^n/(\partial A \cup \partial B \cup \{0\})$. In C one has to observe that the solutions $\lambda^\star$ or $\lambda^\pm$ are infinitely times differentiable functions of x because of the implicit function theorem applied to (1.28). As a consequence, $LBL(x)$ is infinitely times differentiable since composition of infinitely times differentiable functions. Now we want to show that the function is continuous and differentiable even in all the points $x \in A/\{0\}$ when passing from case 1a to 2a of Lemma 1.4.1. The other cases can be proven in a similar way. Formula (1.29) and its partial derivative are continuous if $\lambda_+ \to \infty$ as x approaches to A from negative values. In order to show that this holds true we show that the minimum point λ_m of $\ell(\lambda)$ goes to infinity as x approaches to A from negative values which, considering that $\lambda_+ > \lambda_m$, brings to the result. Suppose, without any loss of generality, that only the first component of x is negative, the equation of λ_m is

$$|\tilde{\sigma}_1|\omega_1 e^{-\frac{1}{2}\tilde{\sigma}_1^2+|\tilde{\sigma}_1|\lambda} = \sum_{i=2}^{n} \tilde{\sigma}_i\omega_i e^{-\frac{1}{2}\tilde{\sigma}_i^2-\tilde{\sigma}_i\lambda}$$

Now consider $\tilde{\lambda}$ solution of the equation

$$|\tilde{\sigma}_1|e^{|\tilde{\sigma}_1|\lambda} = N(x)e^{-<\tilde{\sigma}>\lambda}$$

where $< \tilde{\sigma} >= \frac{1}{N(x)} \sum_{i=2}^{n} \omega_i e^{-\frac{1}{2}\tilde{\sigma}_i^2} \tilde{\sigma}_i$ and $N(x) = \sum_{i=2}^{n} \tilde{\sigma}_i \omega_i e^{-\frac{1}{2}\tilde{\sigma}_i^2}$. The solution $\tilde{\lambda}$ has the property $\tilde{\lambda} < \lambda_m$ since the first term of the second equation is trivially greater of first term of the first equation while the second term of the second equation is smaller of the respective in the first equation because obtained via the geometric average. Solving the second equation one has

$$\tilde{\lambda} = \frac{1}{|\tilde{\sigma}_1| + < \tilde{\sigma} >} \log \left(\frac{N(x)}{|\tilde{\sigma}_1|} \right)$$

which goes to infinity as x_1 approaches to 0. The case of more than one component negative is simply a generalization of the result above. Finally we consider the origin. With a similar argument it can be shown that if $K \neq 1$ then $LBL(x)$ is a continuous and differentiable function in $x = 0$. When $K = 1$ the solution $\lambda^\star$ goes to 0 as x approaches to 0 and therefore the function is continuous but not differentiable in the origin.

Proof of Lemma 2.2.2. Let $\mathcal{B}$ be the cube $\mathcal{B} = \{x \in \mathbb{R}^n : |x_i| < 1 \ \forall i\}$. For the implicit function theorem the solution ϑ_λ is a continuous function of x in $\mathcal{B}$ and $ICUB(x)$ is an infinitely times differentiable function. When instead $|x_i| = 1$ for some i it is continuous but not differentiable because of the square root term.

Proof of Lemma 2.2.4. From eq.(1.19) one can immediately see that each component of x is less or equal than 1. The point $x = \rho_i$ belongs to $\mathcal{Q}$ because $\rho_i \cdot (\rho^{-1}\rho_i) = \rho_i \cdot e_i = 1$ and has the i^{th} component which equals one, so it has to be tangent to the plane $x_i = 1$. Otherwise via direct computation one has that the equation of the tangent plane to $\mathcal{Q}$ in a point x_a is $x \cdot (\rho^{-1} x_a) = 1$. If $x_a = \rho_i$ then the tangent plane is described by the equation $x_i = 1$.

Proof of Lemma 2.2.5. Fixed a certain direction v, the set of points $x \in \mathcal{Q}$ in which a straight line parallel to v is tangent to Q are the intersection between $\mathcal{Q}$ itself and the plane $\pi = \{x \in \mathbb{R}^n : x \cdot (\rho^{-1} v) = 0\}$. If $v = e_i$ then $\pi = \{x \in \mathbb{R}^n : x \cdot \rho_i^{-1} = 0\}$ and one can notice that the points $\{\rho_j\}_{j \neq i}$ belong to π. Furthermore even the origin belongs to π and so one can parameterize π as a linear combination of $\{\rho_j\}_{j \neq i}$ which means $\pi = \pi_i$ and its intersection with the quadric is by definition $\mathcal{Q}_i$.

Proof of Proposition 2.3.1. $\mathcal{Q}_+ \cup \partial\mathcal{Q}_+$ is a compact set, so $LBL(x)$ attains extreme values in it. Now let's consider a point $x_b \in \partial\mathcal{Q}_+$. By Lemma 2.2.5 the derivative of $LBL(x)$ along the direction e_i is also the derivative of the restricted function, furthermore $\frac{\partial LBL(x)}{\partial x_i} > 0$ and so the function is increasing along the direction e_i which goes inside $\mathcal{Q}_+$. This means that for each $x_b \in \partial\mathcal{Q}_+$ there exists a point x_b' such that $LBL(x_b') > LBL(x_b)$ and so the maximum point is reached in the interior part of $\mathcal{Q}_+$ and it must be a solution of eq.(1.24). At the same time this argument shows that outside $\mathcal{Q}_+$ the function attains a local minimum.

Proof of Lemma 2.3.1. Simply observing that the equation $x = \psi(x)$ is equivalent to the Lagrange equation (1.24) and $\psi(x) \in \mathcal{Q}$.

Proof of Lemma 2.3.2. Via direct computation the Jacobian matrix has the form

$$\mathrm{J}(x) = \frac{\partial \psi(x)}{\partial x} = \frac{1}{\alpha(x)^{3/2}}[\rho - \psi(x)\psi(x)']\mathrm{H}(x)$$

where $\mathrm{H}(x)$ is the hessian matrix of the lower bound $\mathrm{H}(x)_{ij} = \frac{\partial^2 LBL(x)}{\partial x_i \partial x_j}$. The vector $v(x) = \mathrm{H}^{-1}(x)\nabla LBL(x)$ is such that $\mathrm{J}(x)v(x) = 0$ and so $\mathrm{J}(x)$ is singular. For $n = 2$ the other eigenvalue must be real. The value of l is found via direct computation using that $v_l = [-\frac{\partial LBL(x)}{\partial x_2}, \frac{\partial LBL(x)}{\partial x_1}]$ is the eigenvector relative to l and substituting the explicit expression of the hessian.

Proof of Proposition 2.3.2. The condition is a direct consequence of the contraction principle. A sufficient condition for the map to be a contraction is that $\max_{x\in\mathcal{Q}} |\mathrm{J}(x)| < 1$ where the norm of the matrix is defined as $|\mathrm{J}(x)| \equiv \sup_{v:|v|=1} |\mathrm{J}(x)v|$. Due to Lemma 2.3.2 this condition simply becomes $l^\star < 1$ where $l^\star = \sup_{x\in\mathcal{Q}} |l|$. In what follows we find an upper bound for $|l|$ which gives the desired condition on the parameters.
From $2\sigma_1\sigma_2\hat{\omega}_1(x)\hat{\omega}_2(x) \leq \sigma_1^2\hat{\omega}_1(x)^2 + \sigma_2^2\hat{\omega}_2^2(x)$ follows that $\frac{\sigma_1^2\sigma_2^2\hat{\omega}_1(x)\hat{\omega}_2(x)}{\hat{\alpha}(x)^{3/2}} \leq \frac{\sigma_1\sigma_2}{\sqrt{\hat{\omega}_1(x)^2\sigma_1^2 + \hat{\omega}_2(x)^2\sigma_2^2}}$.
Now, using that $\hat{\omega}_1(x) + \hat{\omega}_2(x) = 1$ one has $\sqrt{\hat{\omega}_1(x)^2\sigma_1^2 + \hat{\omega}_2(x)^2\sigma_2^2} \geq \sqrt{\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}$, which by substitution gives $\frac{\sigma_1^2\sigma_2^2\hat{\omega}_1(x)\hat{\omega}_2(x)}{\hat{\alpha}(x)^{3/2}} \leq \sqrt{\sigma_1^2 + \sigma_2^2}$.
Now we have to deal we the term in brackets. The first term is always positive while the solution $\lambda^\star$ can have both signs, so

$$\max_{x\in\mathcal{Q}} \left\{ \left| \frac{\tilde{\sigma}_1\tilde{\sigma}_2}{\tilde{\sigma}_1\hat{\omega}_1(x) + \tilde{\sigma}_2\hat{\omega}_2(x)} + \lambda^\star \right| \right\} \leq \max_{x\in\mathcal{Q}} \frac{\tilde{\sigma}_1\tilde{\sigma}_2}{\tilde{\sigma}_1\hat{\omega}_1(x) + \tilde{\sigma}_2\hat{\omega}_2(x)} + \max_{x\in\mathcal{Q}} |\lambda^\star|$$

Using the normalization as above one has $\tilde{\sigma}_1\hat{\omega}_1(x) + \tilde{\sigma}_2\hat{\omega}_2(x) > \tilde{\sigma}_m$ where $\tilde{\sigma}_m$ is the smaller between $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ and so $\frac{\tilde{\sigma}_1\tilde{\sigma}_2}{\tilde{\sigma}_1\hat{\omega}_1(x)+\tilde{\sigma}_2\hat{\omega}_2(x)} < \tilde{\sigma}_M < \sigma_2$.

The upper bound for $|\lambda^\star|$ is given by $|\lambda'|$ solution of the equation

$$e^{-\tilde{\sigma}_m\lambda'} \sum_{i=1}^{2} \omega_i e^{-\frac{1}{2}\tilde{\sigma}_i^2} - K = 0$$

Simply solving the equation above one has

$$\lambda' = \frac{1}{\tilde{\sigma}_m} \log\left(\frac{K}{\sum_{i=1}^{2} \omega_i e^{-\frac{1}{2}\tilde{\sigma}_i^2}}\right)$$

The final result is finally obtained with inequality

$$|\lambda'| \leq \frac{1}{\rho\sigma_1} \max\left\{-\log(K), \left|\log\left(\frac{\sum_{i=1}^{2} \omega_i e^{-\frac{1}{2}\sigma_i^2}}{K}\right)\right|\right\}$$

If $K \geq 1$ the condition for uniqueness can be improved first noticing that $\lambda^\star$ is negative and so

$$\max_{x\in\mathcal{Q}}\left\{\left|\frac{\tilde{\sigma}_1\tilde{\sigma}_2}{\tilde{\sigma}_1\hat{\omega}_1(x)+\tilde{\sigma}_2\hat{\omega}_2(x)} + \lambda^\star\right|\right\} \leq \max\left\{\max_{x\in\mathcal{Q}} \frac{\tilde{\sigma}_1\tilde{\sigma}_2}{\tilde{\sigma}_1\hat{\omega}_1(x)+\tilde{\sigma}_2\hat{\omega}_2(x)}, \ -\min_{x\in\mathcal{Q}} \lambda^\star\right\}$$

The lower bound for the solution $\lambda^\star$ required above can be found using that $\lambda^\star \geq \lambda_{GA}$ where λ_{GA} is the solution obtained using the geometric average instead of $\mathcal{S}$

$$|\lambda_{GA}(x)| = \frac{\log K}{<\tilde{\sigma}>} + \frac{1}{2}\frac{<\tilde{\sigma}>^2}{<\tilde{\sigma}>} \leq \frac{\log K}{\bar{\sigma}} + \frac{1}{2}\frac{\sigma_2^2}{\bar{\sigma}}$$

where $<\tilde{\sigma}>= \sigma_1\omega_1 + \sigma_2\omega_2$, $<\tilde{\sigma}^2>= \sigma_1\omega_1^2 + \sigma_2\omega_2^2$.

Proof of Proposition 2.3.3 First we show that for $x = \rho_i$ it is possible to compute $PECUB(x,d)$ for some d. The associate conditioning variable is $\Lambda = g_i$. The value of d is found via the trivial inequality $\mathcal{S} \geq \omega_i e^{-\frac{1}{2}\sigma_i^2 - \sigma_i g_i}$ which implies $d = -\frac{\ln(K/\omega_i)}{\sigma_i} - \frac{\sigma_i}{2}$. Now consider $ICUB(\rho_i)$. From Lemma 2.3.3 follows $d = \lambda_i^*$, furthermore if $\lambda \leq \lambda_i^*$ by Theorem 1.5.1 $c(\lambda) = \ell(\lambda)$ which implies the result.

Proof of Proposition 2.3.4. For $n = 2$ basket the true price is

$$C = \mathbb{E}\left[\omega_1 e^{-\frac{\sigma_1^2}{2} - \sigma_1\alpha_1} + \omega_2 e^{-\frac{\sigma_2^2}{2} - \sigma_2\alpha_2} - K\right]_+$$

where α_1 and α_2 are two normal correlated variables with zero mean, unitary variance and $\mathrm{Corr}[\alpha_1, \alpha_2] = \rho$, while $ICUB(x)$ reads as

$$ICUB(x) = \mathbb{E}\left[\omega_1 e^{-\frac{\sigma_1^2}{2} - \tilde{\sigma}_1\lambda - \hat{\sigma}_1\theta} + \omega_2 e^{-\frac{\sigma_2^2}{2} - \tilde{\sigma}_2\lambda - \hat{\sigma}_2\theta} - K\right]_+$$

where λ and θ are two uncorrelated standard normal variables. Applying now the linear transformation

$$\begin{cases} x_1\lambda + \sqrt{1-x_1^2}\,\theta = \lambda' \\ x_2\lambda + \sqrt{1-x_2^2}\,\theta = \theta' \end{cases}$$

the argument of the expectation value is the same for both the formulas above, while λ' and θ' are normal with zero mean, unitary variance and correlation $\mathrm{Corr}[\lambda',\theta'] = x_1x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}$. Now using the constraint one has $\mathrm{Corr}[\lambda',\theta'] = x_1x_2 \pm |\rho - x_1x_2|$ that equals ρ in $\mathcal{Q}/\mathcal{Q}_+$ which means that $ICUB(x)$ and the exact price coincide in that zone.

Proof of Proposition 2.3.5. We start showing that the first derivative of $ICUB(x)$ along an arbitrary coordinate direction e_j is discontinuous in ρ_i. Around ρ_i we can write $x_i = q(x_j)$, where $q(x_j) = 1 - \frac{\gamma_{ij}^2}{2}(x_j-\rho_{ij})^2$ with $\gamma_{ij}^2 = 1/(1-\rho_{ij}^2)$. Considering the restriction to the curve

$$c_{ij}(t) = \begin{cases} x_j &= t + \rho_{ij} \\ x_i &= 1 - \dfrac{\gamma_{ij}^2 t^2}{2} \\ x_{k\neq i,j} &= \rho_{ik} \end{cases}$$

we can compute the first derivative along the direction e_j

$$\lim_{t\to 0^\pm} \frac{d}{dt} ICUB(c_{ij}(t)) = -\frac{\rho_{ij}}{\sqrt{1-\rho_{ij}^2}} \psi_j^{(1)}(\rho_i) - \psi_j^{(2)}(\rho_i) \pm \gamma_{ij}\psi_i^{(1)}(\rho_i)$$

where

$$\begin{aligned} \psi_j^{(1)}(x) &= \sigma_j\omega_j \int_{\lambda_i^*}^{\infty} d\lambda\ \phi(\lambda+\tilde{\sigma}_j)\phi(\vartheta_\lambda+\hat{\sigma}_j) \\ \psi_j^{(2)}(x) &= \sigma_j\omega_j \int_{\lambda_i^*}^{\infty} d\lambda\ \phi(\lambda+\tilde{\sigma}_j)\phi(\vartheta_\lambda+\hat{\sigma}_j)\frac{\partial\theta_\lambda}{\partial\lambda} \end{aligned}$$

and $\psi_j^{(1,2)}(x) > 0$. Finally we show that the right and left limits are finite also along a generic direction. We consider the curve $c(t) = \alpha_1 c_{ij}(t) + \alpha_2 c_{ik}(t)$ with $\alpha_1 + \alpha_2 = 1$. From direct computation it follows:

$$\lim_{t\to 0^\pm} \frac{d}{dt} ICUB(c(t)) = \alpha_1 \nabla_j(\rho_i) + \alpha_2 \nabla_k(\rho_i) \pm \frac{\alpha_1\gamma_{ij}^2 + \alpha_2\gamma_{ik}^2}{\gamma}\psi_i^{(1)}(\rho_i)$$

where $\gamma = \alpha_1\gamma_{ij}^2 + \alpha_2\gamma_{ik}^2$ and ∇_j the j^{th} component of the gradient.

Proof of Proposition 2.3.6. Consider the set of curves of the proposition above. In the particular case in which all the out of diagonal elements of the correlation matrix are equal to $\hat{\rho}$, substituting the term $\partial\vartheta_\lambda/\partial\lambda$ via the implicit function theorem we have

$$\lim_{t\to 0^\pm} \frac{d}{dt} ICUB(c(t)) = \frac{\sigma_i\omega_i}{\sqrt{1-\hat{\rho}^2}} \int_{\lambda_i^*}^{\infty} d\lambda\ \phi(\lambda+\sigma_i)\phi(\vartheta_\lambda) \left\{ \pm 1 + \frac{\omega_j^\lambda}{\sum_{k\neq i}\omega_k^\lambda} \right\}$$

where

$$\omega_k^\lambda \equiv \omega_k \; \sigma_k \; \phi(\lambda + \tilde{\sigma}_k)\phi(\vartheta_\lambda + \hat{\sigma}_k) \qquad \forall k \neq i \; .$$

Being the ratio in braces always positive and lower than 1 it follows that the the ρ columns are local minima for the restricted function along the coordinate directions. Furthermore being $\gamma_{ij} = \gamma_{ik}$ we have

$$\lim_{t \to 0^\pm} \frac{d}{dt} ICUB(c(t)) = \alpha_1 \lim_{t \to 0^\pm} \frac{d}{dt} ICUB(c_{ij}(t)) + \alpha_2 \lim_{t \to 0^\pm} \frac{d}{dt} ICUB(c_{ik}(t))$$

which implies the result.

CHAPTER 3

Optimal bounds: a new algorithm

3.1 Introduction

In this chapter we describe the new algorithm we design for solving the two optimization problems (1.20) and (1.36). From a numerical point of view the main issue of both the problems is to handle together the equality constraint and local optimal points. Two common algorithms used for global optimization are the Simulated Annealing (SA from now on) originally proposed by Kirkpatrick et al. [24] and the family of Genetic Algorithms (GA from now on), for which we refer to [38] for a survey on the topic. These tools cannot be easily adapted to constrained optimization problems where the main issue is how to handle the constraint. The most common technique is to employ the penalization method to transform the constrained problem into an unconstrained one, but in practice setting a good penalty factor represents a significant disadvantage of this approach, in particular for equality constraints. Indeed using GA of Matlab for computing the optimal bounds reveals to be extremely slow and in some cases the global maximum (or minimum) is not reached. Here we propose a new procedure based on the SA. The key element is that we tackle the constraint $\mathcal{Q}$ by directly exploring *only* the feasible points. Accuracy of the solution obtained from SA is then refined through a fixed point map, developed *ad hoc* for optimal bounds, which after fews iterations reaches the nearest optimal point. The introduced technique revealed to be fast enough to obtain optimal solutions in a reasonable time for financial applications. In particular the performance of the proposed procedure is also tested and compared with GA of Matlab with numerical examples.

3.2 Dicretization of ICUB(x)

In this section we explicitly write the objective function we used to approximate the solution of the optimization problem (1.36). In first instance we consider the discretization of the integral on a given fixed grid (i.e. not dependent on x and $\theta(\lambda)$). Defining the grid $\{\lambda_k\}_{k=1}^N$, $\theta_k = \theta(\lambda_k)$ and the integration weights $\{h_k\}_{k=1}^N$ the discretized UB_{opt}, reads as

$$\begin{aligned}\mathrm{UB}_d &= \min_{x\in\mathcal{Q}} \sup_{\theta\in\mathbb{R}^N} \sum_{i=1}^{n}\sum_{k=1}^{N} h_k\omega_i\varphi(\lambda_k)\left[e^{-\frac{1}{2}\sigma_i^2x_i^2-\sigma_ix_i\lambda_k}\phi\left(\sigma_i\sqrt{1-x_i^2}+\theta_k\right)-\frac{K}{n}\phi(\theta_k)\right]\\ &:= \min_{x\in\mathcal{Q}} \sup_{\theta\in\mathbb{R}^N} ICUB_d(x,\underline{\theta})\end{aligned} \tag{3.1}$$

where we have defined the objective function $ICUB_d(x,\theta)$ and the vector $\underline{\theta}\in\mathbb{R}^N$ with components θ_k. The maximization over $\underline{\theta}$ can be further investigated explicitly writing the first order conditions $\frac{\partial ICUB_d(x,\theta)}{\partial\theta_k} = 0$. Using (B.8) it is immediate to verify that solving the first order equation is equivalent to solve $z_k(x,\theta_k) = 0$ for $k = 1,\dots,N$, where

$$z_k(x,\theta_k) := \sum_{i=1}^{n}\omega_i e^{-\frac{1}{2}\sigma_i^2-\sigma_ix_i\lambda_k-\sigma_i\sqrt{1-x_i^2}\theta_k} - K \tag{3.2}$$

The function is a sum of convex decreasing functions in θ_k and hence it is convex itself, which implies that the equation admits only one solution that corresponds to the supremum (when finite). Moreover, thanks to convexity, optimization over $\underline{\theta}$ can be efficiently performed using the Newton method exploiting the explicit expression for the derivative

$$\frac{\partial z_k(x,\theta_k)}{\partial\theta_k} = \sum_{i=1}^{n}\omega_i\sigma_i\sqrt{1-x_i^2}e^{-\frac{1}{2}\sigma_i^2-\sigma_ix_i\lambda_k-\sigma_i\sqrt{1-x_i^2}\theta_k} \tag{3.3}$$

Solving problem (3.1) however still require to compute the supremum over $\underline{\theta}$ for each value of x.

3.3 Algorithm

The following section is devoted to describe in detail the implementation of our procedure.

3.3.1 Objective functions and initialization of SA

The objective function we minimize are

- $f_u(x,\theta) := ICUB_d(x,\theta)^1$ with $x \in \mathcal{Q}$ and $\theta \in \mathbb{R}^n$
- $f_\ell(x,y) := -LB(x,y) + (y_1 - y_2)^2 \mathbf{1}_{\{(y_1-y_2)<0\}}$ with $x \in \mathcal{Q}$ and $y \in \mathbb{R}^2$

The second term of $f_\ell(x,y)$ is just a penalization term which prevent the function being flat when $y_2 < y_1$. The first stage of our scheme is SA which requires as input a first guess solution. For both upper and lower bounds we set the same starting guess x_0 which is the vector such that

$$(x_0)_i = \frac{\sum_{i=1}^n \rho_{ij}\sigma_i\omega_i}{\sqrt{x \cdot \rho^{-1} x}} \tag{3.4}$$

originally proposed in [12] as a good guess of the optimal solution for both Lower and Upper Bounds. The vector y_0 of the lower bound is initialized as $y_0 = (-1,1)$. For the upper bound θ is initialized as the vector which maximises $f_u(x_0,\theta)$, given x_0:

$$\theta_0 = \underset{\theta \in \mathbb{R}^N}{\arg\max}\, f_u(x_0,\theta) \tag{3.5}$$

which can be efficiently computed performing a Newton procedure to $z_k(x_0,\theta_k) = 0$ for $k = 1,\ldots,N$, where $z_k(x,\theta_k)$ and its derivative are given by (3.2) and (3.3).

3.3.2 Generator and acceptance criterion

The SA generator is the function $g(p^{(i)})$ which generate a new proposal $p^{(p)}$ point given $p^{(i)}$, $p^{(p)} = g(p^{(i)})$. In the following we define the functions which generates the couples $(x^{(p)},\theta^{(p)})$ for the Upper Bound and $(x^{(p)},y^{(p)})$ for the Lower Bound. The point $x^{(p)}$ is generated according to the same procedure for both $f_u(x,\theta)$ and $f_\ell(x,y)$ as follows: denoting with M the Cholesky matrix of ρ^{-1}, i.e. such that $M^T M = \rho^{-1}$, we compute the vector $z = Mx$, then we generate a new point z' uniformly on the sphere of radius δ centered in z. Then we project this point on the sphere centered in the origin with unitary radius $z'' = z'/\sqrt{(z' \cdot z')}$ (hence not breaking the spherical symmetry) and the proposal point is $x^{(p)} = M^{-1}z''$. The procedures for generating $y^{(p)}$ and $\theta^{(p)}$ are different. The vector $y^{(p)}$ of the lower bound is generated on the sphere of radius δ and centered in $y^{(i)}$. The point $\underline{\theta}^{(p)}$ is generated performing one Newton step from the point $(x^{(p)},\theta^{(i)})$, i.e.

$$\theta_k^{(p)} = \theta_k^{(i)} - \frac{z_k(x^{(p)},\theta_k^{(i)})}{\frac{\partial z_k(x^{(p)},\theta_k^{(i)})}{\partial \theta_k}} \tag{3.6}$$

[1] Actually we will not minimize over θ otherwise we would not obtain an upper bound. However in the following we will explain how to handle the problem as it was a joint minimization over x and θ.

Then the new point is accepted with Metropolis criterion at a certain temperature T_k, i.e. the probability of acceptance is $p(x^{(p)}, y^{(p)}|x^{(i)}, y^{(i)}) = \min\left(1, \exp\left(-\frac{f_\ell(x^{(p)},y^{(p)})-f_\ell(x^{(i)},y^{(i)})}{T_k}\right)\right)$ for the lower bound and $p(x^{(p)}, y^{(p)}|x^{(i)}, \theta^{(i)}) = \min\left(1, \exp\left(-\frac{f_u(x^{(p)},\theta^{(p)})-f_u(x^{(i)},\theta^{(i)})}{T_k}\right)\right)$ for the upper bound. Before going on with the description of the algorithm we want to make a couple of remarks on the generator of the upper bound. Essentially the $\underline{\theta}$ generator computes $\sup_{\underline{\theta}\in\mathbb{R}^N} f_u(x^{(p)}, \theta)$ performing one single Newton step using as intelligent starting guess the vector $\theta^{(i)}$ of the previous step. However, since at each step we generate the couple $(x^{(p)}, \underline{\theta}^{(p)})$ from $(x^{(i)}, \underline{\theta}^{(i)})$ the procedure proposed requires the same computational effort as a joint optimization over x and $\underline{\theta}$. We want to stress that the generator is the key element in boosting the performance of our procedure. This generator, in fact, is designed to explore only the feasible points $x \in \mathcal{Q}$ consequently it gives two advantages with respect to the penalty method: first the area explored is far less and second we overcome the issue of setting the penalization factor.

3.3.3 Cooling schedule and stopping criterion

The temperature is decreased according to the schedule $T_{k+1} = f(T_k)$ where $f(T) = \frac{T}{1+T}$ with $T_0 = 1$, which corresponds to linearly decrease the inverse of the temperature with $\Delta\frac{1}{T} = 1$. SA is then stopped after M consecutive attempt of accepting a worse point failed. In particular we set $M = 20$. Using this criterion we stop SA when the probability of exploring "worse" points is negligible and the system cannot escape anymore from the (eventually local) nearest minimum. Then it remains only to tune the cooling considering the trade off between computational time and convergence properties.

3.3.4 The fixed point scheme

Accuracy of the solution is obtained via a fixed point scheme, we first describe the procedure for the lower bound and then the corresponding for the upper bound.

Lower Bound

In this part we consider optimization problem (1.20). The first order conditions are given in eq.(1.24). The construction of the fixed point map relies on the following two propositions

Proposition 3.3.1. *Defining* $\psi_\ell : \mathbb{R}^n \to \mathcal{Q}$ *as*

$$\psi_\ell(x) = \frac{\rho\nabla LBL(x)}{\sqrt{\nabla LBL(x)\cdot\rho\nabla LBL(x)}} \tag{3.7}$$

then $\hat{x}$ is a fixed point of ψ_ℓ iff is a solution of (1.24).

Proof. Via direct calculation equations (1.24) can be written as $\hat{x} = \rho \nabla LBL(\hat{x})/\alpha$. Obtaining α from the second equation and substituting we get $\hat{x} = \psi(\hat{x})$ which is the result. □

Now if we define our iterative scheme as

$$x^{(i+1)} = \psi_\ell(x^{(i)}) \tag{3.8}$$

The following proposition ensure that we can use use this scheme for the purpose of maximization

Proposition 3.3.2. *Let $\hat{x}$ be a solution of* (1.24) *for some* $\alpha \neq 0$ *and suppose that* $\lim_{i\to\infty} x^{(i)} = \hat{x}$*. Then $\hat{x}$ is a maximum.*

Proof. We call $\mathcal{L}(x, \alpha)$ the Lagrangian function of the problem

$$\mathcal{L}(x, \alpha) = LBL(x) - \frac{\alpha}{2}\left(x \cdot \left(\rho^{-1} x\right) - 1\right) \tag{3.9}$$

and we define the sequence $\mathcal{L}^{(i)}$ as

$$\mathcal{L}^{(i)} = \mathcal{L}(x^{(i)}, \alpha^{(i)}) = LBL(x^{(i)}) - \frac{\alpha^{(i)}}{2}\left(x^{(i)} \cdot \left(\rho^{-1} x^{(i)}\right) - 1\right) \tag{3.10}$$

where

$$\alpha^{(i)} = \left(\nabla f(x^{(i)}) \cdot \left(\rho^{-1} \nabla f(x^{(i)})\right)\right)^{\frac{1}{2}} \tag{3.11}$$

Since $x^{(i)} \in \mathcal{Q}$ it holds true that $\mathcal{L}^{(i)} = f(x^{(i)})$ and hence

$$\begin{aligned}
LBL(x^{(i+1)}) - LBL(x^{(i)}) &= \mathcal{L}^{(i+1)} - \mathcal{L}^{(i)} \\
&\simeq \nabla \mathcal{L}(x^{(i)}, \alpha^{(i)}) \cdot (x^{(i+1)} - x^{(i)}) \\
&= (\nabla LBL(x^{(i)}) - \alpha^{(i)} \rho^{-1} x^{(i)}) \cdot \left(\frac{\rho \nabla LBL(x^{(i)})}{\sqrt{\nabla f(x^{(i)}) \cdot \rho \nabla LBL(x^{(i)})}} - x^{(i)}\right) \\
&= (\nabla LBL(x^{(i)}) - \alpha^{(i)} \rho^{-1} x^{(i)}) \cdot \left(\frac{\rho \nabla LBL(x^{(i)})}{\alpha^{(i)}} - x^{(i)}\right) \\
&= \alpha^{(i)} \left(\frac{\nabla LBL(x^{(i)})}{\alpha^{(i)}} - \rho^{-1} x^{(i)}\right) \cdot \rho \left(\frac{\nabla LBL(x^{(i)})}{\alpha^{(i)}} - \rho^{-1} x^{(i)}\right) \\
&> 0
\end{aligned} \tag{3.12}$$

□

Then our recipe is: starting from the solution x_{SA} obtained from SA we iterate (3.7) until a certain accuracy is obtained.

Upper bound

For the upper bound we define the fixed point map in complete analogy with the lower bound. Starting from the first order optimal equations

$$\begin{cases} \nabla ICUB(x) = \alpha\rho^{-1}x \\ x \cdot (\rho^{-1}x) = 1 \end{cases} \tag{3.13}$$

we state the following propositions

Proposition 3.3.3. *Defining $\psi_u : \mathbb{R}^n \to \mathcal{Q}$ as*

$$\psi_u(x) = -\frac{\rho\nabla ICUB(x)}{\sqrt{\nabla ICUB(x) \cdot \rho\nabla ICUB(x)}} \tag{3.14}$$

then $\hat{x}$ is a fixed point of ψ_u iff is a solution of (3.13).

Proof. Completely analogous to Prop.3.3.1 □

Then defining the scheme

$$x^{(i+1)} = \psi_u(x^{(i)}) \tag{3.15}$$

the following proposition holds

Proposition 3.3.4. *Let $\hat{x}$ be a solution of* (3.13) *for some $\alpha \neq 0$ and suppose that $\lim_{i\to\infty} x^{(i)} = \hat{x}$. Then $\hat{x}$ is a minimum.*

Proof. Completely analogous to Prop.3.3.2 □

Then likewise the lower bound our recipe is to iterate the scheme (3.15) until a certain accuracy is obtained.
Remark. In most of the cases, as we will show in the next section, the solution of problem (1.5.2) is one columns of the correlation matrix $x^\star = \rho_k$ (which means $x_i^\star = \rho_{ik}$, $i = 1, \dots, n$). However these points are angular points and hence gradient based methods like SQP may fail in actually computing the minimum. The map we defined, even though depends on the gradient of the function, does not suffer this problem because it holds true that

$$\lim_{x\to\rho_k} \psi_u(x) = \rho_k \tag{3.16}$$

which means that ψ_u "sees" the columns of the correlation matrix as regular fixed points.

3.3.5 Summary of the algorithm

The procedure for optimizing upper and lower bounds can be summarized as follows:

Step 1: Set $x_0 =$ as in (3.4), $y_0 = (-1, 1)$ for the lower bound, θ_0 as in (3.5) for the upper bound and start the SA procedure as described previously, with objective functions $f_\ell(x, y)$ (or $f_u(x)$ for the upper bound).

Step 2: Stop SA when stopping criterion is met and use the rough approximation x_{SA} given by SA as starting point for the iterative procedure $x^{(i+1)} = \psi_\ell(x^{(i)})$ (or $x^{(i+1)} = \psi_u(x^{(i)})$ for the upper bound).

Step 3: Stop the iterative procedure when desired accuracy μ is achieved, i.e. when $LBL(x^{(i+1)}) - LBL(x^{(i)}) < \mu$ (or $ICUB(x^{(i)}) - ICUB(x^{(i+1)}) < \mu$ for the upper bound).

3.4 Numerical examples

In this section we will test the effectiveness of the algorithm on different examples of basket options comparing the performance with a benchmark algorithms: the Genetic Algorithm, the function "ga" of "MATLAB and Global Optimization Toolbox Release 2016b, The MathWorks, Inc., Natick, Massachusetts, United States". For the benchmark algorithm the constraint is handled employing the penalization method (with constant penalty) and the size of the initial population has been raised until the global minimum is always reached. All the codes are implemented in "MATLAB Release 2016b, The MathWorks, Inc., Natick, Massachusetts, United States" on an intel i7 with processor frequency 2.8 GHz and 8 GB of RAM memory.

3.4.1 A significant high dimensional numerical example: symmetric basket

Here we consider an high dimensional example in which it is possible to explicitly show the existence of a local minimum for the lower bound located exactly in x_0 and hence test the capability of our algorithm to escape from the local minimum and converge to the global one. The basket under investigation is composed by n assets with the following financial parameters:

- $\omega_i = \frac{1}{n}\ \forall i$
- $\sigma_i = \sigma\ \forall i$
- $\rho = (1 - r)I + r\eta\eta^T$

where I is the identity matrix and η is such that $\eta_i = \pm 1$ for $i = 1, \ldots, n$, depending on the sign of the correlation of the i^{th} asset with the first one. The eigenvectors of ρ are η with eigenvalue $1 + (n - 1)r$ and its orthogonal subspace with eigenvalue $1 - r$. The starting guess is $x_0 = \sqrt{\frac{1-r}{n}}v$ where v is the the vector which has all the components equal to 1. By symmetry arguments (or via direct computation) we know that x_0 is a stationary point. In the following we show that x_0 is a maximum of $LBL(x_0)$. The Hessian matrix (B.6) in x_0 reads as

$$\mathrm{H}(x_0) = -\frac{\sigma^2}{n} h(y_1^\star + \tilde{\sigma}) \left[I + \frac{y_1^\star + \tilde{\sigma}}{n\tilde{\sigma}} vv' \right] \tag{3.17}$$

where $\tilde{\sigma} = \sigma\sqrt{\frac{1-r}{n}}$. Let's for now focus on the matrix in brackets. This matrix is of the same type of the correlation matrix and its eigenvectors are v with eigenvalue $1 + \frac{y_1^\star + \tilde{\sigma}}{\tilde{\sigma}}$ and its orthogonal space with eigenvalue 1. However the normal vector to the constraint $\mathcal{Q}$ in x_0 is $\rho^{-1}x_0$ which is proportional to x_0 (since it is an eigenvector) and hence the direction v must be excluded in the study of the convexity. The eigenvalues are then all equal to $-\frac{\sigma^2}{n}h(y_1 + \tilde{\sigma})$ and their sign is determined by the term $y_1 + \tilde{\sigma}$. By explicit computation we have that $y_1^\star = -\frac{\log K}{\tilde{\sigma}} - \frac{1}{2}\tilde{\sigma}$ and substituting we have that the condition for x_0 to be a maximum is $\log K < \frac{1}{2}\sigma^2\frac{1-r}{n}$. We can note that for large values of n we have that the point is maximum when the option is ITM while it's a minimum for the ATM or the OTM. In Tab.3.1 and Tab.3.2 we compare the performance of our algorithm with the benchmark reporting also the true benchmark price with a confidence interval of 95% computed performing MC simulation with 10^5 simulations and employing the antithetic variable method.

n	MC price (%)	LB (%)	$LBL(x_0)$	OP time (s)	GA time (s)
50	[1.9690,2.0113]	1.9380	1.3385e-07	1.9	116
100	[1.9484,1.9907]	1.9321	6.6913e-08	2.3	1074
250	[1.9073,1.9489]	1.9347	2.6763e-08	2.7	8318
500	[1.9340,1.9764]	1.9342	1.3381e-08	9.8	-

Table 3.1: *Comparison of the performance of our procedure (OP) with GA for computing the lower bound. The correlation is set to $r = 0.9$, the volatility is $\sigma = 0.3$ and the moneyness is $K = 1$.*

3.5 Conclusions

In this chapter we introduced a new ad hoc technique for the computation of optimal upper and lower bounds to the price of a basket call. The key element of our procedure is that our algorithm is able to efficiently explore the feasible points directly moving on the constraint. The advantages of this approach are two. First we do not face the problem of setting the penalization term and then we explore only the feasible points with a relevant boost in performance. Another relevant element introduced is the deterministic map. A possible

n	MC price (%)	ICUB (%)	$ICUB(x_0)$ (%)	OP time (s)	GA time (s)
50	[1.9690,2.0113]	2.7761	11.9235	18	-
100	[1.9484,1.9907]	2.7922	11.9235	32	-
250	[1.9073,1.9489]	2.8002	11.9235	63	-
500	[1.9340,1.9764]	1.9342	11.9235	95	-

Table 3.2: *Comparison of the performance of our procedure (OP) with GA for computing the upper bound. The correlation is set to $r = 0.9$, the volatility is $\sigma = 0.3$ and the moneyness is $K = 1$.*

direction of research is to test our technique on different objective functions $f(x)$ with constraints defined via a quadratic form of the type $x \cdot Mx = 1$ where M is a symmetric positive definite matrix.

APPENDIX $\mathcal{B}$

Explicit derivatives

Derivatives of the $LB(x,y)$

$$\frac{\partial LB(x,y)}{\partial x_i} = \omega_i \sigma_i \left[\phi(y_1 + \sigma_i x_i) - \phi(-y_2 - \sigma_i x_i)\right] \tag{B.1}$$

$$\frac{\partial LB(x,y)}{\partial y_1} = \sum_{i=1}^{n} \omega_i \phi(y_1 + \sigma_i x_i) - K\phi(y_1) \tag{B.2}$$

$$\frac{\partial LB(x,y)}{\partial y_2} = -\sum_{i=1}^{n} \omega_i \phi(-y_2 - \sigma_i x_i) + K\phi(-y_2) \tag{B.3}$$

$$\frac{\partial^2 LB(x,y)}{\partial x_i \partial x_j} = -\omega_i \sigma_i^2 \left[h(y_1 + \sigma_i x_i) + h(-y_2 - \sigma_i x_i)\right] \delta_{ij} \tag{B.4}$$

Derivatives of $LBL(x)$

$$\frac{\partial LBL(x)}{\partial x_i} = \omega_i \sigma_i \left(\phi(y_1^\star + \sigma_i x_i) - \phi(y_2^\star + \sigma_i x_i)\right) \tag{B.5}$$

$$\begin{aligned}\frac{\partial^2 LBL(x)}{\partial x_i \partial x_j} =& \sigma_i\sigma_j\omega_i\omega_j \left(\frac{h(y_1^\star+\sigma_i x_i)h(y_1^\star+\sigma_j x_j)}{\sum_{k=1}^n \omega_k\sigma_k x_k \phi(y_1^\star+\sigma_k x_k)} - \frac{h(-y_2^\star-\sigma_i x_i)h(-y_2^\star-\sigma_j x_j)}{\sum_{k=1}^n \omega_k\sigma_k x_k\phi(-y_2^\star-\sigma_k x_k)}\right)\dots \\ & -\sigma_i^2\omega_i\left(h(y_1^\star+\sigma_i x_i) - h(-y_2^\star-\sigma_i x_i)\right)\end{aligned} \tag{B.6}$$

where $h(x) = x\phi(x)$.

First derivatives of $ICUB_d(x,\underline{\theta})$

$$\begin{aligned}\frac{\partial ICUB_d(x,\theta)}{\partial x_i} = -\sum_{k=1}^{N} h_k\omega_i\sigma_i\phi(\lambda_k+\sigma_i x_i)\Bigg[&\frac{x_i}{\sqrt{1-x_i^2}}\phi\left(\sigma_i\sqrt{1-x_i^2}+\theta_k\right)+ \\ &(\lambda_k+\sigma_i x_i)\Phi\left(\sigma_i\sqrt{1-x_i^2}+\theta_k\right)\Bigg]\end{aligned} \tag{B.7}$$

$$\frac{\partial ICUB_d(x,\theta)}{\partial \theta_k} = \sum_{i=1}^{n} h_k\omega_i\left[\phi\left(\theta_k+\sigma_i\sqrt{1-x_i^2}\right)\phi\left(\lambda_k+\sigma_i x_i\right) - K\phi\left(\lambda_k\right)\phi\left(\theta_k\right)\right] \tag{B.8}$$

Second Derivatives

$$\frac{\partial^2 ICUB_d(x,\theta)}{\partial \theta_k^2} = -\sum_{i=1}^{n} h_k\omega_i\left[h\left(\theta_k+\sigma_i\sqrt{1-x_i^2}\right)\phi\left(\lambda_k+\sigma_i x_i\right) - K\phi\left(\lambda_k\right)h\left(\theta_k\right)\right] \tag{B.9}$$

Part II

Pricing illiquid corporate bonds

CHAPTER 4

A closed formula for illiquid corporate bonds and an application in the European market

4.1 Introduction

The natural question that arises when dealing with liquidity is: “How long does it take to liquidate a given position?". Despite the relevance of this question, not only a unique modeling framework, but even a standard language for addressing liquidity has not appeared in the financial industry yet. Unfortunately liquidity problems are - in general - really complicated; several are the aspects of asset liquidity including: tightness (i.e. bid-ask spread, the transaction cost incurred in case of a small liquidation), market impact (i.e. the average response of prices to a trade, see e.g. [7]), market elasticity (i.e. how rapidly a market regenerates the liquidity removed by a trade) and time-to-liquidate a position. In this part we focus on corporate bonds: in the literature it has been observed a large evidence of a component in corporate bond spread due to illiquidity in addition to credit spreads, see e.g. [14, 29].

Traditional liquidity measures have been developed for the equity market within Market Impact Models (see e.g. [7,16,27] and references therein) with a particular focus on the stocks with larger capitalization; execution typically takes place in timeframes of minutes to hours, only in some cases can have horizons of few days. The theoretical framework is the one described by [1] who assume - for the asset of interest - the presence of a marginal supply-demand curve and the knowledge of its dynamics. However these liquidity measures are not applicable to securities that do not trade on regular basis as illiquid corporate bonds. In this case a complete representation of asset liquidity could be meaningless for several reasons as i) the market is still largely OTC and bid-ask quotes are not available for many corporate bonds[1] ii) time-to-liquidate a position can be of some weeks and even of some months in some cases[2] and iii) trading costs often decrease with trade size, see e.g. [15] Moreover the bond market can be very differentiated even for the same issuing institution: some bonds can be very illiquid while some others, even with similar characteristics (e.g. the same time to maturity), are trading every day, with trading activity far from being uniform over time but mostly concentrated on recently issued bonds ('on-the-run' issues).

In view of the above situation observed in the corporate bond market, it is clear the reason why it is crucial to have a model price for illiquid coupon bonds that takes into account a precise measure of illiquidity. It can be extremely useful for investors who could rely on it i) to decide how much to pay for an illiquid security compared to a liquid security for which the price can be found in the market and ii) to determine the value of the bonds they either own or receive as collateral.

In this chapter we simplify significantly the problem for corporate coupon bonds addressing just one single aspect of market liquidity - the time-to-liquidate a given position (hereinafter ttl)- and we propose a closed formula for the liquidity component of corporate bond spreads defined as the difference in bond yields between a bond with limited liquidity and a very liquid bond of the same issuer. Several are the advantages with respect to the existing techniques (see e.g. [2] and references therein): liquidity is considered an intrinsic characteristic of each single issue, it can vary over time and it depends from the size. Liquidity is expressed in terms of a price discount (or equivalently in terms of a liquidity spread) as a simple closed formula of a single one-dimensional parameter, the ttl, the time lag that - at a given value date and for a given size - an experienced trader needs to liquidate the position.

This problem reminds the celebrated work of [28] on non-marketability of some non-dividend-paying shares in IPOs. More recently [25] tackle a similar problem in the case of a risk-free zero-coupon (ZC) bond with maturity T. Unfortunately in the most frequent situations either coupon payments or credit risk are present and it is not straightforward to extend [25] methodology to these cases. In this chapter we propose an alternative modeling

[1]Practitioners well know that publicly disclosed quotes are often not true commitments to trade at that price but rather just indications (i.e. 'indicative' quotes).

[2]In the corporate bond market a difference of some orders of magnitude with respect to large cap stocks is observed: "a typical US large cap stock, say Apple as of November 2007, had a daily turnover of around 8bn USD" with an "average of 6 transactions per second and on the order of 100 events per second affecting the order book", see [7] pag.76.

approach that allows this extension.

We consider an application in the European market, where the problem of pricing illiquidity is even more relevant than in the U.S. market, due to regulatory differences. European companies had the equivalent of about €8.4 trillion of bonds outstanding in various currencies in May 2014, up from €6.3 trillion at the beginning of 2008 [17]. Even if European bond market is almost as large as the U.S. one, unfortunately the former is more opaque than the latter. In U.S.A. starting from the 1^{st} of July 2002 information on the prices and the volumes of completed transactions was publicly disclosed for a significant set of corporate bonds. The National Association of Security Dealers (NASD, and after July 2007 the Financial Industry Regulatory Authority, FINRA) mandated transparency in the corporate bond market through the Trade Reporting and Compliance Engine (TRACE) program; under TRACE, all trades for corporate bonds in USD must be reported within 15 minutes of execution, see e.g. [5, 14] and references therein. Also the European Union is seeking to make the credit market more transparent by publicly disclosing bond-trading prices. After several years of haggling between policy makers, European Parliament approved an update of its Markets in Financial Instruments Directive (also known as “MiFID II") in April 2014, and then the European Securities and Markets Authority was given the task of defining the new terms and rules, proposing changes to the European Commission by December 2014 [17]. Unfortunately up to now no transparent public information is available on bonds’ prices in Europe, mainly because large dealers are concerned they will suffer the same fate observed in the U.S. market after the introduction of the TRACE reporting scheme, with slumps in their fixed-income revenues and declines in profits, see e.g. [5]. Moreover in Europe, it is relatively frequent to observe private placements to institutional investors, where a single issue is detained by a very limited pool of bondholders, and, especially in the financial sector, there are several bonds with small issue size aimed either at retail investors or at private-banking clients of a banking institution. Often no market price is available in these cases.

For these reasons calibration of a liquidity model, especially in Europe, is often a challenging task. The proposed formula, besides bond characteristics (maturity, coupon, sinking features, etc...), depends on standard market quantities as i) the observed risk-free interest curve ii) issuer’s credit spread term-structure and iii) bond volatility. We show in detail model calibration at a given value date t_0 for two European issuers in the financial sector and the liquidity spread curves that are obtained for different values of the time-to-liquidate.

The contributions of this chapter to the existing literature on illiquid coupon bonds are threefold. First it provides a simple closed formula for illiquid corporate coupon bonds that relates the time-to-liquidate a position to the price difference with respect of the corresponding liquid bond. Second it clarifies, via a detailed calibration on some examples in the European market, the relative importance of model parameters in liquidity spread like volatility and time-to-maturity compared to some others as credit spreads and bonds characteristics (e.g. coupons, payments dates). Third it allows to quantify the liquidity impact in terms of prices for corporate debt of a not sufficiently transparent market and it suggest some policy implications: this study highlights the importance to implement a post-trade transparency in Europe similar to TRACE in U.S.A. where also information dissemination is extended to all corporate bonds (while in the TRACE case some restrictions are still present). Having a transparent market information on both liquid and less liquid bonds with similar characteristics would allow a complete quantification of liquidity impact on corporate prices.

The remaining of the chapter is organized as follows. In section 4.2 we briefly describe the model set up and the liquidity problem formulation. In section 3 we deduce the closed formula and in section 4 we show in detail how to calibrate model parameters on real market data for two European bond issuers. In section 5 we state some concluding remarks.

4.2 The model

The modeling framework includes two main sets of financial ingredients: we should i) specify the dynamics for corporate bonds introducing the interest rate and the credit component dynamics and ii) describe how illiquidity affects corporate bond prices. Our aim is to consider a model set up as parsimonious as possible due to the presence of not abundant accurate data sources, as discussed in the introduction.

The next subsection is devoted to describe corporate bonds' dynamics, while in the following we focus on the illiquidity modeling.

4.2.1 The model set up

We model interest rates and credits according to a simplified version of the model in [35]. Under the usual hypotheses, we consider the background filtration $(\mathcal{G}_t)_{(t\geq 0)}$ generated by a d-dimensional Brownian Motion W_t, with $dW_t^{(j)}\, dW_t^{(l)} = \rho_{jl}\, dt$ for $j, l = 1, \cdots, d$ and $\rho \in \Re^{d\times d}$ the instantaneous correlation matrix. The risk-free interest rate r_t and the intensity λ_t are processes adapted to $(\mathcal{G}_t)_{(t\geq 0)}$. Default for an obligor C is modeled via a Cox process $\mathcal{N}_t$ with intensity λ_t i.e., conditional on the background filtration $(\mathcal{G}_t)_{(t\geq 0)}$, $\mathcal{N}_t$ is an inhomogeneous Poisson process with intensity λ_t. The quantity $d\mathcal{N}_t$ indicates the number of jumps between t^- and t; it is equal to 1 if a jump occurs and zero otherwise. We define $\left(\mathcal{F}_t^{\mathcal{N}}\right)_{(t\geq 0)}$ the filtration generated by $\mathcal{N}_t$, see e.g. [36]. The full filtration is obtained by combining this one and the background filtration:

$$(\mathcal{F}_t)_{(t\geq 0)} = (\mathcal{G}_t)_{(t\geq 0)} \vee \left(\mathcal{F}_t^{\mathcal{N}}\right)_{(t\geq 0)} \qquad . \tag{4.1}$$

Market practitioners model corporate bond spread via Zeta-spreads; this is equivalent - from a modeling perspective - to consider zero recovery and to state that default probability models the whole credit risk for the obligor C. We consider a dynamic version of a Zeta-spread modeling where default follows a Cox process. This assumption is not limiting: it is straightforward to generalize the results in order to include a finite recovery as considering a Fractional Recovery model adding one additional parameter, see e.g. [36].

In this study we focus on fixed rate bonds that are not callable, puttable or convertible. A corporate coupon bond of the obligor C at value date $t_0 \geq 0$ is

$$\overline{P}(t_0, T; \mathbf{c}, \mathbf{t}) := \sum_{i=1}^{N} c_i \overline{B}(t_0, t_i) \tag{4.2}$$

where the defaultable ZC (with zero recovery) $\overline{B}(t_0, T)$ is related to default time t_d and stochastic discount $\mathfrak{D}(t_0, T) := \exp - \int_{t_0}^{T} r_s ds$ via

$$\overline{B}(t_0, T) := \mathbb{E}\left[\mathfrak{D}(t_0, T) \mathbf{1}_{t_d > T} | \mathcal{F}_{t_0}\right] \qquad . \tag{4.3}$$

In corporate coupon bond definition (4.2), price depends on the set of flows $\mathbf{c} := \{c_i\}_{i=1,\cdots,N}$ and the set of payment dates $\mathbf{t} := \{t_i\}_{i=1,\cdots,N}$. The i^{th} payment c_i at time t_i for $i < N$ is the coupon payment with the corresponding daycount, while the last payment at $t_N = T$ has bond face value added to the coupon payment. Corporate coupon bond $\overline{P}$ always indicates invoice (or dirty) prices as in standard fixed income modeling.

Proposition 4.2.1. *The following expressions are equivalent to a defaultable ZC $\overline{B}(t_0, T)$ defined in (4.3)*

i) $\overline{B}(t_0, T) = \mathbb{E}\left[\overline{\mathfrak{D}}(t_0, T) | \mathcal{G}_{t_0}\right]$

ii) $\overline{B}(t_0, T) = \mathbb{E}\left[\mathfrak{D}(t_0, \tau) \mathbf{1}_{t_d > \tau} \overline{B}(\tau, T) | \mathcal{F}_{t_0}\right] = \mathbb{E}\left[\overline{\mathfrak{D}}(t_0, \tau) \overline{B}(\tau, T) | \mathcal{G}_{t_0}\right] \quad \forall \tau \text{ s.t. } t_0 \leq \tau \leq T$

where $\overline{\mathfrak{D}}(t_0, T) := \exp\left(-\int_{t_0}^{T} (r_s + \lambda_s) ds\right)$ *is called defaultable stochastic discount.*

Proof. See Appendix C. □

Definition 4.2.1. *The forward defaultable ZC bond $\overline{B}(t; \tau, T)$ with $t \leq \tau \leq T$ is defined as the price i) established in t, ii) that should be paid in τ if the obligor C has not defaulted up to time τ, iii) in order to get 1 in T if the obligor C has not defaulted up to time T (and zero otherwise)*

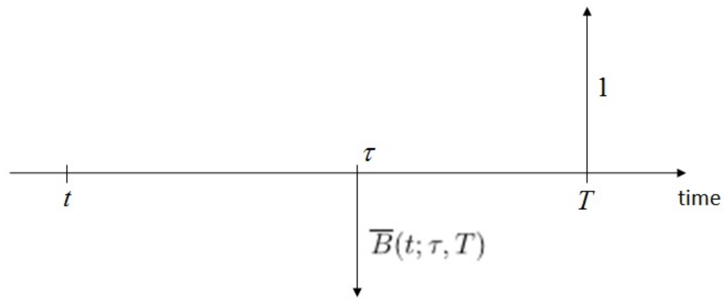

Figure 4.1: *We show the flows that characterize a forward defaultable ZC bond paid in τ if no default event occurs up to τ, whose price is established at time t. The contract gives the right to receive 1 if no default event occurs up to T.*

Proposition 4.2.2. *The forward defaultable ZC bond is related to defaultable ZC via*

$$\overline{B}(t;\tau,T) = \frac{\overline{B}(t,T)}{\overline{B}(t,\tau)} \quad . \tag{4.4}$$

Proof. See Appendix C □

From def.4.2.1 of forward defaultable ZC we observe that $\overline{B}(\tau;\tau,T) = \overline{B}(\tau,T)$ and the forward defaultable bond price tends to the default bond price as time t tends to τ. We indicate with $\overline{P}(t;\tau,T;\mathbf{c},\mathbf{t})$ the forward defaultable coupon bond corresponding to (4.2); clearly in the forward $\overline{P}(t;\tau,T;\mathbf{c},\mathbf{t})$ only coupons with payment date $t_i > \tau$ appear.

Remark 1. A defaultable ZC with Fractional Recovery (FR) is the price of a defaultable ZC where, if a default occurs in t_d, then the value of the defaultable asset is $1-q$ times its pre-default value, with $0 < q < 1$ i.e.

$$\overline{B}_{FR}(t_d,T) = (1-q)\overline{B}_{FR}(t_d-,T) \quad .$$

It is useful to remind that a defaultable ZC with zero recovery can be seen as a particular case of a ZC with FR when q tends to 1 from below, see e.g. [36]. In some cases it is simpler to use this modeling perspective for a generic q and then considering the limit for q close to 1.

Assumption 1. The following dynamics under the risk-neutral measure are assumed for the risk-free ZC and the defaultable ZC for every $t \in (t_0, T]$

$$\left\{ \begin{array}{lcl} \dfrac{dB(t,T)}{B(t,T)} & := & r_t\, dt + \sigma(t,T) \cdot dW_t \\ \dfrac{d\overline{B}(t,T)}{\overline{B}(t^-,T)} & := & (r_t + q\lambda_t)dt + \overline{\sigma}(t,T) \cdot dW_t - q\, d\mathcal{N}_t \end{array} \right. \tag{4.5}$$

with $B(t_0,T)$ and $\overline{B}(t_0,T)$ their initial conditions at value date t_0; the instantaneous rate r_t and the intensity λ_t satisfy

$$\left\{ \begin{array}{rcl} r_t & := & -\dfrac{\partial \ln B(t_0,t)}{\partial t} + \dfrac{1}{2}\displaystyle\int_{t_0}^{t} \dfrac{\partial}{\partial t}\left[\sigma(t',t) \cdot \rho\sigma(t',t)\right] dt' - \displaystyle\int_{t_0}^{t} \dfrac{\partial}{\partial t}\sigma(t',t) \cdot dW(t') \\ r_t + q\lambda_t & := & -\dfrac{\partial \ln \overline{B}(t_0,t)}{\partial t} + \dfrac{1}{2}\displaystyle\int_{t_0}^{t} \dfrac{\partial}{\partial t}\left[\overline{\sigma}(t',t) \cdot \rho\overline{\sigma}(t',t)\right] dt' - \displaystyle\int_{t_0}^{t} \dfrac{\partial}{\partial t}\overline{\sigma}(t',t) \cdot dW(t') \end{array} \right. \tag{4.6}$$

where we consider the case with $q = 1^-$. The volatilities $\sigma(t,T)$ and $\overline{\sigma}(t,T)$ are d-dimensional vectors of deterministic functions of time with $\sigma(T,T) = \overline{\sigma}(T,T) = \mathbf{0} \in \Re^d$ and $x \cdot y$ indicates the scalar product between two vectors $x, y \in \Re^d$.[3] $\diamondsuit$

The above model is a generalization of [19] model to the defaultable case (see e.g. [36]) and it is named Defaultable (multifactor) HJM model (hereinafter DHJM). Considering a realization of the processes between t_0 and t and using the Generalized Itô lemma (see e.g. Appendix C), we get that the value of ZC in t starting from the initial condition is

$$\overline{B}(t,T) = \overline{B}(t_0,T)\ (1-q)^{\mathcal{N}_t}\ \exp\left\{ \int_{t_0}^{t} \left[r_s + q\lambda_s - \frac{1}{2}\overline{\sigma}^2(s,T) \right] ds + \int_{t_0}^{t} \overline{\sigma}(t,T) \cdot dW_s \right\}$$

where we have defined

$$\overline{\sigma}^2(t,T) := \overline{\sigma}(t,T) \cdot \rho\overline{\sigma}(t,T) \qquad .$$

With the above assumption it is possible to model exactly the features of a default: the default of $\overline{B}(t,T)$ occurs when the Poisson process jumps and the jump size is

$$\Delta\overline{B}(t,T) = \overline{B}(t,T) - \overline{B}(t^-,T) = -\overline{B}(t^-,T)\, q\, d\mathcal{N}_t$$

i.e. the ZC loses a fraction q of its pre-default value; in particular the case with $q = 1^-$ describes the zero-recovery model.

[3] It is straightforward to generalize these results considering the volatilities $\overline{\sigma}(t,T)$ and $\sigma(t,T)$ as vectors of adapted processes to the filtration $\mathcal{G}$: the following Lemma 1 and 2 hold even in this case. However the assumption of deterministic volatilities is adequate for the liquidity model we describe in the next subsection.

Lemma 4.2.1. *Given $\overline{B}(t,T)$ following (4.5), the dynamics for the forward defaultable ZC is*

$$\frac{d\overline{B}(t;\tau,T)}{\overline{B}(t^-;\tau,T)} = \frac{d\overline{B}(t;\tau,T)}{\overline{B}(t;\tau,T)} = [\overline{\sigma}(t,T) - \overline{\sigma}(t,\tau)] \cdot [dW_t + \rho\,\overline{\sigma}(t,\tau)\,dt] \tag{4.7}$$

or equivalently $\forall\, t$ *s.t.* $t_0 \le t \le \tau$

$$\overline{B}(t;\tau,T) = \overline{B}(t_0;\tau,T)\exp\left\{-\frac{1}{2}\int_{t_0}^{t}\left[\overline{\sigma}^2(s,T) - \overline{\sigma}^2(s,\tau)\right]ds + \int_{t_0}^{t}[\overline{\sigma}(s,T) - \overline{\sigma}(s,\tau)]\cdot dW_s\right\} \tag{4.8}$$

Proof. See Appendix C. □

The above lemma states that the dynamics of the forward defaultable ZC bond is, *mutatis mutandis*, the same of the corresponding dynamics for a risk-free ZC (see e.g. [32]) and it is continuous. This result could seem paradoxical at a first glance, however being the forward ZC bond a ratio of two defaultable ZC bond with different maturities of the same issuer (as shown in prop.4.2.1), in case of default before τ both terms are reduced by a fraction q of their pre-default value generating no jump in the value of the ratio, whatever is the FR model considered.

Lemma 4.2.2. *Given equation (4.6), the defaultable stochastic discount between* t *and* $\tau \ge t$ $\overline{\mathfrak{D}}_q(t,\tau) := \exp\left(-\int_t^\tau (r_s + q\,\lambda_s)ds\right)$ *is related to the corresponding defaultable ZC via the relation*

$$\overline{\mathfrak{D}}_q(t,\tau) = \overline{B}(t,\tau)\exp\left\{-\frac{1}{2}\int_t^\tau \overline{\sigma}^2(s,\tau)ds + \int_t^\tau \overline{\sigma}(s,\tau)\cdot dW_s\right\}$$

and in particular the relation holds for $\overline{\mathfrak{D}}(t,\tau)$ *in the limit* $q \nearrow 1$.

Proof. See Appendix C. □

These two lemmas correspond to the two standard properties of HJM models (see e.g. lemma 1 and lemma 2 in [20] for the one factor Gaussian HJM case and references therein); these properties hold also in the defaultable bonds' case described by **assumption 1**. Hereinafter we consider uniquely lemma 4.2.1 and 4.2.2 in the limit $q \nearrow 1$ that corresponds to the Zero recovery model we are interested in.

A consequence of these two lemmas is that it is possible to introduce a τ-defaultable-forward measure (hereinafter also $\overline{\tau}$-forward measure), s.t. the process

$$W_t^{(\overline{\tau})} := W_t + \int_{t_0}^{t}\rho\,\overline{\sigma}(s,\tau)\,ds$$

is a d-dimensional Brownian Motion under the new forward measure. We indicate with $\mathbb{E}^{(\overline{\tau})}[\bullet]$ the expectation under the $\overline{\tau}$-forward measure. In the $\overline{\tau}$-forward measure $\overline{B}(t;\tau,T)$ is martingale and the dynamics for the forward defaultable ZC has a particularly simple form

$$d\overline{B}(t;\tau,T) = \overline{B}(t;\tau,T)\, v(t;\tau,T) \cdot dW_t^{(\overline{\tau})} \tag{4.9}$$

with $v(t;\tau,T) := \overline{\sigma}(t,T) - \overline{\sigma}(t,\tau)$.

In the next subsection we describe the modeling framework on how illiquidity affects corporate coupon bonds (4.2).

4.2.2 Problem formulation

Let us consider a hypothetical investor in t_0 who holds an illiquid corporate bond (4.2), i.e. he needs some time in order to liquidate a position with a given size. We assume that this investor is an experienced trader with a complete information on that particular corporate market segment (e.g. he knows all features on bonds of that issuer and all potential clients that could be interested in buying the bond he holds) at value date t_0 and he is able to sell a position of given size on the illiquid bond after a time-to-liquidate $(\tau - t_0)$ at the same price of a liquid bond with the same characteristics (issuer, coupons, payment dates).

This problem reminds the celebrated work of [28], where the author compares the value in t_0 of an illiquid security and of a liquid one with equal future cash flows after $(\tau - t_0)$. An additional feature characterizes the hypothetical investor in [28]: he is able to sell the liquid security "with perfect timing" during $[t_0, \tau]$. The additional value of the liquid security over the illiquid one is calculated by regarding the optimal strategy of this hypothetical investor. As an example [28] focuses his attention on a non-dividend paying stock; in this chapter we consider a coupon bearing bond that could have several payment dates (even before τ) with different flows over time. Also for this reason when dealing with fixed income securities, in order to compare two assets with the same future cash flows in τ it is better to consider the corresponding forward security. This requirement is equivalent to the prescription in the paper of Longstaff (see e.g. eq.(2) in [28]) for a non-dividend-paying security with deterministic interest rates and in absence of default. The selling price for this hypothetical investor, able to sale with optimal timing the forward coupon bond, is

$$M_\tau := \max_{t_0 \le t \le \tau} \overline{P}(t;\tau,T;\mathbf{c},\mathbf{t}) \ \ ;$$

this price is paid at time τ as in the forward defaultable bond case. Longstaff idea is very intuitive: the main limitation of holding an illiquid bond, compared with a comparable issue of the same corporate entity, is related to the impossibility for a while to sell the bond and convert its value into cash. The time-to-liquidate is the main exogenous model parameter: it models the liquidity restriction as an opportunity cost for this hypothetical investor.

More recently Longstaff results have been extended by [25] who tackle a similar problem in the case of a risk free zero coupon bond with maturity T. The authors consider the illiquidity premium in the case of a risk-free short rate which follows a [41] model. The ZC can be traded at a given set of dates, established at value date t_0; illiquidity price in their study is obtained via a (numerically intense) Monte Carlo technique. Unfortunately, as already mentioned in the introduction, [25] approach could not be extended easily to the case of interest; in this chapter we consider coupon bond prices in presence of credit risk via a simple closed formula.

Assumption 2. The illiquidity price Δ_τ is defined as the value in t_0 of the difference $M_\tau - \overline{P}(\tau, T; \mathbf{c}, \mathbf{t})$. Its present value equals

$$\Delta_\tau := \mathbb{E}\left[\mathfrak{D}(t_0,\tau)\mathbf{1}_{t_d>\tau}M_\tau|\mathcal{F}_{t_0}\right] - \mathbb{E}\left[\mathfrak{D}(t_0,\tau)\mathbf{1}_{t_d>\tau}\overline{P}(\tau,T;\mathbf{c},\mathbf{t})|\mathcal{F}_{t_0}\right] \tag{4.10}$$

$\diamondsuit$

Let us briefly comment Assumption 2. When considering the expected value at value date of future cash flows according to Longstaff criteria, we have to include the indicator function $\mathbf{1}_{t_d>\tau}$, i.e. uniquely a time-to-default t_d larger than τ gives a contribution to the illiquidity price Δ_τ in case of zero recovery. In fact, it should be considered in the left term of the difference in (4.10) because all forward ZC bonds are paid only if $t_d > \tau$ and in the right term in (4.10), due to the statement ii) in prop.4.2.2 that holds for each defaultable ZC that appear in coupon bond price $\overline{P}$ defined in (4.2).

Remark 2. The above definition does not consider hypothetical coupon payments between the value date t_0 and τ. We remind that the time-to-liquidate is, even in the most illiquid cases, of few months, and then at most one coupon payment could be present in the time interval (t_0, τ). In practice corporate bond traders consider that payment, i.e. within a short lag in the future, equivalent to cash. We assume that the first coupon, if paid before τ, maintains its credit risk but it gives the same contribution to both the liquid and illiquid coupon bonds.[4]

Lemma 4.2.3. *Within the DHJM model of Assumption 1, the price of illiquidity is equal to*

$$\begin{aligned}\Delta_\tau &= \mathbb{E}\left[\overline{\mathfrak{D}}(t_0,\tau)M_\tau|\mathcal{G}_{t_0}\right] - \mathbb{E}\left[\overline{\mathfrak{D}}(t_0,\tau)\overline{P}(\tau,T;\mathbf{c},\mathbf{t})|\mathcal{G}_{t_0}\right] = \\ &= \overline{B}(t_0,\tau)\left\{\mathbb{E}^{(\overline{\tau})}\left[M_\tau|\mathcal{G}_{t_0}\right] - \mathbb{E}^{(\overline{\tau})}\left[\overline{P}(\tau,T;\mathbf{c},\mathbf{t})|\mathcal{G}_{t_0}\right]\right\} \quad .\end{aligned} \tag{4.11}$$

Proof. See Appendix C. $\square$

[4] It could be possible to include even this coupon in the model description, complicating remarkably the notation, without adding a significant contribution to the final result.

Remark 3. Let us observe that in the price of illiquidity Δ_τ all quantities of interest do not depend separately from r_t and λ_t, but depend only from their combination

$$\overline{r}_t := r_t + \lambda_t \qquad .$$

The above properties hold for all DHJM model is selected (i.e. whatever $\sigma(t,T)$ and $\overline{\sigma}(t,T)$ are chosen) for the dynamics (4.5) of the risk-free ZC curve $B(t,T)$ and the defaultable ZC curve $\overline{B}(t,T)$. As discussed in the introduction the main driver for model selection is parsimony when dealing with illiquid corporate bonds, due to the poor data set and model calibration issues. One of the simplest model within this set was proposed by [35] where both r_t and λ_t follow two correlated 1-dimensional [21] models

$$\begin{cases} r_t &= \varphi_t + x_t^{(1)} \\ \lambda_t &= \psi_t + x_t^{(2)} \end{cases}$$

where $x_t^{(1)}$ and $x_t^{(2)}$ are two correlated Ornstein-Uhlenbeck (OU) processes with zero mean and zero initial value. This model has the main advantage to allow an elementary separate calibration on the zero-rates (via φ_t) and the Zeta-spread (via ψ_t). However the above remark suggests to consider an even simpler model as stated in the following Assumption.

Assumption 3. We model the rate $\overline{r}_t$ as a Hull-White (HW) model

$$\overline{r}_t = \varphi_t + \psi_t + x_t$$

with φ_t, ψ_t two deterministic functions of time and x_t an Ornstein-Uhlenbeck (OU) process with zero mean and initial value

$$\begin{cases} dx_t &= -\hat{a}\, x_t\, dt + \hat{\sigma}\, dW_t \\ x_{t_0} &= 0 \end{cases}$$

where $\hat{a}, \hat{\sigma}$ are two positive constant parameters.[5] $\diamondsuit$

This assumption is in line with day-by-day practice: as we discuss in section 4, in the market place, generally one cannot observe derivative instruments that allow to calibrate separately the volatility of the risk-free curve and the volatility of the credit spread and there is not enough information to discriminate the two dynamics. Conversely, the two initial curves (risk-free and defaultable) can be easily calibrated separately on market data and the integrals of φ_t, ψ_t between t_0 and a given maturity T are related to these two curves up to T. We do not report these relations here because we provide final formulas in terms of $B(t_0,T)$ and $\overline{B}(t_0,T)$.

[5] Also a non-constant $\hat{\sigma}(t)$, chosen as a deterministic function of time, allows to replicate all the results in this study. Once again it is preferred the more parsimonious choice.

Remark 4. The HW model is a one factor DHJM with volatility

$$\overline{\sigma}(t,T) = \frac{\hat{\sigma}}{\hat{a}}\left(1 - e^{-\hat{a}(T-t)}\right) \in \Re \qquad t \leq T \tag{4.12}$$

and then the volatility $v(t;\tau,t_i)$ is a separable function in the times t and t_i, i.e.

$$v(t;\tau,t_i) = \xi_i\,\nu(t) \tag{4.13}$$

with $\xi_i = (\hat{\sigma}/\hat{a})\left[1 - e^{-\hat{a}(t_i-\tau)}\right]$ and $\nu(t) = e^{-\hat{a}(\tau-t)}$ with $t_0 \leq t \leq \tau \leq t_i$, see e.g [32].

In the next section we show that, within Assumption 1, 2 and 3, it is possible to compute the price of illiquidity Δ_τ via a closed formula and it is possible to associate a liquidity spread as a component of corporate bond spread in addition to credit spread, as observed in econometric studies, see e.g. [14, 29].

4.3 A closed formula for illiquid corporate coupon bonds

In this section we show the main result of this part: the illiquidity price Δ_τ of definition (4.10) can be evaluated directly via a simple closed-form solution.

This result is far from being obvious. A defaultable forward coupon bond $\overline{P}(t;\tau,T;\mathbf{c},\mathbf{t})$ is the sum of forward defaultable ZCs $\{\overline{B}(t;\tau,t_i)\}_{i=1,\cdots,N}$, each one following the dynamics (4.7) and then described as a Geometric Brownian Motion (GBM) process (4.8). No known closed formula exists for the running maximum of a sum of GBMs.

In order to get the closed formula we proceed taking the following steps. First we prove that a lower and an upper bound of (4.10) can be computed via closed formulas. Then we show, calibrating model parameters for two European issuers, that the difference between upper and lower bounds is negligible for all practical purposes. We can then use one of the two bounds as the closed-form solution we are looking for; in this section we prove the existence of these bounds while in the next section we show the tightness of their difference.

Lemma 4.3.1. *The following inequalities hold:*

$$\sum_i c_i\overline{B}(t^*;\tau,t_i) \leq \max_{t\in[t_0,\tau]}\left\{\sum_i c_i\overline{B}(t;\tau,t_i)\right\} \leq \sum_i c_i \max_{t\in[t_0,\tau]}\overline{B}(t;\tau,t_i) \;\; \forall t^* \in [t_0,\tau],\; t_i \geq \tau$$

where the sum over i is limited to all coupons with payment date t_i larger than τ.

Proof. The left inequality is obvious since the maximum value of a function on the time interval $[t_0,\tau]$ is greater than the same function valued in any other time t^* in the interval. The right inequality is due to the fact that the maximum of a sum is lower or equal to the sum of maxima. □

In particular we can choose t^* equal to the time-location

$$t^* = \min \left\{ t' \;\middle|\; \overline{B}(t';\tau,t_N) = \max_{t\in[t_0,\tau]} \overline{B}(t;\tau,t_N) \right\} \quad .$$

The idea is that a coupond bond (4.2) is the sum of ZCs who have different weights c_i with the last one c_N (that contains the face value) generally two orders of magnitude larger than the others. In most cases the forward coupon bond reaches the maximum when the N^{th} ZC reaches its maximum.

Furthermore, as already observed in **remark 4**, all ZCs follow a GBM driven by the same Wiener process with a deterministic volatility that differs only for a multiplicative constant ξ_i but it has exactly the same time dependency $\nu(t)$ for all $i = 1, \ldots, N$ s.t. $t_i > \tau$

$$\overline{B}(t;\tau,t_i) = \overline{B}(t_0;\tau,t_i) \exp\left[\xi_i \left(-\frac{\xi_i}{2} \int_{t_0}^{t} \nu^2(s)\, ds + \int_{t_0}^{t} \nu(s)\, dW^{(\overline{\tau})}(s) \right)\right]$$

and then when computing the maximum a different time-location can only be due to the drift term $-\xi_i/2 \int_{t_0}^{t} \nu^2(s)\, ds$.

For these reasons it is quite rare that the time-location when the coupon bond reaches the maximum is different from the time when the last ZC $\overline{B}(t;\tau,t_N)$ reaches its maximum; the contribution to the expected value in (4.10) of these cases is negligible.

In the next theorem we show that the expected values of these lower and upper bounds have simple form; in section 4 that they can be considered equal for all practical purposes.

Theorem 4.3.1. *Lower and upper bounds for the illiquidity price (4.10) are:*

$$\sum_{i=1}^{N} c_i \overline{B}(t_0,t_i) \left(\pi_i^L(\tau) - 1\right) \le \Delta_\tau \le \sum_{i=1}^{N} c_i \overline{B}(t_0,t_i) \left(\pi_i^U(\tau) - 1\right)$$

with

$$\begin{aligned}
\pi_i^U(\tau) &:= \frac{4 + \Sigma_i^2(\tau)}{2}\, \Phi\left(\frac{\Sigma_i(\tau)}{2}\right) + \frac{\Sigma_i(\tau)}{\sqrt{2\pi}}\, \exp\left(-\frac{\Sigma_i^2(\tau)}{8}\right) \\
\pi_i^L(\tau) &:= \int_0^1 d\eta \frac{e^{-\frac{1}{8}\Sigma_N^2(\tau)}}{\pi\sqrt{1-\eta}\sqrt{\eta}}\, e^{-\frac{\eta}{2}\Sigma_i(\tau)\,(\Sigma_i(\tau)-\Sigma_N(\tau))} \\
&\quad \left\{ 1 + \sqrt{\frac{\pi\,(1-\eta)}{2}}\, \Sigma_N(\tau)\, e^{\frac{1-\eta}{8}\Sigma_N^2(\tau)}\, \Phi\left[\frac{\sqrt{1-\eta}}{2}\, \Sigma_N(\tau)\right] \right\} \\
&\quad \left\{ 1 + \sqrt{\frac{\pi\,\eta}{2}}\, (2\,\Sigma_i(\tau) - \Sigma_N(\tau))\, e^{\frac{\eta}{8}(2\,\Sigma_i(\tau)-\Sigma_N(\tau))^2}\, \Phi\left[\frac{\sqrt{\eta}}{2}\, (2\,\Sigma_i(\tau) - \Sigma_N(\tau))\right] \right\}
\end{aligned}$$

(4.14)

if $t_i > \tau$ *and* $\pi_i^U(\tau) = \pi_i^L(\tau) = 1$ *otherwise. The cumulate volatility is*

$$\Sigma_i^2(\tau) := \int_{t_0}^{\tau} v^2(s;\tau,t_i)\,ds \quad .$$

Proof. See Appendix C □

This theorem is the key result of this part: it indicates a lower and an upper bound for the price of illiquidity Δ_τ. As we show in section 4.4 these bounds are very tight, their difference can be considered negligible for all practical purposes. In practice it can be used indifferently one of the two closed form solutions and in particular the simplest espression among the two bounds, i.e. the upper bound. This fact allows to define in an elementary way a Liquidity basis as done in the next subsection.

4.3.1 Liquidity basis

A consequence of the above theorem and of the tightness of the difference between the two bounds is that the illiquid corporate coupon price is

$$\overline{\overline{P}}_\tau(t_0,T;\mathbf{c},\mathbf{t}) := \overline{P}_\tau(t_0,T;\mathbf{c},\mathbf{t}) - \Delta_\tau = \sum_{i=1}^{N} c_i \overline{B}(t_0,t_i)\,\left(2-\pi_i^U(\tau)\right) \tag{4.15}$$

where $\pi_i^U(\tau)$ in defined in (4.14). We can also define an illiquid ZC as

$$\overline{\overline{B}}_\tau(t_0,t_i) := \overline{B}(t_0,t_i)\,\left(2-\pi_i^U(\tau)\right)$$

and the Liquidity basis

$$L_\tau(t_i) := -\frac{1}{t_i-t_0}\ln\frac{\overline{\overline{B}}_\tau(t_0,t_i)}{\overline{B}(t_0,t_i)} = -\frac{1}{t_i-t_0}\ln\left(2-\pi_i^U(\tau)\right) \quad . \tag{4.16}$$

We can then decompose the illiquid ZC bond in the three components of risk-free discount, credit and liquidity

$$\overline{\overline{B}}_\tau(t_0,T) = \underbrace{e^{-R(T)(T-t_0)}}_{risk-free}\;\underbrace{e^{-Z(T)(T-t_0)}}_{credit}\;\underbrace{e^{-L_\tau(T)(T-t_0)}}_{liquidity}$$

where $R(T)$ is the Zero rate and $Z(T)$ is the Zeta spread. This corresponds to what is done by practitioners in their day-by-day activities: they add a basis related to liquidity to bond's credit spread. The main advantage of the model presented in this study is that,

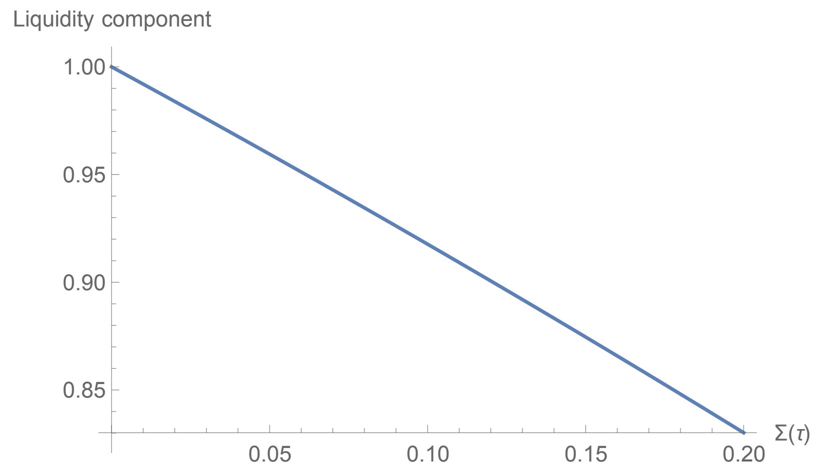

Figure 4.2: *We show the liquidity component $\left(2-\pi_i^U(\tau)\right)$ in an illiquid defaultable ZC bond as a function of the cumulate volatility $\Sigma_i(\tau)$ for $t_i > \tau$.*

given the ttl for the illiquid corporate bond position of interest, it allows to associate a Liquidity basis given some bond characteristics (e.g. coupon payment dates) and the two parameters ($\hat{a}$ and $\hat{\sigma}$) related to the volatility of the corresponding liquid bond.

It is useful to underline that the Liquidity basis depends only from volatility parameters and it is impacted by neither the rates component nor the credit component. The liquidity component in ZC price $\left(2-\pi_i^U(\tau)\right)$ is just a function of the cumulate volatility $\Sigma_i(\tau)$. Its plot is shown in figure 4.2.

In particular modeling $\overline{r}_t$ according to **assumption 3**, we get that the cumulated volatility is

$$\Sigma_i(\tau) = \xi_i \sqrt{\frac{1-e^{-2\hat{a}(\tau-t_0)}}{2\hat{a}}}$$

where ξ_i has been defined in equation (4.13).

4.4 An application to the financial sector in the European bond market

In this section we illustrate the impact of illiquidity applying formula (4.15) to obligations with different maturities issued by two main financial institutions in Europe. We also show that the difference between upper and lower bounds is negligible for all practical purposes.

4.4.1 The dataset

The two European financial institutions in Europe that we consider in this study are BNP Paribas S.A. (hereinafter BNPP) and Banco Santander S.A. (Santander) on the 10^{th} of

September 2015 (value date). The settlement date is the 14^{th} of September 2015.[6] At value date the two issuers have rating A for BNPP and A- for Santander according to S&P.

In order to construct the Zeta-spread curve, i.e. the (liquid) credit component in the spread we consider all senior unsecured benchmark issues (i.e. with issue size larger than €500 millions) with maturity lower or equal 10 years. Coupons are paid annually with Act/Act day-count convention for all bonds in both sets. Closing day mid-prices are reported in tables 4.1 and 4.2.

maturity	coupon (%)	clean price	dirty price
27-Nov-2017	2.875	105.575	107.845
12-Mar-2018	1.500	102.768	103.522
21-Nov-2018	1.375	102.555	103.667
28-Jan-2019	2.000	104.536	105.782
23-Aug-2019	2.500	106.927	107.070
13-Jan-2021	2.250	106.083	107.583
24-Oct-2022	2.875	110.281	112.850
20-May-2024	2.375	106.007	106.779

Table 4.1: *BNPP bond data. Coupons are annual with day-count convention Act/Act. Prices are end-of-day mid prices on the* 10^{th} *of September 2015. We show both clean and dirty prices.*

maturity	coupon (%)	clean price	dirty price
27-Mar-2017	4.000	105.372	107.208
04-Oct-2017	4.125	107.358	111.224
15-Jan-2018	1.750	102.766	103.913
20-Apr-2018	0.625	99.885	100.132
14-Jan-2019	2.000	103.984	105.306
13-Jan-2020	0.875	99.500	100.083
24-Jan-2020	4.000	112.836	115.382
14-Jan-2022	1.125	98.166	98.916
10-Mar-2025	1.125	93.261	93.848

Table 4.2: *Santander bond data. Coupons are annual with day-count convention Act/Act. Prices are end-of-day mid prices at value date.*

The risk-free curve is the OIS curve as market standard; it has been bootstrapped from OIS quoted rates. Their quotes at value date are reported in table 4.3 (with market conventions, i.e. annual payments and Act/360 day-count); in the same table we report also swap rates (annual fixed leg with 30/360 day-count). In table 4.4 we show the FRA rates of interest and the Euribor 6m fixing on the same value date (both with Act/360 day-count).

[6] Settlement date is equal to two business days after value date for both interest rate and credit products in the Euro-zone.

Unfortunately prices on liquid options on BNPP and Santander bonds are not available in the market at value date. We consider a proxy in order to calibrate volatilities' parameters (HW parameters $\hat{a}$ and $\hat{\sigma}$); we notice that at value date both banks are Systemically Important Financial Institutions (SIFI) and belong to the panel of banks contributing to Euribor rate. The dynamics of the spread between Euribor and OIS curve can be considered a good proxy of the dynamics of the average credit spread for financial institutions with the above characteristics. ATM swaptions on Euribor swap rates are very liquid in Europe: we can then use these OTC option contracts at t_0 as a proxy in order to calibrate volatilities' parameters.

We show swaption ATM volatilities in basis points (bps) in table 4.5 and swaption market prices are obtained according to the new market standards (normal model) that allow for negative interest rates. All market data are provided by Bloomberg.

	OIS rate (%)	swap rate *vs* 6m (%)
1w	-0.132	-
2w	-0.132	-
1m	-0.132	-
2m	-0.133	-
3m	-0.136	-
6m	-0.139	-
1y	-0.147	0.044
2y	-0.135	0.080
3y	-0.083	0.154
4y	0.008	0.259
5y	0.122	0.377
6y	0.254	0.512
7y	0.392	0.652
8y	0.529	0.786
9y	0.655	0.909
10y	0.766	1.016
11y	0.866	1.109
12y	0.957	1.195
15y	1.160	1.383

Table 4.3: *OIS rates and swap rates vs Euribor 6m in %: end-of-day mid quotes (annual 30/360 day-count convention for swaps vs 6m, Act/360 day-count for OIS) on the 10^{th} of September 2015.*

4.4.2 Calibration of model parameters

As discussed in section **3**, the closed formula for illiquid bond prices, besides bond characteristics (maturity, payment dates, coupons, sinking features, time-of-liquidate, etc...),

	rate (%)
Euribor 6m	0.038
FRA 1 × 7	0.038
FRA 2 × 8	0.041
FRA 3 × 9	0.043

Table 4.4: *Euribor 6m fixing rate and FRA in % (day-count Act/360). FRA rates are end-of-day mid quotes at value date.*

	2y	3y	4y	5y	6y	7y	8y	9y	10y
1m	15.16	19.22	27.79	34.90	41.05	47.88	54.35	60.07	63.71
2m	15.38	20.11	29.26	36.31	42.92	50.04	55.76	61.69	65.55
3m	14.89	20.26	29.34	37.10	44.06	51.61	57.56	63.37	67.02
6m	19.40	25.59	32.85	40.02	46.81	53.01	58.23	63.36	67.18

Table 4.5: *ATM swaptions normal volatilities in bps with expiry 1m up to 6m and tenor 2y up to 10y on the* 10^{th} *of September 2015.*

includes the observed i) zero-rate curve, ii) credit spread term-structure for the issuer of interest and iii) bond volatility.

These “ingredients" can be calibrated on market data following standard techniques.

Discount curves $B(t_0, T)$ are bootstrapped following the procedure in [4]. For each one of the two issuers, its time-dependent Zeta-spread curve

$$Z(T) := -\frac{1}{T-t_0} \ln \frac{\overline{B}(t_0,T)}{B(t_0,T)}$$

can be bootstrapped from liquid bond dirty prices as in [36]. We consider $Z(t_0) = 0$ and a linear interpolation rule; day-count convention for Zeta-spreads is Act/365 as market standard.

Finally, the volatility parameters (HW parameters $\hat{a}$ and $\hat{\sigma}$) should be calibrated on options on corporate bonds. Unfortunately, prices on liquid options on BNPP and Santander bonds are not available in the market at value date. We consider a proxy in order to calibrate the volatility parameters; we notice that at value date both banks are Systemically Important Financial Institutions (SIFI) and belong to the panel of banks contributing to the Euribor rate. The dynamics of the spread between the Euribor and the OIS curve can be considered a good proxy of the dynamics of the average credit spread for

nancial institutions with the above characteristics. As mentioned in [18], this spread models the risk related to the Euro interbank market, and default risk is one important component of this interbank risk. Let we underline that we use this proxy to calibrate only volatility parameters, while credit spreads are calibrated on issuer liquid bond market.

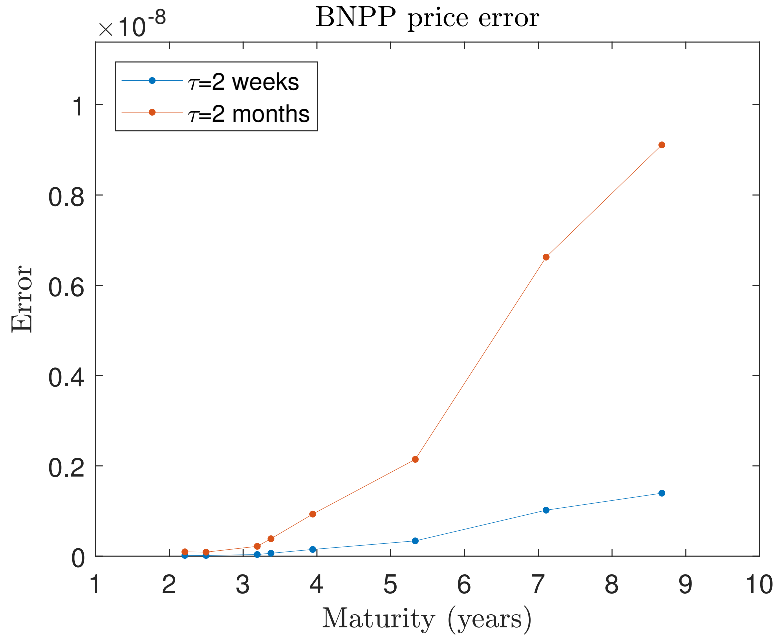

Figure 4.3: *Difference between the upper and lower bounds for the illiquidity price Δ_τ for BNPP bonds. We conside illiquid bonds with the same characteristics (e.g. coupons, payment dates) of the bonds in table 4.1 with ttl equal to 2 weeks (red line) and ttl equal to 2 months (yellow line). This difference is of the order of 10^{-7} times the face value and then negligible for all practical purposes.*

ATM swaptions on Euribor swap rates are very liquid in Europe: we can use these OTC option contracts at t0 as a proxy, in order to calibrate the volatility parameters. Swaption ATM normal volatilities are provided by Bloomberg; their values in t0 and the calibration procedure are reported in [3]. Calibrated values are $\hat{a} = 13,31\%$ and $\hat{\sigma} = 1,27\%$.

4.4.3 Illiquid bond prices

In this section we show that, considering two sets of illiquid bonds (one set for each issuer) with the same characteristics of liquid bonds (e.g. coupons and payments dates) and ttl equal to either 2 weeks or 2 months, the difference between lower and upper bound for the illiquidity price Δ_τ is of the order of 10^{-7} times the face value. In figure 4.3 we show this difference for BNPP and in figure 4.4 for Santander.

Moreover we consider the bond with longest maturity within Santander set (i.e. the one with the largest difference) and in figure 4.5 we plot the difference between the two bounds with values of $\hat{a} \in (0, 10\%)$ and $\hat{\sigma} \in (0, 5\%)$. We observe that this difference is always of the order of 10^{-6} times the face value (or lower). This difference is the maximum error we commit if we value Illiquidity price Δ_τ with one of these bounds: it is negligible for all practical purposes. This fact allows us to consider indifferently the lower or the upper

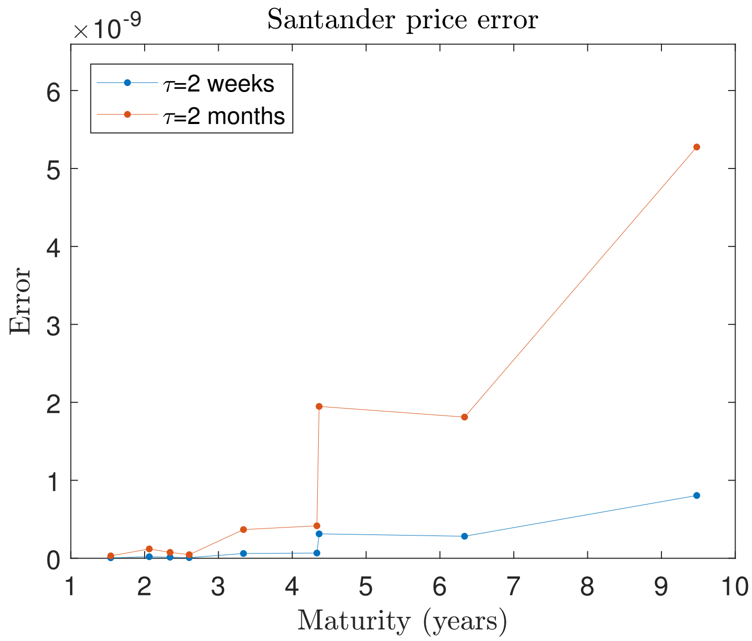

Figure 4.4: *Difference between the upper and lower bounds for the illiquidity price Δ_τ for Santander bonds. We conside illiquid bonds with the same characteristics (e.g. coupons, payment dates) of the bonds in table 4.2 with ttl equal to 2 weeks (red line) and ttl equal to 2 months (yellow line). This difference is of the order of 10^{-7} times the face value.*

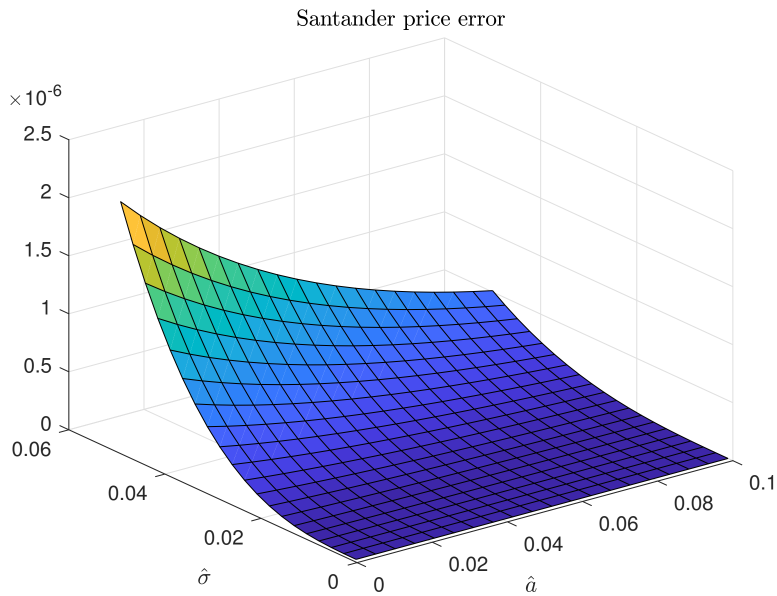

Figure 4.5: *Difference between the upper and lower bounds for the illiquidity price Δ_τ for Santander bond with the longest maturity in table 4.2 varing $\hat{a} \in (0, 10\%)$ and $\hat{\sigma} \in (0, 5\%)$. We conside an illiquid bonds with the same characteristics (e.g. coupons, payment dates) with ttl equal to either 2 weeks or 2 months.*

bound as closed-form solution for Δ_τ.

In section 3 we have shown that a Liquidity basis (4.16) could be added to each ZC in order to take into account liquidity. Practitioners often consider a Liquidity yield spread as the term that should be added to the yield in order to obtain the illiquid bond price (4.16)

$$\overline{\overline{P}}_\tau(t_0, T; \mathbf{c}, \mathbf{t}) =: \sum_{i=1}^{N} c_i \; e^{-[y(T)+\mathfrak{L}_\tau(T)]\,(t_i - t_0)}$$

where $y(T)$ is the yield of the corresponding liquid bond $\overline{P}(t_0, T; \mathbf{c}, \mathbf{t})$.

In the figures 4.6 and 4.7 we show the Liquidity yield spread for BNPP and Santander for different bond maturities and ttl equal to two weeks and two months.

Finally it is useful to observe that Liquidity spreads, obtained with the technique described in this part, are of the same order of magnitude of the ones observed in econometric studies in the U.S. market for bonds of similar maturity (2y up to 10y) and similar ratings. For example Dick-Nielsen et al. report Liquidity spreads post-subprime crisis in the U.S. market between 24.7 bps and 105.4 bps for A rated issuers and between 55.0 bps and 175.1 bps for BBB, see e.g. table B2, panel B in [14].

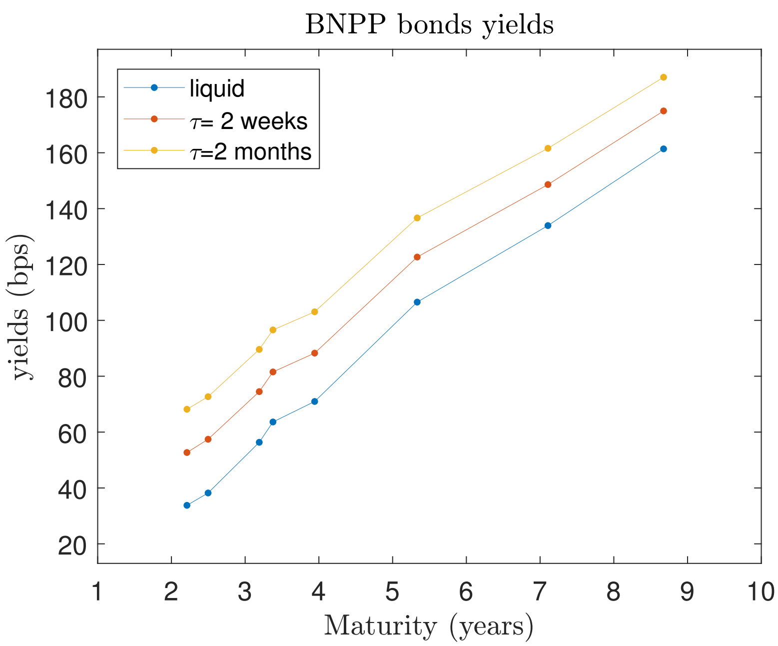

Figure 4.6: *BNPP bond yields. We consider all benchmark issues with maturity lower than 10y described in table 1 and their yields (blue line). We show also the yield obtained for illiquid bonds with the same characteristics (e.g. coupons, payment dates) with ttl equal to* 2 *weeks (red line) and ttl equal to* 2 *months (yellow line).*

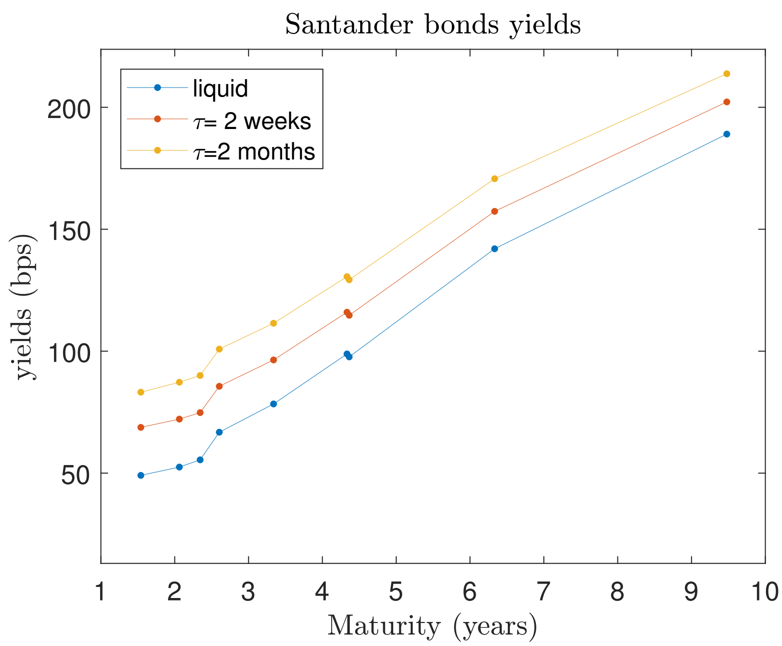

Figure 4.7: *Santander bond yields. We consider all benchmark issues with maturity lower than 10y in table 2 and their yields (blue line). We show also the yield obtained for illiquid bonds with the same characteristics (e.g. coupons, payment dates) with ttl equal to 2 weeks (red line) and ttl equal to 2 months (yellow line).*

4.5 Conclusions

In this chapter we have proposed a closed formula (4.15) for illiquid corporate coupon bonds when the corresponding liquid credit curve can be observed in the market for the same issuer. This formula is obtained bounding from above and below the illiquidity price (10). Calibrating model parameters on market data, we have shown that these two bounds coincide for all practical purposes.

This formula clarifies that illiquidity is an intrinsic component of bond spread and then of bond price. In presence of the liquid credit curve it is possible to detangle the two components of credit and liquidity in the observed spread over risk free rate. In particular we have shown that the Liquidity spread depends mainly from bond volatility and from the time-to-liquidate a given position (via a cumulated volatility).

This closed formula (4.15) is very simple: besides a set of parameters that can be easily calibrated on liquid market data, the model includes just an additional parameter the “time-to-liquidate". It can be used by practitioners for different possible applications; let us mention some of them.

This model can support traders in their day-by-day activities. On the one hand, ttl parameter can be evaluated *ex-ante* by an experienced trader with a deep knowledge of the characteristics of that particular illiquid market (concentration, frequency for similar trades with similar characteristics observed in the recent past) who desires to liquidate a given position; the formula gives a theoretical background to market practice of adding a Liquidity spread to bond yields either when pricing illiquid issues or when receiving them as collateral. On the other hand, the formula can be used also in order to get an “implied time-to-liquidate" from market quotes if both liquid and illiquid prices are available, translating observable spreads into a time lag for liquidating a position and then providing an interesting piece of information to market participants.

Moreover the model can be useful also to risk managers. Ttl can be easily backtested *ex-post* by risk managers, who can measure the average time needed for liquidating a position in an illiquid corporate bond of a given size. It also gives a theoretical background for setting Vega limits on illiquid bonds' positions; the proposed approach clarifies that the cumulated volatility is the key driver of the Liquidity basis.

Finally this study also allows to draw some policy implications for the European bond market. TRACE reports, on a selected set of bonds, information about executed trades (trade time, volume and price) in the US market but it does not reveal quotes. In the bond market the time-to-liquidate is the natural quantity to be estimated from traded volumes per unit time; on the contrary, bid-ask spread requires quotes and, in many situations, bid-ask could be a relevant information only for small size trades.

This study gives some insights into the design of the post-trade transparency process in order to be truly informative and helpful for investors in the European market, since it clarifies which are the key ingredients for an effective transparency: collection should include the same TRACE dataset (trade time, volume and price) but dissemination of these data should be extended to all executed trades in bonds in Euro, not being limited to a selection of them as in the US case. This simple policy rule would have implications not only to liquidity but also to transparency: it can reduce significantly market opaqueness and then the total cost of debt for corporate issuers.

APPENDIX C

Proofs

Proof of Proposition 1. Point i) is a straightforward consequence of the filtration (4.1), see e.g. [36]. Using definition (4.3), well known properties of stochastic discounts and the law of iterated expectations we get

$$\begin{aligned}\overline{B}(t_0,T) :=& \mathbb{E}\left[\mathfrak{D}(t_0,\tau)\mathbf{1}_{t_d>\tau}\mathfrak{D}(\tau,T)\mathbf{1}_{t_d>T}|\mathcal{F}_{t_0}\right] = \mathbb{E}\left[\mathfrak{D}(t_0,\tau)\mathbf{1}_{t_d>\tau}\mathbb{E}\left[\mathfrak{D}(\tau,T)\mathbf{1}_{t_d>T}|\mathcal{F}_\tau\right]|\mathcal{F}_{t_0}\right] = \\ & \mathbb{E}\left[\mathfrak{D}(t_0,\tau)\mathbf{1}_{t_d>\tau}\overline{B}(\tau,T)|\mathcal{F}_{t_0}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathfrak{D}(t_0,\tau)\mathbf{1}_{t_d>\tau}\overline{B}(\tau,T)|\mathcal{F}^{\mathcal{N}}_{t_0}\right]|\mathcal{G}_{t_0}\right] = \\ & \mathbb{E}\left[\mathfrak{D}(t_0,\tau)\ \exp\left(-\int_{t_0}^{\tau}\lambda_s ds\right)\ \overline{B}(\tau,T)|\mathcal{G}_{t_0}\right] \quad ,\end{aligned}$$

and then also point ii) is proven.

Proof of Proposition 2. It is enough to use forward defaultable ZC definition and to impose that the NPV is zero at time t (see also figure 4.1)

$$\mathbb{E}\left[\mathfrak{D}(t,\tau)\mathbf{1}_{t_d>\tau}\overline{B}(t;\tau,T)|\mathcal{F}_t\right] = \mathbb{E}\left[\mathfrak{D}(t,T)\mathbf{1}_{t_d>T}|\mathcal{F}_t\right] \qquad .$$

The proposition is proven after observing that $\overline{B}(t;\tau,T)$ is a known quantity in t, since the forward price is established at time t, and using definition (4.3).

In order to prove Lemma 4.2.1 we need to remind the Generalized Itô's Lemma.

Generalized Itô's Lemma. Let $\mathbf{X}(t) := \{X_1(t), \cdots, X_d(t)\}$ a d-dimensional semi-martingale process with a finite number of jumps, $\mathbf{X}^c(t)$ it's continuous part and a function $f : \Re^d \to \Re$

$$df(\mathbf{X}(t)) = \sum_{i=1}^{d} \frac{\partial f(\mathbf{X}^-(t))}{\partial X_i} dX_i^c(t) + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f(\mathbf{X}^-(t))}{\partial X_i \, \partial X_j} \langle dX_i^c(t) \, dX_j^c(t) \rangle + \Delta f(\mathbf{X}(t))$$

where

$$\Delta f\left(\mathbf{X}(t)\right) = \lim_{h \to 0^+} f\left(\mathbf{X}(t)\right) - f\left(\mathbf{X}(t-h)\right) \qquad .$$

Proof. A formal proof can be found in [22].

Proof of Lemma 4.2.1. A direct application of the Generalized Itô's Lemma, using dynamics (4.5) and equation (4.4).

Proof of Lemma 4.2.2. Given the definition of $\overline{\mathfrak{D}}_q(t_0, T)$ and definitions (4.6) in DHJM models, it is a straightforward computation after integrating $r_s + q\,\lambda_s$ for s between t_0 and T.

Proof of Lemma 4.2.3 . It is an application of the Girsanov theorem on the risk neutral measure and the $\overline{\tau}$-forward measure. It allows to obtain, using Lemma 4.2.2, that

$$\mathbb{E}\left[\overline{\mathfrak{D}}(t,\tau) \bullet \right] = \overline{B}(t,\tau)\, \mathbb{E}^{(\overline{\tau})}\left[\, \bullet \,\right]$$

proving the lemma.

The following technical lemma is needed in order to prove Theorem 4.3.1

Lemma. The joint probability of i) the maximum $y = \max[x(t); t \in (0,T)]$ and ii) its time-location $\theta \in (0,T)$, where $x(t) = c\,t + W(t)$ is a 1-dimensional Wiener process with drift $c\,t$ where $c \in \Re$, is

$$p(\theta, y; c) = \frac{1}{\pi} \frac{y}{\sqrt{T-\theta}\,\theta^{3/2}}\, e^{-\frac{c^2 T}{2} - \frac{y^2}{2\theta} + cy} \left\{ 1 - \sqrt{2\pi\,(T-\theta)}\, c\, e^{\frac{c^2 (T-\theta)}{2}}\, \Phi\left[-c\sqrt{T-\theta} \right] \right\}$$

with $y = x(\theta) > 0$ and $\theta \in (0,T)$.

Proof. Consider the density $p(\theta, y, x; c, \sigma^2)$ in equation (1.5) in [37], where x is the endpoint $x(T)$. The marginal distribution is obtained by setting $\sigma = 1$ and by integrating on $x < y$.

Proof of Theorem 1. The upper bound is obvious given Lemma 4.2.3 and after observing that each ZC $\overline{B}(t;\tau,t_i)$ in equation (4.9) is martingale under the $\overline{\tau}$-forward measure and follows a GBM with volatility $v(t;\tau,t_i)$.

The lower bound is the sum over i of the expected value of $\overline{B}(t^*;\tau,t_i)$ computed at the time t^* s.t. $\overline{B}(t^*;\tau,t_N)$ reaches its (first) maximum for a given realization of the process. In this case, using the separability property of the volatility observed in **Remark 3**, we get

$$\mathbb{E}^{(\overline{\tau})}\left\{\overline{B}(t^*;\tau,t_i)\right\} = \overline{B}(t_0;\tau,t_i)\,\mathbb{E}^{(\overline{\tau})}\left\{\exp\left[\xi_i\left(-\frac{1}{2}\xi_i\int_{t_0}^{t^*}\nu^2(s)\,ds + \int_{t_0}^{t^*}\nu(s)\,dW^{(\overline{\tau})}(s)\right)\right]\right\} \ .$$

By means of the change of time

$$\tilde{t} := \tilde{t}(t) := \int_{t_0}^{t}\nu^2(s)\,ds \ \in \ (0,\tilde{\tau})$$

where $\tilde{\tau}$ stands for $\tilde{t}(\tau)$. We get $dW^{(\overline{\tau})}(\tilde{t}) = \nu(t)\,dW^{(\overline{\tau})}(t)$ and

$$\begin{aligned}\mathbb{E}^{(\overline{\tau})}\left\{\overline{B}(t^*;\tau,t_i)\right\} &= \overline{B}(t_0;\tau,t_i)\,\mathbb{E}^{(\overline{\tau})}\left\{\exp\left[\xi_i\left(-\frac{1}{2}\xi_i\,\theta + W^{(\overline{\tau})}(\theta)\right)\right]\right\} \\ &= \overline{B}(t_0;\tau,t_i)\,\mathbb{E}^{(\overline{\tau})}\left\{\exp\left[\xi_i\left(-\frac{1}{2}(\xi_i-\xi_N)\,\theta + x(\theta)\right)\right]\right\}\end{aligned}$$

where we have defined $x(\theta) := -\xi_N\,\theta/2 + W^{(\overline{\tau})}(\theta)$ the maximum value reached by the drifted Brownian Motion $x(\tilde{t})$ and

$$\theta := \tilde{t}(t^*) \ \in \ (0,\tilde{\tau}) \ \ .$$

Let us observe that

$$\mathbb{E}^{(\overline{\tau})}\left\{\exp\left[\xi_i\left(-\frac{1}{2}(\xi_i-\xi_N)\,\theta + x(\theta)\right)\right]\right\} = \int_0^{\tilde{\tau}} d\theta \int_0^{+\infty} dy\, p\left(\theta, y; -\frac{\xi_N}{2}\right) e^{\xi_i\left(-\frac{1}{2}(\xi_i-\xi_N)\,\theta + y\right)}$$

where $p(\theta, y; -\xi_N/2)$ has been deduced in previous technical lemma with a generic drift c. After computing the integral w.r.t. y and some algebra we get the result.

Let us observe that, from a technical point of view, we have shown that it is possible to compute the maximum of a GBM with time dependent parameters even when the GBM is not a martingale process (with a drift of a particular form), contrary to what is generally considered in the literature where the GBM is always assumed to be martingale in order to get a running maximum.

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