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An auction model for bankruptcy games: cheating prevention and collusion analysis

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Abstract

The bankruptcy game is a cooperative game useful to model situations in different contexts. It consists of dividing a resource that is not sufficient for the satisfaction of all questions among many agents and in a fair manner. Each participant requires a fraction of the available resource. Some examples of application are the sharing of bandwidth or data within the context of networking, the division of the remaining money after the bankruptcy of a bank among creditors, the cache sharing, etc. This model can be reinterpreted as an auction in which, for simplicity, we assume there is only one seller, both the owner of the resource and the auctioneer, and more buyers interested in the purchase. Each individual buyer reports his demands to the seller at the same time as all the others so that the bids are private information known only to the bidder itself and the auctioneer. Aim of the latter is to regulate the auction so that it is a fair procedure and avoiding participants' cheating behaviours. For this reason, it is useful to define a price based on the amount of resource assigned. Once established a rule for the resource allocation problem, it is necessary to define a price mechanism in such a way none of the participants has the incentive to lie or cheat in order to obtain a greater quantity of resource. Within this context, we study various auction mechanisms and in particular the work focuses on the mechanism described by Myerson. This type of auction satisfies the property called *truthfulness*: none of the participants benefits from lying by asking for more or less than necessary. We analized also the possibility of collusion in this particular mechanism: we asked ourselves whether or not players are incentivates in forming coalitions. Since other mechanisms, as second-price auctions, already analyzed in the literature, are sensitive to collusion, it is quite reasonable to think that the participants in a Myerson auction are also inclined towards collaboration. Through the use of numerical simulations, we confirm the starting hypothesis by verifying that also the Myerson's price system is not robust to agreements between players.

Sommario

Il gioco di bancarotta è un modello di gioco cooperativo che si presta a modellizzare situazioni in contesti anche molto diversi tra loro. Esso consiste nel dividere tra più agenti ed in modo equo una risorsa non sufficiente alla soddisfazione di tutte le domande. Ogni partecipante richiede una frazione della risorsa a diposizione. Alcuni esempi di applicazione sono la condivisione di banda o dati all'interno del contesto di networking, la divisione del denaro restante dopo il fallimento di una banca tra i diversi creditori, la condivisione di memoria ecc. Questo modello può essere reinterpretato come un'asta in cui, per semplicità, ipotizziamo ci sia un solo venditore, sia proprietario della risorsa, che banditore dell'asta, e più compratori interessati all'acquisto. Ogni singolo compratore riferisce la sua domanda al venditore in contemporanea a tutti gli altri, in modo che le domande siano informazioni private conosciute solo dall'interessato e dal banditore. Obiettivo di quest'ultimo è regolamentare l'asta affinchè sia un procedimento equo e senza imbrogli da parte dei partecipanti. Per questo motivo, è utile la definizione di un prezzo in base alla quantità di risorsa assegnata. Stabilita quindi una regola per l'assegnazione della risorsa, è necessaria la definizione di un meccanismo di prezzo in modo che nessuno dei partecipanti sia incentivato a mentire o imbrogliare per poter ottenere una maggiore quantità di risorsa. All'interno di questo contesto, vengono studiati diversi meccanismi di aste e in particolare il lavoro si focalizza sul meccanismo descritto da Myerson. Questo tipo di asta infatti soddisfa la proprietà chiamata *truthfulness*: nessuno dei partecipanti trae vantaggio dal mentire chiedendo di più o di meno del necessario. Viene inoltre analizzata la possibilità di collusione in questo particolare meccanismo di asta: ci si è domandati se sia o meno conveninete ai gicatori unirsi formando coalizioni. Siccome altri meccanismi come le aste al secondo prezzo, già analizzati in letteratura, sono sensibili alla collusione, è abbastanza ragionevole pensare che anche i partecipanti ad un'asta di Myerson siano propensi alla collaborazione. Attraverso l'utilizzo di simulazioni numeriche, l'ipotesi di partenza è stata confermata appurando che anche il sistema di prezzo definito da Myerson non è robusto ad accordi trasversali tra i giocatori.

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Introduction

Resource allocation is a common problem emerging when a resource has to be shared between agents that have different demands for the available resource. Some examples are: cache allocation to content providers, spectrum sharing, money division, etc. There are multiple approaches to solve the problem in the particular case in which the resource is scarce and not able to fulfill the user demands. The most well-known solutions are obtained using the classical proportional rule and Max-Min Fair rule. The first one consists simply in dividing the resourse proportinally to the demand, while the second one firstly maximizes the minimum allocation, then maximizes the second lowest and so on. Other ways to solve allocation problems are game-theoretic modelization through bankruptcy game or through an auction. In this way, the solution of the allocation problem coincides with the solution of the correspondent game. For example, we can use the nucleolus, the Shapley value or the mood value solution. The latter one derives from the tau-value and it is not a standard resolution but has some interesting fairness properties. The auction is a specific type of market wherein a given agent sells a given set of commodities to interested buyers. For this reason, it is a suitable model for allocation problems. Since usually the resource is scarce, agents are in competition to obtain a bigger fraction of the total estate and each of them has a demand that is a private information, known only by himself and the seller. In this competitive environment, each agent tries to cheat in order to obtain more resource.
In this work we are interested into analyze mechanisms that prevent agents' cheating behaviour. In partiular, a partecipant can lie about his true demand, asking for more or less resource, or can make agreements with some other agents. Classically, in order to avoid the first type of cheating, strategy-proof rules as the MMF are preferred, alternatively the resource allocation problem is designed as a truthful auction mechanisms (e.g. the Vickrey-Clarke-Groves (VCG) mechanism or the Myerson inspired auction). While classically the auction mechanism most used is the VCG one, in this work we use the Myerson's Lemma, to extend the work in [6].

Myerson's mechanism associates to each allocation rule a price such that partecipants are not incentivate to lie about their demands, in order to obtain strategy-proof one-shot allocation rules and to avoid the use of complex mechanism. This is possible thanks to the Revelation Principle [10] that guarantees an equivalence between an auction mechanism and an incomplete information resource allocation problem. Furthermore, we analyze the other possible way for users to cheat, i.e. the possibility of making agreement merging the demands. A subset of users can submit a single demand equal to the sum of their demands trying to improve their utility, that is the difference between the value of the obtained resource and the price. In particular, we are interested into numerically compare the users utility, defined by the Myerson lemma, for the most important allocation rules.
The work is structured as follows:

- In Chapter 1 we present the theoretical notions on cooperative and non-cooperative games.

- In Chapter 2 we focuse on a particular subclass of cooperative games, that is the bankruptcy game.

- In Chapter 3 we illustrate the auction mechanism. We start from the simple single-item auction, arriving to Myerson's mechanism.

- In Chapter 4 we present the pricing mechanism given by Myerson and its application considering different allocation rules. We resume the work done in [6] and we extend the analysis to the mood value solution.

- In Chapter 5 we analize another possible way that users have to cheat: demands aggregation. We define a new TU game and we study its characteristic fuction.

Chapter 1

Cooperative and Non Cooperative Games

In this chapter we present all the basic theoretical concepts and definitions related to cooperative and non-cooperative game theory, useful for the next analysis. In our work we focuse on a particular type of games called *Bankrupcy games*, belonging to the class of cooperative ones. These games consist of players claiming a resource that is scarce and it is not enough to satisfy all the requests. We are interested in the study ways to solve bankruptcy games, thus in this chapter we illustrate first the general concept of solution of a cooperative game: we define the core of a game and how it can be characterized. Moreover, we present some well known one point solutions: the nucleolus, the shapley value and the tau value.
Since bankruptcy problems are essentially allocation problems, they can be reinterpreted using auctions, models that belong to the class of non-cooperative games. Thus, at the end of the chapter, we illustrate some basic definitons related to non-cooperative models.

1.1 Introduction to Game Theory

Game theory is a branch of mathematics that deals with the analysis of the optimal decision making in a context with two or more decision-makers. Decision-makers are usually called agents or players. Game theory studies not only the choice of a single agent but also the interactions among agents' decisions: a game is an interactive decision making process. The two main assumptions at the basis of game theory are that players are *egoistic* and *rational*, meaning that they care only about their own preference, no matter what other players do. The first assumption however has no ethical meaning:

a player can decide "egoistically" that the happines of another person is his goal. The key point is that this altruistic behaviour can be satisfactory for a player whose aim is to maximize his utility function, usually indicated with u, achiving his own fulfilment. The assumption of rationality is much more complicated to explain. The crucial point is that players have preferences on outcomes, so they are able to give an order on them. For this reason, it's important to introduce a real value function u defined over the set of alternatives, representing how much a player is satisfied. This function, mentioned before, is called *utility function*. Despite the aim of a player is to maximize u, his payoff depends on other players' choice and behaviour.
Many situations can be modeled as games. There are two main classes of games: the cooperative and the non-cooperative ones. As the name indicates, the distinction is based on the possibility or not of cooperations among agents. In this work we focus our attention on the cooperative setting. We present some examples to give an idea of some possible applications of cooperative game theory.

Example (Buyers and Seller)**.** There are one seller and two potential buyers for an important indivisible good. The seller (player one) evaluates the good a. The buyers (players two and three) evaluate it b and c respectively. Suppose $a < b < c$.

Example (Children game)**.** Three players must vote a name of them. If one gets at least two votes, he win 1000 Euros. They can make agreement that are binding. If no one gets more than one vote, the 1000 Euros are lost.

Example (Glove game)**.** N players have a glove each, some of them a right glove, some other a left glove. They need to reach an agreement in order to obtain pair of gloves.

Example (Airport game)**.** Three flying companies need a new landing lane in a city. The first company needs 1 km long landing field which costs c_1. The second one needs 2 km which cost c_2, while the third one needs 3 km which cost c_3. Realistically $c_1 < c_2 < c_3$, $c_2 < 2c_1$ and $c_3 < 3c_1$. How do they share the cost?

All the examples illustrated before require cooperation or encourage players to cooperate in order to reach a comprimise. We assume that the agreements among agents are binding meaning that no player can break a pact.

1.2 Cooperative Games

It is fundamental to define the chacteristic function, meaning to define how a game is represented. Let N denote the set of players and S any subset of N. S is called *coalition*. The cardinality of these two sets is denoted by n and s respectively. Let 2^N denote the set of all possible coalitions.

Definition 1.1. The **characteristic function** of a game is a function

$$v : 2^N \to \mathbb{R}$$

such that $v(\emptyset) = 0$.

This function assignes to each coalition $S \subseteq N$ the value $v(S)$, that is the value that S can get once formed. The condition $v(\emptyset) = 0$ is a normalization condition. Recalling the previous examples:

Example (Buyers and Seller)**.** For this game a reasonable characteristic function could be:

$$v(\{1\}) = a \quad v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0 \quad v(\{1,2\}) = b \quad v(\{1,3\}) = c \quad v(N) = c$$

Example (Children game)**.** In this case, we assume v of this form:

$$v(A) = \begin{cases} 1000 \quad if \quad |A| \geq 2 \\ 0 \quad \text{otherwise} \end{cases}$$

Example (Glove game)**.** Denoting with $\{L, R\}$ the partition of N that divides players having a right glove from players having a left glove, the characteristic function is:

$$v(S) = \min\{|L \cap S|, |R \cap S|\}$$

Example (Airport game)**.** In this case, v can be written as:

$$v(S) = \min\{c_i : i \in S\}$$

The cooperative game defined by $v : 2^N \to \mathbb{R}$ is indicated by the pair (N, v). The function v is also called *side payment* or *Transferable Utility* (TU). These terms highlight the fact that the amount $v(A)$ can be freely divided among the members of A, without restrictions. Let $\mathcal{G}(N)$ be the class of all possible cooperative games with N as set of players.

Definition 1.2. A TU game is said to be:

- **Additive** if $\forall S, T \subseteq N \quad s.t. \quad S \cap T = \emptyset \quad v(S) + v(T) = v(S \cup T)$

- **Superadditive** if $\forall S, T \subseteq N \quad s.t. \quad S \cap T = \emptyset \quad v(S) + v(T) \leq v(S \cup T)$
- **Convex** if $\forall S \subseteq T \subseteq N \setminus i$, then $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$
- **Monotone** if $\forall S \subseteq T \in 2^N, v(S) \leq v(T)$
- **Cohesive** if $\forall\, S_1, \ldots, S_k \; s.t. \; S_i \cup S_j = \emptyset \; and \; \bigcup_i S_i = N$, then $\sum_i v(S_i) \leq v(\bigcup_i S_i) = v(N)$

  Some properties are correlated each other. Trivially:

- Additivity $\Rightarrow$ Superadditivity
- Superadditivity $\Rightarrow$ Cohesivity
- Convexity $\Rightarrow$ Superadditivity

Superadditive games are the most interesting ones, since the players are encouraged to aggregate and cooperate.

Remark. The examples illustrated before (buyers and seller, children game, glove game and the airport game) are all superadditive games. The airport game is also a convex game, while the children game is not.

An interesting subclass of games is singled out from the following definition.

Definition 1.3. A game (N, v) is called **simple** provided v is valued on $\{0, 1\}$, $A \subset C$ implies $v(A) \leq v(C)$ and $v(N) = 1$.

Coalitions for which $v(A) = 1$ are called *winning coalitions*. Recalling the children game in the example, it can be reinterpreted as simple game:

Example (Children game)**.** This game is equivalent to a three players-game (N, v) with v given by

$$v(A) = \begin{cases} 1 & if \quad |A| \geq 2 \\ 0 & otherwise \end{cases}$$

In this case the winning coalitions are :

$$\{1, 2\}, \{1, 3\}, \{2, 3\}, N$$

An important concept is the one of balanced game.

Definition 1.4. A **balanced map** is a function $\lambda : 2^N \setminus \emptyset \to \mathbb{R}_+$ such that $\sum_{C \subseteq N} \lambda(C) \chi_C = \chi_N$.

Here we denote with $\chi_A : N \to \{0,1\}$ the function:

$$\chi_A = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases}$$

By definition, a map is balanced when the amount received over all the coalitions containing an agent i sums up to 1.

Definition 1.5. A game is **balanced** if, for each balanced map λ, we have $\sum_{C\subseteq N, C\neq\emptyset} \lambda(C)v(C) \leq v(N)$.

Definition 1.6. A family $(S_1, \ldots, S_m)$ of coalitions is called **balanced** provided there exists $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\lambda_i > 0 \quad \forall i$ and

$$\sum_{k: i \in S_k} \lambda_k = 1 \quad \forall i \in N.$$

λ is called *balancing vector*

Example 1.1. Given N players, a partition of N is a balanced family, with balancing vector made by all ones.

Example 1.2. Consider a game with four players. The family

$$(\{1,2\},\{1,3\},\{2,3\},\{4\})$$

is balanced with balancing vector

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)$$

Example 1.3. Consider a game with three players. The family

$$(\{1\},\{2\},\{3\},N)$$

is balanced with every vector of the form

$$(p,p,p,1-p) \quad p \in (0,1)$$

Example 1.4. Consider again a game with three players. The family

$$(\{1,2\},\{1,3\},\{3\})$$

is not balanced.

Definition 1.7. A **minimal** balancing family is a balanced family such that no subfamily is balanced.

Example 1.5. Consider the case $N = \{1, 2, 3\}$. The minimal balancing families are:

$$(1,1,1,0,0,0,0) \quad \text{with balanced family} \quad (\{1\},\{2\},\{3\})$$

$$(1,0,0,0,0,1,0) \quad \text{with balanced family} \quad (\{1\},\{2,3\})$$
$$(0,1,0,0,1,0,0) \quad \text{with balanced family} \quad (\{2\},\{1,3\})$$
$$(0,0,1,1,0,0,0) \quad \text{with balanced family} \quad (\{3\},\{1,2\})$$
$$(0,0,0,0,0,0,1) \quad \text{with balanced family} \quad (N)$$
$$\left(0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0\right) \quad \text{with balanced family} \quad (\{1,2\},\{1,3\},\{2,3\})$$

1.2.1 Solutions of Cooperative Games

The Core

Given a game (N, v), a solution is a vector $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ assigning each player i a utility x_i. More in general, a set of solution vectors is called solution concept. For each cooperative game it is not possible to define a solution concept providing a reasonable outcome for all the situations that a game can model. For this reason, a solution should satisfy some requirements. Two basic requisites are:

1. $x_i \geq v(\{i\})\ \forall i \in N$ (*individual rationality*): no player should receive less than what he can get by his own.
2. $\sum_{i=1}^{n} x_i = v(N)$ (*efficiency*): the total amount should be splitted among players.

Solutions satisfying these two properties are called imputations. Formally:

Definition 1.8. Given a cooperative game (N, v), the **imputation** is a solution that satisfies individual rationality and efficiency.

We denote the set of all imputations with the symbol $\mathcal{I}(v)$. This set is large, therefore we consider subsets of $\mathcal{I}(v)$ containing significant solutions. The first subset we consider is the core which is interesting for the stability property.

Definition 1.9. Given a TU game (N, v), the **core** $\mathcal{C}(v)$ is the set of solutions such that

$$\mathcal{C}(v) = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N) \wedge \sum_{i \in S} x_i \geq v(S) \quad \forall S \subset N \right\}$$

The core, as said before, is a subset of the imputation set: $\mathcal{C}(v) \subset \mathcal{I}(v)$. While imputations are feasible solutions accepted by single players, the core contains feasible solutions accepted by all coalitions. This is due to the condition

$$\sum_{i \in S} x_i \geq v(S) \quad \forall S \subset N$$

imposed in definition 1.9. In fact, it guarantees that no player has incentive to leave the coalition S. In this sense $\mathcal{C}(v)$ is the set of feasible, efficient and stable solutions. The core of a game is not easy to compute or describe with the exception of some simple cases.

Example 1.6. Consider a game with two players $N = \{1, 2\}$ and a characteristic function given by: $v(\{1\}) = 5$, $v(\{2\}) = 5$ and $v(\{1, 2\}) = 20$.

Then the core is the set

$$\mathcal{C}(v) = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 5, x_2 \geq 5, x_1 + x_2 = 20 \right\}$$

that can be represented by the area of the red triangle in the center of the picture.

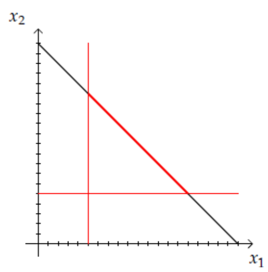

Figure 1.1: Core of the game

Example (Buyers and seller)**.** The core is described by the system:

$$\begin{cases} x_1 \geq a, \;\; x_2 \geq 0, \;\; x_3 \geq 0 \\ x_1 + x_2 \geq b, \;\; x_1 + x_3 \geq c, \;\; x_2 + x_3 \geq 0 \\ x_1 + x_2 + x_3 = c \end{cases}$$

The result is the set:

$$\mathcal{C}(v) = \{(x, 0, c - x) : b \leq x \leq c\}$$

The core of a game can be empty.

Example (Children game)**.** Using for simplicity the characteristic function given by:

$$v(A) = \begin{cases} 1 & if \quad |A| \geq 2 \\ 0 & otherwise \end{cases}$$

To guarantee stability, we should impose:

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_3 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 1 \end{aligned}$$

summing up all the tree inequalities, we get $2x_1 + 2x_2 + 2x_3 \geq 3$. Imposing the efficiency condition, we obtain

$$x_1 + x_2 + x_3 = 1.$$

The core is empty since the system has no solution.

The following theorems give a characterization of the core of a game if it satisfy some properties.

Theorem 1.1. Given a game (N, v), if it is convex then the core is always non empty $\mathcal{C}(v) \neq \emptyset$.

Theorem 1.2 (Bondareva-Shapley)**.** A TU game (N, v) has non empty core if and only if it is balanced.

This means that convexity is only a sufficient condition for the non emptiness of the core of a TU game, while the balancedness is both necessary and sufficient. Despite the fact that the Borandeva-Shapley theorem completely characterizes the set of games with non empty core, it is not always possible or feasible to check the balancedness condition. Another possible way to characterize the core is to use results from linear optimization exploiting the fact that, by definition, it can be identified by a set of linear contraints.

Given a game (N, v), we can consider the following Linear Programming (LP) problem:

$$\begin{aligned} &\min \sum_{i=1}^{n} x_i \\ &\sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N \end{aligned} \tag{1.1}$$

Theorem 1.3. The LP problem, desciribed in 1.1, has always non empty set of solutions C. The core $C(v)$ is non empty and $C = C(v)$ if and only if the value of the LP is $v(N)$.

Using the dual formulation of the problem 1.1, we have the following theorem:

Theorem 1.4. Given a game (N, v), the core $C(v)$ is not empty if and only if every vector $(\lambda_S)_{S \subseteq N}$ fulfilling the conditions

$$\begin{cases} \lambda_S \geq 0 \quad \forall S \subseteq N \\ \sum_{S \in N i \in S \subseteq N} \lambda_S = 1 \quad \forall i = 1, \dots n \end{cases} \tag{1.2}$$

verifies also

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

Proof. The LP problem 1.1 has the following matrix form

$$\begin{cases} \min \mathbf{c}^T \mathbf{x} \\ A\mathbf{x} \geq \mathbf{b} \end{cases}$$

where

- $\mathbf{c} = (1 \dots 1)^T \in \mathbb{R}^n$
- A is a $(2^n - 1) \times n$ matrix:

$$(A)_{i,j} = \begin{cases} 1 & j \in S_i \\ 0 & \text{otherwise} \end{cases}$$

- $\mathbf{b} = (v(\{1\}), v(\{2\}), \dots, v(N))^T$

The dual problem takes the form:

$$\begin{cases} \max \sum_{S\subseteq N} \lambda_S v(S) \\ \lambda_S \geq 0 \\ \sum_{S:i\in S\subseteq N} \lambda_S = 1 \quad \forall i \end{cases}$$

Since the primal has solution, the fundamental duality theorem states that also the dual has solution and there is no duality gap. Thus, the core $C(v)$ is non-empty if and only if the value V of the dual problem is such that $V \leq v(N)$. □

Usually, the dual problem is easier to solve than the primal, even if at a first reading could look uninteresting. The coefficient λ_S has the interpretation of how much in percentage a coalition represents the players. This meaning is suggested by the constraint

$$\sum_{S:i\in S\subseteq N} \lambda_S = 1 \quad \forall i$$

together with nonnegativity constraints. Hence, the theorem states that no matter the players decide their quota in the coalitions, the corresponding weighted values must not exceed the available amount of utility $v(N)$.
The geometry of the set of λ_S is simple to describe: it is the intersection of various planes with the cone made by the first octant. As result, we get a convex polytope with a finite number of extreme points that they correspond to possible and valid solutions. These extreme points are characterized by the theory throught the following theorem:

Theorem 1.5. The positive coefficient of the extreme points of the constraint set in 1.2 are the balancing vectors of the minimal balanced coalitions.

The core of a game is not easy to characterize but for superadditive games the following theorem gives a necessary and sufficient condition:

Proposition 1.6. Given a superadditive game (N, v) with $N = \{1, 2, 3\}$. The core $C(v)$ is non empty if and only if

$$v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\}) \leq 2v(N)$$

An elegant result about the non emptiness of the core is the following.

Definition 1.10. In a simple game (N, v), a player i is a **veto player** if $v(A) = 0$ for all A such that $i \notin A$.

Theorem 1.7. Let (N, v) be a simple game. Then $C(v) \neq \emptyset$ if and only if there is at least a veto player.

Proof. If there is no veto player, then for every i there is A_i such that $i \notin A_i$ and $v(A_i) = 1$. Suppose $(x_1, \ldots, x_n) \in C(v)$. It follows that

$$\sum_{j \neq i} x_j \geq \sum_{j \in A_i} x_j = 1$$

for all $i = 1, \ldots, n$. Summing up the above inequalities as i runs from 1 to n, it provides:

$$(n-1) \sum_{j=1}^{n} x_j = n.$$

This is a contraddiction since $\sum_{j=1}^{n} x_j = 1$. Conversely, any imputation assigning 0 to all the non-veto players is inside the core. $\square$

Moreover, if there is at least a veto player the core is the convex polytope having as extreme points the vectors $(0, 0, \ldots, 1, 0, \ldots, 0)$ where 1 corresponds to the position of a veto player.

Example 1.7. Consider the game (N, v) with $n = 3$ and v given by:

$$v(A) = \begin{cases} 1 & if \quad 2 \in A \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that player two is a veto player. Thus the core is not empty and it is exactly the singleton:

$$C(v) = \{(0, 1, 0)\}$$

One-point Solutions

As said before, there are many possible solution concepts for a game. They differentiate each other in terms of fairness or/and according to properties that they satisfy. Given a TU game (N, v), we introduce the most common and well known one-point solutions illustrating for each one the properties they satisfy.
The first important solution concept for coopertive games is the nucleolus. To introduce it, we had to define the excess.

Definition 1.11. Given a TU game (N, v) and an imputation $\mathbf{x} \in \mathcal{I}(v)$, the **excess** $e(A, \mathbf{x})$ of a coalition A is

$$e(A, \mathbf{x}) = v(A) - \sum_{i \in A} x_i$$

The excess is a measure of the dissatisfaction of a coalition since it is defined as the difference of what a coalition can get and what it gets with respect to the imputation. Lower is the excess, happier is the coaltion.

Definition 1.12. Given a TU game (N, v) and an imputation $\mathbf{x}$, the **lexicographic vector** attached to the imputation is the vector $\theta(\mathbf{x}) \in \mathbb{R}^{2^n-1}$ such that:

- $\theta(\mathbf{x}) = e(A, \mathbf{x})$ for some A
- $\theta_1 \geq \theta_2 \geq \dots \theta_{2^n-1}$

Definition 1.13. Given a TU game (N, v), the **nucleolus** is the solution $\nu(v)$ such that

$$\nu(v) = \{\mathbf{x} \in \mathcal{I}(v) : \quad \theta(\mathbf{x}) \leq_L \theta(\mathbf{y}) \quad \forall \mathbf{y} \in \mathcal{I}(v)\}$$

With $x \leq_L y$ we want to denote that x is greater in the lexicographic order than y, meaning that one of these two alternatives holds:

- $x = y$
- $\exists j \geq 1$ such that $x_i = y_i \quad \forall i < j$ and $x_j < y_j$

Thus, the nucleolus is the lexicographically minimal imputation. In other words, it is the imputation that minimizes the excess.

Example (Children game)**.** We have already seen that the core of this game is empty. In order to find the nucleolus, suppose $\mathbf{x} = (a, b, 1 - a - b)$ with $a, b \geq 0$ and $a + b \leq 1$. The excesses are given by

$$\begin{aligned} e(\{1,2\}) &= 1 - (a + b) \\ e(\{1,3\}) &= b \\ e(\{3,2\}) &= a \end{aligned}$$

We want to minimize the quantity: $\max\{1 - a - b, b, a\}$. The result is

$$\nu(v) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

Theorem 1.8. Let (N, v) be a TU game. If $\mathcal{I}(v) \neq \emptyset$, then the nucleolus $\nu(v)$ is a singleton.

Theorem 1.9. Let (N, v) be a TU game. If $\mathcal{C}(v) \neq \emptyset$, then $\nu(v) \in \mathcal{C}(v)$

Proof. Take $x \in C(v)$. Then $\theta_1(x) \leq 0$. Thus $\theta_1(\nu) \leq 0$. $\square$

Example (Buyers and Seller)**.** As we had already seen, the core of this game is described by the set

$$\mathcal{C}(v) = \{(x, 0, c - x) : b \leq x \leq c\}$$

For the theorem 1.9, the nucleolus belongs to the core. Take a general solution of the core $(x, 0, c - x)$ with $x \in [b, c]$. We had to find x such that all the excesses are minimized. The relevant excesses are

$$e(\{1, 2\}) = b - x \quad e(\{3, 2\}) = x - c$$

Thus

$$\nu(v) = \left(\frac{b + c}{2}, 0, \frac{c - b}{2}\right)$$

The second solution we present is the Shapley value. It is a very important solution concept, having a wide range of applications. The idea at the basis of its definition was to find a reasonable list of properties charactering the solution, meaning that the only solution fulfilling the list of properties was the proposed one.

Formally, the formula is the following one:

Definition 1.14. Given a TU game (N, v), the **Shapley value** is the solution $\phi(v)$ such that

$$\phi_i(v) = \sum_{S \in N \setminus \{i\}} \frac{s!\,(n - s - 1)!}{n!} \{v(S \cup \{i\}) - v(S)\} \tag{1.3}$$

Theorem 1.10 (Shapley theorem)**.** The Shapley value defined in 1.14 is the unique solution having the following properties:

1. *Efficiency*: $\sum_{i=1}^{n} \phi_i(v) = v(N)$
2. *Simmetry*: if $i, j \in N \quad v(A \cup \{i\}) = v(A \cup \{j\}) \quad \forall A \in 2^{N \setminus \{i,j\}} \Rightarrow \phi_i = \phi_j$
3. *Null player property*: if $i \in N \quad v(A \cup \{i\}) = v(A) \quad \forall A \in 2^N \Rightarrow \phi_i = 0$
4. *Linearity*: $\forall v, w \in \mathcal{G}(N) \Rightarrow \phi(v + w) = \phi(v) + \phi(w)$

The solution given by Shapley distributes the total amount $v(N)$ to players. The formula is made by two terms:

- The Shapley coefficient: $\frac{s!(n-s-1)!}{n!}$, that depends only on the cardinality of the coalition $S \in 2^N$

- The marginal contribution of player i to the coalition S: $v(S \cup \{i\}) - v(S)$

  The result is a weigthed sum of all marginal contributions of players.

The Shapley coefficient, appearing in the formula 1.3, has a probabilistic interpretation. Suppose the players plan to meet at a certain place at a fixed hour and suppose the order of arrival is equally likely. Moreover, suppose player i enters into coalition S if and only if he finds all the members of S and only them when arriving. The coefficient represents exactly the probability that player i enters in S.

In case of a simple games, the Shapley value assumes the form:

$$\phi_i(v) = \sum_{A \in A_i} \frac{(|A|-1)!\,(n-|A|)!}{n!}$$

where A_i is the set of the coalitions such that:

- $i \in A$
- A is a winning coalition
- $A \setminus \{i\}$ is not a winning coalition

Example (Children game)**.** Suppose we want to calculate the Shapley value for the children game. $N = \{1, 2, 3\}$ and the characteristic function is given by $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0 \quad v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(N) = 1$. Player one gives a marginal contribution of 1 to coalition $\{3\}$ and $\{2\}$, thus applying the formula we get:

$$\phi_1 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Since all players are symmetric and the Shapley value gives equal value to symmetric players, the solution is

$$\phi(v) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

In simple games, Shapley value assumes the meaning of measuring the fraction of power every player had. The Shapley value is an example of the so called *power indeces*: it takes into account marginal contributions of player i to any coalition S, indicated as $m_i(v, S)$, and weights them according to a probabilistic coefficient. This can be generalized:

Definition 1.15. A power index ψ is called a **probabilistic index** provided for each player i there exists a probability measure p_i on $2^{N\setminus\{i\}}$ such that

$$\psi_i(v) = \sum_{S \in 2^{N\setminus\{i\}}} p_i(S)\ m_i(v, S)$$

In the set of the probabilistic indeces, there is an important subfamily, that one of the *semivalues*. For this subclass, the coefficient $p_i(S)$ does not depend from the player i but only from the size of the coalition S:

$$p_i(S) = p_i(s)$$

Furthermore, if $p_i(s) > 0$ for all s, then the semivalue is called *regular semivalue*. It is easy to see that the Shapley value is a regular semivalue with

$$p(s) = \frac{1}{n\binom{n-1}{s}}$$

Another well known regular semivalue is the **Banzhaf index** β. This is defined as

$$\beta_i = \sum_{S \in 2^{N\setminus\{i\}}} \frac{1}{2^{n-1}}(v(S) - v(S \setminus \{i\}))$$

In this case the coefficient $p(s)$ is constant: it is assumed that player i has the same probability to joint any coalitions. Banzhaf value satisfies all properties characterizing the Shapley value except one: the efficiency.

Example (Children game)**.** Suppose we want to calculate the Banzhaf index for the children game. $N = \{1, 2, 3\}$ and the characteristic function is given by $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0 \quad v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(N) = 1$. Recalling that player one gives a marginal contribution of 1 to coalition $\{3\}$ and $\{2\}$. We get:

$$\beta_1 = \frac{1}{3}$$

Since all players are symmetric:

$$\beta(v) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

Thus in this case, Banzhaf index coincides with the Shapley value.

A third solution we present is the tau-value solution. It was introduced the first time by Tijs in 1981. This is based on the idea of a compromise between an upper and a lower value for each player in the game. Given the TU game (N, v), we can define as $\mathbf{m}(v)$ the minimum right payoff, that is the vector with entries equal to

$$m_i(v) = v(\{i\})$$

and we can indicate with $\mathbf{M}(v)$ the utopia point, that is the marginal contribution of player i to the grand coalition N that utopistically could be assigned to i:

$$M_i(v) = v(N) - v(N \setminus \{i\})$$

It makes sense to consider a compromise between the lower and the upper vectors if the following two conditions are satisfied:

1. $\mathbf{m}(v) \leq \mathbf{M}(v)$
2. $\sum_i m_i(v) \leq v(N) \leq \sum_i M_i(v) \quad \forall i \in N$

A game (N, v) satisfying these two conditions is said to be *quasi balanced.*

Definition 1.16. The τ-value is defined as a convex combination of the minimum right payoff and the utopia point:

$$\tau(v) = \alpha\mathbf{m}(v) + (1-\alpha)\mathbf{M}(v) \quad \alpha \in [0, 1]$$

Note that it is defined only for quasi balanced games. This class of games contains all games with non-empty core.

1.3 Non Cooperative Games

To proceed in our analysis, we need to formally define some basic concepts related to the non-cooperative games. Since our aim is to analyze deeply the auction mechanism, we formalize the main concepts referring to *one-shot simultaneous move games.* This is a specific type of non-cooperative game in which all players choose simultaneously an action from the set of all possible actions.

One-shot simultaneous move games consist of a set of players $N = \{1, 2, \ldots, n\}$, each of them has his own set of possible strategies S_i. A strategy must not be confused with a move: with the term strategy we indicate any options that a player chooses with respect to other players' actions.

Example 1.8. Consider for simplicity the known game "rock, scissor, paper". The game is described by the following bimatrix:

	Rock	Scissor	Paper
Rock	0,0	1,-1	-1,1
Scissor	-1,1	0,0	1,-1
Paper	1,-1	-1,1	0,0

where the first player selects the row, while the second player the column. A strategy is defined by a vector

$$\mathbf{s} = (\text{Scissor, Paper, Rock})$$

specifying an action for each other players' possible move.

Player i has a set of possible strategies denoted with S_i. Let $S = \times_i S_i$ denote the set of all possible ways in which players can pick strategies. In order to specify the game, it is necessary to give each player an ordere of preference on the set of possible outcomes. As seen in the previous section, the easiest way to specify an order is to define a utility function on the set of alternatives. Let $u_i(s_i, \mathbf{s}_{-i})$ denote the utility of player i when the strategy s_i is played and the strategies of all other players are described by the vector $\mathbf{s}_{-i}$. Using this notation, we can define the concept of dominant strategy solution.

Definition 1.17. A solution $\bar{\mathbf{s}}$ is a **weakly dominant strategy solution** if for each player i, for each strategy s_i and each startegy vector $\mathbf{s}_{-i} \in S_{-i}$, we have that

$$u_i(\bar{s}_i, \mathbf{s}_{-i}) \geq u_i(s_i, \mathbf{s}_{-i}) \tag{1.4}$$

The definiton of strong dominant strategy solution is the same but has a strict inequality in 1.4, formally:

Definition 1.18. A solution $\bar{\mathbf{s}}$ is a **strongly dominant strategy solution** if for each player i, for each strategy $s_i \neq \bar{s}_i$ and each startegy vector $\mathbf{s}_{-i} \in S_{-i}$, we have that

$$u_i(\bar{s}_i, \mathbf{s}_{-i}) > u_i(s_i, \mathbf{s}_{-i}) \tag{1.5}$$

Example 1.9. Consider the well known prisoner's dilemma. Two prisoners are on a trial for a crime and each one faces a choice of confessing to the crime or remaining in silent. If they both remain silent, the authorities will not be able to prove charges against them and they will both serve a short prisoner term of two years. If only one of them confesses, his term will be reduced to one year and he will be used as a witness against the other, who in turn will get a sentence of five years. Finally if they both confess, the both will get a small break for cooperating and will have to serve prison sentence of four years each.

We can summarize the game with the following bimatrix:

	Confess	Silent
Confess	4,4	1,5
Silent	5,1	2,2

It is easy to see that they have a strictly dominant strategy that it consists in confessing. The unique stable solution, assuming both prisoners rational, is the one with outcome $(4,4)$.

It is important to notice that a dominant strategy solution may not give an optimal payoff to any player. This is the case illustrated in the example 1.9: it is possible to improve the payoffs of all players simultaneously achieving the more appealing outcome $(2,2)$. Having a single dominant strategy for each players is an extremely stringent requirement for a game and very few games satisfy it. Thus, we need to seek a less stringent solution concept. A desirable game-theoretic solution is one in which players act in accordance with their incentives, maximizing their own payoff. This idea is captured by the notion of Nash equilibrium.

Definition 1.19. A strategy vector $\mathbf{s} \in S$ is said to be a **Nash equilibrium** if for all players i and each alternate strategy $\bar{s}_i \in S_i$, we have that

$$u_i(s_i, \mathbf{s}_{-i}) \geq u_i(\bar{s}_i, \mathbf{s}_{-i})$$

In other words, a strategy is a Nash equilibrium if no player i can change his chosen strategy s_i to $\bar{s}_i$ and improve his payoff, assuming all other players' strategy fixed. Such a solution is self-enforcing in the sense that it is in every player's interest to persist in his strategy once players are playing such a solution. Clearly a dominant strategy solution is a Nash equilibrium.

The next chapters start from the definitions illustrated above. We first analize a particular class of cooperative games that are the bankrupcty games and then we proceed illustrating the auction mechanism.

Chapter 2

Bankruptcy Games

Bankruptcy games belong to a specific class of games modeling situations in which a number of agents claim a certain resource that cannot satisfy the total demand. They find many application in reality describing several situations in which a limited resource has to be shared. Some examples are given by sharing internet data, manage the cache of a device, distributing the money between creditors after a bank failure etc. Bankruptcy games can be solved in many different ways. One is to use the allocation method.

In this chapter, we first illustrate the mathemathical model, then we differentiate bankruptcy games from bankruptcy problems. Finally, we present three main solutions that are used in next analysis: the proportional rule, the adjusted proportional rule and the constrained equal award rule. For each of them we give the main properties and a characterization.

2.1 Mathematical model

A bankruptcy problem is defined by the pair $(\mathbf{c}, E) \in \mathbb{R}_+^n \times \mathbb{R}$, where $\mathbf{c}$ is the vector of the demands and E is the total amount of avaiable resource. Furthermore, by definition, the following condition holds

$$\sum_i c_i > E > 0$$

in order to model a non trivial bankruptcy situation. In fact, if the sum of all claims is less than the total estate E, the solution consists in giving all claimants a quantity equal to their demand. If $E = 0$ meaning that there is no resource available, all claimants receive 0.

Let $\mathbb{B}^N$ denotes the class of all bankrupticy problems with $N = \{1, 2, 3, \ldots, n\}$ as set of players. A division rule is a function $f : \mathbb{B}^N \rightarrow \mathbb{R}^n$ assigning each player $i \in N$ a nonnegative real value. In particular, the division rule $f(\mathbf{c}, E) = (x_1, x_2, x_3, \ldots, x_n)$, called also allocation rule, provides solutions satisfying properties as the following ones:

- *Efficiency*: $\sum_{i=1}^{n} x_i = E$
- *Demand boundendess*: $x_i \leq c_i \quad \forall i \in N$
- *Individual rationality*: $x_i \geq 0 \quad \forall i \in N$

Bankruptcy problems are not necessarily cooperative games and can be solved without a game-theoretical approach. One method, for example, is to use a proportional division rule, dividing the estate proportionally to the claim. This means that each agent receives $x_i = \frac{c_i}{\sum_j c_j} E$. The proportional allocation rule is not a game theoretical method since it is not invariant under strategic equivalence. To better understand what it means, we can consider the following example.

Example 2.1. Consider two bankruptcy problems with two claimants:

- First problem: $(\mathbf{c}, E) = ((60, 40), 60)$
- Second problem: $(\mathbf{c}, E) = ((80, 40), 80)$

The proportional rule gives $\mathbf{x} = (36, 24)$ and $\mathbf{x} = (53.3\bar{3}, 26.6\bar{6})$ respectively. The claim of the second player is equal in both problems $c_2 = 40$ and one may argue that the resource allocated to him should be equal. It is cleaar that the proportional rule do not give the same amount to player two. Thus, it is not invariant under strategic equivalence.

2.1.1 Division Rules

As shown in the example 2.1, the proportional rule could not be the best choice to solve a bankruptcy problem. However, it is the most used and known one since it is the simplest way to divide a resource.
We proceed in the analysis illustrating the most common division rules. The first one, already mentioned, is the proportional rule. Its formal definition is the following:

Definition 2.1. Given a backruptcy problem $(\mathbf{c}, E) \in \mathbb{B}^N$, the **proportional rule P** is defined as $P(\mathbf{c}, E) = \lambda \mathbf{c}$ where the parameter λ is chosen in such a way $\sum_{i \in N} \lambda c_i = E$

This rule is the easiest one since it is simple to understand and to apply in many practical situations.

Example 2.2. Consider the problem with three claimants:

$$(\mathbf{c}, E) = ((3, 5, 7), 10)$$

The parameter λ can be computed as $\lambda = \frac{E}{\sum_i c_i} = \frac{2}{3}$. The proportional allocation results multiplying λ and the vector $\mathbf{c}$:

$$\mathbf{x} = \frac{2}{3}(3, 5, 7) = \left(2, \frac{10}{3}, \frac{14}{3}\right) \simeq (2, 3.3, 4.7)$$

A modification of the proportional rule is the so called *adjusted proportional.* This rule takes into account the minimal right of an agent. Formally, for each bankruptcy problem $(\mathbf{c}, E) \in \mathbb{B}^N$ and each $i \in N$, let

$$m_i(\mathbf{c}, E) = \max\left\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\right\}$$

be the minimal right of claimant i, that is the minimum that he can asks after all other claimants have been satisfied. Denote with $\mathbf{m}(\mathbf{c}, E)$ the vector in $\mathbb{R}^n$ having as entries the minimum rights of agents.

Definition 2.2. Given a backruptcy problem $(\mathbf{c}, E) \in \mathbb{B}^N$, the **adjusted proportional rule A** is defined as

$$A(\mathbf{c}, E) = \mathbf{m}(\mathbf{c}, E) + P\left(\left(\min\left\{c_i - m_i(\mathbf{c}, E), E - \sum_j m_j(\mathbf{c}, E)\right\}\right)_{i \in N}, E - \sum_j m_j(\mathbf{c}, E)\right)$$

To better understand how calculate the adjust proportional value using the definition, we report some examples.

Example 2.3. Consider the same problem of the example 2.2:

$$(\mathbf{c}, E) = ((3, 5, 7), 10)$$

The vector of minimum rights is $\mathbf{m}(\mathbf{c}, E) = (0, 0, 2)$. The adjusted proportional gives

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mathbf{P}\left(\begin{pmatrix} \min\{3 - 0, 8\} \\ \min\{5 - 0, 8\} \\ \min\{7 - 2, 8\} \end{pmatrix}, 10 - 2\right) = \mathbf{m} + \mathbf{P}(\bar{\mathbf{c}}, \bar{E})$$

We want to find the proportional allocation for the new problem $(\bar{\mathbf{c}}, \bar{E}) = ((3,5,5),8)$. The new parameter λ is given by $\frac{8}{13}$ and the solution is $\mathbf{P} = \left(\frac{24}{13}, \frac{40}{13}, \frac{40}{13}\right)$. The final result for the adjusted proportional rule is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{24}{13} \\ \frac{40}{13} \\ \frac{40}{13} \end{pmatrix} = \begin{pmatrix} \frac{24}{13} \\ \frac{40}{13} \\ \frac{66}{13} \end{pmatrix} \simeq \begin{pmatrix} 1.8 \\ 3.1 \\ 5.1 \end{pmatrix}$$

From the comparison of this result with the one in the example 2.2, the adjusted proportional gives a different outcome with respect the simple proportional rule. In fact, it considers the *type* of each claimant, meaning that it takes into account if player i's demand is lower or greater than the estate E, and his minimum right.

For backruptcy problems in which all claimants have minimum right equal to zero and in which all the demands are lower than the estate (called *zero-normalized problems*), the adjusted proportional reduces to the proportional rule.

Example 2.4. Consider the bankruptcy problem $(\mathbf{c}, E) = ((4,6,6),10)$. It is easy to see that is a zero normalized problem:

1. All players' claims are lower than E: $c_i < E \quad \forall i \in N$
2. The minimum right is equal to zero for all players, since $4+6 = 10$ and $6+6 > 10$.

With some calculation, we can verify that

$$\mathbf{P}(\mathbf{c}, E) = \mathbf{A}(\mathbf{c}, E) = \left(\frac{5}{2}, \frac{15}{4}, \frac{15}{4}\right) = (2.5, 3.75, 3.75)$$

Another classical allocation rule, especially used in networking application, is the one which gives every claimant the same amount until his demand is not satisfied and the estate is not finished.

Definition 2.3. Given a backruptcy problem $(\mathbf{c}, E) \in \mathbb{B}^N$, the **constrained equal award rule CEA** is defined as $CEA(\mathbf{c}, E) = \min\{\lambda, c_i\}$ where the parameter λ is chosen in such a way that $\sum_{i \in N} \min\{\lambda, c_i\} = E$.

This rule is known in networking with the name of *Max-Min Fair* allocation. Re-ordering the demands in a non decreasing order $c_1 \leq c_2 \leq c_3 \leq \ldots \leq c_n$, the Max-Min allocation for player i can be also defined as

$$MMF_i(\mathbf{c}, E) = \min\left(c_i, \frac{E - \sum_{j=1}^{i-1} MMF_j(\mathbf{c}, E)}{n - i + 1}\right)$$

Example 2.5. Consider the same problem of the example 2.2:

$$(\mathbf{c}, E) = ((3,5,7), 10)$$

First, the constrained equal award rule assignes 3 to all players $\mathbf{x} = (3,3,3)$. We obtain the problem $(\bar{\mathbf{c}}, \bar{E}) = ((0,2,4), 1)$. The first claimant is fully satisfied, so the CEA rule assignes 0.5 to the other two agents. Thus, the solution is given by the vector $\mathbf{x} = (3, 3.5, 3.5)$.

As we can see from the example, the constrained equal award protects players having a lower demand. An easy interpretation of the proportional and the CEA allocation can be given through the idea of communicating vessels.

The proportional rule is the most intuitive one, so, if we imagine the resource E as a liquid in a tank, we have that each player is represented by a container whose section is equal to the demand c_i. All the containers have the inferior basis at the same level.

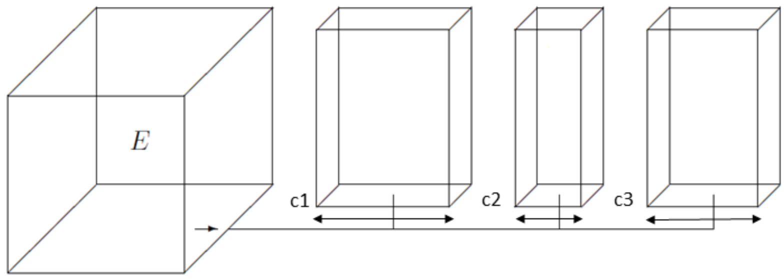

Figure 2.1: Interpretation of the proportional using communicating vessels

Consider the constrained equal award rule. Each claimant can be represented by a container with unitary section but having height equal to the demand c_i. As for the proportional rule, they have the inferior basis at the same level.

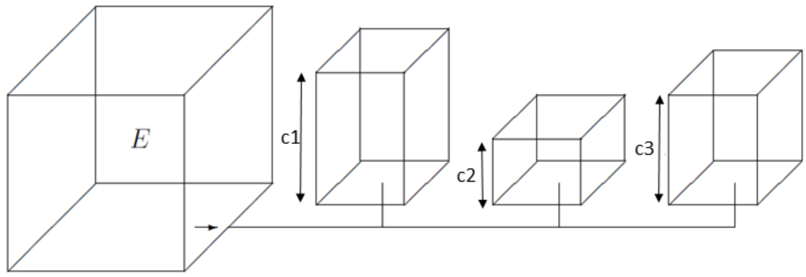

Figure 2.2: Interpretation of the CEA rule using communicating vessels

2.2 Game Theoretic Division Rules

We can model bankruptcy problems as bankruptcy game, solving them with classical solution of TU games. In order to fully describe a bankrupcty game, we should define the characteristic function $v : 2^N \to \mathbb{R}$. There are two natural ways to define it starting from the problem $(\mathbf{c}, E)$: the first one is called *pessimistic* since it consists in assigning to a coalition S the remaining estate after all players in $N \setminus S$ have been satisfied; the second one is called *optimistic* and it gives to the coalition S a quantity equal to the demand if the latter is less than the estate E.
Formally, the pessimistic characteristic function is defined by

$$v_P(S) = \max\left(0, E - \sum_{i \in N \setminus S} c_i\right)$$

while the optimistic one is given by

$$v_O(S) = \min\left(E, \sum_{i \in S} c_i\right)$$

The pessimistic description is more realistic then the optimistic one. To better understand the reason, we give the following example:

Example 2.6. Consider the problem $(\mathbf{c}, E) = ((6, 4), 8)$.

The pessimistic function gives

$$v(\{1\}) = 4, v(\{2\}) = 2, v(\{1, 2\}) = 8$$

While, the optimistic one gives

$$v(\{1\}) = 6, v(\{2\}) = 4, v(\{1,2\}) = 8$$

If player one and player two do not coalize, the optimistic description gives each one a quantity equal to the demand but the sum $c_1 + c_2$ is greater than the available resource $E = 8$.

A bankruptcy problem can be modeled as a TU game having the characteristic function v_P defined above. An important property of these type of games is that they are convex.

Theorem 2.1. Bankruptcy games are convex.

Proof. Let (N, v) be a bankruptcy game. Let $S \subseteq T \subseteq N \setminus \{i\}$. Recalling the definition of convex game, we want to show that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

or equivalently that

$$v(S \cup \{i\}) + v(T) \leq v(T \cup \{i\}) + v(S).$$

For all $C \subseteq N$, we donete $c(C) = \sum_{j \in C} c_j$, then we can write:

$$E - \sum_{j \in N \setminus C} c_j = E - \sum_{j \in N} c_j - \sum_{j \in C} c_j = E - c(N) + c(C).$$

Let $\Delta = E - \sum_{j \in N} c_j = E - c(N)$. We have $E - \sum_{j \in N \setminus C} c_j = \Delta + c(C)$. First, observe that $\forall (x, y) \in \mathbb{R}^2$, $\max\{0, x\} + \max\{0, y\} = \max\{0, x, y, x+y\}$.

$$\begin{aligned} v(S \cup \{i\}) + v(T) &= \max\left\{0, E - \sum_{j \in N \setminus (S \cup \{i\})} c_j\right\} + \max\left\{0, E - \sum_{j \in N \setminus T} c_j\right\} \\ &= \max\{0, \Delta + c(S) + c_i\} + \max\{0, \Delta + c(T)\} \\ &= \max\{0, \Delta + c(S) + c_i, \Delta + c(T), 2\Delta + c(S) + c(T) + c_i\} \end{aligned}$$

In the same way

$$v(T \cup \{i\}) + v(S) = \max\{0, \Delta + c(T) + c_i, \Delta + c(S), 2\Delta + c(S) + c(T) + c_i\}$$

We can notice that since $S \subseteq T$, $c(S) \leq c(T)$. Thus

$$\max\{0, \Delta + c(T) + c_i, \Delta + c(S), 2\Delta + c(S) + c(T) + c_i\} =$$
$$\max\{0, \Delta + c(T), \Delta + c(T) + c_i, 2\Delta + c(S) + c(T) + c_i\}$$

We also have

$$\Delta + c(S) + c_i \leq \Delta + c(T) + c_i$$

$$\Delta + c(T) \leq \Delta + c(T) + c_i$$

It follows that:

$$\max\{0, \Delta + c(S) + c_i, \Delta + c(T), 2\Delta + c(S) + c(T) + c_i\}$$
$$\leq \max\{0, \Delta + c(T) + c_i, 2\Delta + c(S) + c(T) + c_i\}$$

which proves that $v(S \cup \{i\}) + v(T) \leq v(T \cup \{i\}) + v(S)$

$\square$

As consequence, the core of these games is always non empty by theorem 1.1.
The crucial point is to establish how the total resource should be splitted among players, meaning find a solution of the TU game (N, v) associated. Curiel et al. in [1] call a division rule f for a bankruptcy problem a *game theoretical division rule* if it is possible to construct a solution concept F such that $f(\mathbf{c}, E) = F(v_{(\mathbf{c},E)})$ for all bankruptcy problems $(\mathbf{c}, E)$. Let $\mathbf{c}^T$ denote the vector having as entries $c_i^T = \min\{c_i, E\}$. Since the game corresponding to $(\mathbf{c}, E)$ and $(\mathbf{c}^T, E)$ is the same, a necessary conditions for a division rule f to be a game theoretic division rule is that $f(\mathbf{c}, E) = f(\mathbf{c}^T, E)$. Curiel et al. proved that is also a sufficient condition:

Theorem 2.2. A division rule f for a bankrupty problem $(\mathbf{c}, E)$ is a game theoretic division rule if and only if $f(\mathbf{c}, E) = f(\mathbf{c}^T, E)$.

As we have already noticed, the proportional rule is not a game theoretical division rule. Conversely, the Max-Min allocation (or CEA) is a game theoretic division rule. In fact, $CEA_i = \min\{c_i, \alpha\}$ where α is determined by imposing efficiency. It follows that $\alpha \leq E$, hence $CEA(\mathbf{c}^T, E) = \min\{c_i, E, \alpha\} = \min\{c_i, \alpha\} = CEA(\mathbf{c}, E)$.
The adjusted proportional rule is also a game theoretic division rule (see the proof given by Curiel et al. in [1]). Indeed can be shown that, for a banckruptcy problem, the adjusted proportional yields the τ-value for the

corresponding bankruptcy game [1]. Given a bankruptcy game (N, v), we can give an easier formula for the τ-value:

$$x_i = min_i + \frac{E - \sum_i min_i}{\sum_i max_i - \sum_i min_i}(max_i - min_i)$$

where $min_i = v(\{i\})$ is the minimal right and $max_i = v(N) - v(N \setminus \{i\})$ is the maximal right for player i. Following the article [4], we call this allocation rule **mood value**.

An interesting fact is that the core of a bankruptcy game (N, v), having as characteristic function $v(S) = \max\Big(0, E - \sum_{i \in N \setminus S} c_i\Big)$, coincides with the set of all admissible solutions.

Proposition 2.3. Let $(\mathbf{c}, E) \in \mathbb{B}^N$ be a bankruptcy problem and (N, v) be the associated bankrupcty game with the pessimistic characteristic function. Then, $x \in \mathcal{C}(v)$ if and only if we have that:

$$\begin{cases} \sum_{i \in N} x_i = E \\ 0 \leq x_i \leq c_i \quad \forall i \in N \end{cases}$$

Proof. Proof of the only if part. The first condition is equal to the efficiency condition. For the second one, we have $\forall i \in N$, $0 \leq v(\{i\}) \leq x_i$ and $E - x_i = \sum_{j \in N \setminus \{i\}} x_j \geq v(N \setminus \{i\}) \geq E - c_i$ which implies $x_i \leq c_i$.

Proof of the if part. The efficiency condition is satisfied. Moreover for every possible coalition $S \in 2^N$, we have two possibilities:

1. $v(S) = 0 \leq \sum_{i \in S} x_i$
2. $v(S) = E - \sum_{i \in N \setminus S} c_i \leq E - \sum_{i \in N \setminus S} x_i \leq \sum_{i \in S} x_i$

□

From this theorem, it follows that for a bankruptcy problem $(\mathbf{c}, E)$ every game theoretical division rule provides allocations inside the core of the associated cooperative game. Hence, given a problem $(\mathbf{c}, E)$, the constrained equal award rule and the adjusted proportional provide outcomes belonging to the core $\mathcal{C}(v)$ of the game (N, v).

[1] In the specific case of bankruptcy games, the τ-value is uniquely determined by choosing the parameter α in such a way results an efficient solution

2.3 Rules' Properties

We have seen three division rules: the proportional rule, that is the most natural way to divide a resource, the adjusted proportional and the constraint equal award, that are based on the idea of equity. These rules differentiate each other for the properties that they satisfy. Specifying which property a rule satisfies is an important starting point to understand which one use to model a situation.

2.3.1 Characterization

Referring to the work of Herrero and Villar (2001) [5] and to the work of Curiel (1987) [1], in this section we present the most useful properties characterizing the division rules seen so far. Some properties may be required from allocation rules. Probably, the most basic requirement is the equity: agents with identical claims should be treated identically. Formally:

Property 2.1 (**Equal treatment of equals**)**.** For all $N \in \mathcal{N}$, all $(\mathbf{c}, E) \in \mathbb{B}^N$ and all $i, j \in N$ we have that $c_i = c_j$ implies

$$f_i(\mathbf{c}, E) = f_j(\mathbf{c}, E)$$

Equal treatment of equals imposes impartiality, since it establishes that all agents with the same claims will receive the same amount. It excludes differentiating agents on the basis of names, gender, religion etc. A second requirement is that rules should be invariant with respect changements in scale, meaning that division rules do not depend on the unit of measure.

Property 2.2 (**Scale invariance**)**.** For all $N \in \mathcal{N}$, all $(\mathbf{c}, E) \in \mathbb{B}^N$ and all $\gamma > 0$ we have that

$$f(\gamma\mathbf{c}, \gamma E) = \gamma f(\mathbf{c}, E)$$

We can observe that the scale invariance property implies that the size of the estate and the claims are not important. Thus, we cannot distinguish analytically between a change of measurement units and a proportional change in the estate and the claims. A third property states that a banckruptcy problem can be solved with two steps which consist in solving two partial bankruptcy problems.

Property 2.3 (**Composition**)**.** For all $N \in \mathcal{N}$, all $(\mathbf{c}, E) \in \mathbb{B}^N$ and all $E_1, E_2 \in \mathbb{R}_+$ such that $E = E_1 + E_2$, we have:

$$f(\mathbf{c}, E) = f(\mathbf{c}, E_1) + f(\mathbf{c} - f(\mathbf{c}, E_1), E_2)$$

The first corresponds to a problem with initial claims equal to $\mathbf{c}$ and a fraction E_1 of the estate; the second problem is made out of the outstanding claims and the reminder estate. The fourth property is the following one:

Property 2.4 (Path independence). For all $N \in \mathcal{N}$, all $(\mathbf{c}, E) \in \mathbb{B}^N$ and all $E' > E$, we have that

$$f(\mathbf{c}, E) = f(f(\mathbf{c}, E'), E)$$

This means that if we first solve the problem $(\mathbf{c}, E')$ and then, using as claims the result obtained, we solve a problem with estate E less than E', the solution coincides with the solution of the original problem $(\mathbf{c}, E)$.
It is easy to see that if a rule satisfies either composition or path-independence, it is monotonic with respect to the estate. That is:

Property 2.5 (Monotonicity). For all $N \in \mathcal{N}$, for any two problems $(\mathbf{c}, E)$ and $(\mathbf{c}, E') \in \mathbb{B}^N$, $E \leq E'$ implies that

$$f_i(\mathbf{c}, E) \leq f_i(\mathbf{c}, E')$$

for all $i \in N$

The property of consistency links the solution of a problem for a given society N with the solutions of the problems corresponding to its sub-societies. Denoting with S any proper subset of N and with $c_S = (c_i)_{i \in S}$, we have the following formal definition:

Property 2.6 (Consistency). For all $N \in \mathcal{N}$, for all $S \subset N$ and all $(\mathbf{c}, E, N) \in \mathbb{B}$, all $i \in S$, we have that

$$f_i(\mathbf{c}, E, N) = f_i(\mathbf{c}_S, \sum_{i \in S} f_i(\mathbf{c}, E, N), S)$$

for all $i \in N$

The property of consistency is related to stability. In fact, it prevents subgroups of agents to renegotiate once there is a solution proposed for the society.
By the theorem stated in the article [5], the proportional rule and the constrained equal award rule satisfy simultaneously equal treatment of equals, scale invariance, composition, path-independence and consistency. Thus, they are monotonic rules. It is easy to see that the adjusted proportional satisfies only scale invariance and equal treatment of equals since it allocates resource according to the type of a player, meaning if he demands more or

less than the state. Even if it does not satisfy composition and path independence, it can be proved that the adjusted proportional rule is monotonic (see [1] for the proof).
After enumerating five common properties between the proportional and the CEA rule, we present some other characterizing each single rule. The CEA rule gives priority in the distribution of the resource to agents with smaller claims, while the proportional does not give any priority in the distribution.

The first property we illustrate represents principles of claims enforceability. It focuses on the case in which there is some agent whose individual claim exceeds the available estate. It proposes to scale down this unfeasible claim to reality. Formally:

Property 2.7 (Independence of claim truncation). For all $N \in \mathcal{N}$, for all $(\mathbf{c}, E) \in \mathbb{B}^N$, we have

$$f(\mathbf{c}, E) = f(\mathbf{c}^T, E)$$

where $c_i^T = \min c_i, E$ for all $i \in N$.

This property establishes that if an individual claim exceeds the total to be allocated, the excess claim should be considered irrelevant. We have the following theorem characterizing the CEA rule:

Theorem 2.4 (Dagan 1996). For all $N \in \mathcal{N}$ the constrained equal awards rule is the only rule satisfying equal treatment of equals, composition and independence of claims truncation.

To characterize the proportional rule, we need to define the concepts of duality and self-duality. These properties introduce symmetry in the behaviour of the solution with respect to awards and losses.

Definition 2.4. For all $N \in \mathcal{N}$ and all $(\mathbf{c}, E) \in \mathbb{B}^N$, given a division rule f, its dual is defined as

$$f^*(\mathbf{c}, E) = \mathbf{c} - f(\mathbf{c}, L)$$

where $L = \sum_{i=1}^n c_i - E$.

Property 2.8 (Self duality). For all $N \in \mathcal{N}$, for all $(\mathbf{c}, E) \in \mathbb{B}^N$, we have $f(\mathbf{c}, E) = f^*(\mathbf{c}, E)$.

Even for the proportional, we have a theorem characterizing it:

Theorem 2.5 (Young 1988). For all $N \in \mathcal{N}$ the proportional rule is the only rule satisfying equal treatment of equals, composition and self duality.

Finally, we consider the adjusted proportional rule. The first property characterizing it is called composition from minimal right and it guarantees each player his minimal right m_i. This property is a special case of composition and the formally it is defined as:

Property 2.9 (Composition from minimal rights). For all $N \in \mathcal{N}$, for all $(\mathbf{c}, E) \in \mathbb{B}^N$, we have

$$f(\mathbf{c}, E) = m(\mathbf{c}, E) + f(\mathbf{c} - m(\mathbf{c}, E), E - \sum_{i \in N} m_i(\mathbf{c}, E))$$

where $m_i(\mathbf{c}, E) = \max\left\{0, E - \sum_{i \neq j} c_j\right\}$.

A second property is called the additivity of claims property.

Property 2.10 (Additivity of claims). For all $N \in \mathcal{N}$, for all $(\mathbf{c}, E) \in \mathbb{B}^N$, a division rule f is said to satisfy addivity of claims if for every zero-normalized simple claims bankruptcy problem $(\mathbf{c}, E) = ((c_1, c_2, c_3, \ldots, c_n), E)$ with changes by splitting up c_i in $c_{i,1}, c_{i,2}, \ldots, c_{i,k}$ into a bankruptcy problem

$$(\mathbf{c}', E) = ((c_1, \ldots, c_{i-1}, c_{i,1}, c_{i,2}, \ldots, c_{i,k}, c_{i+1}, \ldots, c_n), E)$$

we have that

$$f_j(\mathbf{c}', E) = f_j(\mathbf{c}, E) \quad \text{for every} \quad j \in N \setminus \{i\}$$

Suppose that one of the claimants dies leaving behind parts of his claim to different heirs. These heirs become new claimants, each one claiming the part of the original claim he received. Their claims together sum up to the original claim. The additivity property states that if the bankruptcy problem is a zero-normalized simple claims problem, nothing changes for the other claimants. This is called even *strategy-proofness* and it is satisfied also by the CEA rule. Finally, it is trivial to prove that the adjusted proportional is a simmetric rule and it satisfies equal treatment of equals. The following theorem characterizes the adjusted proportional rule:

Theorem 2.6. The adjusted proportional rule is the unique game theoretic division rule for bankruptcy problems which satisfies the composition from minimal right, equal treatment of equal and additivity of claims.

We sum up all the properties in a table 2.1.

Property	Prop.	Adj. Prop.	CEA
Equal treatment of equals	✓	✓	✓
Scale invariance	✓	✓	✓
Composition	✓		✓
Path independence	✓		✓
Monotonicity	✓	✓	✓
Consistency	✓		✓
Composition from minimal right		✓	
Additivity of claims		✓	✓
Independence of claim truncation			✓
Self duality	✓		

Table 2.1: Table summarizing the properties

Chapter 3

Auction Design

The best known type of auction is the one we see in movies, where there is an auctioneer selling a good and there are many buyers trying to win it progressively increasing the bid. This is the simplest model: it is a single-item auction, since there is one indivisible good that has to be allocated, and we call the mechanism *english auction*. Nowadays there are many types of auctions, for example are used even auctions in which the winner offers less then the others. In this chapter we present the most known type of auction, starting from the simple single item auction and arriving to more complex mechanisms. We also give the mathematical description and we show many applications of this model. In particular, the auction can be used to solve bankruptcy problems, extending the usual single-item auction. In this case, the seller has a resource that should be divided among the bidders that propose him a valuation of the resource. We start our analysis with a general introduction on mechanism design and then we introduce the auction model. From the single-item, we arrive to more complex mechanisms as the second price auction.

3.1 Introduction to Mechanism Design

Mechanism design is the subfield of microeconomics and game theory that deals with the implementation of a good system to solve problems involving multiple self-interested agents, assuming that they act rationally in a game theoretic sense. In recent years mechanism design has found many important applications. Some examples are scheduling and resource allocation problems. In general, the rules are given and the goal is to find a possible outcome of the game, while here we face the inverse problem: design the rules in order to achieve an appealing result. The main issue is to create incentives in order to extract the relevant private information from the players to reach a socially efficient outcome. The goals of mechanism design are called with the abstract term of *social choice* that is simply an aggregation of participants' preferences toward a single joint decision. We give some basic examples:

- *Elections.* In political elections each voter has his own preference among many candidates. The outcome of an election is a single social choice.
- *Auctions.* Consider an auction involving multiple buyers and a single sellers. Its rules define the social choice that in this specific case is the identity of the winner.
- *Market.* Generally in reality there are multiple buyers and multiple sellers. Each participant has his own preferences but the outcome is a single social choice that is the allocation of money and goods.

In order to better understand the importance of designing systems to reach desirable results, we report an episode regarding women's badminton in 2012 at the London Olympics.

The tournament design used is the usual one used even in the World Cup soccer. There are four groups (A, B, C, D) of four teams each. The tournament has two phases. In the first phase, each team plays against the other three teams in its group, and does not play teams in other groups. The top two teams from each group advance to the second phase, while the bottom two teams from each group are eliminated. In the second phase, the remaining eight teams play a standard knockout tournament. There are four quarterfinals, with the losers eliminated, followed by two semifinals, with the losers playing an extra match to decide the bronze medal. The winner of the final gets the gold medal, the loser the silver one.

To understand the issue, we had to explain how the eight winners are paired up in the quarterfinals. The team with best score from group A plays with the second best team from goup C in the first quarterfinal, similarly the best team from group C plays with the second best one from group A in the third quarterfinal. The top two best ones from B and D are paired up analogously in the second and fourth quarterfinals. The problems arises when the Danish team (PJ) beated the Chinese team (TZ), and as a result PJ won group D with TZ coming in second. Both teams advanced to the knockout stage of the tournament. The first controversial match involved another team from China (WY), and the South Korean team (JK). Both teams had a 2-0 record in group A play. Thus, both were headed for the knockout stage. The issue was that the group A winner would likely meet the fearsome TZ team in the semifinals of the knockout stage, where a loss means a bronze medal at best, while the second-best team in group A would not face TZ until the final, with a silver medal guaranteed. Then both the WY and JK teams try to deliberately lose the match. This unappealing spectacle led to scandal and the disqualification of the two teams.

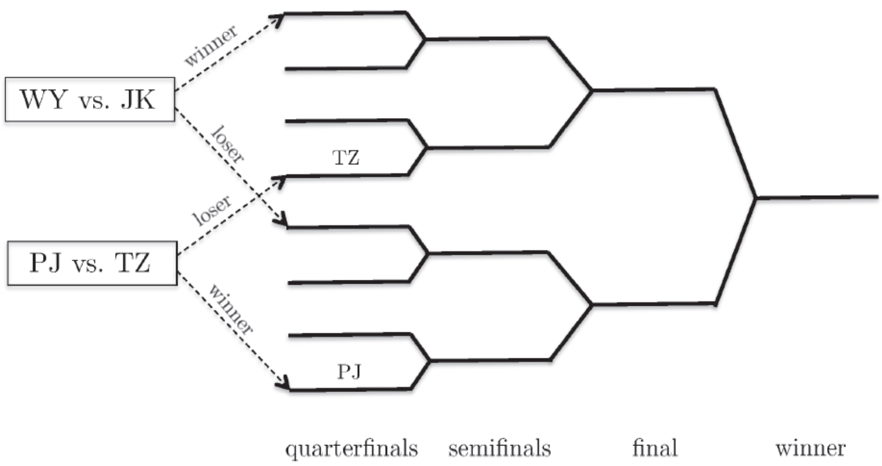

Figure 3.1: Women's badminton tournament at the 2012 Olympics

The episode shows that rules matter and poorly designed systems can lead to undesirable behaviour. Usually rules are not designed from scratch but the designer wants to understand a game that already exists in order to add or modify rules to obtain an appealing outcome. In particular the designer studies the *equilibria*. In most games, the best action that a participant could play depends on what the other players do and, informally, an equilibrium is a steady state of a system where each participant, assuming everything else stays the same, wants to remain as it is. The goal of mechanism design is to give rules in order to obtain the desirable outcomes as equilibria of the game. Often mechanism designers try to achieve outcomes satisfying properties as truthfulness, budget balance etc. Recalling the example of auction made before, one may want to know what are the rules that incentivate players to reveal their real valuation of the good. In next sections, we firstly formally explain the auction model and then we give the rules that make an auction truthful.

3.2 Auction Design Basics

Auctions are a widely used model in economic theory. They are a form of market where the roles of participants are the following:

1. *Seller*: is the agent willing to sell the goods it possesses. An auction can have a single or multiple sellers. We limit our analysis to the case of a single seller but all the definitions and theorems can be easily extended to multiple sellers' case.

2. *Bidders*: is the set of potential buyers that will be participating in the auction and competing for the goods. In general, we assume bidders to be selfish and capable of lying about their valuation of the goods, in order to make the maximum profit.

3. *Auctioneer*: is the third party between buyers and seller. An auctioneer initiates the auction and decides about the winner and the pricing that winner has to pay. In the scenario we consider there is a single seller that is itself the auctioneer, so we use the term auctioneer or seller with no difference.

Figure 3.2: Example of participants in an auction

There are many types of auctions different for the mechanism that chooses the winner and the price the winner should pay. Some of the most known are the english auction, the dutch auction, the first price and the second price auctions.

- *English auction.* This method is called "ascending price auction" since, starting from a low first bid, the auctioneer solicits increasingly higher bids. The process continues in case of a single item until that item is sold to the last and highest bidder.

- *Dutch auction.* This method is called "descending price auction" and it is an auction in which the auctioneer begins with a high asking price, and lowers it until some participant accepts the price, or it reaches a predetermined reserve price.

- *First price auction.* In this type of auction, the bidders propose to the seller a private valuation and the seller gives the item to the highest bid. The winner has to pay an amount equal to his bid.

- *Second price auction.* In this case, the winner is determined in the same way of first price auction but he pays an amount equal to the second highest bid.

The first price and second price auctions are types of sealed-bid auction, as opposed to the first two formats that belong to the category of open auctions.
The auction model is simply a method for allocating goods. For this reason it is very used and some examples of applications are cache allocation, spectrum sharing etc.
We start our analysis from the single-item auction that is the easiest mechanism one could think almost.

Single Item Auctions. A single-item auction is an auction where a seller has an item and there are n bidders that are potentially interested in buying it. The goal is to analyze bidder's behaviour.
The first step is to model what a generic bidder wants. The first assumption is that each participant i has a *nonnegative valuation* v_i of the item: v_i is his maximum willingness-to-pay for the item being sold. The second assumption is that this valuation is *private*, meaning that v_i is unknown to all other bidders and initially to the seller too. The third assumption is that bidders have a *quasi-linear utility*: if a bidder lose the auction then his utility is 0, while, if he wins, it is $v_i - p$, where p is the price he must pay to obtain the item. To describe the utility function, we can introduce a binary variable x_i for each player. x_i is 1 if player i wins, 0 otherwise. A feasibility constraint is $\sum_i x_i \leq 1$. The utility formally becomes:

$$u_i = \begin{cases} 0 & \text{if } x_i = 0 \\ v_i - p_i & \text{if } x_i = 1 \end{cases}$$

Since our aim is to analyze the second price auction mechanism that belongs to the class of the sealed-bid auctions, we formally describe these ones.

Sealed-Bid Auctions. The sealed-bid auction can be summarized by the following mechanism:

1. Each bidder communicates a bid b_i to the seller
2. The seller decides who gets the item
3. The seller decides the price the winner has to pay

Remark. Recalling the concepts of non-cooperative game theory illustrated in chapter one, an auction can be modelize as a non-cooperative game where players are the potential buyers and a strategy is simply the evaluation b_i declared by bidder i to the seller. This must not be confused with the real evaluation v_i, that is a private information known only by the bidder i. This distinction is necessary to analyze players' cheating behaviour.

Procedures of this type are called *direct-revelation mechanism*, because in the first step agents are asked to reveal directly their private evaluations. The direct revelation mechanism can be described by a pair (f, p) where

- $f : V \to A$ is a function defined on the set $V = \times_i V_i$ of all possible players' valuations and it is called as **action rule**. It models the step two, namely the decision of the seller.
- $p : V \to \mathbb{R}^n$ takes real value and is called **payment rule**, determining the payment each player has to pay to the seller.

Remark. There exists indirect mechanisms: some examples are the dutch and english auction explained before.

In the second step of the direct mechanism we trivially assumed that the seller choose the bidder with the highest bid, as in the case of first and second price auctions. The third step, instead, can be implemented in multiple ways. For example, the seller could fix the price to be 0 or to be fixed (constant). The pricing rule affects significantly bidder behaviors. Consider two natural choices:

- *No payment.* In this version we give the item for free to the player with the highest b_i. Clearly, this method is easily manipulated: every player will benefit by exaggerating his b_i, reporting a much higher $\bar{b}_i \gg b_i$ that can cause to him to win the item, even if his real evaluation is not the highest.

- *Pay the bid.* An attempt to avoid this behaviour could be having the winner pay the declared bid. However, also this system is open to manipulations: a player bidding b_i and paying b_i has 0 utility. Thus, he should attempt to declare a lower value $\bar{b}_i < b_i$ that still wins. In this case he can win the item and get a positive utility $u_i = b_i - \bar{b}_i$. To determine the value $\bar{b}_i$, player i should only know the value of the second highest bid and declare a value just above it.

  Example 3.1. Consider an auction with four bidders whose bids are given by the vector $\mathbf{b} = (100, 80, 75, 50)$. Suppose that the vector of bids corresponds to their real evaluation: $\mathbf{b} = \mathbf{v}$. If the first player knows the evaluation of the second player, he can obtain a non negative utility bidding a value slightly higher than 80, for example $b_1 = 81$. Clearly he wins the auction with utility $u_1 = 100 - 81 > 0$.

In order to prevent player's cheating behaviour and to assign the item to the bidder with highest real evaluation, a solution could be use the method already mentioned in the previous section, called second price auction.

3.3 Second Price Auctions

Second price auctions are the most common ones in practice. These are sealed-bid auctions in which the highest bidder wins and pays a price equal to the second highest bid. This mechanism is also known as Vickrey auction from the name of the economist William Vickrey that first analyzed the mechanism in the article "Counterspeculation, auctions and competitive sealed tenders" in 1961.
Suppose n players, each one having a non negative valuation of the object $v_i \geq 0$, and suppose that $v_1 > v_2 > \cdots > v_n$. In a first price auction the winner is always player one bidding $v_2 \leq b_1 \leq v_1$. So the seller gives the item to the player whose valuation is higher. In second price auctions, it does not always happen. In fact, $(v_2, v_1, 0, 0, \ldots, 0)$ is a Nash equilibrium. In this case the winner is player two and all agents have zero utility. This example show that second price auctions are more complex than first price auctions.

Despite this fact, Vickrey's mechanism has an interesting property. In fact, it is truthful meaning since no player has incentive in lying on his valuation. This result is the reason why second price auctions are widely used. A bidder when deciding a strategy by selecting a bid does not need to speculate about other bidders valuation. This is completely different from first price auctions where, instead, the optimal strategy depends on other players' behaviour.

Proposition 3.1. In a second price auction, a weakly dominant strategy for every bidder i is to set the bid b_i equal to his private valuation v_i.

Proof. Consider an arbitrary bidder i with valuation v_i. Assume that all bids of the other bidder are fixed and denote with $\mathbf{b}_{-i}$ the vector of all bids $\mathbf{b}$ but with the i-th component removed. We want to show that $v_i = b_i$. Let $B = \max_{j \neq i} b_j$ denote the highest bid by some other bidder. Since we are considering a second price outcome, the possible outcomes are only two:

1. $b_i < B$, meaning that i loses and receives utility equal to 0
2. $b_i \geq B$, then i wins at price B and receives utility equal to $v_i - B$

Finally, consider two cases:

1. $v_i < B$: in this case the maximum utility bidder i can obtain is $\max\{0, v_i - B\} = 0$. But the same result is achieved by bidding truthfully.
2. $v_i \geq B$: in this case the maximum utility bidder i can obtain is $\max\{0, v_i - B\} = v_i - B$. But the same result is achieved by bidding truthfully.

Thus, declare $b_i = v_i$ is a weakly dominant strategy. □

Example 3.2. Consider again the case of an auction with four bidders and bid vector given by $\mathbf{b} = (100, 80, 75, 50)$. Consider the second player with bid $b_2 = 80$ and suppose $v_2 = b_2$. Moreover, suppose other bids to be fixed. If second player declares less than 100, he loses. Instead declaring a bid $\bar{b}_2 \geq 100$ makes him win with utility $u_2 = \max\{v_2 - 100\} = \max\{80 - 100, 0\} = 0$. But the same utility is obtained bidding truthfully $v_2 = 80$. Thus the second player has no interest in lying on his evaluation.

The second important property is that each truthful bidder never regrets partecipating in a second price auction. Formally:

Proposition 3.2. In a second price auction, every truthful bidder is guaranteed a nonnegative utility.

Proof. Each bidder in a second price auction can lose or win. If a player loses, he receives 0 utility. If a bidder i wins, then he receives utility equal to $v_i - p$, where p is the second highest bid. Since i is the winner (hence, b_i is the highest evaluation) and he bids truthfully $b_i = v_i$ then $p \leq v_i$ and $v_i - p \geq 0$ □

Despite the truthfulness property, second price auctions are susceptible to *collusion*. With this term, we indicate the propensity of players to form coalitions in order to obtain more at a lower price. To better understand, we give the following example:

Example 3.3. Consider the auction with four players described by the vector of bids $\mathbf{b} = (100, 80, 75, 50)$. By the theorem 3.1, we know that bids correspond to the real evaluations $v_i = b_i \quad \forall i = 1, 2, 3, 4$. This means that player one wins paying a price of 80. Player two, knowing he loses, can negotiate with player one to declare a smaller evaluation $\bar{b}_2 = 75$ and to split half and half of the 5 extra units of utility that player one will save.

As shown, the players can lie on their evaluation and can make deals with the winner to achieve a positive utility.

3.3.1 Properties of second price auctions

Second price single-item auctions are *ideal* since they satisfy three desiderable properties. The first is formalized by the following definition:

Definition 3.1. An auction is *Dominant-Strategy Incentive Compatible* (DSIC) if truthful bidding is always a weakly dominant strategy for every bidder and if truthful bidders always obtain nonnegative utility.

The condition of non negative utility is usually considered a separated requirement and is called *individual rationality*. We define the *social welfare* of an outcome of a single-item auction as

$$\sum_{i=1}^{n} v_i x_i$$

where x_i is 0 if bidder i lose, 1 if he win. Since we are considering single-item auctions, $\sum_{i=1}^{n} x_i \leq 1$ is a feasibility constraint. The social welfare in this case is equal to winner's valuation or 0 if there is no winner.

Definition 3.2. An auction is *welfare maximizing* if, when bids are truthful, the auction outcome has the maximum possible social welfare.

Welfare maximization states that even if the bids are a priori unknown to the seller, the designed auction mechanism identifies the bidder with the highest valuation. The properties for a second price auction can be summarized by the following theorem.

Theorem 3.3. A second price single-item auction satisfies the following:

1. **Strong incentive guarantees.** It is a DSIC auction

2. **Strong performance guarantees.** It is a welfare maximizing auction

3. **Computational efficiency.** It can be implemented in time polynomial in the size of the input, meaning the number of bits necessary to represent the numbers $v_1, v_2, \ldots, v_n$.

The proof of the points 1 and 2 of the theorem derives from the propositions 3.1 and 3.2. Truthfulness makes easy for a bidder to choose a strategy, moreover guarantees to the seller that no one lies in evaluating the object allowing to assign the item to player whose valuation is the most. The third point 3 derives from theorem 11.24 in [11], since a snigle item auction is a simpler case of a combinatorial auction, a generalized model that we do not consider in the analysis. The importance of the third property can be seen in applications since an auction should run in a reasonable time. This can be better understand with an example of real application: the sponsored search auction.

Example 3.4. Every time we use search engine on web typing a query, an auction is run in real time to decide which advertisers' link are shown, how these links are arranged visually and what the advertisers are charged. These types of auction are called *spondored search auctions*.

As we already said, the goal is to desing a truthful auction whose equilibria lead to an efficient action. This is satisfied by second price auctions. Until now, we have analyzed single-item auctions. From now on, we consider a more general setting assuming that a resource has to be sold to the participants of an auction. Thus, recalling the binary variable x_i introduced to take into account the winner, from now on, we have that x_i is a continuos variable taking values in $[0, 1]$ and representing the fraction of resource assigned to player i. This is a useful generalization that allows us to model more complex mechanisms such as bankruptcy games.
The next sections introduce a well-known model: the VCG mechanism. It is used in practice because it has many advantageous properties.

3.3.2 VCG Auction

The Vickrey-Clarke-Groves mechanism (VCG) is a generalization of the second price auction mechanism. Reference articles are [19], [16] and [3]. The formal definition is the following one:

Definition 3.3. A **VCG mechanism** is a direct mechanism (f, p) such that:

- It implements a social optimum

$$f(v_1, v_2, \ldots, v_n) \in \arg\max_{a \in A} \sum_i v_i(a)$$

- For some functions $h_i : V_{-i} \to \mathbb{R}$, we have that for all $v_1 \in V_1, \ldots, v_n \in V_n$

$$p_i(v_1, \ldots, v_n) = h_i(\mathbf{v}_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n))$$

Note that the terms $\sum_{j \neq i} v_j(f(v_1, \ldots, v_n))$ is exactly the social welfare minus the value of the player i, $v_i(f(v_1, \ldots, v_n))$. Thus, this mechanism aligns all players' incentives with the goal of maximizing social welfare, which is exactly obtained telling the truth. The other term in the payment formula $h_i(\mathbf{v}_{-i})$ has no strategic implication for player i since it does not depend in any way on his valuation. Hence, for player i, $h_i(\mathbf{v}_{-i})$ is just a constant. Nevertheless, the choice of this term affects significantly how much player i has to pay and in which direction (depending on the sign, we have that either player gives money to the system or the reverse). We have the following result:

Theorem 3.4 (Vickrey-Clarke-Groves)**.** Every VCG mechanism is incentive compatible, meaning that truthful bidding is always a weak dominant strategy.

Proof. Fix i, $\mathbf{v}_{-i}$, v_i and $\bar{v}_i$. We need to show that for a generic player i with true valuation given by v_i, the utility when declaring v_i is not less than the utility when declaring $\bar{v}_i$. Denote $a = f(v_i, \mathbf{v}_{-i})$ and $\bar{a} = f(\bar{v}_i, \mathbf{v}_{-i})$. The utility of i, when declaring v_i, is

$$v_i(a) + \sum_{j \neq i} v_j(a) - h_i(\mathbf{v}_{-i})$$

while, when declaring $\bar{v}_i$, is

$$v_i(\bar{a}) + \sum_{j \neq i} v_j(\bar{a}) - h_i(\mathbf{v}_{-i})$$

Since $a = f(v_i, \mathbf{v}_{-i})$ maximizes social welfare over all alternatives,

$$v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(\bar{a}) + \sum_{j \neq i} v_j(\bar{a})$$

and thus the same inequality holds when subtracting the term $h_i(\mathbf{v}_{-i})$ from both sides. $\square$

The problem is now to determine the term h_i. One trivial possibility is choosing $h_i = 0$. This has the advantage to be simple but does not make sense because the mechanism pays here a great amount to the players, since the price has negative sign. Intuitively, we would prefer that players pay the seller. Two requests seem resonable:

1. Non negative utilities for all players: $u_i \geq 0$
2. Non negative payments from the players to the system: $p_i \geq 0$

The following choice provides the two properties:

Definition 3.4. The choice

$$h_i(\mathbf{v}_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b)$$

is called the **Clarke pivot payment**.

Under this rule, the payment of player i is

$$p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_i(a)$$

where $a = f(v_1, \ldots, v_n)$.
Intuitively, player i pays an amount equal to the difference between the social welfare of the other players without i's partecipation and with i's partecipation. With this choice of the functions h_i, we have the following theorem:

Theorem 3.5. A VCG mechanism with Clarke pivot payments makes no positive transfers to the seller. If $v_i(a) \geq 0$ for every $v_i \in V_i$ and $a \in A$, then the utility is always nonnegative.

Proof. Let $a = f(v_1, \ldots, v_n)$ be the alternative maximizing $\sum_j v_j(a)$ and b be the alternative maximizing $\sum_{j \neq i} v_j(b)$. To show that the utilities are nonnegative, note that utility for a generic player i is given by:

$$v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_j v_j(a) - \sum_j v_j(b) \geq 0$$

where the first inequality holds because $v_i(b) \geq 0$ and the second because a was chosen in such a way $\sum_j v_j(a)$ is maximimum. To show that there are not positive transfers from players to the system, namely $p_i \geq 0$, note that

$$p_i(v_1, \ldots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \geq 0$$

since b was chosen in such a way that $\sum_{j \neq i} v_j(b)$ is maximum. $\square$

In other words, the mechanism is DSIC.
The Clarke pivot rule does not fit many situations where valuation are negative, meaning when alternatives have a cost for the players. In such cases the function h_i can be properly modified.

3.4 The Revelation Principle

Until now, we have consider only direct mechanisms. We now formalize the general notion of mechanism. The idea is that each player has some private information $t_i \in T_i$ that captures his preferences over a set of alternatives A. This means that $v_i(t_i, a)$ is the value that player i assignes to a when his private information is t_i. The aim is to implement some function $F : T_1 \times \ldots \times T_n \to A$ that aggregates these preferences. We denote with X_i the set of all possible actions for player i. The outcome function $a : X_1 \times \ldots \times X_n \to A$ choses an alternative in A for each profile of actions. The payment function $p : X_1 \times \ldots \times X_n \to \mathbb{R}$ specifies the payment of each player for every profile of actions. The general formal definitions are the following ones:

Definition 3.5. A mechanism for n players is given by:

1. players' type spaces $T_1, \ldots, T_n$
2. players' action spaces $X_1, \ldots, X_n$
3. a set of alternatives A

4. players' valuation functions $v_i : T_i \times A \to \mathbb{R}$
5. an outcome function $a : X_1 \times \ldots \times X_n \to A$
6. payment functions $p_1, \ldots, p_n$

The game induced by the mechanism is given by using the type spaces T_i, the action spaces X_i and the utilities

$$u_i(t_i; x_1, \ldots, x_n) = v_i(t_i; a(x_1, \ldots, x_n)) - p_i(x_1, \ldots, x_n)$$

Definition 3.6. The mechanism implements $f : T_1 \times \ldots \times T_n \to A$ in dominant strategies if for some dominant strategy equilibrium $s_1, \ldots, s_n$ of the induced game, where $s_i : T_i \to X_i$, we have that for all t_i

$$f(t_1, \ldots, t_n) = a(s_1(t_i), \ldots, t_n(s_n))$$

The introduction of a more general definition of mechanism seems to be reasonable and apparently allows us to do more than what is possible using incentive compatible direct revelation mechanisms. But this is not true: any general mechanism that implements a function in dominant strategies can be converted into an incentive compatible one. The result is summarized in the theorem:

Proposition 3.6 (Revelation principle)**.** If there exists an arbitrary mechanism that implements f in dominant strategies, then there exists an incentive compatible mechanism that implements f. The payments of the players in incentive compatible mechanism are identical to those, obtained at equilibrium, of the original mechanism.

Proof. Let $s_1, \ldots, s_n$ be a dominant strategy equilibrium of the original mechanism. Define a new direct revelation mechanism:

$$f(t_1, \ldots, t_n) = a(s_1(t_1), \ldots, s_n(t_n))$$

and

$$\bar{p}_i(t_1, \ldots, t_n) = p_i(s_1(t_1), \ldots, s_n(t_n)).$$

Since each s_i is a dominant strategy for player i, then for every t_i, $\mathbf{x}_{-i}, \bar{x}_i$, we have that

$$v_i(t_i, a(s_i(t_i), \mathbf{x}_{-i})) - p_i(s_i(t_i), \mathbf{x}_{-i}) \geq v_i(t_i, a(\bar{x}_i, \mathbf{x}_{-i})) - p_i(\bar{x}_i, \mathbf{x}_{-i})$$

Thus in particular it is true for all $\mathbf{x}_{-i} = \mathbf{s}_{-i}(\mathbf{t}_{-i})$ and any $\bar{x}_i = s_i(\bar{t}_i)$ which gives the definition of incentive compatibility of the mechanism $(f, \bar{p}_1, \ldots, \bar{p}_n)$. $\square$

As consequence, without loss of generality, we restrict our analysis to the case of direct incentive compatible mechanisms.

3.4.1 Characterizations

Since our analysis is focused on incentive compatible mechanisms, we give some useful characterizations without reporting the proof, that can be found in [11].

Proposition 3.7. A mechanism is incentive compatible if and only if it satisfies the following conditions for every i and for every v_{-i}:

1. The payment p_i does not depend on v_i, but only on the alternative chosen $f(v_i, \mathbf{v}_{-i})$. That is, for every $\mathbf{v}_{-i}$, there exists prices $p_a \in \mathbb{R}$ for every $a \in A$, such that for all v_i with $f(v_i, \mathbf{v}_{-i}) = a$, we have that $p(v_i, \mathbf{v}_{-i}) = p_a$

2. The mechanism optimizes for each payer. That is, for every v_i, we have that $f(v_i, \mathbf{v}_{-i}) \in \arg\max_a (v_i(a) - p_a)$, where the quantification is over all alternatives in the range of $f(., \mathbf{v}_{-i})$

Definition 3.7. A **single parameter domain** V_i is defined by a publicy known subset of winning alternatives $W_i \subset A$ and a range of values $[t^0, t^1]$. V_i is the set of v_i such that for some $t^0 \leq t \leq t^1$, $v_i(a) = t$ for all $a \in W_i$ and $v_i(a) = 0$ for all $a \notin W_i$. In such setting we will abuse notation and use v_i as the scalar t.

For this setting it is easy to completely characterize incentive compatible mechanisms.

Definition 3.8. A social choice function f on a single parameter domain is monotone in v_i if for every $\mathbf{v}_{-i}$ and every $v_i \leq \bar{v}_i \in \mathbb{R}$ we have that

$$f(v_i, \mathbf{v}_{-i}) \in W_i \quad \Rightarrow \quad f(\bar{v}_i, \mathbf{v}_{-i}) \in W_i$$

Meaning that if v_i makes i win, then so will every higher valuation $\bar{v}_i$.

For a monotone function f, for every $\mathbf{v}_{-i}$ for which player i can both win and lose, is called *critical value* the value below which i loses and above which he wins.

Example 3.5. In second price auctions the critical value for each player is the highest value declared by the others.

Consider the single parameter domain. We call *normalized* a mechanism if the established payment for losing is always 0. It is easy to see that every incentive compatible mechanism can be turned into a zero normalized one.

Theorem 3.8. A normalized mechanism $(f, p_1, \ldots, p_n)$ on a single parameter domain is incentive compatible if and only if the following conditions holds:

1. f is monotone in every v_i.
2. Every winning bid pays the critical value to the winner.

It can be proved that the payment function is uniquely determined by the social choice function f. From these results, we get that the only incentive compatible mechanism that maximizes the social welfare are those with VCG payements.

3.5 Myerson's Auction

Another possible way to design truthful auctions whose equilibria are efficient solutions is the Myerson's mechanism. It was introduced for the first time in 1981 by Myerson in the article "Optimal auction design" [10]. Here we want to give the mathematical framework and state the main properties of the mechanism. This type of auction is better analyzed in the fourth chapter.
Recalling the sealed-bid auction, even in this case the seller has to make two choices: the action rule, in order to allocate the resource, and the pricing rule. Consider the direct revelation mechanism described by the pair (f, p). Here we denote with $\mathbf{x}$ the result of the action function, meaning $\mathbf{x}$ is a vector containing the resource allocated for each player. We denote with $\mathbf{p}$ the result of the payment function, meaning that $\mathbf{p}$ is a vector with entries equal to the price players have to pay to the seller. Thus, the direct revelation mechanism can be described by the pair $(\mathbf{x}, \mathbf{p})$.
The difference from VCG auction mechanism is the designer's goal. While the VCG auction maximizes the social welfare, the Myerson's auction maximizes the seller's profit. All the mechanisms that achieve this goal are called *optimal mechanisms*. We focus our attention on the design of profit-maximizing auctions in which an auctioneer is selling (or buying) a set of services. As usual, we assume n agents. Each one is *single parameter*, i.e. agent i's valuation for receiving service is v_i and the valuation for no service is normalized to zero. The mechanism takes as inputs sealed bid from agents $\mathbf{b}$ and computes an outcome consisting of an allocation $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and prices $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. We assume quasi linear utilities expressed by

$$u_i = v_i x_i - p_i$$

where x_i takes value in $[0, 1]$. Thus agent's goal is to maximize the difference between his valuation and his payment. To generalize, we assume that there is a cost $c(\mathbf{x})$ in producing the outcome $\mathbf{x}$, which must be paid by the mechanism. The goal is then to design a mechanism so that the auctioneer's profit, defined as

$$\text{Profit} = \sum_i p_i - c(\mathbf{x}),$$

is maximized and the mechanism is truthful.

The next example shows that the VCG mechanism maximizing the social welfare is a poor mechanism if the goal is profit maximization.

Example 3.6. In a digital goods auction, an auctioneer is selling multiple units of an item, such as downloadable audio file or a pay-per-view television broadcast, to consuler each interested in one unit. Since the marginal cost of duplicating a digital good is negligible and digital goods are free disposal, we can assume that the auctioneer has unlimited supply of unit for sale. Thus, in this case we have $c(\mathbf{x}) = 0$ for all $\mathbf{x}$. The profit of the VCG mechanism for digital goods auction is 0. Indeed, since items are available in unlimited supply, no bidder places any externality on any other bidder, meaning that an allocation does not influence any other future allocation and the other valuations.

Myerson's auction belongs to the family of Bayesian auctions, assuming that agents' private valuations are drawn from a known a priori distribution. The Bayesian optimal mechanism is the one that maximizes the auctioneer's expected profit, where the expectation is taken over the randomness agent's evaluation. We do not analyze this type of models but in the next chapter we provide an easier characterization of Myerson's auction without the introduction of bayesian games.

Chapter 4

Myerson's Pricing Mechanism

This chapter illustrates the pricing mechanism introduced by Myerson in 1981 [10]. Following the article [6], we explain how this mechanism can be applied in the case of the proportional rule, the Max-Min fair, the nucleolus solution and the shapley value. Moreover, we add the application for the mood value, illustrated in chapter one and in the article [4].
The proportional and the Max-Min are more used in application for example in networking, the mood value solution is derived from the theoretical tau-value solution, while the last two rules are classical solutions of TU cooperative games. All these allocations can be used to solve bankruptcy games. Thus, we can use an auction mechanism to solve it. The application of Myerson's mechanism to bankruptcy games is useful because it exploits the property of being truthful: every claimant is encouraged to give his true evaluation of the resource.

Given a bankruptcy problem, we fix the allocation rule to divide the estate, we apply the pricing mechanism given by Myerson and we determine the cost every player have to pay. The basic assumption is that the demands are private information, even if we consider a bankruptcy problem. We apply the revelation principle analyzed in chapter three. Finally, to better analyze the pricing function obtained, we run some simulations and we compare results for the five mentioned allocation rules.

4.1 Myerson's theorem

Recalling Myerson's setting illustrated in the previous chapter, we assume quasi-linear utility for each player. Private valuations per unit of staff are denoted with $v_i \quad \forall i \in N$. Utilities are given by the following formula

$$u_i = v_i x_i(\mathbf{b}) - p_i(\mathbf{b})$$

with the restriction $p_i(\mathbf{b}) \in [0, b_i x_i(\mathbf{b})]$ so that truthful agents receive nonnegative utility and the seller cannot pay an agent. We assume $x_i(\mathbf{b})$ takes values in $[0, 1]$. The setting is a single parameter environment with n agents.

Definition 4.1. An allocation rule $\mathbf{x}$ for a single-parameter environment is **implementable** if there is a payment rule $\mathbf{p}$ such that the direct-revelation mechanism ($\mathbf{x}$, $\mathbf{p}$) is DSIC.

We recall the definition of monotonicity:

Definition 4.2. An allocation rule $\mathbf{x}$ for a single-parameter environment is **monotone** if for every agent i and bids b_{-i} by the other agents, the allocation $x_i(z, \mathbf{b}_{-i})$ to i is nondecreasing in his bid z.

Remark. The proportional rule, the max-min fair, the mood value, the nucleolus and the shapley value are monotone rules.

We now state and prove the Myerson's theorem, useful to understand the computations of prices for the five allocations.

Theorem 4.1 (Myerson)**.** Fix a single-parameter environment.

1. An allocation rule $\mathbf{x}$ is implementable if and only if it is monotone.

2. If $\mathbf{x}$ is monotone, then there is a unique payment rule for which the direct-revelation mechanism ($\mathbf{x}$, $\mathbf{p}$) is DSIC and $\mathbf{p}_i(b) = 0$ whenever $b_i = 0$.

3. The payment rule in (2) is given by an explicit formula.

Proof. Fix a single-parameter environment and consider an allocation rule x, which may or may not be monotone. Fix i and $\mathbf{b}_{-i}$ arbitrarily. For simplicity, denote with $\mathbf{x}(z)$ and $\mathbf{p}(z)$ the allocation $\mathbf{x}_i(z, \mathbf{b}_{-i})$ and the payment $p_i(z, \mathbf{b}_{-i})$ of agent i when she bid z, respectively.
Suppose $(\mathbf{x}, \mathbf{p})$ is DSIC and consider any $0 \leq y < z$. Since agent i might well have private valuation z and is free to submit the false bid y, DSIC demands that

$$zx(z) - p(z) \geq zx(y) - p(y) \tag{4.1}$$

Similarly, since agent i might well have the private valuation y and could submit the false bid z, $(\mathbf{x}, \mathbf{p})$ must satisfy

$$yx(y) - p(y) \geq yx(z) - p(z) \tag{4.2}$$

Rearranging inequalities 4.1 and 4.2 yields

$$z\left[x(y)-x(z)\right] \leq p(y)-p(z) \leq y\left[x(y)-x(z)\right] \tag{4.3}$$

This implies that every allocation rule $\mathbf{x}$ is monotone.
Consider the case where x is a piecewise constant function. Then the graph of x is flat except for a finite number of "jumps". Fix z and let y tend to z from above. Taking the limit $y \downarrow z$, the left and the right-hand side of 4.3 becomes 0 if there is no jump in x at z. If there is a jump of magnitude h at z, then the left and the right-hand side both tends to $h\ z$. This implies the following constraint on p:

$$\forall \quad z \qquad jump\ in\ p\ at\ z = z\left[jump\ in\ x\ at\ z\right] \tag{4.4}$$

Combining this with the initial condition $p(0) = 0$, the payment formula is the following:

$$p_i(b_i, \mathbf{b}_{-i}) = \sum_{j=1}^{l} z_j \ \left[jump\ in\ x_i(\cdot, \mathbf{b}_{-i})\ at\ z_j\right] \tag{4.5}$$

where $z_1, \ldots, z_l$ are the breakpoints of the allocation function $x_i(\cdot, \mathbf{b}_{-i})$ in the range $[0, b_i]$.
Similarly, consider x monotone and differentiable [1]. Divide all terms in 4.3 by $y - z$ and let $y \downarrow z$, then

$$p'(z) = zx'(z) \tag{4.6}$$

Combining this with the initial condition $p(0) = 0$ yields the payment formula:

$$p_i(b_i, \mathbf{b}_{-i}) = \int_0^{b_i} z \frac{d}{dz} x(z, \mathbf{b}_{-i}) dz \quad \forall i \in N, b_i\ and\ \mathbf{b}_{-i} \tag{4.7}$$

This formula gives the only possible payment rule that has a chance of extending the given allocation rule $\mathbf{x}$ into a DSIC mechanism. Thus, for every allocation rule $\mathbf{x}$, there is at most one payment rule $\mathbf{p}$ such that $(\mathbf{x}, \mathbf{p})$ is DISC.
We give a proof by picture that, when $\mathbf{x}$ is monotone and piecewise constant and $\mathbf{p}$ is defined by 4.5, then (x; p) is a DSIC mechanism. The same argument works more generally for monotone allocation rules that are not piecewise constant, with payments defined as in 4.7. This will complete the

[1] With some additional facts from calculus, the proof extends to general monotone functions. The details are omitted.

proof.

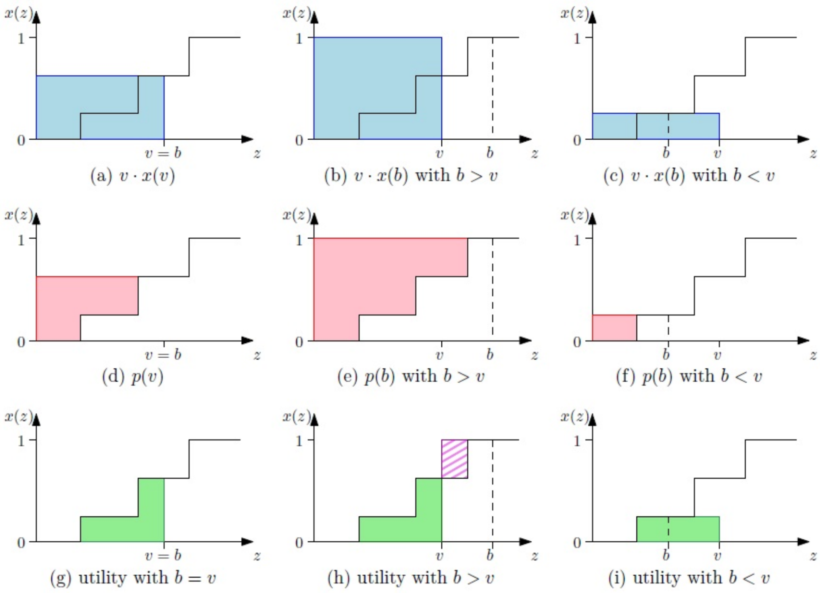

Figure 4.1: Payment rule in the tree cases

Figure 4.1 depicts the utility of a bidder when player i bids truthfully, overbids, and underbids. The allocation curve $x(z)$ and the private valuation v of the bidder are the same in all three cases. We depict the first term $v \quad x(b)$ as a shaded rectangle of width v and height $x(b)$. Using the formula 4.7, we see that the payment $p(b)$ can be represented as the shaded area to the left of the allocation curve in the range $[0, b]$. The bidder's utility is the difference between these two terms. When the bidder bids truthfully, its utility is precisely the area under the allocation curve in the range $[0, v]$. When the bidder overbids, its utility is this same area, minus the area above the allocation curve in the range $[v, b]$. When the bidder underbids, its utility is a subset of the area under the allocation curve in the range $[0, v]$. Since the bidder's utility is the largest in the first case, the proof is complete.

□

Following the calculation done in the proof of Myerson's thorem and assuming continuous allocation rules $\bar{x}_i \in [0, 1]$ with continuous derivatives, we can apply the formula for the integration by parts, obtaining

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \frac{x_i(b_i, \mathbf{b}_{-i})}{E} - \frac{1}{E} \int_0^{b_i} x_i(z_i, \mathbf{b}_{-i}) dz$$

Since the allocation rule is normalized and divided by the available resource E, then we obtain the fraction of resource $\bar{x}_i$ given to each player i.

Remark. All the analyzed allocation rules are continous with continuous derivatives. Thus, the formula 4.1 can be applied.

The pricing function has an easy interpretation: it is the area above the curve $\bar{x}_i(b_i, \mathbf{b}_{-i})$. For example, taking a generic monotone allocation, the price is the area in purple in the following picture

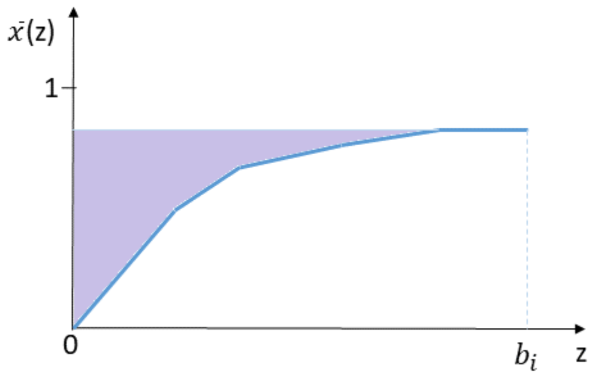

Figure 4.2: Price as the area above the curve

Since we have assumed the utility $u_i = v_i x_i(\mathbf{b}) - p_i(\mathbf{b})$, u_i can be interpreted as the area between 0 and z, that is the area under the curve. These are useful results for the implementation of pricing functions in R.

4.2 Pricing Functions

In this section, following the article [6], we report how to calculate the price for the proportional rule, the max-min fair, the nucleolus and the Shapley value. We add calculation for the mood value, an allocation solution having interesting fairness property.
Given a bankruptcy problem $(\mathbf{c}, E)$, we can solve it assuming that every claimant i is a partcipant of an auction described by a direct revelation mechanism $(\mathbf{x}, \mathbf{p})$, for a fixed allocation rule $\mathbf{x}$. Furthermore, we assume that each player bids $b_i = c_i$ and that the valuations are private. The price p_i is determined following the theorem 4.1.

Proportional Rule

We recall that the proportional allocation rule function for every player i, condering all other players' bids $\mathbf{b}_{-i}$ fixed, has the form:

$$x_i(z, \mathbf{b}_{-i}) = \frac{Ez}{z + \sum_{j \neq i} b_j}$$

All the hypotheses are satisfied, so we can apply the formula 4.1 integrating from zero to b_i. We obtain the following pricing function:

$$p_i = \frac{b_i^2}{\sum_j b_j} - b_i + \left(\sum_{j \neq i} b_j \right) \log \left(\frac{\sum_j b_j}{\sum_{j \neq i} b_j} \right) \tag{4.8}$$

Max-Min Fair

For the Max-Min fair allocation, assuming that the bids are in incresing order $b_1 \leq b_2 \leq \ldots \leq b_n$, we recall that the function for player i has the form:

$$x_i(z) = \min \left(z, \frac{E - \sum_{j=1}^{i-1} x_j(\mathbf{b}, E)}{n - i + 1} \right)$$

For simplicity, we have omitted $\mathbf{b}_{-i}$ among the arguments of the allocation function, since the vector is considered fixed. The equation shows that as we increase z from 0 to b_i we have

$$x_i(z) = \begin{cases} z & \text{if} \quad z \leq C_i \\ C_i & \text{if} \quad z > C_i \end{cases}$$

where C_i is the critical point where the curve becomes constant. To determine C_i, we can calculate x_i for any sufficiently large z. This point can be chosen to be E because $\frac{E - \sum_{j=1}^{i-1} x_j(\mathbf{b}, E)}{n-i+1} \leq E$ for any vector $\mathbf{b}$. Then $C_i = x_i(E)$. The allocation function is shown in the figure:

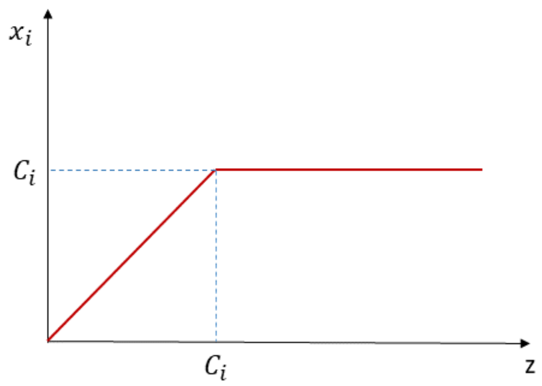

Figure 4.3: Allocation function for the Max-Min

Thus, the integral can be calculated as the area below the curve:

$$\int_0^{b_i} x_i(z)dz = \begin{cases} \frac{b_i^2}{2} & \text{if} \quad b_i \leq C_i \\ \frac{C_i^2}{2} + (b_i - C_i)C_i & \text{if} \quad b_i > C_i \end{cases}$$

And the final pricing function has the form:

$$p_i = b_i \frac{\min(b_i, x_i(E))}{E} - \frac{\min^2(b_i, x_i(E))}{2E}$$

Mood Value

Recalling the definition of mood value

$$x_i = min_i + m(max_i - min_i)$$

where

$$min_i = \max\left\{E - \sum_{j \neq i} b_j, 0\right\} \qquad \text{and} \qquad max_i = \min\{b_i, E\}$$

we can write the allocation rule as

$$x_i(z) = \begin{cases} min_i + \frac{E - \sum_j min_j}{\sum_{j \neq i} max_j - \sum_j min_j + z}(z - min_i) & if \quad z < E \\ min_i + \frac{E - \sum_j min_j}{\sum_{j \neq i} max_j - \sum_j min_j + E}(E - min_i) & if \quad z \geq E \end{cases}$$

Since supponsing $\mathbf{b}_{-i}$ fixed implies that player i kwnows the value of his minimum allocation, the function can be consequently modified as follows:

$$x_i(z) = \begin{cases} z \quad if \quad z < min_i \\ min_i + \frac{E-\sum_j min_j}{\sum_{j\neq i} max_j - \sum_j min_j + z}(z - min_i) \quad if \quad min_i < z < E \\ min_i + \frac{E-\sum_j min_j}{\sum_{j\neq i} max_j - \sum_j min_j + E}(E - min_i) \quad if \quad z \geq E \end{cases}$$

The integral $\int_0^{b_i} x_i(z, \mathbf{b}_{-i})dz$ inside the pricing function has to be calculated in three different cases:

1. $b_i \leq min_i$:

$$\int_0^{b_i} x_i(z)dz = \frac{b_i^2}{2}$$

2. $min_i < b_i < E$:

$$\int_0^{b_i} x_i(z)dz = \int_{min_i}^{b_i} x_i(z)dz + \frac{min_i^2}{2} = \\ min_i(b_i - min_i) + \left(E - \sum_j min_j\right)(b_i - min_i) - \\ \left(E - \sum_j min_j\right) \sum_{j\neq i} (max_j - min_j)\, ln\left(\frac{\sum_j (max_j - min_j)}{\sum_{j\neq i}(max_j - min_j)}\right) + \\ \frac{min_i^2}{2}$$

3. $b_i \geq E$:

$$\int_0^{b_i} x_i(z)dz = \frac{min_i^2}{2} + \int_{min_i}^{E} x_i(z)dz + \int_E^{b_i} x_i(z)dz$$

where the first integral is

$$\int_{min_i}^{E} x_i(z)dz = min_i(E - min_i) + \left(E - \sum_j min_j\right)(E - min_i) - \\ \left(E - \sum_j min_j\right) \sum_{j\neq i} (max_j - min_j)\, ln\left(\frac{\sum_j (max_j - min_j)}{\sum_{j\neq i}(max_j - min_j)}\right)$$

and the second is

$$\int_E^{b_i} x_i(z)dz = \left[min_i + \frac{E-\sum_j min_j}{\sum_j max_j - \sum_j min_j}(E - min_i)\right](b_i - E)$$

From these calculations we derive the pricing formula.

Nucleolus

The nucleolus solution has not a closed and simple formula. We use the package [14] to calculate the nucleolus for given a bankruptcy problem, while for the price we approximate the integral inside the formula 4.1. Knowing that the slope of the function $x_i(z)$ cannot change more than 2^{n-1} times, we divide the interval $[0, b_i]$ into sub intervals of length given by

$$\Delta = \frac{b_i}{2^{n-1}+1}$$

In this way, the integral can be discretized as follows:

$$\int_0^{b_i} x_i(z)dz \approx \sum_{k=0}^{2^{n-1}} \left(x_i(k\Delta)\Delta + \frac{\Delta}{2}[x_i((k+1)\Delta) - x_i(k\Delta)] \right)$$

The formula above is a second order approximation [2] and the error for $\Delta \to 0$ is $err = O(\Delta^2)$, since $x_i(z)$ is a regular function. We consequently calculate the pricing function.

Shapley Value

We recall the formula of the Shapley value:

$$x_i(\mathbf{b}) = \sum_{S \in N \setminus \{i\}} \frac{s!\,(n-s-1)!}{n!} \{v(S \cup \{i\}) - v(S)\}$$

where

$$v(S) = \max\left(0, E - \sum_{i \in N \setminus S} b_i\right)$$

for each coalition $S \subseteq N$.

The allocation x_i given by Shapley on the interval $[0, b_i]$ is piece-wise linear with respect player i's bid b_i. We konw that $x_i(0) = 0$ and the derivative is a stepwise function given by:

$$\frac{\partial x_i(z)}{\partial z} = \begin{cases} \sum_{j=1}^{2^n-1} \hat{\Theta}_j \quad \text{for} \quad 0 < z < \hat{\Phi}_1 \\ \sum_{j=k+1}^{2^n-1} \hat{\Theta}_j \quad \text{for} \quad \hat{\Phi}_k < z < \hat{\Phi}_{k+1} \quad k = 1, 2, \dots 2^{n-1} - 1 \\ 0 \quad \text{for} \quad \hat{\Phi}_{2^{n-1}} < z < b_i \end{cases}$$

[2] Trapezoid rule

where

- $\Phi \in \mathbb{R}^{2^n-1}$ is the vector having as entries the image of the function

$$q(S) = \max\left\{0, E - \sum_{j \in N \setminus \{S,i\}} b_j\right\} \quad \forall S \in N \setminus \{i\}$$

- Θ is the corresponding vector having as elements all the shapley coefficients:

$$\alpha_S = \frac{s!\,(n-s-1)!}{n!}$$

- $\hat{\Phi}$ is the vector Φ sorted in increasing order
- $\hat{\Theta}$ is the vector of coefficients which corresponds to $\hat{\Phi}$

Thus, the integral can be calculated as the area under the curve summing up all the areas of triangles and rectangles.

4.3 Price comparison

In order to have an idea of the magnitude and trend of the prices, we use the software R to simulate bankruptcy games and to calculate the price for the five allocation solutions mentioned before.
First, we write functions that implements the price for the allocation rule illustrated before and then we run some simulations. For simplicity, we simulate games with a number of players equal to $n = 3$. Each bid b_i for $i = 1, 2, 3$ is assumed to be independent from all the others. It is drawn from a uniform distribution $\mathcal{U}\{1, 2, 3, \ldots, 100\}$. Furthermore, we assume bids to be integer numbers. The hypothesis of independence is not restrictive since we suppose that all valuations are private information and that players do not communicate each other. To ensure that the simulated game is a bankrupcty game, we choose E to be a fraction of the sum of the bids. We call this fraction *congestion* C since it indicates the percentage of demand satisfied by the resource:

$$E = C \sum_i b_i$$

Greater is the congestion, greater is the fraction of requests that can be satisfy. We choose $C = 0.05,\ 0.35,\ 0.65$ and 0.95. Since all the prices are normalized, we can compare them for different allocation solutions and different players. These are the results for 280 simulated games, differentiating games with respect to congestion.

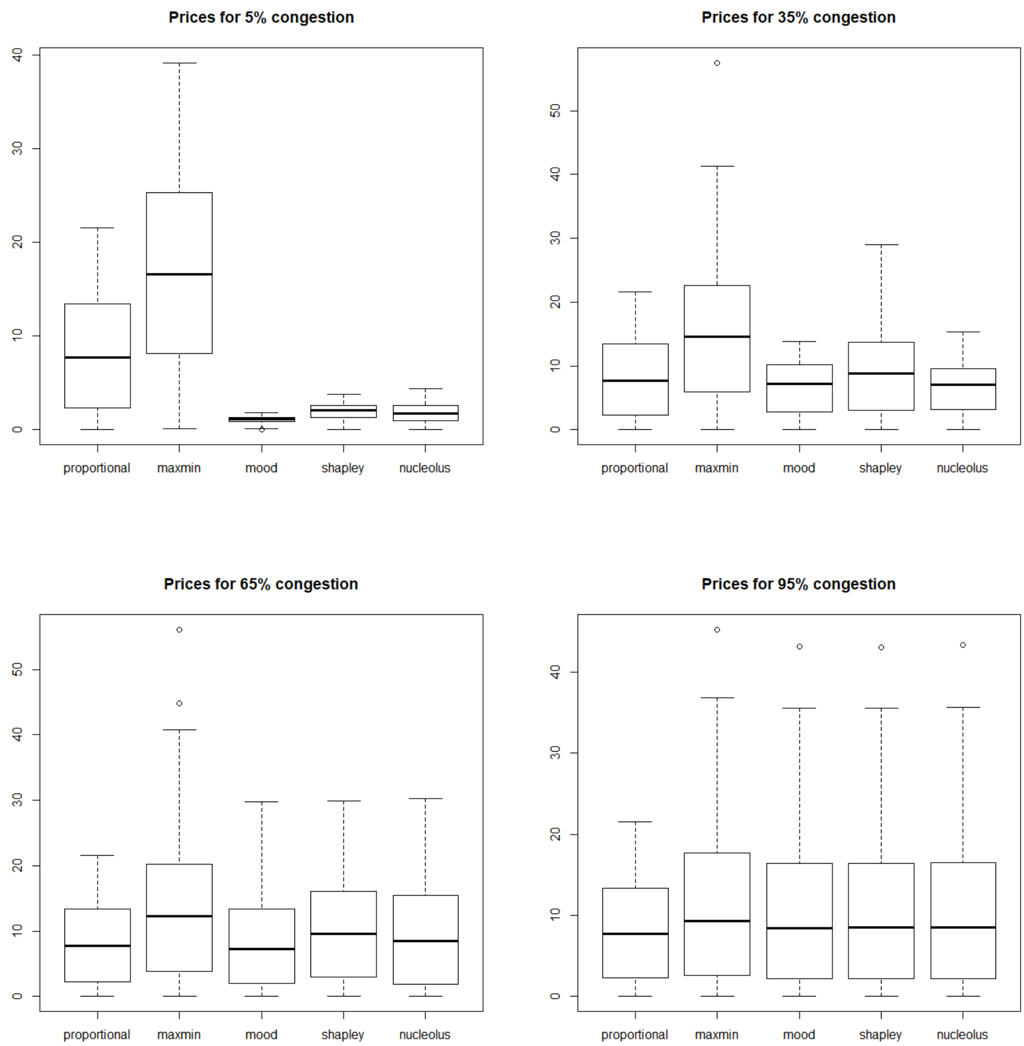

Figure 4.4: Prices for different congestion

As we can see from the boxplots, the prices are not so different if C is high. This is a reasonable result since values of C near to one give a non-meaningful bankruptcy game

$$\sum_i c_i \approx E$$

Going to the limit $C \to 1$, all the solutions gives the same result: all players receive a quantity equal to the demand.
The price obtained from the Max-Min allocation presents higher variability in the magnitude with respect to the others. This is because Max-Min fair allocation depends strictly on player's type.
Considering a bankruptcy game, we can classify a player i into four categories, named: MM, MG, GM and GG. If a player asks less than the total amount of resource, we call it moderate player (M) while, if he asks more, he is called greedy player (G). In a similar way, if the sum of the demand of the remaining $n-1$ players exceeds the total resource E, meaning that $v(\{i\}) = 0$, the group is a group of greedy players (G), otherwise we have a group of moderate players (M). This classification can be summarized in the table:

	$b_i < E$	$b_i \geq E$
$\sum_{j \neq i} b_j < E$	MM	MG
$\sum_{j \neq i} b_j \geq E$	GM	GG

Table 4.1: Types of player

The first letter refers to the group of players while the second one refers to the player itself (G=greedy or M=moderate).
This classification arises in a natural way looking at the formula of the mood value $x_i = min_i + m(max_i - min_i)$ because, depending on the case MM, MG, GM or GG, we have that min_i or max_i can be equal to 0.
Considering all possible combinations of $n = 3$ players, it is possible to characterize the type of game. In the article [4] it is proved that only six combinations are possible and the game can be:

- GG: all players are in scenario GG
- MM: all players are in scenario MM
- GM: all players are in scenario GM
- GM-MG: some players are in scenarion GM, others in scenario MG
- GM-GG: some players are in scenarion GM, others in scenario GG
- GM-MM: some players are in scenarion GM, others in scenario MM

This characterization can be useful to compare prices in a different way and see if there are some interesting patterns. We report the results for the simulations analyzed before.

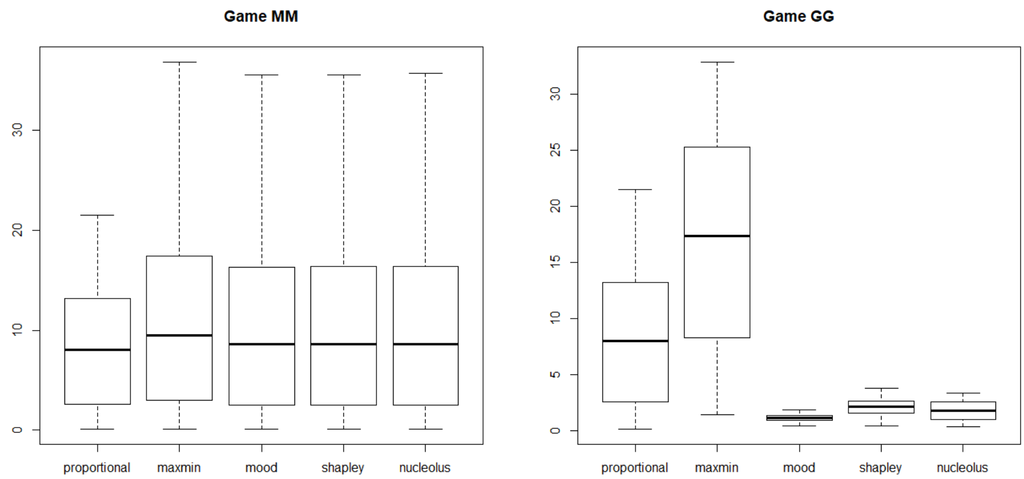

Figure 4.5: Prices in game MM and GG

As we expect, boxplots for games MM and GG are not so different from the ones for congestion $C = 0.95$ and $C = 0.05$ respectively. In games where there are two types of players, we differentiate them with different colors. The point in the center is the median, the other two are the first and the third quantile, similarly to a boxplot.

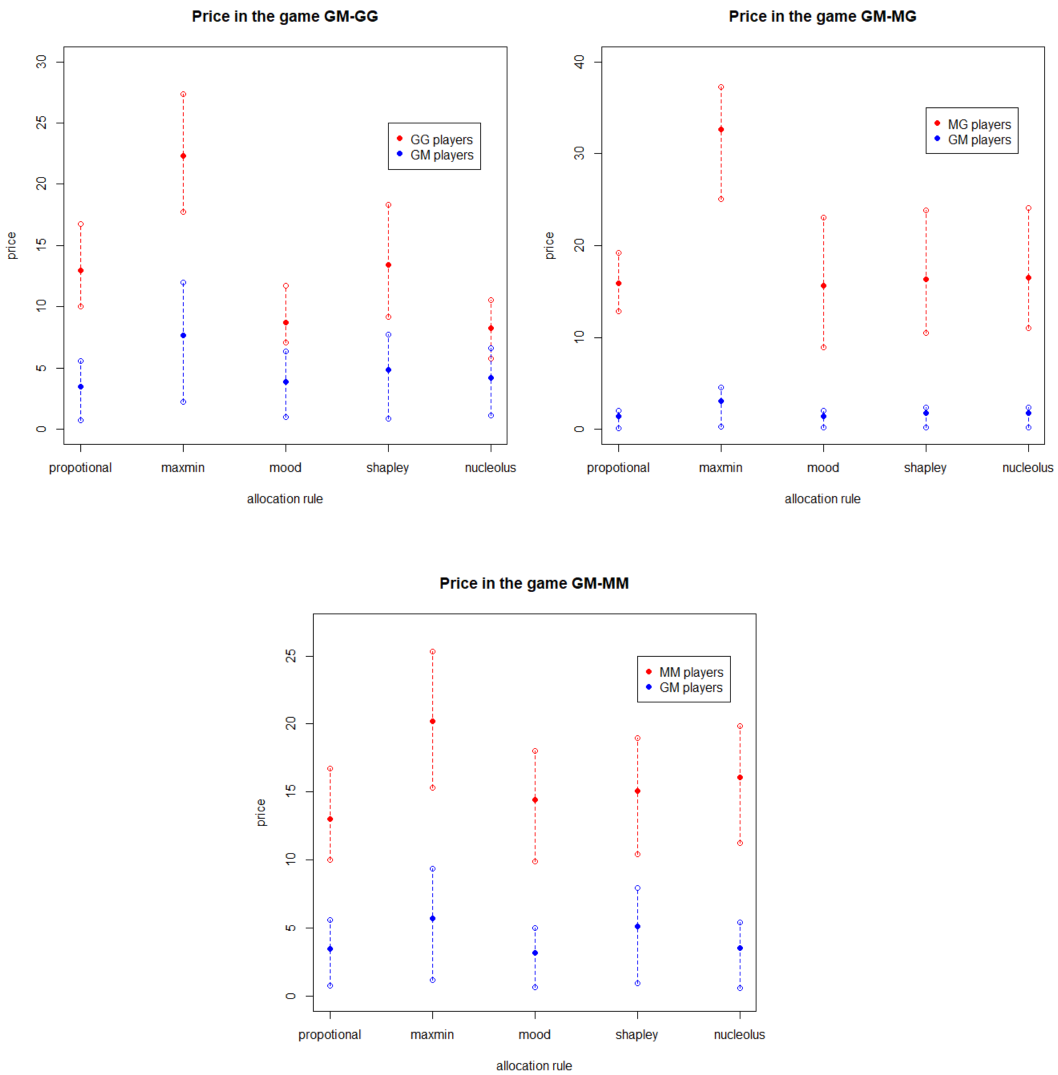

Figure 4.6: Prices in game GM-GG, GM-MG and GM-MM

As we can see from figure 4.6, there are some type of players that pays more. This happens for all allocation rules. We omit the plots for the game GM since to obtain a large number of games of this type, we should run more simulations or we should construct games ad hoc to obtain all GM players inside a game. Simulations with a greater number of players $n > 3$ are expensive and an increase in n let the number of MG players become negligible, as shown by the results obtained in [4].

Chapter 5

Collusion Analysis

In this chapter, we analyze the possibility of collusion in Myerson's auction. Even if the pricing rule ensures that there is no incentive to lie about the true demand, users can aggregate each other trying to obtain more at a lower price. It is important to determine if there is this possibility in order to prevent users' cheating behaviour. We already know that second price auctions, as the VCG one, are not robust to collusion, thus it is reasonable to suppose that even Myerson's auction is sensitive to aggregation. We run some simulations in order to prove or disprove this hypothesis. In order to compare the case of single players with the case of coalitions, we first consider the price given by Myerson and then we contruct a new cooperative game: for each allocation rule, we define the characteristic function. Since the core of a game represents the chance for players to reach an agreement, we try to characterize it using theorems illustrated in chapter one. Finally, we present an application that, with simple calculations, confirms our hypothesis.

5.1 Introduction

As first step of our analysis, we want to define more precisely what we mean with the term *collusion*. We have already used it in the third chapter, saying that it indicates the propensity of players to form coalitions. This is not entirely accurate since this term has a broader meaning. The general definition is the following:

> Collusion is an agreement, usually illegal and therefore secretive, which occurs between two or more persons to limit open competition by deceiving, misleading, or defrauding others of their legal rights, or to obtain an objective forbidden by law typically by defrauding or gaining an unfair advantage.

In general, collusion at an auction can involve bidders, sellers, and/or auctioneers. We focus our attention on buyers' collusion, since is the easiest to analyze, considering auctions with only one auctioneer and many potential buyers. The auction model assumes that all players have independent strategies. This is no longer valid as soon as some bidders make prior agreements, as is often seen in practice. Facing a well-defined bidding procedure, potential buyers cooperate sharing information and making agreements on a predetermined behavior during the auction. We analyze the propensity of players to aggregate forming coalitions. It is therefore essential to study the stability of these ones.
Classic auction mechanisms, such as the second-price auction, are not designed to withstand collusion among actors. The aim of our analysis is to indagate if Myerson's auction mechanism is robust to buyers' collusion.

5.2 Collusion analysis

To analyze collusion, we have to compare utility functions. Utilities give an order of preferences for players on the set of possible outcomes. This means that if we want to know what result is better between $\mathbf{x}_1$ and $\mathbf{x}_2$ for player i, we have to compare utility calculated at these two points: $u(\mathbf{x}_1)$ and $u(\mathbf{x}_2)$.
The auction is a non-cooperative model, thus aggregation among agents is not considered. To compare the case of single players with the case in which there are coalitions, we had to define a new cooperative game, allowing, as the name said, cooperations among participants. We start our analysis with the definition of a new game (N, v), starting from the given bankruptcy one $(\mathbf{c}, E)$.
This section is structured as follows:

- First part: theoretical framework introducing the new cooperative game (N, v).
- Second part: simulations and results.

5.2.1 Theoretical setting

We have already seen in the previous chapter that Myerson's auctions satisfy the truthfulness property: no player has incentive in lying on his valuation. This property ensures that all the bids presented to the auctioneer are equal to the true valuation of the resource. Nevertheless, there is another possibility that agents have to cheat the system. This consists in aggregating each others trying to reach a more attractive outcome, that in our case is a higher value of allocation and/or a lower price. The preferences over outcomes are described, as said before, by utility functions $u_i \quad \forall i \in N$. Given a bankruptcy problem $(\mathbf{c}, E)$, we recall that the utility in case of Myerson's mechanism is

$$u_i = v_i x_i(\mathbf{b}) - p_i(\mathbf{b})$$

where

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \frac{x_i(b_i, \mathbf{b}_{-i})}{E} - \frac{1}{E} \int_0^{b_i} x_i(z_i, \mathbf{b}_{-i}) dz$$

and recalling that 4.1 $v_i = \frac{b_i}{E}$, then we obtain:

$$u_i(b_i, \mathbf{b}_{-i}) = \frac{1}{E} \int_0^{b_i} x_i(z_i, \mathbf{b}_{-i}) dz \tag{5.1}$$

Thus, the utility is exactly the area below the curve [1] represented in the following picture

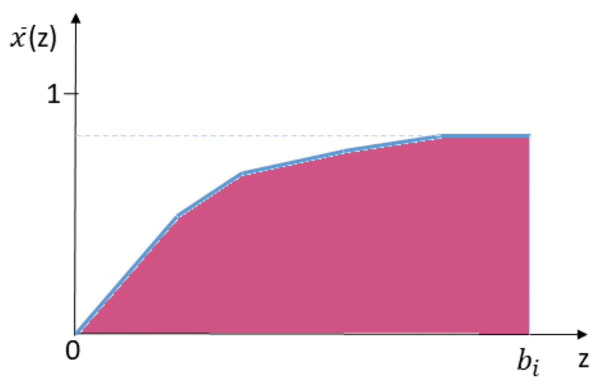

Figure 5.1: Utility as the area below the curve

Inside the contest of Myerson mechanism, once the allocation rule is fixed and the bankruptcy problem $(\mathbf{c}, E)$ is given, we want to know if it is convenient for two or more players form a coalition aggregating demands and partecipating as a single player. In order to analyze player's behaviour, we

[1]Recall that the allocation rule is normalized in such a way $\bar{x}_i \in [0, 1]$

define a new cooperative game. Given n agents, this new game is characterized as usual by the pair (N, v) where v is the characteristic function defined in the following way:

$$v(S) = u_S \quad \forall S \subset N \tag{5.2}$$

where the utility is given by the formula 5.1. In particular:

- For coalitions of two or more players, the characteristic function corresponds to the utility of an agent having as demand the sum of the demands, considering fixed all other players outside the coalition. This is better explained by the example below 5.1.

- For singletons $\{i\}$, it corresponds to the utility in Myerson's auction.

- For the grand coalition, it corresponds exactly to the the sum of all demands, since we are considering a single bidder auction. This is because all players partecipate together as a single player and, as consequence, the price is zero and all the available resource is allocated for every possible value of the demand. Recalling that the utility is the area below the curve 5.1, the area in this case is a rectangle where the base is $\sum_i c_i$ and the height is E. Thus, for a bankruptcy game $(\mathbf{c}, E)$, $v(N) = \sum_i c_i$, since the area is normalized and divided by E.

With these assumptions, it seems profitable to form the grand coalition since no money is due to the seller and the utility is higher than in all other cases.

$$u_S = \frac{b_S}{E} \times x_S(\mathbf{b}) - p_S(\mathbf{b}) \leq u_N = v_N \times E - 0 = \sum_{i \in N} b_i \quad \forall S \subseteq N$$

However, if this happens, the problem becomes how to divide fairly the resource among players bringing us back to the initial problem. Moreover, thinking to applications as the networking one (e.g. spectrum allocation), the grand coalition of all players is a rare event: claimants are spread on a large region and it is unconvient for faraway players to coalize and share the band.

We assume the demand of a coalition to be equal to the sum because the mechanism is truthful and each participant already know that there is no gain in giving a higher or a lower bid. Reasonably, each one gives to the seller the true valuation and the same happens when players form coalitions. Hence, we simply consider the case in which agents add up together the claims.

To better understand the construction of the characteristic function v, we provide an example.

Example 5.1. Consider a bankruptcy game with three players $N = \{1, 2, 3\}$ described by the couplet

$$(\mathbf{c}, E) = \left(\begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}, 10 \right)$$

Suppose the allocation rule $x(\mathbf{b})$ to be fixed. For singletons

$$\begin{aligned} v(\{1\}) = u_1 &= \frac{1}{E} \int_0^{b_1} x_1(z) dz \\ v(\{2\}) = u_2 &= \frac{1}{E} \int_0^{b_2} x_2(z) dz \\ v(\{3\}) = u_3 &= \frac{1}{E} \int_0^{b_3} x_3(z) dz \end{aligned}$$

where $\mathbf{b} = (b_1, b_2, b_3) = (2, 5, 7)$. For the grand coalition $v(N) = \sum_i c_i = 14$, while for coalitions of two players $v(\{i, j\}) = u_{\{i,j\}}$ where $u_{\{i,j\}}$ is the utility of the first agent in the game characterized by $\mathbf{b} = (b_i + b_j,\ b_z)$; z is the third player outside the coalition. For instance, if we consider $\{1, 2\}$, then

$$v(\{1, 2\}) = u_{\{1,2\}} = \frac{1}{E} \int_0^{b_1 + b_2} x_{\{1,2\}}(z) dz$$

with $\mathbf{b} = (5 + 2, 7) = (7, 7)$ and $E = 10$.
We give the precise result for all the allocation rules we consider in the analysis:

Proportional

$$\begin{array}{ccc} v(\{1\}) = 1,15 & v(\{2\}) = 1,02 & v(\{3\}) = 2,15 \\ v(\{1,2\}) = 2,15 & v(\{1,3\}) = 3,85 & v(\{2,3\}) = 8,11 \end{array}$$

Max-Min

$$\begin{array}{ccc} v(\{1\}) = 0,2 & v(\{2\}) = 0,8 & v(\{3\}) = 0,8 \\ v(\{1,2\}) = 1,25 & v(\{1,3\}) = 1,25 & v(\{2,3\}) = 3,2 \end{array}$$

Nucleolus

$$\begin{array}{ccc} v(\{1\}) = 0,1 & v(\{2\}) = 0,97 & v(\{3\}) = 2,17 \\ v(\{1,2\}) = 2,02 & v(\{1,3\}) = 3,6 & v(\{2,3\}) = 6,6 \end{array}$$

Mood value

$$\begin{array}{ccc} v(\{1\}) = 0,13 & v(\{2\}) = 1,01 & v(\{3\}) = 2,21 \\ v(\{1,2\}) = 2,14 & v(\{1,3\}) = 3,74 & v(\{2,3\}) = 6,72 \end{array}$$

Shapley value

$$\begin{array}{ccc} v(\{1\}) = 1,33 & v(\{2\}) = 0,95 & v(\{3\}) = 2,15 \\ v(\{1,2\}) = 2,05 & v(\{1,3\}) = 3,65 & v(\{2,3\}) = 6,7 \end{array}$$

5.2.2 Simulations

We run 200 simulations of bankruptcy games. As done in chapter four, we consider $n = 3$ players, bids independent and drawn from a uniform distribution $\mathcal{U}\{1, 2, 3, \dots, 100\}$ on integer numbers and the resource E chosen as a percentage of the sum of the demands, called congestion. Then, having these initial data, we calculate the value of the characteristic function v of the cooperative game (N, v) described in 5.2. To compare utilities, we calculate the difference between the marginal value of player i, that is

$$v(S) - v(S \setminus \{i\}) \quad \forall S \subseteq N,$$

and the value of the single player $v(\{i\})$. We use boxplot to show results and we differentiate games on the congestion rate. We repeat the procedure using all the allocation rules seen in the previous analysis: the proportional rule, the mood value, the Max-Min fair allocation, the nucleolus and the Shapley value.
Note that all the values have been normalized in order to better compare them. The boxplots show the difference divided by the total resource:

$$\frac{v(S) - v(S \setminus \{i\}) - v(\{i\})}{E}$$

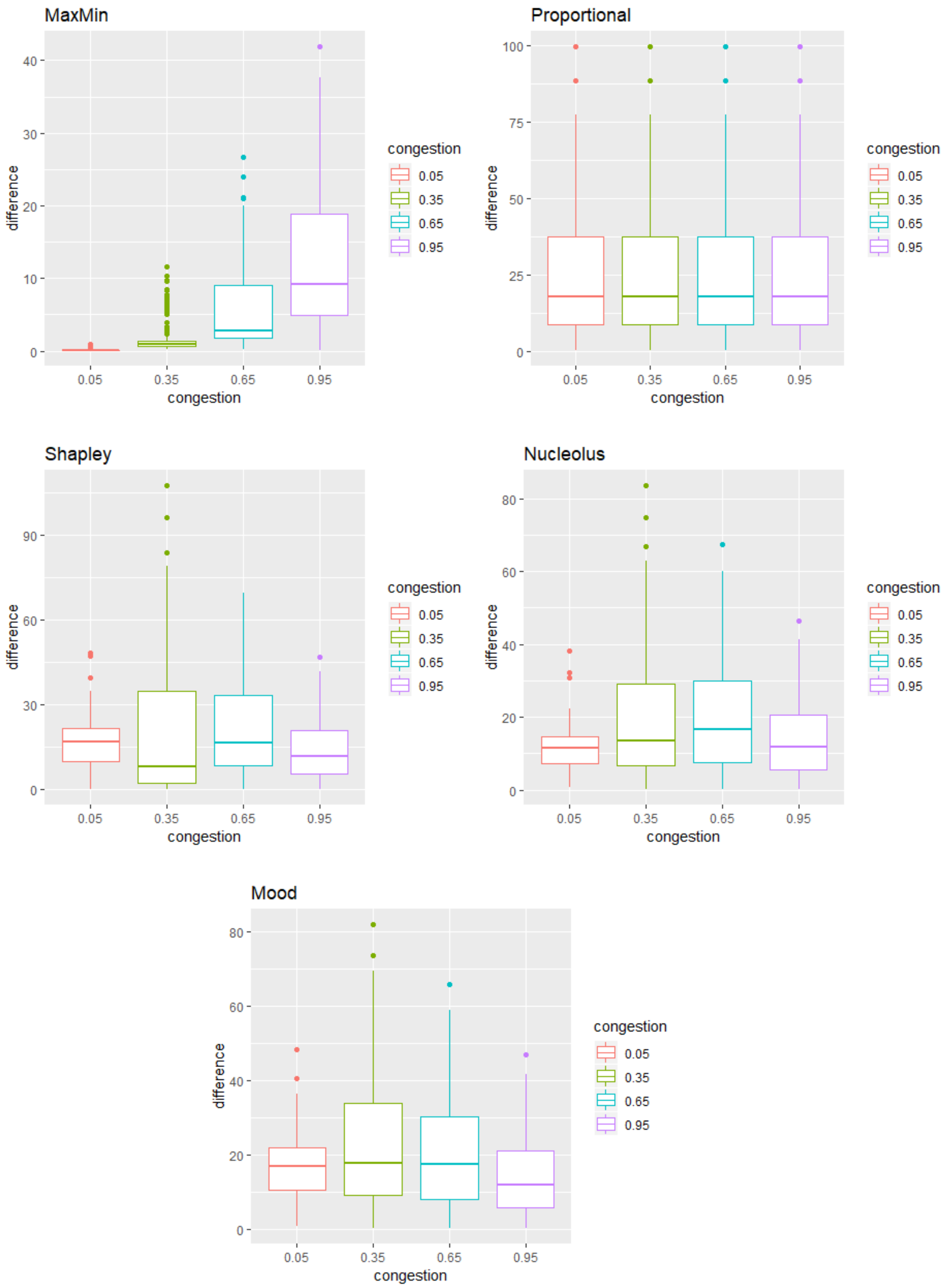

Figure 5.2: Comparison of utilities

As we can see from the figure 5.2, the differences are equal or greater than 0 for all allocations and for all congestion rates. In particular, we can observe that the first quartile is always greater than 0. This means that coalize is a weakly dominant strategy for most of the simulated games: the utility is greater or equal if players are aggregated than in the case they play alone. Looking at the boxplots for the proportional allocation rule, they are all equal. This is due to the construction of simulations: we randomly draw **c** and we let the congestion varying, moreover, for this allocation rule, the utility does not depend on the resource E but only on the bids **b**. As we already noticed in the analysis of pricinig function done in chapter four, the Max-Min allocation is the one for which we see the highest variability.
This result is not a theoretical result, since it is the outcome of a numerical analysis. Moreover, all simulations are done in case of three players since the number of possible coalitions grows exponentially with respect the number of players n. For these reasons we would like to verify the result using another method, in particular exploiting the interpretation of core of the game defined in 5.2.

5.3 Core characterization

Given a cooperative game (N, v), we recall the definition of the core:

$$\mathcal{C}(v) = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N) \wedge \sum_{i \in S} x_i \geq v(S) \quad \forall S \subset N \right\}$$

Considering the contest of bankruptcy problems and interpreting $\mathbf{x}$ as allocation, the core has the interpretation of set of solutions accepted by coalitions. Allocations inside the core are solutions for which cooperation is convenient and aggregation among players is encouraged. Thus, we can conclude that if a game has non empty core, then exists the possibility for players to cheat the system by aggregating in groups.

Taking into account this meaning, we try to characterize the core using the R package *CoopGame* [14]. Since the direct characterization is difficult, we use theorems proved in chapter one to prove or disprove the emptiness of the core. We simulate games with the procedure already described: we run 200 simulations of bankruptcy games for each allocation rule considering $n = 3$ players and bids independently drawn from $\mathcal{U}\{1, 2, 3, \ldots, 100\}$; we chose E as a percentage of the sum of the demands. Then, we calculate the

value of v described in 5.2.
After obtaining the game (N, v), we proceed in our analysis exploiting theorem 1.6. We first check superadditivity, using the function implemented in the R package *isSuperadditiveGame* that takes as argument the vector having as entries v calculated for all possible coalitions. Then, for superadditive games, we check the condition

$$v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \leq 2v(N)$$

Once fixed the allocation rule, the data are summarized in a table: each simulation is a row and in every column there is $v(S) \quad \forall S \subseteq N$. Moreover, we add two columns showing the results: the first gives information on the superadditivity of the game and the entry is yer or no; the second gives information on the core and the variable is:

- 1 if the core is nonempty
- 0 if the core is empty
- NA, i.e. Not Available, if the game is not superadditive, meaning that we cannot use the characterization given by the theorem. This means that we have not any information on the core

We give below an example of header of the resulting table in case of mood value allocation.

	v(1)	v(2)	v(1,2)	v(3)	v(1,3)	v(2,3)	v(N)	superadditive	core
1	3.081272737	31.0812727	31.747939	52.7699625	53.76996	95.76996	206	yes	1
2	1.716539211	8.5820421	27.335034	19.7797945	45.37694	68.35998	162	yes	1
3	0.004482473	20.5672722	29.982359	21.2919792	30.79198	116.46894	146	yes	1
4	3.703467115	13.1282850	26.659387	37.5522196	58.01951	89.91220	198	yes	1
5	1.492981634	10.2676269	31.080196	21.3391941	48.77997	78.70772	182	yes	1
6	3.176799908	4.8970503	17.997419	15.9692719	36.30603	41.67851	134	yes	1
7	1.583873015	7.4712427	26.707551	15.5921881	43.59219	56.77464	117	yes	1
8	13.728707885	20.3953746	22.728708	58.2813827	61.78138	71.78138	209	yes	1
9	5.617764808	8.6180689	10.399710	34.9513947	38.85434	47.05203	128	yes	1
10	1.716809729	21.0274046	31.270260	34.6184045	47.61840	106.01727	171	yes	1
11	7.335866665	15.4075929	23.634821	52.2394938	65.88813	87.40342	230	yes	1
12	4.409342313	11.6881966	15.223638	30.7055349	35.70553	53.80283	131	yes	1
13	2.116174475	21.8711699	23.475329	41.0686174	43.23999	94.09660	213	yes	1

Figure 5.3: Example of results in the case of Mood Value allocation.

Results for n=3

The simulations show that in case of three players all games are superadditive games for all allocation rules condidered. Thus, the theorem 1.6 holds and, checking the condition, we obtain a characterization of the core. We find that for all simulated games, the core is non empty. This means that there is convenience for a player to form coalitions with other players. We run more simulations obtaining the same result. However, the non emptiness of the core is not a rule: the outcome is obtained through a numerical analysis, thus we cannot say the core is empty in all three players bankruptcy games. Despite this fact, it is unlikely that in many simulations there is not any game with empty core. Moreover, since even second price auctions are not designed to withstand collusion among actors, it is reasonable to suppose that this holds also for Myerson's auctions.

Since for core characterization only the table is needed, we can extend our analysis to the four players' case. To obtain information on the core, we use a different procedure than in the three players' case. In particular, for $n = 4$ we use the theorem 1.2. This one gives a necessary and sufficient condition for the non emptiness of the core: the core is non empty if and only if the game is balanced. We recall that if the core is non empty, there is the conveninece for players to form coalitions. For this reason, we check the balancedness with the function implemented in the R package [14] and called *isBalancedGame*. The input is a TU cooperative game in the same form as the one for the function *isSuperadditiveGame*, while the output is TRUE if the game is balanced, FALSE otherwise.
As in the three players' case, the result can be described by a table. We give an example of header in the case of proportional allocation.

	v(1)	v(2)	v(3)	v(4)	v(1,2)	v(1,3)	v(1,4)	v(2,3)	v(2,4)	v(3,4)	v(1,2,3)	v(1,2,4)	v(1,3,4)	v(2,3,4)	v(N)	balanced
1	0.443231809	5.32666139	19.0153088	20.161802	9.4475238	27.917233	29.038396	53.404017	55.09609	102.89154	67.003813	68.98347	125.10961	203.15295	261	yes
2	4.799384223	7.15953178	8.1717065	15.387243	24.9391793	27.071085	46.679642	33.492776	55.66266	59.19889	71.776367	108.35899	114.20967	131.78602	259	yes
3	1.271176830	1.63519529	6.4731566	6.964614	10.7719336	18.986942	20.302622	22.382296	23.84193	36.86494	43.248485	45.54923	66.13646	74.62347	139	yes
4	0.211018899	1.55373597	7.2493613	20.263106	13.0921912	19.925303	42.369229	24.281929	49.31552	64.43363	46.626048	84.14948	106.97204	121.39943	212	yes
5	0.652699542	2.08188613	5.3359438	12.928745	2.0410901	5.720396	24.067114	7.895973	29.28342	45.42068	11.663075	37.98025	58.17236	69.74423	95	yes
6	0.076801774	9.62869185	20.2062821	22.129480	7.2079934	19.620342	22.659625	51.299841	56.83810	95.54861	54.947890	60.75596	101.40329	208.81029	229	yes
7	0.204882393	3.12942302	5.5219587	27.101658	9.1397953	12.682147	32.758383	34.144706	68.48001	80.59097	41.646297	80.59097	94.37640	177.18090	218	yes
8	1.513641942	4.58660501	16.5621074	17.947515	12.5671360	29.689991	32.147609	43.249445	46.36434	81.71037	67.280371	71.49881	119.59003	157.56405	244	yes
9	1.116222420	3.78606377	18.3318548	18.777989	12.3057115	32.121343	32.718504	50.781231	51.58057	94.96702	73.442676	74.47454	130.73335	183.69722	268	yes
10	0.472692103	7.70972238	20.6270739	26.804938	3.5579154	18.993104	31.772215	34.545033	52.66522	111.96769	38.989178	58.55157	122.89464	188.49889	216	yes
11	3.594437284	4.22493737	6.9363472	17.094404	17.5495101	23.180392	42.966856	24.881969	45.47894	55.55283	54.578149	88.53572	105.22043	110.24305	216	yes
12	0.591102952	6.38958440	8.4470697	14.325599	7.4193515	10.175676	21.041112	23.814201	41.35917	49.22565	31.763302	52.92858	62.45363	109.18978	146	yes

Figure 5.4: Example of results in the case of proportional allocation.

Results for n=4

The simulations for $n = 4$ players show that all games are balanced for all allocation rules. This means that by theorem 1.2, all games have non empty core. Thus, cooperation is convenient for agents that have the possibility to cheat the designed system. As said before, we can not take these results as a general rule. Using the function already implemented, we obtain that there are some simulated games not superadditive and/or not convex. This fact underlines that an increase in the number of players n could show different behaviours, obtaining games with different properties. Unfortunatly, the simulations with more than four players are computational expensive.

5.4 A simple application

Considering the context of networking, the possibility of collusion inside a Myerson's auction can be viewed from another perspective. In particular, we can consider an allocation problem in which each player has an account and we wonder if it is convenient for a player to have multiple accounts instead of one single account. In this case, the possibility of collusion is positive, meaning that a user has not incentive in the creation of multiple accounts, so that he has not incentive in cheating the system.
Since calculations for the comparison of utilities are very complex, we consider the case of the proportional allocation. Thus, we fix the rule to be the PF and, recalling that the price is given by 4.8, player i's utility is given by:

$$u_i = b_i - \sum_{j \neq i} b_j \; ln \left(\frac{\sum_j b_j}{\sum_{j \neq i} b_j} \right)$$

Suppose the two player's case $N = \{1, 2\}$, we want to know if player one has interest in having two accounts. His utility in case of one unique account is given by:

$$u_1 = b_1 - b_2 \; ln \left(\frac{b_1 + b_2}{b_2} \right)$$

Suppose he divides his demand into two accounts: with the first, he asks $b_1 - x$, while using the second, he asks x. For simplicity, we assume $x \in [0, b_1]$. Is it convenient?

The utility of player one, considering his first account, is

$$\bar{u}_1(x) = (b_1 - x) - (b_2 + x)\ ln\left(\frac{b_1 + b_2}{b_2 + x}\right) \tag{5.3}$$

The question is where the function $\bar{u}_1$ with respect to x, described by the formula above 5.3, has its maximum: if it has maximum for $x = 0$ means that player one has no convenience in having a fake identity and in dividing his demand; if the maximum is reach for $x = b_1$, then player one has interest in owning more accounts.
The derivative is

$$\bar{u}_1'(x) = -ln\left(\frac{b_1 + b_2}{b_2 + x}\right)$$

It is a decreasing function, thus player 1's utility has maximum for $x = 0$. This confirms our thesis: there is incentinve in collusion. In this case can be view as a positive property since it makes sure that none of the users has multiple accounts.

Conclusion and Further Developments

The aim of this work was to analyze a different way to solve bankruptcy problems. In particular, we reviewed the usual bankruptcy situation approaching it as a non-cooperative game. We rewrote the allocation problem as an auction using the game theoretical model. We focused our attention on the Myerson's auction mechanism. As second price auctions, it is described by an allocation rule and a princing rule. It is proved that this type of auction is truthful, so users are not incentivate in lying on their true evaluation. For this reason, we fixed an allocation rule and we calculated the pricing function that makes this property hold. Thus, thinking about the bankruptcy situation, it ensures that players' demand is the real one. Then, we wondered if it exists another possibility to cheat the system. For this reason, we analyzed if partecipants are incentivated in aggregation, forming coalitions and obtaining a larger fraction of the available resource and/or a lower price. Following literature, we called this possibility collusion. It can be proved that second price auctions are not robust, hence we supposed that the same holds for Myerson's mechanism.
To prove or disprove our assumption, we run numerical simulations. For all the resulting games, the plots we have obtained show that aggregation is convenient for all the allocation rules considered. Thus, since the behaviour chosing a proportional rule is not different from the one obtained chosing the mood value or any other among the considered, we can fix the rule according the problem we have to model. The analysis done seems to confirm the hypothesis that Myerson's auction, as the second price one, is not designed to withstand collusion among players. However, this result is not a general rule. In order to prove that collusion is always possible, we have to verify it in a theoretical way, starting from the definition in 5.2. We tried to analyze the marginal value in the simplest possible case that is considering the proportional allocation. Since the utility given by Myerson's mechanism is complicated even in this case, we could not reach a conclusion. Despite this fact, it seems unlikely that in many simulations there is not any game in which collusion is not possible.

To conclude, we report some points that can be better analyzed in further analysis.

- First of all, with the use of more powerful calculator, we can extend the simulations to more than three or four players.
- Second, the simulations can be constructed in another way for example chosing E at random and then consequently chose the vector $\mathbf{c}$, in order to see if some results or some games' properties are different.
- Extend the analysis to all semivalues, described in chapter one, starting from the pricing function already implemented for the Shapley solution.
- Apply the resulting pricing function to a real applicated problem.
- Extend the model considering more than one seller.
- Finally, we can try to obtain theoretical results on the possibility of collusion using for example an allocation rule made ad hoc in such a way that calculations are easier.

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