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Trembling-Hand Perfection in Extensive Form Correlated Equilibria

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[...] *Noi siamo sul promontorio dei secoli! ...*
Perchè dovremmo guardarci alle spalle,
se vogliamo sfondare le misteriose porte dell'impossibile?
Il tempo e lo spazio morirono ieri.
Noi viviamo già nell'assoluto,
perchè abbiamo già creata l'eterna velocità onnipresente. [...]

Filippo Tommaso Marinetti,
Manifesto del Futurismo, 1909.

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Sommario

Sin dalla definizione del concetto di equilibrio di Nash, lo studio dei suoi principali raffinamenti e delle sue generalizzazioni è stato centrale nell'ambito della Teoria dei Giochi e ha permesso di sviluppare strumenti molto utili nelle applicazioni pratiche legate all'Intelligenza Artificiale. Il principale raffinamento dell'equilibrio di Nash è il Perfect Equilibrium definito da Selten, che irrobustisce i risultati ottenuti identificando quegli equilibri che rimangono stabili anche se qualcuno dei giocatori commette un errore. Uno dei principali risultati che generalizzano il lavoro di Nash è il Correlated Equilibrium definito da Aumann, che introduce la figura di un moderatore del quale i giocatori hanno convenienza a fidarsi e che prescrive loro le azioni da eseguire per raggiungere un punto di equilibrio. Nel presente lavoro si affrontano questi due concetti congiuntamente e specificatamente per i giochi in forma estesa, inserendo il concetto di perfezione nel contesto dei giochi correlati, con l'obiettivo di fornire un irrobustimento del Correlated equilibrium. In tal modo, dopo uno studio sulla natura delle perturbazioni che si possono verificare quando si giocano giochi correlati, si arriva alla definizione di *Extensive Form Perfect Correlated Equilibrium (EFPCE)*, che coniuga definitivamente i due concetti sopra esposti. Per fornire poi anche una implementazione, si arriva alla definizione del problema della ricerca di un EFPCE formulato in forma sequenza per giochi a due giocatori, dimostrando l'esistenza di una soluzione al problema indipendentemente dalla perturbazione scelta (analogamente a quanto succede per il perfect equilibrium). Infine, ci si interessa dei rapporti che intercorrono tra i vari tipi di equilibrio con particolare attenzione a quello appena definito. Ciò che si ottiene è pertanto una tassonomia completa dei più importanti solution concepts disponibili per i giochi in forma estesa che dimostra come, dato che l'Extensive Form Correlated equilibrium (EFCE) è una generalizzazione del Nash equilibrium (NE), anche l'EFPCE sia una generalizzazione dell' Extensive Form Perfect Equilibrium (EFPE). Inoltre, si dimostra che l'intersezione tra gli insiemi NE e EFPCE è esattamente l'insieme degli EFPE.

Abstract

Since the definition of the concept of Nash equilibrium, the study of its main refinements and generalizations has been central in the field of Game Theory and has allowed to develop very useful tools for practical applications related to Artificial Intelligence. The main refinement of Nash equilibrium is the Perfect Equilibrium defined by Selten, which strengthens the equilibrium requirements obtained by identifying those equilibria that remain stable even if some of the players may make a mistake. One of the main results that generalizes Nash's work is the Correlated Equilibrium defined by Aumann, which introduces the figure of a trustworthy moderator who prescribes the actions to be performed to reach an equilibrium. In this work, these two concepts are jointly analyzed, specifically for games in extensive form, by inserting the concept of perfection in the context of correlated games, with the aim of providing a refinement of the Correlated equilibrium. In this way, after a study on the nature of the perturbations that can occur when playing games with a mediator, we formulate the definition of *Extensive Form Perfect Correlated Equilibrium (EFPCE)*, which definitively combines the two concepts described above. From a computational perspective, we formulate the problem of finding an EFPCE in sequence form for two-player games, showing the existence of a solution to this problem independently from the chosen perturbation scheme (similarly to what happens for the perfect equilibrium). Finally, we are interested in the relationships between the various kinds of equilibria with particular attention to the one just defined. Therefore, we obtain a complete taxonomy involving the most important solution concepts available for games in extensive form, which shows how, given that the Extensive Form Correlated equilibrium (EFCE) is a generalization of the Nash equilibrium (NE), also the EFPCE is a generalization of the Extensive Form Perfect Equilibrium (EFPE). Furthermore, it is shown and proved that the intersection between the NE and EFPCE sets is exactly the set of EFPEs.

Contents

Sommario **IV**

Abstract **VI**

1 Introduction **1**
1.1 A general overview 1
1.2 Structure of the Thesis 3

2 Preliminaries **5**
2.1 Games And Representation Forms 5
2.1.1 Games And Game Theory 5
2.1.2 Normal Form Games 6
2.1.3 Extensive Form Games 9
2.2 Nash Equilibria And Refinements 16
2.2.1 Nash Equilibrium 16
2.2.2 Perfect Equilibrium 17
2.2.3 Quasi-Perfect Equilibrium 19
2.3 Correlated Equilibria And Refinements 20
2.3.1 Correlated Equilibrium 21
2.3.2 Extensive Form Correlated Equilibrium 22
2.3.3 Perfect Correlated Equilibrium 25
2.4 Sequence Form and Linear Programming Characterization 26
2.4.1 More on the Sequence Form 26
2.4.2 Characterizing Nash Equilibria 26
2.4.3 Characterizing Correlated Equilibria 29

3 Extensive Form Perfect Correlated Equilibria **37**
3.1 Perturbations Analysis 37
3.1.1 Perturbing the Correlation Device 38
3.1.2 Perturbing the Observation 40

3.1.3 Perturbing the Execution 41
3.2 Equilibrium Definition 42
3.2.1 The Main Result 42
3.2.2 Sequence Form Characterization 44

4 Characterization **47**
4.1 Probabilistic Approach 47
4.1.1 Nash And Perfect Equilibria Distribution 47
4.1.2 Correlated Equilibria Distribution 49
4.2 A Complete Taxonomy 50
4.2.1 Before EFPCE 50
4.2.2 EFPCE vs EFCE 53
4.2.3 EFPCE vs NE 56
4.2.4 EFPCE vs PE 59

5 Conclusions And Future Work **63**
5.1 Conclusions 63
5.2 Future Work 64

Bibliography **67**

List of Figures

2.1 Extensive Form Representation of Chicken Game 10
2.2 Example of a Game in Agent Form 12
2.3 Sequence Form Game of Example 3 14
2.4 Example of Extensive Form Game 18
2.5 Perturbed Game of Figure 2.4 with $l(\varepsilon) = \varepsilon$ 19
2.6 Extended Chicken Game with μ_1 as a CD 24
2.7 Extended Chicken Game with μ_2 as a CD 24

3.1 Extended Game obtained using $\widetilde{\mu}_1$ as a CD 40
3.2 An Extended Perturbed version of the Chicken Game 43

4.1 Set Diagram representation of relation 4.1 52
4.2 Set Diagram representation of relation 4.2 53
4.3 Asymmetric Heads or Tails Game in Extensive Form 54
4.4 Extended Asymmetric Heads or Tails Game with μ as a CD . 55
4.5 Extended Perturbed Asymmetric Heads or Tails Game 55
4.6 Set Diagram representation of relation 4.3 56
4.7 Extended Perturbed Chicken Game 57
4.8 Extended Asymmetric Heads or Tails Game with $\widetilde{\mu}$ as a CD . 58
4.9 Set Diagram representation of relation 4.4 59
4.10 Set Diagram of the complete taxonomy 62

List of Tables

2.1 Chicken Game 8
2.2 A Game of Candies 8

4.1 Asymmetric Heads or Tails 54

Chapter 1

Introduction

1.1 A general overview

The present work moves in the context of Artificial Intelligence, especially in the research field concerning multi-agent systems where each agent is rational. Consider a multi-agent system in which individuals must confront each other (without cooperation) to achieve a result, having different points of view regarding the convenience of every possible final outcome. Case studies like the one described above can be analyzed and solved using mathematical models provided by *Game Theory*.

During the last decade, the Algorithmic Game Theory community has achieved important milestones in the computational study of solution concepts inherited from Game Theory. The most acclaimed results are those concerning the computation of the *Nash Equilibrium* (Daskalakis et al., 2009; Conitzer and Sandholm, 2008). These problems have been tackled using different approaches, such as regret minimization techniques (Zinkevich et al., 2008; Cesa-Bianchi and Lugosi, 2006) and local search (Ceppi et al., 2010; Gatti et al., 2012). Recently, other works addressed the computation of other solution concepts, such as the Stackelberg equilibrium (Conitzer and Sandholm, 2006; Coniglio et al., 2017, 2018; Basilico et al., 2017d; De Nittis et al., 2018; Farina et al., 2018c; Marchesi et al., 2018, 2019). Moreover, algorithmic game theory has been applied to various practical applications, such as auctions (i.e., mechanism design) (Myerson, 1989; Ceppi et al., 2011; Gatti et al., 2015; Lopes and Coelho, 2014).

The extensive-form representation for non-cooperative games with imperfect information and perfect recall with multiple players represents one of

the most studied models in the field of game theory and has been repeatedly analyzed from many points of view. One of the major achievements following the Nash Equilibrium defined in (Nash, 1951) was the development of its main refinement, the Extensive Form Perfect Equilibrium defined in (Selten, 1975), whose practical application makes possible to greatly extend the degree of resilience of real-world implementations of this category of games, because it makes the implemented solutions remarkably robust to trembles when dealing with a wide range of errors, regardless of their origin.

One of the most important generalizations of the Nash equilibrium is the correlated equilibrium proposed in (Aumann, 1974), which introduced a mediator who can act in the context of the game and on which the players have previously agreed, that deals first with the random selection of an equilibrium point to reach and then privately informs the players of which is the right strategy to follow when playing the game in order to get to that point. This type of game, while remaining in the class of non-cooperative games, actually introduces a minimum element of coordination whose strength lies in the fact that players are privately informed of the choices made by the mediator, and must therefore trust the moderation system in order to avoid encountering improper results. This result, extensively studied and analyzed in the literature, was defined for games in extensive form in (von Stengel, 1996).There is an increasing amount of works dedicated to the computation of correlated equilibria (see, e.g., Jiang and Leyton-Brown (2011); Papadimitriou and Roughgarden (2008); Celli et al. (2019)). In particular, the possibility of modeling some form of communication between players allows one for extending this domain to new interesting settings. Two examples, with ubiquitous practical application, are *team games* (see, e.g., Basilico et al. (2017c,b); Celli and Gatti (2018); Farina et al. (2018a)) and *security games* (see, e.g., Tambe (2011); Basilico et al. (2017a, 2016, 2017e); Munoz de Cote et al. (2013)).

Even if the two equilibrium concepts just described are supporting pillars of the game theory context and have been individually studied in a very thorough way, there is no joint discussion of these themes in the literature. This work, therefore, aims to define, on the basis of the concept of Correlated Equilibrium, a refinement that exploits the concept of perfection and can strengthen the correlated equilibria results already present in the literature.

Once the aforementioned analysis has been carried out, it is clear that

another missing result in the literature is the description of the relationships between the most important types of equilibrium. Some of these relationships have already been analyzed by von Stengel and Forges (2008), but the bigger picture, including our newly defined solution concept, is still missing. A complete taxonomy of these equilibria is particularly useful when comparing solutions in terms of the utility they provide to players. This is the case, for example, of *Price-of-Anarchy* analyses (Roughgarden, 2005, 2009).

1.2 Structure of the Thesis

This work is organized as follows:

Chapter 2 introduces the fundamentals of game theory, describing the concepts of Representation Forms, Nash Equilibrium, Correlated Equilibrium and their most important refinements, with particular emphasis on those concepts that are needed to understand the rest of this work. It also contains a mathematical formulation for the problem of searching for an equilibrium in a generic extensive-form game.

Chapter 3 describes how to perturb a correlated extensive form game and contains the definition of the Extensive Form Perfect Correlated Equilibrium. It also presents a mathematical formulation for the problem of searching for such an equilibrium in a generic extensive-form game.

Chapter 4 describes the complete taxonomy that sums up the relations between the various equilibria defined in the previous chapters. It also contains the description of a possible probabilistic approach to this taxonomy.

Chapter 5 summarizes the main results of this thesis and illustrates the future works that can be developed building on this one.

Chapter 2

Preliminaries

In this chapter, we will provide the reader with an appropriate theoretical background to understand the following work by introducing the main concepts that will be used further in the development of the main results presented below.

2.1 Games And Representation Forms

This section concerns the concept of Game and is used to introduce the main representation forms of a Game.

2.1.1 Games And Game Theory

According to the definition in (Myerson, 1997), Game Theory is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers". It is important to notice that agents are supposed to be *rational*, meaning that each agent is aware of his alternatives, forms expectations about any unknowns, has clear preferences over the outcomes and chooses his actions in order to maximize his expected reward. In the absence of uncertainty, the following elements provide a basic model of rational choice (Osborne and Rubinstein, 1994):

- A set of *actions* A available to the agent;
- A set C of possible *consequences* (outcomes) to these actions;
- A *consequence function* $g : A \to C$ that determines the consequence of each action;
- A *preference relation* $\succsim$ over the set of possible outcomes C.

Usually decision-makers's preferences are expressed with a *utility function* $U : C \to \mathbb{R}$, which defines a preference relation over the outcomes by the condition $x \succsim y \Leftrightarrow U(x) \geq U(y)$. To model situations of decision-making under uncertainty this model of rationality is adapted, according to (von Neumann and Morgenstern, 1944), by letting each decision-maker maximize her expected utility.

Given one or more intelligent individuals (called *Players*) and a set of available *Actions*, we define a *Game* as a mathematical model that describes the player's interests in choosing one of the available actions, also considering the choices of other players (if any). This model only describes the *Outcome* that a Player can get by choosing a certain action in a certain moment, but does not specify which is the best action to choose. Since this model is only a description, a *solution concept* is needed to decide how to play a game. Some of the main known solution concepts will be presented in section 2.2.

2.1.2 Normal Form Games

Each game can be represented in numerous forms, with the aim of adapting in the best possible way to the theoretical analysis that must be performed. Furthermore, each of these forms can be used to better represent some information, since no form can represent the totality of information in a game. As will be shown later the transition from one representation form to another can often cause the loss of such information. It is therefore essential to choose the most appropriate form of representation when dealing with a game. The following is the most used form of representation and describes the game considering that at the beginning and simultaneously each player chooses a strategy (a probability distribution over her actions) to follow.

First of all, consider a generic Game G played by m Players, represented by the set $N = \{1, \ldots, m\}$. Every Player n is associated with a set of k_n available actions $A_n = \{a_1^n, \ldots, a_{k^n}^n\}$. We can now procede with the following definitions:

Definition 1 (Strategy)**.** *Consider a player $n \in N$. A* Strategy *is defined as function $s_n : A_n \to [0, 1]$, with $\sum_{x=1}^{k_n} s_n(a_x^n) = 1$, that returns the probability with which player n plays each action of A_n.*

A Strategy s_n in which for one a_x^n holds that $s_n(a_x^n) = 1$ is called *Pure Strategy*. Pure Strategies are therefore those in which a Player plays with certainty an available action.

Definition 2 (Strategy Profile)**.** *Given a set of m Players N defined as above, a* Strategy Profile *is a tuple $s = (s_1, \ldots, s_m)$, of Strategies, each one associated with a Player.*

A Strategy Profile is called *Pure* if it only contains Pure Strategies. If we consider the set S_n containing all the Pure Strategies of n, in a Pure Strategy Profile s holds that $s_1 \in S_1, \ldots, s_m \in S_m$. A Strategy Profile that is not Pure is called *Mixed*, if no s_n in s is Pure and each s_n in s is played with probability $p(s_n) > 0$, s is called *Fully Mixed (Completely Mixed).*

Before proceeding with the definition of the Game models, it is necessary to define how the Players orient themselves in the selection of the Actions. In a generic Game each Player gets a certain result (*Outcome*) based on the strategy profile selected by all Players. This result is quantified in a real number to be easily compared with others.

Definition 3 (Payoff Function)**.** *A* Payoff Function *for a Player $n \in N$ is a function $f_n : S_1 \times \cdots \times S_m \to \mathbb{R}$ that maps each Pure Strategy Profile with a payoff for every Player.*

It is now possible to introduce one of the most important Representation Forms of a Game:

Definition 4 (Normal Form Game)**.** *A* Normal Form Game *is a triplet $G = (N, S, F)$ where:*

- *$N = \{1, \ldots, m\}$ is the set of Players*
- *$S = \{S_1, \ldots, S_m\}$ is the set containing all the Pure Strategies*
- *$F = \{f_1, \ldots, f_m\}$ is the set containing all the Payoff Functions*

Games in normal form are usually represented as m-dimensional matrices that contain tuples of Outcomes of the ordered Players in the cells. Each player controls a dimension of the matrix, therefore every possible Pure Strategy Profile also identifies a cell of the matrix. The followings are Examples of Games that will be used later:

Example 1 (Chicken Game)**.** *The chicken-dare game can be thought of as two drivers racing towards an intersection. A player can chose to dare (Go) and pass through the intersection or chicken out and (Stop). The game results in a draw when both players chicken out and the worst possible outcome if they both dare. A player wins when he dares while the other chickens out.*

	Go	Stop
Go	$(6,6)$	$(2,7)$
Stop	$(7,2)$	$(0,0)$

Table 2.1: Chicken Game

Example 2 (A Game of Candies)**.** *Consider a two player game in which two little cousins wants to ask to their grandfather some candies. They know that child* 1 *is grandfather's favourite so if he asks (action b) him both the players will gain 3 candies. If child* 2 *wants to ask when his cousin does not want (action a), he must decide wheter to mention (action c) the fact that the candies will be shared with his cousin or not (action d). If he does then he gains 2 candies, otherwise only 1 while the first child always gains 2.*

	c	d
a	$(2,2)$	$(2,1)$
b	$(3,3)$	$(3,3)$

Table 2.2: A Game of Candies

The following refinement of the normal form will be widely used in the following chapters:

Definition 5 (Reduced Normal Form Game)**.** *Given a Normal Form Game $G = (N, S, F)$ as defined in Definition 4, the corresponding* Reduced Normal Form Game *is a triplet $G^r = (N, S^r, F)$ where S^r has the following characteristics:*

- *There are no $s_n, s_n^r \in S_n^r$, with $s_n \neq s_n^r$ that are realization equivalent (s_n and s_n^r are realization equivalent iff $\forall n \in N, f_n((s_1, \ldots, s_n, \ldots, s_m)) = f_i((s_1, \ldots, s_n^r, \ldots, s_m))$)*
- *$\forall s_n \in S_n \setminus S_n^r, s_n$ is realization equivalent to some $s_n^r \in S^r$*

The Reduced Normal Form is a refinement of the Normal Form which consists in the unification of the moves of a Player that, fixing the moves of the others, result in the same Outcomes for all the players. As an example of this, if we consider two 2-Player Games G and G^r defined as follows:

$$G = \begin{pmatrix} (1,1) & (0,0) \\ (1,1) & (0,0) \\ (0,0) & (1,0) \end{pmatrix}, G^r = \begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (1,0) \end{pmatrix}$$

It is obvious that G^r is the Reduced Normal Form of G obtained by unifying the first and second rows.

2.1.3 Extensive Form Games

Consider now the situation in which Players face each other in turn in a Game with successive steps in which each player is called to select a move. This class of Games models a wide set of real situations (chess, checkers, all board games, etc..) but it is impossible to preserve information about time using the Normal Form as defined above. It is therefore necessary to define the class of *Extensive-Form Games*.

An extensive-form game is usually represented as a tree, called the Game Tree. The Game starts at the root node of the tree and ends at one of its leaves. The internal nodes of the tree, also called nonterminal decision nodes, represent steps of the Game where one player has to take an action. The actions available to the player correspond to the edges following that node. We also allow for particular nonterminal nodes, called chance nodes, which represent places where a fictitious player, called nature, makes a move following a specified probability distribution over the actions available at the node. Finally, players' payoffs are specified over the leaves of the game tree, which are called the terminal nodes. Formally:

Definition 6 (Extensive-Form Game)**.** *An* Extensive-Form Game with imperfect information *is a tuple*

$$\Gamma = (N, H, Z, A, \rho, \chi, \pi_0, \{f_n\}_{n\in N}, \{\mathfrak{I}_n\}_{n\in N}),$$

where:

- $N = \{0, 1, \ldots, m\}$ *is the set of Players where* 0 *is the Nature Player*
- $H = \bigcup_{n\in N} H_n$*, where* H_n *denotes the set of* n*'s nonterminal nodes and* H_0 *is a finite set of chance nodes.*
- Z *is a finite set of terminal nodes*
- A *is a finite set of actions*
- $\rho : H \to 2^{|A|}$ *is a function that assigns a set of available actions to each nonterminal node*
- $\chi : H \times A \to H \cup Z$ *is the successor function that defines the terminal node reached when performing an action from a nonterminal node*

- $\forall h \in H_0, \pi_0(h)$ *defines a probability distribution over* $\rho(h)$
- $f_n : Z \to \mathbb{R}$ *is the Payoff function as defined in Definition 3 (*f_n *is not defined for the Nature Player)*
- $\forall n \in N, \mathfrak{I}_n$ *defines a partition of* H_n *into* Information Sets

Considering the notation used above, we can better define an *Information Set* $I_n \in \mathfrak{I}$ as the set of decision nodes that are indistinguishable to the player. For this reason, the action played at different decision nodes within the same information set are played according to the same probability distributionn. Figure 2.1 shows the extensive form representation of the Chicken Game of Example 1. Every Information Set is therefore referred to

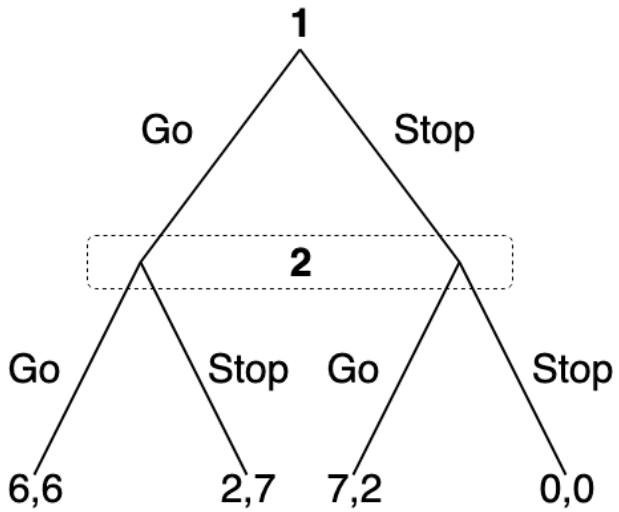

Figure 2.1: Extensive Form Representation of Chicken Game

a single Player. In the following chapters the concept of perfect recall will be very important:

Definition 7 (Games With Perfect Recall)**.** *An Extensive-Form Game* Γ *has* Perfect Recall *if, for every Player* $n \in N$ *and Information Set* $I_n \in \mathfrak{I}_n$*, all the decision nodes belonging to* I_n *share the same sequence of player* n*'s moves on their paths from the root.*

Hence, Perfect Recall means that no player forgets what he did or knew in the past. In other words, all the actions available in a certain information set are visible from all the nodes contained in that information set.

Let us now see how the concept of Strategy changes in Games in extensive form compared to those in normal form. A player's pure strategy, also called *Plan*, specifies an action at each information set in which it is the player's turn to move. A mixed strategy is a probability distribution over a player's pure strategies. When working with games that have perfect recall it is possible to refine the concept of strategy in the following way:

Definition 8 (Behavioral Strategy)**.** *A* Behavioral Strategy *defines, for every player $n \in N$ and information set $I_n \in \mathfrak{I}_n$, a probability distribution over the actions $\rho(I_n)$ available at I_n. For $n \in N$, let $\pi_n \in \Pi_n$ be a player n's behavioral strategy, where $\pi_n : A_n \to [0,1]$ such that*

$$\forall I_n \in \mathfrak{I}_n, \quad \sum_{a \in \rho(I_n)} \pi_n(a) = 1.$$

The fixed Behavioral strategy of the nature player is denoted by π_0.

We call *Behavioral Strategy Profile* the tuple $\pi = (\pi_0, \pi_1, \ldots, \pi_m)$, where $\forall n \in N, \pi_n \in \Pi_n$.

Another representation form that must be taken into account when working with games in extensive form is the *Agent Form*. In the representation in agent form, each nonterminal decision node is considered as belonging to a different player. In other words, in the agent form each player is divided into as many agents as the information sets associated with that player and each of these agents operates separately from the others, playing as an individual in its own right. Formally:

Definition 9 (Agent Form)**.** *Given an Extensive Form Game*

$$\Gamma = (N, H, Z, A, \rho, \chi, \pi_0, \{f_n\}_{n \in N}, \{\mathfrak{I}_n\}_{n \in N}),$$

its Agent Form *Representation is the triple $\Gamma_a = (\hat{N}, \hat{A}, \hat{F})$ where:*

- *$\hat{N} = N \times I$ is the set of Agents such that $|\hat{N}| = \sum_{n \in N} |\mathfrak{I}_n|$. Each agent is labelled with the pattern n.h (player label . information set label).*
- *$\hat{A} = \{A_{1.1}, \ldots, A_{n.|H_n|}\}$ is the set of the Actions of the agents where $A_{n.h} = \{a \in A_n : a \in h, h \in H_n\}$.*
- *$\hat{F} = \{F_{1.1}, \ldots, F_{n.|H_n|}\}$ is the set of utility functions of the agents.*

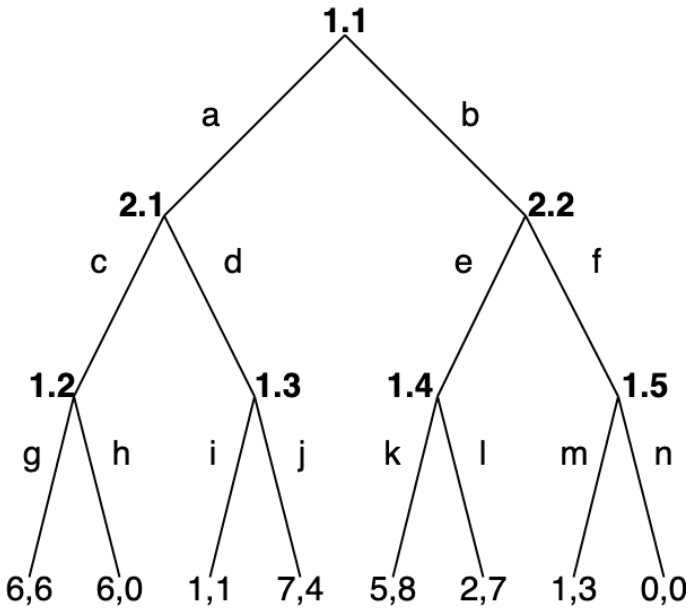

Figure 2.2: Example of a Game in Agent Form

Figure 2.2 shows an example of a two-player game in *Agent Form* where the first player's information sets are divided into 5 Agents and the second player's information sets into 2 Agents.

Consider again the games in extensive form with perfect recall defined above. It is possible to obtain an equivalent representation form starting from the concept of *Sequence*. A sequence is an ordered set of moves of a player that describes a part of the path between the root node and one of the decision nodes. Formally:

Definition 10. *Given a node w in which it is the player n's turn to move, a sequence q is an ordered set of actions of player n in the path from the root node to w. We say that action $a \in q$ iff a is somewhere in q. The set of all sequences is denoted as $Q = \{Q_1, \ldots, Q_m\}$, where Q_n is the set of player n's sequences.*

A sequence that, together with some sequences of the other players, leads to a terminal node is called *Terminal Sequence*. The sequence containing only the root node's information is denoted with $q_\emptyset$. A *Sequence Profile* is a tuple containing one sequence per player. Hence, it applies the following definition:

Definition 11 (Sequence Form)**.** *Given an Extensive Form Game*

$$\Gamma = (N, H, Z, A, \rho, \chi, \pi_0, \{f_i\}_{i \in N}, \{\mathfrak{I}_i\}_{i \in N}),$$

its Sequence Form *Representation is the tuple* $\Gamma_s = (N, Q, \{f_n\}_{n \in N})$ *where:*

- N *is the set of Players*
- $Q = \bigcup_{n \in N} Q_n$ *is the set of sequence profiles, which are collection of sequences, one per player*
- *For each player* $n \in N \setminus \{0\}$*, the function* $f_n : \bigtimes_{n \in N} Q_n \to \mathbb{R}$ *specifies the utility of player* n *only if the argument is leading to a terminal node.*

We now define the equivalent of the strategy profile for the sequence form:

Definition 12 (Realization Plan)**.** *A* Realization Plan (sequence form strategy) *is a probability function* $r_n : Q_n \to [0, 1]$ *with the constraint that* $r_n(q_\emptyset) = 1$ *and that in each information set* $I \in \mathfrak{I}_n$ *and sequence* q *leading to* I*, it holds that* $r_n(q) = \sum_{a \in \rho(I)} r_n(q|a)$*, where* $q|a$ *is the sequence obtained by choosing* a *in* I*, only if* q *is non-terminal.*

Finally, from this last definition, it follows that:

Definition 13 (Sequence Form Strategy Profile)**.** *Given a Game* Γ_S *in sequence form, a* Sequence Form Strategy Profile r *is a tuple* $r = (r_0, r_1, \ldots, r_m)$ *containing one strategy per player. The Strategy profile* r_{-n} *is used to denote a tuple with one strategy per player, except for player* n*.*

We now see an example of how to derive the sequence form, characterized in matrix form, starting from the extensive form and using the constraints imposed by Definition 12 of Realization Plan.

Example 3 (Game in Sequence Form)**.** *In figure 2.3 a game in sequence form is shown. Given this representation, we report the equations that defines the constraints of the realization plan* r_n *for each of the two players* ($N = \{1, 2\}$)*:*

$$\begin{aligned} &r_1(q_\emptyset) = 1, \quad r_1(q_\emptyset) = r_1(a) + r_1(b), \qquad r_1(b) = r_1(b|c) + r_1(b|d) \\ &r_2(q_\emptyset) = 1, \quad r_2(q_\emptyset) = r_2(A) + r_2(B) \end{aligned}$$

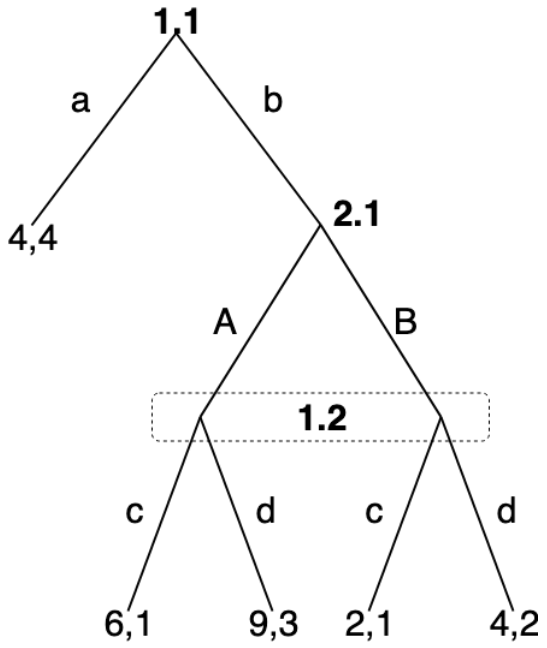

Figure 2.3: Sequence Form Game of Example 3

Those constraints are better represented in matrix form by the equation $F_n r_n = f_n$*, where:*

$$r_1 = \begin{bmatrix} r_1(q_\emptyset) \\ r_1(a) \\ r_1(b) \\ r_1(b|c) \\ r_1(b|d) \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}, \qquad f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$r_2 = \begin{bmatrix} r_2(q_\emptyset) \\ r_2(A) \\ r_2(B) \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \qquad f_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The matrix definition just presented will be more appropriately discussed in 2.4.1.

Since for the continuation of this work it will often be necessary to compare various representation forms for the same game, we provide below a series of tools for this comparison, paying particular attention to the comparison with the sequence form, which will be central in the subsequent formulations because of the facilitations that are obtained from the computational point of view. First of all, the concept of Realization Equivalence:

Definition 14 (Realization Equivalence)**.** *Given an extensive form game, two strategies for a certain player n, even defined on different representations, are* Realization Equivalent *if, for every strategy of the opponents ($N \setminus \{n\}$), they induce the same probability distribution over the outcomes of the game.*

On the basis of this definition we describe how to derive the realization equivalent representation in sequence form starting from the others previously described:

Normal Form → Sequence Form Given a game and a player n, a sequence form strategy r_n and a normal form strategy σ_n are *realization equivalent* if and only if the following property holds:

$$\forall q \in Q_n, \quad r_n(q) = \sum_{s \in S_n : action(q) \in s} \sigma_n(s).$$

Where $action(q) \in s$ means that all the actions composing sequence q are also in s with non-zero probability. The operation of deriving the realization plan of a generic sequence starting from the normal form game turns out to be extremely simple, this is due to the fact that each normal form game is associated to only one sequence form, while the opposite is not true. In any case, conversion operations starting from the representation in sequence form will not be particularly useful in the following and therefore are not described in this work.

Extensive Form → Sequence Form Given a game in extensive form Γ and a player n, a sequence form strategy r_n and a behavioral strategy π_n are *realization equivalent* if and only if the following property holds:

$$\forall q \in Q_n, \quad r_n(q) = \prod_{a \in A_n : a \in q} \pi_n(a).$$

In other words, the realization plan of a sequence is the product of the probabilities with which the individual actions that makes the sequence are played. This fact becomes particularly relevant when dealing with terminal sequences. It should be noted that the probability with which a certain strategy profile $s = (s_1, \ldots, s_m)$ is played corresponds, for the definition of realization equivalece, to the product of the individual realization plans of the individual players of the terminal sequences associated with that strategy profile. Let q_n^T be one of the terminal sequences associated to s_n for each player $n \in N$ then with an abuse of notation, holds that:

$$\pi(s) = \prod_{n \in N} r_n(q_n^T).$$

2.2 Nash Equilibria And Refinements

After presenting the concept of Game and how it is represented, it is now necessary to introduce the concept of a solution to a game. The followings are some of the most important results in Game Theory.

2.2.1 Nash Equilibrium

One of the most important results of game theory is due to John F. Nash, published in (Nash, 1951), and is in his honor called *Nash Equilibrium*. Nash Equilibrium (NE) discourages each player's deviation from the equilibrium strategy, given that other players play their Equilibrium Strategies.

First of all, let's define one of the main concepts of Game Theory:

Definition 15 (Best Response)**.** *Consider a generic game $G = (N, S, F)$ in normal form with m players. Having fixed the strategies of players in $N \setminus \{n\}$, it is possible to define for the remaining player n the* Best Reply *function $BR : \bigtimes_{i \in N \setminus \{n\}} S_i \to S_n$ that returns the strategy $\hat{s}_n \in S_n$ that maximizes n's outcome. In other words:*

$$BR(s_{-n}) = \max_{s_n} f_n(s_{-n}, s_n).$$

The *Best Response* function is therefore the function that is used by the players to select the most profitable strategy. A fundamental situation is reached when all the players play their best response strategy at the same time:

Definition 16 (Nash Equilibrium (Best Response))**.** *Given a Game $G = (N, S, F)$ in Normal Form, the strategy profile $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_m)$ is a* Nash Equilibrium *iff it holds that:*

$$\forall n \in N, \quad \hat{s}_n = BR(\hat{s}_{-n}).$$

Thus, there is a *Nash Equilibrium* in situations where the strategy of each player is best response to the strategies of the others. This situation can also be described without the Best Response concept.

Definition 17 (Nash Equilibrium in Normal Form Games)**.** *Given a NF Game $G = (N, S, F)$ defined as in 2.1.2, a Strategy Profile $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_m)$ is a* Nash Equilibrium *iff it holds that*

$$\forall n \in N, \forall s_n \in S_n, \quad f_n(\hat{s}) \geq f_n((\hat{s}_1, \ldots, s_n, \ldots, \hat{s}_m)).$$

As it can be easily seen in the definition above, a Strategy Profile $\hat{s}$ is a Nash Equilibrium for a game in NF if every player, having fixed the strategy of the others, has convenience in choosing the strategy contained in $\hat{s}$.

Things in the case of Games in Extensive Form (EF), in Agent Form (AF) and in Sequence Form (SF) are slightly different due to the different nature of this representations. We must now consider the tree structure and the fact that the strategy of each player is made up of different actions carried out in successive steps. Formally:

Definition 18 (Nash Equilibrium in Extensive Form Games)**.** *A Behavioral Strategy Profile* $\hat{\pi} \in \Pi$ *is a* Nash Equilibrium of an Extensive-Form Game $\Gamma = (N, H, Z, A, \rho, \chi, \pi_0, \{f_n\}_{n\in N}, \{\mathfrak{I}_n\}_{n\in N})$ *iff it holds that:*

$$\forall n \in N \setminus \{0\}, \forall \pi_n \in \Pi_n, \quad f_n(\hat{\pi}) \geq f_n(\hat{\pi}/\pi_n).$$

where $\hat{\pi}/\pi_n$ *is the strategy profile in which all the players but* n *plays according to* $\hat{\pi}$ *while player* n *plays* π_n*.*

2.2.2 Perfect Equilibrium

The main result of refining Nash equilibrium, published in (Selten, 1975), will be illustrated below. Given the Game Γ of Figure 2.4 in extensive form, there cannot be any mistakes if the players are absolutely rational. Nevertheless, a satisfactory interpretation of equilibrium points in extensive-form games seems to require that the possibility of mistakes is not completely excluded. This can be achieved by a point of view which looks at complete rationality as a limiting case of incomplete rationality. Consider the game of Figure 2.4 corresponding to the extensive form of the Game of Candies of Example 2, there are two Nash equilibria $\pi_1 = (b, c), \pi_2 = (b, d)$. If Player 1 does not make any mistake both of those strategy profiles are equally profitable, but what happens if, with a tending to zero probability ε, Player 1 chooses the action a? In such a case, Player 2 would certainly select the action c, since it provides a better outcome. Thus, π_1 is somehow a better result when playing the game Γ. To generalize that result, we must consider the possibility in which an error can be committed in any information set of the game by any player by imposing that each action is played with non-zero probability. Such a Game is defined in the following way:

Definition 19 (Selten's Perturbed Game)**.** *Given a Game* Γ *in extensive form, a* Perturbed Game *is a pair* $(\Gamma, l(\varepsilon))$ *where* $l(\varepsilon) : A \to (0, 1]$ *is a* Lower

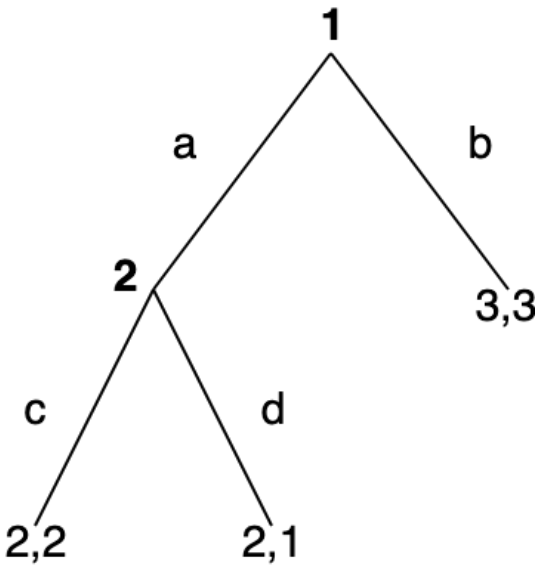

Figure 2.4: Example of Extensive Form Game

Bound Function *defining a positive lower bound on the probability of playing each action, such that:*

$$\forall n \in N, \forall I \in \mathfrak{I}_n, \quad \sum_{a \in A_I} l(\varepsilon, a) < 1.$$

The *Perturbed Game* thus obtained is the extensive form game with perfect recall that has the same structure of Γ in which each action $a \in A_I$ for all $I \in \mathfrak{I}$ is played with a probability $p(a) \geq l(\varepsilon, a)$. Such a game allows us to analyze all the possible errors that may occur during the course of the game, albeit with minimal probability. Figure 2.5 shows the Game of Figure 2.4 perturbed with

$$l(\varepsilon) = \varepsilon, \quad \forall a \in A.$$

Starting from this formulation it is therefore necessary to define the following result:

Definition 20 (Extensive Form Perfect Equilibrium)**.** *Given a game* Γ *in extensive form and the corresponding perturbed game* $(\Gamma, l(\varepsilon))$*, perturbed with the lower bound* $l(\varepsilon)$*, the Strategy Profile* $\hat{\pi}$ *is an* Extensive Form Perfect Equilibrium *iff it is a limit point of a sequence of Nash Equilibria of* $(\Gamma, l(\varepsilon))$ *when* $\varepsilon \to 0$*.*

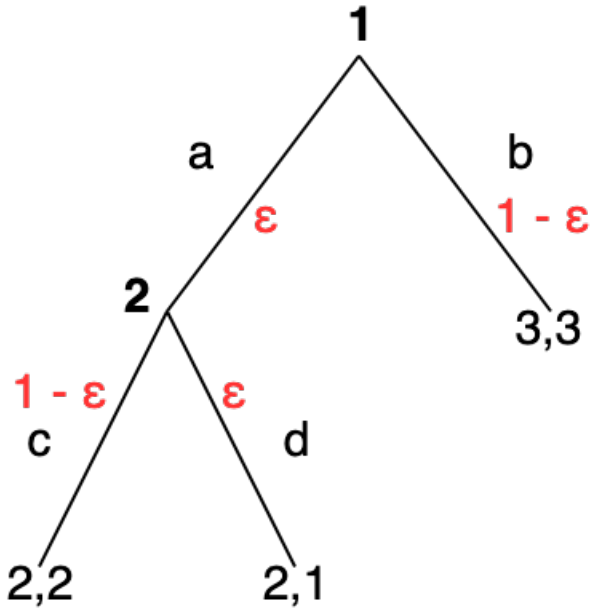

Figure 2.5: Perturbed Game of Figure 2.4 with $l(\varepsilon) = \varepsilon$

In other words, among all the Nash equilibria of an extensive-form game, *Perfect Equilibria (PE)* are those in which in each information set the player chooses the move that provides the greatest expected outcome. It is also easy to see that, since the Perfect Equilibrium of a Game is a Nash Equilibrium of its perturbed game, every game has at least one perfect equilibrium, i.e., it is always possible to find a strategy profile that selects the most profitable move at every information set for all the players.

2.2.3 Quasi-Perfect Equilibrium

In the literature, many refinements of the concept of Nash equilibrium have been proposed. Most of them are based on the idea of introducing trembles, which account for the possibility that players may make mistakes with low but non zero probability, playing actions that they would never play in an equilibrium. In an extensive-form perfect equilibrium, introduced by Selten (see 2.2.2), each player takes into account the possibility that both herself and her opponents may tremble in future. On the other hand, in a *quasi-perfect equilibrium*, introduced by Van Damme (van Damme, 1984), players only consider the possibility of opponents' trembles.

We are now ready to formally define quasi-perfect equilibria, following the approach of van Damme. First, we introduce the idea of best responses at information sets, which is crucial for the definition of quasi-perfect equilibria.

Definition 21 (I_n-Best Response)**.** *Given an extensive-form game Γ, a completely mixed behavioral strategy profile $\pi \in \Pi$, a player $n \in N \setminus \{0\}$ and an information set $I_n \in \mathfrak{I}_n$, a behavioral strategy $\pi_n \in \Pi_n$ is an* I_n-Best Response *to $\pi \in \Pi$, written $\pi_n \in BR_n(\pi)$, if the following equation holds:*

$$\max_{\hat{\pi}_n \in \Pi : \pi_n = \hat{\pi}_n} f_n(\pi /_{I_n} \hat{\pi}_n | I_n) = \max_{\hat{\pi}_n \in \Pi_n} f_n(\pi /_{I_n} \hat{\pi}_n | I_n).$$

Intuitively, π_n is an *I_n-Best Response ($BR_{h_n}(\pi)$)* to π whenever playing as prescribed by π_n at information set I_n is part of some player n's optimal strategy in the game following I_n, given that I_n has been reached and the other players behave as prescribed by π. In other words, π_n is a best response to π whenever player n ends up playing at information set I_n. The formal definition of quasi-perfect equilibrium follows:

Definition 22 (Quasi-Perfect Equilibrium)**.** *Given an extensive-form game Γ, $\pi \in \Pi$ is a* Quasi-Perfect Equilibrium *of Γ if there exists a sequence $\{\hat{\pi}(\varepsilon_k)\}_{k \in N}$ of completely mixed behavioral strategy profiles such that:*

- *π is a limit point of $\{\hat{\pi}(\varepsilon_k)\}_{k \in N}$ for $\varepsilon_k \to 0$.*
- *for every player $n \in N \setminus \{0\}$ and information set $I_n \in \mathfrak{I}_n, \pi_n \in BR_{I_n}(\hat{\pi}(\varepsilon_k))$ holds eventually (i.e. for sufficiently large $k \in \mathbb{N}$).*

In words, provided that $\varepsilon \in \mathbb{R}^+$ is sufficiently small, for every player n and information set I_n, π_n prescribes player n to play an optimal strategy against $\hat{\pi}(\varepsilon)$, given that I_n has been reached and she will play optimally in the rest of the game (assuming opponents follow strategies in $\hat{\pi}(\varepsilon)$). As it is clear, this condition encodes the fact that a player only considers the possibility of opponents' future trembles, while she does not care about hers.

2.3 Correlated Equilibria And Refinements

A further class of equilibria introduced in (Aumann, 1974), central in the context of this work, is discussed below, which extends the potentialities of the Nash Equilibrium by adding the figure of an impartial trustworthy mediator who makes recommendations according to a known probability distribution. This avoids that the randomization in the selection of players' strategy is uncoordinated and leads to undesiderable results.

2.3.1 Correlated Equilibrium

Consider the Chicken Game of Example 1, represented in Normal Form:

$$G = \begin{pmatrix} (6,6) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

As we have already seen in the previous sections, there are two Nash equilibria in pure strategies $(2,7), (7,2)$ and one in mixed strategies (consisting in playing the first row and the first column with probability $p = \frac{2}{3}$). It can be easily observed that even if players play the equilibrium in mixed strategies the maximum payoff expected would be equal to $P_E = \frac{14}{3}$. We now ask ourselves if and how it is possible to increase this expected payoff. If, for example, the outcomes $(6,6), (7,2), (2,7)$ could be played with a probability $p = \frac{1}{3}$, an expected payoff $P = 5$ would be obtained, higher than the one obtained by playing the equilibrium mixed strategy. However, given that the probability distribution described above is not marginalizable (i.e. probability distributions can not be obtained from the product of individual strategy profiles), it is necessary to introduce new elements in the way in which the game described above is played.

Consider now a game in which players, before starting to play, agree on the selection of an external mediator, who randomly select one of the player's profitable pure strategy profiles and privately communicate to each player a pure strategy to follow. During the game, if none of the players has convenience to deviate from what is suggested, a new type of Equilibrium is achieved. Such a mediator is defined in the following way:

Definition 23 (Correlation Device)**.** *Given a Game in Normal Form $G = (N, S, F)$, $M = \bigtimes_{n \in N} S_n$ is the set containing all the joint pure strategy profiles. A* Correlation Device *is a Mechanism based on a probability distribution over M that randomly selects one of the pure strategy profiles and privately communicates to each player $n \in N$ the chosen corresponding strategy $\hat{s}_n \in S_n$.*

A *Correlation Device(CD)* is therefore based on a probability distribution on the cells of an m-dimensional matrix. In the case of the Chicken Game of Example 1, the following matrices are examples of Correlation Devices:

$$\mu_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}, \mu_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

In the following, with an abuse of notation, the symbol μ will be used to indicate a correlation device rather than the probability distribution it is

based on.

First of all, let us define the game played after the recommendation:

Definition 24 (Extended Normal Form Game)**.** *Given a Game $G = (N, S, F)$ in normal form and a correlation device μ on which the players have previously agreed, the Game G_μ in which the players are first informed about the recommendations drawn according to μ and then play G is called* Extended Normal Form Game *of G via μ.*

At this point it is possible to define this further class of equilibria as follows:

Definition 25 (Correlated Equilibrium)**.** *Given a game $G = (N, S, F)$ in Normal Form and a correlation device μ on which all the players have previously agreed, consider the corresponding extended game G_μ described in Definition 24. The* Correlated Equilibria *of G are all the Strategy Profiles recommended with a non zero probability from which no player has incentive to deviate in the course of the extended Game.*

Each *Correlated Equilibrium(CE)* is therefore associated with one or more correlation devices. The set of correlated Equilibria of a game is obtained by considering all the possible correlation devices. It should be noted that a correlation device is obtained upon acceptance by all the players. Therefore, having fixed a game, all the strategy profiles recommended with a strictly positive probability from at least one correlation device are Correlated Equilibria. In Example 1, since

$$\hat{\mu} = \begin{pmatrix} \frac{4}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

is a correlation device, all the strategy profiles are Correlated Equilibria.

2.3.2 Extensive Form Correlated Equilibrium

The definition of the Correlated Equilibrium is considerably more complicated in the case of games in extensive form and it is due to (von Stengel and Forges, 2008). First of all note that the following definitions refer to games in extensive form with perfect recall and without chance nodes. Given a game Γ in extensive form with the above characteristics, we denote with I_n the set containing the information sets of player $n \in N$ and with A_{I_n} the set of available moves for the player n in the information set $I \in \mathfrak{I}_n$. A

sequence of moves of a Player is a sequence along the path from the root to some node in the game tree. The Set of Pure Strategies for a player n is

$$\Sigma_n = \bigtimes_{I \in \mathfrak{I}_n} A_{I_n}$$

and the set of all pure Strategy Profile is therefore

$$\Sigma = \bigtimes_{n \in N} \Sigma_n.$$

It is now possible to redefine the concept of Correlation Device, in a form compatible with games in extensive form that does not conflict with the definition given above.

Definition 26 (Correlation Device for Extensive Form Games)**.** *Given a Game in Extensive Form* Γ*, a* Correlation Device *is a mechanism based on a probability distribution* μ *over* Σ*. This device makes private recommendations to the players by picking a strategy profile* $s = (s_1, \ldots, s_m)$ *according to* μ*. Then, whenever a Player* n *reaches an information set* $I \in \mathfrak{I}_n$*, he receives the move* a *at* I *specified in* s_n *as a signal, interpreted as a recommendation to play* a*.*

After defining a correlation device, we now define a support structure starting from the concepts of correlation device and game in extensive form. Its purpose is to integrate the variations introduced by the correlation within the Extensive Form structure.

Definition 27 (Extended Game)**.** *Given an Extensive Form Game* Γ *and a Correlation Device* μ*, the* Extended Game Γ_μ *of* Γ *is the Extensive-Form Game that has as its root a chance node that selects the move according to* μ *(with the meaning of giving a recommendation). From this point on the subtrees are all replicas of* Γ *played following the recommendations given above (depending on the recommendations given, some Information Sets might cross).*

Consider the Chicken Game of Example 1, Figure 2.6 and 2.7 shows respectively the corresponding extended game with the following μ_1 and μ_2 as a CD:

$$\mu_1 = \begin{pmatrix} \frac{4}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} \end{pmatrix}, \mu_2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}.$$

The extended tree thus obtained has obviously a very strong bond with the correlation device that has been chosen. A structure defined in this way consists in the grouping of the possible phases of the game and allows the

analysis of a correlated game, which by definition has elements external to the model of the game in extensive form, reducing it to the problem of finding a Nash equilibrium. It is now time to define the equivalent of the Correlated Equilibrium for games in extensive form:

Definition 28 (Extensive Form Correlated Equilibrium)**.** *Given an Extensive-Form Game* Γ *and a Correlation Device* μ*, an* Extensive Form Correlated Equilibrium *of* Γ *is an Extensive Form Nash Equilibrium of* Γ*'s Extended Game* Γ_μ *in which the players follow their recommendations.*

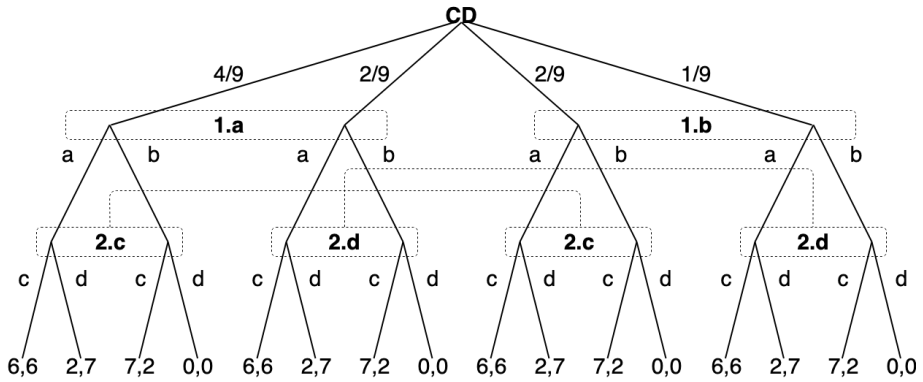

Figure 2.6: Extended Chicken Game with μ_1 *as a CD*

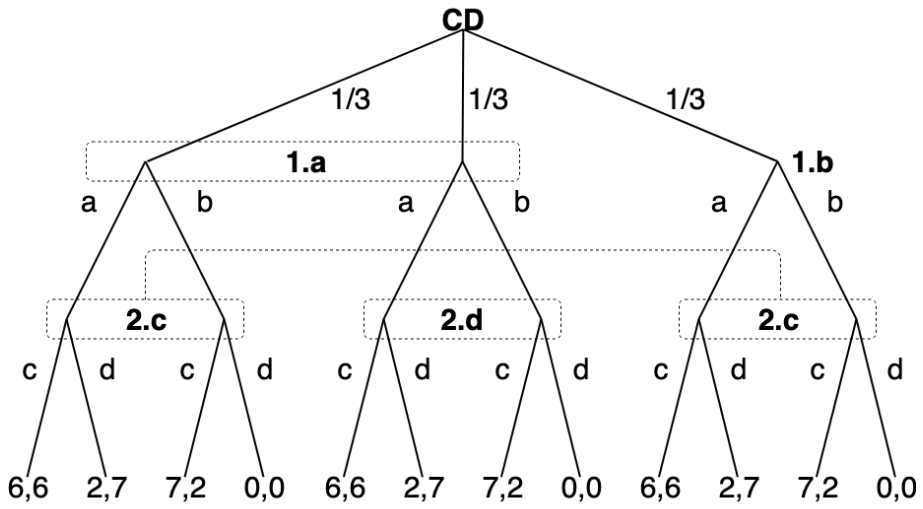

Figure 2.7: Extended Chicken Game with μ_2 *as a CD*

It is easy to see in Figure 2.6 that for the Chicken Game all the available

strategy profiles are *Extensive Form Correlated Equilibria(EFCE)*. Note that if the same correlation device would be used in a normal form context, the same result could be obtained, but this relation does not hold for all games. Hence, having fixed a game, the sets of CE and EFCE are generally different.

2.3.3 Perfect Correlated Equilibrium

We study the analogue of perfection for Normal Form Correlated equilibria defined in (Dhillon and Mertens, 1996). This time the refinement does not involve the modification of the correlation device, but is based, following the model of Selten, on the perturbation of the normal form game. It follows that the next definition is obtained simply by introducing the concept of perfection in place of the simple nash equilibrium. Hence:

Definition 29 (Normal Form Perfect Correlated Equilibrium)**.** *Given a game $G = (N, S, F)$ in Normal Form and a correlation device μ on which all the players have previously agreed, consider the corresponding extended game G_μ in which the players are informed about the recommendation from the correlation device before playing. Given also a lower bound function, let us call $(G_\mu, l(\varepsilon))$ the perturbed game based on G_μ in which each strategy profile s is played with probability $p(s) \geq l(\varepsilon, s)$. The* Perfect Correlated Equilibria *of G are all the Strategy Profiles recommended with a non zero probability from which no player has incentive to deviate in the course of the Game $(G_\mu, l(\varepsilon))$.*

We note in the definition above that, unlike what is done in 2.2.2, the lower bound of probability is assigned to the entire strategy profile directly starting from the normal form representation and not to the individual actions of the extensive form. However, this difference does not change the nature of the definition, since the imposition of a lower bound of probability on an entire strategy profile of a game in (reduced) normal form G necessarily implies that, calling Γ any game in extensive form having G as a reduced normal form representation, every action $\hat{a} \in A$ on the path $\hat{\pi}$ corresponding to the strategy profile $\hat{s}$ is still played with non-zero probability. This fact can be much more easily seen with the help of the sequence form representation and will be better discussed in 2.4.

2.4 Sequence Form and Linear Programming Characterization

In this section we will see how the characterization in sequence form of the above described equilibria assumes particular importance from the computational point of view, because by its nature the sequence form representation turns out to be the most compact method to represent a game in extensive form. We will then provide a description of how starting from this representation form it is possible to derive a (linear) mathematical formulation to solve the problem of the search for an equilibrium.

2.4.1 More on the Sequence Form

We now deal with the characterization of the results previously obtained from the sequence form point of view. The sequence form, defined in 2.1.3, represents a trivial solution to adapt the theory described above to the computation systems. In fact, unlike the extensive form, the sequence form can be represented in matrix form considering the flow constraints imposed by the definition of realization plan, thus the sequence form representation is of the same size as the game tree (von Stengel, 1996). In particular, for every player $n \in N \setminus \{0\}$, we can express player n's utility function as a multidimensional matrix $U_n \in \mathbb{R}^{|Q_0| \times \cdots \times |Q_m|}$ where each entry is indexed by a sequence profile $q = (q_0, \ldots, q_m) \in Q$ and it is equal to $u_n(q)$. Additionally, for $n \in N$, the constraints defining a realization plan $r_n \in R_n$ can be expressed as the matrix inequality $F_n r_n = f_n$ where F_n is a matrix with size $(|\mathfrak{I}_n| + 1) \times |Q_n|$, r_n is intended as a column vector of realization plan values and f_n is a column vector of $|\mathfrak{I}_n| + 1$ positions. A clear example of this matrix representation has been prematurely provided in 2.1.3, to expose the potential of the sequence form to the full.

2.4.2 Characterizing Nash Equilibria

Given a game Γ in sequence form as in Definition 11 and considering Definitions 12 and 13, the following is one of the possible formulations of the

problem of finding a Nash Equilibrium of Γ:

$$\underset{r=(r_0,\dots,r_m)}{\arg\max} \quad \sum_{n\in N}\left[\sum_{q\in Q}\prod_{n'\in N} r_{n'}(q_{n'})\right]U_n(q) \tag{2.1}$$

$$s.t. \quad \sum_{q\in Q}\left[\prod_{n\in N} r_n(q_n)\right]U_n(q) \geq \sum_{q\in Q}\hat{r}_n(q_n)\left[\prod_{n'\in N\setminus\{n\}} r_{n'}(q_{n'})\right]U_n(q)$$

$$\forall n \in N, \forall \hat{r}_n \in \hat{R}_n \tag{2.2}$$

$$F_n r_n = f_n \quad \forall n \in N \tag{2.3}$$

$$r_n \geq 0 \quad \forall n \in N \tag{2.4}$$

Where $\hat{R}_n$ is the set containing all the pure realization plans that are assigning either probability 1 or 0 to each sequence. The purpose of the problem, expressed in the objective function, is the search for the strategy profile that maximizes the overall expected utility of the players (social welfare utility). This strategy profile (characterized by the realization plan $\hat{r}$), in order to be a Nash equilibrium, must necessarily respect the first constraint, which is nothing other than the matrix translation of the constraints expressed in Definition 17. In practice it is required that, for any player $n \in N$, once the realization plans r_{-n} of the others are set, the one selected by player n will provide maximum utility. The second constraint instead is that imposed by the formulation in sequence form as illustrated above and defines the constraints of the realization plan of the various players. Finally, the last of the constraints imposes that no sequence is associated with a negative realization probability, an obvious but not otherwise codifiable feature.

We now deal, similarly to what has been done for the Nash equilibrium, with the characterization of its main refinement, the perfect equilibrium that has been described in 2.2.2. Before proceeding with this characterization, however, we need to adapt the concept of perturbation to the sequence form, which will be central in the following. We introduce the idea of trembles in the sequence form representation of extensive-form games. In order to do so, we define what it means to perturb (according to some perturbation schemes) a game in sequence form. This is strictly connected with the concept of trembles, i.e., the idea of forcing each player to take every action with low-but-non-zero probability. In particular, we introduce a specific type of perturbation scheme, which turns out to be also useful in the characterization of quasi perfect equilibria.

Definition 30 (Lower Bound Function)**.** *Given a game in sequence form Γ_S and a player $n \in N \setminus \{0\}$, let $l_n : \mathbb{R}^+ \times Q_n \to [0,1]$ be a function that*

maps any positive number $\varepsilon \in \mathbb{R}^+$ *and sequence* $q_n \in Q_n$ *to a lower-bound* $l_n(\varepsilon, q_n)$ *such that:*

- $l_n(\varepsilon, q_n)$ *is a polynomial in the variable* ε*, for every* $q_n \in Q_n$
- *there exists* $\hat{\varepsilon} \in \mathbb{R}^+$ *such that, for* $\varepsilon \in (0, \hat{\varepsilon}], l_n(\varepsilon, q_n) > 0$ *for every* $q_n \in Q_n$
- $\lim_{\varepsilon \to 0^+} l_n(\varepsilon, q_n) = 0$*, for every* $q_n \in Q_n \setminus \{q_\emptyset\}$*, while* $l_n(\varepsilon, q_\emptyset) = 1$
- $\lim_{\varepsilon \to 0^+} \frac{l_n(\varepsilon, q_n|a)}{l_n(\varepsilon, q_n)} = 0$*, for every* $q_n \in Q_n$ *and* $a \in A_n$ *such that* $q_n|a \in Q_n$*.*

We define an l_n*-quasi perfect* perturbation scheme for R_n as a function $\varepsilon \to R_n(\varepsilon)$ defined over $\varepsilon \in \mathbb{R}^+$ in which $R_n(\varepsilon)$ is the set of all the realization plans $r_n \in R_n$ such that $r_n(q_n) \geq l_n(\varepsilon, q_n)$ for every $q_n \in Q_n$. In words, an l_n-quasi perfect perturbation scheme constraints player n to play every sequence $q_n \in Q_n$ with probability grater than or equal to the lower bound $l(\varepsilon, q_n)$. Furthermore, the lower bounds enjoy particular properties:

- they can be expressed as polynomials in the variable ε
- they are positive for sufficiently small values of ε
- they approach zero as ε goes to zero
- for every sequence $q_n \in Q_n$ and action $a \in A_n$ such that $q_n|a \in Q_n$, $l_n(\varepsilon, q_n|a)$ approaches zero faster than $l_n(\varepsilon, q_n)$.

Now we introduce the concept of perturbed sequence form of a game:

Definition 31 (Perturbed Sequence Form)**.** *Given an extensive-form game* Γ*, we call perturbed sequence form of* Γ *the sequence form representation* Γ_S *together with an* l_n*-quasi perfect perturbation scheme for* R_n*, for every player* $n \in N \setminus \{0\}$*.*

In the following, the tuple $(\Gamma_S, \{l_n\}_{n \in N \setminus \{0\}})$ denotes a perturbed sequence form of Γ, defined for some l_n-quasiperfect perturbation schemes. Moreover, we let $\Gamma_S(\varepsilon)$ be a particular perturbed game instance, which is obtained from Γ_S by restricting each set of realization plans R_n to be $R_n(\varepsilon)$. In matrix notation, we can write the constraints defining $R_n(\varepsilon)$ as $r_n \geq l_n(\varepsilon)$ for all $n \in N \setminus \{0\}$, where, overloading notation, $l_n(\varepsilon)$ is a column vector that contains the valued functions of ε, defined over $\varepsilon \in \mathbb{R}^+$, whose components are the functions $l_n(\varepsilon, q_n)$ for $q_n \in Q_n$. In the following, we use $r(\varepsilon)$ to denote a valid realization plan profile for a perturbed game instance $\Gamma_S(\varepsilon)$,

i.e, $r_n(\varepsilon) \in R_n(\varepsilon)$ for all $n \in N \setminus \{0\}$.

Note that such a perturbation scheme is not the most general, since the sequence form imposes a lack of control on the tremble on each action composing a sequence and therefore is not possible to specify a perturbation for each action of each player in a general way. However, this is not an issue for our purpose since we are interested in a formulation of the problem of searching for some equilibria with the goal of finding one of them.

Hence, similarly to what has been done in (2.1) for the Nash equilibrium, the following is a possible formulation for the problem of searching for a Quasi-Perfect Equilibrium as defined in Definition 22:

$$\underset{r=(r_0,\dots,r_m)}{\arg\max} \quad \sum_{n\in N}\Big[\sum_{q\in Q}\prod_{n'\in N} r_{n'}(q_{n'})\Big]U_n(q) \tag{2.5}$$

$$s.t. \quad \sum_{q\in Q}\Big[\prod_{n\in N} r_n(q_n)\Big]U_n(q) \geq \sum_{q\in Q}\hat{r}_n(q_n)\Big[\prod_{n'\in N\setminus\{n\}} r_{n'}(q_{n'})\Big]U_n(q)$$

$$\forall n \in N, \forall \hat{r}_n \in \hat{R}_n \tag{2.6}$$

$$F_n r_n = f_n \quad \forall n \in N \tag{2.7}$$

$$r_n \geq l_n(\varepsilon) \quad \forall n \in N \tag{2.8}$$

Where the objective function and the first two constraints are exactly the same ones used in (2.1) when searching for a Nash equilibrium, while the third constraint is slightly different, since it is induced by the fact that each $r_n(q)$ must be greater than or equal to the lower bound $l_n(\varepsilon, q)$ assigned to the sequence q.

2.4.3 Characterizing Correlated Equilibria

Now consider the games we talked about in 2.3.2. Our goal is to build a characterization also for this type of games and to formulate the problem of searching for an extensive-form correlated equilibrium as presented in Definition 28 that allows the maximization of the expected utility of the players. Compared to what has been done with the Nash equilibrium, there is now the problem of formulating constraints which must be compatible with the presence of an element that is external to the development of the game such as a correlation device as in Definition 26. We have already seen in Definition 27 how in extensive form correlated equilibria this problem was overcome through the construction of an extended game that incorporates the beginning phase in which the selection of strategies to be recommended

occurs within a wider game. Starting from this definition we will see that it is therefore possible to derive incentive constraints for the definition of the problem. This type of analysis has already been formulated in (von Stengel, 1996) for the two-player case only and then in (von Stengel and Forges, 2008) for any number of players. When dealing with games with more than two players, or with chance moves, the consistency constraints of the sequences on the marginal probabilities of moves that are correlated across information sets are only necessary conditions. For this reason, the characterization given in (von Stengel and Forges, 2008) for games with more than two players does not use the sequence form. Note that this is not an issue, since, as we will see, such a formulation of the problem of finding an extensive form correlated equilibrium will result in a linear programming problem even without the sequence form representation. However, since we are interested in analyzing the properties of the equilibrium just defined, we will examine the two player case using the sequence form, keeping in mind that the parallelism that exists beetween the two representation forms makes the following properties adjustable basing on the context.

Let us start with some definitions that will be important in the following. First of all, it is necessary to have references regarding the positioning of the nodes and of the information sets of the game in extensive form, hence:

Definition 32 (Precedence Relation)**.** *An information set $I_1 \in \mathfrak{I}_{n_1}$* precedes *another information set $I_2 \in \mathfrak{I}_{n_2}$ if and only if $n_1 = n_2 = n$ and there are nodes $k \in I_1$ and $v \in I_2$ such that k is on the path from the root to v. Furthermore, $I_1 \in \mathfrak{I}_n$ immediately precedes $I_2 \in \mathfrak{I}_n$ when I_1 precedes I_2 and there is no information set $I_x \in \mathfrak{I}_n$ that succeeds I_1 and precedes I_2.*

For notational convenience the *precedes* relation will be written in the following as $pre(a, b)$ meaning that *a precedes b*. The *Preceedes* relation is therefore defined only between two information sets of the same player. Without further definitions, the inverse relation, called *succeeds* beetween I_1 and I_2 has the following meaning: I_1 *succeeds* I_2 ($suc(I_1, I_2)$) iff

$$I_1 \neq I_2 \quad AND \quad pre(I_2, I_1) = False.$$

Definition 33 (Agrees With Relation)**.** *A move $a \in A_I$ agrees with a strategy profile s if and only if the information set I is reached and the move a is chosen when players play according to s.*

For notational convenience the just defined relation will be written in the following as $agr(a, s)$ meaning that *a agrees with s*.

Definition 34 (Terminates Relation)**.** *A move $a \in A_I$ of player n* terminates *a strategy profile s_{-n} if and only if I is reached and no further information set of player n is reached if n plays the moves leading to I at all preceding information sets and plays a at I, and the other players play s_{-n}.*

For notational convenience the *terminates* relation will be written in the following as $ter(a, s)$ meaning that *a terminates s*.

Definition 35 (Reachability Relation)**.** *An information set $I \in \mathfrak{I}_n$ is* reachable *by a strategy profile $s \in S$ if and only if the player n reaches I at a certain stage if all players choose the strategies in s.*

For notational convenience the just defined relation will be written in the following as $rea(I, s)$ meaning that I is *reachable* by s.

We are now ready to define the constraints of the problem. First of all it is necessary, at each information set $I \in \mathfrak{I}_n$, to have a measure of how a move $a \in I$ that is going to be selected can contribute to the player n's utility. Such a function $u : \mathfrak{I} \to \mathbb{R}$ is defined in the following fashion:

$$u(a) = \sum_{s \in S : agr(a,s)} f_n(s) z(s). \tag{2.9}$$

Where $f_n(s)$ is the expected payoff of player n if all the players choose the strategy s and $z(s) : S \to [0, 1]$ is the probability according to which s is recommended by the Correlation Device. Note that, let q be the sequence profile that corresponds to s in extensive form and $r(q) = \prod_{n \in N} r_n(q_n)$, it holds that $r(q) = z(s)$. Basing on the evaluation of $u(a)$ for every a in the information set, the player decides whether to follow the recommendation or to deviate. Given a move $a \in I$ and an information set $\hat{I} \in \mathfrak{I}_n$ such that $\hat{I} = I$ or $suc(\hat{I}, I)$, we use $v(\hat{I}, a)$ to denote the optimal expected payoff at $\hat{I}$ given that the player's recommended move is a at I. Consider the following inequality:

$$v(\hat{I}, a) \geq \sum_{s \in S : agr(a,s), ter(d, s_{-n})} f_n(s_d^{\hat{I}}) z(s) + \sum_{\widetilde{I} \in \mathfrak{I}_n : pre(\hat{I}, \widetilde{I}) via\ d} v(\widetilde{I}, a), \quad d \in A_{\hat{I}}. \tag{2.10}$$

Where $s_d^{\hat{I}}$ is the strategy profile that specifies moves leading to $\hat{I}$ at information sets preceding I, and d at $\hat{I}$, and the same moves as s at all other information sets. Finally, we must impose that, given a recommended move a, the player does not gain anything by deviating from this recommendation:

$$u(a) = v(I, a), \quad \forall a \in A_I. \tag{2.11}$$

In (von Stengel and Forges, 2008) it is stated that in a perfect recall extensive-form game, a probability distribution z that fulfills the incentive constraints (2.9),(2.10) and (2.11) for all players defines an EFCE. The number of constraints that describe the set of EFCE is polynomial in the size of the game tree.

We now consider the constraints set out above and derive their correspondent in sequence form to best fit in the case description of two players sequence form games. In this form of representation some of the properties that we have used above in defining constraints are intrinsic and therefore the reformulation should be easier to exploit. Before formalizing the constraints in sequence form, we consider the following definitions:

Definition 36 (Connected Sequences)**.** *Given an extensive-form game* Γ *and* $h_1, h_2 \in H$*, if* h_1 *is on the path from the root of* Γ *to* h_2 *and* $h_1 \in I_1, h_2 \in I_2$*, we say that* I_1 *and* I_2 *are* Connected *(i.e.,sharing a path).*

Definition 37 (Relevant Sequences)**.** *Given an extensive-form game* Γ_S *in sequence form and two sequences of different players* $q_1 \in Q_{n_1}$ *and* $q_2 \in Q_{n_2}$*,* (q_1, q_2) *is a* Relevant Pair *if and only if the followings hold:*

- $q_1 = q_\emptyset$ *or* $q_2 = q_\emptyset$
- $q_1 = q_I|a$ *and* $q_1 = q_J|b$*,* $I \in \mathfrak{I}_1, a \in \rho(I), J \in \mathfrak{I}_2, b \in \rho(J)$ *and* I, J *are connected.*

For notational convenience, in the following we will write $rel(q_1, q_2)$ if the pair (q_1, q_2) is relevant. We now note that in the formulation above $z(s)$ is used to denote the probability with which the strategy profile s is played. We need to define a corresponding formulation in sequence form. Such a formulation in (von Stengel, 1996) is called *Correlation Plan* and represents the equivalent of a realization plan for the sequence profiles.

Definition 38 (Correlation Plan)**.** *Consider a two player extensive-form game without chance moves and with perfect recall. A* Correlation Plan *is a partial function* $z : Q_1 \times Q_2 \to \mathbb{R}$ *so that there exists a probability distribution* $\hat{\mu}$ *on the set of reduced strategy profiles* Σ^* *(corresponding to a probability distribution* μ *over* $Q_1 \times Q_2$*) such that, for each relevant sequence pair* (q_1, q_2)*,* $z(q_1, q_2)$ *is defined and it holds that*

$$z(q_1, q_2) = \sum_{(s_1,s_2)\in\Sigma^*,(s_1,s_2)\ agrees\ with\ (q_1,q_2)} \mu(q_1, q_2).$$

Where (s_1, s_2) *agrees with* (q_1, q_2) *if* s_1 *chooses all the moves in* q_1 *and* s_2 *all the moves in* q_2*.*

If we consider the correlation plans of the two players r_1 and r_2, it turns out that $z(q_1, q_2) = r_1(q_1)r_2(q_2)$. Once s is defined, we can use it to generate, as a random variable (in place of μ), a pair of reduced pure strategies to be recommended to the players. The moves in that reduced strategy pair are generated inductively starting from the root of the tree.

It is now time to formulate the above constraints in sequence form to derive the formulation of the problem of searching for an Extensive Form Correlated Equilibrium. We start from the equivalent of the first constraint defined in (2.9), which expresses the expected utility of a player when playing a certain move at a certain information set. The corresponding sequence form formulation is the following:

$$u(q_n) = \sum_{q_{-n} \in Q_{-n}} z(q_n, q_{-n}) f_n(q_n, q_{-n}) + \sum_{I \in \mathfrak{I}_n : q_{n,I} = q_n} \sum_{a \in \rho(I)} u(q_{n,I}|a). \quad (2.12)$$

Where $q_{-n} \in Q_{-n}$ is obviously the sequence played by the other player. The first part of the sum defines the contribution when q_n and suitable sequences q_{-n} are defined by leaf. The second (double) sum considers the information sets reached when playing q_n and $u(q_{n,I}|a)$ is the expected utility obtained when playing the recommended move a at the following information set I, given the new posterior information obtained there ($q_{n,I}$ denotes the sequence of player n leading to the information set I). It is obvious that the following holds:

$$u(q_\emptyset, q_\emptyset) = \sum_{(q_1, q_2) \in Q_1 \times Q_2} z(q_1, q_2) f_n(q_1, q_2).$$

Consider now the second constraints in (2.10) that express for each information set $I \in \mathfrak{I}_n$ in which the sequence q_n is defined the maximum available payoff for the possible moves. This constraints is expressed in sequence form in the following way:

$$v(I, q_n) \geq \sum_{q_{-n} \in Q_{-n}} z(q_n, q_{-n}) f_n(q_{n,I}|a, q_{-n}) + \sum_{J \in \mathfrak{I} : q_n = q_{n,I}|a} v(J, q_n), \quad a \in A_I \quad (2.13)$$

The first sum is obviously well defined and concerns only the case in which I is the last information set before a leaf, while the second sum denotes the maximum expected payoff obtained at the following information set when following a recommendation. As shown in (2.11), the third constraint in sequence form is nearly the same:

$$v(I, q_{n,I}|a) = u(q_{n,I}|a) \quad (2.14)$$

which says that the recommended move a at information set is I is optimal. As an illustration of the incentive constraints, consider an information set I that precedes no further information sets of player $n \in N = \{1, 2\}$, then from (2.12),(2.13) and (2.14), it follows the following equation, which says that a player cannot gain by changing his move from a to $\hat{a}$ at I:

$$u(q_{n,I}|a) = \sum_{q_{-n} \in Q_{-n}} z(q_{n,I}|a, q_{-n}) f_n(q_{n,I}|a, q_{-n}) \geq \sum_{q_{-n} \in Q_{-n}} z(q_{n,I}|a, q_{-n}) f_n(q_{n,I}|\hat{a}, q_{-n}), \quad \hat{a} \in A_I.$$

Having identified the necessary constraints, we can now exploit them for the formulation of the problem of searching for an Extensive Form Correlated Equilibrium of a two player extensive-form game. The problem starts from (2.15) with the objective function, that regards the searching for a correlation plan z that maximizes the social welfare function, expressed as the sum of the expected utility of both the players when they follow the recommendations starting from the empty sequence (and exploiting the definition of u_n given above). The first pair of constraints is the direct application of what we saw in (2.12), where the payoff function f_n is replaced by the utility matrix U_n that we have defined above in 2.4.2 and which has the same meaning. The second pair of constraints is the direct transcription of what has been defined in (2.13) and presents, for each sequence and each information set that agrees with the sequence, the maximum payoff that can be obtained choosing one of the moves available at that information set. In such a formulation the information set $\hat{I}$ is the one that precedes I and obviously belongs to the player that is selecting a move. The fifth constraint of the problem corresponds to the one given in (2.14), and prescribes that, at each information set of each player, the recommended move is always optimal. With this last one, the set of the three constraints defined in (von Stengel, 1996) is completely considered. Differently from what has been done in the previous characterizations, in the formulation of this problem the concept of realization plan has not been used and therefore also the typical constraints of the sequence form are useless. We have introduced above the concept of Correlation Plan, which replaces that of realization plan and expands it, therefore the last constraints of the problem are dedicated to the formulation of the flow constraints according to Definition 38. Hence, the following is the definition of the problem:

$$\underset{z\in\mathbb{R}^{|Q_1|\times|Q_2|}}{\arg\max} \quad u_1(q_\emptyset) + u_2(q_\emptyset) \tag{2.15}$$

$$s.t. \quad u_1(q_1) = \sum_{q_2\in Q_2} z(q_1,q_2)U_1(q_1,q_2) + \sum_{I\in\mathfrak{I}_1:q_{1,I}=q_1} \sum_{a\in\rho(I)} u_1(q_{1,I}|a), \quad \forall q_1 \in Q_1 \tag{2.16}$$

$$u_2(q_2) = \sum_{q_1\in Q_1} z(q_1,q_2)U_2(q_1,q_2) + \sum_{I\in\mathfrak{I}_2:q_{2,I}=q_2} \sum_{a\in\rho(I)} u_2(q_{2,I}|a), \quad \forall q_2 \in Q_2 \tag{2.17}$$

$$v_1(I,q_1) \geq \sum_{q_2\in Q_2} z(q_1,q_2)U_1(q_{1,I}|a,q_2) + \sum_{J\in\mathfrak{I}_1:q_1=q_{1,I}|a} v_1(J,q_1),$$
$$\forall q_1 = q_{1,I}|a \in Q_1, \forall I \in \mathfrak{I}_1 : \hat{I} = I \vee pre(\hat{I},I), a \in \rho(I) \tag{2.18}$$

$$v_2(I,q_2) \geq \sum_{q_1\in Q_1} z(q_1,q_2)U_2(q_1,q_{2,I}|a) + \sum_{J\in\mathfrak{I}_2:q_2=q_{2,I}|a} v_2(J,q_2),$$
$$\forall q_2 = q_{2,I}|a \in Q_2, \forall I \in \mathfrak{I}_2 : \hat{I} = I \vee pre(\hat{I},I), a \in \rho(I) \tag{2.19}$$

$$v_n(I,q_{n,I}|a) = u_n(q_{n,I}|a), \quad \forall n \in \{1,2\}, \forall I \in \mathfrak{I}_n, \forall a \in \rho(I) \tag{2.20}$$

$$z(q_{1,I},q_2) = \sum_{a\in\rho(I)} z(q_{1,I}|a,q_2),$$
$$\forall I \in \mathfrak{I}_1, \forall q_2 \in Q_2 : rel(q_{1,I}|a,q_2), \forall a \in \rho(I) \tag{2.21}$$

$$z(q_1,q_{2,I}) = \sum_{a\in\rho(I)} z(q_1,q_{2,I}|a),$$
$$\forall I \in \mathfrak{I}_2, \forall q_1 \in Q_1 : rel(q_1,q_{2,I}|a), \forall a \in \rho(I) \tag{2.22}$$

$$z(q_\emptyset,q_\emptyset) = 1 \tag{2.23}$$

$$z(q_1,q_2) \geq 0, \quad \forall (q_1,q_2) \in Q_1 \times Q_2 \tag{2.24}$$

Chapter 3

Extensive Form Perfect Correlated Equilibria

This chapter is divided into two sections and contains the main result of this document, that is the definition of a refinement of the Correlated Equilibrium that makes it robust against errors. The very theme of the error is dealt with in the first section, which analyzes the various types of errors that may occur when playing a game. This type of analysis is subsequently used to formulate the definition of Extensive Form Perfect Correlated Equilibrium.

3.1 Perturbations Analysis

In this section we will analyze three different types of perturbation that may occur when playing an extensive-form Game Γ that sees the participation of a mediator (defined as a Correlation Device). The following are the phases of the game:

1. The players agree on the distribution μ to use as a correlation device. In this phase all the strategy profiles are selected from which no player should have the interest to deviate in case they are recommended by the correlation device and a subset of the found strategy profiles is taken in consideration, each element weighed with a probability. The probability distribution μ will assign probability greater than zero to the selected strategy profiles. Note that the individual strategies refer only to the information sets that are on the path associated to the strategy profile chosen by the correlation device. No recommendation is defined for all other information sets.

2. Once the correlation device has been selected, this mechanism is in-

structed to select a strategy profile to be recommended to the players according to the probability distribution μ. This strategy profile $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_m)$ is divided into the strategies $\hat{\pi}_n$ to be communicated to the individual players.

3. The individual strategies $\hat{\pi}_n$ obtained are communicated privately to each player. The reading of the signals communicated will take place as the game unfolds, so players only read the move related to the initial information set.

4. The game is played as a normal Extensive-Form Game with the distinction that at each information set on the recommended path the player who must make the move reads first the signal relative to the recommendation, evaluates whether to follow the recommendation and then acts. The evaluation is based on the quantification of the expected payoff associated to each of the available moves.

5. The game ends when one of the players makes a move from a terminal node. The players earn a payoff equal to the label associated with the chosen move.

After observing how a game with a mediator takes place, let's move on to the analysis of the three possible types of errors that may occur, generating a perturbation. The idea behind the definition of the refinement of the correlated equilibria that we are looking for follows perfectly what Selten has done. It is therefore expected that, once defined how to perturb the game described above and imposed a probability lower bound to every possible action (strategy profile), it is sufficient to bring the solution to its limit points to obtain the result we are looking for.

3.1.1 Perturbing the Correlation Device

Let us now look at the first two points of the game described above and consider the critical steps in which errors may occur. The class of errors that can be identified is that of *errors on the Correlation Device*. In detail there are two types of errors in this class:

Perturbed Correlation Device In the first phase in which the players agree on the correlation device, it is possible that, for an error of all the players, a probability distribution that is selected as the correlation device could recommend to a player $n \in N$ a strategy $\hat{\pi}_n$ in which there is a recommended move in a certain $I \in \mathfrak{I}_n$ from which the player n has

convenience to deviate. As a result of this erroneous recommendation, it would no longer be possible to achieve the correlated equilibrium as planned by the Correlation Device.

Wrong Strategy selection Once the players have agreed on the probability distribution to be used for the correlation device, the device proceeds to the random selection of the strategy profile to be recommended. This selection, for reasons not further definable, could be subject to perturbations that lead the correlation device to erroneously choose other strategies for the players. The strategies selected in this way could therefore identify different correlated equilibria from the one indicated by the correlation device.

Regarding the first type of error (*Perturbed Correlation Device*), it is necessary to consider that, in order to correctly model the situation, we need to respect the constraint imposed by the fact that the object that is being perturbed is a probability distribution. Therefore, the perturbation consists in the variation of the probability on $\hat{\mu}$ associated to a certain strategy profile (or to more than one), compensated by an opposite variation on another one (sum of variations on others), keeping in mind that any strategy profile must still be recommended with a probability not lower than the imposed lower bound. Given the correlation device $\hat{\mu}$ used in 2.3.1 referring to Example 1 ad using $l(\varepsilon) = \varepsilon$ as the lower bound function, the followings are examples of perturbed correlation devices:

$$\widetilde{\mu}_1 = \begin{pmatrix} \frac{1}{3} + \varepsilon & \frac{1}{3} - \varepsilon \\ \frac{1}{3} - \varepsilon & \varepsilon \end{pmatrix}, \widetilde{\mu}_2 = \begin{pmatrix} \varepsilon & \frac{1}{2} - \varepsilon \\ \frac{1}{2} - \varepsilon & \varepsilon \end{pmatrix}$$

The introduction of a lower bound probability associated with each strategy profile essentially allows the correlation device, based on the perturbed distribution, to recommend to the players, albeit with probability tending to zero, all those strategy profiles that had been excluded when they had agreed on the distribution due to the fact that one of them would have been interested in deviating from the equilibrium path if he had received such a recommendation. Figure 3.1 shows the extended game that is obtained starting from the chicken game and using $\widetilde{\mu}_1$ as a correlation device. Since the perturbed distribution described above does not satisfy the incentive constraints of the game, it is impossible to consider it for the search for correlated equilibria, we can therefore conclude that the type of error described above should not be taken into account for our modeling since it violates the structure rules of the game itself.

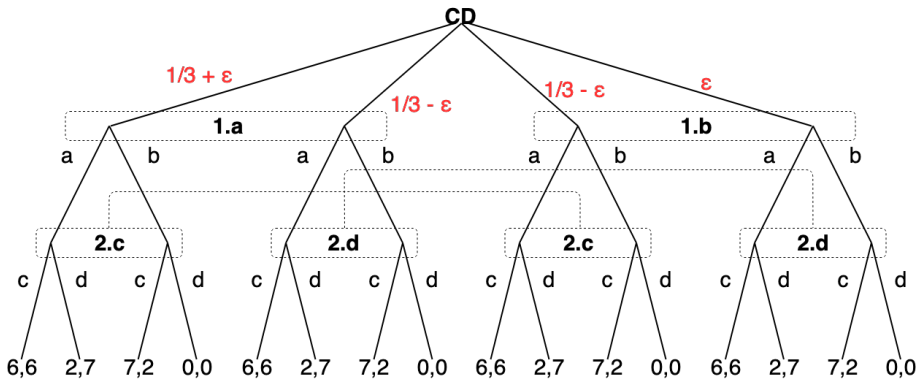

Figure 3.1: Extended Game obtained using $\widetilde{\mu}_1$ as a CD

As regards the second type of error (*Wrong Strategy Selection*), it should be specified that it is impossible to model it in terms of perturbations. We must also clarify that from the point of view of the players it is indifferent which of the correlated equilibria is reached, since in the initial phase everyone accepted the distribution, provided that the selection is made according to the distribution specified for the correlation device. This class of errors will not be further considered because it is not particularly interesting for the work that follows.

3.1.2 Perturbing the Observation

We now carefully observe the third point of the game as described at the beginning of this section. In this phase of the game the single strategies drawn by the correlation device according to its own distribution are transmitted individually to the players in form of signals. Even though the whole strategy $\hat{\pi}_n$ is transmitted before the game begins, a player n will interpret the signal relative to the recommended move to a certain information set $I \in \mathfrak{I}_n$ only after the information set I is reached while playing the game. The error that can occur in the context of this phase is called *Observation Error* and can occur while the signal is transmitted. If this signal is somehow corrupted during the transmission, it may happen that the strategy is misinterpreted. Since the player n is receiving a $\hat{\pi}_n$ strategy from the correlation device, in the worst case it can happen that, basing on the signal structure, the strategy $\widetilde{\pi}_n$ can be perceived by the player, which differs from the first in the generic information set I^* due to the fact that, given two moves $a_1, a_2 \in I^*$ with $a_1 \neq a_2$, it holds that $a_1 \in \hat{\pi}_n$ and $a_2 \in \widetilde{\pi}_n$. In other words the two strategies differ at least in an information set. This divergence between the

strategy developed by the correlation device, which matches with those of the other players, and the one perceived by the player could easily cause the exit from the equilibrium path, preventing the achievement of the outcome designated by the correlation device.

Let us now focus on how to model the error described above. Even though the signal is received at one time, since the recommendation to be followed is known by the player only once the interested information set is reached, the perturbation of the game consists in imposing a lower bound of probability to all actions in the information set on the recommended equilibrium path. It can therefore easily be observed that this perturbation pattern is intuitively identical to that obtained by examining the errors that can be committed in the last phases of the game. However, this topic will be discussed in more detail below.

3.1.3 Perturbing the Execution

The last type of error we consider is the one that can occur in the last two remaining phases of the game described at the beginning of this section. Suppose that the correlation device selection and then the strategies to be recommended have been correctly carried out and that the game has started without incurring errors. Whenever a certain player's information set $I \in \Im_n$ is reached, the player $n \in N$ discovers the received recommendation and decides whether to follow it by analysing the expected outcomes resulting from the selection of each possible move. If no errors have occurred in the initial stages of the game, the player has the certainty of having no interest in deviating but must still carry out the analysis above. The mistake that can be made is evidently a consequence of a defect in this analysis. If, in fact, for some calculation errors, it is considered that a move $a \in I$ is more convenient than the recommended one $\hat{a} \in I, a \neq \hat{a}$, due to its rationality, the player would be interested in deviating from the path of equilibrium. A deviation of this kind would then bring all players from the next information set I' to I to play in best response. From that moment on the game results corresponding to the game in extensive form that has the I' as the root node and in which the terminal node reached will be the one corresponding to a Nash equilibrium of this sub-game. Any errors in this sub-game must therefore be modeled exactly as described by Selten.

Let us now see how it is possible to model the perturbation described above in the manner of Selten. As regarding the information set on the

recommended path, once a player arrives at the information set I the possibility that the analysis conducted is erroneous must be considered assigning in the perturbed game a lower bound probability $l(\varepsilon)$ to each action. As regarding the information set out of the path instead we must consider that, given that all the actions are played with non zero probability, they are now all reachable and therefore it is necessary to model all the sub-games out of path as for Extensive Form Perfect Equilibria.

3.2 Equilibrium Definition

After understanding how we should consider the perturbation of a game with mediator in an extensive form, we can now move on to the mathematical formalization and to the search for the result we set ourselves at the beginning of the chapter. In this regard, in the following we will provide the formal definition of this type of equilibrium and then, since we are working on extensive-form games, we will proceed, as for the other equilibria, to give a formulation of the result found in the context of the sequence form.

3.2.1 The Main Result

In 3.1 we have seen that the only type of perturbation that makes sense to consider when dealing with moderated extensive form games is that of the third type described in 3.1.3. Perturbing the execution of a game means to consider all the possible deviations that may be encountered during its development. We can then proceed with the following definition, which adapts Definition 19 to the case examined in 3.1.3 and merges it with Definition 27:

Definition 39 (Perturbed Extended Game)**.** *Given a Game* Γ *in Extensive Form, a Correlation Device* μ *and a lower bound function* $l(\varepsilon) : A \to (0, 1]$*, let's call* Γ_μ *the Extended Game based on* μ*. The* Perturbed Extended Game $(\Gamma_\mu, l(\varepsilon))$ *is the Perturbed Game associated to* Γ_μ *obtained using the lower bound function* $l(\varepsilon)$*.*

In other words, the *Extended Perturbed Game* is obtained by composing the operations of extension and perturbation. This definition is perfectly compliant with the perturbation method that we had identified in the previous section. The only possible perturbation method is therefore the one used for games in extensive form, but adapted to extended games. It should be noted that, as far as the actions available for the first information set of the Extended Game are concerned, there is no need to define a perturbation

since each of the actions, which corresponds to the selection of the correlation device, is played with non-zero probability. Considering therefore a sufficiently low lower bound one can neglect this detail and consider the Extended Perturbed Game thus obtained as a perturbed game as in Definition 19. Figure 4.7 shows the Extended Perturbed version of the chicken game of Example 1 perturbed with the lower bound function $l(\varepsilon) = \varepsilon$. The game thus obtained is the perturbed version of the game in Figure 2.7.

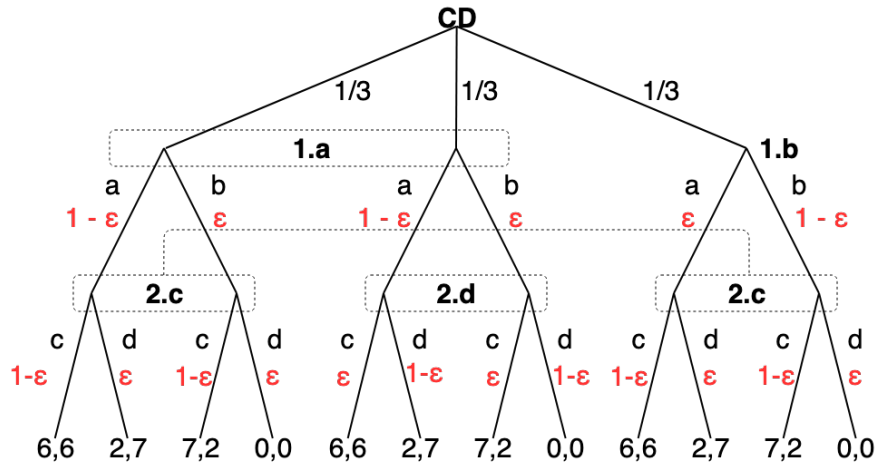

Figure 3.2: An Extended Perturbed version of the Chicken Game

Once the nature of the perturbation is defined, we can use the result of Definition 39 to proceed to the formalization of the main result of this work. We know from (von Stengel and Forges, 2008) the concept of Extensive Form Correlated Equilibrium which, however, suffers, like any solution concept that we have previously treated, from poor precision for what concerns situations outside the recommended path. In a similar way to what is done in (Dhillon and Mertens, 1996) for the normal form correlated we can state the following definition:

Definition 40 (Extensive Form Perfect Correlated Equilibrium)**.** *Given an Extensive-Form Game* Γ*, a Correlation Device* μ *on which players have agreed and a generic lower bound function* $l(\varepsilon) : A \to (0, 1]$*, let* $(\Gamma_\mu, l(\varepsilon))$ *be the Extended Perturbed Game of* Γ *using* μ *and* $l(\varepsilon)$*. The pure strategy profile* $\hat{s} = (\hat{s}_0, \ldots, \hat{s}_m)$ *that follows the recommendations at each information set is an* Extensive Form Perfect Correlated Equilibrium *of* Γ *iff it is a limit point of a* $(\Gamma_\mu, l(\varepsilon))$*'s Nash Equilibrium when* $\varepsilon \to 0$*.*

Given the Extended Perturbed Game of Figure 4.7, it is easy to see

that all the strategy profiles $(Go, Go), (Go, Stop)$ and $(Stop, Go)$ are *Extensive Form Perfect Correlated Equilibria(EFPCE)*, since the imposition of a lower bound does not encourage a deviation for a player. In other words, an EFPCE is a recommended strategy profile that is deviation-resistant against the possible perturbations. It therefore represents in all respects the analogous of the concept of perfection for the extensive-form games.

3.2.2 Sequence Form Characterization

After providing the definition of Extensive Form Perfect Correlated Equilibrium we also deal with constructing its characterization in sequence form, with the aim of also responding to the practical problem of finding such an equilibrium point given a certain game Γ in extensive form. To fullfil this purpose we must define, similarly to what has been done above, the type of tremble to be used and the way in which it has to be formulated inside the research problem. We consider the definition of lower bound $l(\varepsilon)$ that we used in 2.4.2 for the problem of finding an extensive form perfect equilibrium, we know that such a perturbation scheme is perfectly responsive to the perturbation requirements for the definition of perfection and therefore, since the extensive form perfect correlated equilibrium of a Γ is also definable as a limit point of the corresponding perturbed extended game Γ_S. It is important to remark that the perturbation of the sequence form implies the loss of some information. A complete perturbation scheme should in fact provide for the possibility of specifying a different tremble for each action of each information set of each player, while the proposed one allows assigning a different tremble to each set of action of the same player inside a strategy profile. However, this difference in representation does not entail particular consequences since our practical goal is the search for a single equilibrium with an acceptable computational effort, as long as we keep in mind that the set of extensive form perfect correlated equilibria that can be obtained using such a perturbation scheme could potentially be narrower than the real (theoretical) one. We need also to note that a correlation plan as defined in Definition 38 specifies the probability with which each sequence profile is played, then such a perturbation scheme can be imposed as the lower bound of a correlation plan. Hence, consider now a two player extensive-form game with perfect recall and without chance nodes, given the lower bound function $l(\varepsilon) : Q_1 \times Q_2 \to \mathbb{R}^+$ as in Definition 30, the formulation of the problem of finding an Extensive Form Perfect Correlated Equilibrium is the same given in (2.15) of 2.4.3, with a variation only on the last constraint, in order to impose that the correlation plan of each pair (q_1, q_2) is greater or equal to

the corresponding lower bound. Therefore, the formulation is the following:

$$\underset{z\in\mathbb{R}^{|Q_1|\times|Q_2|}}{\arg\max} \quad u_1(q_\emptyset) + u_2(q_\emptyset) \tag{3.1}$$

$$s.t. \quad u_1(q_1) = \sum_{q_2\in Q_2} z(q_1,q_2)U_1(q_1,q_2) + \sum_{I\in\mathfrak{I}_1:q_{1,I}=q_1} \sum_{a\in\rho(I)} u_1(q_{1,I}|a), \quad \forall q_1 \in Q_1 \tag{3.2}$$

$$u_2(q_2) = \sum_{q_1\in Q_1} z(q_1,q_2)U_2(q_1,q_2) + \sum_{I\in\mathfrak{I}_2:q_{2,I}=q_2} \sum_{a\in\rho(I)} u_2(q_{2,I}|a), \quad \forall q_2 \in Q_2 \tag{3.3}$$

$$v_1(I,q_1) \geq \sum_{q_2\in Q_2} z(q_1,q_2)U_1(q_{1,I}|a,q_2) + \sum_{J\in\mathfrak{I}_1:q_1=q_{1,I}|a} v_1(J,q_1),$$
$$\forall q_1 = q_{1,I}|a \in Q_1, \forall I \in \mathfrak{I}_1 : \hat{I} = I \vee pre(\hat{I},I), a \in \rho(I) \tag{3.4}$$

$$v_2(I,q_2) \geq \sum_{q_1\in Q_1} z(q_1,q_2)U_2(q_1,q_{2,I}|a) + \sum_{J\in\mathfrak{I}_2:q_2=q_{2,I}|a} v_2(J,q_2),$$
$$\forall q_2 = q_{2,I}|a \in Q_2, \forall I \in \mathfrak{I}_2 : \hat{I} = I \vee pre(\hat{I},I), a \in \rho(I) \tag{3.5}$$

$$v_n(I,q_{n,I}|a) = u_n(q_{n,I}|a), \quad \forall n \in \{1,2\}, \forall I \in \mathfrak{I}_n, \forall a \in \rho(I) \tag{3.6}$$

$$z(q_{1,I},q_2) = \sum_{a\in\rho(I)} z(q_{1,I}|a,q_2),$$
$$\forall I \in \mathfrak{I}_1, \forall q_2 \in Q_2 : rel(q_{1,I}|a,q_2), \forall a \in \rho(I) \tag{3.7}$$

$$z(q_1,q_{2,I}) = \sum_{a\in\rho(I)} z(q_1,q_{2,I}|a),$$
$$\forall I \in \mathfrak{I}_2, \forall q_1 \in Q_1 : rel(q_1,q_{2,I}|a), \forall a \in \rho(I) \tag{3.8}$$

$$z(q_\emptyset,q_\emptyset) = 1 \tag{3.9}$$

$$z(q_1,q_2) \geq l(\varepsilon,q_1,q_2), \quad \forall (q_1,q_2) \in Q_1 \times Q_2 \tag{3.10}$$

Hence, it is easy to see that the problem written above of finding an EFPCE of a given game is a linear programming problem. We now want to consider what is stated in (Farina et al., 2018b) when dealing with a problem that is similar to the one we mentioned above. First, consider a suitable *Trembling Linear Program(TLP)*, i.e., a function that assigns a parameter ε to a linear problem $P(\varepsilon)$ with ε as a parameter such that a limit solution to a TLP is a limit point, as $\varepsilon \to 0$, of optimal solutions for $P(\varepsilon)$. As an example, a limit solution to the QPE TLP are QPEs. It is therefore proved that, given as input a TLP, there exists $\varepsilon^* > 0$ such that for all $0 < \hat{\varepsilon} < \varepsilon^*$,

any optimal basis for the numerical LP $P(\hat{\varepsilon})$ is stable and that such a value ε^* can be computed in polynomial time. This fact has an important impact on the perturbation concept that can be used in the problem described above because it implies that any $\hat{\varepsilon}$ in the range $(0, \varepsilon^*]$ guarantees the termination of the solution algorithm. Therefore it is now proved that for every game and every solution algorithm to the problem formulated in 3.1, an EFPCE can be computed using any suitable perturbation scheme. Moreover, it is also implied that an extensive form perfect correlated equilibria exists for every game Γ in extensive form.

Chapter 4

Characterization

We have previously seen how the correlated equilibrium is in fact a generalization of the Nash equilibrium and we have defined in the previous chapter a new refinement for this type of equilibrium. We now ask ourselves in what relation is this new refinement with those proposed in the preliminaries in Chapter 2. To obtain a correct taxonomy, however, it is first necessary to make the definitions comparable to each other, which we will do in the first section.

4.1 Probabilistic Approach

The purpose of this chapter is to provide a taxonomy of the equilibria described above and of their refinements, specifying their classification and any dependency relationships. To reach this goal, however, it is necessary to make these equilibria comparable to each other by placing them in the most general context, which as we have seen in Chapter 2 is that of correlated games described at the beginning of Chapter 3. A correlated game is completely defined based on the probability distribution used, which is initially agreed by all players. For convenience and following what has already been stated in (Dhillon and Mertens, 1996) and (von Stengel, 1996) we will therefore consider the concept of probability distribution agreed between the players and we will redefine the equilibria described above on the basis of this concept.

4.1.1 Nash And Perfect Equilibria Distribution

Let us now consider the class of Nash equilibria in fully mixed strategies, i.e., those strategy profiles in which each strategy is played with non-zero probability. We know from (Aumann, 1974) that each Nash Equilibrium

is also a Correlated equilibrium and therefore if it is recommended by the correlation device no player will have interest in deviating. Now we are going to understand how, starting from a probability distribution on the single strategies (s_n), it is possible to obtain a probability distribution on the strategy profiles (s).

Definition 41 (Nash Equilibrium Distribution)**.** *Given a Game $G = (N, S, F)$ in reduced normal form and one of its Nash Equilibria in fully mixed strategies defined by the probability $p_n(s_n), \forall s = (s_1, \ldots, s_m) \in S$ for each player $n \in N$, the distribution μ over the set of all the pure strategy profiles S is defined in the following way:*

$$\forall s \in S, \quad \mu(s) = \prod_{n \in N} p_n(s_n).$$

is the Nash Equilibrium Distribution(NED) *associated to the equilibrium.*

It should be noted that the distribution thus obtained can be marginalized in order to obtain the probability distribution on the single strategies p_n from which the definition started. Obviously in the particular case of the Nash Equilibrium in pure strategies $\hat{s}$, the probability distribution will be $\forall s \in S \setminus \{\hat{s}\}, \mu(s) = 0$ and $\mu(\hat{s}) = 1$. We note also that a certain Nash Equilibrium Distribution is generally associated to only one equilibrium. Given a NED μ, each strategy profile s for which $\mu(s)$ is strictly positive is by definition a Correlated Equilibrium. It can then be observed that each Nash Equilibrium is associated with at least one Nash Equilibrium Distribution and vice versa.

It is important at this point to specify that, with an abuse of notation, we can talk about *Nash Equilibrium Distribution* also for equilibria defined through behavioral strategies for games in extensive form, bringing the game from the Extensive Form to the Reduced Normal Form through the simple steps illustrated in 2.1.3, preserving its equilibria's properties.

Let us now focus on the concept of perfection illustrated in 2.2.2 obtained from Selten and look for the equivalent of Nash Equilibrium Distribution for Perfect Equilibria. If we observe that each Perfect Equilibrium (whether in pure strategies or in mixed) is necessarily a Nash Equilibrium we get the following definition:

Definition 42 (Perfect Equilibrium Distribution)**.** *Given a Game $G = (N, S, F)$ in Reduced Normal Form and one of its Nash Equilibria $\hat{s}$. A*

Nash Equilibrium Distribution of G associated to the equilibrium $\hat{s}$ is a Perfect Equilibrium Distribution (PED) *iff $\hat{s}$ is a Perfect Equilibrium of G.*

Consider the chicken game of Example 1 in reduced normal form and its only fully mixed Nash equilibrium defined by the probabilities $p_1(Go) = \frac{2}{3}$ and $p_2(Go) = \frac{2}{3}$, the corresponding *Nash Equilibrium Distribution* is μ_1. The distribution μ_2 is the one associated to the nash equilibrium $(Go, Stop)$ of the same game.

$$\mu_1 = \begin{pmatrix} \frac{4}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} \end{pmatrix}, \mu_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Note that μ_1 results to also be a *Perfect Equilibrium Distribution*, since the equilibrium is perfect (like all the Nash equilibria in fully mixed strategies).

4.1.2 Correlated Equilibria Distribution

Let us now try to apply the concept of distribution highlighted above to the correlated equilibria. this classification regarding games in normal form has already been defined in (Dhillon and Mertens, 1996) and is the following one:

Definition 43 (Correlated Equilibria Distribution)**.** *The* Correlated Equilibrium Distributions (CED) *of a game $G = (N, S, F)$ are the probability distribution on the product*

$$S = \prod_{n \in N} S_n$$

of the pure strategy sets of G which are induced by some Correlated Equilibrium $\hat{s}$ of G.

The previous result has been used as a model for the definitions given in the whole section of the equilibria according to the probability distributions that they generate. The probability distribution generated starting from the refinement of the correlation with the concept of perfection is also defined in (Dhillon and Mertens, 1996). The following definition is substantially identical to the previous one:

Definition 44 (Perfect Correlated Equilibrium Distribution)**.** *The* Perfect Correlated Equilibrium Distributions (PCED) *of a game $G = (N, S, F)$ are the probability distribution on the product*

$$S = \prod_{n \in N} S_n$$

of the pure strategy sets of G what are induced by some Perfect Correlated Equilibrium $\hat{s}$ of G.

Now consider the correlated games in extensive form as in (von Stengel and Forges, 2008) treated in 2.3.2. In that work, the following definition is implied:

Definition 45 (Extensive Form Correlated Equilibrium Distribution)**.** *Given a Game* Γ *in extensive form and a correlation device* μ *on which the players have agreed,* μ *is an* Extensive Form Correlated Equilibrium Distribution (EFCED)*.*

At last, we report below the distribution that is associated to the extensive form perfect equilibrium that we have defined in 3.2.1. Since a generic distribution on which the players have agreed on can recommend both EFCE and EFPCE, we must now impose that all the strategy profiles that can be recommended by such a distribution are EFPCE. Thus, the following definition holds:

Definition 46 (Extensive Form Perfect Correlated Equilibrium Distribution)**.** *Given a Game* Γ *in extensive form and a correlation device* μ *on which the players have agreed,* μ *is an* Extensive Form Perfect Correlated Equilibrium Distribution (EFPCED) *if and only if all the strategy profiles* $\hat{\pi}$ *that can be recommended by* μ *are Extensive Form Perfect equilibria of the corresponding subtree in the extended game* Γ_μ*.*

Note that the distribution μ_1 used in 4.1.1 is also an *Extensive Form Perfect Correlated Equilibrium*, since all the strategy profiles of the chicken game are extensive form perfect correlated equilibria.

4.2 A Complete Taxonomy

The purpose of this section is the construction of a taxonomy as complete as possible of the equilibria examined in previous chapters that can be defined on the class of games in extensive form. During the development of this section a precise definition of the relationships that exist between the different equilibrium concepts and the relationships that exist from the point of view of set theory will be provided. The final analysis will be particularly important since we will see how the Extensive Form Perfect Correlated Equilibrium defined in 3.2.1 is integrated into this context and which are its relationships with the other equilibrium concepts.

4.2.1 Before EFPCE

Let us now consider the class of games in extensive form. We know from (Aumann, 1974) that the concept of correlated equilibrium is a generaliza-

tion of the Nash equilibrium for the class of normal form games. By looking at the definition of the extensive form correlated form equilibrium given in (von Stengel, 1996), we can see how this relation of inclusion is preserved also in the context of games in extensive form. If we take into consideration a correlated extensive-form game with the distribution μ as a correlation device, it is easy to notice how, once the strategy profile $\hat{s}$ has been recommended by μ, if $\hat{s}$ is a Nash equilibrium then no player will be interested in deviating due to the definition of extensive form correlated equilibrium. Furthermore for the definition of Nash equilibrium, for each Nash equilibrium $\hat{s}$, there must necessarily be a distribution for the correlation device on which players can agree that recommends $\hat{s}$. To fully define this relationship it is now appropriate to demonstrate that, given a generic game Γ in extensive form with the μ distribution as a correlation device, there is at least one extensive form correlated equilibrium $\hat{s}$ that is not a Nash equilibrium, or similarly there exists for Γ a $\hat{\mu}$ that recommends $\hat{s}$ with non-zero probability.

Actually we have already seen an example of such a $\hat{\mu}$ in 2.3.1 when we introduced the concept of correlation. If we consider the extensive form of the chicken game of the example 1 shown in Figure 2.1, we notice that

$$\hat{\mu} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$$

Is a correlation device on which the players agree. In $\hat{\mu}$ the strategy profile $\hat{s} = (A, a)$, is recommended with probability $p(\hat{s}) = \frac{1}{3}$, but it is not a Nash equilibrium in pure strategies. Hence, we have just found an extensive form correlated equilibrium which is not a Nash equilibrium, since $\hat{\mu}$ cannot be marginalized in order to obtain the product of some Nash equilibria in mixed strategy, and therefore we can conclude that the following relation is valid:

$$NE \subseteq EFCE. \tag{4.1}$$

The set diagram in Figure 4.1 represents the above defined relation where:

(1) is the point that represents the strategy profile $\hat{s}$

(2) is a generic NE

Let us now consider the concept of perfection introduced in (Selten, 1975) which has been illustrated in 2.2.2. The extensive form perfect equilibrium (or briefly, perfect equilibrium) is by definition a refinement of the Nash equilibrium, we can therefore for the sake of completeness limit ourselves

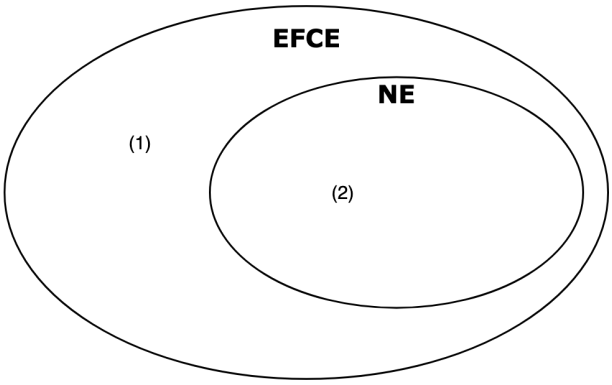

Figure 4.1: Set Diagram representation of relation 4.1

to provide an example of a Nash equilibrium which is not an extensive form perfect equilibrium. We have already seen such an example in 2.2.2. Consider the game Γ in Figure 2.4 of Example 2, both the strategy profiles $\hat{s}_1 = (b, c)$ and $\hat{s}_2 = (b, d)$ are Nash equilibria but only $\hat{s}_1$ is also an extensive form perfect equilibrium. Considering the corresponding reduced normal form

$$G = \begin{pmatrix} (2,2) & (2,1) \\ (3,3) & (3,3) \end{pmatrix}$$

the correlation device

$$\hat{\mu} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$$

recommends both the Nash equilibria with non-zero probability. Since $\hat{s}_2$ can be recommended, we found a Nash equilibrium that is not an extensive form perfect equilibrium. Hence, the following relation, which is an expansion of (4.1), is valid:

$$EFPE \subseteq NE \subseteq EFCE. \tag{4.2}$$

Thus, the set diagram of Figure 4.2 represents the above defined relation where:

(1) is the point that represents the strategy profile $\hat{s}$ defined above

(2) represents the strategy profile $\hat{s}_1$

(3) represents the strategy profile $\hat{s}_2$

With this last example we have concluded the preliminary analysis and that of Figure 4.2 turns out to be the complete taxonomy of the equilibria of the games in extensive form illustrated above (without considering the new solution concept proposed in this work).

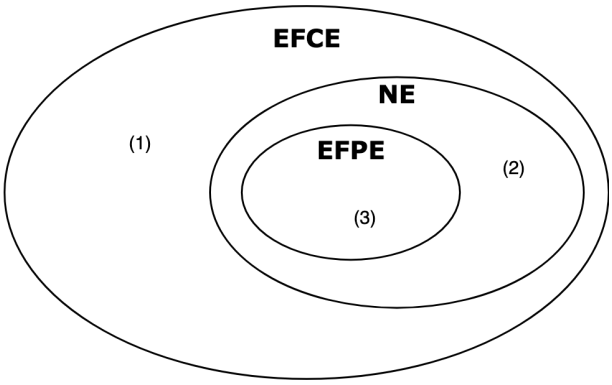

Figure 4.2: Set Diagram representation of relation 4.2

4.2.2 EFPCE vs EFCE

After having accurately illustrated the situation before the introduction of the extensive form perfect correlated equilibrium defined in 3.2.1, let us now consider how the newly introduced refinement is placed in the taxonomy of Figure 4.2 starting from the outermost set.

We introduced already in 3.2.1 the extensive form perfect correlated equilibrium as a refinement of the extensive form correlated equilibrium built using the concept of perfection applied to the tree related to the extended game. Therefore, we intuitively know that their relationship, like the one that exists between Nash equilibria and perfect equilibria, is of inclusion. For each generic game Γ, each EFPCE is defined as perfect equilibrium of the extended game Γ_μ (or as a limit point of Nash equilibria of the extended perturbed game $(\Gamma_\mu, l(\varepsilon))$), while each EFCE as a Nash equilibrium of Γ_μ. Thus, each EFPCE is also an EFCE for the relation already illustrated in (4.2). Let's now look at the following example:

Example 4. *Let us consider the asymmetric game in which two players toss a fair coin at the same time. Player 1 gets a point only if both players get the same result $(H,h),(T,t)$, while Player 2 gets a point only when both players get head as a result (H,h). A reduced normal form representation of this game is represented in the Table below. One of the possible extensive form representations corresponding to the representation in normal form of the game just described is given in Figure 4.3.*

Now consider the game Γ in extensive form of Figure 4.3 presented in Example 4. The strategy profiles $s_1 = (H,h)$ and $s_2 = (T,t)$ are both Nash

	h	t
H	$(1,1)$	$(0,0)$
T	$(0,0)$	$(1,0)$

Table 4.1: Asymmetric Heads or Tails

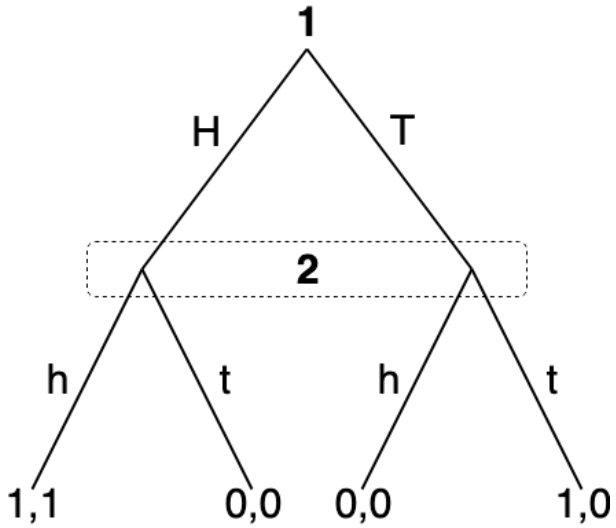

Figure 4.3: Asymmetric Heads or Tails Game in Extensive Form

equilibria but only the first is also a perfect equilibrium. If we now use

$$\mu = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

as a correlation device we get the extended game Γ_μ of Figure 4.4. We can now easily verify that both the strategy profile $s_1 = (H, h)$ and $s_2 = (T, t)$ are EFCE. However, if we perturb the game Γ_μ with the standard lower bound function

$$l(\varepsilon, a) = \varepsilon, \quad \forall a \in A$$

we get the extended perturbed game $(\Gamma_\mu, l(\varepsilon))$ represented in Figure 4.5. Observing $(\Gamma_\mu, l(\varepsilon))$ we can see how, since every action is played with non-zero probability, in the subtree of the perturbed extended game corresponding to the strategy profile s_2, the strategy profile itself is not a Nash equilibrium of the $(\Gamma_\mu, l(\varepsilon))$ game, because player 2 prefers to play the action h anyway in the information set $2.t$ since it returns a greater expected utility

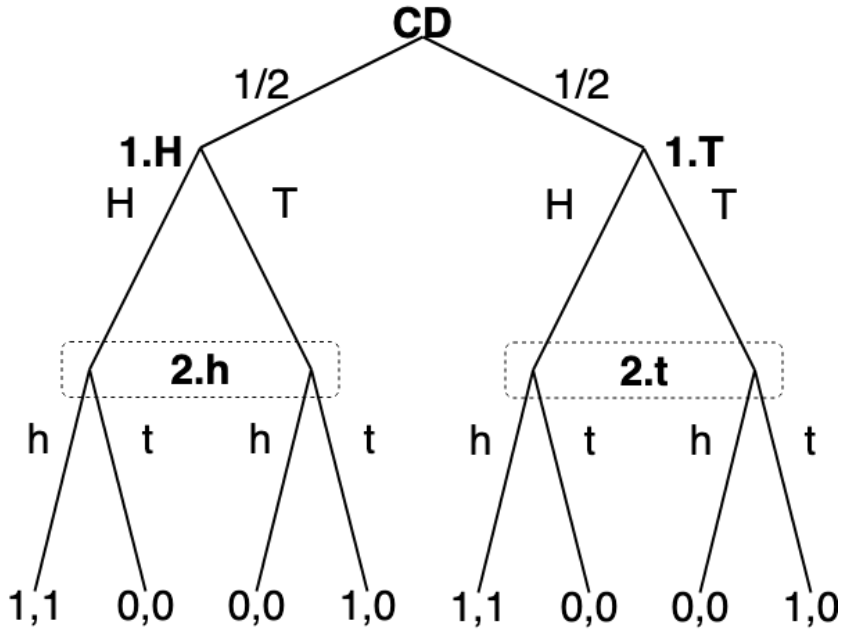

Figure 4.4: Extended Asymmetric Heads or Tails Game with μ as a CD

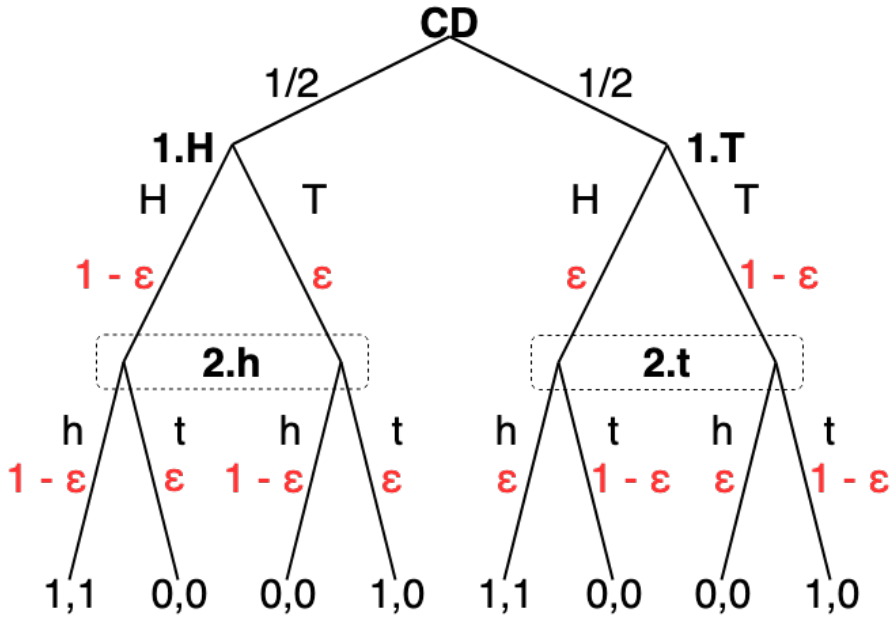

Figure 4.5: Extended Perturbed Asymmetric Heads or Tails Game

($f_2(T,h) \geq f_2(T,t)$). Hence, s_2 is not an EFPCE. We have therefore found an example of a strategy profile that is an EFCE but not an EFPCE, completely defining the relationship between these two equilibria. As a result, the following relation appears to be valid:

$$EFPCE \subseteq EFCE. \tag{4.3}$$

The first step of the taxonomy we are constructing is therefore represented in Figure 4.6 where:

(1) is the point that represents the strategy profile $s_1 = (H,h)$

(2) is the point that represents the strategy profile $s_2 = (T,t)$

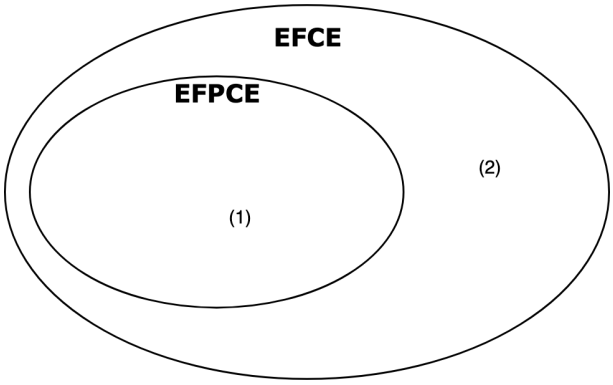

Figure 4.6: Set Diagram representation of relation 4.3

4.2.3 EFPCE vs NE

We now focus on the relationship between the EFPCE and the NE, knowing from relations (4.2) and (4.3) that we are moving in the context of the EFCE. For the very definition of correlation, we intuitively know that neither set can completely include the other. We therefore expect the existence of at least one example of EFPCE that is not a NE and, in a different game, of a NE that is not an EFPCE. However, the two sets cannot be disjoint, since there must certainly be games in which a strategy profile is both a NE and an EFPCE at the same time, so we expect the existence of an intersection.

Now consider the Chicken game of Example 1 introduced in 2.1, of which we have defined the extensive form Γ in 2.1.3 represented in Figure 2.1. In the example that we used to define the concept of correlation the distribution

$$\mu = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$$

has been selected as a correlation device. We know from 2.2.1 that the strategy profiles $(Go, Stop)$ and $(Stop, Go)$ are Nash equilibria in pure strategies. The other recommended strategy profile $s_1 = (Go, Go)$ on the other hand is not a Nash equilibrium but we have shown in 2.3.2 that it is an EFCE by showing in Figure 2.7 its extended game Γ_μ. We now observe in Figure 4.7

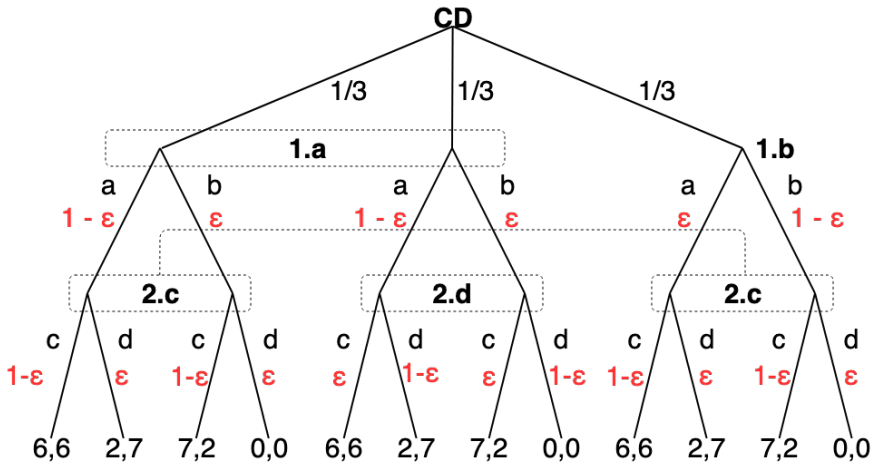

Figure 4.7: Extended Perturbed Chicken Game

the extended perturbed game $(\Gamma_\mu, l(\varepsilon))$ of Γ in the path corresponding to the recommendation $\hat{s} = s_1$. It is clear that neither player in the perturbed game is interested in deviating from the recommendation received, so s_1 is an EFPCE.

As a second example we take the game of Example 4, whose extensive form is highlighted in Figure 4.3, into consideration. Let us now look at the strategy profile $s = (T, t)$ that we know from 4.2.2 which is a Nash equilibrium but not an EFPE and therefore turns out to be a good candidate for the example we are looking for. We use here as a correlation device the distribution

$$\widetilde{\mu} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

that has the particularity to recommend only the pure strategy profile $\hat{s} = s$. The extended game $\Gamma_{\widetilde{\mu}}$ thus obtained is shown in Figure 4.8. We can easily note that $\Gamma_{\widetilde{\mu}}$ is in all respects identical to the extensive form of the game Γ of Figure 4.3. At this point we know with certainty that s is not a PE of $\Gamma_{\widetilde{\mu}}$ and therefore cannot be an EFPCE of Γ.

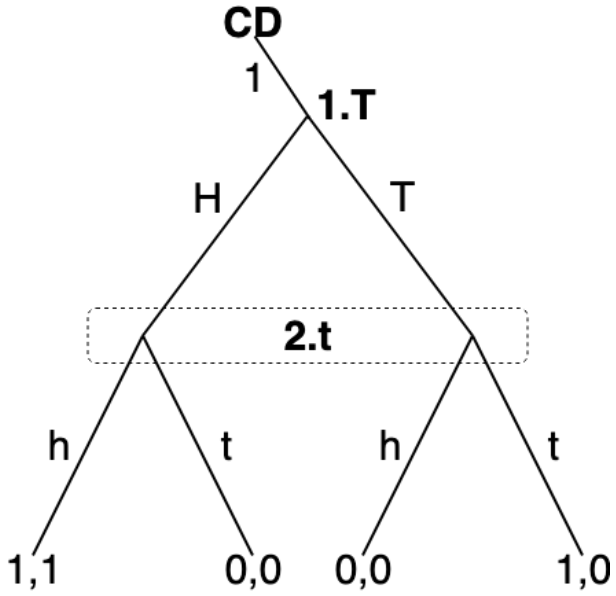

Figure 4.8: Extended Asymmetric Heads or Tails Game with $\widetilde{\mu}$ as a CD

After having verified the relationship between NE and EFPCE, we now provide the proof that the two sets are not completely disjoint, but that they instead present an intersection. This can be easily seen considering the game illustrated in Example 4 that we have just used above. If in the same context we take

$$\mu = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

as a correlation device similarly to what we did in 4.2.2, we observe that the strategy profile $s_1 = (H, h)$ is simultaneously a NE and an EFPCE, since neither player is interested in deviating when s_1 is recommended.

We are now ready to conclude this part of the analysis. From the examples above we can constructively derive the following relationships:

$$\begin{aligned} NE &\not\subset EFPCE \\ EFPCE &\not\subset NE \\ EFPCE \cap NE &\neq \emptyset. \end{aligned} \tag{4.4}$$

Those relationships can be summarized with the set diagram of Figure 4.9, where:

(1) represents the strategy profile (Go, Go) of the chicken game of Example 1

(2) represents the strategy profile (T, t) of the game of Example 4

(3) represents the strategy profile (H, h) of the game of Example 4.

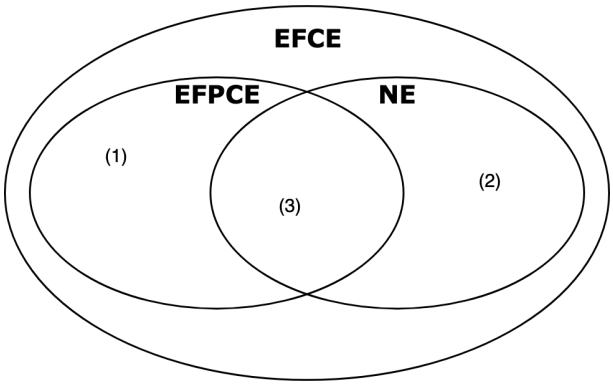

Figure 4.9: Set Diagram representation of relation 4.4

4.2.4 EFPCE vs PE

After defining the relationship between NE and EFPCE, it is now necessary to establish the relationship between EFPCE and EFPE, thus to compare the major refinements of the EFCE and NE respectively. The characterization of EFPE has already been provided in 4.2.1, so we are interested in understanding the arrangement of the EFPCE set in the taxonomy of Figure 4.2.

Consider again the strategy profile (Go, Go) of the chicken game of Example 1, taking as a correlation device any distribution that would recommend it with non-zero probability. Since we know from 4.2.3 that (Go, Go) is an EFPCE but not a NE, we have found an example of a strategy profile that is an EFPCE but not an EFPE.

We now operate within the set of NE. Let us understand whether there is a strategy profile s for some game such that s is an EFPE but not an EFPCE. We now remember from (Dhillon and Mertens, 1996) that the main feature of an EFPE is to play the best move (which guarantees the maximum available expected utility) in every information set from the root node onwards. Intuitively, this feature also seems to be shared by the EFPCE

class, since we defined the EFPCE of a game Γ as the EFPE of the corresponding extended game Γ_μ. The following definition will be useful in the proof of Theorem 1, which states what we have intuitively found out:

Definition 47 (Selten's Test Sequence)**.** *Given an extensive form game* Γ*, a* Selten's Test Sequence *is a sequence of perturbed games* $\{(\Gamma, \eta_k)\}_{k\in\mathbb{N}}$ *such that*

$$\forall a \in A, \quad \lim_{k\to\infty} \eta_k(a) = 0.$$

In a *Selten's Test Sequence*, when $k \in \mathbb{N}$ grows to infinity, the lower bound function η_k tends to zero for all actions. Note that, in such a definition, we use η_k to denote the lower bound function. Since we are not interested here in highlighting its dependency from ε, we could have indifferently used $l_k(\varepsilon) = \eta_k$. We can now therefore define again the concept of *Extensive Form Perfect Equilibrium* using a Test sequence. This definition will be particularly useful in the following:

Definition 48 (Extensive Form Perfect Equilibrium (Test Sequences))**.** *Given an extensive form game* Γ*, a behavioral strategy profile* π *is an* Extensive Form Perfect Equilibrium *iff there exist a test sequence* $\{(\Gamma, \eta_k)\}_{k\in\mathbb{N}}$ *and a sequence of behavioral strategy profiles* $\{\pi^k\}_{k\in\mathbb{N}}$ *such that each* π^k *is a NE of* (Γ, η_k) *and* π *is a limit point of* $\{\pi^k\}_{k\in\mathbb{N}}$*.*

Such a definition is obviously referred to the same concept of Definition 20 and will have a central impact on the proof of the following theorem:

Theorem 1 (EFPE⊆EFPCE)**.** *Given an extensive-form game* Γ*, any EFPE of* Γ *defines a correlated distribution* μ *which is a EFPCED of* Γ *(or, equivalently, which can recommend only EFPCE).*

Proof. Given a behavioral strategy profile $\widetilde{\pi}$ for Γ, if it is an EFPE for Γ then from Definition 48 follows that there exists some Selten's Test Sequence $\{(\Gamma, \eta_k)\}_{k\in\mathbb{N}}$ such that every (Γ, η_k) has a local equilibrium point π^k such that, for $k \to \infty$, the sequence of π^k converges to $\widetilde{\pi}$. This implies that, for every player $n \in N$ and information set $I \in \mathfrak{I}_n$, it holds that

$$f_n(\pi^k/\widetilde{\pi}_{n,I}) = \max_{\pi'_{n,I}} f_i(\pi^k/\pi'_{n,I}),$$

where $\pi'_{n,I}$ is a player n's local strategy at I in the perturbed game (Γ, η_k). Let $\sigma_n(I)$ be the action selected by σ_n at information set I, then define $\mu \in \Delta(\Sigma)$ as

$$\forall \sigma \in \Sigma, \quad \mu(\sigma) = \prod_{n\in N} \prod_{I\in\mathfrak{I}_n} \widetilde{\pi}_I(\sigma_n(I)).$$

We want to show that μ is such that, given $\sigma \sim \mu$, following σ is an EFPE in the extended game Γ_μ. For each n, and $a \in A_n$ we have, in Γ_μ, a set of corresponding actions $\{\hat{a}_j\}_{j=1}^m$. Consider the Selten's test sequence $\{(\Gamma_\mu, \hat{\eta}_k)\}_{k\in\mathbb{N}}$, such that for each action in $\{\hat{a}_j\}_{j=1}^m$ it holds that

$$\hat{\eta}_k(\hat{a}_j) = \eta_k(a).$$

For each n, and $I \in \mathfrak{I}_n$, we need to show that there exists a sequence of NEs of $(\Gamma_\mu, \hat{\eta}_k)$ that has σ as limit point. This follows from the definition of μ. Specifically, the expected utility observed by the player at each information set is the same in both games, and, thus, the local equilibrium conditions are satisfied also in the extended game. □

Hence, we proved in Theorem 1 that the following relation holds for every extensive-form game Γ:

$$EFPE \subseteq EFPCE \subseteq EFCE \tag{4.5}$$

The relation (4.5) shows that the EFPE set is a subset of that of the EFPCEs. We also know from 2.2.2 that it is also a subset of NEs. We can therefore conclude that all the EFPEs are contained in the intersection between NE and EFPCE. We now want to determine the nature of this intersection. the following Corollary 1.1 of the Theorem 1 answers this question in a complete way:

Corollary 1.1. *Given an extensive-form game Γ, there is no NE that is not an EFPE but is an EFPCE.*

Proof. Consider the last sentence of the proof of Theorem 1. We know that, given an extensive-form game Γ, a correlation device μ for Γ and the corresponding extended game Γ_μ, the expected utility $u_n : A_n \to \mathbb{R}$ of each move in each information set in Γ_μ is the same as in the corresponding information set in the game Γ. This relationship also exists between the corresponding perturbed games (Γ, η) and (Γ_μ, η), given the same lower bound function η. Let's now consider a behavioral strategy profile $\hat{\pi}$ which is a NE for Γ but not an EFPE and a correlation device μ that recommends it. Since $\hat{\pi}$ is not an EFPE of Γ, there must exist an information set $I \in \mathfrak{I}_n$ for a player $n \in N$ in which the move $\hat{a} = \hat{\pi}_n(I)$ (the move that $\hat{\pi}$ specifies in I) is not optimal, i.e., it holds that

$$\exists a \in I, \quad u_n(a) > u_n(\hat{a}).$$

We now know that, given the extended game Γ_μ and the information set $\widetilde{I}$ that corresponds to I in Γ_μ, it holds that

$$\exists a \in \widetilde{I}, \quad u_n(a) > u_n(\hat{a})$$

in Γ_μ as well. Therefore the fact that $\hat{\pi}$ is not a NE of Γ_μ implies that $\hat{\pi}$ is not an EFPCE of Γ. □

Hence, we proved that if a strategy profile $\hat{\pi}$ is a NE but is not an EFPE, then $\hat{\pi}$ cannot be an EFPCE and therefore the intersection between the sets of NE and EFPCE does not contain such a strategy profile. Thus, the following relation holds for every extensive form game Γ:

$$NE \cap EFPCE = EFPE \tag{4.6}$$

With this last relation we have completed the description of the complete taxonomy of the main refinements of both Nash equilibria and Correlated equilibria for games in extensive form. Figure 4.10 is therefore the set diagram representation of the main result of this chapter and is obtained by combining relations (4.2),(4.5) and (4.6).

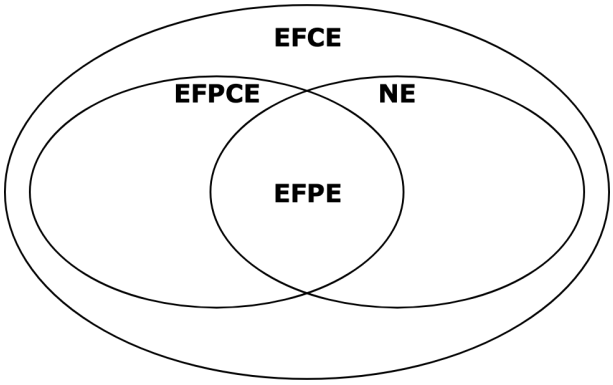

Figure 4.10: Set Diagram of the complete taxonomy

Chapter 5

Conclusions And Future Work

To conclude this work, it is now time to summarize the main results we obtained and highlight some of their possible future developments.

5.1 Conclusions

Even if the present document only consists of a completely theoretical analysis of the discussed solution concepts, their applicability from a practical point of view is extremely wide and can be easily integrated into real-world applications, considering that the innovations introduced consist of improvements and reinforcements of some of the most relevant solution concepts of game theory. This analysis produced three particularly significant results.

The first result, which is the main one, consists in the definition of a refinement of the correlated equilibrium for extensive-form games that includes the concept of tremble in the definition of correlated equilibrium. First of all, we studied the possible perturbations of the class of correlated extensive-form games and we saw that the possible trembles in this class of games are extremely similar to those of EFPE. Starting from this idea we defined the *Extensive Form Perfect Correlated Equilibrium* on the basis of the already known parallelism between the NE and the EFPE. The refinement thus defined takes into consideration all the possible trembles that can occur when searching for an EFCE, resulting in a considerable strengthening of the Von Stengel definition.

The second result obtained in this work is the sequence form formulation

provided for the newly defined EFPCE. Although, in fact, the constraints used were already provided in (von Stengel, 1996), a linear programming formulation that specifies the way in which the search for an EFCE can be implemented was not present in the literature. As regards the EFPCE, the linear programming formulation was constructed by following the aforementioned parallelism between EFPE and EFPCE, defining the concept of tremble at the level of sequence form in a way that is similar to the one used for the EFPE. Thanks to this formulation it was also possible to derive some of the properties of the EFPCE that were not known in advance, as the certainty of the existence of at least one EFPCE for each extensive form game.

The third and last major result obtained consists in the construction of a complete taxonomy of the most important equilibrium concepts that can be used in solving extensive-form games. This taxonomy was completely absent in the literature and allowed us to frame this work and its results in depth in the current scientific context, starting from the definition of the relations between the newly defined EFPCE, the NE, and EFPE in particular. It has been shown that not only the EFPE is the generalization of the EFPCE as the EFCE is of the NE, but also that the intersection of the EFPCE and NE sets is precisely the set of EFPE.

5.2 Future Work

In parallel to the three main results illustrated above we can introduce three interesting proposals of some future works based on what we presented in this thesis.

This work is an important step forward on the path that had already been started in (Dhillon and Mertens, 1996) that explores the jointly conducted analysis of the concepts of perfection and correlation. To continue on this path it would be very interesting to try to build a parallel formulation by applying this time a different refinement to the Correlated equilibrium: the Quasi-Perfection introduced in (van Damme, 1984) and illustrated in 2.2.3. A refinement of the EFCE could thus be built in which each player at any time can select the move to make considering only the future trembles of the opponent, assuming to play without errors in the subsequent information sets.

As far as the formulations of the search problem we have defined in 2.15

and 3.1 are concerned, we can easily see that both have been provided considering only two-player games due to the fact that we only needed those formulations to highlight some useful properties. As regards the formulation concerning the EFCE search for games with a generic number of players, the existence of a polynomial-time algorithm is confirmed in (von Stengel and Forges, 2008) even if the formulation given is not useful in a practical approach. Although the sequence form is not adequate for the representation of this situation since the incentive constraints presented above are only sufficient but not necessary conditions, this search problem should also be analyzed for the newly defined EFPCE considering extensive form games with more than two players.

Starting from the complete taxonomy of the equilibria defined in 4.2, one could extend the analysis by inserting quasi-perfection defined above for games in extensive form and the numerous other refinements available in the literature. Moreover, on the basis of the comparison made between EFPE and PE by Van Damme, it would be interesting to include the equilibria calculated on the normal form of the respective games in the taxonomy, so as to better understand the relationships between the various forms of representation, taking the comparison between the equilibria of a single game as based on its reduced normal form. The theoretical basis from which to start was provided in 4.1 and should be heavily expanded for this purpose.

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