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Analytical estimation of the error on the radius of industrial pipes

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Abstract

The aim of this work is to provide an analytical description of the problem of estimating the minimum error which occurs when measuring the diameter of a perfect cylinder, representing a pipe in the context of industrial production, with a computer vision system composed by m calibrated independent cameras, each one paired with a laser.

The analysis of this problem has been carried out combining the mathematical tools given by affine and euclidean geometry, Lie group theory, statistical parameter estimation and data analysis.

More specifically, we derive the system of equations which define the mathematical model for the deterministic problem. Then we add uncertainty on measurements and we compute a modified version of the classical Cramer Rao Lower Bound adapted to geometric fitting problems, the so called Kanatani Cramer Rao Lower Bound.

Sommario

Lo scopo di questo lavoro è dare una descrizione analitica al problema della stima dell'errore minimo sulla misurazione del diametro di un cilindro perfetto, che modella un tubo ideale nel contesto della produzione industriale, con un sistema di visione artificiale composto da m telecamere calibrate e indipendenti, ognuna delle quali accoppiata con un laser.

L'analisi di questo problema è stata condotta combinando gli strumenti matematici forniti dalla geometria affine ed euclidea, dalla teoria dei gruppi di Lie e dalla statistica applicata alla stima di parametri e all'analisi dei dati.

Nello specifico, deriviamo il sistema di equazioni che definisce il modello matematico per il problema deterministico. Poi includiamo l'incertezza dovuta agli strumenti di misurazione e calcoliamo una versione modificata del classico limite inferiore di Cramer Rao adattato appositamente per i problemi di fitting geometrico, chiamato limite inferiore di Kanatani Cramer Rao.

Chapter 1

Introduction

The object of this work is the study of the error on the construction of pipes in an industrial context, where high precision is an essential requirement.

The main problem is to estimate the minimum error which occurs when measuring the diameter of a perfect cylinder, representing the pipe, with a system composed by m calibrated independent cameras.

Each camera/laser pair can be calibrated with very high precision (meaning in the order of a few millimeters). At the best of our knowledge, there is not a method able to estimate the error occurring when coupling the different measurements coming from each camera/laser pair in order to cover the whole diameter of the cylinder.

Our work aims to evaluate a lower bound for the error, below which any improvement is impossible, according to the features of the components, the setting and the physics of the sensors.

There are various approaches to error analysis problem, e.g. among the most popular there are the ones based on MonteCarlo simulations. However, in this thesis we choose the Cramer Rao approach to search for a lower bound on the covariance matrix of the estimated parameters. Being a parametric approach, the Cramer Rao Lower Bound allows the analysis of the dependancy of the estimated error on the given instrumental and methodological limitations. In this work, we are not going to use the classical approach, but a modified version called Kanatani Cramer Rao lower bound ([21],[22],[23],[24],[25],[26]) which is intended to be specifically adapted for geometric fitting problems.

In the first part of this work we are going to introduce all the theoretical concepts and notations which we use in the last chapters, i.e. the theoretical background needed to develop our approach, that combines different areas of geometry, algebra and statistics.

In the second part of this work, we are going to explain into details the model that we developed to solve our problem. Moreover, we will show the results of our theoretical analysis in a case study. Our computations will be performed in Matlab and R.

The thesis is structured as follows:

- Chapter (2) contains a summary of concepts of affine geometry that are necessary for

the formulation of the deterministic problem;

- Chapter (3) extends the previous concepts to euclidean geometry;
- Chapter (4) reviews the basics of the theory of Lie groups and algebras with particular focus on the two and three dimensional cases;
- Chapter (5) is about the definition of conic curves and quadric surfaces in euclidean spaces;
- Chapter (6) introduces the statistical theory behind methods of parametric performance evaluation and the Kanatani Cramer Rao Lower Bound;
- Chapter (7) describes the method that we use in order to arrive to the equations for the deterministic model describing our problem of interest;
- Chapter (8) focuses on the statistical model and here we apply the Kanatani Cramer Rao Lower Bound to our problem;
- Chapter (9) contains the analysis of the case study;
- Appendix (A.1) contains the most technical computations, while in Appendix (A.2)) we put the codes that we used for our formulations, written in Matlab and R languages.

Chapter 2

Affine geometry

Affine geometry can be seen as a way to generalize classic euclidean geometry in absence of the notions of angles and distances.

For the sake of brevity, we are not going to prove the results that we show, since they are well known and their proof can be found easily in the literature (refer to the source [27]).

2.1 Vector spaces

Definition 2.1.1. Given $M = \{1, \dots, m\}$ and $N = \{1, \dots, n\}$, a matrix of order (m, n) with elements in the field \mathbb{K} is a function

$$\begin{aligned} A: M \times N &\rightarrow \mathbb{K} \\ (i, j) &\mapsto a_{ij} \end{aligned}$$

The n -tuple $A_{R(i)} = (a_{i1}, \dots, a_{in}) \in \mathbb{K}^n$ is called the i^{th} row of A , while the m -tuple $A_{C(j)} = (a_{1j}, \dots, a_{mj}) \in \mathbb{K}^m$ is called the j^{th} column of A . The set of the matrices of the type (n, m) with ground field \mathbb{K} is called $Mat(m, n; \mathbb{K})$.

Definition 2.1.2. Let us consider a system of m linear equations with coefficients a_{ij} , constant terms b_i in the ground field \mathbb{K} and n unknowns x_j :

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{cases}$$

where $1 \leq i \leq m, 1 \leq j \leq n$.

Then the coefficients matrix and the columns of constant terms and unknowns are

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

The complete matrix of the linear system is $[A|B]$, obtained by adding the column B to A on the right. Finally, the linear system defined by the complete matrix of coefficients $[A|0_{[n \times 1]}]$ is called homogeneous system associated to $[A|B]$.

Definition 2.1.3. A linear map (also called a linear mapping, linear transformation or, in some contexts, linear function) is a mapping $f: V \rightarrow W$ between two vector spaces on the same ground field \mathbb{K} that preserves the operations of addition and scalar multiplication. The set of linear maps from V to W is called $Hom(V, W)$.

In the case when $V = W$ the map is called a linear operator or an endomorphism of V .

Example: If A is a real $m \times n$ matrix, then A defines a linear map from $V = \mathbb{R}^n$ to $W = \mathbb{R}^m$ by sending the column vector $\mathbf{x} \in \mathbb{R}^n$ to the column vector $f(\mathbf{x}) = A\mathbf{x} \in \mathbb{R}^m$.

Definition 2.1.4. In linear algebra, the kernel (also known as null space or nullspace) of a linear map $L: V \rightarrow W$ between two vector spaces V and W , is the set of all elements $\mathbf{v} \in V$ for which $L(\mathbf{v}) = 0$, where 0 denotes the zero vector in W . That is $\ker(L) = \{\mathbf{v} \in V | L(\mathbf{v}) = 0\}$.

Definition 2.1.5. Let \mathcal{A} and \mathcal{B} be two ordered bases of a vector space. The matrix of change of basis from \mathcal{A} to \mathcal{B} is the matrix whose columns are the coordinate vectors of the elements of the new basis \mathcal{B} relative to the original basis \mathcal{A} and we call such a matrix $M_{\mathcal{B}\mathcal{A}}$.

Definition 2.1.6. Let V be a vector space and $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n) \in V^n$ be its basis. For any vector $\mathbf{v} \in V$, the linear combination $\mathbf{v} = t_1\mathbf{v}_1 + \dots + t_n\mathbf{v}_n$ is called decomposition of \mathbf{v} with respect to the basis \mathcal{B} . The coefficients of the decomposition t_1, \dots, t_n are called components of \mathbf{v} with respect to \mathcal{B} . Finally, the function

$$\begin{aligned} \Phi_{\mathcal{B}}: V &\rightarrow Mat(n, 1; \mathbb{K}) \\ \mathbf{v} &\mapsto \mathbf{v}|_{\mathcal{B}} = [t_1, \dots, t_n]^T \end{aligned}$$

is called map of the components.

Definition 2.1.7. Let V, W be finitely generated vector spaces on the same ground field \mathbb{K} with bases $\mathcal{B}_V = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ and $\mathcal{B}_W = (\mathbf{w}_1, \dots, \mathbf{w}_m)$. Then the map of the components for the linear maps is

$$\begin{aligned} \Phi_{\mathcal{B}_V\mathcal{B}_W}: Hom(V, W) &\rightarrow Mat(m, n; \mathbb{K}) \\ f &\mapsto F|_{\mathcal{B}_V\mathcal{B}_W} \end{aligned}$$

where $F|_{\mathcal{B}_V\mathcal{B}_W} = \left[F(\mathbf{v}_1)|_{\mathcal{B}_W} \mid \dots \mid F(\mathbf{v}_n)|_{\mathcal{B}_W} \right]$. We say that $F|_{\mathcal{B}_V\mathcal{B}_W}$ represents f with respect to the two chosen bases.

Theorem 2.1.8. Let V, W be finitely generated vector spaces on the same ground field \mathbb{K} with bases $\mathcal{B}_V, \mathcal{B}_W$ and $f \in Hom(V, W)$. Then $F|_{\mathcal{B}_V\mathcal{B}_W}$ is the only matrix for which $f(\mathbf{v})|_{\mathcal{B}_W} = F|_{\mathcal{B}_V\mathcal{B}_W} \mathbf{v}|_{\mathcal{B}_V}$ for any $\mathbf{v} \in V$. Moreover, the map of the components $\Phi_{\mathcal{B}_V\mathcal{B}_W}$:

- is an isomorphism of vector spaces;
- respects the multiplicative structure, in the sense that if U is a vector space on a ground field \mathbb{K} with basis \mathcal{B}_U and $g \in \text{Hom}(U, V)$ then

$$\Phi_{\mathcal{B}_U \mathcal{B}_W}(f \circ g) = \Phi_{\mathcal{B}_V \mathcal{B}_W}(f) \Phi_{\mathcal{B}_U \mathcal{B}_V}(g).$$

We give the notion of affine spaces by using the so called Weyl axioms.

Definition 2.1.9. Let us consider a non-empty set A , a vector space V and a function

$$\begin{aligned} \psi: A \times A &\rightarrow V \\ (P, Q) &\mapsto \overrightarrow{PQ} \end{aligned}$$

The structure $\mathbb{A} = (A, V, \psi)$ is an affine space if

- (i) for any $P \in A$ fixed, the function

$$\begin{aligned} \psi_P: A &\rightarrow V \\ Q &\mapsto \overrightarrow{PQ} \end{aligned}$$

is bijective;

- (ii) the parallelogram rule holds

$$\psi(P, Q) + \psi(P, Q) + \psi(Q, R) = \psi(P, R) \text{ for any } P, Q, R \in A.$$

Definition 2.1.10. Given $\mathbb{A} = (A, V, \psi)$, the elements $P \in A$ are called points, the couple $(P, Q) \in A \times A$ is called oriented segment from P to Q and the vector \overrightarrow{PQ} is called displacement vector from P to Q . The set A and the vector space V are called support and direction of A respectively. The dimension of the affine space is defined as $\dim(A) = \dim(V)$. The space is called affine line if $\dim(A) = 1$ and affine plane if $\dim(A) = 2$.

Example: Let V be a vector space and ψ the function

$$\begin{aligned} \psi: V \times V &\rightarrow V \\ (\mathbf{v}_1, \mathbf{v}_2) &\mapsto \mathbf{v}_2 - \mathbf{v}_1 \end{aligned}$$

The function ψ is bijective: indeed for any $\mathbf{w} \in V$ there exists a unique vector $\mathbf{v}_2 \in V$ such that $\mathbf{w} = \psi(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_2 - \mathbf{v}_1$, i.e. $\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{w}$. Moreover, $\psi(\mathbf{v}_1, \mathbf{v}_2) + \psi(\mathbf{v}_2, \mathbf{v}_3) = \mathbf{v}_2 - \mathbf{v}_1 + \mathbf{v}_3 - \mathbf{v}_2 = \mathbf{v}_3 - \mathbf{v}_1 = \psi(\mathbf{v}_1, \mathbf{v}_3)$ for any $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$. Therefore (V, V, ψ) is an affine space denoted as \mathbb{A}_V . The affine space associated with $(\mathbb{K}^n, (\mathbb{K}^n, \mathbb{K}, +, \cdot), \psi)$ is denoted as $\mathbb{A}_{\mathbb{K}}^n$.

The function ψ associates a couple of points P, Q to the displacement vector between them. This is equivalent to state that it is possible to start from a point P and to reach any

other point in the space through any of the possible translations, and viceversa any possible translation is uniquely determined by the starting point P and the ending point Q . Moreover, if we move from P to Q first and then from Q to R , this is equivalent to moving directly from P to R .

Given an arbitrary point $O \in A$, the request for ψ_O to be a bijective map between the points in A and the vectors in V is fundamental for the study of A . The reason is that, once we choose a basis \mathcal{B} of V with components $\Phi_{\mathcal{B}}$, we can use these components to express ψ_O , obtaining a bijection between A and the matrix space $Mat(n, 1; \mathbb{K})$, where $n = \dim(A)$ and \mathbb{K} is the field over which V is defined. This way any point of A is assigned a unique n-tuple in \mathbb{K}^n , and viceversa.

Definition 2.1.11. An affine reference system of an affine space $\mathbb{A} = (A, V, \psi)$ is a subset $B_O \in A \times V$ constituted by a point $O \in A$ and by the vectors of a basis \mathcal{B} of V . The function

$$\begin{aligned}\Phi_{B_O} : A &\rightarrow Mat(n, 1; \mathbb{K}) \\ P &\mapsto P|_{B_O} = (\Phi_{\mathcal{B}} \circ \psi_O)(P)\end{aligned}$$

is called map of coordinates and the elements of the matrix $(\Phi_{\mathcal{B}} \circ \psi_O)(P)$ are called coordinates of P with respect to B_O . The point O is called the origin of the reference system.

Remark 2.1.12. If B_O is a reference system in \mathbb{A} , then the coordinates of a point $P \in \mathbb{A}$ satisfy the fundamental relation

$$P|_{B_O} = \Phi_{\mathcal{B}}(\psi_O(P)) = \Phi_{\mathcal{B}}(\overrightarrow{OP}) = \overrightarrow{OP}|_{\mathcal{B}},$$

from which we derive $\overrightarrow{PQ}|_{\mathcal{B}} = (\overrightarrow{OQ} - \overrightarrow{OP})|_{\mathcal{B}} = \overrightarrow{OQ}|_{\mathcal{B}} - \overrightarrow{OP}|_{\mathcal{B}} = Q|_{B_O} - P|_{B_O}$.

In the case of an affine space \mathbb{A}_V , if the direction V have a canonical basis \mathcal{B} then $B_O = (0_V, \mathcal{B})$ is called canonical reference system, with respect to which $\mathbf{v}|_{B_O} = (\mathbf{v} - \mathbf{0})|_{\mathcal{B}} = \mathbf{v}$ holds for any $\mathbf{v} \in \mathbb{A}_V$.

Example: Let us consider the case of the affine plane $\mathbb{A}_{\mathbb{R}}^2$ with points $P_0 = (0, 0)$, $P_1 = (1, 0)$, $P_2 = (0, 1)$ and $Q = (1, 1)$. The points P_0, P_1, P_2 determine six possible different reference systems of $\mathbb{A}_{\mathbb{R}}^2$, with respect to each the point Q will have different coordinates.

Let us choose the reference system with origin in P_0 and the basis

$$B_0 = (\overrightarrow{P_0P_1} = P_1 - P_0 = (1, 0), \overrightarrow{P_0P_2} = P_2 - P_0 = (0, 1)).$$

This is the canonical reference system for $\mathbb{A}_{\mathbb{R}}^2$, with respect to which the coordinates of Q are $Q|_{B_{P_0}} = \overrightarrow{P_0Q}|_{B_0} = (Q - P_0)|_{B_0} = (1, 1)|_{B_0} = [1, 1]$.

If instead we choose the origin P_1 and the basis

$$B_1 = \{\overrightarrow{P_1P_0} = P_0 - P_1 = (-1, 0), \overrightarrow{P_1P_2} = P_2 - P_1 = (-1, 1)\},$$

the coordinates of Q with respect to the reference system $B_{P_1} = \{P_1, \overrightarrow{P_1P_0}, \overrightarrow{P_1P_2}\}$ are $Q|_{B_{P_1}} = \overrightarrow{P_1Q}|_{B_1} = (Q - P_1)|_{B_1} = (0, 1)|_{B_1} = [-1, 1]$.

The other reference systems are defined analogously.

2.2 Affine subspaces

Definition 2.2.1. Let $\mathbb{A} = (A, V, \psi)$ be an affine space, let S be a subset of A and $\psi|_{S \times S}$ the restriction of ψ . Then S is called affine subspace of \mathbb{A} if $Im(\psi|_{S \times S})$ is a vector subspace of V and $(S, Im(\psi|_{S \times S}), \psi|_{S \times S})$ is also an affine subspace. If $dim(S) = dim(\mathbb{A}) - 1$, the subspace is called hyperplane of \mathbb{A} .

Proposition 2.2.2. Let $\mathbb{A} = (A, V, \psi)$ be an affine space, let S be a subset of A and P be a point of S . Given the set $U = \overrightarrow{PQ} \in V | Q \in S$, then S is an affine subspace of \mathbb{A} if and only if U is a vector subspace of V . In such a case, U is the direction of S .

Remark (2.2.2) means that an affine subspace S is completely determined if we know a point P and its direction U , since a point $Q \in \mathbb{A}$ belongs to S if and only if $\overrightarrow{PQ} \in U$. Therefore, an affine line is uniquely determined by a point P and a non-zero vector belonging to its direction. Equivalently, an affine plane is uniquely determined by a point P and two independent vectors composing a basis of its direction.

Proposition 2.2.3. Given the vector spaces V and W , let us consider the application $f \in Hom(V, W)$ and the affine space $\mathbb{A}_V = (V, V, \psi)$. Then $f^{-1}(\mathbf{w})$ is an affine subspace of \mathbb{A}_V for any $\mathbf{w} \in Im(f) \subseteq W$.

Also the inverse of remark (2.2.3) holds, i.e. for any affine subspace S of \mathbb{A}_V there exist a linear application f and a vector $\mathbf{w} \in Im(f) \subseteq W$ such that $S = f^{-1}(\mathbf{w})$.

Proposition 2.2.4. Let \mathbb{A} be an affine space with dimension $n < \infty$ on the field \mathbb{K} and B_O be a reference system. Let S be an affine subspace of dimension m and $B_Q = (Q, (\mathbf{u}_1, \dots, \mathbf{u}_m))$ be a reference system.

Let us call $A = [\mathbf{u}_1|_B \ \dots \ \mathbf{u}_m|_B] \in Mat(n, m; \mathbb{K})$, $B = [b_1 \dots b_n]^T = Q|_{B_O}$ and $X = [x_1 \dots x_n]^T = P|_{B_O}$ the column of the coordinates of point $P \in \mathbb{A}$ with respect to B_O . Then

- (i) decomposition $X = B + \sum_{i=1}^m t_i A_{C(i)}$ exists if and only if $P \in S$, and in such a case it is said to be a parametric representation of subspace S with respect to the reference system B_O ;
- (ii) if we reduce the matrix A contained inside the matrix $[A|X - B]$, we obtain the matrix $[S|X']$. Its last $n - m$ rows constitute an algebraic representation of S with respect to the reference system B_O .

Corollary 2.2.5. Let \mathbb{A} be an affine space with dimension $n < \infty$ on the ground field \mathbb{K} and B_O be a reference system. The non-empty subset S of \mathbb{A} is a subspace with dimension m if and only if there exists a matrix $[A|B] \in Mat(n - m, n + 1; \mathbb{K})$ with maximum rank such that $\Phi_{B_O}(S)$ is the set of the solution of the system associated to $[A|B]$. In this case, Φ_B is an homeomorphism between the direction U of S and $ker(A)$.

Example: In the affine space $\mathbb{A}_{\mathbb{R}}^3$, let us search for the plane Π containing the points $P_1 = (1, 0, 0), P_2 = (0, 1, 0), P_3 = (0, 0, 1)$. We know that the plane is uniquely determined by a point, for instance P_1 , and two vectors composing a basis of the direction. Let us consider the vectors $\overrightarrow{P_1P_2} = P_2 - P_1 = (-1, 1, 0)$ and $\overrightarrow{P_1P_3} = P_3 - P_1 = (-1, 0, 1)$.

They are clearly independent, so the plane is the set of all points of $Q \in \mathbb{A}_{\mathbb{R}}^3$ such that $\overrightarrow{P_1Q} = t_1\overrightarrow{P_1P_2} + t_2\overrightarrow{P_1P_3}$ for any $t_1, t_2 \in \mathbb{R}$, while $B_{P_1} = \{P_1, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}\}$ is a reference system for Π .

The canonical reference system for $\mathbb{A}_{\mathbb{R}}^3$ is $B_{P_0} = \{P_0, \overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}, \overrightarrow{P_0P_3}\}$ where the origin is the point $P_0 = (0, 0, 0)$ and the basis associated with the direction is $B_0 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Using the notation of proposition (2.2.4), we have

$$A = \left[\overrightarrow{P_1P_2} \Big|_{B_0} \quad \overrightarrow{P_1P_3} \Big|_{B_0} \right] = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = P_1 \Big|_{B_{P_0}}.$$

So a parametric representation of Π is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

An algebraic representation of the plane is obtained by reducing the matrix

$$\begin{bmatrix} -1 & -1 & x-1 \\ 1 & 0 & y \\ 0 & 1 & z \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & x+y+z-1 \end{bmatrix}$$

from which we derive the equation $x + y + z - 1 = 0$.

2.3 Mutual position of affine subspaces

Definition 2.3.1. Given the affine space \mathbb{A} , let S and T be two subspaces of dimension greater than zero with directions U and W respectively. Then, S and T are called:

- (i) parallel if $U \subseteq W$ or $W \subseteq U$;
- (ii) incident if they are not parallel and they have at least one common point;
- (iii) skewed if they are not parallel nor incident.

In particular, note that two subspaces are parallel even if one is nested in the other.

Remark 2.3.2. Let S, T be two affine parallel subspaces of \mathbb{A} , with dimension $\dim(S) \leq \dim(T) < \infty$. If $S \cap T \neq \emptyset$, then:

- (i) $\dim(S) < \dim(T)$ implies $S \subset T$;

(ii) $\dim(S) = \dim(T)$ implies $S = T$.

Corollary 2.3.3. *Given a finite-dimensional subspace S and a point P of the affine space \mathbb{A} , there exists a unique subspace T parallel to S containing P and satisfying $\dim(T) = \dim(S)$.*

From now on we assume that $\mathbb{A} = (A, V, \psi)$ is an affine space of dimension $n < \infty$, while S and T are subspaces with dimension p and q respectively, $p \geq q$. Given a reference system B_O of \mathbb{A} , let us suppose that the linear systems associated to $[A|B] \in \text{Mat}(n-p, n+1)$ and $[A'|B'] \in \text{Mat}(n-q, n+1)$ are the algebraic representation of T and S respectively. Finally, we define

$$[\tilde{A}|\tilde{B}] = \begin{bmatrix} A & B \\ A' & B' \end{bmatrix} \in \text{Mat}(2n-p-q, n+1).$$

Lemma 2.3.4. *If $S \cap T$ is non-empty, then it is an affine subspace of \mathbb{A} . In such a case, an algebraic representation of $S \cap T$ is the system associated to the reduction of matrix $[\tilde{A}|\tilde{B}]$.*

Theorem 2.3.5. *S and T are:*

- (i) *parallel with $T \subseteq S$ if and only if $r([\tilde{A}|\tilde{B}]) = r(\tilde{A}) = n - q$;*
- (ii) *parallel and disjoint if and only if $r([\tilde{A}|\tilde{B}]) > r(\tilde{A}) = n - q$;*
- (iii) *incident if and only if $r([\tilde{A}|\tilde{B}]) = r(\tilde{A}) > n - q$;*
- (iv) *skewed if and only if $r([\tilde{A}|\tilde{B}]) > r(\tilde{A}) > n - q$.*

Corollary 2.3.6. *If S is a hyperplane of \mathbb{A} , then S and T are either parallel or incident.*

2.4 Pencils of hyperplanes

Definition 2.4.1. Let \mathbb{A} be an affine space of dimension $n \geq 2$. Then:

- (i) if S is an affine subspace with dimension $n-2$, then the set of all hyperplanes containing S is said to be a proper pencil of hyperplanes having S as axis;
- (ii) if T is a hyperplane, then the set of all hyperplanes parallel to T is said to be the improper pencil of hyperplanes parallel to T .

Given a reference system B_O in \mathbb{A} , any algebraic representation of a hyperplane T is given by one linear equation $[A|B] \in \text{Mat}(1, n+1; \mathbb{K})$. The intersection of 2 generic distinct hyperplanes T_1, T_2 of the pencil, with equations $[A_1|B_1], [A_2|B_2] \in \text{Mat}(1, n+1; \mathbb{K})$ is represented by the system

$$[\tilde{A}|\tilde{B}] = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \in \text{Mat}(2, n+1).$$

If it is a pencil, then $T_1 \cap T_2 \neq \emptyset$ and consequently the solution of the system exists and has dimension $n-2$, as stated by the theorem in linear algebra that determines the number

of solutions for a system of linear equations, given the rank of its augmented matrix and coefficient matrix.

This means that the intersection coincides with S . Therefore T_1 and T_2 are sufficient to determine the pencil completely.

Remark 2.4.2. T belongs to the pencil containing the two distinct hyperplanes T_1, T_2 if and only if there exist $t_1, t_2 \in \mathbb{K}$ such that the equation associated to

$$[t_1A_1 + t_2A_2 | t_1B_1 + t_2B_2] \in \text{Mat}(1, n; \mathbb{K})$$

is an algebraic representation of T .

In the case of two-dimensional affine spaces we speak of pencil of lines, in the case of three dimensional affine spaces we speak of pencil of planes.

2.5 Affine transformations

Definition 2.5.1. Let $\mathbb{A} = (A, V, \psi_{\mathbb{A}})$ and $\mathbb{B} = (B, W, \varphi_{\mathbb{B}})$ be two affine spaces on the same field \mathbb{K} . Then the function $f: A \rightarrow B$ is a homomorphism between affine spaces, or affine transformation, if there exists a linear mapping $\bar{f}: V \rightarrow W$ such that for any oriented segment $(P, Q) \in \mathbb{A}$,

$$\varphi_{\mathbb{B}}(f(P), f(Q)) = \bar{f}(\varphi_{\mathbb{A}}(P, Q)).$$

The set of all affine transformations from \mathbb{A} to \mathbb{B} is called $\text{Hom}(\mathbb{A}, \mathbb{B})$.

Definition 2.5.2. Let $\mathbb{A} = (A, V, \psi_{\mathbb{A}})$ and $\mathbb{B} = (B, W, \varphi_{\mathbb{B}})$ be finite-dimensional affine spaces on the same field \mathbb{K} , with reference systems $B_{\mathbb{A}} = (O_{\mathbb{A}}, B_V)$ and $B_{\mathbb{B}} = (O_{\mathbb{B}}, B_W)$ respectively. Then the map of coordinates for affine transformation is the function

$$\begin{aligned} \phi_{B_{\mathbb{A}}B_{\mathbb{B}}}: \text{Hom}(\mathbb{A}, \mathbb{B}) &\rightarrow \text{Mat}(m, n+1; \mathbb{K}) \\ f &\mapsto \left[F|_{B_V B_W} \quad | \quad f(O_{\mathbb{A}})|_{B_{\mathbb{B}}} \right] \end{aligned}$$

where $F|_{B_V B_W}$ is the matrix representing $\bar{f} \in \text{Hom}(V, W)$.

Now we state explicitly the representation theorem for affine transformations.

Theorem 2.5.3. Let $\mathbb{A} = (A, V, \psi_{\mathbb{A}})$ and $\mathbb{B} = (B, W, \varphi_{\mathbb{B}})$ be affine spaces with dimension n and m respectively, on the same ground field \mathbb{K} , with reference systems $B_{\mathbb{A}} = (O_{\mathbb{A}}, B_V)$ and $B_{\mathbb{B}} = (O_{\mathbb{B}}, B_W)$.

Given $f \in \text{Hom}(\mathbb{A}, \mathbb{B})$, then $\left[F|_{B_V B_W} \quad f(O_{\mathbb{A}})|_{B_{\mathbb{B}}} \right]$ is the only matrix for which

$$f(P)|_{B_{\mathbb{B}}} = \left[F|_{B_V B_W} \quad f(O_{\mathbb{A}})|_{B_{\mathbb{B}}} \right] \begin{bmatrix} P|_{B_{\mathbb{A}}} \\ 1 \end{bmatrix}$$

holds for any $P \in \mathbb{A}$.

Definition 2.5.4. Given two affine spaces \mathbb{A} and \mathbb{B} , an affinity is an affine transformation $f \in \text{Hom}(\mathbb{A}, \mathbb{B})$ whose associated linear mapping \bar{f} is an isomorphism. If such an f exists, then \mathbb{A} and \mathbb{B} are affinely equivalent.

Example: Translations are affinities. In Euclidean geometry, a translation is a geometric transformation that moves every point of a figure or a space by the same distance in a given direction. It is a one-to-one correspondence between two sets of points or a mapping from an affine space to itself. For any \mathbf{v} in a direction V of \mathbb{A} there exists an affine transformation $f_{\mathbf{v}}$ called translation, defined as

$$\begin{aligned} f_{\mathbf{v}}: \mathbb{A} &\rightarrow \mathbb{A} \\ P &\mapsto \varphi_P^{-1}(\mathbf{v}) \end{aligned}$$

so if $Q = f_{\mathbf{v}}(P)$ then $\overrightarrow{PQ} = \mathbf{v}$. We observe that it can be shown that $\bar{f}_{\mathbf{v}} = Id_{\mathbf{v}}$ and consequently $f_{\mathbf{v}}$ is an affinity of \mathbb{A} . The matrix representing $f_{\mathbf{v}}$ with respect to a reference system is

$$F_{\mathbf{v}}|_{\mathcal{B}_O} = \begin{bmatrix} \bar{f}_{\mathbf{v}}|_{\mathcal{B}} & f_{\mathbf{v}}(0)|_{\mathcal{B}_O} \end{bmatrix} = f_{\mathbf{v}} = \begin{bmatrix} Id_{\mathbf{v}}|_{\mathcal{B}} & \overrightarrow{Of_{\mathbf{v}}(0)}|_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} I_m & \mathbf{v}|_{\mathcal{B}} \end{bmatrix}.$$

Other examples of affinities are homotheties, i.e. the affine transformations associated to linear mappings of the type $f = cId_V$ for $c \in \mathbb{K}$. From a geometric point of view, affinities can be characterized as transformations that do not change relations of incidence and parallelism. This means that the image of parallel subspaces through affinities are still parallel subspaces of the same dimension as the initial ones, and the equivalent holds for all the other cases of mutual position. Many properties of affinities are the same as the ones of isomorphisms between vector spaces.

Remark 2.5.5. Let \mathbb{A} and \mathbb{B} be two affine spaces. Then:

- (i) \mathbb{A} and \mathbb{B} are affine equivalent if and only if they have the same dimension;
- (ii) the affine transformation $f \in \text{Hom}(\mathbb{A}, \mathbb{B})$ is an affinity if and only if the image of a reference system of \mathbb{A} is a reference system of \mathbb{B} .

Chapter 3

Euclidean geometry

3.1 Euclidean vector spaces

Definition 3.1.1. Let V be a real vector space. A scalar product on V is an external binary operation with real values, denoted with

$$\begin{aligned}\langle \cdot, \cdot \rangle : V \times V &\rightarrow \mathbb{R} \\ (\mathbf{v}_1, \mathbf{v}_2) &\mapsto \langle \mathbf{v}_1, \mathbf{v}_2 \rangle\end{aligned}$$

and satisfying the following properties:

- (i) it is linear with respect to the first argument, meaning

$$\langle t_1 \mathbf{v}_1 - t_2 \mathbf{v}_2, \mathbf{v}_3 \rangle = t_1 \langle \mathbf{v}_1, \mathbf{v}_3 \rangle + t_2 \langle \mathbf{v}_2, \mathbf{v}_3 \rangle$$

for any $t_1, t_2 \in \mathbb{K}$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$;

- (ii) it is symmetric, meaning $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_2, \mathbf{v}_1 \rangle$ for any $\mathbf{v}_1, \mathbf{v}_2 \in V$;

- (iii) it is positive defined, meaning $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ for any $\mathbf{v} \in V$, and its value is exactly 0 only if $\mathbf{v} = 0$.

Definition 3.1.2. A euclidean vector space is the algebraic structure $(V, \langle \cdot, \cdot \rangle)$ given by a real vector space and a scalar product.

Definition 3.1.3. In a euclidean vector space V , the scalar product induces a norm that is the function

$$\begin{aligned}\| \cdot \| : V &\rightarrow \mathbb{R} \\ \mathbf{v} &\mapsto \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}\end{aligned}$$

Definition 3.1.4. Let V be a euclidean vector space and $\mathbf{v} \in V$ be a vector. Then:

- (i) if $\| \mathbf{v} \| = 1$ the vector is called unit vector;

- (ii) if $\|\mathbf{v}\| \neq 0$, the unit vector $\tilde{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is called either normalization of \mathbf{v} or unit vector associated with \mathbf{v} .

A subset U of V is called normalized if it is constituted by unit vectors.

The scalar products definable on a real vector space can be represented through appropriate matrices, once a basis has been chosen.

Definition 3.1.5. Let V be a euclidean vector space and $U = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be a finite subset of V . Then the Gram matrix associated with U is

$$G|_U = [\langle \mathbf{u}_i, \mathbf{u}_j \rangle] \in \text{Mat}(m, m; \mathbb{R}).$$

The determinant of $G|_U$ is called the gramian of U and it is noted as

$$\det(G|_U) = G(U) = G(\mathbf{u}_1, \dots, \mathbf{u}_m).$$

Proposition 3.1.6. Let V be a euclidean finitely generated vector space and $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis. Then:

- (i) the only matrix $A \in \text{Mat}(n, n; \mathbb{R})$ such that $\langle \mathbf{u}, \mathbf{w} \rangle = \mathbf{u}|_B^T A \mathbf{w}|_B$ for any $\mathbf{w}, \mathbf{u} \in V$ is the Gram matrix $G|_B$;
- (ii) the Gram matrix associated to the basis B' is $G|_{B'} = M_{B'B}^T G|_B M_{B'B}$.

Definition 3.1.7. Let V be a euclidean vector space. Two vectors $\mathbf{v}_1, \mathbf{v}_2$ are called orthogonal if $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$. If this is the case, we write $\mathbf{v}_1 \perp \mathbf{v}_2$. A subset U of V is called orthogonal if each one of its elements is orthogonal to all the others, and it is called orthonormal if the vectors are also normalized.

Now let $\mathbf{v}_1, \mathbf{v}_2 \in V$ both be different from zero. Then, the angle between the two vectors is

$$\widehat{\mathbf{v}_1 \mathbf{v}_2} = \arccos \left(\frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\|\mathbf{v}_1\| \|\mathbf{v}_2\|} \right).$$

The angle $\widehat{\mathbf{v}_1 \mathbf{v}_2}$ is well defined and it is a quantity included in the interval $[0, \pi]$. Clearly, it depends on the euclidean structure associated to the vector space.

Definition 3.1.8. Let $B_U = (\mathbf{u}_1, \dots, \mathbf{u}_m)$ be an orthonormal basis of the subspace U . Then the orthogonal projection on U is the map $P_U: \mathbb{R}^m \rightarrow \mathbb{R}^m$ according to the formula

$$P_U(\mathbf{v}) = \sum_{i=1}^m \langle \mathbf{v}, \mathbf{u}_i \rangle \mathbf{u}_i.$$

Definition 3.1.9. Given a finitely generated real vector space, the basis B has the same orientation as B' if $\det(M_{B B'}) > 0$.

The request for the vector space to be real guarantees that the inequality makes sense, since $\det(M_{B B'})$ is necessarily a real number.

Definition (3.1.9) gives a binary relation in the set of the possible bases of real vector space, which is called orientation relation.

Proposition 3.1.10. *Orientation is an equivalence relation over the set of the bases of a real vector space. Moreover, for every real space there exist two orientation classes.*

Definition 3.1.11. In a finitely generated real vector space V , let \mathcal{B} be the quotient set with the 2 classes of orientation of V as its elements. Then:

- (i) an orientation of V is a bijective function $\xi: \mathbb{B} \rightarrow \{-1, +1\}$;
- (ii) the structure (V, ξ) is called oriented real vector space;
- (iii) a basis belonging to $\xi^{-1}(1)$ is called positively oriented;
- (iv) a basis belonging to $\xi^{-1}(-1)$ is called negatively oriented.

The canonical orientation of a vector space is the one for which the canonical basis of the space, if it exists, is positively oriented.

Definition 3.1.12. Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be two euclidean vector spaces. Then the function $f: V \rightarrow W$ is a homeomorphism of euclidean spaces, or isometry, if it is a linear application such that $\langle f(\mathbf{v}_1), f(\mathbf{v}_2) \rangle_W = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_V$ holds for any $\mathbf{v}_1, \mathbf{v}_2 \in V$.

3.2 Euclidean cartesian reference systems and isometries

Definition 3.2.1. A euclidean space $\mathbb{E} = (A, V, \langle \cdot, \cdot \rangle, \psi)$ is an affine space $\mathbb{A} = (A, V, \psi)$ for which $(V, \langle \cdot, \cdot \rangle)$ is a vector euclidean space. The euclidean subspaces S of \mathbb{E} are the affine subspaces of A with euclidean structure inherited from \mathbb{E} . Finally, the dimension of a euclidean space is equal to its affine dimension.

The canonical euclidean space is $\mathbb{E}^n = (\mathbb{R}^n, (\mathbb{R}^n, \mathbb{R}, +, \cdot), \langle \cdot, \cdot \rangle_E, \psi)$ with associated affine space $\mathbb{A}_{\mathbb{R}}^n$ and the euclidean structure is determined by the euclidean scalar product.

Definition 3.2.2. Let P_1, Q_1, P_2, Q_2 be points of a euclidean space \mathbb{E} . Then:

- (i) the distance between two points is the function

$$\begin{aligned} d: A \times A &\rightarrow \mathbb{R} \\ (P_1, Q_1) &\mapsto \overrightarrow{P_1Q_1} \end{aligned}$$

and $d(P_1, Q_1)$ is called the length of the oriented segment (P_1, Q_1) ;

- (ii) the angle between the oriented segments (P_1, Q_1) and (P_2, Q_2) is

$$\overbrace{(P_1, Q_1)(P_2, Q_2)} = \overbrace{P_1Q_1P_2Q_2}.$$

Any euclidean space is also a metric space with respect to the induced distance. Indeed:

Proposition 3.2.3. *Given any points P, Q, R in the euclidean space \mathbb{E} , the distance satisfies these properties:*

- (i) *positivity, i.e. $d(P, Q) \geq 0$, and moreover $d(P, Q) = 0$ if and only if $P = Q$;*
- (ii) *simmetry, i.e. $d(P, Q) = d(Q, P)$;*
- (iii) *triangular, i.e. $d(P, Q) + d(Q, R) \geq d(P, R)$.*

The definition of reference systems (2.1.11) is naturally extended to euclidean spaces by taking into account the metric structure.

Definition 3.2.4. A cartesian reference system of $\mathbb{E} = (A, V, \langle \cdot, \cdot \rangle, \psi)$ is a reference system B_O of the associated affine space \mathbb{A} given by an origin $O \in A$ and an orthonormal basis B of V . The coordinates associated to B_O are called cartesian coordinates.

We know that a euclidean space is oriented whenever the support vector space is euclidean and oriented. In such a case, we distinguish between positively and negatively oriented cartesian reference systems, according to the orientation of the associated bases.

Definition 3.2.5. Let $\mathbb{E}_1 = (A_1, V_1, \langle \cdot, \cdot \rangle_1, \psi_1)$ and $\mathbb{E}_2 = (A_2, V_2, \langle \cdot, \cdot \rangle_2, \psi_2)$ be euclidean spaces and let $f \in \text{Hom}(\mathbb{A}_1, \mathbb{A}_2)$ be an affine transformation between the two correspondent support affine spaces. Then we call f an isometry if the associated linear mapping \bar{f} is an isometry from $(V_1, \langle \cdot, \cdot \rangle_1)$ to $(V_2, \langle \cdot, \cdot \rangle_2)$.

If two euclidean spaces have the same dimension, then f is an isometry if and only if the image of a cartesian reference system is still a cartesian system. If this is the case, isometries can be geometrically characterized as affinities which do not change the angles and the distances between geometric objects. Congruence is a concept strictly close to isometry.

Definition 3.2.6. Let \mathbb{E}_1 and \mathbb{E}_2 be two euclidean spaces and let A_1, A_2 be two subsets. A_1 and A_2 are called congruent if there exists an isometry f such that $f(A_1) = A_2$.

By definition, two sets are congruent if and only if their corresponding points have same angles and distances.

For example, translations are isometries, while homotheties are not.

3.3 Distance between a point and a subspace

Definition 3.3.1. Let S be a non-empty subset of a metric space and P be a point in the space. Then

$$d(S, P) = \inf_{Q \in S} d(Q, P).$$

By definition, the distance between two points is a real valued function bounded from below. In general, the minimum of $d(Q, P)$ for $Q \in S$ may not exist, but its infimum surely exists and therefore $d(S, T)$ is well-defined. On the other hand, if S is an affine subspace of a euclidean space \mathbb{E} then Q is the projection of P on S .

Proposition 3.3.2. *Let \mathbb{E} be a euclidean space, P be a point and S be a subspace of finite dimension. Then:*

$$(i) \quad d(S, P) = \|\overrightarrow{P_S(P)P}\|;$$

(ii) if $Q \in S$ and $B = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis of the direction U of S , then

$$d(S, P) = \sqrt{\frac{G(\mathbf{u}_1, \dots, \mathbf{u}_n, \overrightarrow{QP})}{G(\mathbf{u}_1, \dots, \mathbf{u}_n)}}.$$

Definition 3.3.3. Let S, T be non-empty subspaces of a metric space. The distance between the two subsets is

$$d(S, T) = \inf_{(P, Q) \in S \times T} d(P, Q).$$

We will focus on the study of the distance between two affine subspaces S and T of a euclidean space \mathbb{E} . The way this quantity can be computed depends on the mutual position of subspaces.

Proposition 3.3.4. *Let \mathbb{E} be a euclidean space and S, T be two subspaces of finite dimension of the direction U, W . If the two subspaces are:*

(i) *incident, then $d(S, T) = 0$ holds;*

(ii) *parallel with $W \subseteq U$, then $d(S, T) = d(S, P)$ holds, where P is a generic point of T ;*

(iii) *skewed, then $d(S, T) = d(S, T')$ holds, where T' is the smallest affine subspace containing T and parallel to S .*

Now we can write the general formula for computing the distance between two affine subspaces.

Theorem 3.3.5. *Let \mathbb{E} be a euclidean space and S, T be two affine subspaces of finite dimension and directions U, W . Given $(P, Q) \in S \times T$ and a basis $B = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ of $U + W$, then*

$$d(S, T) = \sqrt{\frac{G(\mathbf{v}_1, \dots, \mathbf{v}_n, \overrightarrow{PQ})}{G(\mathbf{v}_1, \dots, \mathbf{v}_n)}}.$$

Chapter 4

Rotation groups

4.1 Matrix representation of groups

Definition 4.1.1. A group $(G, *)$ is an algebraic structure with the set G and the internal operation

$$\begin{aligned} * : G \times G &\rightarrow G \\ (g_1, g_2) &\mapsto g_1 * g_2. \end{aligned}$$

The operation must satisfy three requirements known as the group axioms:

- Associativity: $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ for any g_1, g_2 and $g_3 \in G$;
- Identity element: there exists an element $e \in G$ such that, for every element $g_1 \in G$, the equation $e * g_1 = g_1 * e = g_1$ holds. Such an element is unique, and is called the identity element;
- Inverse element: for each $g_1 \in G$, there exists an element $g_2 \in G$, such that $g_1 * g_2 = g_2 * g_1 = e$, where e is the identity element.

Definition 4.1.2. Given two groups, $(G, *)$ and (H, \cdot) , a group homomorphism from $(G, *)$ to (H, \cdot) is a function $h: G \rightarrow H$ such that for all u and v in G it holds that $h(u * v) = h(u) \cdot h(v)$. If a group homomorphism is bijective, then the groups G and H are called isomorphic and it is a group isomorphism.

Definition 4.1.3. Given a vector space V , a linear application $f \in \text{Hom}(V, V)$ is called endomorphism of V . The set of all endomorphisms of V is called $\text{End}(V)$. An endomorphism which is also an isomorphism is called automorphism and we call $GL(V)$ the set of all automorphisms of V .

Definition 4.1.4. In a euclidean vector space V , an isometric endomorphism is an application $f \in \text{End}(V)$ which is also an isometry. The set of all isometric endomorphisms of V is called $O(V)$.

Definition 4.1.5. Let (V, ξ) be an oriented euclidean vector space. A rotation of (V, ξ) is an isometric endomorphism f of the space which respect the orientation, meaning that $\xi(f(B)) = \xi(B)$ holds for any basis B of V . The set of all the rotations of (V, ξ) is called $SO(V)$.

In other terms, a rotation is a linear application which maps any positive (negative) oriented orthonormal basis into another positive (negative) oriented orthonormal basis.

Definition 4.1.6. Now let us define the following sets of matrices:

- $GL(n, \mathbb{K}) = \{A \in Rot(n, n; \mathbb{K}), \det(A) \neq 0\}$ is called General Linear group;
- $SL(n, \mathbb{K}) = \{A \in GL(n, \mathbb{K}), \det(A) = 1\}$ is called Special Linear group;
- $O(n, \mathbb{K}) = \{A \in Rot(n, n; \mathbb{K}), AA^T = I_n\}$ is called Orthogonal group;
- $SO(n, \mathbb{R}) = O(n, \mathbb{R}) \cap GL(n, \mathbb{R})$ is called Special Orthogonal group.

Remark 4.1.7. The sets $GL(V), O(V), SO(V)$ are all groups with respect to the operation of composition and $SO(V) \subseteq O(V) \subseteq GL(V)$.

The sets $GL(n, \mathbb{R}), O(n, \mathbb{R}), SO(n, \mathbb{R})$ are all groups with respect to the operation of matrix multiplication and $SO(n, \mathbb{R}) \subseteq O(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$.

Once a proper basis has been chosen, an invertible linear application can be represented as a matrix in the corresponding group.

Theorem 4.1.8. *Let V be a finitely generated space and let \mathcal{B} be its basis. Then the map of the components $\Phi_{\mathcal{B}}$ is an isomorphism of groups from $GL(V)$ to $GL(n, \mathbb{K})$.*

If V is also euclidean and \mathcal{B} is also orthonormal, then the map of the components $\Phi_{\mathcal{B}}$ is an isomorphism of groups from $O(V)$ to $O(n, \mathbb{K})$.

Lastly, if (V, ξ) is an oriented finitely generated euclidean space and \mathcal{B} is an orthonormal basis then the map of the components $\Phi_{\mathcal{B}}$ is an isomorphism of groups from $SO(V)$ to $SO(n, \mathbb{K})$.

As a consequence of theorem (4.1.8), we observe that the set of rotations does not depend from the choice of the orientation ξ of the space, so we can speak simply of rotations of the euclidean space V .

4.2 Lie groups, Lie algebra and exponential map

A comprehensive discussion on this topic falls outside the scope of this thesis, therefore we are going to give an overview focusing only on the aspects of interest for our work.

A differential manifold is a type of manifold that is locally similar enough to a linear space to allow one to do calculus. Any manifold can be described by a collection of charts. If the charts are suitably compatible (namely, the transition from one chart to another is differentiable), then computations done in one chart are valid in any other differentiable chart.

An atlas on a topological space M is a collection of pairs $\{(U_\alpha, \varphi_\alpha)\}$ called charts, where the U_α are open sets that cover M , and $\varphi_\alpha: U_\alpha \rightarrow \mathbf{R}^n$ is a homeomorphism of U_α onto an open subset of n -dimensional real space for each index α .

The transition maps of the charts are the functions

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap U_\beta)}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta).$$

A C^k -atlas is an atlas whose transition maps are C^k . A continuous atlas is a C^0 atlas, a smooth atlas is a C^∞ atlas. If the atlas is at least C^1 , it is also called a differential structure or differentiable structure.

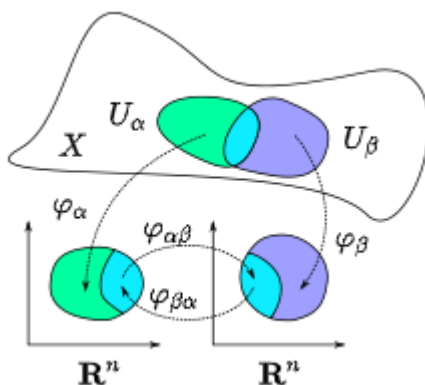


Figure 4.1: Charts on a manifold, ([https://en.wikipedia.org/wiki/Differentiable manifold](https://en.wikipedia.org/wiki/Differentiable_manifold)).

Different atlases can give rise to the same manifold. The circle can be mapped by two coordinate charts, but if the domains of these charts are changed slightly a different atlas for the same manifold is obtained. These different atlases can be combined into a bigger atlas. It can happen that the transition maps of such a combined atlas are not as smooth as those of the constituent atlases. If C^k atlases can be combined to form a C^k atlas, then they are called compatible. Compatibility of atlases is an equivalence relation; by combining all the atlases in an equivalence class, a maximal atlas can be constructed. Each C^k atlas belongs to a unique maximal C^k atlas.

A topological manifold is a second countable Hausdorff space with an equivalence class of atlases.

A differentiable manifold is a topological manifold equipped with an equivalence class of atlases whose transition maps are all differentiable.

A smooth manifold or C^∞ -manifold is a differentiable manifold for which all the transition maps are smooth.

Definition 4.2.1. A Lie group is a group that is also a differentiable manifold, in which the group operations of multiplication and inversion are differentiable maps.

The sets of the type $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ are Lie groups.

Example: Let us consider the case of the group $SO(2, \mathbb{R})$. It is a one-dimensional compact

connected Lie group which is diffeomorphic to the circle. Using the rotation angle φ as a parameter, this group can be parametrized as follows:

$$SO(2, \mathbb{R}) = \left\{ \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} : \varphi \in \mathbb{R}/(2\pi\mathbb{Z}) \right\}.$$

Addition of the angles corresponds to multiplication of the elements of $SO(2, \mathbb{R})$, and taking the opposite angle corresponds to inversion. Thus both multiplication and inversion are differentiable maps.

Definition 4.2.2. A Lie algebra is a vector space g over some field \mathbb{K} together with a binary operation $[\cdot, \cdot]: g \times g \rightarrow g$ called the Lie bracket that satisfies the following axioms:

- bilinearity, $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y]$ for all scalars $a, b \in \mathbb{K}$ and all elements $x, y, z \in g$;
- alternativity, $[x, x] = 0$ for all $x \in g$;
- the Jacobi identity, $[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$ for all $x, y, z \in g$.

We observe that, thanks to the bilinearity, the Lie bracket is determined once we know how it is evaluated on the elements of $\mathcal{B} \times \mathcal{B}$ where \mathcal{B} is a basis of the vector space G .

Examples:

1. The set of all skew-symmetric 2×2 matrices is a Lie Algebra denoted as $so(2)$. Since $so(2)$ is a vectorial space of dimension equal to 1 with respect to the operations of sum and multiplication of matrices, its basis is given by only one element:

$$E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

and the algebra is fixed by the anti-commutation operation $[E, E] = 0$.

2. The set of all skew-symmetric 3×3 matrices is a Lie algebra denoted as $so(3)$. It is spanned by the basis

$$E_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, E_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The commutation relations among this generators are

$$[E_x, E_y] = E_z, [E_y, E_z] = E_x, [E_z, E_x] = E_y.$$

Every Lie group has an associated Lie algebra, which is the tangent space around the identity element of the group. That is, the Lie algebra is a vector space generated by differentiating the

group transformations along chosen directions in the space, at the identity transformation. The tangent space has the same structure as all group elements, though tangent vectors undergo a coordinate transformation when moved from one tangent space to another.

There exists a function, called exponential map, which associates each element of the algebra \mathfrak{g} to an element of the group G .

If \mathfrak{g} and G are sets of matrices, then the exponential map is

$$\begin{aligned} \exp: \mathfrak{g} &\rightarrow G \\ A &\mapsto e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \end{aligned}$$

If G is connected and compact, then the exponential map is surjective, i.e any matrix in the group G can be derived by applying the exponential map to a matrix in the algebra.

Examples:

1. For any skew-symmetric matrix $A \in \mathfrak{so}(2)$, e^A is always in $SO(2)$ and it is a matrix of the type

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = e^{\theta E}.$$

In this case we report also the computations, since they are quite easy: we start by evaluating the powers of the element of the basis E .

$$E^0 = I, E^1 = E, E^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I, E^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -E.$$

Then we write explicitly the exponential of E :

$$\begin{aligned} e^{tE} &= \sum_{i=0}^{\infty} \left[\frac{t^{4i} I}{(4i)!} + \frac{t^{4i+1} E}{(4i+1)!} - \frac{t^{4i+2} I}{(4i+2)!} - \frac{t^{4i+3} E}{(4i+3)!} \right] = \\ &= \sum_{i=0}^{\infty} \left[\frac{t^{4i}}{(4i)!} - \frac{t^{4i+2}}{(4i+2)!} \right] I + \sum_{i=0}^{\infty} \left[\frac{t^{4i+1}}{(4i+1)!} - \frac{t^{4i+3}}{(4i+3)!} \right] E = \\ &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots \right) I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \right) E. \end{aligned}$$

By writing it as the sum of two separate summations, we observe that the first one must converge to the terms on the diagonal while the second to the terms on the antidiagonal of the resulting matrix: by writing explicitly the sum of the terms of each summation we also observe that the first one is the cosine series expansion, while the second is the sine series expansion.

2. For any skew-symmetric matrix $A \in so(3)$, e^A is always in $SO(3, \mathbb{R})$. For instance

$$R_3(\gamma) = e^{\gamma E_z} = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SO(3, \mathbb{R}),$$

$$R_2(\beta) = e^{\beta E_y} = \begin{bmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{bmatrix} \in SO(3, \mathbb{R}),$$

$$R_1(\alpha) = e^{\alpha E_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \in SO(3, \mathbb{R}).$$

Euler stated in his rotation theorem that $SO(3, \mathbb{R})$ can be parametrized with three angles, called Euler angles. Therefore it is possible to imagine any rotation as a sequence of elementary rotations with respect to each of the axes in a reference system.

In particular, a rotation can be decomposed into a rotation with angle α around the x axis, a rotation with angle β around the y axis and a rotation with angle γ around the z axis.

The choice of the order is not unique, since different combinations of elementary rotations can lead to the same final result.

Since $SO(3, \mathbb{R})$ is connected and compact (surjectivity of the exponential map), for any rotation matrix $R \in SO(3, \mathbb{R})$ there exist $A \in so(3)$ such that $R = e^A$.

Since $so(3)$ has dimension equal to 3, we can write $A = \alpha E_x + \beta E_y + \gamma E_z$ so that R is expressed as the exponential of a linear combination of the elements of the basis of $so(3)$:

$$R = e^{\alpha E_x + \beta E_y + \gamma E_z}.$$

This result is compatible with Euler's statement that each rotation is described by three parameters:

$$R = R_3(\gamma')R_2(\beta')R_1(\alpha') = \begin{bmatrix} \cos \beta' \cos \gamma' & -\cos \alpha' \sin \gamma' - \sin \alpha' \sin \beta' \cos \gamma' & \sin \gamma' \sin \alpha' - \cos \gamma' \cos \alpha' \sin \beta' \\ \sin \gamma' \cos \beta' & \cos \gamma' \cos \alpha' - \sin \beta' \sin \alpha' \sin \gamma' & -\cos \gamma' \sin \alpha' - \sin \gamma' \cos \alpha' \sin \beta' \\ \sin \beta' & \sin \alpha' \cos \beta' & \cos \alpha' \cos \beta' \end{bmatrix}.$$

Euler angles give a parametrization of the group, but not necessarily an injective one.

Indeed, a problem called *gimbal lock* happens as a result of the composition of elementary rotations.

If we choose for instance $\beta = \frac{\pi}{2}$, since $\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$ we obtain

$$R = \begin{bmatrix} 0 & -\cos \alpha \sin \gamma - \sin \alpha \cos \gamma & \sin \gamma \sin \alpha - \cos \gamma \cos \alpha \\ 0 & \cos \gamma \cos \alpha - \sin \alpha \sin \gamma & -\cos \gamma \sin \alpha - \sin \gamma \cos \alpha \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\sin(\alpha + \gamma) & -\cos(\alpha + \gamma) \\ 0 & \cos(\alpha + \gamma) & -\sin(\alpha + \gamma) \\ 1 & 0 & 0 \end{bmatrix}$$

There are infinitely many values of α and γ which leads to the same rotation matrix (for example if we choose $\alpha = \beta = \frac{\pi}{3}$ or $\alpha = 0$ and $\beta = \frac{2\pi}{3}$).

Chapter 5

Quadrics and conics

5.1 Quadric hypersurfaces

Throughout this section, we work in an oriented euclidean space \mathbb{E} with dimension equal to n .

Definition 5.1.1. Respectively, the *hypersphere* and the *hyperdisk* with center C and radius R are the subsets of points

$$S_{C,R} = \{P \in \mathbb{E} | d(C, P) = R\} \text{ and } D_{C,R} = \{P \in \mathbb{E} | d(C, P) \leq R\}.$$

If $\dim(\mathbb{E}) = 2$ then S and D are a circumference and a circle respectively, while if $\dim(\mathbb{E}) = 3$ they are a sphere and a disk.

Let us choose a positive-oriented cartesian reference system B_O of \mathbb{E} , with respect to which the generic point P and the center C have coordinates $P|_{B_O} = [x_1 \ \dots \ x_n]^T$ and $C|_{B_O} = [\bar{x}_1 \ \dots \ \bar{x}_n]^T$.

Therefore it is

$$\overrightarrow{CP}|_B = P|_{B_O} - C|_{B_O} = \begin{bmatrix} x_1 - \bar{x}_1 \\ \vdots \\ x_n - \bar{x}_n \end{bmatrix} \text{ and } d(C, P) = \|\overrightarrow{CP}\|.$$

Definition 5.1.2. Let B_O be a positive-oriented cartesian reference system of \mathbb{E} . A quadric hypersurface \mathcal{Q} is the locus of the points with coordinates x_1, \dots, x_n satisfying a second order equation of the type

$$\sum_{i=1}^n a_{ii}x_i^2 + 2 \sum_{i=1}^n \sum_{j>i}^n a_{ij}x_i x_j + 2 \sum_{i=1}^n b_i x_i + c = 0,$$

where a_{ij}, b_i and c are real coefficients for any $1 \leq i \leq j \leq n$. If $\dim(\mathbb{E}) = 2$ the quadric hypersurfaces are called conic curves, while if $\dim(\mathbb{E}) = 3$ they are called quadric surfaces.

The hypersphere is an example of quadric hypersurface.

Definition 5.1.3. Given a hyperquadric, we set:

- (i) the matrix of second order coefficients $A \in \mathbb{S}(n; \mathbb{R})$ with elements

$$(A)_{ij} = \begin{cases} a_{ij} & \text{if } 1 \leq i \leq j \leq n, \\ a_{ji} & \text{otherwise} \end{cases} ;$$

- (ii) the column of the first order coefficients $B \in \text{Mat}(n, 1; \mathbb{R})$ with elements $(B)_{i1} = b_i$ for $1 \leq i \leq n$,

- (iii) the complete matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in \mathbb{S}(n+1; \mathbb{R})$.

Proposition 5.1.4. *Given the columns*

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \text{Mat}(n, 1; \mathbb{R}) \text{ and } Y = \begin{bmatrix} X \\ 1 \end{bmatrix} \in \text{Mat}(n+1, 1; \mathbb{R}),$$

the equation of the quadric hypersurface is $X^T A X + 2B^T X + C = 0$ or equivalently $Y^T C Y = 0$.

Definition 5.1.5. The epipolar plane of a quadric \mathcal{Q} with respect to the point $P = (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{E}^3$ is defined as the plane of equation

$$\begin{bmatrix} \tilde{x} & \tilde{y} & \tilde{z} & 1 \end{bmatrix} C \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0.$$

If P lies on the quadric surface, then the epipolar plane coincides with the plane tangent to the quadric in P , i.e. it is the only plane crossing the surface along a degenerate conic which contains the point P .

5.2 Euclidean affine spaces and euclidean elements

Let us recall the five postulates of the plane euclidean geometry:

- (i) a segment of a line can be drawn between any couple of points;
- (ii) any segment can be extended to a line;
- (iii) given any segment, it is possible to draw a circle with this segment as radius and one of its extreme points as centre;
- (iv) all right angles are congruent;

- (v) given a line and one point not laying on it, there exist one and only one line containing the point and not intersecting the line.¹

In the section (8.4.2), we have defined the concepts of point, segment, line and plane in the context of affine spaces. By construction, each couple of points defines a unique segment that lays on a line, which exists and it is unique. Therefore we have built a model for Euclid's geometry.

In particular, we defined the structure of vector euclidean space in section (8.4.1) in order to define the so-called metric properties of geometric objects, i.e. their length and the angles they form.

5.3 Reduction to canonical form of quadrics

Let B_O and \tilde{B}_O be two positive-oriented cartesian reference systems in the euclidean space \mathbb{E} of dimension n . Then the coordinates of a point P with respect to the two system are related by $P|_{B_O} = M_{\tilde{B}B}P|_{\tilde{B}_O} + \tilde{O}|_{B_O}$ where $M_{\tilde{B}B} \in SO(n; \mathbb{R})$.

From now on, we denote $X = P|_{B_O}$, $\tilde{X} = P|_{\tilde{B}_O}$, $T = \tilde{O}|_{B_O}$ and $Q = M_{\tilde{B}B}$.

Therefore $X = Q\tilde{X} + T$.

Proposition 5.3.1. *Let \mathcal{Q} be a quadric with associated matrices $A \in \mathbb{S}(n, \mathbb{R})$, $B \in \text{Mat}(n, 1; \mathbb{R})$, $C \in \mathbb{S}(n + 1; \mathbb{R})$ and constant term $c \in \mathbb{R}$ with respect to B_O . Then the hypersurface \mathcal{Q} is still described by a second order equation with respect to \tilde{B}_O and its associated matrices are $\tilde{A} = Q^T A Q \in \mathbb{S}(n, \mathbb{R})$, $\tilde{B} = Q^T (AT + B) \in \text{Mat}(n, 1; \mathbb{R})$ and $\tilde{C} = F^T C F$ where*

$$F = \begin{bmatrix} Q & T \\ 0_{[1 \times n]} & 1 \end{bmatrix} \in \text{Mat}(n + 1, n + 1, \mathbb{R}).$$

Finally, the constant term of \mathcal{Q} with respect to \tilde{B}_O is $\tilde{c} = T^T AT + 2B^T T + c$.

Theorem 5.3.2. *Let us suppose that the matrices A and C associated to the quadric \mathcal{Q} have rank equal to r and q respectively. Moreover, let $\lambda_1, \dots, \lambda_r$ be the non-zero eigenvalues of A , eventually overlapping. Then there exists a positive-oriented cartesian reference system \tilde{B} of \mathbb{E} which is canonical for \mathcal{Q} , with respect to which the equation of the quadric is:*

- $\lambda_1 \tilde{x}_1^2 + \dots + \lambda_r \tilde{x}_r^2 = 0$ if $q = r$;
- $\lambda_1 \tilde{x}_1^2 + \dots + \lambda_r \tilde{x}_r^2 + \tilde{c} = 0$ with $\tilde{c} \neq 0$ if $q = r + 1$;
- $\lambda_1 \tilde{x}_1^2 + \dots + \lambda_r \tilde{x}_r^2 + 2p\tilde{x}_{r+1} = 0$ with $p > 0$ if $q = r + 2$.

Corollary 5.3.3. *After the change of coordinates $X = Q\tilde{X} + T$, the matrices \tilde{A} and \tilde{C} still have the same rank as A and C respectively, \tilde{A} has the same eigenvalues of A and \tilde{C} has the same determinant as C .*

¹This is an equivalent version of the fifth of Euclid's postulates.

Definition 5.3.4. Given a representation of the quadric \mathcal{Q} through the matrices A and C , the invariants of the matrices are:

- $I_i = (-1)^{n-i}c_{n-i}$ where c_i are the coefficients of the characteristic polynomial of A for $1 \leq i \leq n$;
- $I_{n+1} = \det(C)$.

Example: Let us consider the conic curve $\mathcal{Q} \in \mathbb{E}^3$ with equation

$$x^2 + 3y^2 - 2y - 1 = 0$$

with respect to a reference system \mathcal{B} . The matrices associated to \mathcal{Q} are

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & -1 \end{bmatrix}.$$

The invariants are $I_1 = 4, I_2 = 3$ and $I_3 = -4$. As we will see in the next theorem (5.4.1), we can deduce that \mathcal{Q} is a real ellipse. The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$, while for the constant term we observe $\tilde{c} = \frac{I_3}{I_2} = -\frac{4}{3}$. Therefore a canonical representation of \mathcal{Q} is $\tilde{x}^2 + 3\tilde{y}^2 - \frac{4}{3} = 0$.

Remark 5.3.5. Let $Q \in SO(3, \mathbb{R})$ be a rotation matrix and $T \in \mathbb{R}^3$ a translation vector, the matrices F of the same type as in proposition 5.3.1 form the Lie group of rototranslations called Special Euclidean group, $SE(3, \mathbb{R})$. The associated Lie algebra is called $se(3)$ and it is spanned by the basis $(E_x, E_y, E_z, T_x, T_y, T_z)$, where

$$T_x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, T_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, T_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The commutators among the subset of generators (E_x, E_y, E_z) are the same as the ones for $so(3)$, while the commutators for (T_x, T_y, T_z) are

$$[T_x, T_y] = 0_{[4 \times 4]}, [T_y, T_z] = 0_{[4 \times 4]}, [T_z, T_x] = 0_{[4 \times 4]}$$

and the mixed commutators between the two subsets are

$$\begin{aligned} [E_x, T_x] &= [E_y, T_y] = [E_z, T_z] = 0_{[4 \times 4]}, [E_y, T_x] = [E_x, T_y] = T_z, \\ [E_z, T_x] &= T_y, [E_x, T_z] = -T_y, [E_y, T_z] = [E_z, T_y] = -T_x. \end{aligned}$$

In $SE(\mathbb{E}^3)$, the group of rototranslations in \mathbb{E}^3 , if we choose a cartesian reference system we have a non-compact Lie group, but it can be shown that the surjectivity holds also for the translations and rototranslations matrices.

5.4 Classification of conic curves and quadric surfaces

Historically, a *conic section* has been defined as any curve that we get by intersecting a cone and a plane. Depending on the way we intersect the plane with the cone we get four different types of the conic. The curves that we get can be either *open* or *closed*.

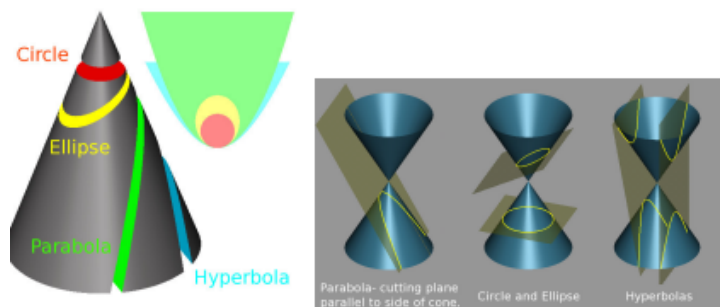
If the plane is parallel to the directrix of the cone, we get a *circle*, which is a closed curve. If the plane is not parallel to the directrix but the curve we get by the intersection is still closed, then the curve is an *ellipse*. When the plane is parallel to a generatrix of the cone (i.e. to the side of the cone) we get an open curve named *parabola*. And when the plane is not parallel to the side of the cone and the curve is open, it is named *hyperbola*.

These are the so called nondegenerate conics. Then there are also the so called *degenerate conics* which are intersections of the cone and the plane going through the tip of the cone (resulting in either a point or two lines).

Recalling the definition (5.1.2), we can observe that the historical definition of conic curve is consistent with the one given in the affine geometry context in the case $\dim(\mathbb{E}) = 2$.

Theorem 5.4.1. *In \mathbb{E}^2 the equation of any nondegenerate conic curve Q can be reduced to one and only one of the following canonic forms:*

$r(C)$	$r(A)$	I_3	I_2	$I_1 I_3$	Canonical form	Modules	Name
3	2	$\neq 0$	>0	<0	$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} = 1$	$\alpha, \beta > 0$	real ellipse
3	2	$\neq 0$	>0	>0	$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} = -1$	$\alpha, \beta > 0$	non real ellipse
3	2	$\neq 0$	<0		$\frac{\tilde{x}^2}{\alpha^2} - \frac{\tilde{y}^2}{\beta^2} = 1$	$\alpha, \beta > 0$	hyperbola
3	1	$\neq 0$	$=0$		$\tilde{x}^2 - 2\rho\tilde{y} = 0$	$\rho > 0$	parabola



: Conic classification in the Euclidean Framework. Images: <http://en.wikipedia.org/wiki/Conic>

Now let us focus on the quadric hypersurfaces in the case $\dim(\mathbb{E}) = 3$, i.e. quadrics. Quadric surfaces in \mathbb{E}^3 can be classified in an equivalent way as conics in \mathbb{E}^2 , as a direct consequence of theorem (5.3.2) and of the definition of the invariants:

Theorem 5.4.2. *In \mathbb{E}^3 the equation of quadric surface Q associated to the matrix C such that $r(C) \in \{3; 4\}$ can be reduced to one and only one of the following canonic forms:*

$r(C)$	$r(A)$	I_4	I_3	I_2	I_1I_3	Canonical form	Modules	Name
4	3	< 0	$\neq 0$	> 0	> 0	$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} + \frac{\tilde{z}^2}{\gamma^2} = 1$	$\alpha, \beta, \gamma > 0$	real ellipsoid
4	3	> 0	$\neq 0$	> 0	> 0	$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} + \frac{\tilde{z}^2}{\gamma^2} = -1$	$\alpha, \beta, \gamma > 0$	non real ellipsoid
4	3	> 0	$\neq 0$	≤ 0		$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} - \frac{\tilde{z}^2}{\gamma^2} = 1$	$\alpha, \beta, \gamma > 0$	hyperbolic
4	3	> 0	$\neq 0$		≤ 0			hyperboloid
4	3	< 0	$\neq 0$	≤ 0		$\frac{\tilde{x}^2}{\alpha^2} - \frac{\tilde{y}^2}{\beta^2} - \frac{\tilde{z}^2}{\gamma^2} = 1$	$\alpha, \beta, \gamma > 0$	elliptic
4	3	< 0	$\neq 0$		≤ 0			hyperboloid
4	2	< 0	$= 0$	> 0	$= 0$	$\frac{\tilde{x}^2}{\alpha^2} + \tilde{y}^2 - 2\rho\tilde{z} = 0$	$\alpha, \rho > 0$	elliptic paraboloid
4	2	> 0	$= 0$	< 0	$= 0$	$\frac{\tilde{x}^2}{\alpha^2} - \tilde{y}^2 - 2\rho\tilde{z} = 0$	$\alpha, \rho > 0$	hyperbolic paraboloid
3	3	$= 0$	$\neq 0$	> 0	> 0	$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} + \tilde{z}^2 = 0$	$\alpha, \beta > 0$	cone with only one real point
3	3	$= 0$	$\neq 0$	≤ 0		$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} - \tilde{z}^2 = 0$	$\alpha, \beta > 0$	real infinite
3	3	$= 0$	$\neq 0$		≤ 0			cone
3	2	$= 0$	$= 0$	> 0	$= 0$	$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} = 1$	$\alpha, \beta > 0$	real elliptic cylinder
3	2	$= 0$	$= 0$	> 0	$= 0$	$\frac{\tilde{x}^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} = -1$	$\alpha, \beta > 0$	non real elliptic cylinder
3	2	$= 0$	$= 0$	< 0	$= 0$	$\frac{\tilde{x}^2}{\alpha^2} - \frac{\tilde{y}^2}{\beta^2} = 1$	$\alpha, \beta > 0$	hyperbolic cylinder
3	1	$= 0$	$= 0$	$= 0$	$= 0$	$\tilde{x}^2 - 2\rho\tilde{y} = 0$	$\rho > 0$	parabolic cylinder

Chapter 6

Performance evaluation

The problem at hand belongs to the class of geometric fitting: we fit a parameterized geometric model expressed as implicit equations to a set of data $(\mathbf{x}_i)_{i=1\dots N}$ which are geometrically constrained to be on predetermined curves, surfaces, and hypersurfaces, and the parameters to be estimated may also be similarly constrained.

Each \mathbf{x}_i is assumed to be perturbed by independent noise from its true value $\underline{\mathbf{x}}_i$ which strictly satisfies an implicit equation

$$F(\underline{\mathbf{x}}_i, \mathbf{p}) = 0, \quad (6.1)$$

where $\underline{\mathbf{x}}_i$ is said to be in the data space domain and \mathbf{p} in the parametric space domain.

In traditional statistical estimation, we can arrange the observable set of data into a high dimensional vector $\mathbf{X} = [\mathbf{x}_1^T \ \dots \ \mathbf{x}_N^T]^T$ and the noise into a vector $\mathbf{E} = [\epsilon_1^T \ \dots \ \epsilon_N^T]^T$ where ϵ_i is the noise on \mathbf{x}_i .

If $\underline{\mathbf{X}}$ is the true value of \mathbf{X} , then we obtain the statistical model

$$\mathbf{X} = \underline{\mathbf{X}} + \mathbf{E} \quad (6.2)$$

where $\underline{\mathbf{X}}$ is the unknown to be estimated.

If $f(\mathbf{E})$ is the probability density of the noise vector \mathbf{E} , the task is to estimate $\underline{\mathbf{X}}$ from \mathbf{X} , i.e. a sampling from $f(\underline{\mathbf{X}} - \mathbf{X})$.

The trouble is that the parameter \mathbf{p} which we really want to estimate is not contained in this model (the existence of the parameters is implicit in the sense that they constrain the mutual relations among the components of \mathbf{x}_i).

In order to make the implicit constraint explicit, the standard approach consists in introducing new parameters \mathbf{t} to turn the implicit constraint equations (7.3) into an explicit form by substituting into the equations of the model (6.2) $\mathbf{x}_i = \mathbf{x}_i(\mathbf{p}, \mathbf{t})$.

Besides the original parameter \mathbf{p} that we want to estimate, there exist the so-called nuisance parameters \mathbf{t} and their number increase with the amount of data (see [21] and [12]): this is referred to as the Neyman-Scott problem, which yields some difficulty in accurately estimating the parameters of interest when conventional Maximum Likelihood Estimation (MLE) is applied.

6.1 Dual approach of Kanatani

Kanatani argued in [22] that MLE is still useful for those problems by considering a different asymptotic sense from the one used in conventional statistics, provided that we take a limit as the noise level goes to zero instead of a limit as the data size N approaches infinity (recalling the large sample consistency property which states that an estimator is consistent if its variance approaches the Cramer Rao lower bound for $N \rightarrow \infty$).

Kanatani showed that the resulting mathematical consequences have corresponding traditional results in a dual form, e.g., the Kanatani Cramer Rao lower bound (KCRLB), which we define in section (6.3), corresponds to the traditional Cramer Rao (CR) lower bound.

The correspondence is dual in the sense that small noise expansions have the form $\dots + O(\epsilon^k)$ for geometric fitting, to which correspond traditional asymptotic expansions in the form $\dots + O(\frac{1}{\sqrt{n^k}})$.

In geometric fitting, data always have the same value however many times we observe them because the true values are assumed to be fixed points, therefore the number n of observations is always 1. Even if we suppose they change their values, if we take their sample mean, its standard deviation is $\frac{1}{\sqrt{n}}$ times that of individual observations.

This means that repeating hypothetical observations n times effectively reduces the noise level ϵ to $\frac{\epsilon}{\sqrt{n}}$.

Thus, the behavior of estimation for $\epsilon \rightarrow 0$ is mathematically equivalent to the asymptotic behavior for $n \rightarrow \infty$ of the number n of hypothetical observations (not the number N of data).

6.2 Cramer-Rao's lower bound for statistical estimators

In a statistical parameter estimation problem, the Cramer-Rao lower bound (CRLB) answers the question: what is the *accuracy attainable* in the estimation of statistical parameters?

If we indicate with $E_x[\cdot]$ the expected value with respect to the distribution of x , we can give the following statements for the CR in the scalar and vectorial case:

- (i) If $\hat{p} = \hat{p}(\mathbf{x})$ is an unbiased estimator for the true parameter $p = p_0$, the variance is lower bounded as $var[\hat{p}(\mathbf{x})] \geq C_{CRB} = I^{-1}(p_0)$ where the term

$$I(p_0) = -E_x \left[\frac{d^2 l(\mathbf{x}|p)}{dp^2} \right]_{p=p_0} = E_x \left[\left(\frac{dl(\mathbf{x}|p)}{dp} \right)^2 \right]_{p=p_0}$$

depends on the loglikelihood function $l(\mathbf{x}|p)$ and it is referred to as Fisher Information term.

- (ii) For multiple parameters, let $\hat{\mathbf{p}} = \hat{\mathbf{p}}(\mathbf{x})$ be an unbiased estimator for the model in $\mathbf{p} = \mathbf{p}_0$, ($E[\mathbf{p}(\mathbf{x})] = \mathbf{p}_0$), the Cramer Rao Bound sets the bound for the covariance as

$cov[\hat{\mathbf{p}}(\mathbf{x})] \geq C_{CRB} = I^{-1}(\mathbf{p}_0)$ where $I(\mathbf{p}_0)$ is the Fisher Information Matrix with entries

$$[I(\mathbf{p}_0)]_{ij} = -E_x \left[\frac{d^2 l(\mathbf{x}|p)}{dp_i dp_j} \right]_{p=\mathbf{p}_0} = E_x \left[\frac{dl(\mathbf{x}|p)}{dp_i} \frac{dl(\mathbf{x}|p)}{dp_j} \right]_{p=\mathbf{p}_0}$$

implying that the difference $cov[\hat{\mathbf{p}}(\mathbf{x})] - I^{-1}(\mathbf{p}_0)$ is positive semidefinite.

In other words, the Cramer-Rao bound (CRB) matrix provides a lower bound on the covariance matrix of any unbiased estimate of a nonrandom parameter vector.

Geometrically, the Cramer-Rao (CR) bound means that constraints on estimation are seen to induce an oblique projection of the columns of the inverse Fisher information matrix onto a linear subspace tangent to the parameter constraint set.

6.3 Kanatani Cramer-Rao Lower Bound

The matrix V , representing the leading terms of the variance, has a natural lower bound (an analogue of the CR bound): there is a symmetric positive semi-definite matrix V_{min} such that for every geometrically consistent estimator $V \geq V_{min}$ in the sense that $V - V_{min}$ is a positive semi-definite matrix.

In fact, the matrix V_{min} coincides with V_{MLE} , i.e. the variance of the Maximum Likelihood Estimator:

$$V_{min} = V_{MLE} = \left(\sum_{i=1}^n \frac{P_{\mathbf{p}} P_{\mathbf{p}}^T}{\|P_{\mathbf{x}_i}\|^2} \right)^{-1}$$

where P is a function of the parameters and of the model variables and

$$P_{\mathbf{p}} = \left[\frac{\partial P}{\partial p_1} \quad \dots \quad \frac{\partial P}{\partial p_k} \right]^T \quad \text{and} \quad P_{\mathbf{x}_i} = \left[\frac{\partial P}{\partial x_1} \quad \dots \quad \frac{\partial P}{\partial x_m} \right]^T$$

stand for the gradient of P with respect to the model parameters p_1, \dots, p_k and for the gradient P with respect to the variables respectively.

This formula has been derived independently in [6] and [10].

Kanatani showed that for any unbiased estimator $V \geq V_{min}$, but this has been proven to hold also for biased estimators in [6].

6.3.1 Noise description and estimators accuracy

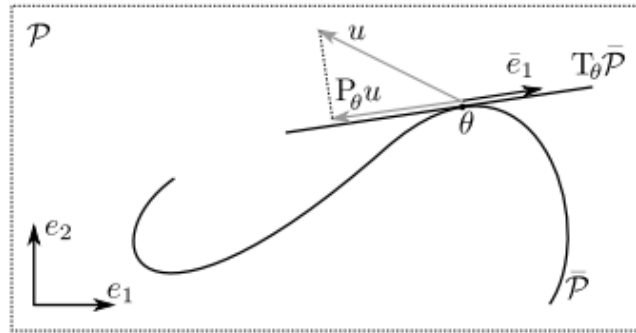


Fig. 1. $\bar{\mathcal{P}}$ is a Riemannian submanifold of \mathcal{P} . We consider estimation problems for which the parameter to estimate is θ , a point of $\bar{\mathcal{P}}$. In this drawing, for simplicity, we chose $\mathcal{P} = \mathbb{R}^2$. The vectors $e = (e_1, e_2)$ form an orthonormal basis of $T_\theta \mathcal{P} \equiv \mathbb{R}^2$, while $\bar{e} = (\bar{e}_1)$ is an orthonormal basis of the tangent space $T_\theta \bar{\mathcal{P}}$. The operator P_θ projects vectors of $T_\theta \mathcal{P}$ orthogonally onto $T_\theta \bar{\mathcal{P}}$. We express the Cramér-Rao bounds for such problems in terms of the basis e , which at times may be more convenient than defining a basis \bar{e} for each point θ .

Since we are focusing only on small noise effects in the dual approach of Kanatani framework, we can simply assume that noise concentrates on a small region around the true value.

As such, noise can be regarded as effectively occurring in the tangent space at that point.

Within this tangent space, the noise distribution can be regarded as Gaussian because the discrepancy at the tail part is of higher order terms.

Accordingly, we define the covariance matrix of \mathbf{x}_i by

$$V[\mathbf{x}_i] = E[(P_{\underline{\mathbf{x}}_i}(\mathbf{x}_i - \underline{\mathbf{x}}_i))(P_{\underline{\mathbf{x}}_i}(\mathbf{x}_i - \underline{\mathbf{x}}_i))^T]$$

where $E[\cdot]$ denotes expectation over the noise distribution, and $P_{\underline{\mathbf{x}}_i}$ denotes projection onto the tangent space to the domain of the data at \mathbf{x}_i .

The estimator of \mathbf{p} computed by solving the geometric fitting problem is $\hat{\mathbf{p}}(\mathbf{x}_1, \dots, \mathbf{x}_N)$.

A natural requirement is that the true value should be obtained in the absence of noise (condition of *consistency*):

$$\lim_{\epsilon \rightarrow 0} \hat{\mathbf{p}} = \mathbf{p}$$

Since $\hat{\mathbf{p}}$ is a function of random variables, it may be constrained too and its domain may not be euclidean and we can measure its accuracy by its covariance matrix: $V[\hat{\mathbf{p}}] = E[(P_{\mathbf{p}}(\mathbf{p} - \hat{\mathbf{p}}))(P_{\mathbf{p}}(\mathbf{p} - \hat{\mathbf{p}}))^T]$ where $P_{\mathbf{p}}$ denotes projection onto the tangent space of the domain of \mathbf{p} .

Kanatani proves in [21] and [22] that if each datum \mathbf{x}_i is an independent Gaussian random variable with mean $\underline{\mathbf{x}}_i$ and covariance matrix $V[\mathbf{x}_i]$, then for an arbitrary unbiased estimator $\hat{\mathbf{p}}$ of \mathbf{p} the following expression holds:

$$V[\hat{\mathbf{p}}] \geq \left(\sum_{i=1}^N \frac{(P_{\mathbf{p}} \nabla_{\mathbf{p}} \bar{F}_i)(P_{\mathbf{p}} \nabla_{\mathbf{p}} \bar{F}_i)^T}{\langle \nabla_{\mathbf{x}} \bar{F}_i, V[\mathbf{x}_i] \nabla_{\mathbf{x}} \bar{F}_i \rangle} \right)^\dagger \quad (6.3)$$

where the superscript \dagger denotes pseudoinverse and the symbols $\nabla_p \bar{F}_i, \nabla_x \bar{F}_i$ denote the gradient of the function $F(\mathbf{x}, \mathbf{p})$, with respect to \mathbf{p}, \mathbf{x} respectively, evaluated at $\mathbf{x} = \underline{\mathbf{x}}_i$.

6.3.2 Linearized constraint optimization example

If the constraint equation (7.3) for the geometric fitting problem can be reduced to a linear form

$$\langle \xi(\mathbf{x}), \mathbf{p} \rangle = 0 \quad (6.4)$$

by changing variables from \mathbf{x} to $\boldsymbol{\xi}$, then the KCRLB (7.8) has the form

$$V[\hat{\mathbf{p}}] \geq \left(\sum_{i=1}^N \frac{\xi(\underline{\mathbf{x}}_i) \xi(\underline{\mathbf{x}}_i)^T}{\langle \mathbf{p}, V[\xi(\underline{\mathbf{x}}_i)] \mathbf{p} \rangle} \right)^\dagger$$

The covariance matrix of $\xi(\mathbf{x}_i)$ has the form $V[\xi(\mathbf{x}_i)] = \nabla_x \xi(\underline{\mathbf{x}}_i)^T V[\mathbf{x}_i] \nabla_x \xi(\underline{\mathbf{x}}_i)$.

If data \mathbf{x}_i are vectors of \mathbb{R}^m and the parameter vector \mathbf{p} is in \mathbb{R}^k , then the Jacobian is a $[m \times k]$ matrix

$$\nabla_x \xi(\underline{\mathbf{x}}_i) = \nabla_x \xi(\mathbf{x})|_{\mathbf{x}=\underline{\mathbf{x}}_i} = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \cdots & \frac{\partial \xi_p}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial \xi_1}{\partial x_m} & \cdots & \frac{\partial \xi_p}{\partial x_m} \end{bmatrix}_{\mathbf{x}=\underline{\mathbf{x}}_i} .$$

We do not need the projection operator here because $\xi(\underline{\mathbf{x}}_i)$ is orthogonal to \mathbf{p} due to equation (6.4).

Chapter 7

The model

Throughout this section, we consider all the involved variables to be deterministic, therefore not affected by errors. We will add and discuss the statistical errors in the model in section 8.

As we already summarized in the introduction, our problem consists in problem estimating the minimum error which occurs when measuring the radius of a perfect cylinder with a system composed by m calibrated independent cameras.

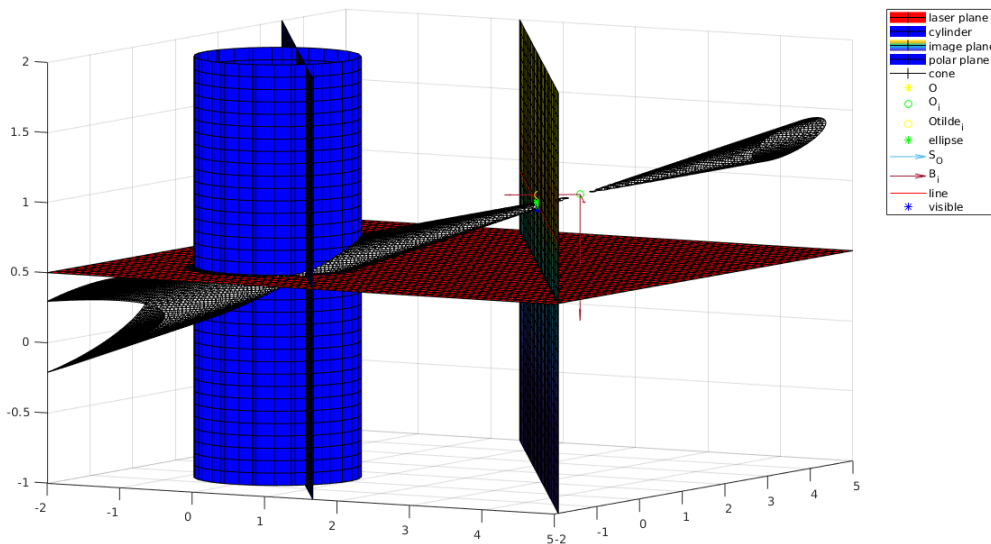


Figure 7.1: output of the Matlab code which simulates the whole system for one camera (placed in the green circle point named O_i): we can see the cylinder, the laser plane, the polar plane, the image plane and the cone whose vertex is the camera and whose intersection with the image plane is an ellipse (projection of the intersection between the cylinder and the laser plane).

7.1 Camera and laser

We have a system composed by a number m of cameras and m lasers pointing in the direction of the cylinder. Each pair of one camera and one laser points towards a slightly different arc section of the cylinder surface, ideally to cover the cylinder diameter.

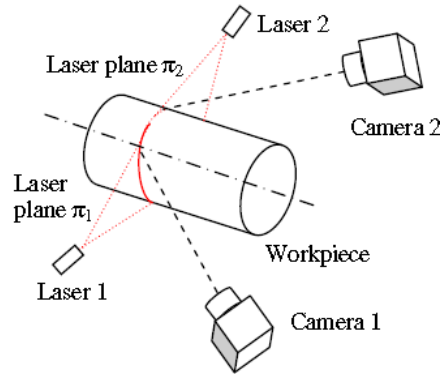


Figure 7.2: Example of a system of two camera/laser pairs, picture from [18]

Our data are the coordinates of the laser points in the image plane of the camera.

Let us now define some concepts we use from 3D computer graphics:

- The image plane of a camera is that plane in the world which is identified with the plane of the monitor. The viewing transformation (conceptual model that describes what steps a graphics system needs to perform to render a 3D scene to a 2D screen) is a projection that maps the world onto the image plane. A rectangular region of this plane, called the viewing window, maps to the monitor. This establishes the mapping between pixels on the monitor and points in the 3D world.
- The camera (or optical) center is the point in 3D space through which all projection lines must pass.
- The distance between the optical center and the image plane is called focal length.

From the mathematical point of view, a camera is a cartesian reference system in \mathbb{E}^3 , a center and focal length parameter: it is an object which *transfers* points from the three-dimensional world to the image plane and it can be represented by a map $H: \mathbb{E}^3 \rightarrow \mathbb{E}^2$ that associates a projection $\mathbf{x} = H\mathbf{X}$ on the image plane (in \mathbb{E}^2) to any point \mathbf{X} in the affine space \mathbb{E}^3 . The map H is therefore a $[3 \times 4]$ matrix and it is called projection matrix.

On the other hand, the plane defined by the intersection between the cylinder surface and the arc section defined by laser scanner is called laser plane and it is mathematically represented with the equation of an affine plane.

We have the following setting: our data are the coordinates of the points in the image plane of each camera, and we know that each point in the image plane is a projection of a point on the surface of the cylinder.

We also know that the set of points seen by each camera define approximately the arc of an ellipse on the surface of the cylinder.

The intersection between the arc (which we parametrize as an ellipse, so we will have to discard the points lying on the surface of the cylinder opposite to the camera, which cannot be seen) and the cylinder results into a plane which we call π_i , $i = 1 \dots m$.

In a reference orthonormal system of π_i , for each camera we obtain a quadric which is a cone with vertex in the camera center and whose flat base is defined by the intersection between the arc of points and the cylinder surface.

In order to set up our approach, for each camera we choose an orthonormal reference system $B_{O_i} = (O_i, (\mathbf{v}_{i1}, \mathbf{v}_{i2}, \mathbf{v}_{i3}))$ and we assume that it follows the left hand rule to be coherent with the reference system $\mathcal{B}_{\tilde{O}_i} = (\mathbf{v}_{i1}, \mathbf{v}_{i2})$ of its image plane $\tilde{\pi}_i$.

We choose $\mathbf{v}_{i1}, \mathbf{v}_{i2}$ to be parallel to the image plane $\tilde{\pi}_i$, while consequently \mathbf{v}_{i3} is perpendicular to $\tilde{\pi}_i$.

We define also the basis corresponding to the image plane reference system as $B_{\tilde{O}_i} = (\tilde{O}_i, (\mathbf{v}_{i1}, \mathbf{v}_{i2}))$.

Let f_i be the focal length of the i th camera.

We can write $\overrightarrow{O_i \tilde{O}_i} = f_i \mathbf{v}_{i3}$.

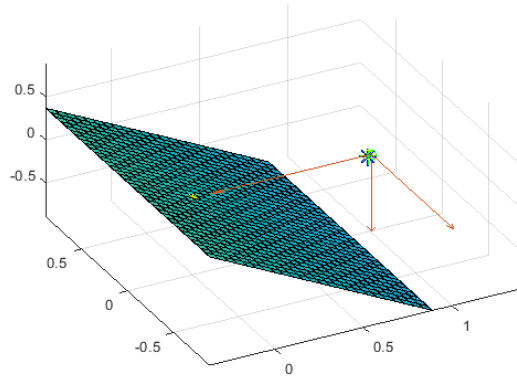


Figure 7.3: the laser plane π_i and the camera reference system \mathcal{B}_{O_i} , Matlab plot

As for the laser plane π_i , we call \mathbf{n}_i the normal to this plane and S_i the point on the plane representing the source.

Now we call S the basis of the direction in the affine space $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ so that we can write

$$\mathbf{X} = \mathbf{P}|_{S_o} = \overrightarrow{O\tilde{P}}|_S = \overrightarrow{O\tilde{O}_i}|_S + \overrightarrow{\tilde{O}_i\tilde{P}}|_S = \overrightarrow{O\tilde{O}_i}|_S + M_{B_i S} \overrightarrow{\tilde{O}_i\tilde{P}}|_{B_i S}$$

where $M_{B_i S}$ is the change of basis matrix between two orthonormal bases with the same orientation, which means that $M_{B_i S}$ represents a rotation and $M_{B_i S} \in SO(3, \mathbb{R})$.

We name the unknowns in the expression of \mathbf{X} as

$$\mathbf{T}_i = \overrightarrow{O\tilde{O}_i}|_S \text{ and } Q_i = M_{B_i S}. \quad (7.1)$$

Notice that if $\mathbf{X}_i = \mathbf{P}|_{B_{O_i}}$ then $\mathbf{X} = Q_i \mathbf{X}_i + \mathbf{T}_i$.

In order to project the points from the laser plane to the image plane, we have to solve the following system of equations for each camera in B_{O_i} :

$$\begin{cases} \langle \mathbf{n}_i, \overrightarrow{S_i P} \rangle = 0 \\ \overrightarrow{O_i P} = t \overrightarrow{O_i Q} \\ \overrightarrow{O_i Q} = t_1 \mathbf{v}_{i1} + t_2 \mathbf{v}_{i2} + f \mathbf{v}_{i3} \end{cases} \quad (7.2)$$

where the first equation describes the laser plane and the third represents the image plane of the camera.

The second equation represents the line which contains the points \mathbf{P} and \mathbf{Q} , or equivalently we may say that it describes any point \mathbf{Q} laying on the line that connects the center of the reference system O_i to a point \mathbf{P} laying on the elliptic intersection between the laser plane and the surface of the cylinder (i.e. the section represented in figure (7.4)).

If $t > 1$ we obtain a point laying on the line connecting point \mathbf{P} to \mathbf{Q} , if $t < 1$ instead we still obtain a point on the same line but farther from the origin than \mathbf{P} .

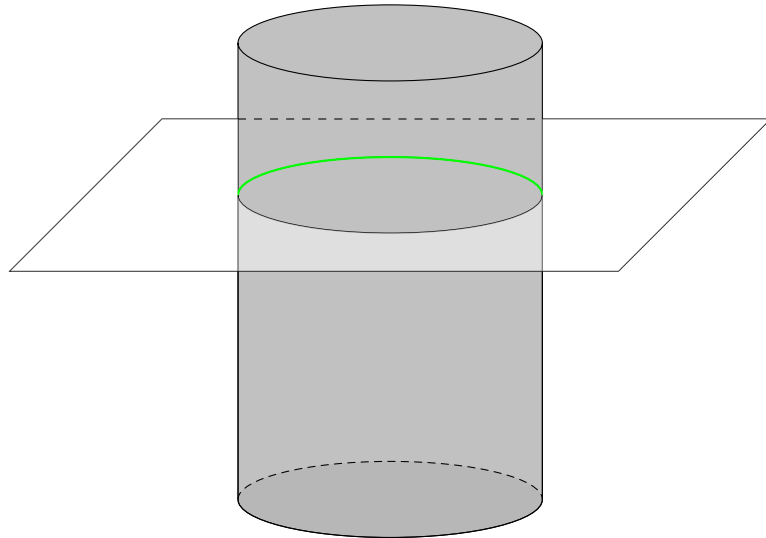


Figure 7.4: The projection of the image plane $\tilde{\pi}_m$ on the laser plane π_m (the white plane in figure) intersects the cylinder defining an elliptic section, which is also the flat base of the cone with vertex in the m^{th} camera center.

7.2 The cylinder

According to the classification of quadric curves given in theorem (5.4), we know the equation of a real elliptic cylinder in \mathbb{E}^3 centered on the z axis and that if we fix the parameters to be equal ($\alpha = \beta$), then the cylinder is circular with radius equal to α . We also know the equation of a real infinite cone from the same theorem.

Let us now consider the canonical form of the equation of a real circular cylinder with radius $\alpha = r$:

$$Q|_{S_o} := x^2 + y^2 = r^2,$$

where $S_o = (O, (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3))$ and r is the only parameter in the equation.

The canonical reference is not unique because there are ∞^2 reference systems in which a cylinder can take that form and that they are independent. Indeed, given a cylinder axis laying along the direction of the vector \mathbf{u}_3 for example, there are infinitely many rotations around the axis \mathbf{u}_3 and infinitely many points on said axis can define the cylinder with the canonical form equation.

Now we define the cylinder matrix as

$$C = \begin{bmatrix} A & B \\ B^T & c \end{bmatrix}, \text{ where } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \mathbf{0}_{[3 \times 1]} \text{ and } c = -r^2. \quad (7.3)$$

If we take a point on the surface of the cylinder, its coordinates are

$$\mathbf{P}_{|S_o} = \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{Y} = \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}.$$

We do not know the canonical reference system, but we know that in a generic reference system a cylinder depends on seven parameters to be estimated: six of them are linked to the rototranslation, three in the unknown \mathbf{T}_i (point cartesian coordinates) and three in the other unknown Q_i , plus the unknown radius that is the scalar quantity r .

The three parameters contained in Q_i are the three Euler angles which we presented and discussed in section (4.2). Using this representation we can write $Q_i \in SO(3, \mathbb{R})$ as the product of three matrices: $Q_i = Q_{i3}Q_{i2}Q_{i1}$ where

$$Q_{i3} = \begin{bmatrix} \cos\gamma_i & -\sin\gamma_i & 0 \\ \sin\gamma_i & \cos\gamma_i & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q_{i2} = \begin{bmatrix} \cos\beta_i & 0 & -\sin\beta_i \\ 0 & 1 & 0 \\ \sin\beta_i & 0 & \cos\beta_i \end{bmatrix}$$

$$\text{and } Q_{i1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i \\ 0 & \sin\alpha_i & \cos\alpha_i \end{bmatrix}$$

As a matter of fact, two of the seven parameters for the cylinder can be excluded from our model: indeed the following theorem holds.

Theorem 7.2.1. *Let Q_i be a rotation matrix and A be the matrix of second order terms of the cylinder of radius r with respect to the cylinder reference system S_o . Then for any three dimensional vector \mathbf{X}_i the equation of the cylinder in the camera reference system B_{O_i} is*

$$\mathbf{X}_i^T \tilde{Q}_i^T A \tilde{Q}_i \mathbf{X}_i + 2\tilde{\mathbf{T}}_i^T \tilde{Q}_i \mathbf{X}_i + \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i - r^2 = 0.$$

where $\tilde{Q}_i \triangleq Q_{i2}Q_{i1}$, $\tilde{\mathbf{T}}_i \triangleq Q_{i3}^T A \mathbf{T}_i$ and

$$\tilde{\mathbf{T}}_i \triangleq Q_{i3}^T A \mathbf{T}_i = \begin{bmatrix} T_{i1}\cos\gamma_i + T_{i2}\sin\gamma_i \\ -T_{i1}\sin\gamma_i + T_{i2}\cos\gamma_i \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{T}_{i1} \\ \tilde{T}_{i2} \\ 0 \end{bmatrix}.$$

Let us divide the proof of theorem (7.2.1) into steps, by stating and proving the following lemma and proposition.

Lemma 7.2.2. *The equation for the cylinder in B_{O_i} can be written as*

$$\mathbf{X}_i^T A_i \mathbf{X}_i + 2B_i^T \mathbf{X}_i + c_i = 0 \quad (7.4)$$

where we call the matrix of the second order terms, the first order terms vector and the zero order term respectively as:

$$A_i = Q_i^T A Q_i, B_i^T = \mathbf{T}_i^T A Q_i \text{ and } c_i = \mathbf{T}_i^T A \mathbf{T}_i + c.$$

Proof. As already pointed out in the notation that we introduced in (7.3), in the canonical reference system for a cylinder $B = \mathbf{0}_{[3 \times 1]}$.

Keeping this in mind, we now write the coordinates of a point \mathbf{X} (with respect to the global reference system) as a function of the unknowns Q_i and \mathbf{T}_i by expressing it as the rototranslation of its coordinates with respect to the image plane $Q_i \mathbf{X}_i + \mathbf{T}_i$. The equation for the cylinder becomes:

$$0 = \mathbf{Y}^T C \mathbf{Y} = \mathbf{X}^T A \mathbf{X} + c = \mathbf{X}_i^T Q_i^T A Q_i \mathbf{X}_i + 2\mathbf{T}_i^T A Q_i \mathbf{X}_i + \mathbf{T}_i^T A \mathbf{T}_i + c.$$

□

Proposition 7.2.3. *The terms in equation (7.4) contain only four unknown parameters and they are $\alpha_i, \beta_i, \tilde{T}_{i1}$ and \tilde{T}_{i2} .*

Moreover, the following equalities are true:

$$(i) \quad \mathbf{X}_i^T A_i \mathbf{X}_i = \mathbf{X}_i^T \tilde{Q}_i^T A \tilde{Q}_i \mathbf{X}_i;$$

$$(ii) \quad Q_i^T A \mathbf{T}_i = \tilde{Q}_i^T \tilde{\mathbf{T}}_i;$$

$$(iii) \quad \mathbf{T}_i^T A \mathbf{T}_i = \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i.$$

Proof. We proceed to prove each of the equalities enumerated:

(i) since it can be proven by simple calculation that $Q_{i3}^T A Q_{i3} = A$, then

$$\begin{aligned} \mathbf{X}_i^T A_i \mathbf{X}_i &= \mathbf{X}_i^T Q_i^T A Q_i \mathbf{X}_i = \mathbf{X}_i^T Q_{i1}^T Q_{i2}^T Q_{i3}^T A Q_{i3} Q_{i2} Q_{i1} \mathbf{X}_i = \\ &= \mathbf{X}_i^T Q_{i1}^T Q_{i2}^T A Q_{i2} Q_{i1} \mathbf{X}_i = \mathbf{X}_i^T \tilde{Q}_i^T A \tilde{Q}_i \mathbf{X}_i; \end{aligned}$$

$$(ii) \quad Q_i^T A \mathbf{T}_i = Q_{i1}^T Q_{i2}^T Q_{i3}^T A \mathbf{T}_i = \tilde{Q}_i^T \tilde{\mathbf{T}}_i;$$

(iii) lastly, from the fact that the matrix A is idempotent and symmetric by definition and that Q_{i3} is an orthonormal matrix, we derive

$$\begin{aligned} \mathbf{T}_i^T A \mathbf{T}_i &= \mathbf{T}_i^T A^2 \mathbf{T}_i = \mathbf{T}_i^T A^T A \mathbf{T}_i = (A \mathbf{T}_i)^T (A \mathbf{T}_i) = (A \mathbf{T}_i)^T Q_{i3}^T Q_{i3} (A \mathbf{T}_i) \\ &= (Q_{i3}^T A \mathbf{T}_i)^T (Q_{i3}^T A \mathbf{T}_i) = \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i. \end{aligned}$$

In order to prove the first statement of the proposition, we observe that the unknown γ_i does not appear in the first term of the equation (7.4) since the matrix \tilde{Q}_i contains only the unknowns α_i and β_i . In the second and third terms of equation (7.4), the unknowns $\tilde{T}_{i1}, \tilde{T}_{i2}$ appear. \square

Proposition (7.2.3) shows that the equation proved in lemma (7.2.2) is equivalent to the one stated in theorem (7.2.1).

We also arrive to the conclusion that the five parameters on which the cylinder depends define the vector:

$$\mathbf{u}_i = (\alpha_i, \beta_i, \tilde{T}_{i1}, \tilde{T}_{i2}, r) \in [0, 2\pi]^2 \times \mathbb{R}^2 \times \mathbb{R}_{\geq 0};$$

and the matrix of the cylinder in each camera reference system as:

$$C_i = \begin{bmatrix} \tilde{Q}_i^T A \tilde{Q}_i & \tilde{Q}_i^T \tilde{\mathbf{T}}_i \\ \tilde{\mathbf{T}}_i^T \tilde{Q}_i & \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i - r^2 \end{bmatrix} = F_i^T C F_i \text{ with } F_i = \begin{bmatrix} \tilde{Q}_i & \tilde{\mathbf{T}}_i \\ \mathbf{0}_{[1 \times 3]} & 1 \end{bmatrix}.$$

If we are able to estimate all the parameters involved here, then we are left to the problem of estimating only the two parameters that we left out of our model thanks to the proposition (7.2.3), which are the third component of the translation and the rotation angle γ_i .

They both can take infinitely many values once we have fixed the first five parameters in our problem.

This way of modeling our problem let us take into consideration all the possible configurations without assuming any particular hypothesis and without having a reference camera as a *privileged* point of view with respect to others, because we simply choose the canonical reference system.

7.3 The ellipse arcs

The laser plane crosses the cylinder along an ellipse arc, which is described by the system

$$\begin{cases} \mathbf{Y}_i^T C \mathbf{Y}_i = 0 \\ \langle \mathbf{n}_i, \overrightarrow{S_i P} \rangle = 0 \\ \overrightarrow{O_i P} = t \overrightarrow{O_i Q} \\ \overrightarrow{O_i Q} = t_1 \mathbf{v}_{i1} + t_2 \mathbf{v}_{i2} + f \mathbf{v}_{i3} \end{cases} \quad (7.5)$$

where the first equation describes the cylinder and the following ones are exactly the same as in system (7.2).

With respect to the image plane, we define the coordinates of \mathbf{Q} and of the normal to the laser plane respectively as

$$\overrightarrow{O_i Q}|_{B_{O_i}} = \begin{bmatrix} t_1 \\ t_2 \\ f \end{bmatrix} = Q|_{B_{O_i}} = \mathbf{Z}_i \text{ and } \mathbf{N}_i \triangleq \mathbf{n}_i|_{B_{O_i}}$$

before stating the following:

Theorem 7.3.1. *The ellipse arc is projected in the image plane in the curve expressed by the equation:*

$$\mathbf{Z}_i^T (\tilde{Q}_i^T A \tilde{Q}_i + 2\mathbf{N}_i \tilde{T}_i^T \tilde{Q}_i + \mathbf{N}_i \tilde{T}_i^T \tilde{T}_i \mathbf{N}_i^T - r^2 \mathbf{N}_i \mathbf{N}_i^T) \mathbf{Z}_i = 0 \quad (7.6)$$

or equivalently

$$(\mathbf{N}_i^T \mathbf{Z}_i)^2 \mathbf{Y}_i C_i \mathbf{Y}_i = \begin{bmatrix} \mathbf{Z}_i^T & \mathbf{N}_i^T \mathbf{Z}_i \end{bmatrix} C_i \begin{bmatrix} \mathbf{Z}_i^T \\ \mathbf{N}_i^T \mathbf{Z}_i \end{bmatrix} = 0 \quad (7.7)$$

Proof. We start from the formula for the distance between point \mathbf{P} and the laser source S_i by decomposing it as

$$\overrightarrow{S_i P} = \overrightarrow{S_i O_i} + \overrightarrow{O_i P} = \overrightarrow{S_i O_i} + t \overrightarrow{O_i Q},$$

where we used the third equation in system (7.2).

Now we substitute it in the second equation in (7.2) and we obtain

$$\langle \mathbf{n}_i, \overrightarrow{S_i O_i} \rangle + t \langle \mathbf{n}_i, \overrightarrow{O_i Q} \rangle = 0, \text{ leading to } t = - \frac{\langle \mathbf{n}_i, \overrightarrow{S_i O_i} \rangle}{\langle \mathbf{n}_i, \overrightarrow{O_i Q} \rangle},$$

where $\langle \mathbf{n}_i, \overrightarrow{O_i Q} \rangle \neq 0$ because we set the laser plane in such a way that it does not pass through the origin of the camera reference system.

Now we impose that the point \mathbf{Q} belongs to the image plane (which is represented by the last equation in system (7.2)), recalling the definition of its coordinates in the image plane as \mathbf{Z}_i . As we wrote the vectorial equation

$$\overrightarrow{O_i P} = t \overrightarrow{O_i Q} = \frac{\langle \mathbf{n}_i, \overrightarrow{O_i S_i} \rangle}{\langle \mathbf{n}_i, \overrightarrow{O_i Q} \rangle} \overrightarrow{O_i Q},$$

we now can write by components the equation for mapping the coordinates of any point P in the laser plane into the corresponding coordinates in the image plane $\tilde{\pi}_i$ in a one to one correspondence:

$$\mathbf{X}_i = \frac{\langle \mathbf{n}_i, \overrightarrow{O_i S_i} \rangle}{\langle \mathbf{n}_i, \overrightarrow{O_i Q} \rangle} \mathbf{Z}_i.$$

Since we know that O_i does not belong to the laser plane π_i by the way we defined our system, we can choose \mathbf{n}_i such that $\langle \mathbf{n}_i, \overrightarrow{O_i S_i} \rangle = 1$.

Then

$$\mathbf{X}_i = \frac{1}{\mathbf{N}_i^T \mathbf{X}_i} \mathbf{Z}_i.$$

By substituting it in the cylinder equation stated by theorem (7.2.1) and eliminating the denominators, we obtain the thesis. \square

Remark 7.3.2. Throughout our work, we are reasonably assuming that the conic is an ellipse. Indeed the intersection between a cylinder and a plane always results in an ellipse except for the case in which the cylinder and the plane are parallel, while the intersection between a cone and a plane may result in any conic as we already discussed in chapter (5.4). For example if the plane is parallel to one of the generatrices of the cone then the result is a

parabola. In our model, this could of course happen from a mathematical point of view, but, since ideal cameras (i.e. cameras with infinite opening) do not exist, a camera would not be able to see the cylinder in such a configuration because of its smaller opening angle.

Such a configuration could be possible only for the case in which the cylinder is very close to the camera, but we can reasonably assume the conic always to be an ellipse.

7.4 Change of coordinates

So far, we have obtained all the equations for our geometric objects in each camera reference system \mathcal{B}_{O_i} . In this chapter, we investigate their relations.

Let us first introduce some useful concepts from computer graphics.

Remark 7.4.1. The *world coordinate system*, also known as the *universe* or the *model* or *global* coordinate system, is the base reference system for the overall model, (generally in 3D), to which all other model coordinates relate.

Remark 7.4.2. The *camera coordinate system*, also known as the *viewpoint*, is based upon the viewpoint of the observer, and changes as they change their view. Moving an object *forward* in this coordinate system moves it along the direction that the viewer happens to be looking at the time.

Remark 7.4.3. The rotation matrix and the translation vector which relate the world coordinate system to the camera coordinate system are called the extrinsic camera parameters.

We now consider the problem of the change of coordinates between the reference system B_{O_i} of a camera and B_{O_j} of an adjacent one in the cameras system.

Let us call the coordinates $\mathbf{X}_i = P|_{B_{O_i}}$ and $\mathbf{X}_j = P|_{B_{O_j}}$ respectively, and call $M_{B_{O_i}B_{O_j}}$ the matrix mapping the change from the reference system B_{O_j} to B_{O_i} .

Therefore we can write

$$\mathbf{X}_i = P|_{B_{O_i}} = \overrightarrow{O_i P}|_{B_{O_i}} = \overrightarrow{O_i O_j}|_{B_{O_i}} + \overrightarrow{O_j P}|_{B_{O_i}} = \overrightarrow{O_i O_j}|_{B_{O_i}} + M_{B_{O_i}B_{O_j}} \overrightarrow{O_j P}|_{B_{O_j}}.$$

Moreover, by definition we have

$$M_{B_{O_i}B_{O_j}} = Q_{ji} \in SO(3, \mathbb{R}) \text{ and } \overrightarrow{O_i O_j}|_{B_{O_i}} = \mathbf{T}_{ij} \text{ thus } \mathbf{X}_i = \mathbf{T}_{ij} + Q_{ji} \mathbf{X}_j.$$

If we recall also the definitions of

$$\mathbf{Y}_i = \begin{bmatrix} \mathbf{X}_i \\ 1 \end{bmatrix}, \mathbf{Y}_j = \begin{bmatrix} \mathbf{X}_j \\ 1 \end{bmatrix} \text{ and } F_{ji} = \begin{bmatrix} Q_{ji} & \mathbf{T}_{ij} \\ \mathbf{0}_{[1 \times 3]} & 1 \end{bmatrix}$$

then we have $\mathbf{Y}_i = F_{ji} \mathbf{Y}_j$.

We must study the transformation of the equation of the conic caused by the change of coordinates.

According to theorem (7.2.1), the equation of the cylinder in the reference system B_{O_i} is $\mathbf{Y}_i^T C_i \mathbf{Y}_i = 0$, where

$$C_i = \begin{bmatrix} \tilde{Q}_i^T A \tilde{Q}_i & \tilde{Q}_i^T \tilde{\mathbf{T}}_i \\ \tilde{\mathbf{T}}_i^T \tilde{Q}_i & \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i - r^2 \end{bmatrix} = F_i^T C F_i \text{ and } F_i = \begin{bmatrix} Q_i & \mathbf{T}_i \\ \mathbf{0}_{[1 \times 3]} & 1 \end{bmatrix}. \quad (7.8)$$

In the reference system B_j we obtain

$$\mathbf{Y}_j^T F_{ji}^T C_i F_{ji} \mathbf{Y}_j = 0.$$

Remark 7.4.4. It is important to notice that with this notation we are using the matrix F_{ji} instead of F_j in order to compute C_j .

In other words, we are stating that using the formulation (7.8) for computing C_j directly from the matrix C of the cylinder in the canonic reference system, as $C_j = F_j^T C F_j$, is equivalent to compute C_j from C_i , i.e. to calculate $C_i = F_i^T C F_i$ and derive $C_j = F_{ji}^T C_i F_{ji}$ consequently. Since it is not obvious, we need to check that this compatibility holds.

Indeed, we assume that $\mathbf{Y} = F_i \mathbf{Y}_i = F_i F_{ji} \mathbf{Y}_j = F_j \mathbf{Y}_j$.

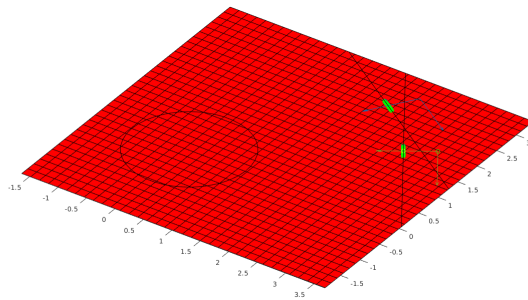
Therefore $F_{ji} = F_i^{-1} F_j$, and if we substitute it into the equation in the system B_{O_i} we obtain $\mathbf{Y}_j^T F_j^T F_i^{-1T} F_i^T C F_i F_i^{-1} F_j \mathbf{Y}_j = 0$ which clearly leads to $\mathbf{Y}_j^T F_j^T C F_j \mathbf{Y}_j = \mathbf{Y}_j^T C_j \mathbf{Y}_j = 0$.

We conclude that $C_j = F_{ji}^T C F_{ji}$.

Consequently, the equation of the conic in the image plane $\tilde{\pi}_j$ is

$$\begin{bmatrix} \mathbf{Z}_j^T & \mathbf{N}_j^T \mathbf{Z}_j \end{bmatrix} F_{ji}^T C_i F_{ji} \begin{bmatrix} \mathbf{Z}_j^T \\ \mathbf{N}_j^T \mathbf{Z}_j \end{bmatrix} = 0. \quad (7.9)$$

Figure 7.5: output of the Matlab code which simulates the whole system for two cameras: we can see the circular cylinder, the image planes of both cameras, their reference systems and the ellipses defined in their image planes (in green).

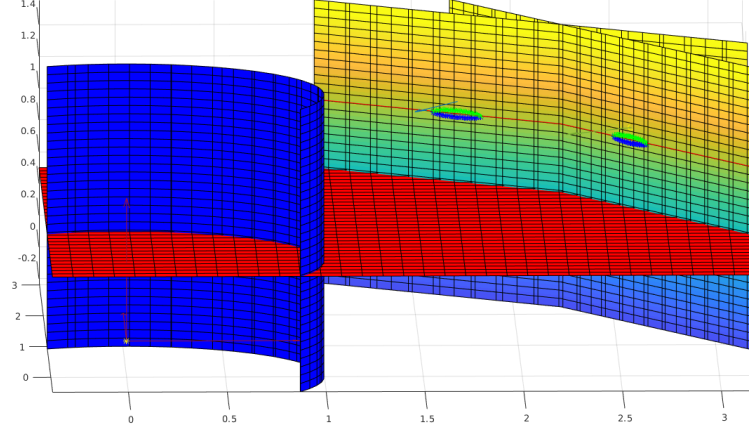


7.5 The visible region

The epipolar plane of the cylinder with respect to a camera divides the space in two regions: one that is visible because it corresponds to the reprojection on the image plane of the points on the cylinder surface which the camera is looking at, and one that is not visible because it is the reprojection of the points that are on the cylinder surface opposite to the camera.

Here we assume the hypothesis that the laser do not limit the visible region any further than the camera does, in other words the coordinates of the points we have are not constrained to be in a subset of the visible region.

Figure 7.6: the same system as in figure (7.5) seen from a different perspective: here we can see the global reference system of the cylinder, the red lines in the image planes that divide the visible region of the corresponding camera from the non visible one (the blue points on the ellipses are the visible ones).



Physically, the visible region is in the bottom half of the image plane.

We recall that

$$P|_{B_{O_i}} = \mathbf{X}_i \text{ and } Y_i = \begin{bmatrix} \mathbf{X}_i \\ 1 \end{bmatrix}$$

and we also know that the coordinates of the origin O_i in the reference system B_{O_i} are $O_i|_{B_{O_i}} = (0, 0, 0)$.

Recalling the equation of the epipolar plane from definition (5.1.5), its equation with respect to B_{O_i} is

$$\begin{bmatrix} O_i|_{B_{O_i}} \\ 1 \end{bmatrix}^T C_i \mathbf{Y}_i = 0,$$

or equivalently, if we recall the definition of C_i given in (7.8) and do the calculation:

$$\begin{bmatrix} \tilde{\mathbf{T}}_i^T \tilde{Q}_i & \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i - r^2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_i \\ 1 \end{bmatrix} = 0.$$

Since we know that O_i belongs to the visible region, by substituting it to \mathbf{Y}_i in the epipolar plane equation we obtain

$$\begin{bmatrix} O_i|_{B_{O_i}} \\ 1 \end{bmatrix}^T C_i \begin{bmatrix} O_i|_{B_{O_i}} \\ 1 \end{bmatrix} = \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i - r^2.$$

We also that the distance between O_i and the axis of the cylinder, i.e. the projection along the cylinder axis, is equal to the constant term in the cylinder equation:

$$\tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i = \mathbf{T}_i^T A \mathbf{T}_i = T_{i1}^2 + T_{i2}^2.$$

Consequently $\begin{bmatrix} O_i|_{B_{O_i}} \\ 1 \end{bmatrix}^T C_i \begin{bmatrix} O_i|_{B_{O_i}} \\ 1 \end{bmatrix} \geq 0$.

Remark 7.5.1. A cylinder is not a centered quadric, therefore there is no origin of the canonical reference system, but the origin may be any point on the axis: indeed in the distance formula the component T_{i3} does not appear.

The subregion that is visible by the camera is $\begin{bmatrix} O_i|_{B_{O_i}} \\ 1 \end{bmatrix}^T C_i \mathbf{Y}_i \geq 0$.

In the image plane $\tilde{\pi}_i$, the visible region is delimited by the line r which is the projection on the image plane of the intersection between epipolar plane and the laser plane.

Therefore we can write the following system:

$$\begin{cases} \begin{bmatrix} O_i|_{B_{O_i}} \\ 1 \end{bmatrix}^T C_i \mathbf{Y}_i = 0 \\ \langle \mathbf{n}_i, \overrightarrow{S_i P} \rangle = 0 \\ \overrightarrow{O_i P} = t \overrightarrow{O_i Q} \\ \overrightarrow{O_i Q} = t_1 \mathbf{v}_{i1} + t_2 \mathbf{v}_{i2} + f \mathbf{v}_{i3} \end{cases} \quad (7.10)$$

which is exactly the same as system (7.2) except for the first equation, where we substitute the equation of the cylinder with the equation of the epipolar plane.

Theorem 7.5.2. *For any vector \mathbf{Z}_i representing the coordinates of a point in the image plane of the i^{th} camera, the visible region is given by the points in the image plane which satisfy the inequality*

$$(\tilde{\mathbf{T}}_i^T \tilde{Q}_i + \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i \mathbf{N}_i^T - r^2 \mathbf{N}_i^T) \mathbf{Z}_i \geq 0.$$

Proof. In the same way as in the proof of theorem (7.3.1), from system (7.10) we arrive to the relation between the points on the laser plane and their reprojection on the image plane

$$\mathbf{X}_i = \frac{1}{\mathbf{N}_i^T \mathbf{Z}_i} \text{ with } \mathbf{Z}_i = \begin{bmatrix} t_1 \\ t_2 \\ f \end{bmatrix} = Q|_{B_{O_i}}.$$

Consequently we can write the equation of the line r delimiting the visible region in the image plane as

$$\begin{bmatrix} \tilde{\mathbf{T}}_i^T \tilde{Q}_i & \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i - r^2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_i \\ 1 \end{bmatrix} = (\tilde{\mathbf{T}}_i^T \tilde{Q}_i + \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i \mathbf{N}_i^T - r^2 \mathbf{N}_i^T) \mathbf{Z}_i = 0.$$

□

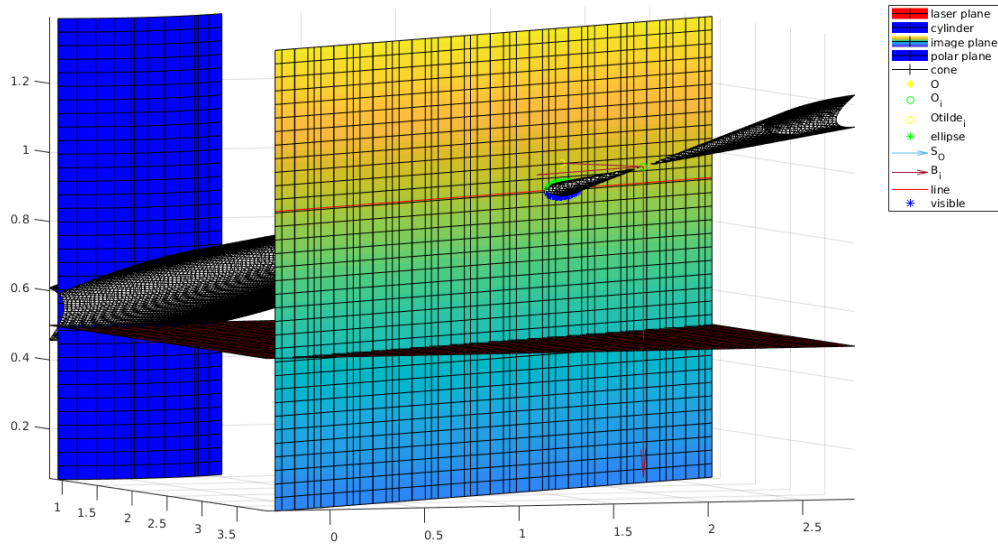


Figure 7.7: the same plot as in figure (7.1) zoomed in and seen from the camera perspective.

7.6 Sampling the conic in the image plane

So far we have derived the equation of the ellipse in the image plane and the portion that corresponds to the visible cylinder surface. In order to compute the Kanatani Cramer Rao Lower Bound (as in the formula we introduced in section (6.3)), we need the coordinates of the points. Therefore we need to develop a way to sample them from the equation.

The procedure that we implemented in Matlab in order to sample the ellipse in the image plane consists in the following steps.

- As a start, we have to choose and fix the position of the cylinder so that we can substitute the parameters in the equation given in theorem (7.3.1) and obtain the numerical equation $Z_i^T \tilde{A}_i Z_i + 2\tilde{B}_i^T Z_i + \tilde{c}_i = 0$ where $Z_i = [t_1 \ t_2]^T$.
- The second step is to derive the coordinates in the canonical reference system with respect to which the equation of the ellipse can be written in the form $\frac{\tilde{t}_1^2}{a^2} + \frac{\tilde{t}_2^2}{b^2} = 1$, which are related to the coordinates in the image plane reference system as:

$$\tilde{Z}_i = \begin{bmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{bmatrix} = Q_i^* Z_i + T_i^*$$

The ellipse can be parametrized, for $\varphi \in [0, 2\pi)$, as

$$\begin{cases} \tilde{t}_1 = a \cos \varphi \\ \tilde{t}_2 = b \sin \varphi \end{cases} \quad (7.11)$$

- The next step is to generate the parameters $\varphi_n \in [0, 2\pi)$ in such a way to have a constant spacing between the sampled points on the ellipse.

The length of the arc on the ellipse between φ_1 and φ_2 is given by the integral of the module of the vector tangent to the ellipse (that is the vector of derivatives with respect to φ of the parametrization, so it is represented with respect to the canonical reference system of the ellipse by the vector $\boldsymbol{\gamma}' = [-a \sin \varphi \quad b \cos \varphi]^T$):

$$L = \int_{\varphi_1}^{\varphi_2} \sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi} d\varphi$$

We used the notation $\Delta\varphi_{12} = |\varphi_2 - \varphi_1|$ to define the angle which is required to move along the ellipse of an arc of constant length $L = \Delta s$.

By imposing this length for the arc and by using an approximation for the integration variable since it is the integral of a special function and it would require a complex analytical computation, we obtain

$$\Delta\varphi_{12} = \frac{\Delta s}{\|\boldsymbol{\gamma}'\| \cos \varphi},$$

where φ is the angle between the tangent vector $\boldsymbol{\gamma}'$ and one of the axis of the reference system $B_{\tilde{O}_i}$ of the image plane according to the following rule:

- if $|\langle \mathbf{v}_1, \boldsymbol{\gamma}' \rangle| \geq |\langle \mathbf{v}_2, \boldsymbol{\gamma}' \rangle|$, then α is taken as the angle between the tangent $\boldsymbol{\gamma}'$ and \mathbf{v}_2 ;
- if instead the inverse inequality is true, then φ is taken as the angle between the tangent $\boldsymbol{\gamma}'$ and \mathbf{v}_1 .

The angle φ is a correction which lets to take into account the cases when the laser plane is either oblique or horizontal with respect to the reference system.

In the end we obtain

$$\begin{aligned} |\cos \varphi| &= \frac{|\langle \mathbf{v}_i, \boldsymbol{\gamma}' \rangle|}{\|\mathbf{v}_i\| \|\boldsymbol{\gamma}'\|} = \frac{|\langle \mathbf{v}_i, \boldsymbol{\gamma}' \rangle|}{\|\boldsymbol{\gamma}'\|} \\ \implies \Delta\varphi_{12} &= \frac{\Delta s}{\begin{bmatrix} -a \sin \varphi & b \cos \varphi \end{bmatrix} Q^{*T} \mathbf{v}_i} \end{aligned}$$

where $Q^{*T} \mathbf{v}_i$ are the coordinates of \mathbf{v}_i with respect to the canonical reference system of the ellipse in the image plane.

Given a starting angle φ_0 , $\varphi_1 = \varphi_0 + \Delta\varphi_{01}$, and so on:

$$\varphi_n = \varphi_{n-1} + \Delta\varphi_{n-1n} \text{ till } \varphi_n < 2\pi.$$

Now we can generate the points on the ellipse.

Recalling the definition of the coordinates in the canonic reference system

$$\tilde{Z}_{in} = \begin{bmatrix} a \cos \varphi_n \\ b \sin \varphi_n \end{bmatrix} = Q_i^* Z_i + T_i^*,$$

by inverse transformation we obtain

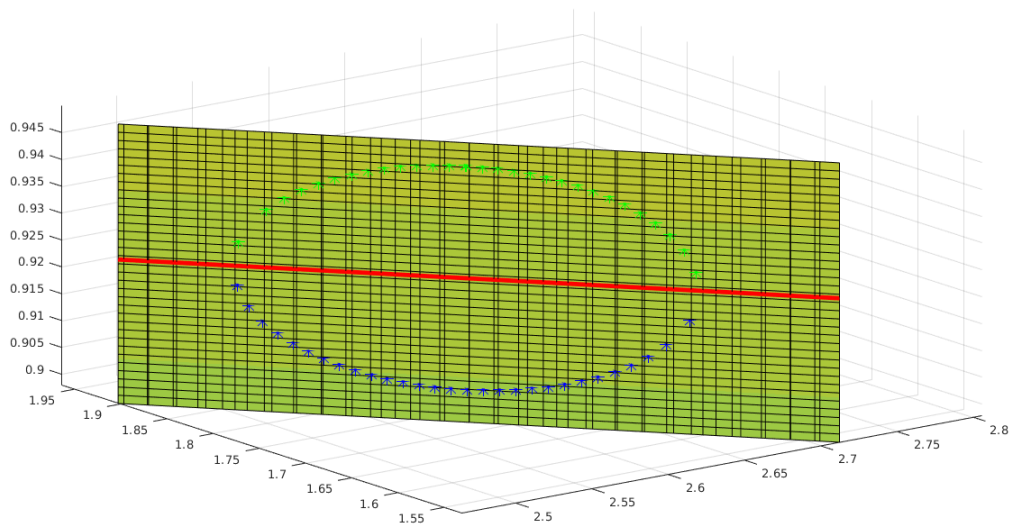
$$Z_{in} = Q_i^{*T} (\tilde{Z}_{in} - T_i^*).$$

- Finally, we exclude the generated points which are not in the visible region, i.e. which do not satisfy the inequality

$$(\tilde{\mathbf{T}}_i^T \tilde{\mathbf{Q}}_i + \tilde{\mathbf{T}}_i^T \tilde{\mathbf{T}}_i \mathbf{N}_i^T - r^2 \mathbf{N}_i^T) \begin{bmatrix} Z_{in} \\ f \end{bmatrix} \geq 0.$$

At the end of this procedure, we get the coordinates of the points to substitute into the Kanatani Cramer Rao Lower Bound formula.

Figure 7.8: the image plane of a camera with the ellipse divided in visible points (blue) and non visible ones (green), with the red line (2D projection of the 3D epipolar plane on the image plane) dividing the two regions.



7.7 Remarks on the code for simulations

In order to perform simulations, we have to choose a global reference system with respect to which we write all of our equations: we choose the canonical reference system of the cylinder, S_O .

We arbitrarily choose the coordinates of the center O_i of the i^{th} camera such that it is reasonable in order for it to look at the cylinder.

While the plots reported in all the previous sections of this report are all defining the camera as perfectly 'centered' with respect to the origin of the global reference system, we also added to the code a set of commands to rotate the camera along the three axis with angles chosen at random to check that the code works with all camera inclinations.

The initial point φ_0 for sampling the ellipse is chosen to be in the non visible region of the image plane (so it is one of the green points in figure (7.8)) in order to avoid eventual dis-

continuities due to the interruption of the sampling when the angle φ_n reaches the value of 2π .

Chapter 8

Kanatani Cramer Rao Lower Bound

In this chapter we add uncertainty on the measurements of our model with the aim of understanding how the addition of random noise affects the accuracy on the estimation of the parameters of the cylinder by computing the Kanatani Cramer Rao Lower Bound.

We proved in section (7.2) that the cylinder has five parameters to be estimated with respect to the reference system of each camera. Moreover, we know that we can measure the rototranslations between each couple of adjacent cameras in the vision system thanks to the calibration. Therefore the parameters to be estimated in the overall live in a space of dimension $5(m - 1)$ on which the parameters of each camera define a $5D$ manifold.

We now choose a reference camera ($i = 1$) so that once we fix the five parameters of one camera in the system, then we can derive all the others as functions of these fixed and independent five parameters.

In this way we are able to compute the KCRLB on all parameters.

8.1 Notation

We now give some remarks on the notations that are going to be useful in the following sections:

- we are going to use the notation $\bar{\cdot}$ to indicate an estimated quantity and the notation $\hat{\cdot}$ to indicate a measured quantity;
- from now on, we will refer to the vector $\mathbf{u}_1 = (\alpha_1, \beta_1, \tilde{T}_{11}, \tilde{T}_{12}, r)$ of the parameters of the cylinder with respect to the reference camera as $\mathbf{u} = (\alpha, \beta, T_1, T_2, r)$, for simplicity of notation;
- we refer to the parameters that have to be estimated for the j^{th} camera as the vector $\mathbf{q}_j = (\alpha_{j1}, \beta_{j1}, \gamma_{j1}, T_{j11}, T_{j12}, T_{j13}, t_{j1}, t_{j2})$;
- recalling the equation (7.7) stated in theorem (7.3.1), for a point k in the image plane

of the camera we can define the quantity

$$\boldsymbol{\xi}_k = \begin{bmatrix} \mathbf{Z}_k^T \\ \mathbf{N}_k^T \mathbf{Z}_k \end{bmatrix};$$

- in a similar way, recalling the equation (7.9) for any point of k in the image plane of the j^{th} camera, we can define the quantity

$$\boldsymbol{\xi}_{jk} = F_{j1} \begin{bmatrix} \mathbf{Z}_{jk}^T \\ \mathbf{N}_j^T \mathbf{Z}_{jk} \end{bmatrix};$$

- we need the following matrices

$$F_T = \begin{bmatrix} 1 & 0 & 0 & T_1 \\ 0 & 1 & 0 & T_2 \\ 0 & 0 & 1 & T_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, F_\beta = e^{\beta M_\beta} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{and } F_\alpha = e^{\alpha M_\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{where } M_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } M_\beta = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in order to define the complete matrix of the cylinder in the reference camera coordinate system: $C_1 = F^T C F = F_\alpha^T F_\beta^T F_T^T C F_T F_\beta F_\alpha$, where C is the complete matrix of the cylinder in the canonical system defined in (7.3).

In this setting, our data must satisfy constraints which can be expressed as quadratic forms by simply re-writing equations (7.7) and (7.9) using the notation that we just introduced:

- $E_k = \boldsymbol{\xi}_k^T C_1 \boldsymbol{\xi}_k = 0$ with respect to the reference camera coordinate system;
- $E_{jk} = \boldsymbol{\xi}_{jk}^T C_1 \boldsymbol{\xi}_{jk}$ for any point k with respect to the coordinate system of the j^{th} camera.

8.2 From the deterministic to the statistical problem

According to the notation that we introduced in the previous section, the KCRLB (6.3) can therefore be written for our problem as

$$V[\bar{\mathbf{u}}] \geq \left(\sum_{j=1}^m \sum_{k=1}^{N_j} \frac{(P_{\bar{\mathbf{u}}} \nabla_{\bar{\mathbf{u}}} E_{jk})(P_{\bar{\mathbf{u}}} \nabla_{\bar{\mathbf{u}}} E_{jk})^T}{(\nabla_{\mathbf{q}_{jk}} E_{jk})^T V[\mathbf{q}_{jk}] \nabla_{\mathbf{q}_{jk}} E_{jk}} \right)^{-1} \quad (8.1)$$

Since relations among the parameters in \bar{u} do not exist, we have that $P_{\bar{u}}$ is equal to the identity matrix.

The term which includes uncertainty in the model is the covariance matrix at the denominator. We are going to discuss this issue first and then describe the calculation of the KCRLB in sections (8.4.2) and (8.4.1).

Under the reasonable hypothesis that the way points are measured and the way rototranslations are measured are independent, we can assume null covariance between the error on the rototranslation parameters and the error on the points coordinates.

Therefore we can write the covariance matrix in blocks:

$$V[\mathbf{q}_{jk}] = \begin{bmatrix} V[\bar{\alpha}_{j1}, \bar{\beta}_{j1}, \bar{\gamma}_{j1}, \bar{T}_{j1}] & \mathbf{0}_{[5 \times 2]} \\ \mathbf{0}_{[2 \times 5]} & V[\hat{t}_{jk}] \end{bmatrix}.$$

If we define $\bar{\mathbf{w}}_{j1} = [\bar{\alpha}_{j1}, \bar{\beta}_{j1}, \bar{\gamma}_{j1}]$ the rotation parameters between the coordinate system of the reference camera and the j^{th} camera in order to simplify the notation, the set of constraints on our data result in a statistical model parametrized with $(\mathbf{w}_{21}, T_{21}, \dots, \mathbf{w}_{mm-1}, T_{mm-1})$.

The associated deterministic model is:

$$\begin{aligned} (\mathbf{w}_{21}, T_{21}, \dots, \mathbf{w}_{mm-1}, T_{mm-1}, \mathbf{w}_{1m}, T_{1m}) = \\ \mathbf{G}(\mathbf{w}_{21}, T_{21}, \dots, \mathbf{w}_{mm-1}, T_{mm-1}) = \\ (\mathbf{w}_{21}, T_{21}, \dots, \mathbf{w}_{mm-1}, T_{mm-1}, H(\mathbf{w}_{21}, T_{21}, \dots, \mathbf{w}_{mm-1}, T_{mm-1})) \end{aligned} \quad (8.2)$$

We use the classic procedure for the lower bound computation.

The matrix $J_{\mathbf{G}}$ of derivatives of the function G with respect to the parameters has $6m$ rows (equal to the total number of measured parameters in the whole system of m cameras) and 6 columns (equal to the number of parameters per camera):

$$J_{\mathbf{G}} = \begin{bmatrix} \frac{\partial G_1}{\partial \alpha_{21}} & \frac{\partial G_1}{\partial \beta_{21}} & \cdots & \frac{\partial G_1}{\partial T_{mm-13}} \\ \vdots & & & \vdots \\ \frac{\partial G_{6m}}{\partial \alpha_{21}} & \frac{\partial G_{6m}}{\partial \beta_{21}} & \cdots & \frac{\partial G_{6m}}{\partial T_{mm-13}} \end{bmatrix} = \begin{bmatrix} I_{[6 \times (m-1)]} \\ J_H \end{bmatrix}.$$

where J_H is the matrix of derivatives of the function H with respect to the parameters.

Now let us call \mathbf{E} the noise vector which perturbs the data in our model. We assume the error distribution is gaussian.

This new formulation of our problem is similar to the one of geometric fitting for which we gave a theoretical overview in section (6), but, if we compare the set of implicit equations (6.1) with the problem that we have now, we notice that they are different in the sense that the true data should satisfy now the set of implicit equations in the form

$$\mathbf{Y} = \mathbf{G}(\mathbf{X}), \quad (8.3)$$

where both \mathbf{Y} and \mathbf{X} are measurements affected by errors.

Indeed, geometric fitting is nothing but a particular case of this class of problems, which are referred to in the literature as *Errors-In-Variables* problems (a treatment of this topic can be found in [5]).

Then we have $\mathbf{Y} + \boldsymbol{\epsilon} = \mathbf{G}(\mathbf{X} + \mathbf{E}) = \mathbf{G}(\mathbf{X}) + J_G \mathbf{E}$, where J_G is the matrix of the derivatives of \mathbf{G} with respect to the parameters involved in the statistical model.

It implies that $\mathbf{E} = J_G^\dagger \boldsymbol{\epsilon}$ where $J_G^\dagger = (J_G^T J_G)^{-1} J_G^T$ is the left inverse of J_G , if it exists.

Then $V[\mathbf{X}] = J_G^\dagger V[\mathbf{Y}] J_G^{\dagger T} = (J_G^T J_G)^{-1} J_G^T V[\mathbf{Y}] J_G (J_G^T J_G)^{-1}$.

Since the error on measurements is known, we only compute the error on the estimated parameters.

In the case of our formulation (8.2), we know that J_G^\dagger exists because the rank of J_G is equal to the number of columns.

Therefore we observe that we can invert a $[6 \times 6]$ matrix instead of a $[6(m-1) \times 6(m-1)]$:

$$\begin{aligned} J_G^\dagger J_G &= \begin{bmatrix} I_{6(m-1)} & J_H^T \\ & J_H \end{bmatrix} \begin{bmatrix} I_{6(m-1)} \\ J_H \end{bmatrix} = I_{6(m-1)} + J_H^T J_H \implies \\ (J_G^T J_G)^{-1} &= I_{6(m-1)} - J_H^T (I_6 + J_H J_H^T)^{-1} J_H \end{aligned} \quad (8.4)$$

where we used the well-known Woodbury Matrix Identity.

Since we want to find the function H which relates the parameters of one camera to all the others, we take advantage of the so-called bundle adjustment technique, according to which F_{j1} is a function of $\hat{F}_{21}, \hat{F}_{32}, \dots, \hat{F}_{mm-1}, \hat{F}_{1m}$.

8.3 The covariance matrix and the bundle adjustment

Remark 8.3.1. Bundle Adjustment (BA) is an iterative process where 3D information is estimated from 2D image measurements. Typically, the position of object points are estimated simultaneously with the position and orientation of the cameras.

In photogrammetry, it is an optimization process that simultaneously refines estimates of 3D object point positions and camera poses (meaning *extrinsic camera parameters*, see remark (7.4.3), in computer vision) from measurements in overlapping images from multiple perspectives.

Any BA problem contains the following elements:

- *projection model* that describes the projection of a 3D object point into 2D image points in an image taken by a camera;
- a set of unknown parameters to be determined, a number of observations;
- a set of known parameters.

In our problem, bundle adjustment plays a role in the computation of the covariance matrix $V[\bar{\mathbf{w}}_{j1}, \bar{T}_{j1}]$ resulting in two constraints on the measurements, which we are now going to explore in more detail, in particular:

1. on the computation of $V[\bar{\mathbf{w}}_{21}, \bar{T}_{21}, \dots, \bar{\mathbf{w}}_{mm-1}, \bar{T}_{mm-1}, \bar{\mathbf{w}}_{1m}, \bar{T}_{1m}]$, taking into account the bundle adjustment constraint

$$F_{21} F_{32} F_{43} \dots F_{mm-1} F_{1m} = I_4;$$

2. on computation of $V[\bar{\mathbf{w}}_{j1}, \bar{T}_{j1}]$ by using the fact that

$$F_{j1} = F_{21} \dots F_{j-1j-2} F_{jj-1} \text{ for } j = 2, \dots, m.$$

8.3.1 First step

According to how we expressed our statistical problem at the end of section (8.2), we know that

$$\begin{aligned} J_G^T V[\hat{\mathbf{w}}_{21}, \hat{T}_{21}, \dots, \hat{\mathbf{w}}_{mm-1}, \hat{T}_{mm-1}, \hat{\mathbf{w}}_{mm-1}, \hat{T}_{1m}] J_G = \\ V[\hat{\mathbf{w}}_{21}, \hat{T}_{21}, \dots, \hat{\mathbf{w}}_{mm-1}, \hat{T}_{mm-1}] + \\ J_H^T V[\hat{\mathbf{w}}_{21}, \hat{T}_{21}, \dots, \hat{\mathbf{w}}_{mm-1}, \hat{T}_{mm-1}; \hat{\mathbf{w}}_{1m}, \hat{T}_{1m}] + \\ V[\hat{\mathbf{w}}_{21}, \hat{T}_{21}, \dots, \hat{\mathbf{w}}_{mm-1}, \hat{T}_{mm-1}; \hat{\mathbf{w}}_{1m}, \hat{T}_{1m}]^T J_H + J_H^T V[\hat{\mathbf{w}}_{1m}, \hat{T}_{1m}] J_H. \end{aligned}$$

Under the hypothesis that the errors on calibration are independent: if

$$V[\hat{\mathbf{w}}_{21}, \hat{T}_{21}, \dots, \hat{\mathbf{w}}_{mm-1}, \hat{T}_{mm-1}; \hat{\mathbf{w}}_{1m}, \hat{T}_{1m}]$$

is our $[6 \times 6(m-1)]$ matrix of observed values, then

$$\begin{aligned} J_G^T V[\hat{\mathbf{w}}_{21}, \hat{T}_{21}, \dots, \hat{\mathbf{w}}_{mm-1}, \hat{T}_{mm-1}, \hat{\mathbf{w}}_{1m}, \hat{T}_{1m}] J_G = \\ V[\hat{\mathbf{w}}_{21}, \hat{T}_{21}, \dots, \hat{\mathbf{w}}_{mm-1}, \hat{T}_{mm-1}] + J_H^T V[\hat{\mathbf{w}}_{1m}, \hat{T}_{1m}] J_H \end{aligned}$$

where the last six measurements are corrected by the error distribution.

Therefore we can write the error on the estimated parameters as function of the error on the measured ones:

$$\begin{aligned} V[\bar{\mathbf{w}}_{21}, \bar{T}_{21}, \dots, \bar{\mathbf{w}}_{mm-1}, \bar{T}_{mm-1}; \bar{\mathbf{w}}_{1m}, \bar{T}_{1m}] = \\ (I_{6(m-1)} - J_H^T (I_6 + J_H J_H^T)^{-1} J_H) (V[\hat{\mathbf{w}}_{21}, \hat{T}_{21}, \dots, \hat{\mathbf{w}}_{mm-1}, \hat{T}_{mm-1}] + \\ + J_H^T V[\hat{\mathbf{w}}_{1m}, \hat{T}_{1m}] J_H) (I_{6(m-1)} - J_H^T (I_6 + J_H J_H^T)^{-1} J_H). \end{aligned}$$

We are left with the determination of J_H .

Recalling the discussion on model formulation which we provide in section (8.2), let us give the following

Definition 8.3.2. L is the relation between the camera calibrations given by the bundle adjustment:

$$\begin{aligned} L : [0, 2\pi]^{[3 \times m]} \times \mathbb{R}^{[3 \times m]} &\longrightarrow Mat(4, 4; \mathbb{R}) \simeq \mathbb{R}^{16} \\ L(\mathbf{X}, \mathbf{Y}) = F_{21} F_{32} F_{43} \dots F_{mm-1} F_{1m} - I_4 = 0_{[4 \times 4]}, &\text{ where} \\ \mathbf{X} = (\mathbf{w}_{21}, T_{21}, \dots, \mathbf{w}_{mm-1}, T_{mm-1}) &\text{ and} \\ \mathbf{Y} = (\mathbf{w}_{1m}, T_{1m}) = H(\mathbf{w}_{21}, T_{21}, \dots, \mathbf{w}_{mm-1}, T_{mm-1}). & \end{aligned}$$

In brief: $L(\mathbf{X}, \mathbf{Y}) = 0$ if $\mathbf{Y} = \mathbf{H}(\mathbf{X})$.

Then, in order to determine the matrix J_H , we use the rule for the derivative of composed functions, if $J_{L, \mathbf{Y}}^\dagger$ exists:

$$\begin{aligned} J_{L, \mathbf{X}}(\mathbf{X}, \mathbf{H}(\mathbf{X})) + J_{L, \mathbf{Y}}(\mathbf{X}, \mathbf{H}(\mathbf{X})) J_H(\mathbf{X}) = 0 \\ \implies J_H(\mathbf{X}) = -J_{L, \mathbf{Y}}(\mathbf{X}, \mathbf{H}(\mathbf{X}))^\dagger J_{L, \mathbf{X}}(\mathbf{X}, \mathbf{H}(\mathbf{X})). \end{aligned} \tag{8.5}$$

Now we have to study

$$J_{L,X} = \begin{bmatrix} \frac{\partial L_{11}}{\partial \alpha_{21}} & \cdots & \frac{\partial L_{11}}{\partial T_{mm-13}} \\ \vdots & & \vdots \\ \frac{\partial L_{44}}{\partial \alpha_{21}} & \cdots & \frac{\partial L_{44}}{\partial T_{mm-13}} \end{bmatrix} \text{ and } J_{L,Y} = \begin{bmatrix} \frac{\partial L_{11}}{\partial \alpha_{1m}} & \cdots & \frac{\partial L_{11}}{\partial T_{1m3}} \\ \vdots & & \vdots \\ \frac{\partial L_{44}}{\partial \alpha_{1m}} & \cdots & \frac{\partial L_{44}}{\partial T_{1m3}} \end{bmatrix},$$

which are respectively a $[16 \times 6(m-1)]$ and a $[16 \times 6]$ matrix.

Let us start with $J_{L,X}(\mathbf{X}, \mathbf{H}(\mathbf{X}))$.

Theorem 8.3.3. *The elements of the matrix $J_{L,X}(\mathbf{X}, \mathbf{H}(\mathbf{X}))$ have the following expressions:*

- $\partial_{\alpha_{ji}}(F_{21}F_{32}F_{43} \cdots F_{mm-1}F_{1m} - I_4)|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = F_{j1}M_{\alpha}F_{j1}^{-1}$;
- $\partial_{\beta_{ji}}(F_{21}F_{32}F_{43} \cdots F_{mm-1}F_{1m} - I_4)|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = F_{j1} \begin{bmatrix} 0 & \sin \alpha_{j1} & -\cos \alpha_{j1} & 0 \\ \sin \alpha_{j1} & 0 & 0 & 0 \\ \cos \alpha_{j1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F_{j1}^{-1}$;
- $\partial_{\gamma_{ji}}(F_{21}F_{32}F_{43} \cdots F_{mm-1}F_{1m} - I_4)|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = F_{j1} \begin{bmatrix} 0 & -\cos \beta_{j1} \sin \alpha_{j1} & \cos \beta_{j1} \sin \alpha_{j1} & 0 \\ \cos \alpha_{j1} \cos \beta_{j1} & 0 & -\sin \beta_{j1} & 0 \\ \sin \alpha_{j1} \cos \beta_{j1} & \sin \beta_{j1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F_{j1}^{-1}$;
- $\partial_{T_{jil}}(F_{21}F_{32}F_{43} \cdots F_{mm-1}F_{1m} - I_4)|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = M_l$.

Proof. We have that

$$\partial_{\alpha_{21}}(F_{21}F_{32}F_{43} \cdots F_{mm-1}F_{1m} - I_4) = (\partial_{\alpha_{21}}F_{21})F_{32}F_{43} \cdots F_{mm-1}F_{1m}$$

and in order to evaluate it in $(\mathbf{X}, \mathbf{H}(\mathbf{X}))$ we use the fact that $\mathbf{Y} = \mathbf{H}(\mathbf{X})$ implies $F_{1m} = (F_{21}F_{32}F_{43} \cdots F_{mm-1})^{-1} = F_{mm-1}^{-1} \cdots F_{43}^{-1}F_{32}^{-1}F_{21}^{-1}$.

Therefore $\partial_{\alpha_{21}}(F_{21}F_{32}F_{43} \cdots F_{mm-1}F_{1m} - I_4)|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = (\partial_{\alpha_{21}}F_{21})F_{21}^{-1}$ and the procedure is the same for any of the other components of $J_{L,X}$:

$$\partial_{\alpha_{32}}L|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = F_{21}(\partial_{\alpha_{32}}F_{32})F_{32}^{-1}F_{21}^{-1} \text{ and so on.}$$

More detailed computations can be found in appendix (A.1). □

Theorem 8.3.4. *The elements of the matrix $J_{L,Y}(\mathbf{X}, \mathbf{H}(\mathbf{X}))$ have the following expressions:*

- $\partial_{\alpha_{1m}}(F_{21}F_{32}F_{43} \cdots F_{mm-1}F_{1m} - I_4) = M_{\alpha}$;
- $\partial_{\beta_{1m}}(F_{21}F_{32}F_{43} \cdots F_{mm-1}F_{1m} - I_4) = \begin{bmatrix} 0 & -\sin(\alpha_{1m}) & -\cos(\alpha_{1m}) & 0 \\ \sin(\alpha_{1m}) & 0 & 0 & 0 \\ \cos(\alpha_{1m}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

- $\partial_{\gamma_{1m}}(F_{21}F_{32}F_{43}\dots F_{mm-1}F_{1m} - I_4) = \begin{bmatrix} 0 & -\cos(\alpha_{1m})\sin(\alpha_{1m}) & \cos(\alpha_{1m})\sin(\beta_{1m}) & 0 \\ \cos(\alpha_{1m})\cos(\beta_{1m}) & 0 & -\sin(\beta_{1m}) & 0 \\ -\cos(\alpha_{1m})\sin(\beta_{1m}) & \sin(\beta_{1m}) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$
- $\partial_{T_l}(F_{21}F_{32}F_{43}\dots F_{mm-1}F_{1m} - I_4) = F_{1m}^{-1}M_l.$

Proof. See computations in appendix (A.1). \square

Finally, recalling the equation (8.4), we must invert $I_{30} + J_{\mathbf{H}}^T J_{\mathbf{H}}$:

$$\begin{aligned} (I_{30} + J_{\mathbf{H}}^T J_{\mathbf{H}})^{-1} &= (I_{30} + J_{\tilde{\mathbf{L}},\mathbf{X}}^T (J_{\tilde{\mathbf{L}},\mathbf{Y}} J_{\tilde{\mathbf{L}},\mathbf{Y}}^T) J_{\tilde{\mathbf{L}},\mathbf{X}})^{-1} = \\ &= I_{30} - J_{\tilde{\mathbf{L}},\mathbf{X}}^T (J_{\tilde{\mathbf{L}},\mathbf{Y}} J_{\tilde{\mathbf{L}},\mathbf{Y}}^T + J_{\tilde{\mathbf{L}},\mathbf{X}} J_{\tilde{\mathbf{L}},\mathbf{X}}^T)^{-1} J_{\tilde{\mathbf{L}},\mathbf{X}}. \end{aligned}$$

8.3.2 Second step

We still have to find out how to estimate the covariance matrix $V[\bar{\mathbf{w}}_{j1}, \bar{T}_{j1}]$ that we need to put into the first block of the covariance matrix $V[\mathbf{q}_{jk}]$ in the denominator of the KCRLB formula. If we define

$$F_{j1} = F_{21}\dots F_{j-1j-2}F_{jj-1} = \mathbf{K}_1(\mathbf{w}_{21}, T_{21}, \dots, \mathbf{w}_{jj-1}, T_{jj-1}) = \mathbf{K}_2(\mathbf{w}_{j1}, T_{j1}),$$

we know that $\mathbf{K}_1(\mathbf{x}) = \mathbf{K}_2(\mathbf{y})$ implies that $\mathbf{K}_1(\mathbf{x} + \delta_{\mathbf{x}}) = \mathbf{K}_2(\mathbf{y} + \delta_{\mathbf{y}})$, therefore $\mathbf{K}_1(\mathbf{x}) + J_{\mathbf{K}_1}\delta_{\mathbf{x}} = \mathbf{K}_2(\mathbf{y}) + J_{\mathbf{K}_2}\delta_{\mathbf{y}} \implies \delta_{\mathbf{y}} = J_{\mathbf{K}_2}^\dagger J_{\mathbf{K}_1}\delta_{\mathbf{x}}$.

Then $V[\bar{\mathbf{w}}_{j1}, \bar{T}_{j1}] = J_{\mathbf{K}_2}^\dagger J_{\mathbf{K}_1} V[\bar{\mathbf{w}}_{21}, \bar{T}_{21}, \dots, \bar{\mathbf{w}}_{jj-1}, \bar{T}_{jj-1}] J_{\mathbf{K}_1}^T J_{\mathbf{K}_2}^T$, where

$$J_{\mathbf{K}_1} = \begin{bmatrix} \frac{\partial K_{111}}{\partial \alpha_{21}} & \dots & \frac{\partial K_{111}}{\partial T_{jj-13}} \\ \vdots & & \vdots \\ \frac{\partial K_{144}}{\partial \alpha_{21}} & \dots & \frac{\partial K_{144}}{\partial T_{jj-13}} \end{bmatrix}, J_{\mathbf{K}_2} = \begin{bmatrix} \frac{\partial K_{211}}{\partial \alpha_{j1}} & \dots & \frac{\partial K_{211}}{\partial T_{j13}} \\ \vdots & & \vdots \\ \frac{\partial K_{244}}{\partial \alpha_{j1}} & \dots & \frac{\partial K_{244}}{\partial T_{j13}} \end{bmatrix}.$$

The independent equations are given by the first three rows of the rototranslation matrices, so that we can define $\tilde{K}_l = (F_{j11}, \dots, F_{j134})$ to obtain

$$V[\bar{\mathbf{w}}_{j1}, \bar{T}_{j1}] = J_{\tilde{K}_2}^\dagger J_{\tilde{K}_1} V[\bar{\mathbf{w}}_{21}, \bar{T}_{21}, \dots, \bar{\mathbf{w}}_{jj-1}, \bar{T}_{jj-1}] J_{\tilde{K}_1}^T J_{\tilde{K}_2}^T.$$

where $J_{\tilde{K}_2}^\dagger = (J_{\tilde{K}_2}^T J_{\tilde{K}_2})^{-1} J_{\tilde{K}_2}^T$.

The derivatives appearing in this formulas can be computed in the same way as the ones already mentioned in the previous section and reported in the appendix (A.1).

8.4 Computation of the KCRLB

We report the expressions of the terms appearing in the KCRLB formula (8.1) for our problem. Computations are reported with more details in appendix (A.1).

Proposition 8.4.1. *The partial derivatives of C_1 with respect to the parameters (see appendix (A.1) for a more detailed insight in computations) are:*

- $\partial_\alpha C_1 = C_1 M_\alpha - M_\alpha C_1$;
- $\partial_\beta C_1 = F^T M_\beta F$;
- $\partial_{T_l} C_1 = F^T M_l + M_l^T F$, where $M_l = \begin{bmatrix} 0 & 0 & 0 & \delta_{l1} \\ 0 & 0 & 0 & \delta_{l2} \\ 0 & 0 & 0 & \delta_{l3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, for $l = 1, 2, 3$;
- $\partial_r C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2r \end{bmatrix}$.

8.4.1 The numerator

As for the term appearing in the numerator in KCRLB formula (8.1), we get

$$(P_{\bar{u}} \nabla_{\bar{u}} E_{jk}) = \begin{bmatrix} \boldsymbol{\xi}_{jk}^T \partial_\alpha C_1 \boldsymbol{\xi}_{jk} \\ \vdots \\ \boldsymbol{\xi}_{jk}^T \partial_r C_1 \boldsymbol{\xi}_{jk} \end{bmatrix},$$

which results in $[(P_{\bar{u}} \nabla_{\bar{u}} E_{jk})(P_{\bar{u}} \nabla_{\bar{u}} E_{jk})^T]_{uv} = \boldsymbol{\xi}_{jk}^T \partial_u C_1 \boldsymbol{\xi}_{jk} \boldsymbol{\xi}_{jk}^T \partial_v C_1 \boldsymbol{\xi}_{jk}$.

8.4.2 The denominator

Regarding the denominator instead, since $\nabla_{\boldsymbol{\xi}_{jk}} E_{jk} = 2C_1 \boldsymbol{\xi}_{jk}$, we get

$$(\nabla_{\boldsymbol{q}_{jk}} E_{jk})^T V[\boldsymbol{q}_{jk}] \nabla_{\boldsymbol{q}_{jk}} E_{jk} = 4\boldsymbol{\xi}_{jk}^T C_1 V[\boldsymbol{\xi}_{jk}] C_1 \boldsymbol{\xi}_{jk}.$$

Since $\boldsymbol{\xi}_{jk} = \boldsymbol{\xi}_{jk}(\boldsymbol{q}_{jk})$, the covariance matrix of $\boldsymbol{\xi}_{jk}$ is computed as

$$V[\boldsymbol{\xi}_{jk}] = \nabla_{\boldsymbol{q}_{jk}} \boldsymbol{\xi}_{jk}^T V[\boldsymbol{q}_{jk}] \nabla_{\boldsymbol{q}_{jk}} \boldsymbol{\xi}_{jk},$$

where $J_{\boldsymbol{\xi}} = \begin{bmatrix} \partial_{\alpha_{j1}} \boldsymbol{\xi}_{jk1} & \cdots & \partial_{t_{j2}} \boldsymbol{\xi}_{jk1} \\ \cdots & & \cdots \\ \partial_{\alpha_{j1}} \boldsymbol{\xi}_{jk4} & \cdots & \partial_{t_{j2}} \boldsymbol{\xi}_{jk4} \end{bmatrix}$.

Proposition 8.4.2. *The derivatives appearing in $J_{\boldsymbol{\xi}}$ have the following expressions (again, see appendix (A.1) for detailed computations):*

- $\partial_{\alpha_{j1}} \boldsymbol{\xi}_{jk} = F_{j1} \begin{bmatrix} 0 \\ -f \\ t_{jk2} \\ 0 \end{bmatrix}$;

$$\begin{aligned}
\bullet \partial_{\beta_{j1}} \boldsymbol{\xi}_{jk} &= F_{j1} \begin{bmatrix} -\sin(\alpha_{j1})t_{jk2} - f \cos(\alpha_{j1}) \\ \sin(\alpha_{j1})t_{jk1} \\ \cos(\alpha_{j1})t_{jk1} \\ 0 \end{bmatrix}; \\
\bullet \partial_{\gamma_{j1}} \boldsymbol{\xi}_{jk} &= F_{j1} \begin{bmatrix} \cos(\beta_{j1})(f \sin(\alpha_{j1}) - t_{jk2} \cos(\alpha_{j1})) \\ \cos(\alpha_{j1}) \cos(\beta_{j1})t_{jk1} - f \sin(\beta_{j1}) \\ t_{jk2} \sin(\beta_{j1}) - \sin(\alpha_{j1}) \cos(\beta_{j1})t_{jk1} \\ 0 \end{bmatrix}; \\
\bullet \partial_{T_{j11}} \boldsymbol{\xi}_{jk} &= F_{j1} \begin{bmatrix} \mathbf{N}_j^T \mathbf{Z}_{jk} \\ 0 \\ 0 \\ 0 \end{bmatrix}; \\
\bullet \partial_{T_{j12}} \boldsymbol{\xi}_{jk} &= F_{j1} \begin{bmatrix} 0 \\ \mathbf{N}_j^T \mathbf{Z}_{jk} \\ 0 \\ 0 \end{bmatrix}; \quad \partial_{T_{j13}} \boldsymbol{\xi}_{jk} = F_{j1} \begin{bmatrix} 0 \\ 0 \\ \mathbf{N}_j^T \mathbf{Z}_{jk} \\ 0 \end{bmatrix}; \\
\bullet \partial_{t_{j1}} \boldsymbol{\xi}_{jk} &= F_{j1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ N_{j1} \end{bmatrix}; \quad \partial_{t_{j2}} \boldsymbol{\xi}_{jk} = F_{j1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ N_{j2} \end{bmatrix}.
\end{aligned}$$

Chapter 9

Simulations

In this chapter, we consider a system of $m = 3$ cameras for which we compute the Kanatani Cramer Rao Lower Bound for the radius of the cylinder.

We take the canonical reference system of the cylinder, $S_O = (O, (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3))$, as the reference system, as we did for the plots reported in section (7).

We choose the three cameras to be coplanar on the plane $z = 0$ and to be equally spaced, with an angle of 120° between each other, and equally distant from the cylinder axis.

The focal length and opening angle (i.e. the field of view) of each camera are assumed to be known, as well as their distance from the center of the reference system, i.e. the camera center O_i .

Therefore we can fix the camera reference system $B_{O_i} = (O_i, (\mathbf{v}_{i1}, \mathbf{v}_{i2}, \mathbf{v}_{i3}))$ and we choose to do it in the following way: $\mathbf{v}_{i3} = O - O_i$ is the vector perpendicular to the image plane and pointing towards the cylinder such that $v_{i3}(3) = 0$, $\mathbf{v}_{i1} = -\mathbf{u}_1 + v_{i3}(3)\mathbf{v}_{i3}$ is the vector parallel to the cylinder axis and \mathbf{v}_{i2} is derived as the cross product of \mathbf{v}_{i1} and \mathbf{v}_{i3} . The three vectors are then normalized with their norms in order to be unitary.

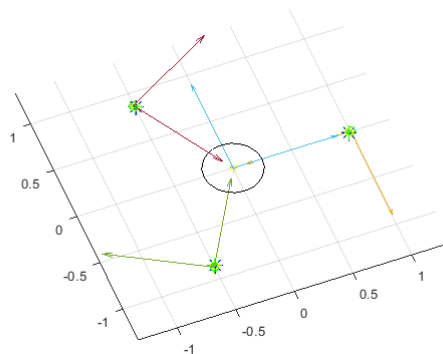


Figure 9.1: the system with the three cameras (each camera center corresponds to one of the green points), seen from a top perspective, equally spaced and equally distant from the cylinder

For each camera, we fix the following parameters:

- $f = 20cm$, the focal length;
- $\Delta s = 5\mu m$, the pixel size;
- the angular field of view $2\tan^{-1}(\frac{b}{f})$ with the horizontal field of view $b = 1024\Delta s$;
- the total number of pixels in the image plane is two times the horizontal field of view, $2b = 2048$;
- the image plane reference system $B_{\tilde{O}_i} = (\tilde{O}_i, (\mathbf{v}_{i1}, \mathbf{v}_{i2}))$, where \mathbf{v}_{i1} represents the pixel rows and \mathbf{v}_{i2} the columns;
- the laser source to be the point $S = O_i + d\mathbf{v}_1$, where d is the distance from the camera center to the cylinder axis, and the normal to the laser plane to be $\mathbf{n}_i = k(\frac{\mathbf{v}_{i1}}{\sqrt{2}} + \frac{\mathbf{v}_{i2}}{\sqrt{2}})$ in order to form an angle of 45 degrees with respect to the image plane, where $k = \frac{1}{\langle \mathbf{v}_{i1}, O_i \tilde{S} \rangle}$ is the same normalizing constant seen in proof of theorem (7.3.1).

We also fix the true value of the cylinder radius to be equal to $25cm$ and the cylinder axis to be parallel to \mathbf{u}_3 .

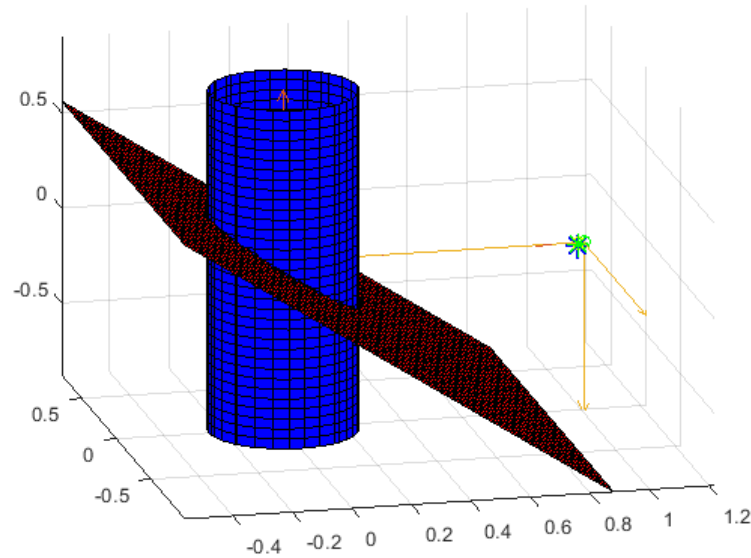


Figure 9.2: example of one of the cameras in the simulation (represented as its center and reference system and its paired laser plane in red which crosses the cylinder)

As for the values of the errors on the extrinsic parameters, we choose to put ourselves in the most conservative case: the covariance matrix is diagonal. In order to make simulation, we set the errors on the parameters to take values in a range between a reasonable maximum and minimum values and we initialize them as vectors of equally spaced positive numbers:

σ	min	max	step	measure unit
angles	0.1	1	0.1	deg
Δ_{xy}	0.5	5	0.5	micron
Δ_z	5	50	5	micron
pixels	12	40	2.5	micron

Table 9.1: values of the standard deviations of the error with which we perturb the five parameters of the model

where the angles are the euler angles α, β, γ of each rotation matrix, Δ_{xy} indicates the translation along the axis x and y , Δ_z indicates the translation along the axis z , i.e. parallel to the cylinder axis, and the error in the last row is the one on the pixel coordinates in the image plane.

With this setting, we run a total of 11000 simulations (one for each of the possible combinations of the values in table) and obtain the following maximum and minimum error on the radius:

σ	min	max	measure unit
radius	0.00397	0.01537	m

Table 9.2: minimum and maximum standard deviation of the error obtained on the radius

Lastly, the following plots show how the error in the radius is related to each of the given errors on the parameters: in each of the following four simulations, the three errors that do not appear in the plot are set to be fixed on their mean value:

parameter	σ_{angles}	σ_{xy}	σ_z	σ_{pixel}
mean value	0.0096	2.75	27.5	27.5
measure unit	deg	micron	micron	micron

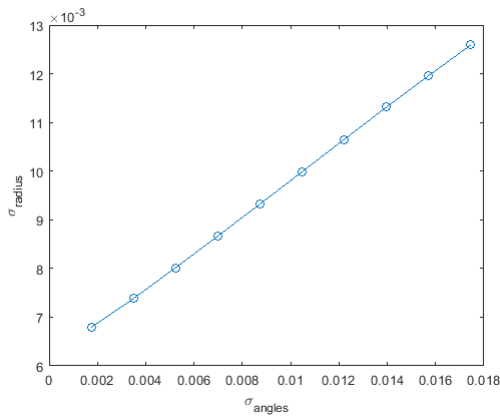


Figure 9.3: σ_{radius} vs σ_{angles} , [m]

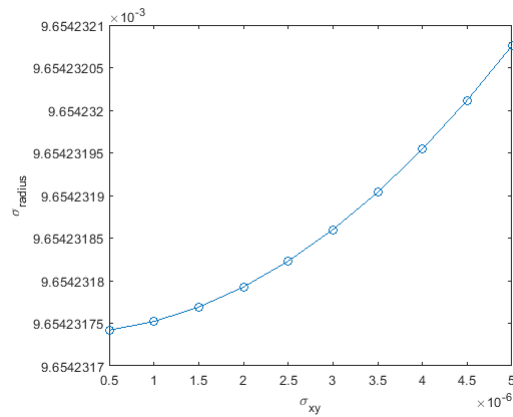
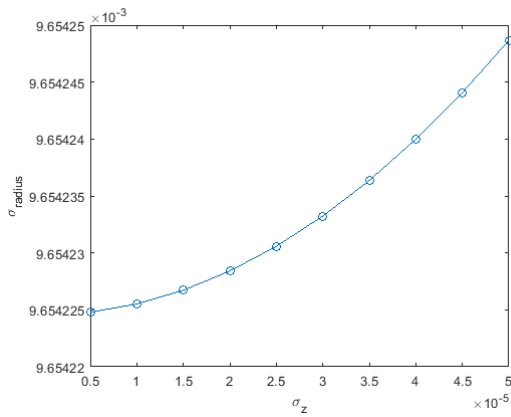
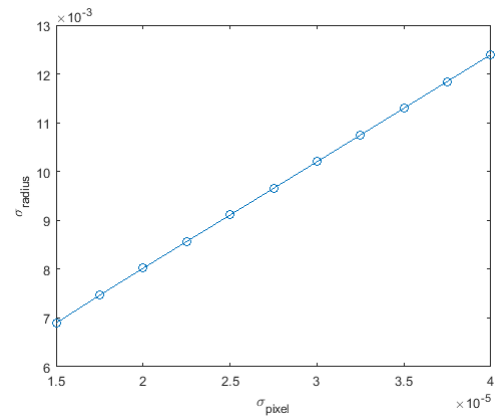


Figure 9.4: σ_{radius} vs σ_{xy} , [m]

Figure 9.5: σ_{radius} vs σ_z , [m]Figure 9.6: σ_{radius} vs σ_{pixel} , [m]

We observe that the error on the radius seems to have a linear dependence on the error on the rotation Euler angles and on the error on the pixels, while the slope of its curve increases more rapidly if plotted versus the errors on the rototranslations.

We can use this information in order to perform some data analysis.

9.1 Data analysis

We now consider the data we have as a dataset composed by 11000 rows corresponding to each simulated configuration (i.e. all the possible combinations of all the values taken by the errors shown in table (9.1)) and 5 columns corresponding to each of the variables involved. According to the plots at the end of the previous chapter, it seems reasonable to investigate a way to express the relation between the error on the radius and the given errors on parameters. Let us start with some descriptive statistical plots of our dataset and then proceed to some basic analysis in order to explore the results that we obtained.

9.1.1 Descriptive plots

A histogram is a graphical display of data using bars of different heights: on the horizontal axis there are the values taken by data (which are referred to as *cells* or *bins*), while on the vertical axis the height of the bars in correspondence of each cell value is proportional to the number of points falling into the cell. If data are normally distributed, their histogram should recall the bell shape of the normal distribution.

Quantile-quantile (QQ) plots are instead graphs on which every observed value is plotted against a standard normal distribution with the same number of points. Therefore we standardize the data before performing the following plots.

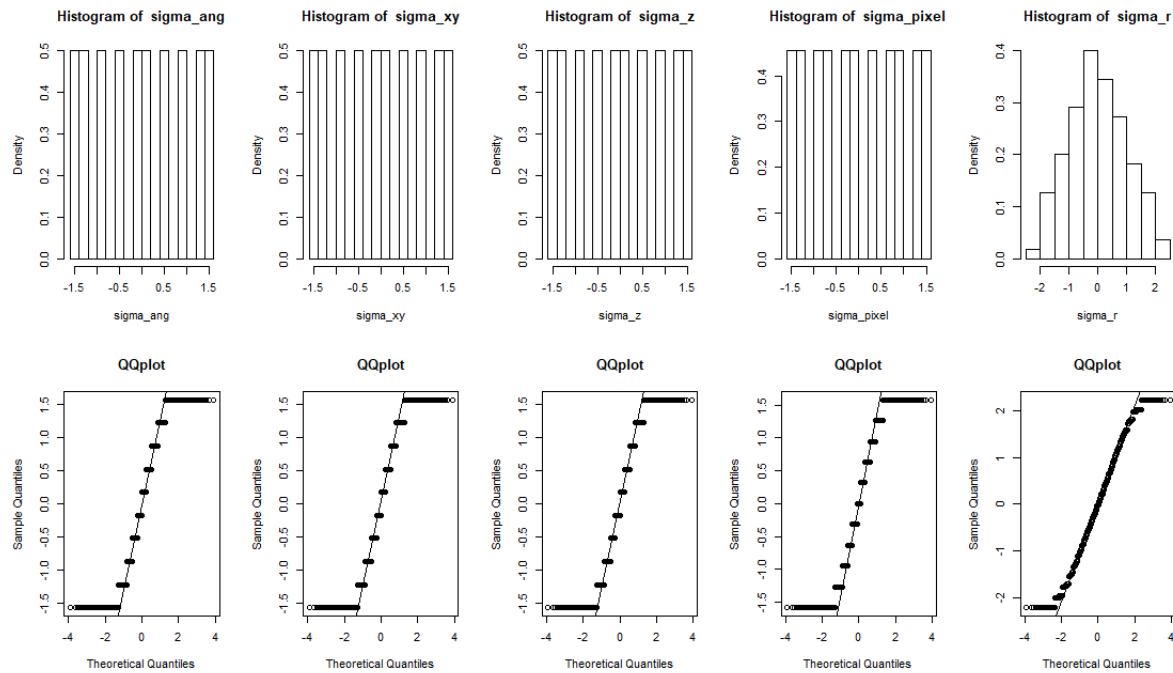


Figure 9.7: histograms and qqplots of the entire dataset

By looking at the plot above, we can see that the first four variables do not match perfectly the normal distribution, which is of course expected since we fixed their values numerically as vectors of equally spaced points, while the normal probability density function is continuous and centered on its mean value (in the sense that it does not give the same probability to all the points and is usually represented with a characteristic symmetric bell shape with the peak in the mean point, both in three and two dimensions).

On the other hand, we can say that the variable σ_{radius} , which is computed as the output of our simulations, is normally distributed since its histogram takes the characteristic bell shape and its points match very well the underlined normal distribution in the qqplot.

9.1.2 Linear regression

The classical linear regression model states that a response variable is composed of a mean, which depends in a continuous manner on the observations, and a random error, which accounts for measurement error and the effects of other variables not explicitly considered in the model. The values of the predictor variables recorded from the experiment or set by the investigator are treated as fixed.

The error (and hence the response) is viewed as a random variable whose behaviour is characterized by a set of distributional assumptions, in particular it must be a set of independently distributed samples with zero mean and constant variance (for a more extensive treatment on linear regression models, we refer to [20]).

Since we are interested in understanding which of the parameters most influences the error

on the radius, it would be proper if we could perform a linear regression analysis of the type

$$\begin{cases} \sigma_{radius} = \beta_1\sigma_{pixel} + \beta_2\sigma_{angles} + \beta_3\sigma_z + \beta_4\sigma_{xy} + \epsilon \\ \epsilon \sim \mathcal{N}(0, s^2) \\ \sigma_{radius} | \sigma_{angles}, \sigma_{xy}, \sigma_z, \sigma_{pixel} \sim \mathcal{N}(\mu_{\sigma_{radius}}, S_{\sigma_{radius}}) \\ \sigma_{angles}, \sigma_{xy}, \sigma_z, \sigma_{pixel} \text{ are fixed.} \end{cases}$$

where we indicated with $\mu_{\sigma_{radius}}$ and $S_{\sigma_{radius}}$ the mean and the variance of σ_{radius} and ϵ is the random noise normally distributed with mean 0 and unknown constant variance S^2 .

This scalar formulation is meant to hold for each one of the configurations that we simulated, therefore we could also write the model in a vector form where the involved normal distributions would have a vector of 11000 elements and a $[11000 \times 11000]$ diagonal covariance matrix as parameters.

Let us consider each of the simulated values for σ_{radius} as the response or dependent variable in our regression model and each of the combinations of $\sigma_{angles}, \sigma_{xy}, \sigma_z, \sigma_{pixel}$ as independent observations or predictors.

Therefore we try to fit a linear regression model without intercept (i.e. with no constant term to be added to the mean beside the linear combination of predictors) in order to see which of the covariates influences the response variable the most.

Table 9.3: summary of the linear model where in the Estimate column we can find the estimated values $\hat{\beta}_i, i = 1, \dots, 4$ for the coefficients $\beta_i, i = 1, \dots, 4$

	Estimate	Std. Error	t value	Pr(> t)
σ_{pixel}	0.6858	0.0002	3729.68	<2e-16
σ_{angles}	0.7264	0.0002	4012.37	<2e-16
σ_z	1.268e-05	0.0002	0.07	0.9438
σ_{xy}	7.822e-06	0.0002	0.04	0.9653

Beside the column of estimated coefficients, the most important column in table (9.3) is the last one, which reports the *pvalue* or *significance level* of the variable in the regression model, i.e. the minimum value for which there is statistical evidence to reject the null hypothesis of the test $H_0 : \beta_i = 0$ against the alternative hypothesis $H_1 : \beta_i \neq 0$.

If the *pvalue* is less than a chosen significance level (conventionally the threshold for statistical significance is 0.05), then there is no statistical evidence to accept the null hypothesis of the test.

We see from the summary in table (9.3) that the variables which are statistically significant are σ_{pixel} and σ_{angles} because their *pvalue* is way below the threshold, i.e. our sample of observations gives reasonable evidence to support the alternative hypothesis $H_1 : \beta_i \neq 0$ for the correspondent coefficients.

As for the other columns reported in table (9.3), the third one shows the values of the t-statistic used for the hypothesis test on each of the estimated coefficients and can therefore

be interpreted as a measure of how much the correspondent coefficient is far from zero. Lastly, the coefficient *standard error* in the second column measures the average amount that the coefficient estimates vary from the actual average value of our response variable, i.e. the smaller this value is, the better the coefficient estimation is precise.

The conclusion on statistical significance acquired from table (9.3) seems to be compatible with the plots at the end of the previous chapter which show that σ_{radius} has a linear relation with σ_{pixel} and σ_{angles} but the relation with σ_z and σ_{xy} seems to be polynomial or exponential.

Anyway, we need to verify the reliability of the linear model fitting by checking the assumptions of normality and homoscedasticity (homogeneity of variance) for the residuals.

This is very important because residuals contain all the information about the lack of goodness-of-fit in the model: indeed they are defined as the difference between the true value of the response and the estimated one, i.e. the quantity $\hat{\epsilon} = \sigma_{radius} - \hat{\sigma}_{radius}$.

The estimated regression coefficient values $\hat{\beta}_i, i = 1, \dots, 4$ give the estimated response variable of the regression model as $\hat{\sigma}_{radius} = \hat{\beta}_1\sigma_{pixel} + \hat{\beta}_2\sigma_{angles} + \hat{\beta}_3\sigma_z + \hat{\beta}_4\sigma_{xy}$.

In order to verify the assumptions, we use the R function which plots the residual of a regression model which as output the following plot:

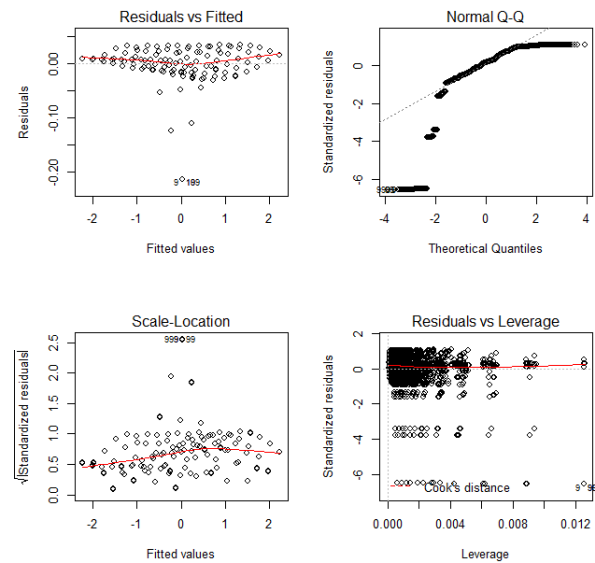


Figure 9.8: Residuals plot

These plots are useful indicators of departures from the assumptions of the model because of the following reasons:

- the left top and bottom plots shows the dependence of the residuals on the predicted value: this dependance should not exist, therefore if we can not recognize any straightforward pattern in this plot, then we can suppose that the model is good. Moreover, the residuals should ideally form a horizontal band because this indicates equal variances

and no dependence on the response.

- the qqplot is useful to determine if the residuals are normally distributed, as we already mentioned. Anyway, if the number of observations is large, minor departures from normality will not greatly affect inferences about the parameters.
- as for the bottom right plot, it is usually considered less important than the others in evaluating the goodness of a model, but it is more useful to determine the presence of outliers (the points in the plots that are very far from the others according to a similarity metric) in the model. Here we limit ourselves to say that leverages are the elements of the matrix H which is involved in the formula of the covariance matrix of the residuals $S^2(I - H)$ (which is not diagonal). A more in-depth explanation of leverages can be found in [20].

As we can see in the plot above, conditions do not seem to be fulfilled at the first look at the residuals plot, so we try some standard procedures to fix this issue.

We now discuss the possible causes of failing the assumptions for goodness-of-fit in the linear model and what we conclude after trying their solutions:

1. We may use the mahalanobis distance as similarity metric to discard some outliers, but since we are dealing with simulated data it does not seem reasonable to seek for outliers nor to lose some of the information by eliminating them.
2. We observe that the term *linear* refers to the fact that the mean is a linear function of the unknown parameters β_1, \dots, β_4 , hence the predictor variables may or may not enter the model as first-order terms.
Hence we try different functions of the predictors in the regression models which seem to fit the curves in the plots of σ_{radius} versus σ_z and σ_{xy} better than the linear one (in particular we tried the polynomial with power 3 and the exponential). Anyway the result remains the same in terms of statistical significance, so we do not report them.
3. We might conclude that residuals are not homoscedastic or that σ_z and σ_{xy} are less significant in explaining the response variable.

The fact that the response variable σ_{radius} is significantly more affected by σ_{ang} and σ_{pixel} seems to be confirmed also from the fact that if we take an extreme case, for instance all the configurations which give the minimum error on the radius (collected in the following table), we see that the values for the two significant variables are stuck to their minimum, while the other two variables are present also with errors slightly higher than their minimum:

Table 9.4: table collecting all the simulated configurations which result in the minimum error on the radius (all values have to be adjusted by a multiplier of 10^{-6})

σ_{ang}	σ_{xy}	σ_z	σ_{pixel}
1745.33	1.00	5.00	15.00
1745.33	1.50	5.00	15.00
1745.33	2.00	5.00	15.00
1745.33	2.50	5.00	15.00
1745.33	3.00	5.00	15.00
1745.33	3.50	5.00	15.00
1745.33	4.00	5.00	15.00
1745.33	4.50	5.00	15.00
1745.33	5.00	5.00	15.00
1745.33	0.50	10.00	15.00
1745.33	1.00	10.00	15.00
1745.33	1.50	10.00	15.00

4. The problems of our regression model may be due to an issue which frequently occurs in statistics and it is called multicollinearity, i.e. casual or intrinsic correlation patterns among the predictors. A sign of the presence of multicollinearity is the fact that if we fit a linear model considering only one predictor of the ones which resulted significant in the complete model (for instance, we report the case of σ_{angles} in the following table), we obtain a slightly different coefficient:

	Estimate	Std. Error	t value	Pr(> t)
σ_{angles}	0.8674	0.0030	293.61	<2e-16

Table 9.5: summary for the model $\sigma_{radius} = \beta\sigma_{angles}$. If we compare this with table (9.3), we notice that $\hat{\beta} = 0.8674 \neq \hat{\beta}_2 = 0.7264$.

Collinearity is a greatly debated issue. One way to eliminate collinearity is to substitute the predictors with their linear combinations, for instance performing principal components regression, as we do in the following chapter.

9.1.3 Principal Component Regression

We now the dataset of predictors, meaning that we now consider a data matrix with 11000 rows and 4 columns and perform a Principal Component Regression.

The principal components regression (PCR) approach involves constructing the principal components and then using them as the predictors in a classical linear regression model that is fit using least squares.

In other words, we assume that the directions in which the observations show the most

variation are the directions that are associated with the response. This assumption is in fact what reduces the multicollinearity effect.

For a more detailed insight in PCR method, we refer to the source [19].

As usual for PCA, all the analysis in this section is performed on standardized data.

If we fit a linear regression model for the response using the 4 principal components as predictors:

$$\sigma_{radius} = \beta_1^{pc} pc_1 + \beta_2^{pc} pc_2 + \beta_3^{pc} pc_3 + \beta_4^{pc} pc_4, \quad (9.1)$$

we find that they are all statistical significant except for the second one:

Table 9.6: summary of the linear model where in the Estimate column we can find the estimated values $\hat{\beta}_i^{pc}$, $i = 1, \dots, 4$ for the coefficients β_i^{pc} , $i = 1, \dots, 4$

	Estimate	Std. Error	t value	Pr(> t)
pc_1	0.0010	0.0001	11.03	<2e-16
pc_2	-0.0002	0.0001	-1.68	0.0923
pc_3	0.0015	0.0001	16.18	<2e-16
pc_4	-0.0018	0.0001	-19.68	<2e-16

Therefore we discard the second principal component from our model and obtain the following summary table and residuals plot, from which we can see that we finally found a seemingly good fit for our regression model:

	Estimate	Std. Error	t value	Pr(> t)
pc_1	0.0010	0.0001	11.03	<2e-16
pc_3	0.0015	0.0001	16.18	<2e-16
pc_4	-0.0018	0.0001	-19.68	<2e-16

Table 9.7: summary table of the reduced principal component regression model for which the fitted values are $\hat{\sigma}_{radius} = \hat{\beta}_1^{rpc} pc_1 + \hat{\beta}_2^{rpc} pc_3 + \hat{\beta}_3^{rpc} pc_4$

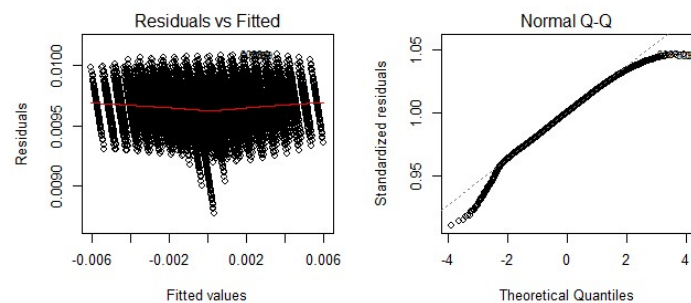


Figure 9.9: Residuals plot of the linear model with the 1st, 3rd and 4th principal components as predictors

Now let us denote with $\bar{\sigma}_{angles}, \bar{\sigma}_{xy}, \bar{\sigma}_z, \bar{\sigma}_{pixel}$ the mean values of each of the columns of the dataset of predictors.

The coefficients of the linear combinations of predictors which form the 4 principal components for our dataset are shown in the following plot:

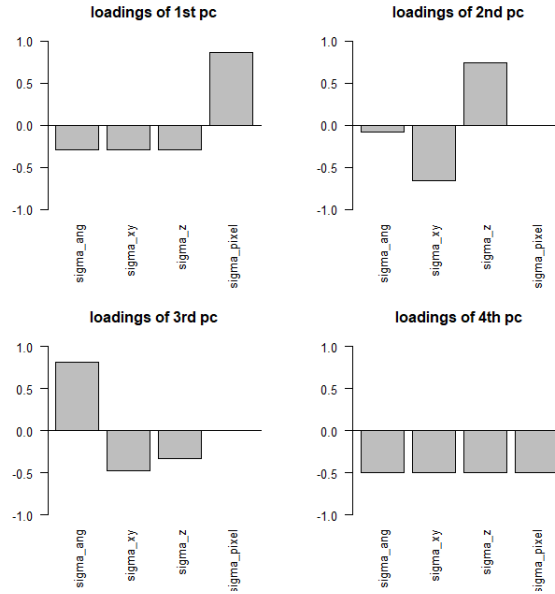


Figure 9.10: loadings of each principal component

Since we know the coefficients of the linear combinations of the observed predictors which let us compute the principal components:

$$\begin{cases} pc_1 = -0.29(\sigma_{xy} - \bar{\sigma}_{xy}) - 0.29(\sigma_{angles} - \bar{\sigma}_{angles}) - 0.29(\sigma_z - \bar{\sigma}_z) + 0.865(\sigma_{pixel} - \bar{\sigma}_{pixel}) \\ pc_3 = 0.812(\sigma_{angles} - \bar{\sigma}_{angles}) - 0.333(\sigma_z - \bar{\sigma}_z) - 0.479(\sigma_{xy} - \bar{\sigma}_{xy}) \\ pc_4 = -0.499(\sigma_{angles} - \bar{\sigma}_{angles}) - 0.499(\sigma_z - \bar{\sigma}_z) - 0.499(\sigma_{xy} - \bar{\sigma}_{xy}) - 0.502(\sigma_{pixel} - \bar{\sigma}_{pixel}), \end{cases}$$

by adding to this system the equation from the model fitted in table (9.7):

$$\hat{\sigma}_{radius} = 0.001pc_1 + 0.0015pc_3 - 0.0018pc_4$$

with a simple computation we obtain a way to compute the estimated error on the radius of the cylinder from the given errors on camera parameters with the formula

$$\begin{aligned} \hat{\sigma}_{radius} = & 0.001826(\sigma_{angles} - \bar{\sigma}_{angles}) - 0.0001103(\sigma_{xy} - \bar{\sigma}_{xy}) \\ & + 0.0001087(\sigma_z - \bar{\sigma}_z) + 0.001768(\sigma_{pixel} - \bar{\sigma}_{pixel}). \end{aligned}$$

9.2 Summary

The analysis which we performed in this chapter were meant to find relations among the data, in particular focusing on how the error on the radius depends on the errors on the measured

camera parameters.

Firstly, we find that the error on the radius has a linear dependency from the errors on the rotation angles and the pixel coordinates, while it has a non-linear dependency on the translation parameters.

Then, in order to express the error on the radius as a function of the parameters, we evaluate the fitting of a model both via linear regression and principal components regression.

In the end we conclude that the most appropriate method is PCR because, by projecting the data along the directions in which they show the most variation, it mitigates the non-linearity and multicollinearity effects in the data. We also compute the coefficients and report the complete formulation of the model in equation (9.1.3). This result seems better than any of the ones we obtained from the simple linear regression models because it permits to reach statistical significance without eliminating any of the predictors.

Chapter 10

Conclusions and future development

In this work, we developed a method for computing the Kanatani Cramer Rao Lower Bound (modified version of the classical Cramer Rao approach in statistical parameter estimation) for the variance of the error on the radius in the production of industrial pipes, when measurements are made by a system of m independent calibrated cameras and each camera is paired with a laser.

We first provided a deterministic formulation for the problem, then we included uncertainties on the measurements and we obtained a statistical parametric formulation of the problem. Finally, we computed the Kanatani Cramer Rao Lower Bound on the standard deviation of the error in the estimation of the cylinder parameters, in particular the radius.

In the end, we test our analytical approach by simulating a particular configuration of the camera/laser and cylinder system.

Lastly, we performed some statistical data analysis in order to obtain a relation between the errors on the parameters and the estimated error.

Future developments of this work will be:

- the inclusion of other sources of errors in the Kanatani Cramer Rao Lower Bound, such as the uncertainty on the laser plane position and on the intrinsics parameters of the cameras;
- the experimental estimation of the input errors of the Kanatani Cramer Rao Lower Bound;
- finally, a simulative and experimental validation of the method that we proposed in this thesis.

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Appendices

Appendix A

A.1 Computations

In this appendix, we report the calculations which lead us to compute all the derivatives appearing in the KCRLB formula as we write it for our problem.

Remark A.1.1. Firstly, it is necessary to compute the partial derivatives of $F = F_T F_\gamma F_\beta F_\alpha$ with respect to the parameters:

- $F_\alpha = e^{\alpha M_\alpha}, \implies \partial_\alpha F_\alpha = M_\alpha F_\alpha = F_\alpha M_\alpha;$
- $F_\beta = e^{\beta M_\beta}, \implies \partial_\beta F_\beta = M_\beta F_\beta = F_\beta M_\beta;$
- $F_\gamma = e^{\gamma M_\gamma}, M_\gamma = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \partial_\gamma F_\gamma = M_\gamma F_\gamma = F_\gamma M_\gamma;$
- $\partial_{T_l} F_T = \begin{bmatrix} 0 & 0 & 0 & \delta_{l1} \\ 0 & 0 & 0 & \delta_{l2} \\ 0 & 0 & 0 & \delta_{l3} \\ 0 & 0 & 0 & 0 \end{bmatrix} = M_l \implies \partial_{T_l} F_T = M_l = M_l F_T = F_T M_l;$
- $\partial_\alpha F = F_T F_\gamma F_\beta \partial_\alpha F_\alpha = F_T F_\gamma F_\beta M_\alpha F_\alpha = F_T F_\gamma F_\beta M_\alpha F_\beta^T F_\gamma^T F_{-T} F;$
- $\partial_\beta F = F_T F_\gamma \partial_\beta F_\beta F_\alpha = F_T F_\gamma F_\beta M_\beta F_\alpha = F F_\alpha M_\beta F_\alpha;$
- $\partial_\gamma F = F_T \partial_\gamma F_\gamma F_\beta F_\alpha = F_T F_\gamma M_\gamma F_\beta F_\alpha = F F_\alpha^T F_\beta^T M_\gamma F_\beta F_\alpha = F_T M_\gamma F_{-T} F;$
- $\partial_{T_l} F = \partial_{T_l} F_T F_\gamma F_\beta F_\alpha = F_T M_l F_\gamma F_\beta F_\alpha = F F_\alpha^T F_\beta^T F_\gamma^T M_l F_\gamma F_\beta F_\alpha = F \begin{bmatrix} Q^T & \mathbf{0}_{[3 \times 1]} \\ \mathbf{0}_{[1 \times 3]} & 1 \end{bmatrix} M_l = M_l.$

Remark A.1.2. For the computation of the numerator in the KCRLB formula as described in section (8.4.1), it is necessary to compute the partial derivatives of C_1 with respect to the parameters:

- $\partial_\alpha C_1 = F_\alpha^T F_\beta^T F_T^T C F_T F_\beta \partial_\alpha F_\alpha + \partial_\alpha F_\alpha^T F_\beta^T F_T^T C F_T F_\beta F_\alpha = F_\alpha^T F_\beta^T F_T^T C F_T F_\beta F_\alpha M_\alpha + M_\alpha^T F_\alpha F_\beta^T F_T^T C F_T F_\beta F_\alpha = C_1 M_\alpha - M_\alpha C_1$;
- $\partial_\beta C_1 = F_\alpha^T F_\beta^T F_T^T C F_T \partial_\beta F_\beta F_\alpha + F_\alpha^T \partial_\beta F_\beta^T F_T^T C F_T F_\beta F_\alpha = F^T C F_T M_\beta F_\beta F_\alpha + F_\alpha^T F_\beta^T M_\beta^T F_T^T C F =$ (here we use scalar product property)
 $F^T C M_\beta F_T F_\beta F_\alpha + F_\alpha^T F_\beta^T F_T^T M_\beta^T C F =$

$$F^T \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F + F^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F = F^T \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F$$

(where for the last equality we use $T_3 = 0$);
- $\partial_{T_l} C_1 = F_\alpha^T F_\beta^T F_T^T C \partial_{T_l} F_T F_\beta F_\alpha + F_\alpha^T F_\beta^T \partial_{T_l} F_T^T C F_T F_\beta F_\alpha = F^T C M_l F_\beta F_\alpha + F_\alpha^T F_\beta^T M_l^T C F = F^T M_l + M_l^T F$;
- $\partial_r C_1 = F^T \partial_r C F = F^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2r \end{bmatrix} F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2r \end{bmatrix}$.

Remark A.1.3. $F_{ji}^{-1} = (F_{T_{ji}} F_{\gamma_{ji}} F_{\beta_{ji}} F_{\alpha_{ji}})^{-1} = F_{T_{ji}}^{-1} F_{\gamma_{ji}}^{-1} F_{\beta_{ji}}^{-1} F_{\alpha_{ji}}^{-1} = F_{-\alpha_{ji}} F_{-\beta_{ji}} F_{-\gamma_{ji}} F_{-T_{ji}}$.

Remark A.1.4. For the computation of the denominator in the KCRLB formula as described in section (8.4.2), it is necessary to compute the following derivatives:

- $\partial_{\alpha_{j1}} \boldsymbol{\xi}_{jk} = F_{j1} M_\alpha \begin{bmatrix} \mathbf{Z}_{jk} \\ \mathbf{N}_j^T \mathbf{Z}_{jk} \end{bmatrix} = F_{j1} \begin{bmatrix} 0 \\ -f \\ t_{jk2} \\ 0 \end{bmatrix}$;
- $\partial_{\beta_{j1}} \boldsymbol{\xi}_{jk} = F_{j1} \begin{bmatrix} 0 & -\sin(\alpha_{j1}) & -\cos(\alpha_{j1}) & 0 \\ \sin(\alpha_{j1}) & 0 & 0 & 0 \\ \cos(\alpha_{j1}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{jk} \\ \mathbf{N}_j^T \mathbf{Z}_{jk} \end{bmatrix} =$
 $= F_{j1} \begin{bmatrix} -\sin(\alpha_{j1}) t_{jk2} - f \cos(\alpha_{j1}) \\ \sin(\alpha_{j1}) t_{jk1} \\ \cos(\alpha_{j1}) t_{jk1} \\ 0 \end{bmatrix}$;
- $\partial_{\gamma_{j1}} \boldsymbol{\xi}_{jk} = M_\gamma F_{j1} \begin{bmatrix} I_3 & \mathbf{0}_{[3 \times 1]} \\ \mathbf{0}_{[3 \times 1]}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{jk} \\ \mathbf{N}_j^T \mathbf{Z}_{jk} \end{bmatrix} = M_\gamma F_{j1} \begin{bmatrix} \mathbf{Z}_{jk} \\ 0 \end{bmatrix} =$
 $= F_{j1} \begin{bmatrix} \cos(\beta_{j1})(f \sin(\alpha_{j1}) - t_{jk2} \cos(\alpha_{j1})) \\ \cos(\alpha_{j1}) \cos(\beta_{j1}) t_{jk1} - f \sin(\beta_{j1}) \\ t_{jk2} \sin(\beta_{j1}) - \sin(\alpha_{j1}) \cos(\beta_{j1}) t_{jk1} \\ 0 \end{bmatrix}$;

$$\begin{aligned}
& \bullet \partial_{T_{j11}} \boldsymbol{\xi}_{jk} = M_1 \boldsymbol{\xi}_{jk} = F_{j1} \begin{bmatrix} \mathbf{N}_j^T \mathbf{Z}_{jk} \\ 0 \\ 0 \\ 0 \end{bmatrix}; \\
& \bullet \partial_{T_{j12}} \boldsymbol{\xi}_{jk} = F_{j1} \begin{bmatrix} 0 \\ \mathbf{N}_j^T \mathbf{Z}_{jk} \\ 0 \\ 0 \end{bmatrix}; \partial_{T_{j13}} \boldsymbol{\xi}_{jk} = F_{j1} \begin{bmatrix} 0 \\ 0 \\ \mathbf{N}_j^T \mathbf{Z}_{jk} \\ 0 \end{bmatrix}; \\
& \bullet \partial_{t_{j1}} \boldsymbol{\xi}_{jk} = F_{j1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ N_{j1} \end{bmatrix}; \partial_{t_{j2}} \boldsymbol{\xi}_{jk} = F_{j1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ N_{j2} \end{bmatrix}.
\end{aligned}$$

Remark A.1.5. Let us compute the expressions of the elements of the matrix $J_{L,\mathbf{X}}(\mathbf{X}, \mathbf{H}(\mathbf{X}))$ (most of this computation have been carried out numerically in Matlab Symbolic), starting with $j = 2, i = 1$:

$$\begin{aligned}
& \bullet \partial_{\alpha_{21}} (F_{21} F_{32} F_{43} \dots F_{mm-1} F_{1m} - I_4)|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = (\partial_{\alpha_{21}} F_{21}) F_{21}^{-1} = F_{21} M_\alpha F_{21}^{-1} = \\
& \begin{bmatrix} 0 & -\sin \beta_{21} & \cos \beta_{21} \sin \gamma_{21} & T_{212} \sin \beta_{21} - T_{213} \cos \beta_{21} \sin \gamma_{21} \\ \sin \beta_{21} & 0 & -\cos \beta_{21} \sin \gamma_{21} & T_{213} \cos \beta_{21} \sin \gamma_{21} - T_{211} \sin \beta_{21} \\ -\cos \beta_{21} \sin \gamma_{21} & \cos \beta_{21} \sin \gamma_{21} & 0 & \cos \beta_{21} (T_{211} \sin \gamma_{21} - T_{212} \cos \gamma_{21}) \\ 0 & 0 & 0 & 0 \end{bmatrix}; \\
& \bullet \partial_{\beta_{21}} (F_{21} F_{32} F_{43} \dots F_{mm-1} F_{1m} - I_4)|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = (\partial_{\beta_{21}} F_{21}) F_{21}^{-1} = \\
& F_{21} \begin{bmatrix} 0 & \sin \alpha_{21} & -\cos \alpha_{21} & 0 \\ \sin \alpha_{21} & 0 & 0 & 0 \\ \cos \alpha_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F_{21}^{-1}; \\
& \bullet \partial_{\gamma_{21}} (F_{21} F_{32} F_{43} \dots F_{mm-1} F_{1m} - I_4)|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = (\partial_{\gamma_{21}} F_{21}) F_{21}^{-1} = \\
& \begin{bmatrix} 0 & -1 & 0 & T_{212} \\ 1 & 0 & 0 & T_{211} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = F_{21} \begin{bmatrix} 0 & -\cos \beta_{21} \sin \alpha_{21} & \cos \beta_{21} \sin \alpha_{21} & 0 \\ \cos \alpha_{21} \cos \beta_{21} & 0 & -\sin \beta_{21} & 0 \\ \sin \alpha_{21} \cos \beta_{21} & \sin \beta_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F_{21}^{-1}; \\
& \bullet \partial_{T_{211}} (F_{21} F_{32} F_{43} \dots F_{mm-1} F_{1m} - I_4)|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = (\partial_{T_{211}} F_{21}) F_{21}^{-1} = M_l.
\end{aligned}$$

The other derivatives can be calculated with the same approach, for instance for $j = 3, i = 2$:

$$\partial_{\alpha_{32}} L|_{(\mathbf{X}, \mathbf{H}(\mathbf{X}))} = F_{21} (\partial_{\alpha_{32}} F_{32}) F_{32}^{-1} F_{21}^{-1} =$$

$$F_{21} \begin{bmatrix} 0 & -\sin \beta_{32} & \cos \beta_{32} \sin \gamma_{32} & T_{212} \sin \beta_{32} - T_{213} \cos \beta_{32} \sin \gamma_{32} \\ \sin \beta_{32} & 0 & -\cos \beta_{32} \sin \gamma_{32} & T_{213} \cos \beta_{32} \sin \gamma_{32} - T_{211} \sin \beta_{32} \\ -\cos \beta_{32} \sin \gamma_{32} & \cos \beta_{32} \sin \gamma_{32} & 0 & \cos \beta_{32} (T_{211} \sin \gamma_{32} - T_{212} \cos \gamma_{32}) \\ 0 & 0 & 0 & 0 \end{bmatrix} F_{21}^{-1}$$

$$= F_{31}M_\alpha F_{31}^{-1}.$$

Remark A.1.6. Now let us consider $J_{\mathbf{L},\mathbf{Y}}(\mathbf{X}, \mathbf{H}(\mathbf{X}))$ and let us fix the number of cameras in the system to be $m = 6$ to show all the computations:

- $\partial_{\alpha_{16}}(F_{21}F_{32}F_{43}F_{54}F_{65}F_{16} - I_4) = F_{21}F_{32}F_{43}F_{54}F_{65}\partial_{\alpha_{16}}F_{16} = F_{21}F_{32}F_{43}F_{54}F_{65}F_{16}M_\alpha = I_4M_\alpha = M_\alpha;$
- $\partial_{\beta_{16}}(F_{21}F_{32}F_{43}F_{54}F_{65}F_{16} - I_4) = F_{21}F_{32}F_{43}F_{54}F_{65}\partial_{\beta_{16}}F_{16} = F_{21}F_{32}F_{43}F_{54}F_{65}F_{16} \begin{bmatrix} 0 & -\sin(\alpha_{16}) & -\cos(\alpha_{16}) & 0 \\ \sin(\alpha_{16}) & 0 & 0 & 0 \\ \cos(\alpha_{16}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\sin(\alpha_{16}) & -\cos(\alpha_{16}) & 0 \\ \sin(\alpha_{16}) & 0 & 0 & 0 \\ \cos(\alpha_{16}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$
- $\partial_{\gamma_{16}}(F_{21}F_{32}F_{43}F_{54}F_{65}F_{16} - I_4) = F_{21}F_{32}F_{43}F_{54}F_{65}\partial_{\gamma_{16}}F_{16} = F_{21}F_{32}F_{43}F_{54}F_{65}M_\gamma F_{16} \begin{bmatrix} I_3 & \mathbf{0}_{[3 \times 1]} \\ \mathbf{0}_{[3 \times 1]}^T & 0 \end{bmatrix} = F_{16}^{-1}M_\gamma F_{16} \begin{bmatrix} I_3 & \mathbf{0}_{[3 \times 1]} \\ \mathbf{0}_{[3 \times 1]}^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\cos(\alpha_{16})\sin(\alpha_{16}) & \cos(\alpha_{16})\sin(\beta_{16}) & 0 \\ \cos(\alpha_{16})\cos(\beta_{16}) & 0 & -\sin(\beta_{16}) & 0 \\ -\cos(\alpha_{16})\sin(\beta_{16}) & \sin(\beta_{16}) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$
- $\partial_{T_l}(F_{21}F_{32}F_{43}F_{54}F_{65}F_{16} - I_4) = F_{21}F_{32}F_{43}F_{54}F_{65}\partial_{T_l}F_{16} = F_{21}F_{32}F_{43}F_{54}F_{65}M_l F_{T_{16}} F_{\gamma_{16}} F_{\beta_{16}} F_{\alpha_{16}} = F_{16}^{-1}M_l.$

Remark A.1.7. By looking at the symbolic computation in Matlab of the matrices $J_{\tilde{\mathbf{L}},\mathbf{X}}(\mathbf{X}, \mathbf{H}(\mathbf{X}))$ and $J_{\tilde{\mathbf{L}},\mathbf{Y}}(\mathbf{X}, \mathbf{H}(\mathbf{X}))$, we observe that in the equation system (8.5) a set of independent equations is given by the six components (1, 2), (1, 3), (2, 3), (3, 1), (3, 2), (3, 3).

Therefore we define $\tilde{\mathbf{L}}(\mathbf{X}, \mathbf{Y}) = (L_{12}, L_{13}, L_{23}, L_{31}, L_{32}, L_{33})(\mathbf{X}, \mathbf{Y})$ so that we can write $J_{\tilde{\mathbf{L}},\mathbf{X}}(\mathbf{X}, \mathbf{H}(\mathbf{X})) + J_{\tilde{\mathbf{L}},\mathbf{Y}}(\mathbf{X}, \mathbf{H}(\mathbf{X}))J_{\mathbf{H}}(\mathbf{X}) = 0.$

It follows from remark (A.1.7) that $J_{\mathbf{H}}(\mathbf{X}) = -J_{\tilde{\mathbf{L}},\mathbf{Y}}^{-1}J_{\tilde{\mathbf{L}},\mathbf{X}}$, where

$$J_{\tilde{\mathbf{L}},\mathbf{Y}} = \begin{bmatrix} 0 & -\sin(\alpha_{16}) & -\cos(\alpha_{16})\cos(\beta_{16}) & & & \\ 0 & -\cos(\alpha_{16}) & \sin(\beta_{16})\cos(\beta_{16}) & 0_{[3 \times 3]} & & \\ -1 & 0 & -\sin(\beta_{16}) & & & \\ & 0_{[3 \times 3]} & & Q_{16}^T & & \end{bmatrix} \text{ and, if } \beta_{16} \neq \frac{\pi}{2}, \frac{3\pi}{2},$$

$$J_{\tilde{\mathbf{L}},\mathbf{Y}}^{-1} = \frac{1}{\cos(\beta_{16})} \begin{bmatrix} \cos(\alpha_{16})\sin(\beta_{16}) & -\sin(\alpha_{16})\sin(\beta_{16}) & -\cos(\beta_{16}) & & & \\ -\sin(\alpha_{16})\cos(\beta_{16}) & -\cos(\alpha_{16})\cos(\beta_{16}) & 0 & 0_{[3 \times 3]} & & \\ -\cos(\alpha_{16}) & \sin(\beta_{16}) & 0 & & & \\ & 0_{[3 \times 3]} & & & \cos(\beta_{16})Q_{16}^T & \end{bmatrix}.$$

A.2 Matlab and R codes

We report in this appendix the codes the Matlab functions and scripts to simulate the system.

```

1  %script cylinder.m for plotting the cylinder
    O=[0;0;0];
3  e1=[1;0;0];
    e2=[0;1;0];
5  e3=[0;0;1];
    syms x y z real
7  X=[x;y;z];
    Y=[x;y;z;1];
9  EPS=[eps;eps;eps];
    EPS=EPS'*X;
11 A=[1 0 0;0 1 0;0 0 0];
    C=[1 0 0 0;0 1 0 0;0 0 0 0;0 0 0 -r^2];
13 cilindro=@(Y) Y'*[A zeros(3,1);zeros(1,3) -r^2]*Y;
    fimplicit3(cilindro(Y)+EPS,'b')%
15
    % function receiveing the camera reference system and parameters in input and
17 %returning the coordinate points in the image plane and the normal to the paired
    %laser plane, also plotting the system camera/laser
19 function [zcc,N]=camera_system(Oi,0,e1,e2,e3,f,r,Ds,A,C,EPS,X,Y,d)
    v3=O-Oi;
21 v3(3)=0;
    v3=v3/norm(v3);
23 v1=-e3+v3(3)*v3;
    v1=v1/norm(v1);
25 v2=-[v3(3)*v1(2)-v1(3)*v3(2);-(v1(1)*v3(3)-v3(1)*v1(3));v1(1)*v3(2)-v1(2)*v3(1)];
    Q=[v1 v2 v3];
27 T=Oi;
    % Piano lama laser nel sistema di riferimento canonico
29 S=Oi+d*v1;
    k=1./(v1'*(S-Oi));
31 n=k*(sqrt(.5)*v1+sqrt(.5)*v3);
    N=Q'*n; %Q'=M_SBi
33 laser=@(X) n'*(X-S);
    fimplicit3(laser(X)+EPS,'r')
35 hold on
    % Piano immagine rispetto al sistema canonico
37 N_imm=v3; %normale al piano immagine v1v2
    p=T+f*v3; %punto del piano immagine
39 immagine=@(X) N_imm'*(X-p);
    fimplicit3(immagine(X)+EPS)
41 %Piano polare
    pp=@(Y) [Oi' 1]*C*Y;
43 fimplicit3(pp(Y)+EPS,'b')%piano polare Q'*(1-T)=0
    % Cono
45 cono=@(X) ((n'*(X-Oi))*Oi+X-Oi)'*A*((n'*(X-Oi))*Oi+X-Oi)-r^2*(n'*(X-Oi))^2;
    fimplicit3(cono(X)+EPS,'w')
47 plot3(0,0,0,'y')
    plot3(T(1),T(2),T(3),'go')
49 plot3(p(1),p(2),p(3),'yo') %Otilde_i
    Ctil=Q'*A*Q+2*Q'*n*T'*A*Q+Q'*n*T'*A*T*n'*Q-r^2*Q'*n*n'*Q;
51 [a, b, Qc, Tc] = canonicalcoeff(Ctil,f);
    V1=Qc*[a;0]+Tc; %teta=0
53 V2=Qc*[0;b]+Tc; %teta=pi/2
    V3=Qc*[-a;0]+Tc; %teta=pi

```

```

55 V4=Qc*[0;-b]+Tc; %teta=3/2*pi
    visibile=(T'*A'*Q+T'*A'*A*T*n'*Q-r^2*n'*Q)*[V1 V2 V3 V4;f f f f];
57 for i=1:4
        if visibile(i)<0
59             teta0=(i-1)*pi/2; break
        end
61 end
    teta=teta0;
63 teta_n=teta0;
    while(teta_n<teta0+2*pi)
65         prodotto=max(abs([-a*sin(teta_n) b*cos(teta_n)]*Qc'*[1;0]),abs([-a*sin(teta_n)
            b*cos(teta_n)]*Qc'*[0;1]));
67         Dteta=Ds/prodotto;
            teta_n=teta_n+Dteta;
69         teta=[teta teta_n];
    end
71 teta=teta(1:end-1);
    t1=a.*cos(teta);
73 t2=b.*sin(teta);
    zc=[t1;t2];
75 zp=Qc*zc+Tc;
    zcc=[zp;f*ones(length(t1),1)'];
77 zp=Q*zcc+T;
    plot3(zp(1,:),zp(2,:),zp(3,),'g*','Markersize',10)%ellisse
79 % plot sistemi di riferimento e1e2e3 e v1v2v3
    starts = zeros(3,3);
81 ends = [e1';e2';e3'];
    quiver3(starts(:,1), starts(:,2), starts(:,3), ends(:,1), ends(:,2), ends(:,3))
83 starts = [T';T';T'];
    ends = [v1';v2';v3'];
85 quiver3(starts(:,1), starts(:,2), starts(:,3), ends(:,1), ends(:,2), ends(:,3))
    starts = zeros(3,3);
87 ends = [N';N';N'];
    quiver3(starts(:,1), starts(:,2), starts(:,3), ends(:,1), ends(:,2), ends(:,3))
89 %retta che separa la regione visibile e non visibile nel piano immagine
    %retta nel piano della lama laser
91 Npp=[0i;1]*C; %componenti vettore normale del piano polare nel sdr globale
    Mcrp11=[Npp(1:3);n']; %matrice dei coeff della retta nel piano lama laser
93 tn=[-Npp(4);n'*S]; %vettore termini noti
    P11=Mcrp11\tn; %sol particolare = coordinate di un punto P della retta
95 vr=null(Mcrp11); %sol omogenea = componenti vettore parallelo alla retta
    %calcolo proiezione del punto P nel piano immagine
97 t1t2=[v1 v2 -(P11-0i)]\[-f*v3];
    Ppi=0i+Q*[t1t2(1:2);f];
99 %plot della retta
    t=linspace(-10,10);
101 Prpi=vr*t+Ppi;
    plot3(Prpi(1,:),Prpi(2,:),Prpi(3,),'r','linewidth',3)
103 visibile=(T'*A'*Q+T'*A'*A*T*n'*Q-r^2*n'*Q)*zcc;
    for i=1:length(t1)
105         if visibile(i)>=0
            plot3(zp(1,i),zp(2,i),zp(3,i),'b*','Markersize',10)
107         end
    end
109
    % function used in camera_system.m to compute the canonical equation of the ellipse
111 %in the image plane
    function [a, b, Q, T] = canonicalcoeff(Ctil,f)
113 Ctil=.5*(Ctil+Ctil');

```

```

Atil=Ctil(1:2,1:2);
115 Btil=Ctil(1:2,3);
    ctil=Ctil(3,3);
117 Btil=Btil*f;
    ctil=ctil*(f^2);
119 I2=det(Atil);%invarianti
    Ctil=[Atil Btil;Btil' ctil];
121 I3=det(Ctil);
    [Q,eigAtil]=eig(Atil);%Q diagonalizza Atil
123 eigAtil=diag(eigAtil);%lambda1x^2+lambda2y^2+terminenoto=0
    c_m=I3/I2;
125 coeff_m=c_m./[eigAtil;c_m];
    a=sqrt(abs(coeff_m(1)));
127 b=sqrt(abs(coeff_m(2)));
    T=Atil\(-Btil);
129 return

```

We report also the Matlab codes to calculate the KCRLB for our problem.

```

1 %function for computing the numerator of the KCRLB formula
function [F,Q,num]=numerator(C,alpha,beta,gamma,T,N,zp,f,DrNum,M1,M2,Malpha)
3 %input: rototranslation parameters wrt reference camera, normal to laser plane
%coordinates of points in the image plane, focal length, dervative of C_1
5 %wrt the first 2 components of translation
%output: rototranslation matrix, rotation matrix, term in the numerator
7 %corresponding to the camera (the summation on points is already computed)
Q1=[1 0 0;0 cos(alpha) -sin(alpha);0 sin(alpha) cos(alpha)];
9 Q2=[cos(beta) 0 -sin(beta);0 1 0;sin(beta) 0 cos(beta)];
Q3=[cos(gamma) -sin(gamma) 0;sin(gamma) cos(gamma) 0;0 0 1];
11 Q=Q3*Q2*Q1;
F=[eye(3) T;0 0 0 1]*[Q2 zeros(3,1);0 0 0 1]*[Q3 zeros(3,1);0 0 0 1];
13 C1=F'*C*F;
    DalphaNum=C1*Malpha-Malpha*C1; %4x4
15 DbetaNum=F'*[0 0 0 0;0 0 0 0;-1 0 0 0;0 0 0 0]*F;
    DT1Num=F'*M1+M1'*F;
17 DT2Num=F'*M2+M2'*F;
    Numsum=0;
19 for i=1:size(zp,2)
    csi=F*[zp(1,i);zp(2,i);f;N'*[zp(1,i);zp(2,i);f]]; %4x1
21    Num=[csi'*DalphaNum*csi; csi'*DbetaNum*csi; csi'*DT1Num*csi; csi'*DT2Num*csi;
        csi'*DrNum*csi]; %5x1
23    %Num=simplify(Num);
    Numsum=Numsum+Num*Num';
25 end
num=Numsum;
27
%function for computing the denominator of KCRLB
29 function Den=denominator(C,F,N,zp,f,Vest,Vpunti,alpha,beta,Q,JK1,JK2)
%input: cylinder matrix, rototranslation wrt reference camera, the estimated
31 %covariance matrix with the current configuration of measurement errors, coordinates of the
%points in the image plane, matrices J_K1 and J_K2
33 %output: denominator corresponding to the camera (the summation on points is
%already computed)
35 JK2inv=(JK2'*JK2)\JK2';
    Vest=JK2inv*JK1'*Vest*JK1'*JK2inv';
37 V=[Vest zeros(6,2);zeros(2,6) Vpunti];
    Cm=F'*C*F;
39 Dt1=F*[1;0;0;N(1)];
    Dt2=F*[0;1;0;N(2)];

```

```

41 Dt=[Dt1 Dt2];
Den=zeros(1,size(zp,2));
43 for i=1:size(zp,2)
Dalpa=F*[0;-f;zp(2,i);0];
45 Dbeta=F*[-sin(alpha)*zp(2,i)-f*cos(beta);sin(alpha)*zp(1,i);cos(alpha)*zp(1,i);0];
Dgamma=F*[cos(beta)*(-cos(alpha)*zp(2,i)+f*sin(alpha));cos(alpha)*sin(beta)*zp(1,i)
47 -f*sin(beta);-sin(alpha)*cos(beta)*zp(1,i)+sin(beta)*zp(2,i);0];
DT=[N'*zp(:,i)*Q';0 0 0];
49 omega=[Dalpa Dbeta Dgamma DT Dt];
Den(i)=4*[zp(:,i)' N'*zp(:,i)]*Cm*omega*V*omega'*Cm*[zp(:,i);N'*zp(:,i)];
51 end
Den=sum(Den);
53
%function for computing the derivatives which are meant to give the matrices J_K1,
55 %J_K2, J_LX and J_LY
function [DalpaDen,DbetaDen,DgammaDen,DalpaF,DbetaF,DgammaF]=derivate(F,alpha,
57 beta,Malpa)
%calcolo derivate rispetto a parametri di rototraslazione
59 invF=inv(F);
DalpaDen=F*Malpa*invF;
61 DbetaDen=F*[0 -sin(alpha) -cos(alpha) 0;sin(alpha) 0 0 0;cos(alpha) 0 0 0;0 0 0 0]
*invF;
63 DgammaDen=F*[0 -cos(alpha)*cos(beta) sin(alpha)*cos(beta) 0;cos(beta)*cos(alpha) 0
-sin(beta) 0;-sin(alpha)*cos(beta) sin(beta) 0 0;0 0 0 0]*inv(F);
65 DalpaF=DalpaDen(:,1:3)*F(1:3,:);
DbetaF=DbetaDen(:,1:3)*F(1:3,:);
67 DgammaF=DgammaDen(:,1:3)*F(1:3,:);

69
%the following script is the one which computes the KCRLB for the radius in all
71 %the possible configurations of the errors on the measurements collected in the
%vectors named d_ang, d_xy, d_z, d_pixel
73 [ca, cb, cc, cd] = ndgrid(d_ang, d_xy, d_z, d_pixel);
combs = [ca(:), cb(:), cc(:), cd(:)];
75 ris=zeros(size(combs,1),1);
for j=1:size(combs,1)
77 %scorrendo i vettori di dev std dei parametri calcola la matrice di KCRLB
V21=diag([combs(j,1)^2 combs(j,1)^2 combs(j,1)^2 combs(j,2)^2 combs(j,2)^2
79 combs(j,3)^2]);
V32=V21;
81 V13=V32;
Vall=[V21 zeros(size(V21));zeros(size(V21)) V21];%cov matrix di tutti i parametri
83 %di tutte le telecamere
Vest=invJG*(Vall+JH'*V21*JH)*invJG;%Vall e V13 sono dati
85 Vpunti=diag([combs(j,4)^2 combs(j,4)^2]);
Den1=denominator(C,F11,N1,zp1,f,zeros(6),Vpunti,0,0,Q11,eps*eye(size(JK21)),
87 eps*eye(size(JK21)));
Den2=denominator(C,F21,N2,zp2,f,Vest(1:6,1:6),Vpunti,alpha21,beta21,Q21,JK21,JK21);
89 Den3=denominator(C,F31,N3,zp3,f,Vest(1:12,1:12),Vpunti,alpha31,beta31,Q31,JK132,
JK232);
91 dainv=num1/Den1+num2/Den2+num3/Den3;
kcr=inv(dainv);
93 ris(j)=sqrt(kcr(5,5));
end
95 dataset=[combs ris];

```

Finally, we report the R script that we used for the data analysis.

```
1 setwd("C:/Users/Sara/Documents/tesi/marcon/teoria per modello/articles/kanatani/matlab")
```

```

data <- read.table('errori.txt', header=T)
3 dimnames(data)[[2]]<-c('sig_ang','sig_xy','sig_z','sig_pixel','sig_r')
  attach(data)
5 #extreme configurations
  dataconfmin<-data[which(data$sig_r==min(data$sig_r)),1:4]
7 dataconfmax<-data[which(data$sig_r==max(data$sig_r)),1:4]
  detach(data)
9 ##### Normality of the components, descriptive plots
  x11(width=12)
11 par(mfcol=c(2,5))
  for(i in 1:5)
13 {
      hist(data[,i], prob=T, main=paste('Histogram of ', dimnames(data)[[2]][i], sep=''))
15      lines(900:2800, dnorm(900:2800,mean(data[,i]),sd(data[,i])), col='blue', lty=2)
      qqnorm(data[,i], main=paste('QQplot of ', dimnames(data)[[2]][i], sep=''))
17      qqline(data[,i])
  }#
19 data<-as.data.frame(scale(data))
  attach(data)
21 fit0 <- lm(sig_r ~ 0+sig_ang+sig_pixel+sig_z+sig_xy)
  summary(fit0)
23 detach(data)
  x11()
25 par(mfrow=c(2,2))
  plot(fit0)
27 #####PCA
  X<-data[,1:4]
29 n <- dim(X)[1]
  p <- dim(X)[2]
31 pc <- princomp(X, scores=T)
  summary(pc)
33 load <- pc$loadings
  load
35 X11()
  par(mfrow=c(2,2))
37 barplot(load[,1], ylim = c(-1, 1),las=2,main='loadings of 1st pc')
  abline(h=0)
39 barplot(load[,2], ylim = c(-1, 1),las=2,main='loadings of 2nd pc')
  abline(h=0)
41 barplot(load[,3], ylim = c(-1, 1),las=2,main='loadings of 3rd pc')
  abline(h=0)
43 barplot(load[,4], ylim = c(-1, 1),las=2,main='loadings of 4th pc')
  abline(h=0)
45 # scores (projected data)
  Y<- pc$scores
47 #####PCR
  data <- read.table('errori.txt', header=T)
49 dimnames(data)[[2]]<-c('sig_ang','sig_xy','sig_z','sig_pixel','sig_r')
  fitpc<-lm(data$sig_r ~ 0+Y[,1]+Y[,2]+Y[,3]+Y[,4])
51 summary(fitpc)
  ridpc<-lm(data$sig_r ~ 0+Y[,1]+Y[,3]+Y[,4])
53 summary(ridpc)
  x11()
55 par(mfrow=c(2,2))
  plot(ridpc)

```
