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Hawking Radiation in Bose-Einstein Condensates: Analytical Results

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Abstract

The purpose of this paper is to treat, with purely analytic method, the study of the analogue Hawking effect in Bose-Einsten condensates (BEC), assuming smooth velocity fields. This work is based on recent analysis which allow a more complete study about the behavior of the modes in a neighborhood of the horizon (turning point). Thanks to an analytical treatment, we can recover an explicit expression for the greybody factor.
The first part of the analysis is standard and it can be found in literature; it consists in a system of one-dimensional second order coupled equations, which can be decoupled in two fourth order ordinary equations.
We find the WKB solutions, far from the horizon. Then, thanks to recent achivements, it has been possible to recover approximations of the solutions near the turning point and, by means of matched asymptotic expansion method, to glue the two approximations.
Finally, by the conservation of current, we will evaluate the termality and the greybody factor.

Estratto

L'obiettivo di questo elaborato è quello di trattare, tramite metodi puramente analitici, lo studio analogo dell'effetto Hawking nei condensati di Bose-Einstein (BEC), assumendo campi di velocità lisci. Il lavoro è basato su recenti analisi che permettono uno studio più completo per quanto riguarda il comportamento dei modi in un intorno dell'orizzonte (turning point). Grazie al fatto di mantenere una trattazione analitica, si è potuto ricavare l'espressione del fattore di corpo grigio.
La prima parte dell'analisi è standard e può esser trovata in letteratura; essa consiste in un sistema di equazioni differenziali del second'ordine, unidimensionali e accoppiate, che possono esser disaccoppiate ottenendo due equazioni differenziali ordinarie del quarto ordine.
Si ricavano le soluzioni WKB, lontano dall'orizzonte. Quindi, grazie a recenti risultati, è stato possibile trovare un'approssimazione delle soluzioni vicino al turning point e, per via di metodi di matched asymptotic expansion, incollare le due approssimazioni.
Infine, usando la conservazione delle correnti, si è ricavato la termalità e il fattore di corpo grigio.

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Chapter 1

Introduction

In a seminal paper, William Unruh [Unr81], described, through a formal equivalence, the behavior of sound waves in a fluid cascade and that of light in a black hole space-time. Essentially, when the fluid velocity crosses the speed of sound, waves cannot propagate upstream anymore. This leads to the creation of the so-called sonic horizon, analogous to the event horizon of a black hole, predicted by general relativity. Stated this analogy, researchers have found many different ways to create it in laboratory.
Beyond the pure analogy, Unruh presented two important ideas. The first is that analogue systems earning a long-lived [1] horizon have to exhibit Hawking radiations, since this phenomenon is kinematic and independent from any gravitational dynamics. On the other hand, Unruh pointed out, almost immediately after Hawking proposed his evaporation theory, that Hawking's calculations suffers from the 'trans-Planckian' problem [2].
In this frame, analogue black holes seems to be a fantastic test to understand how high-energy processes might come into play in Hawking's effect.
Analogue systems provide a clarification of how a continuous low-energy relativistic space-time diffuses when approaching the atomic high-energy description. Even if the importance of Hawking's proposal of black holes evaporation has already been validated by the relevance of the development that it has inspired, in physics any theoretical prediction has to be confronted with nature. Here we face an elementary problem: there are no clear prospects to verify Hawking's effect by gravitational black holes, not in the near future. Are long-lived trapping horizons produced naturally in astrophysical scenarios? If they are, do these horizons radiate?
Assuming the previous statements to be true, the importance of analogues systems can reduce the lack of the observations of the evaporation which is almost impossible to study by direct investigations. We need to show that

[1] Long-lived is used to refer to an horizon which exists from a longer time with respect to the ones characteristics of the system.

[2] Hawking modes close to the horizon region, posseses huge frequency components, beyond the Planck scale.

setting up an horizon leads to a spontaneous particle production, in particular Hawking particles. The importance in experimental development is to show that approximations, simplifications and additional factors, not apparently present in the initial theoretical description, do not destroy the global effects.
In its first work, Unruh used water to reproduce an analogue event horizon, but, unfortunately, water flume experiments are unable to probe the quantum spontaneous creation aspects of Hawking radiation, since thermal effects are dominant. One needs a quantum analogue system such as a Bose-Einstein condensate (BEC) [GACZ00].
In a BEC, it is possible to achieve a background temperature of the order of the Hawking temperature. The typical dispersion relation for achoustic phonons in BEC is:

$$(\omega - vk)^2 = c^2 \left(k^2 + \frac{k^4}{k_0^2}\right),$$

where c is the speed of sound.
In 2010, Jeff Steinhauer and its group produced a sonic horizon in a BEC [LIB$^+$10, MdNGKS19]. They created an horizon by making the condensate flowing down a step-like potential cascade. In this way, the fluid condensate accelerates to velocity higher than the speed of sound in a BEC.
From this considerations a rich phenomenology arises, connected to the analogue gravity, based on the possibility of investigating Hawking radiations in analogous systems.
During the years, many analogous black holes descriptions have been produced and we deal with one of them: analogue black hole in BEC with smooth velocity fields and superluminal dispersion relation.
By using purely analytical computations and without assuming step-like velocity fields, it has been possible to study the characteristic modes of analogous black holes, to compute the temperature and to evaluate analitically the greybody factor. In most of the other cases these analytical expressions cannot be evaluated, since some initial approximations do not allow to derive the expressions of the necessary modes, involved in the calculations of the greybody factor.
This work can be divided in four steps: firstly, we found the asymptotic solutions, at first order, through the WKB method, far from the horizon. This solutions allowed us to recover the expressions of the wavevectors of the modes, and their group velocities.
Secondly, we compute the solutions in the near horizon region, fundamental step which had allowed us to connect the two approximations through matched asymptotic expansion, which is the third step.
Finally, once recovered the matching coefficients and evaluated the analytical expressions of the modes, it has been possible to find an analytical expression for the thermality and the greybody factor, through conserved current.

Chapter 2

Quantum field theory

2.1 Classical mechanics

Let us introduce the Lagrangian formalism [CCP82,GP80]. Let $x_1, ..., x_N$, be N points in $\mathbb{R}^3$. We introduce the notion of Lagrangian coordinates $q_1, ..., q_f$ such that

$$x_j = x_j(q_1, ..., q_f, t), \qquad j = 1, ..., N.$$

Here f is the number of degrees of freedom, such that $f = 3N - p$, and p is the number of constraints on our system. These coordinates are general coordinates that describe our system (can be cartesian, polar, cylindrical, ecc.).
We introduce also the concept of *Material derivative* of a point:

$$\dot{x}_j = \frac{\mathrm{d}x_j}{\mathrm{d}t} = \frac{\partial x_j}{\partial q_s}\delta\dot{q}_s + \frac{\partial x_j}{\partial t},$$

and therefore, the Kinetic energy, can be rewrite as

$$T = \frac{1}{2}m_j\dot{x}_j = \frac{1}{2}m_j\left(\frac{\partial x_j}{\partial q_s}\delta\dot{q}_s + \frac{\partial x_j}{\partial t}\right).$$

Let us define now the Lagrangian function:

$$\mathcal{L}(q, \dot{q}, t) = T(q, \dot{q}, t) - U(q, t),$$

where T is the kinetic energy of the system and U is the potential energy. The Lagrange equations are defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0,$$

which represent a set of f second order ordinary equations.
The conjugate momentum p_s relative to the general coordinate q_s is defined as

$$p_s = \frac{\partial \mathcal{L}}{\partial \dot{q}_s}.$$

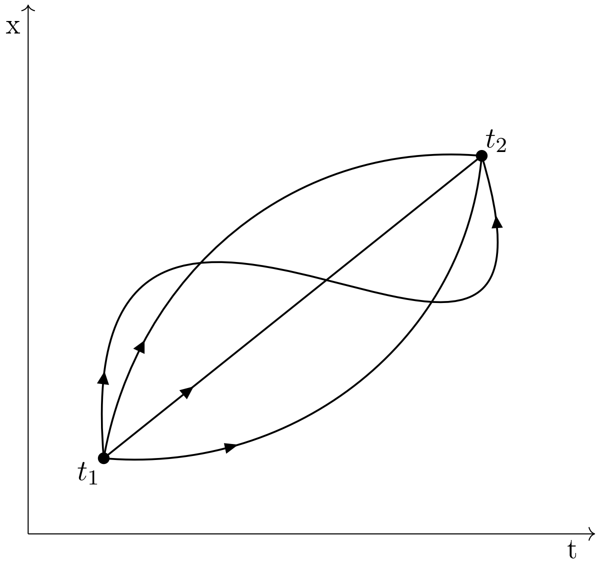

Figure 2.1: Some of the infinite number of paths between two points.

We also introduce the notion of action S as

$$S = \int \mathcal{L}\, \mathrm{d}t,$$

where the integral is taken over an entire path, from t_1 to t_2:

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x})\, \mathrm{d}t.$$

As the particle moves from $x(t_1)$ to $x(t_2)$, there is an infinite number of possible paths it may take, some of which are shown in Figure 2.1. The specific path choosed by the particle is determined by the principle of least action, which states that the actual motion is the one for which S is minimum.
Another essential way to describe our system, widely used in classical mechanics but especially in quantum mechanics, is the Hamiltonian one. The Hamilton function is defined as

$$H(q, p, t) = T(q, \dot{q}, t) + U(q, t)$$

and represents the total energy E_{tot} of the system. In point-particle mechanics, the relation between the Hamilton and Lagrangian function is

$$H(q, p, t) = \sum_s p_s \dot{q}_s - \mathcal{L}(q, \dot{q}, t),$$

where we recall that $p_s = \partial\mathcal{L}/\partial\dot{q}_s$. We have now a set of $2f$ first order ordinary equations:

$$\dot{p} = \frac{\partial H}{\partial \dot{q}}, \qquad \dot{q} = -\frac{\partial H}{\partial p}.$$

2.2 Lagrangian formulation for scalar fields

2.2.1 Relativistic notation

Any theory of the fundamental nature of matter must be consistent with relativity, as well as with quantum theory [Ryd96, Wal84].
Consider two events in space-time (x, y, z, t) and $(x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z, t+\mathrm{d}t)$. We want to generalize the notion of distance between two points in space, to the 'interval' $\mathrm{d}s$ between two events in space-time. Since $\mathrm{d}s$ has to be the same for all the inertial observers, it must be invariant under Lorentz transformations and rotations, and so it is given by [1]

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \qquad (2.2.1)$$

We call the vector which connects the two events *timelike* if they are separated by an interval such that $\mathrm{d}s^2 > 0$; *spacelike* those with $\mathrm{d}s^2 < 0$; *null* or *lightlike* the ones with $\mathrm{d}s^2 = 0$.
The generalization to 4-dimensional space-time is not straightforward due to the invariant interval no longer being positive definite, as in 3-dimensional space. We therefore define

$$\begin{aligned} x^\mu &= (x^0, x^1, x^2, x^3) = (ct, x, y, z), \\ x_\mu &= (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z) \end{aligned} \qquad (2.2.2)$$

and redefine the invariant as:

$$\mathrm{d}s^2 = \sum_{\mu=0}^{3} \mathrm{d}x^\mu \mathrm{d}x_\mu = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2.$$

A 4-vector like x^μ, with upper index, is called *contravariant vector* and one like x_μ, with a lower index, is called *covariant vector*. The inner product of a covariant and a contravariant vector is an invariant (scalar)[2].
The relation between x^μ and x_μ may be given by introducing a *metric tensor* $g_{\mu\nu}$:

$$\begin{aligned} x_\mu &= g_{\mu\nu} x^\nu \\ &= g_{\mu 0} x^0 + g_{\mu 1} x^1 + g_{\mu 2} x^2 + g_{\mu 3} x^3. \end{aligned}$$

By inspection of (2.2.2), the metric tensor $g_{\mu\nu}$ may be written as a diagonal matrix

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

[1] Of course, we could have defined $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$; we choose (2.2.1) for later convenience.

[2] To simplify the notation, we adopt the *summation convention*: an index appearing once in an upper and once in a lower position is implicitly summed (in this case, from 0 to 3).

where $g^{\mu\nu}$ is the inverse [3] of $g_{\mu\nu}$ (that exists, since the metric tensor has non-zero determinant).
Concerning differential operators, we define

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \partial_1, \partial_2, \partial_3) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

and, as above, we can recover the *contravariant* form through the *covariant* and viceversa thanks to the metric tensor and its inverse as

$$\partial^\mu = g^{\mu\nu}\partial_\nu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right).$$

From this definitions, we recover the Lorentz invariant second-order differential operator

$$\Box = \partial^\mu\partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{2.2.3}$$

called *D'Alambert operator*.
The energy-momentum 4-vector of a particle is

$$p^{\,\mu} = \left(\frac{E}{c}, \mathbf{p}\right), \qquad p_\mu = \left(\frac{E}{c}, -\mathbf{p}\right)$$

giving the invariant

$$p^2 = p^{\,\mu}p_\mu = \frac{E^2}{c^2} - \mathbf{p}\cdot\mathbf{p} = m^2c^2. \tag{2.2.4}$$

2.2.2 Klein-Gordon equation

We are now able to write a wave equation for a particle with no spin, a scalar particle [Ryd96, Sch14]. Since it has no spin, it has only one component, which we denote by ϕ. The wave equation is obtained from equation (2.2.4) by substituting differential operators for E and $\mathbf{p}$, in the standard fashion of quantum theory

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \mathbf{p} \to -i\hbar\nabla. \tag{2.2.5}$$

Equation (2.2.4) then gives

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi + \frac{m^2c^2}{\hbar^2}\phi = 0$$

which becomes, in units $\hbar = c = 1$ and using (2.2.3)

$$\left(\Box + m^2\right)\phi = 0. \tag{2.2.6}$$

This is known as the Klein-Gordon equation.

[3] The inverse of the metric tensor has the same values as $g_{\mu\nu}$ in Minkowski space (in Cartesian coordinates), but this equality does not hold in general.

2.3 The real scalar field

2.3.1 Variational principle and Noether's theorem

The passage from a point particle at position $x(t)$ to a field $\phi(x^\mu) = (x, y, z, t)$ can be visualized as the 'replacement' of x by ϕ and of t by x^μ [Ryd96]. The scalar field obeys the Klein-Gordon equation (2.2.6).
We now show how to derive the Euler-Lagrange equation applying a variational principle to an action

$$S = \int \mathcal{L}(\phi, \partial_\mu\phi, x^\mu)\, \mathrm{d}^4x.$$

The Lagrangian is defined as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 \\ &= \frac{1}{2}[(\partial_0\phi)^2 - (\nabla\phi)^2 - m^2\phi^2]. \end{aligned} \tag{2.3.1}$$

The field ϕ traces a 4-dimensional region R of space-time. We now subject both x^μ and ϕ to a variation, which vanishes on the boundary ∂R

$$\begin{aligned} x^\mu &\rightarrow x'^\mu = x^\mu + \delta x^\mu, \\ \phi(x) &\rightarrow \phi'(x) = \phi(x) + \delta\phi(x). \end{aligned} \tag{2.3.2}$$

It is important to note that $\delta\phi$, as defined above, is the functional variation of ϕ and that ϕ' is compared with ϕ at the same event x^μ in space-time.
We define also the *total variation* $\Delta\phi$ of ϕ as

$$\phi'(x') = \phi(x) + \Delta\phi(x)$$

and then it follows, to first order in $\delta x = x' - x$, that

$$\begin{aligned} \Delta\phi &= \phi'(x') - \phi(x') + \phi(x') - \phi(x) \\ &= \delta\phi + (\partial_\mu\phi)\delta x^\mu. \end{aligned}$$

The varation in the action then is

$$\delta S = \int \mathcal{L}(\phi', \partial_\mu\phi', x'^\mu)\, \mathrm{d}^4x' - \int \mathcal{L}(\phi, \partial_\mu\phi, x^\mu)\, \mathrm{d}^4x,$$

where $\mathrm{d}^4x' = J(x'/x)\mathrm{d}^4x$, and $J(x'/x)$ is the Jacobian of the transformation from x to x'. From (2.3.2)

$$\frac{\partial x'^\mu}{\partial x^\lambda} = \delta^\mu_\lambda + \partial_\lambda\delta x^\mu \quad \Rightarrow \quad J\left(\frac{x'}{x}\right) = \det\left(\frac{\partial x'}{\partial x}\right) = 1 + \partial_\mu(\delta x^\mu).$$

So, we can rewrite the variation of the action as

$$\delta S = \int (\delta \mathcal{L} + \mathcal{L}\partial_\mu \delta x^\mu)\, \mathrm{d}^4 x,$$

where

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu.$$

We can notice from (2.3.2) that we can commute the variation with the derivative as $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$ and then, observing that we can rewrite

$$\mathcal{L}\partial_\mu \delta x^\mu + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu = \partial_\mu(\mathcal{L}\delta x^\mu)$$

as the total divergence, the variation of the action becomes

$$\delta S = \int_R \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \partial_\mu(\mathcal{L}\delta x^\mu)\right] \mathrm{d}^4 x.$$

The second term may be rewritten so as to introduce a total divergence

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right] - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right]\delta\phi,$$

and the resulting integral over R can be written as a surface integral over ∂R, using the 4-dimensional generalization of Gauss's theorem, giving us

$$\delta S = \int_R \left\{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right]\right\}\delta\phi\, \mathrm{d}^4 x + \int_{\partial R}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \mathcal{L}\delta x^\mu\right] \mathrm{d}\sigma_\mu.$$

Since we took two variations that vanish on the boundary, i.e. $\delta\phi = 0$ and $\delta x^\mu = 0$ on ∂R, the second integral vanishes and the condition for a stationary action is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] = 0.$$

This is known as *Euler-Lagrange equation* for ϕ. It is the equation that describes the motion of the field ϕ (like Newton's equation for point masses). From here, we can also recover the Klein-Gordon equation: rewriting (2.3.1) as

$$\mathcal{L} = \frac{1}{2}g^{k\lambda}(\partial_k \phi)(\partial^\lambda \phi) - \frac{m^2}{2}\phi^2$$

gives

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = g^{\mu\nu}(\partial_\nu \phi) = \partial^\mu \phi$$

and then the Euler-Lagrange equation gives

$$\partial_\mu \partial^\mu \phi + m^2\phi \equiv \Box\phi + m^2\phi = 0$$

which is the Klein-Gordon equation.
We now explore another consequence of the use of a variational principle. If the action is unchanged under a re-parametrization of x^μ or ϕ, i.e. is invariant under some group of transformations on x^μ or ϕ, there exist one or more conserved quantities that are combinations of the field and its derivative. This result is known as *Noether's theorem*. Let us rewrite the surface term in the variation in the action as follows:

$$\delta S = \int_R \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \right\} \delta\phi \, \mathrm{d}^4 x + \int_{\partial R} \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} [\delta\phi + (\partial_\nu \phi)\delta x^\nu] - \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right] \delta x^\nu \right\} \mathrm{d}\sigma_\mu,$$

having added and subtracted one term.
The term in the first square bracket, in the surface integral, is the total variation

$$\Delta\phi = [\delta\phi + (\partial_\nu \phi)\delta x^\nu],$$

while we defined as *energy-momentum tensor* θ^μ_ν the term in the second square bracket:

$$\theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}.$$

We have then

$$\delta S = \int_R \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \right\} \delta\phi \, \mathrm{d}^4 x + \int_{\partial R} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \theta^\mu_\nu \delta x^\nu \right] \mathrm{d}\sigma_\mu$$

We now suppose that S is invariant under a group of infinitesimal transformations on x^μ and ϕ:

$$\Delta x^\mu = X^\mu_\nu \delta\omega^\nu, \qquad \Delta\phi = \Phi_\mu \delta\omega^\mu$$

characterised by an infinitesimal parameter $\delta\omega^\nu$. We highlight that in our case X^μ_ν is a matrix, Φ_μ is a set of number and ν is a single index (everything can be generalized to multiple indices). If we assume the previous transformation, the requirement that $\delta S = 0$ gives us

$$\int_{\partial R} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - \theta^\mu_k X^k_\nu \right] \delta\omega^\nu \, \mathrm{d}\sigma_\mu = 0$$

or, since $\delta\omega^\nu$ is arbitrary

$$\int_{\partial R} J^\mu_\nu \, \mathrm{d}\sigma_\mu = 0$$

where

$$J^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - \theta^\mu_k X^k_\nu.$$

It follows from Gauss's theorem and form the fact that R is arbitrary, that

$$\int_R \partial_\mu J^\mu_\nu \, \mathrm{d}^4x = 0 \qquad \Rightarrow \qquad \partial_\mu J^\mu_\nu = 0. \tag{2.3.3}$$

We therfore have a *conserved (divergenceless) current* J^μ_ν, whose existence follows from the invariance of the action under the previous tranformations. This give rise to a *conserved (time indipendent) charge* Q_ν, defined by

$$Q_\nu = \int_\sigma J^\mu_\nu \, \mathrm{d}\sigma_\mu,$$

where the integral is taken over a spacelike hypersurface σ. If the points in the surface are at constant time, then

$$Q_\nu = \int_V J^0_\nu \, \mathrm{d}^3x$$

where the integral is taken over a 3-dimensional volume V. Then the conservation of the charge can be stated integrating over V, as

$$\int_V \partial_0 J^0_\nu \, \mathrm{d}^3x + \int_V \partial_i J^i_\nu \, \mathrm{d}^3x = 0.$$

The second term can be transformed into a surface integral through the 3-dimensional Gauss's theorem, and vanishes, as the surface goes to infinity, leaving,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V J^0_\nu \, \mathrm{d}^3x = \frac{\mathrm{d}Q_\nu}{\mathrm{d}t} = 0.$$

This is *Noether's theorem.*

2.4 Canonical Quantization and Particle Interpretation

2.4.1 The real Klein-Gordon field

Now we consider the Klein-Gordon equation, describing a field $\phi(x)$ [Ryd96, Sch14]. Since the equation has no classical analogue, $\phi(x)$ is a strictly quantum field and then, we treat it as an operator, subjected to various commutation relations. This process is also known as 'second quantization'. To begin, let us find the energy of the 'classical' Klein-Gordon field. The Hamilton function, starting from the definition of the Lagrangian given in (2.3.1), is defined as

$$H = \frac{1}{2} \int \left[(\partial_0 \phi)^2 + \nabla\phi \cdot \nabla\phi + m^2\phi^2 \right] \mathrm{d}^3x. \tag{2.4.1}$$

For the complex scalar field, we have

$$H = \int \left[(\partial_0 \phi^*)(\partial_0 \phi) + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi \right] \mathrm{d}^3 x$$

and in each case, the Hamiltonian and thus the energy, is positive definite. The field $\phi(x)$ is regarded as an *Hermitian operator*, whose Fourier expansion may be written

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega_k} \left[a(k) e^{-ik\cdot x} + a^\dagger(k) e^{ik\cdot x} \right], \tag{2.4.2}$$

with $\omega_k = (\mathbf{k}^2 + m^2)^{\frac{1}{2}}$ and $c = 1$. The coefficients $a(k)$ and $a^\dagger(k)$ are also operators. The measure of the integrand has been so chosen because it is relativistically invariant. For the Klein-Gordon field we have the 'mass-shell' condition $k^2 = k_0^2 - \mathbf{k}^2 = m^2$ ($\hbar = c = 1$), so an invariant element in phase space is, with $k_0 > 0$ (positive energy condition)

$$\begin{aligned}
\frac{\mathrm{d}^4 k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\theta(k_0) &= \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta(k_0^2 - \omega_k^2)\theta(k_0) \\
&= \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta[(k_0 + \omega_k)(k_0 - \omega_k)]\theta(k_0) \\
&= \frac{\mathrm{d}^4 k}{(2\pi)^3} \frac{1}{2k_0} [\delta(k_0^2 + \omega_k) + \delta(k_0^2 - \omega_k)]\theta(k_0) \\
&= \frac{\mathrm{d}^4 k}{(2\pi)^3} \frac{dk_0}{2k_0} \delta(k_0^2 - \omega_k)\theta(k_0) = \frac{d^3 k}{(2\pi)^3 2\omega_k}
\end{aligned}$$

where $\theta(k_0)$ is the heaviside function and $\mathrm{d}^4 k = \mathrm{d}^3 k \, \mathrm{d}k_0$.
The quantity $\phi(x,t)$ plays a role in field theory analogous to that played by $\mathbf{x}$, the position vector, in classical mechanics.
The quantization of mechanics follows from the Heisenberg commutation relations

$$\begin{aligned}
&[x_i, p_j] = i\delta_{ij} \qquad (i,j = 1,2,3), \\
&[x_i, x_j] = [p_i, p_j] = 0,
\end{aligned} \tag{2.4.3}$$

where the momentum p_j is defined canonically as $\partial L/\partial \dot{x}_j$. $\mathbf{x}$ and $\mathbf{p}$ refer to the position and momentum of a particle, measured at the same time!
In a scalar field theory $\phi(\mathbf{x}, t)$ plays a role analogous to $\mathbf{x}(t)$, and describes a system with an infinite number of degrees of freedom, since, at each time, ϕ has an indipendent value at each point in space. To approach this continuum case, we divide the space into cells, each of volume δV_r, and we define $\phi_r(t)$ as the average value of $\phi(x)$ in cell r at time t. The average Lagrangian density in each cell is then $\mathcal{L}_r$. Then the momentum variable p_r, conjugate to ϕ_r is

$$p_r(t) = \frac{\partial L}{\partial \dot{\phi}_r(t)} = \delta V_r \frac{\partial \mathcal{L}_r}{\partial \dot{\phi}_r(t)} = \delta V_r \pi_r(t) \tag{2.4.4}$$

where the field $\pi(x,t)$ is defined by

$$\pi(\mathbf{x},t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\mathbf{x},t)} \tag{2.4.5}$$

and $\pi_r(t)$ is the average value in cell r.
The Heisenberg commutation relation give

$$\begin{aligned} &[\phi_r(t), p_s(t)] = i\delta_{rs}, \\ &[\phi_r(t), \phi_s(t)] = [p_r(t), p_s(t)] = 0 \end{aligned} \tag{2.4.6}$$

and then substituting the (2.4.4) into the (2.4.6) gives

$$[\phi_r(t), \pi_s(t)] = \frac{1}{\delta V_s} i\delta_{rs}.$$

In the continuum limit $\delta V_r \to 0$, we obtain

$$\begin{aligned} &[\phi(\mathbf{x},t), \pi(\mathbf{x}',t)] = i\delta(\mathbf{x}-\mathbf{x}'), \\ &[\phi(\mathbf{x},t), \phi(\mathbf{x}',t)] = [\pi(\mathbf{x},t), \pi(\mathbf{x}',t)] = 0 \end{aligned} \tag{2.4.7}$$

These are known as *equal-time commutation relations* (**ETCR**), and we now use them to find the commutation relations between $a(k)$ and $a^\dagger(k)$, in equation (2.4.2). First of all, from the definition (2.4.5) and the definition of the Lagrangian (2.3.1), we have

$$\pi(x) = \dot{\phi}(x).$$

Now we will check that the positive energy (also known as positive frequency) solution

$$f_k(x) = \frac{1}{[(2\pi)^3 2\omega_k]^{\frac{1}{2}}} e^{-ikx} \tag{2.4.8}$$

form an orthonormal set

$$\int f_k^*(x) i \overleftrightarrow{\partial_0} f_{k'}(x)\, \mathrm{d}^3x = \delta^3(\mathbf{k}-\mathbf{k}'), \tag{2.4.9}$$

where $\overleftrightarrow{\partial_0}$ is defined by

$$A(t)\overleftrightarrow{\partial_0} B(t) = A(t)\frac{\partial B(t)}{\partial t} - \frac{\partial A(t)}{\partial t} B(t).$$

This operator permits to define a scalar product which is positive definite.
The field expansion (2.4.2) may then be written as

$$\phi(x) = \int \frac{\mathrm{d}^3k}{[(2\pi)^3 2\omega_k]^{\frac{1}{2}}} [f_k(x)a(k) + f_k^*(x)a^\dagger(k)]. \tag{2.4.10}$$

Inverting the previous expression, using (2.4.9), we get

$$\begin{aligned} a(k) &= \int \mathrm{d}^3x[(2\pi)^3 2\omega_k]^{\frac{1}{2}} f_k^*(x) i \overleftrightarrow{\partial_0} \phi(x), \\ a^\dagger(k') &= \int \mathrm{d}^3x'[(2\pi)^3 2\omega_{k'}]^{\frac{1}{2}} \phi(x') i \overleftrightarrow{\partial_0} f_{k'}(x'). \end{aligned} \tag{2.4.11}$$

From the definition of the commutation relation in the continuum limit, and the fact that $\pi(x) = \dot{\phi}(x)$, after some calculations, we have

$$[a(k), a^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\mathbf{k} - \mathbf{k}'). \tag{2.4.12}$$

Similarly, we get

$$[a(k), a(k')] = 0, \qquad [a^\dagger(k), a^\dagger(k')] = 0.$$

The operators $a(k)$ and $a^\dagger(k)$ play a crucial role in the particle interpretation of the quantized field theory. First we construct the operator

$$(2\pi)^3 2\omega_k \delta^3(0) N(k) = a^\dagger(k) a(k). \tag{2.4.13}$$

It is simple to show that $N(k)$ and $N(k')$ commute:

$$[N(k), N(k')] = 0,$$

so the eigenstates of these operators may be used as a basis. Let the eigenvalues be denoted by $n(k)$:

$$N(k) \left| n(k) \right\rangle = n(k) \left| n(k) \right\rangle.$$

Using the relations:

$$\begin{aligned} [N(k), a^\dagger(k)] &= a^\dagger(k), \\ [N(k), a(k)] &= -a(k), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} N(k) a^\dagger(k) &= a^\dagger(k) N(k) + a^\dagger(k), \\ N(k) a(k) &= a(k) N(k) - a(k), \end{aligned}$$

lead us to find

$$N(k) a^\dagger(k) \left| n(k) \right\rangle = [n(k) + 1] a^\dagger(k) \left| n(k) \right\rangle \tag{2.4.14}$$

and

$$N(k) a(k) \left| n(k) \right\rangle = [n(k) - 1] a(k) \left| n(k) \right\rangle. \tag{2.4.15}$$

These equations tell us that, if the state $\left| n(k) \right\rangle$ has eigenvalue $n(k)$, the states $a^\dagger(k) \left| n(k) \right\rangle$ and $a(k) \left| n(k) \right\rangle$ are eigenstates of $N(k)$ with respective eigenvalues $n(k) + 1$ and $n(k) - 1$. $N(k)$ is a *particle number* operator, or more precisely, a *number density* operator. To justify the name, we compute the field energy, through the Hamiltonian, found by substituting the (2.4.2) into the (2.4.1), obtaining

$$\begin{aligned} H &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \frac{k_0}{2} [a^\dagger(k)a(k) + a(k)a^\dagger(k)] \\ &= \int \mathrm{d}^3 k\, k_0 \left[N(k) + \frac{1}{2} \right] \end{aligned} \tag{2.4.16}$$

with $\omega_k = k_0$ and, similarly, the field momentum is

$$\mathbf{P} = \int \mathrm{d}^3 k\, \mathbf{k} \left[N(k) + \frac{1}{2} \right].$$

These expressions suggest the interpretation that $N(k)$ is the operator for the number of particles with momentum $\mathbf{k}$ and energy k_0. This means that $N(k)$ never becomes negative! To justify that, we can note that the state $a(k)\left|n(k)\right\rangle$ must have non negative norm, as in all the Hilbert space:

$$[a(k)\left|n(k)\right\rangle]^\dagger [a(k)\left|n(k)\right\rangle] = \left\langle n(k)\right| a^\dagger a(k) \left|n(k)\right\rangle = n(k) \left\langle n(k)|n(k)\right\rangle > 0.$$

So that, if $\left|n(k)\right\rangle$ has non-negative norm, $n(k)$ must be positive or zero. On the other hand, from the equation (2.4.15), $a(k)$ reduces $n(k)$ by 1 and, repeated application, will continue to reduce it. The only way to avoid $n(k)$ becoming negative is to have a ground state $\left|0(k)\right\rangle$, or $\left|0\right\rangle$ for short, with

$$a(k)\left|0\right\rangle = 0 \quad \Rightarrow \quad N(k)\left|0\right\rangle = a^\dagger(k)a(k)\left|0\right\rangle = 0. \tag{2.4.17}$$

This relation tells us that the ground state in vacuum contains no particle with momentum $\mathbf{k}$. On the contrary, $a^\dagger(k)$ increase $N(k)$ by one at time. This is why $N(k)$ is called number operator. We can also recover the connection between the above analysis and the quantum mechanical armonic oscillator. In fact, our Hamiltonian (2.4.16) is equivalent to the harmonic oscillator form

$$H = \int \mathrm{d}^3 k \left[\frac{1}{2} P^2(k) + \frac{\omega_k^2}{2} Q^2(k) \right]$$

by substituting

$$P(k) = \left(\frac{\omega_k}{2} \right)^{\frac{1}{2}} [a(k) + a^\dagger(k)], \qquad Q(k) = \frac{i}{(2\omega_k)^{\frac{1}{2}}} [a(k) - a^\dagger(k)].$$

The Klein-Gordon field is then equivalent to an infinite sum of oscillators. The operators $a(k)$ and $a^\dagger(k)$ are called *annihilation* and *creation* operators for the field. Moreover, the fact that $N(k)$ is non-negative, implies that the energy of the quantized field is non-negative, according to what we found for the classical Klein-Gordon field. Our Hamiltonian contains an infinite contribution from all the oscillator ground state, but, since the zero of the energy is arbitrary, this may be subtracted with no physical consequences and redefine the Hamiltonian as

$$H = \int \mathrm{d}^3 k \omega_k N(k),$$

with the property

$$\langle 0| H |0\rangle = \int \mathrm{d}^3 k \, k_0 \langle 0| a^\dagger(k) a(k) |0\rangle = 0,$$

thanks to the (2.4.17). This means that the mean value of the Hamiltonian, and then the mean energy of the system, is equal to zero.
Formally, the fact that the annihilation operator is on the right of the creation operator is called *normal ordering*, and it is denoted as : :. This has been introduced for the purpose to rescale the ground state energy to zero. Thus, decomposing the field $\phi(x)$ into positive and negative frequency parts, from equation (2.4.10)

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

with

$$\phi^{(+)}(x) = \int \frac{d^3 k}{[(2\pi)^3 2\omega_k]^{\frac{1}{2}}} a(k) f_k(x),$$
$$\phi^{(-)}(x) = \int \frac{d^3 k}{[(2\pi)^3 2\omega_k]^{\frac{1}{2}}} a^\dagger(k) f^*_k(x),$$

we have

$$: \phi(x)\phi(y) : = \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x) + \phi^{(-)}(x)\phi^{(-)}(y).$$

From the equation (2.4.14) we can see that the state $a^\dagger(k) |n(k)\rangle$ is proportional to the state $|n(k)+1\rangle$, since both solves the eigenvalue equation, with eigenvalue $n(k)+1$, so we write

$$a^\dagger(k) |n(k)\rangle = c_+(n(k)) |n(k)+1\rangle$$

or, to be more precise

$$a^\dagger(k_i) |n(k_1), \ldots, n(k_i), \ldots\rangle = c_+(n(k_i)) |n(k_1), \ldots, n(k_i)+1, \ldots\rangle,$$

where the creation operator had increased the counting of particle with momentum k_i. The coefficient $c_+(n(k))$ may be found by imposing the normalization condition, which gives

$$|c_+(n(k))|^2 = n(k) + 1.$$

So, within a phase factor, $c_+(n(k)) = [n(k)+1]^{\frac{1}{2}}$. Then, by the same argument, using $a(k)$ and the corresponding coefficient $c_-(n(k))$, we find

$$c_-(n(k)) = [n(k)]^{\frac{1}{2}}.$$

The vacuum state constains no particle of any momentum, i.e.

$$|0\rangle = |0,0,\ldots\rangle\,,$$

and an aritrary normalized state containing $n(k_1)$ particle with momentum k_1, $n(k_2)$ particle with momentum k_2, etc., may be written as

$$|n(k_1),n(k_2),\ldots\rangle = \frac{1}{(n(k_1)!n(k_2)!\ldots)^{\frac{1}{2}}}[a^\dagger(k_2)]^{n(k_2)}[a^\dagger(k_2)]^{n(k_2)}\ldots|0\rangle\,.$$

From the last formula, there is evidently no restriction on $n(k)$ since any number of particle can exist in the same momentum state. The particle are called *bosons* and this relations follows directly from the commutation relations (2.4.7) [4]. With $|k\rangle = a^\dagger(k)\,|0\rangle = \langle 0|\,a(k)$, we are saying that from zero particle of momentum k, we have created one with that momentum, since we have

$$\begin{aligned}\langle k|k'\rangle &= \langle 0|\,a(k)a^\dagger(k)\,|0\rangle\\ &= \langle 0|\,[a(k),a^\dagger(k)]\,|0\rangle + \underbrace{\langle 0|\,a^\dagger(k)a(k)\,|0\rangle}_{0}\\ &= (2\pi)^3 2k_0\delta^3(\mathbf{k}-\mathbf{k}'),\end{aligned}$$

and this normalization is also covariant.

2.5 Quantum field theory (QFT) in curved space

In this section we will show what experiences an observer (paticle detector) that uniformly accelerates through the Minkowski vacuum state. The results will be that accelerating observers perceives a bath of thermal radiations. In other words, this means that an accelerating observer see a different vacuum state with respect a static Minkowski observer [BD84].

2.5.1 Cylindrical two dimensional space-time

Let define the usual Minkowski metric for two dimensional space-time, a cylinder $\mathbb{R}^1 \times \mathbb{S}^1$, as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$$

[4] If we want to describe *fermions*, we must modify these commutation relations.

and apply a coordinates transformation

$$\begin{cases} u = t - x \\ v = t + x \end{cases} .$$

We can now rewrite the metric in function of null coordinates as

$$\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}v.$$

The modes restricted to our cylinder are

$$u_k(t,x) = \frac{1}{\sqrt{2L\omega}} e^{i(kx-\omega t)},$$

where $k = 2\pi n/L$, $n = 0, \pm 1, \pm 2, \ldots$, and L is the length of the cylinder's circumnference.
In our case we can impose two types of boundary conditions, *periodic* or *antiperiodic*. In the first case we fix

$$u_k(t,x) = u_k(t, x + nL),$$

while in the second

$$u_k(t,x) = (-1)^n u_k(t, x + nL),$$

which are also-called *twisted modes*. In the last case, k is given by

$$k = 2\pi \left(n + \frac{1}{2}\right)\frac{1}{L}, \qquad n = 0, \pm 1, \pm, 2, \ldots$$

Because of the field modes are forced into a discrete set, this implies that the field energy is distributed. We now define the *stress-energy tensor* as

$$T_{\alpha\beta} = \partial_\alpha \phi\, \partial_\beta \phi - \frac{1}{2}\eta_{\alpha\beta}\left(\eta^{\lambda\delta}\partial_\lambda \phi\, \partial_\delta \phi\right) + \frac{1}{2} m^2 \phi^2 \eta_{\alpha\beta},$$

where we called $\eta_{\alpha\beta}$ the Minkowski metric tensor.
In cartesian coordinates the stress-energy tensor can be written as

$$\begin{aligned} T_{tt} = T_{xx} &= \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2, \\ T_{xt} = T_{tx} &= (\partial_t \phi)(\partial_x \phi). \end{aligned}$$

We shall now evaluate $\langle 0_L | T_{\mu\nu} | 0_L \rangle$, where clearly $|0_L\rangle$ is the vacuum state associated to our space. The Minkowski vacuum state $|0\rangle$ can be recovered if

$$L \to \infty \quad \Rightarrow \quad |0_L\rangle \to |0\rangle\,.$$

Defining

$$\langle 0|\, T_{\alpha\beta}\, |0\rangle = \sum_k T_{\alpha\beta}[u_k, u_k^*],$$

we encounter a problem when we evaluate the following expression

$$\langle 0_L|\, T_{\alpha\beta}\, |0_L\rangle = \frac{1}{2L}\sum_{n=-\infty}^{+\infty} |k| = \frac{2\pi}{L^2}\sum_{n=-\infty}^{+\infty} n \to +\infty.$$

In fact, the series diverges. The compactified spatial section, is able to modify the long wavelengths modes but the ultraviolet behaviour is unchanged. In Minkowski space vacuum, the ultraviolet divergence was removed by normal ordering with respect to the creation and annihilation operator. Taking a general state $|\psi\rangle$

$$\langle \psi| : T_{\alpha\beta} : |\psi\rangle = \langle \psi|\, T_{\alpha\beta}\, |\psi\rangle - \langle 0|\, T_{\alpha\beta}\, |0\rangle$$

we recover that

$$\langle 0| : T_{\alpha\beta} : |0\rangle = 0.$$

If we consider the Minkowski space as a covering space for $\mathbb{R}^1 \times \mathbb{S}^1$, then $|0_L\rangle$ can be considered as the vacuum state in the above space. We can then remove the divergence behaviour.

2.5.2 Quantum field theory in Rindler space

Uniformly accelerating detector perceives the usual Minkowski space vacuum state to be a thermal bath of radiation [BD84]. Inertial detector responds to a thermal flux of radiation streaming away from a mirror that recedes along a non-uniformly accelerating trajectory.
We now consider a two-dimensional Minkowski space with metric

$$\mathrm{d}s^2 = \mathrm{d}\bar{u}\mathrm{d}\bar{v} = \mathrm{d}t^2 - \mathrm{d}x^2.$$

Under the following coordinates transformation

$$t = \frac{1}{a}e^{a\xi}\sinh a\eta, \qquad x = \frac{1}{a}e^{a\xi}\cosh a\eta,$$

or equivalently

$$\bar{u} = \frac{1}{a}e^{-au}, \qquad \bar{v} = \frac{1}{a}e^{av}$$

where a is a positive constant, $-\infty < \eta, \xi < +\infty$ and $u = \eta - \xi$, $v = \eta + \xi$, the metric can be written as:

$$\mathrm{d}s^2 = e^{2a\xi}\mathrm{d}u\mathrm{d}v = e^{2a\xi}(\mathrm{d}\eta^2 - \mathrm{d}\xi^2). \tag{2.5.1}$$

In the last coordinate system, η and ξ cover only one quadrant of Minkowski space, the region for which $|x| > t$, called also region ($\mathbf{R}$).
In Figure 2.2 we have that:

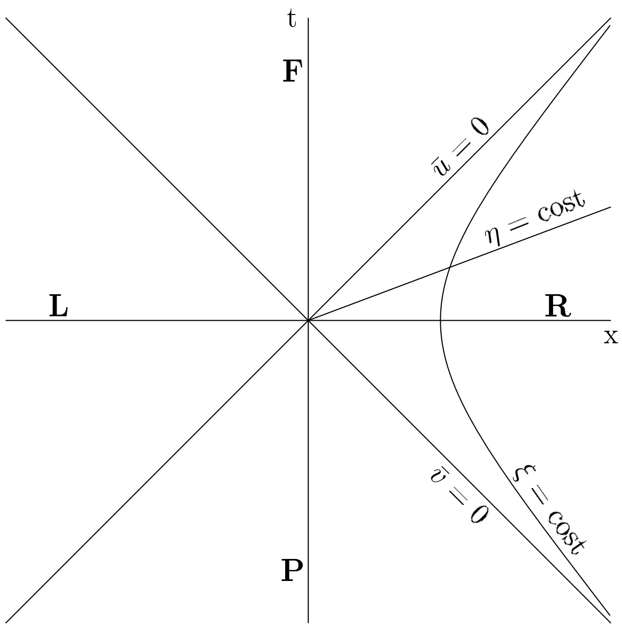

Figure 2.2: Representation of the Rindler space.

- lines of constant η, where $x \propto t$, are straight, passing from the origin;
- lines of constant ξ are hyperbolae. In this case we have that

$$x^2 - t^2 = \frac{1}{a^2} e^{2a\xi} = cost$$

are the worldline of uniformly accelerated observers.
Moreover lines of large positive ξ (far from $x = t = 0$) represent weakly accelerated observers.

The principal problem is that Rindler coordinates (η, ξ) are non-analytic across $\bar{u} = \bar{v} = 0$.
All hyperbolae are asymptotic to the null rays $\bar{u} = \bar{v} = 0$ (or $u = +\infty$ and $v = -\infty$), which means that accelerated observers approach the speed of light as $\eta \to \pm\infty$.
This observers proper time is related to η, ξ by:

$$\tau = e^{a\xi}\eta.$$

A second Rindler wedge can be obtained by reflecting the coordinates defined in region (**R**) by changing the sign of the above coordinates tranformation on the right hand side, obtaining the wedge (**L**). In (**L**) the direction of time is reflected, i.e. increasing t means decreasing η. The Rindler observer, with constant spatial ξ, approaches but do not cross the null rays $u = +\infty$ and $v = -\infty$, which act like an horizon. This means that no events in (**L**) can be

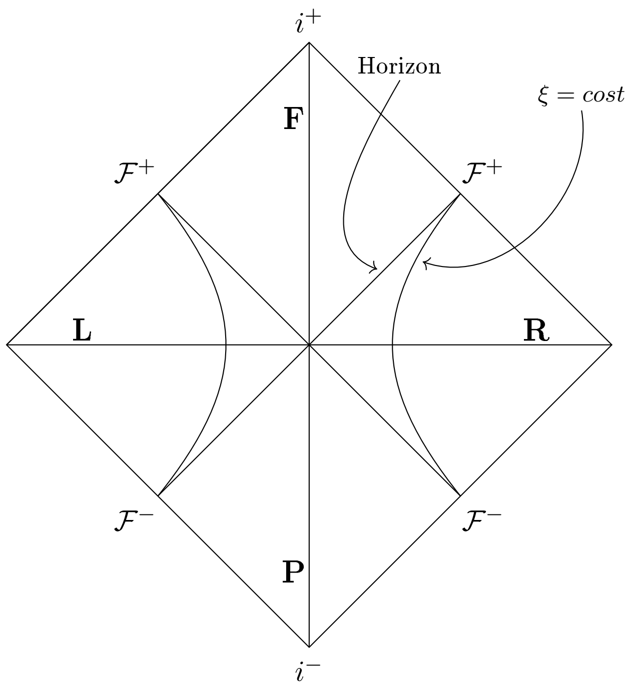

Figure 2.3: Conformal diagram of Rindler space.

looked by (**R**) and vice-versa. Events in (**L**) can be connected with events in (**R**) only through spacelike line and this means that the two region represents two causally disjoint universes.
(**F**) and (**P**) stays respectively for *future* and *past*. Events in both (**F**) and (**P**) can be connected by null-rays to both (**L**) and (**R**).
The Rindler observer intersect $\mathcal{F}^{\pm}$, rather than $i^{\pm}$, as do asymptotically inertial observers. The null ray $u = +\infty$ acts as a future event horizon. Events in the portion (**F**) cannot causally influence events in (**R**).
We consider now the quantization of a massless scalar field ϕ in a two-dimensional Minkowski space-time. The wave equation

$$\Box\phi = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi = \frac{\partial^2\phi}{\partial\bar{u}\partial\bar{v}} = 0$$

posses orthonormal modes solutions

$$\bar{u}_k = \frac{1}{\sqrt{4\pi\omega}}e^{i(kx-\omega t)},$$

with $\omega = |k| > 0$ and $-\infty < k < +\infty$. This modes are positive frequency with respect to the timelike Killing vector ∂_t, satifying

$$\mathcal{L}_{\partial_t}\bar{u}_k = -i\omega\bar{u}_k.$$

We can distinguish the two cases

$$\begin{cases} k > 0 \Rightarrow \text{right moving waves: } \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\bar{u}}, \text{ along } \bar{u} = cost, \\ k < 0 \Rightarrow \text{left moving waves: } \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\bar{v}}, \text{ along } \bar{v} = cost. \end{cases} \tag{2.5.2}$$

The Minkowski vacuum state $|0_M\rangle$ is constructed by expanding ϕ in terms of $\bar{u}_k$. In Rindler region (**R**) and (**L**), we can base our quantization on u_k, instead of $\bar{u}_k$. The metric (2.5.1) is conformal to the whole of Minkowski, under the conformal transformation:

$$g_{\mu\nu} \to e^{-2a\xi} g_{\mu\nu}$$

and the metric reduces to $d\eta^2 - d\xi^2$ with $-\infty < \eta, \xi < +\infty$. Then, since the wave equation is conformally invariant, we can rewrite it in Rindler coordinates as

$$e^{2a\xi} = \Box\phi = \left(\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\xi^2} \right) \phi = \frac{\partial^2 \phi}{\partial u \partial v} = 0,$$

for which exist modes solution of the form

$$u_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi - \pm i\omega\eta}$$

where $\omega = |k| > 0$, $-\infty < k + \infty$ and at the exponent, the sign $\pm$ are referred to (**L**) and (**R**) region respectively, due to the time reversal in (**L**) and the fact that the modes are right moving in (**R**) and left moving in (**L**) towards increasing ξ.
The modes are positive frequency with respect to the timelike Killing vector ∂_η in (**R**) and $-\partial_\eta$ in (**L**) such that

$$\mathcal{L}_{\pm\partial_\eta} u_k = -i\omega u_k.$$

We then define

$$\begin{aligned} u_k^R &= \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{i(k\xi - \omega\eta)}, & \text{in } (\mathbf{R}) \\ 0, & \text{in } (\mathbf{L}) \end{cases}, \\ u_k^L &= \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{i(k\xi + \omega\eta)}, & \text{in } (\mathbf{L}) \\ 0, & \text{in } (\mathbf{R}) \end{cases}, \end{aligned} \tag{2.5.3}$$

which are respectively complete in the right and left Rindler region. Both sets toghether are complete. The modes can be analitically continued into the future and past regions, which means extend a to be immaginary. Then this modes are good to quantize ϕ as the Minkowski space basis. We can expand ϕ through the different basis in either

$$\phi = \sum_{k=-\infty}^{+\infty} (a_k \bar{u}_k + a_k^\dagger \bar{u}_k^*)$$

or

$$\phi = \sum_{k=-\infty}^{+\infty} \left(b_k^{(1)L} u_k + b_k^{(1)\dagger L} u_k^* + b_k^{(2)R} u_k + b_k^{(2)\dagger R} u_k^* \right),$$

yelding two alternative Fock spaces and two alternative vacuum states $|0_M\rangle$, $|0_R\rangle$, such that,

$$a_k |0_M\rangle = 0,$$
$$b_k^{(1)} |0_R\rangle = b_k^{(2)} |0_R\rangle = 0.$$

This vacuum states will be not equivalent, because of the changing in sign in the exponent at $\bar{u} = \bar{v} = 0$ (crossover point between (**L**) and (**R**)).
The functions u_k^R do not go over smoothly to u_k^L, passing from (**R**) to (**L**)). This means that passing from $\bar{u} < 0$ to $\bar{u} > 0$ (or $\bar{v} < 0$ to $\bar{v} > 0$), the right- (or left-) mooving modes are not analytic at this point. On the other hand, the positive frequency Minkowski modes in (2.5.2) are analytic not only on the real $\bar{u}$ (or $\bar{v}$) axis, but also analytic and bounded in the entire lower half of the complex $\bar{u}$ (or $\bar{v}$) plane. This analiticity property remains true for any pure positive frequency Minkowski modes. Then, Rindler modes, by their non-analiticity at $\bar{u} = \bar{v} = 0$, cannot be combination of pure positive frequency Minkowski modes, but must also contain negative frequencies and this implies that the vacuum state cannot be the same. This means that the vacuum of one set of modes contains particles associated with the other set of modes.
To determine what Rindler particles are present in the Minkowski vacuum state, one must determine the Bogoliubov tranformations between the two set of modes (Fourier transform of Rindler modes) [ABFP13]. To do that, we can use the *Unruh method*, noting that the two modes u_k^R and u_k^L are not analytic. The two un-normalized combination are analytic and bounded, such that

$$\begin{cases} u_k^R + e^{-\frac{\pi\omega}{aL}} u_k^* \\ u_k^{*R} + e^{\frac{\pi\omega}{aL}} u_k \end{cases},$$

where the first mode is proportional to

$$\begin{cases} \bar{u}^{\frac{i\omega}{a}}, & \text{k>0} \\ \bar{v}^{-\frac{i\omega}{a}}, & \text{k<0} \end{cases}$$

while the second is proportional to

$$\begin{cases} \bar{v}^{\frac{i\omega}{a}}, & \text{k>0} \\ \bar{u}^{-\frac{i\omega}{a}}, & \text{k<0} \end{cases}.$$

In this cases $\omega = |k|$ and $-\infty < \bar{u}, \bar{v} < +\infty$.

Expanding ϕ as

$$\phi = \sum_{k=-\infty}^{+\infty} \left[2\sinh\left(\frac{\omega\pi}{a}\right)\right]^{-\frac{1}{2}} \left[d_k^{(1)} \left(e^{\frac{\pi\omega}{2a}} u_k^R + e^{-\frac{\pi\omega}{2a}} u_k^{*L}\right)\right] + \left[2\sinh\left(\frac{\omega\pi}{a}\right)\right]^{-\frac{1}{2}} \left[d_k^{(2)} \left(e^{-\frac{\pi\omega}{2a}} u_k^{*R} + e^{\frac{\pi\omega}{2a}} u_k^L\right)\right] + h.c.$$

where now

$$d_k^{(1)} |0_M\rangle = d_k^{(2)} |0_M\rangle = 0.$$

In the above expression, we have also introduced a normalization factor. We can also relate $b_k^{(1,2)}$ and $d_k^{(1,2)}$ thanks to the inner product, such that

$$\begin{aligned} b_k^1 = (\phi, u_k^R) &= \left[2\sinh\left(\frac{\omega\pi}{a}\right)\right]^{-\frac{1}{2}} \left[e^{\frac{\pi\omega}{2a}} d_k^{(2)} + e^{-\frac{\pi\omega}{2a}} d_{-k}^{(1)\dagger}\right], \\ b_k^2 = (\phi, u_k^L) &= \left[2\sinh\left(\frac{\omega\pi}{a}\right)\right]^{-\frac{1}{2}} \left[e^{\frac{\pi\omega}{2a}} d_k^{(1)} + e^{-\frac{\pi\omega}{2a}} d_{-k}^{(2)\dagger}\right]. \end{aligned} \tag{2.5.4}$$

The Bogoliubov transformations provides the transformations between $|0_R\rangle$ and $|0_M\rangle$.
If we now consider an accelerating Rindler observer at $\xi = cost$, then he has a proper time which is proportional to η. Proper observer in (**L**) detect particles counted by number operator $b_k^{(1)}$ and $b_k^{(1)\dagger}$, while in (**R**) through $b_k^{(2)}$ and $b_k^{(2)\dagger}$.
If the field is in the state $|0_M\rangle$, using (2.5.4), we obtain

$$\langle 0_M | \, b_k^{(1,2)\dagger} b_k^{(1,2)} \, |0_M\rangle = \frac{e^{\frac{-\pi\omega}{a}}}{2\sinh\frac{\pi\omega}{a}} = \frac{1}{e^{2\frac{\pi\omega}{a}} - 1},$$

which represent particle in mode k. This represents precisely the *Planck spectrum* for radiation at temperature

$$T_0 = \frac{a}{2\pi k_B}.$$

The temperature T seen by the accelerated observer is given by the *Tolman relation*

$$T = \frac{1}{\sqrt{g_{00}}} T_0.$$

Particles detected by Rindler observer are called *Rindler particles.*

Chapter 3

Black hole geometry

This chapter aims at introducing the fundamentals of black hole geometry, focusing in particular on the simplest case of a non-charged and non-rotating black hole, the so-called Schwarzschild black hole.

3.1 Spherically Symmetric Gravitational Field

We start the analysis of the physical properties of black holes with the simplest case in which both the black hole and its gravitational field are spherically symmetric [FN98, CB14].
Let us write the metric[1] in a region far from strong gravitational fields, where special relativity is valid. In cartesian coordinates we have the Minkowski metric:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}\ell^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

where c is the speed of light and $\mathrm{d}\ell^2$ is the distance in three-dimensional space.
The metric tensor is then

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using spherical spatial coordinates system (r, θ, ϕ):

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}\ell^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

Since $\mathrm{d}s^2$ can be negative, the space-time separation between two events is not the usual distance. If we take two events $P = (t, x, y, z)$ and $Q =$

[1] We use $(-, +, +, +)$, signature convention for the space-time metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. From now on, Greek indices μ, ν take values 0,1,2,3, while small Latin indices $i, j = 1, 2, 3$ enumerate spatial coordinates.

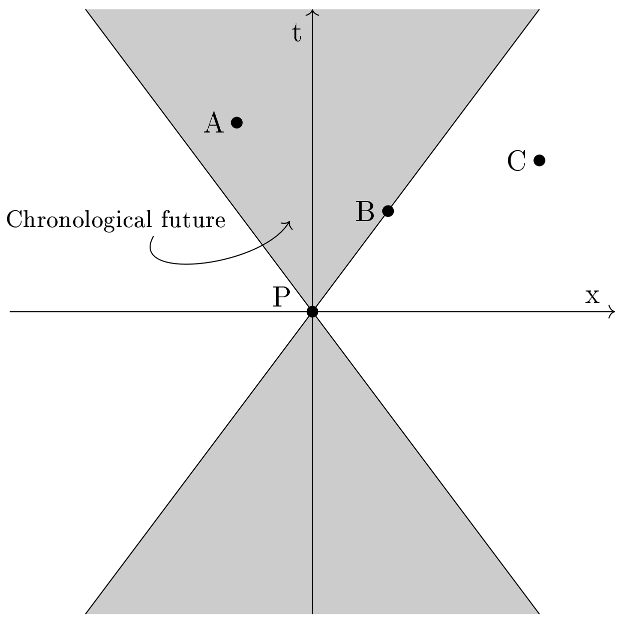

Figure 3.1: Representation of the Minkowski light cone with only one spatial variable. The points A, B and C are connected to P by a timelike, lightlike and spacelike vector respectively.

(t', x', y', z'), the interval between them can be positive, null or neagtive. The vector $\overrightarrow{PQ}$ is said to be:

- timelike if $\mathrm{d}^2(P, Q) < 0$;
- lightlike (or null) if $\mathrm{d}^2(P, Q) = 0$;
- spacelike if $\mathrm{d}^2(P, Q) > 0$.

The set of outgoing lightlike vectors from P forms the so-called Minkowski light cone, see Figure 3.1.
Given then a D-dimensional differential manifold $\mathcal{M}$ and a lorentzian metric g, we will say that a space-time ($\mathcal{M}$, g) is time-orientable if there exists a non-spacelike continuous vector field, defined everywhere (it has to be timelike or lightlike in each point of $\mathcal{M}$). This means that a preferred direction for time can be defined.
In a time-orientable space-time, a curve is called timelike (or spacelike/lightlike) if in each point of the curve the tangent vector is timelike (or spacelike/lightlike). A non-spacelike curve is said to be future (past) directed, in a time-orientable space-time, if, at each point of the curve, the tangent vector is future (past) directed.
Given now two subsets A, B $\subset \mathcal{M}$, we will say that

- the chronological future (past) of A relative to B is the set $\mathcal{I}^{+(-)}$ (A,

B) of points $p \in$ B that can be reached from A along future (past) directed timelike curves

- the causal future (past) of A relative to B, is the set $\mathcal{J}^{+(-)}(\mathrm{A}, \mathrm{B})$ of points $p \in$ B that can be reached from A along future (past) directed non-spacelike curves

3.2 Schwarzschild metric

The Schwarzschild black hole represents the general solution of the Einstein equations in vacuum, with spherical symmetry [BCF18]. The metric reads

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right) c^2 \mathrm{d}t_s^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2, \tag{3.2.1}$$

where t_s represents the Schwarzschild time coordinate, $r_s = 2GM/c^2$ is the Schwarzschild radius, G is the Newton gravitational constant, M is the mass of the field source and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2$ is the angular line element. The metric is the standard one on the 2D round sphere with $(t_s, r) \in (-\infty, +\infty) \times (0, +\infty)$. The coordinates (t_s, r, θ, ϕ) are called *Schwarzschild coordinates.*
As $r_s/r \to 0$ the Schwarzschild metric approaches the flat Minkowski metric, so the coordinats (t_s, r, θ, ϕ) correspond to the usual spherical coordinates of flat space-time, for an observer situated at infinity [Rob12]. The equation (3.2.1) contains two singularities at $r = 0$ and $r = r_s$. Since the Schwarzschild metric is valid only in vacuum, these singularities are relevant only when the entirety of the mass is confined to a radius smaller than r_s. In this case, it will inevitably collapse to a single point of infinite density at $r = 0$. Such objects are called *black holes.* The point $r = 0$ is a genuine singularity of Schwarzschild space-time and we shall not be concerned with it. We are instead interested in the surface for $r = r_s$, the so-called *event horizon.*
Let us briefly examine the effect of the event horizon on light trajectories, or null curves, with $\mathrm{d}s^2 = 0$. For simplicity, we shall consider radial trajectories, so we have also $\mathrm{d}\Omega^2 = 0$. This leaves us with a differential equation for the radial null curves:

$$\frac{\mathrm{d}t_s}{\mathrm{d}r} = \pm \frac{1}{c} \left(1 - \frac{r_s}{r}\right)^{-1}.$$

Far from the Schwarzschild radius, where $r \gg r_s$, we have that $|\mathrm{d}t_s/\mathrm{d}r| \to 1/c$, so that light behaves just as it does in flat space-time. However, if we approach the Schwarzschild radius, $|\mathrm{d}t_s/\mathrm{d}r|$ diverges in such a way that light takes longer and longer to travel any distance, and, if travelling towards the event horizon, can never reach it in a finite time t. The singularity at $r = r_s$ is only an artifact generated by the coordinates system that we used and

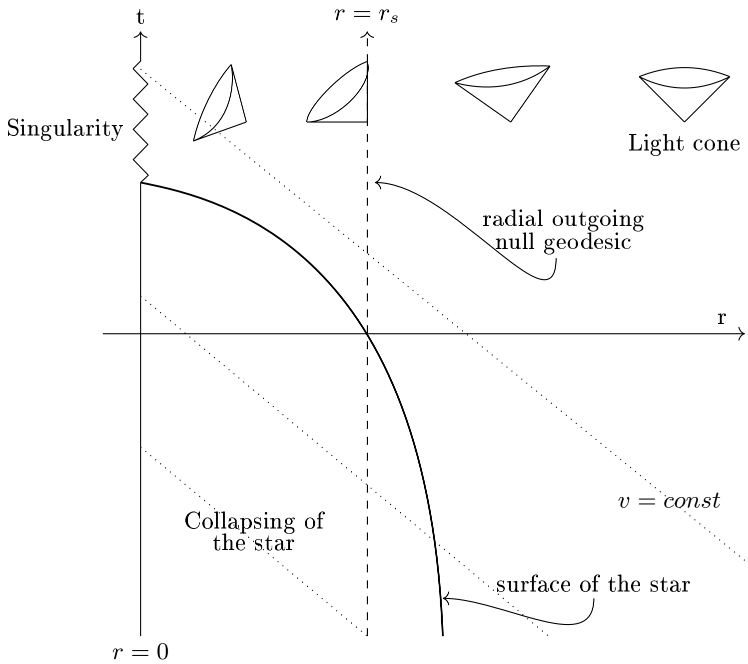

Figure 3.2: Representation of a collapsing star and black hole creation. The light cone bends approaching the horizon and, once it is crossed, the cone points towards the singularity. Then timelike and lightlike curves are destined to fall into the singularity in $r = 0$ and this is the reason why nothing can ever escape from a black hole.

can be eliminated by a coordinate transformation. Using the time advanced Eddington-Finkelstein coordinate

$$v = t_s + r^*,$$

where

$$r^* = \int \frac{\mathrm{d}r}{1 - r_s/r} = r + r_s \ln\left|\frac{r}{r_s} - 1\right|,$$

the Schwarzschild line element can be written as

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2.$$

Then to examine as before the effect of the event horizon on the radial null curves ($\theta = const, \phi = const$), we set again $\mathrm{d}s^2 = 0$. Therefore

$$v = const$$

describes the motion of ingoing spherical light fronts, while

$$\frac{\mathrm{d}r}{\mathrm{d}v} = \frac{1}{2}\left(1 - \frac{r_s}{r}\right) \tag{3.2.2}$$

is related to the outgoing ones. Inspecting the last expression, we can see that only outgoing rays at $r > r_s$ manage to move outwards, since $\mathrm{d}r/\mathrm{d}v > 0$. For $r < r_s$ they move inwards ($\mathrm{d}r/\mathrm{d}v < 0$). At $r = r_s$ they are blocked forever. These considerations tell us that neither light not matter (since nothing can propagate faster than light) can escape from the region $r < r_s$, which represent the black hole.

3.3 Analogue gravity

According to General Relativity, we can say that black holes are space-time regions where the gravitational field is so strong that not even light can escape [BFFP05, BLV05, SU08, ABFP14, LRCP12]. Hawking, in 1974, discovered that if we take into account Quantum Mechanics, black holes are no longer black. He showed that they thermally radiate with a temperature which is inversely proportional to their mass. If we consider a black hole with mass M, it emits, like a black body, with a temperature

$$T_H = \frac{\hbar c^3}{8\pi G k_B M},$$

where G is the Newton gravitational constant and k_B is the Boltzmann constant. The maximal power emission is given by the Stefan law

$$P = A\sigma_S T_H^4,$$

where σ_S is the Stefan constant given by

$$\sigma_S = \frac{\pi^2 k_B^4}{60 c^2 \hbar^3}$$

and A is the area of the horizon.

If we coonsider for example a Schwarzschild black hole with a solar mass ($M = 1.989 \cdot 10^{30} Kg$) we have that the temperature is $T_H \simeq 6.17 \cdot 10^{-8} K$. This temperature is far below the Cosmic Microwave Background temperature, which is approximately $2.7K$. There is then no hope to detect Hawking radiations from astrophysical black holes resulting from gravitational collapse, since those have a greater mass than the solar one, and emit at an even lower temperature. However, it is possible to detect Hawking radiation emitted by light black holes produced in the early universe. Unfortunately it is improbable to find such black holes since they have been subjected to a period of inflation, which has diluted them away.

An alternative to direct observation is to reproduce the principal characteristics of an event horizon in a simple enough system, in laboratory. The prototype for this configuration was discovered by Unruh [Unr81], who stated the analogy between light propagation in a curved space-time and sound propagation in a non-uniformly moving fluid. Later, the analogy has been generalized with the study of wave propagation in non-homogeneous condensed matter systems. This analogy has given rise to a new line of research, the *Analogue Gravity*.

3.4 Black hole evaporation in laboratory

Let us consider a non relativistic, irrotational, barotropic and non-viscous fluid (water), described by a velocity field $\vec{v} = \nabla\psi$, a density ρ and a pressure p (since it is barotropic $p = p(\rho)$) [BLV05, FM11]. The equations describing the fluid are

$$\begin{aligned} \rho\left[\partial_t\vec{v} + (\vec{v}\cdot\nabla)\vec{v}\right] &= -\nabla p - \rho\nabla\phi, \\ \partial_t\rho + \nabla\cdot(\rho\vec{v}) &= 0, \end{aligned}$$

where ϕ represents the gravitational potential. The irrotationality of $\vec{v}$ allow us to rewrite the Euler equation as

$$\vec{0} = \nabla\left(\partial_t\psi + \frac{v^2}{2} + \phi\right) + \frac{1}{\rho}\partial_\rho p\,\nabla\rho.$$

If there exists a primitive $G(\rho)$ of the funtion $\partial_\rho p(\rho)$, we can write

$$\partial_t\psi + \frac{v^2}{2} + \phi + G = 0. \tag{3.4.1}$$

Decomposing ψ and ρ in a background and perturbation part as

$$\begin{aligned} \psi &= \psi_0 + \delta\psi, \\ \rho &= \rho_0 + \delta\rho, \end{aligned}$$

we can rewrite the continuity equation and equation (3.4.1), at first order in the perturbations, as

$$\begin{aligned} \partial_t\delta\rho + \nabla\left[\delta\rho\vec{v}_0 + \rho_0\nabla\delta\psi\right] &= 0, \\ \partial_t\delta\psi + \vec{v}_0\cdot\nabla\delta\psi + \frac{1}{\rho_0^2}\partial_\rho p_0\delta\rho &= 0, \end{aligned} \tag{3.4.2}$$

where $\vec{v}_0 = \nabla\psi_0$ and $\partial_\rho p_0 = \partial_\rho p|_{\rho=\rho_0}$. From the system of equations (3.4.2) we can get a unique equation in $\delta\psi$. After some standard calculations, we write

$$\begin{aligned} \frac{1}{\rho_0}\Bigg[\partial_t\left(\frac{\rho_0}{\partial_\rho p_0}\partial_t\delta\psi\right) + \partial_t\left(\frac{\rho_0}{\partial_\rho p_0}\vec{v}_0\cdot\nabla\delta\psi\right) + \nabla\cdot\left(\frac{\rho_0}{\partial_\rho p_0}\vec{v}_0\partial_t\delta\psi\right) \\ -\nabla\cdot(\rho_0\nabla\delta\psi) + \nabla\cdot\left(\vec{v}_0\frac{\rho_0}{\partial_\rho p_0}\vec{v}_0\cdot\nabla\delta\psi\right)\Bigg] = 0. \end{aligned}$$

The main observation is that the previous equation is exactly the Klein-Gordon equation for a massless scalar field on a non trivial space-time metric

$$g_{\mu\nu}\nabla^\mu\nabla^\nu\mathbf{\Phi} = 0,$$

where $g_{\mu\nu}$ are the components of the metric

$$\mathrm{d}s^2 = \frac{\rho}{c_0}\left[\left(c_0^2 - v_0^2\right)\mathrm{d}t^2 + 2\mathrm{d}t\,\vec{v}_0\cdot\mathrm{d}\vec{x} - \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}\right] \tag{3.4.3}$$

and $c_0 = \sqrt{\partial_\rho p_0}$ is the local sound velocity.
From the irrotationality of $\vec{v}_0$ we can define a new time coordinate

$$\tau = t + \int_\gamma \frac{\vec{v}_0\cdot\mathrm{d}\vec{x}}{c_0^2 - v_0^2},$$

since the inegral depends only on the initial and final points of γ. Introducing also the coordinate

$$\mathrm{d}q = \frac{\vec{v}_0\cdot\mathrm{d}\vec{x}}{v_0},$$

we can rewrite (3.4.3) as

$$\mathrm{d}s^2 = \frac{\rho_0}{c_0}\left[\left(c_0^2 - v_0^2\right)\mathrm{d}\tau^2 - \frac{v_0^2\mathrm{d}q^2}{c_0^2 - v_0^2} - \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}\right].$$

3.5 Analogue gravity in Bose-Einstein condensates

The Hamiltonian describing a many-body system composed by N interacting bosons confined in an external potential $V_{ext}(\mathbf{x})$, in a second quantized formalism, can be written as [GACZ00, BLV05, FBC$^+$13, ABFP13]

$$\hat{H} = \int \mathrm{d}^3x\left[\hat{\Psi}^\dagger\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}\right)\hat{\Psi} + \frac{g}{2}\hat{\Psi}^\dagger\hat{\Psi}^\dagger\hat{\Psi}\hat{\Psi}\right], \tag{3.5.1}$$

where $\hat{\Psi}(t,\mathbf{x})$ is the field operator which annihilates an atom at position $\mathbf{x}$ and obeys the bosonic equal time commutation relation

$$[\hat{\Psi}(\mathbf{x}), \hat{\Psi}^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x}-\mathbf{x}').$$

In our description g is a coupling constant related to the atom-atom scattering length by $g = 4\pi\hbar^2 a/m$.
At sufficiently low temperature, a macroscopic fraction of the atoms accumulates in a single particle lowest energy state, described by a macroscopic wavefunction $\Psi_0(\mathbf{x})$, whose time evolution equation is given by the Gross-Pitaevskii equation

$$i\hbar\frac{\partial\Psi_0}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V - ext + g|\Psi_0|^2\right)\Psi_0.$$

Using the density-phase representation, the condensate wavefunction can be written as $\Psi_0 = \sqrt{n}e^{i\theta}$, the Gross-Pitaevskii equation can be rewritten as a pair of real equations

$$\begin{cases}\partial_t n + \nabla(n\mathbf{v}) = 0,\\ \hbar\,\partial_t\theta = -\frac{\hbar^2}{2m}(\nabla\theta)^2 - gn - V_{ext} - V_q,\end{cases}$$

where the first equation is the continuity equation with an irrotational condensate velocity $\mathbf{v}_0 = \hbar\nabla\theta/m$ while the second is the analogous of the Euler equation fon an irrotational inviscid fluid, with an additional term of quantum pressure

$$V_q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}.$$

In the density-phase representation, dividing the density and the phase in a mean value and a perturbative one, the field operator can be rewritten as

$$\hat{\Psi} = \sqrt{n+\hat{n}_1}e^{i(\theta+\hat{\theta}_1)} \simeq \Psi_0\left(1+\frac{\hat{n}_1}{2n}+i\hat{\theta}_1\right)$$

and the two equations of motion for the fluctuations are

$$\begin{cases} \partial_t\hat{n}_1 = -\nabla\left(\mathbf{v}_0\hat{n}_1 + \frac{\hbar n}{m}\nabla\hat{\theta}_1\right), & (3.5.2a)\\ \hbar\,\partial_t\hat{\theta}_1 = -\hbar\mathbf{v}_0\nabla\hat{\theta}_1 - \frac{mc^2}{n}\hat{n}_1 + \frac{mc^2}{4n}\xi^2\nabla\left[n\nabla\left(\frac{\hat{n}_1}{n}\right)\right] = 0. & (3.5.2b)\end{cases}$$

We have introduced a foundamental length scale called *healing length* defined as $\xi = \hbar/(mc)$ and the local speed of sound $c = \sqrt{ng/m}$.
In the so-called hydrodynamic approximation, for length scale much larger than ξ, the last term in the evolution equation for $\hat{\theta}_1$ (3.5.2b) can be neglected and $\hat{n}_1$ can be decoupled as

$$\hat{n}_1 = -\frac{\hbar n}{mc^2}\left[\mathbf{v}_0\nabla\hat{\theta}_1 + \partial_t\hat{\theta}_1\right].$$

When this form is inserted in the evolution equation of $\hat{n}_1$ (3.5.2a), we obtain

$$-\left(\partial_t + \nabla\mathbf{v}_0\right)\frac{n}{mc^2}\left(\partial_t + \mathbf{v}_0\nabla\right)\hat{\theta}_1 + \nabla\frac{n}{m}\nabla\hat{\theta}_1 = 0 \qquad (3.5.3)$$

which in matrix form reads as

$$\partial_\mu\left(f^{\mu\nu}\partial_\nu\hat{\theta}_1\right) = 0,$$

where the matrix element f^{ij} is defined as

$$f^{00} = -\frac{n}{c^2}, \qquad f^{ij} = \frac{n}{c^2}\left(c^2\delta^{ij} - v_0^i v_0^j\right),$$

and $i,j = 1,2,3$.
In any Lorentzian manifold, the d'Alambertian operator is

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right),$$

where $g_{\mu\nu}$ is the metric and $g^{\mu\nu}$ its inverse and $g = \det(g_{\mu\nu})$. Using this definition, the equation (3.5.3) can be rewritten as

$$\Box\hat{\theta}_1 = 0,$$

once one identifies

$$\sqrt{-g}g^{\mu\nu} = f^{\mu\nu}.$$

Inverting this definition, we obtain the effective metric

$$g_{\mu\nu} = \frac{n}{mc}\begin{pmatrix} -(c^2 - \mathbf{v}_0^2) & -v_0^i \\ -v_0^j & \delta_{ij} \end{pmatrix}.$$

We have then shown that under hydrodynamical approximation, the equation of motion for the phase fluctuation in a BEC can be written as a Klein-Gordon equation for a massless scalar field, propagating in a space-time with metric $g_{\mu\nu}$.

Chapter 4

Black hole radiation in Bose-Einsten condensates

4.1 Introduction

In this chapter we will study a superluminal disperive BEC model. After some manipulations of the model, to bring us back to a known framework in literature, we will find the asymptotic solutions, through the WKB method, far from the horizon (turning point).
After that, we will compute the solutions in the near horizon region, fundamental step that will allow us to connect the two approximations through matched asymptotic expansion.
Finally, once recovered the matching coefficients and evaluated the asymptotic expressions of the modes, we will find an analytical expression for the thermality and the greybody factor, thanks to a current balance.

4.2 Settings

4.2.1 Dilute gases

We will give in this paragraph the basic ingredients to describe the condensates and their perturbations [MP09].
In a second quantized fashion, atoms are described by a field operator $\Psi(t, \mathbf{x})$, which annihilates an atom at time t and position $\mathbf{x}$. The field obeys the equal time commutator

$$[\Psi(t, \mathbf{x}), \Psi^\dagger(t, \mathbf{x}')] = \delta^3(\mathbf{x} - \mathbf{x}').$$

The time evolution of Ψ is given by the Heisenberg equation

$$i\hbar\partial_t \Psi(t, \mathbf{x}) = [\Psi(t, \mathbf{x}), H], \tag{4.2.1}$$

where the Hamiltonian is defined as

$$H = \int \mathrm{d}^3x \left\{ \frac{\hbar^2}{2m} \nabla_x \Psi^\dagger \nabla_x \Psi + V \Psi^\dagger \Psi + \frac{g}{2} \Psi^\dagger \Psi^\dagger \Psi \Psi \right\},$$

where m is the mass of the set of atoms, V is the external potential and g represents the coupling constant which describes the scattering of atoms.
Approaching the 0K (typical scales are of 300nK for 10^4 atoms), a large amount of the atoms condenses in a common state. We can then separate this common state from its perturbations, decomposing the field operator into a constant part Ψ_0, which describes the condensed atoms, and a fluctuation part, as

$$\Psi = \Psi_0 + \widetilde{\phi}. \tag{4.2.2}$$

In this approximation, Ψ_0 satisfies the Gross-Pitaevskii equation

$$i\hbar \partial_t \Psi_0 = [T + V + g\rho_0]\Psi_0, \tag{4.2.3}$$

where T is the kinetic operator defined as $T = -\hbar^2 \nabla_x^2/(2m)$ and $\rho_0 = |\Psi_o|^2$ is the condensate atom density, which obeys the continuity equation:

$$\partial_t \rho_0 + \mathrm{div}(\rho_0 \mathbf{v}) = 0,$$

where $\mathbf{v}$ represents the condensate velocity.

4.2.2 Stationary condensates

We will work with a stationary condensate. This means that, since in the general case V, g and ρ_0 depend both on $\mathbf{x}$ and t, there exists a Galilean frame in which V, g and ρ_0 are only funcion of $\mathbf{x}$.
In this frame, the condensate wave function can be written as

$$\Psi_0(t, \mathbf{x}) = e^{\frac{-i\mu t}{\hbar}} \sqrt{\rho_0(\mathbf{x})} e^{iW_0(\mathbf{x})} \tag{4.2.4}$$

where μ is the chemical potential. The condensate wave vector is $\mathbf{k}_0 = \partial_x W_0$.
In the case of a one-dimensional stationary condensate, the continuity equation is reduced to

$$\rho_0(x) v(x) = const, \tag{4.2.5}$$

where x is the longitudinal coordinate and $v = \hbar k_0(x)/m$ is the condensate velocity. From now on, we will imagine that the condensate is flowing from the right to the left (so that $v(x)$ will be negative, with respect an x axis directed to the right). Substituting (4.2.4) into (4.2.3) we obtain

$$\mu = \left[\frac{mv^2(x)}{2} + \rho_0^{-1/2} T \rho_0^{1/2} + V(x) + g(x)\rho_0(x) \right].$$

Because of (4.2.5), we can characterize the non-homogeneity of the background by means of two functions whose product must remain constant. We will consider the condensate velocity $v(x)$ and the x-dependent speed of sound

$$c^2(x) = g(x)\frac{\rho_0(x)}{m}. \tag{4.2.6}$$

4.2.3 Bogoliubov-de Gennes equation

We aim now at showing that the relative fluctuation, at the linear order, obeys a fourth-order equation which does not depend on the external potential [CW18]. Putting (4.2.2) into equation (4.2.1) and linearizing in $\widetilde{\phi}$, we obtain

$$i\hbar\,\partial_t\widetilde{\phi} = [T + V + 2g\rho_0]\widetilde{\phi} + g\Psi_0^2\widetilde{\phi}^\dagger. \tag{4.2.7}$$

For mathematical convenience, we define the relative fluctuation $\phi = \widetilde{\phi}/\Psi_0$ and we rewrite the field operator as

$$\Psi = \Psi_0(1 + \phi). \tag{4.2.8}$$

Substituting now $\widetilde{\phi} = \Psi_0\phi$ in (4.2.7) and using the equations (4.2.4) and (4.2.6), we get

$$i\hbar(\partial_t + v\partial_x)\phi = T_\rho\phi + mc^2[\phi + \phi^\dagger], \tag{4.2.9}$$

where we have defined the 'dressed' kinetic operator as

$$T_\rho = -\frac{\hbar^2}{2m\rho_0}\partial_x\rho_0\partial_x = -\hbar^2\frac{v}{2m}\partial_x\frac{1}{v}\partial_x,$$

which accounts for the non-homogeneity of the condensate density. This expression is valid for stationary condensates only and is obtained by using equation (4.2.5).

We can now express the field operator as a superposition of the form

$$\phi_\omega(t,x) = a_\omega e^{-i\omega t}\phi_\omega(x) + a_\omega^\dagger[e^{-i\omega t}\varphi_\omega(x)]^*, \tag{4.2.10}$$

where a_ω and $a_\omega^\dagger$ are respectively phonon annihilation and creation operators.

Substituting (4.2.10) into equation (4.2.9) and taking the commutator first with a_ω and then with $a_\omega^\dagger$ we obtain a system of second order coupled equations:

$$\begin{cases} [\hbar(\omega + iv\partial_x) - T_\rho - mc^2]\phi_\omega = mc^2\varphi_\omega, \\ [-\hbar(\omega + iv\partial_x) - T_\rho - mc^2]\varphi_\omega = mc^2\phi_\omega. \end{cases} \tag{4.2.11}$$

4.3 Analysis of fourth order equations

It is possible to decouple the equations (4.2.11) by dividing, for example, the first by c^2 and substituting the expression in the second one, and viceversa. After some manipulations, we obtain

$$\begin{aligned} &\left\{[\hbar(\omega+iv\partial_x)+T_\rho]\frac{1}{c^2}[-\hbar(\omega+iv\partial_x)+T_\rho]+2mT_\rho\right\}\phi_\omega=0,\\ &\left\{[-\hbar(\omega+iv\partial_x)+T_\rho]\frac{1}{c^2}[\hbar(\omega+iv\partial_x)+T_\rho]+2mT_\rho\right\}\varphi_\omega=0, \end{aligned} \tag{4.3.1}$$

where we recall that both v and c are function of the longitudinal coordinate x. We have obtained two fourth order decoupled equations in ϕ_ω and φ_ω. From now on, we will work and develop calculations exclusively for ϕ_ω, since the same procedures can be extended to φ_ω.
Expanding the expressions (4.3.1), after standard computations [1] and dividing by $\hbar^2$, we get

$$\bar{c}_4\ \partial_x^4\phi_\omega+\bar{c}_3\ \partial_x^3\phi_\omega+\bar{c}_2\ \partial_x^2\phi_\omega+\bar{c}_1\ \partial_x\phi_\omega+\bar{c}_0\ \phi_\omega=0. \tag{4.3.2}$$

We now need to eliminate the third order derivative term. This decision will be useful to obtain a more manageable form of the fourth order equation, which is also more studied in literature. To do that, without loss of generality, we write $\phi_\omega=h(x)\zeta(x)$ (and the same for φ_ω), where $h(x)$ and $\zeta(x)$ are two generic functions depending on x, and we substitute the factorization into (4.3.2). Performing the derivative of the product for each term and grouping by $\zeta(x)$ we obtain

$$\begin{aligned} &\bar{c}_4h(x)\zeta^{(4)}(x)\\ +&\Big(\bar{c}_3h(x)+4\bar{c}_4h'(x)\Big)\zeta^{(3)}(x)\\ +&\Big(\bar{c}_2h(x)+3\bar{c}_3h'(x)+6\bar{c}_4h''(x)\Big)\zeta''(x)\\ +&\Big(\bar{c}_1h(x)+2\bar{c}_2h'(x)+3\bar{c}_3h''(x)+4\bar{c}_4h^{(3)}(x)\Big)\zeta'(x)\\ +&\Big(\bar{c}_0h(x)+\bar{c}_1h'(x)+\bar{c}_2h''(x)+\bar{c}_3h^{(3)}(x)+\bar{c}_4h^{(4)}(x)\Big)\zeta(x)=0. \end{aligned} \tag{4.3.3}$$

Now we can take the coefficient of $\zeta^{(3)}(x)$ and set it equal to zero. Replacing the expressions of the coefficients $\bar{c}_4$ (A.1.1) and $\bar{c}_3$ (A.1.2) and simplifying, we get the equation

$$h'(x)+\left(\frac{c'(x)}{c(x)}+\frac{v'(x)}{2v(x)}\right)h(x)=0,$$

[1] the extended coeffiecients can be found in the Appendix A.1

whose solution is

$$h(x) = c(x)\sqrt{v(x)}.$$

Now we can substitute the above expression of $h(x)$ in the equation (4.3.3). Since $h(x) \neq 0$, we can divide by it, obtaining a new equation without the third order term [2]

$$\widetilde{c}_4\ \partial_x^4\zeta(x) + \widetilde{c}_2\ \partial_x^2\zeta(x) + \widetilde{c}_1\ \partial_x\zeta(x) + \widetilde{c}_0\ \zeta(x) = 0. \tag{4.3.4}$$

At this point, we can start looking for some solutions of the equation (4.3.4). To find them, we want to use the WKB approximation [Hol12]. We will use as expansion parameter the *healing length*, which we define as

$$\eta = \frac{\hbar}{\sqrt{2}m\bar{c}}, \tag{4.3.5}$$

where

$$\bar{c} = \inf_x c(x).$$

This way the healing length is independent from the longitudinal coordinate since taking the $\inf_x c(x)$ corresponds to obtain the $\sup_x \eta(x)$. Therefore, the healing length we have defined represents the minimum length scale at which dispersion becomes relevant. We are also sure that, if $\eta \to 0$, since it is defined as the sup_x, then $\eta(x) \to 0, \forall x$.
We have now to substitute, wherever it is possible in (4.3.4), the healing length, so that we can make a series expansion in η. For example, we can rewrite $\widetilde{c}_4$ (A.2.1) as

$$\widetilde{c}_4 = \frac{\hbar^2}{4m^2c^2(x)} = \frac{1}{2}\eta^2\frac{\bar{c}^2}{c^2(x)}.$$

Doing the same for the other coefficients, we obtain a new equation [3]

$$c_4\ \partial_x^4\zeta(x) + c_2\ \partial_x^2\zeta(x) + c_1\ \partial_x\zeta(x) + c_0\ \zeta(x) = 0. \tag{4.3.6}$$

We are now in the optimal setup to use the WKB method [4]. We want to find a solution, at first order, of the type

$$\zeta(x) = e^{\frac{\theta(x)}{\eta}}\left(y_0(x) + \eta y_1(x)\right). \tag{4.3.7}$$

[2]the extended coeffiecients can be found in the Appendix A.2
[3]the extended coeffiecients can be found in the Appendix A.3
[4]see Appendix B

Putting then (4.3.7) into (4.3.6), computing the derivatives and performing a series expansion in η, we get

$$\begin{aligned}&\frac{1}{\eta}\left(\frac{2\bar{c}^2\theta'(x)^3}{c^2(x)}+\frac{2v^2(x)\theta'(x)}{c^2(x)}-2\theta'(x)\right)y_0'(x)\\&+\frac{1}{\eta}\left(\frac{\bar{c}^2\theta'(x)^4}{2c^2(x)}+\frac{v^2(x)\theta'(x)^2}{c^2(x)}-\theta'(x)^2\right)y_1(x)\\&+\left[\frac{1}{\eta^2}\left(\frac{\bar{c}^2\theta'(x)^4}{2c^2(x)}+\frac{v^2(x)\theta'(x)^2}{c^2(x)}-\theta'(x)^2\right)\right.\\&+\frac{1}{\eta}\left(\frac{3\bar{c}^2\theta'(x)^2\theta''(x)}{c^2(x)}-\frac{i\sqrt{2}\bar{c}v(x)c'(x)\theta'(x)^2}{c^3(x)}\right.\\&-\frac{2c'(x)\theta'(x)}{c(x)}+\frac{2v(x)v'(x)\theta'(x)}{c^2(x)}+\frac{i\sqrt{2}\bar{c}v'(x)\theta'(x)^2}{c^2(x)}\\&\left.\left.+\frac{v^2(x)\theta''(x)}{c^2(x)}-\frac{2i\omega v(x)\theta'(x)}{c^2(x)}-\theta''(x)\right)\right]y_0(x)=0.\end{aligned} \tag{4.3.8}$$

Balancing now the term in $1/\eta^2$, we recover the eikonal equation

$$\begin{aligned}&\frac{\bar{c}^2}{2c^2(x)}\theta'(x)^4+\left(\frac{v^2(x)}{c^2(x)}-1\right)\theta'(x)^2=\\&=\ \theta'(x)^2\left[\frac{\bar{c}^2}{2c^2(x)}\theta'(x)^2+\left(\frac{v^2(x)}{c^2(x)}-1\right)\right]=0.\end{aligned} \tag{4.3.9}$$

Since it is a fourth order equation, it has four solutions, two of which are identically equal to zero, so that $\theta'(x)=0$. These solutions will give rise, as we will see later on, to an equation with a singularity, called in literature fuchsian singularity. The other two solutions are instead regular.

4.3.1 WKB large wavevector solutions

We start studying the two solutions generated by

$$\theta'_{1,2}(x)=\pm\frac{i}{\bar{c}}\sqrt{2(v^2(x)-c^2(x))}. \tag{4.3.10}$$

We remark that, since these solutions are purely immaginary and, in this case, $v(x)>c(x)$, we have that the modes related to $\theta'_{1,2}$ will live inside the black hole.

Substituting now $\theta'_1(x)=+i\sqrt{2(v^2(x)-c^2(x))}/\bar{c}$ in equation (4.3.8) we obtain

$$\begin{aligned}&\left(1-\frac{v^2(x)}{c^2(x)}\right)y_0'(x)\\&+\left(-\frac{\sqrt{v^2(x)-c^2(x)}v'(x)}{c^2(x)}+\frac{3c'(x)}{2c(x)}\right.\\&\left.+\frac{v(x)\sqrt{v^2(x)-c^2(x)}c'(x)}{c^3(x)}-\frac{3v(x)v'(x)}{2c^2(x)}-\frac{i\omega v(x)}{c^2(x)}\right)y_0(x)=0,\end{aligned}$$

which is a first order ordinary equation, having solution

$$y_{0,1}(x) = \frac{1}{(v^2(x) - c^2(x))^{3/4}} \exp\left(-i\omega \int^x \mathrm{d}x' \frac{v(x')}{v^2(x') - c^2(x')}\right) \left(\sqrt{\frac{v^2(x)}{c^2(x)} - 1} + \frac{v(x)}{c(x)}\right)^{-1}. \tag{4.3.11}$$

Doing the same computations with $\theta_2'(x) = -i\sqrt{2(v^2(x) - c^2(x))}/\bar{c}$, we get

$$y_{0,2}(x) = \frac{1}{(v^2(x) - c^2(x))^{3/4}} \exp\left(-i\omega \int^x \mathrm{d}x' \frac{v(x')}{v^2(x') - c^2(x')}\right) \left(\sqrt{\frac{v^2(x)}{c^2(x)} - 1} + \frac{v(x)}{c(x)}\right). \tag{4.3.12}$$

Then the two complete solutions are

$$\begin{aligned}
\zeta_1(x) =& \exp\left(\frac{\theta_1(x)}{\eta} - i\omega \int^x \frac{v(x')}{v^2(x') - c^2(x')} \mathrm{d}x'\right) \\
& \frac{1}{(v^2(x) - c^2(x))^{3/4}} \left(\sqrt{\frac{v^2(x)}{c^2(x)} - 1} + \frac{v(x)}{c(x)}\right)^{-1}, \\
\zeta_2(x) =& \exp\left(\frac{\theta_2(x)}{\eta} - i\omega \int^x \frac{v(x')}{v^2(x') - c^2(x')} \mathrm{d}x'\right) \\
& \frac{1}{(v^2(x) - c^2(x))^{3/4}} \left(\sqrt{\frac{v^2(x)}{c^2(x)} - 1} + \frac{v(x)}{c(x)}\right),
\end{aligned}$$

where, by integrating (4.3.10), we have defined in the previous two equations $\theta_{1,2}$ as

$$\theta_{1,2}(x) = \pm \frac{i\sqrt{2}}{\bar{c}} \int^x \mathrm{d}x' \sqrt{(v^2(x') - c^2(x'))}$$

We can now study the group velocity of these modes, to understand their motion. We remark that, since we will derive k as a function of ω, we will compute $\mathrm{d}k/\mathrm{d}\omega$ which corresponds to the inverse of the group velocity. To do that we extract the wavevectors from the solutions in the previous equations, for $x \to -\infty$. In this region, the two velocities $v(x)$ and $c(x)$ field are constant:

$$v(x) = v_*, \qquad c(x) = c_*.$$

The two wavevectors are then

$$k_{1,2}(\omega) = \pm \frac{1}{\eta} \frac{\sqrt{2}}{\bar{c}} \sqrt{v_*^2 - c_*^2} - \omega \frac{v_*}{v_*^2 - c_*^2}. \tag{4.3.13}$$

From the above equation, since η is small, the wavevectors will be large. From (4.3.13) we can evaluate the inverse of the group velocity as

$$\left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right)^{-1} = \frac{\mathrm{d}k_{1,2}}{\mathrm{d}\omega} = -\omega\frac{v_*}{v_*^2 - c_*^2}$$

which is equal for the two modes. Moreover, since $v < 0$, we have also a positive group velocity. The two modes are then counter-propagating with respect to the background $v(x)$, i.e. they move from the left to the right.

4.3.2 WKB small wavevector solutions

We want now to deal with the case $\theta'(x) = 0$, that is θ equal to a constant. Before any calculation, we are going to introduce an important parameter, the *surface gravity*, defined as

$$\kappa = \left.\frac{\mathrm{d}(v(x) + c(x))}{\mathrm{d}x}\right|_{\mathcal{H}} \tag{4.3.14}$$

where $\mathcal{H}$ represents the position of the sonic horizon, that we localize in $x = 0$ for simplicity (we recall that, in our reference frame, $v(x)$ is flowing from the right to the left, being then negative, and, evaluated on the horizon, assume the value $v(0) = -c_0$; in our case we have also that $c(0) = c_0$). Using the definition (4.3.14) we have then, near the horizon,

$$(v(x) + c(x)) \sim (v(0) + c(0)) + x\left.\frac{\mathrm{d}(v(x) + c(x))}{\mathrm{d}x}\right|_{x=0} = \kappa x, \tag{4.3.15}$$

while

$$(v(x) - c(x))|_{x=0} = -2c_0,$$

and so

$$(v^2(x) - c^2(x))|_{x=0} = [(v(x) + c(x))(v(x) - c(x))]\,|_{x=0} \sim -2c_0\kappa x. \tag{4.3.16}$$

Now, to compute the other two solutions, we insert again (4.3.7) into (4.3.6). Once we have performed the calcuations, we substitute $\theta'(x) = 0$. Expanding in series as before with respect to η, we obtain

$$\begin{aligned} &\left(\frac{v^2(x)}{c^2(x)} - 1\right)y_0''(x) \\ &+\left(-\frac{2c'(x)}{c(x)} + \frac{2v(x)v'(x)}{c^2(x)} - \frac{2i\omega v(x)}{c^2(x)}\right)y_0'(x) \\ &+\left(\frac{v^2(x)c''(x)}{c^3(x)} - \frac{c''(x)}{c(x)} + \frac{v(x)c'(x)v'(x)}{c^3(x)}\right. \\ &-\frac{2v^2(x)c'(x)^2}{c^4(x)} + \frac{v(x)v''(x)}{2c^2(x)} - \frac{i\omega v'(x)}{c^2(x)} + \frac{v'(x)^2}{4c^2(x)} \\ &\left.-\frac{\omega^2}{c^2(x)} - \frac{v''(x)}{2v(x)} + \frac{3v'(x)^2}{4v^2(x)}\right)y_0(x) = 0, \end{aligned} \tag{4.3.17}$$

where, for brevity, we will denote the coefficient of y_0 with $f(x)$. Dividing now by the coefficient of y_0'' and evaluating the expression on the horizon, in $x = 0$, we get

$$y_0''(x) - \frac{c_0}{2\kappa x}\left(2i\omega\frac{1}{c_0} - \frac{1}{c_0^2}(2c'(x)c(x) - 2v'(x)v(x))|_{x=0}\right) - \frac{c_0}{2\kappa x}f(x)y_0(x)$$
$$= y_0''(x) - \frac{c_0}{2\kappa x}\left(2i\omega\frac{1}{c_0} - \frac{1}{c_0^2}(2c_0\kappa)\right) - \frac{c_0}{2\kappa x}(f(x))y_0(x)$$
$$= y_0''(x) + \frac{1}{x}\left(-\frac{i\omega}{\kappa} + 1\right)y_0' + \frac{1}{x}(f(x))y_0(x) = 0,$$

where, in the last line, we have multiplied $f(x)$ by $-c_0/(2\kappa)$ without relabeling the function. We remark that, since we used a linear approximation across the horizon in our calculations, the solutions will be valid only in a neighborhood of $x = 0$.
To solve this equation, we will use an integration by series, since $x = 0$ is a singular point for the coefficients of y_0' and y_0. Nevertheless, we are in a particular case, in which the singularity is a pole of the first order and we can apply results from the theory of differential equations with fuchsian singularities [CCP82]. To do that, we rewrite the last equation as

$$y_0''(x) + \frac{a_0}{x}y_0'(x) + \frac{b_0}{x^2}y_0(x) = 0 \tag{4.3.18}$$

and we look for a solution of the type

$$y_0(x) = x^\alpha \sum_{n=0}^{\infty} c_n x^n. \tag{4.3.19}$$

Substituting (4.3.19) into (4.3.18), after some manipulations, we get

$$\alpha(\alpha - 1) + a_0\,\alpha + b_0 = 0, \tag{4.3.20}$$

known as *determinant equation*.
In our case, if we assume

$$a_0 = \left(-\frac{i\omega}{\kappa} + 1\right), \qquad b_0 = 0,$$

equation (4.3.20) becomes

$$\alpha^2 - \frac{i\omega}{\kappa}\alpha = 0.$$

The solutions are

$$\alpha_1 = 0, \qquad \alpha_2 = \frac{i\omega}{\kappa}.$$

Then, substituting α_1 in (4.3.19) we find

$$y_{0,3}^N(x) = y_0(x) = c_0 + \cdots = 1 + \ldots$$

where we have chosen $c_0 = 1$ and the apex N stands for 'near' the horizon. In the second case, with α_2 we obtain

$$y_{0,4}^N(x) = y_0(x) = x^{i\omega/\kappa}\left(\sum_{n=0}^{\infty} c_n x^n\right) \sim x^{i\omega/\kappa} + \ldots$$

This solution is related to the Hawking mode, and, the analogy becomes more explicit rewriting the last expression as

$$y_{0,4}^N(x) = e^{\frac{i\omega}{\kappa}\log x},$$

where the logarithmic behavior was highlighted.
Also in this case, we want to study the group velocity. Then, we have to find an approximation for the two solutions for $x \to +\infty$. To recover the two modes, we compute the reduced equation by putting $\eta = 0$ in equation (4.3.4). In this case, since we are far from the horizon, the two velocity fields can be assumed constant and all the derivatives can be considered equal to zero. The resulting equation becomes

$$\left(\frac{v^2(x)}{c^2(x)} - 1\right)\zeta''(x) - \frac{2i\omega v(x)}{c^2(x)}\zeta'(x) - \frac{\omega^2}{c^2(x)}\zeta(x) = 0.$$

With standard computations, the two solutions in the far horizon region are

$$\zeta_3^F(x) = \exp\left(-i\frac{\omega}{c-v}x\right),$$
$$\zeta_4^F(x) = \exp\left(i\frac{\omega}{c+v}x\right),$$

where F stands for 'far' from the horizon.
The two wavevectors

$$k_3 = -\frac{\omega}{c-v}, \qquad k_4 = \frac{\omega}{c+v}, \tag{4.3.21}$$

lead us to the following relation for the (inverse of) the group velocity

$$\frac{\mathrm{d}k_3}{\mathrm{d}\omega} = -\frac{\omega}{c-v}, \qquad \frac{\mathrm{d}k_4}{\mathrm{d}\omega} = \frac{\omega}{c+v}.$$

In view of the previous relations, we can say that the solution $\zeta_3(x)$ is a co-propagating mode (move from the left to the right, as the background flow) while $\zeta_4(x)$ is a counter-propagating mode outside the black hole.
It is possible, heuristically, to find interpolant solutions that recover the

behavior of the previous solutions for $x \to 0^+$ and $x \to +\infty$. The two interpolant solutions can be written as

$$\zeta_3^{INT}(x) = \exp\left(-i\omega \int^x \mathrm{d}x' \frac{1}{c(x') - v(x')}\right)$$
$$\zeta_4^{INT}(x) = \exp\left(i\omega \int^x \mathrm{d}x' \frac{1}{c(x') + v(x')}\right).$$

4.3.3 WKB solutions

From now on, we will call the four solutions of equation (4.3.6) $\phi_i^{(WKB)}$, for $i = 1, \dots, 4$. Collecting them, we have that

$$\begin{aligned}
\phi_1^{(WKB)}(x) &= \exp\left(\frac{\theta_1(x)}{\eta} - i\omega \int^x \mathrm{d}x' \frac{v(x')}{v^2(x') - c^2(x')}\right) \\
&\quad \frac{1}{(v^2(x) - c^2(x))^{3/4}} \left(\sqrt{\frac{v^2(x)}{c^2(x)} - 1} + \frac{v(x)}{c(x)}\right)^{-1} \\
\phi_2^{(WKB)}(x) &= \exp\left(\frac{\theta_2(x)}{\eta} - i\omega \int^x \mathrm{d}x' \frac{v(x')}{v^2(x') - c^2(x')}\right) \\
&\quad \frac{1}{(v^2(x) - c^2(x))^{3/4}} \left(\sqrt{\frac{v^2(x)}{c^2(x)} - 1} + \frac{v(x)}{c(x)}\right) \\
\phi_3^{(WKB)}(x) &= \exp\left(-i\frac{\omega}{c - v} x\right) \\
\phi_4^{(WKB)}(x) &= \exp\left(i\frac{\omega}{c + v} x\right).
\end{aligned} \tag{4.3.22}$$

where the equations for $\phi_3^{(WKB)}, \phi_4^{(WKB)}$ hold rigorously only for $x \to +\infty$. In the asymptotic region, the general solutions of (4.3.1) are the superposition of plane waves $e^{ik(\omega)x}$, with constant amplitude. The asymptotic roots $k(\omega)$ satisfy the dispersion relation [MP09, FBC+13]

$$(\omega - v_\infty k)^2 = c_\infty^2 \left(k^2 + \frac{\eta^2 k^4}{2}\right), \tag{4.3.23}$$

where the asymptotic velocities v_∞, c_∞ are equal to v_*, c_* for $x \to -\infty$ and to v, c for $x \to +\infty$. This relation produces a superluminal propagation of high frequency modes [Cor98]. For this reason, the two modes $\phi_{1,2}^{(WKB)}$ are situated inside the horizon and propagate inside it.
Then, the global solution can be written as linear combination [Cor98] of the four modes as

$$\phi_\omega(x) = \sum_{i=1}^{4} c_i\, \phi_i^{(WKB)}(x). \tag{4.3.24}$$

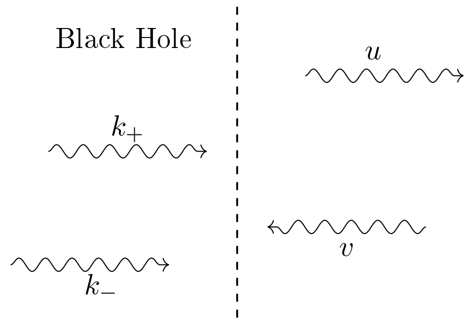

Figure 4.1: The left half represent the region inside the black hole. The modes k_+ and k_- are respectively $\phi_1^{(WKB)}$ and $\phi_2^{(WKB)}$. These two waves are counter-propagating modes that will not be able to escape from the black hole. The co-propagating mode v is $\phi_3^{(WKB)}$ while the Hawking mode u represents ϕ_4^{WKB}.

We have to compute the value of the coefficients c_i. Since we found the WKB-solutions in a region which is sufficiently far from the horizon, our goal is to find solutions of the equation near the sonic horizon, with a range of validity of the approximation that overlaps with that of the WKB. We have to find a Near Horizon Expansion (NHE).

4.3.4 Near horizon approximation

Near the horizon we have to solve the following equation

$$\frac{\mathrm{d}^4\phi}{\mathrm{d}z^4} - \left(z\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + \lambda\frac{\mathrm{d}\phi}{\mathrm{d}z} \right) = 0, \tag{4.3.25}$$

where $\lambda = 1 - i\omega/\kappa$ and z is defined as

$$z = \left(p'_{30}(0)\right)^{1/3} \ \eta^{-2/3} \ x. \tag{4.3.26}$$

To find p'_{30} we took equation (4.3.6) and, after having divided by c_4/η^2, we set $\eta = 0$. Then p_{30} is the coefficient in front of the second order term:

$$p_{30}(x) = \frac{2}{\bar{c}^2}\left(v^2(x) - c^2(x)\right) \qquad \rightarrow \qquad p'_{30}(0) = -\frac{4c_0\kappa}{\bar{c}^2}.$$

We can immediatly notice that a constant ϕ satisfies (4.3.25). Since the constant is arbitrary, for later convenience, we assume it equal to 1 and we label the corresponing solution with subscript 3:

$$\phi_3^{(NHE)}(z) = 1.$$

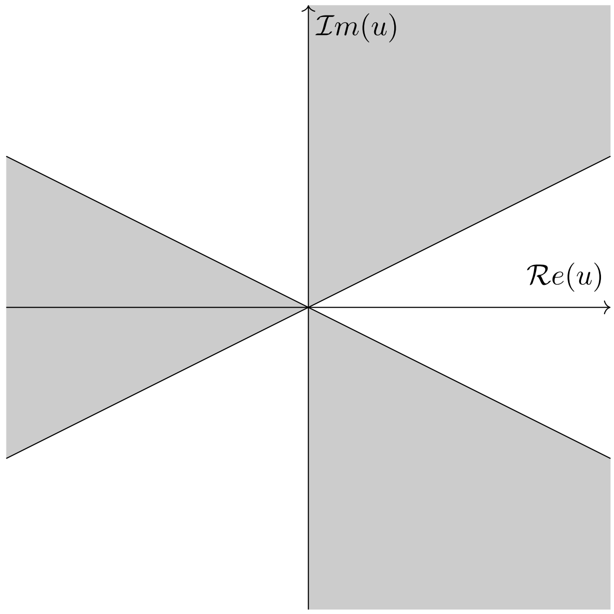

Figure 4.2: Acceptable regions for the solution (in white).

For the other solutions, after computing the Laplace transform (4.3.25) we find

$$\phi_i(z) = \frac{1}{2\pi i}\int_{C_i}\frac{\mathrm{d}t}{t}t^{\lambda-1}\,\exp\{zt - t^3/3\}, \tag{4.3.27}$$

where C_i are adequate circuits in the complex plane. Since we want to distinguish between the inner ($z < 0$) and the outer ($z > 0$) region with respect to the horizon, we can write the variable z as a modulus times its sign, as

$$z = |z|\mathrm{sgn}(z).$$

At this point we perform a change of variable. In particular, we substitute $t = \sqrt{|z|}u$ in (4.3.27), obtaining

$$\phi_i(z) = \frac{1}{2\pi i}|z|^{\frac{\lambda-1}{2}}\int_{C_i}\frac{\mathrm{d}u}{u}u^{\lambda-1}\,\exp\{|z|^{3/4}h_{\pm}(u)\}, \tag{4.3.28}$$

where we have defined

$$h_{\pm}(u) = u\,\mathrm{sgn}(z) - \frac{u^3}{3},$$

and $\pm$ stands for 'out' or 'in' with respect to the horizon. We have to study the function $h_{\pm}(u)$, as the power u^3 generates forbidden zones at infinity. Writing u^3 as complex number

$$u^3 = \rho^3 e^{3i\theta} = \rho^3\left(\cos(3\theta) + i\sin(3\theta)\right),$$

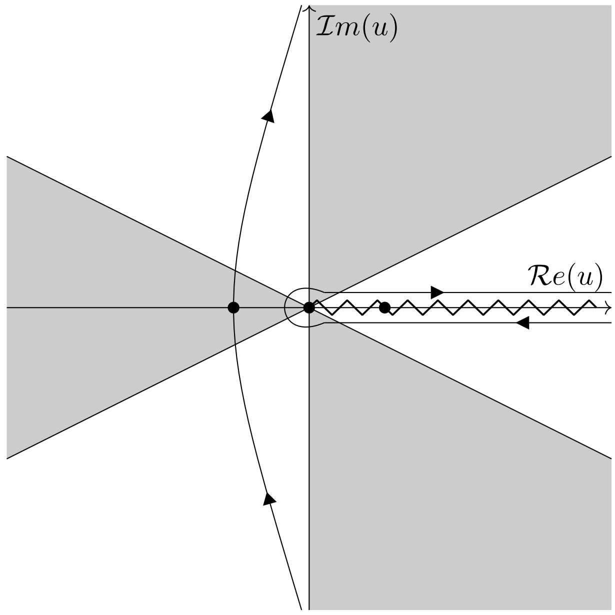

Figure 4.3: Cut-integral with two real pole in ± 1

we have that

$$-u^3 \quad \rightarrow \quad \cos(3\theta) > 0.$$

We get three acceptable zones:

$$\begin{array}{lccc}
(1) & -\frac{\pi}{2} < 3\theta < \frac{\pi}{2} & \rightarrow & -\frac{\pi}{6} < \theta < \frac{\pi}{6}, \\
(2) & \frac{3\pi}{2} < 3\theta < \frac{5\pi}{2} & \rightarrow & \frac{\pi}{2} < \theta < \frac{5\pi}{6}, \\
(3) & \frac{7\pi}{2} < 3\theta < \frac{9\pi}{2} & \rightarrow & \frac{7\pi}{6} < \theta < \frac{3\pi}{2},
\end{array}$$

which are represented in Figure 4.2.
For future purpose, we need also to compute the values of u that generate a saddle point for $h(u)$. To do this, we set the first derivative equal to zero. We have that

$$h'_{out}(u) = 1 - u^2, \qquad \text{for } z > 0$$
$$h'_{in}(u) = -1 - u^2, \qquad \text{for } z < 0,$$

and, putting them equal to zero we find

$$\begin{aligned} u_{max,out} &= \pm 1, \\ u_{max,in} &= \pm i. \end{aligned} \tag{4.3.29}$$

To study the integral (4.3.28) we want to use the steepest-descent method (also known as saddle-point method) [Mil]. Since we have a pole on the acceptable zone of the real axis, we must procede by making a cut on the

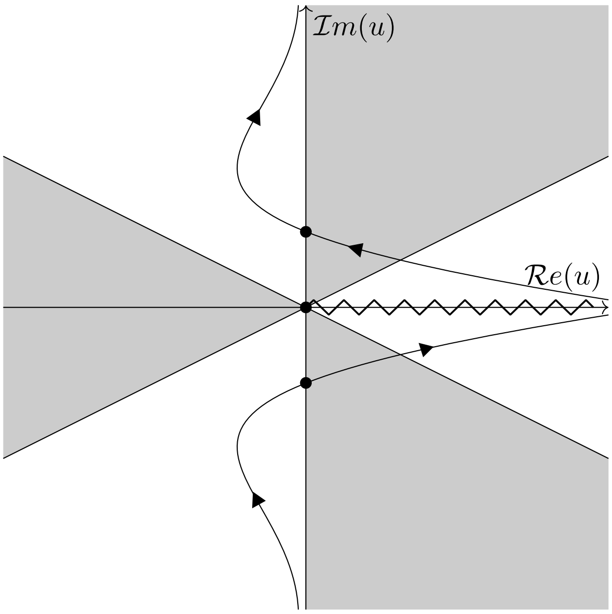

Figure 4.4: Circuits along the two immaginary pole $\pm i$

real positive semi-axis. Computing the cut integral [5], represented in Figure 4.3, we get

$$\begin{aligned} \phi_4^{(NHE)}(z) &= -\frac{1}{2\pi i}\,\Gamma\left(-\frac{i\omega}{\kappa}\right)|z|^{i\omega/\kappa}\;2\sinh\Big(\frac{\pi\omega}{\kappa}\Big) \\ &= -\frac{1}{\pi i}\,\Gamma\left(-\frac{i\omega}{\kappa}\right)\;\exp\left(\frac{i\omega}{\kappa}\log|z|\right)\;\sinh\Big(\frac{\pi\omega}{\kappa}\Big). \end{aligned} \tag{4.3.30}$$

This is the expression of the Hawking mode, in the near horizon region. We can indeed appreciate the characteristic spacial logarithmic dependence of the phase.

The two solutions arising from the two immaginary pole, see Figure 4.4, for $z \to +\infty$, are

$$\begin{aligned} \phi_{out}^{(NHE)}(z) =& \frac{\sqrt{\pi}}{2\pi i}|z|^{-3/4}\Bigg[\exp\left(i\theta(u_{max,out})\right)\exp\left(|z|^{3/2}(\pm\frac{2}{3})\right) \\ & \exp\left(-\frac{i\omega}{2\kappa}\log|z|\right)(\pm 1)^{(-1-i\omega/\kappa)}\Bigg] \\ \phi_{in}^{(NHE)}(z) =& \frac{\sqrt{\pi}}{2\pi i}|z|^{-3/4}\exp\left(\frac{\pi\omega}{2\kappa}\right)\Bigg[\exp\left(i\theta(u_{max,in})\right) \\ & \exp\left(|z|^{3/2}(\mp\frac{2}{3}i)\right)\exp\left(-\frac{i\omega}{2\kappa}\log|z|\right)\exp\left(\mp\frac{\pi}{2}i\right)\Bigg], \end{aligned} \tag{4.3.31}$$

[5]see Appendix C

where $\theta(u) := \arg(u')$ and $\theta(u_{max})$ is the angle of inclination of the oriented tangent to the circuit in the points that we obtain in (4.3.29). We can find a numerical value of the angle of incidence [BH86] if we evaluate h'' in the saddle point, such that

$$h''_{out}(\pm 1) = \mp 2$$
$$h''_{in}(\pm i) = 2(\mp i) = 2e^{\mp i\frac{\pi}{2}} = 2e^{i\alpha_{1,2}},$$

where

$$\alpha_1 = -\frac{\pi}{2}, \qquad \alpha_2 = \frac{\pi}{2}.$$

Then the two angles are

$$\theta_1(+i) = -\frac{\alpha_1}{2} + \frac{\pi}{2} = \frac{3\pi}{4}, \qquad \theta_2(-i) = -\frac{\alpha_2}{2} + \frac{\pi}{2} = \frac{\pi}{4}. \tag{4.3.32}$$

For our purposes, we will concentrate only on $\phi_{in}^{(NHE)}(z)$, since it will be the one we are going to match with the two WKB solutions $\phi_{1,2}^{(WKB)}(x)$, in the next section. Regarding $\phi_{out}^{(NHE)}(z)$ consists in a decaying mode (the one with the exponential function with real negative argument), essential to close the circuit in Figure 4.3, while the growing mode (the other solution) will be irrelevant for our studies.

4.3.5 Matching of the solutions

To be able to sew the WKB and NHE solutions, we have to compute the limit in the first for $x \to 0$ and in the second for $z \to +\infty$, for each of the four solutions. This way, we are able to overlaps the two domains in the linear region, see Figure 4.5, where the two approximations holds. The aim is to find the coefficients c_i of equation (4.3.24), writing the relation

$$c_i\ \phi_i^{(WKB)}(x) = \phi_i^{(NHE)}(z),$$

for each of the four modes, recalling that z is function of x and that the equality in the previous relation must holds for x.

The simplest of the modes is

$$c_3\ \phi_3^{(WKB)}(x) = \phi_3^{(NHE)}(z),$$

for which, since we have chosen $\phi_3^{(WKB)}(x) = \phi_3^{(NHE)}(z) = 1$, then $c_3 = 1$. The coefficient c_4 is also rather simple to compute:

$$c_4\ \phi_4^{(WKB)}(x) = \phi_4^{(NHE)}(z) \tag{4.3.33}$$

where we have [6]

$$c_4 \exp\left(\frac{i\omega}{\kappa}\log|x|\right) = -\frac{1}{2\pi i}\,\Gamma\left(-\frac{i\omega}{\kappa}\right)\ \exp\left(\frac{i\omega}{\kappa}\log|z|\right)\ 2\sinh\Big(\frac{\pi\omega}{\kappa}\Big).$$

[6] Note that, since here $x > 0$, then $x = |x|$

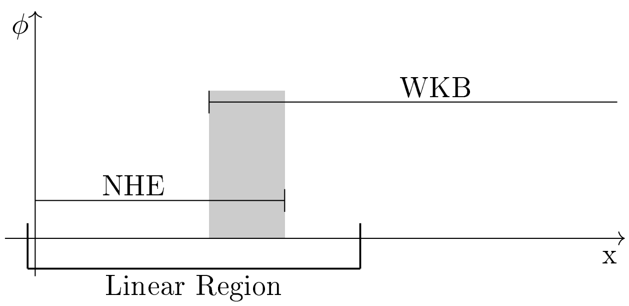

Figure 4.5: Subdivision of the regions (in this case, outside the black hole) where the single approximations are valid. The linear region is the one where we can write $v(x) + c(x) \sim \kappa x$.

Recalling the definition of z in (4.3.26) and substituting it into (4.3.30), for similarity we get

$$c_4 = -\frac{1}{\pi i}\left(\left(\frac{4c_0\kappa}{\bar{c}^2}\right)^{1/3}\eta^{-2/3}\right)^{i\omega/\kappa}\Gamma\left(-\frac{i\omega}{\kappa}\right)\sinh\left(\frac{\pi\omega}{\kappa}\right).$$

Regarding the two solutions $\phi_{1,2}^{(WKB)}$ we have to perform an asymptotic expansion for $x \to 0$, taking special care since the two modes are inside the black hole and then $x < 0$. Recalling that

$$\theta'_{1,2}(x) = \pm\frac{\sqrt{2}i}{\bar{c}}\sqrt{(v^2(x) - c^2(x))},$$

as x is approaching the horizon and using (4.3.16), we get

$$\begin{aligned}
\theta_{1,2}(x) &= \pm\frac{\sqrt{2}i}{\bar{c}}\int_x^0 \mathrm{d}x'\sqrt{(v^2(x') - c^2(x'))}\\
&= \pm\frac{\sqrt{2}i}{\bar{c}}\sqrt{2c_0\kappa}\int_x^0 \mathrm{d}x'\sqrt{-x'}\\
&= \pm\frac{2i}{\bar{c}}\sqrt{c_0\kappa}\left[\frac{2}{3}(-x')^{3/2}\right]_x^0\\
&= \pm\frac{2i}{\bar{c}}\sqrt{c_0\kappa}\left(-\frac{2}{3}(-x)^{3/2}\right)\\
&= \frac{2}{\bar{c}}\sqrt{c_0\kappa}\,|x|^{3/2}\left(\mp\frac{2}{3}i\right).
\end{aligned}$$

Then, by (4.3.22), the two WKB modes goes as

$$\phi_{1,2}^{(WKB)}(x) = -\frac{|x|^{-3/4}}{(2c_0\kappa)^{3/4}}\exp\left(\frac{1}{\eta}\frac{2}{\bar{c}}\sqrt{c_0\kappa}\,|x|^{3/2}\left(\mp\frac{2}{3}i\right) - \frac{i\omega}{2\kappa}\log|x|\right).$$

Taking now the solution NHE from (4.3.31) and stating the equality

$$c_{1,2}\ \phi_{1,2}^{(WKB)}(x) = \phi_{in}^{(NHE)}(z)$$

we obtain, substituting again (4.3.26) in (4.3.31)

$$\begin{aligned} c_{1,2} = \pm\, 2^{-\frac{3}{4}} \sqrt{\frac{c_0 \kappa \bar{c}}{\pi}}\, \eta^{\frac{1}{2}} \left(\left(\frac{4 c_0 \kappa}{\bar{c}^2} \right)^{\frac{1}{3}} \eta^{-\frac{2}{3}} \right)^{-\frac{i\omega}{2\kappa}} \\ \exp\left(\pm \frac{\pi}{2} \frac{\omega}{\kappa} + i\,\theta(\pm i) \right), \end{aligned} \tag{4.3.34}$$

where $\theta(\pm i)$ is $3\pi/4$ and $\pi/4$ respectively for c_1 and c_2, according to (4.3.32). At this point, since we have found the four coefficients for ϕ_ω, is the moment to reintroduce the other solution φ_ω [7] to proceed for the analytic evaluation of the temperature T_H and the greybody factor Γ_ω.

4.4 Thermality and greybody factor

To proceed in our calculations, we need a mode basis which is orthonormal and complete [MP09, FBC+13]. The orthonormality is defined with respect to the scalar product in the space of the solutions of equations (4.2.11). Considering the doublet $W = (\phi_\omega, \varphi_\omega)$, the scalar product is defined as

$$\langle W' | W \rangle = \langle (\phi^*_{\omega'}, \varphi_{\omega'}) | (\phi_\omega, \varphi^*_\omega) \rangle = \int \mathrm{d}x \frac{1}{\rho_0(x)} \left(\phi_\omega \phi^*_{\omega'} - \varphi^*_{\omega'} \varphi_\omega \right)$$

where the presence of ρ_0 is due to the fact that we have used the relative fluctuation in equation (4.2.8).
Since near the horizon a scattering process takes place, the modes we have to use to evaluate the norm have to be chosen sufficiently far, i.e. for $|x| \to +\infty$. In the far horizon region, after the transient due to mode conversion, the four modes ϕ_j, φ_j behave like a plane wave multiplied by an amplitude factor [MFR11]:

$$\phi_j = D_j(\omega)\, e^{-i\omega t} e^{i k_j(\omega) x}, \qquad \varphi_j = E_j(\omega)\, e^{-i\omega t} e^{i k_j(\omega) x}, \tag{4.4.1}$$

where $D_j(\omega)$ and $E_j(\omega)$ are defined as

$$D_j(\omega) = D(\omega; k_j) c_{j,\phi}, \qquad E_j(\omega) = E(\omega; k_j) c_{j,\varphi}.$$

In this case $c_{j,\phi}$ and $c_{j,\varphi}$ are the matching coefficients of ϕ_j and φ_j respectively, whereas $D(\omega; k_j)$ and $E(\omega; k_j)$ are the two normalization factors in

[7] The four solutions WKB, NHE and the matching coefficients can be found in appendix A.3.3

which the coupling between ϕ_j and φ_j resides and they are defined [FBC+13] as

$$D(\omega;k_j)=\frac{\omega-v_0k_j+\frac{c_0\eta k_j^2}{\sqrt{2}}}{2\sqrt{\sqrt{2}\pi\hbar\rho_0c_0\eta k_j^2\left|(\omega-v_0k_j)\left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1}\right|}},$$

$$E(\omega;k_j)=-\frac{\omega-v_0k_j-\frac{c_0\eta k_j^2}{\sqrt{2}}}{2\sqrt{\sqrt{2}\pi\hbar\rho_0c_0\eta k_j^2\left|(\omega-v_0k_j)\left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1}\right|}},$$

where v_0, c_0 are the asymptotic velocities at $\pm\infty$, depending on the modes we are considering [8], while the k_j are the corresponding wavevectors.
Evaluating then the norm squared of the four modes, sufficiently far from the horizon, we obtain that

$$\langle(\phi^*_\omega,\varphi_\omega)|(\phi_\omega,\varphi^*_\omega)\rangle\propto\left[|D_j(\omega)|^2-|E_j(\omega)|^2\right],$$

and, since we are only interested in the sign of the norm, we have that,

$$\left[|D_j(\omega)|^2-|E_j(\omega)|^2\right]\propto(\omega-v_0k_j)\,.$$

Substituting now the expression of the wavevector k_j (4.3.13) and (4.3.21), we get that the squared norms of the modes $1,3,4$ are positive while the mode 2 is negative:

$$\begin{aligned}
\langle(\phi^*_1,\varphi_1)|(\phi_1,\varphi^*_1)\rangle &\propto -v_*\frac{1}{\eta}\frac{\sqrt{2}}{\bar{c}}\sqrt{v_*^2-c_*^2}>0 \quad\to\quad P\\
\langle(\phi^*_2,\varphi_2)|(\phi_2,\varphi^*_2)\rangle &\propto v_*\frac{1}{\eta}\frac{\sqrt{2}}{\bar{c}}\sqrt{v_*^2-c_*^2}<0 \quad\to\quad N_*\\
\langle(\phi^*_3,\varphi_3)|(\phi_3,\varphi^*_3)\rangle &\propto 1-\frac{v}{c-v}>0 \quad\to\quad V\\
\langle(\phi^*_4,\varphi_4)|(\phi_4,\varphi^*_4)\rangle &\propto 1-\frac{v}{c+v}>0 \quad\to\quad H
\end{aligned}$$

where we have identified the negative norm mode N with a subscript.
If we change the point of view and we look at the four modes as particles, we can say that the ones with positive norm are particles while the one with negative norm is an antiparticle. From this perspective, we can say that the three particles that point against the horizon are scattered in one particle that escapes from the black hole:

$$P+N_*+V\quad\to\quad H.$$

[8] Recall that we have called v_* and c_* the asymptotic velocities for $x\to-\infty$ and v,c the ones for $x\to+\infty$

The physical process can be seen and treated as a current balance of the type

$$|J_x^{u,out}| = |J_x^{v,in}| + |J_x^{+,in}| - |J_x^{-,in}| \tag{4.4.2}$$

where the generic current J_x is defined as

$$\begin{aligned} J_x &= -\frac{\hbar}{2mi}\left[\phi \overleftrightarrow{\partial_x} \phi^* + \varphi \overleftrightarrow{\partial_x} \varphi^*\right] \\ &= -\frac{\hbar}{2mi}\left[\phi\, \partial_x \phi^* - \phi^* \partial_x \phi + \varphi\, \partial_x \varphi^* - \varphi^* \partial_x \varphi\right]. \end{aligned} \tag{4.4.3}$$

In equation (4.4.2) the negative sign in the last term is due to the corresponding negative norm of the mode. As for the norms, the currents in the balance have to be evaluated far from the horizon. In this way, subsituting (4.4.1) in (4.4.3), after some manipulations, the generic current can be written as

$$J_x = \frac{\hbar}{m}\left[|D_j(\omega)|^2 + |E_j(\omega)|^2\right]\, k(\omega).$$

The thermality can be obtained from the ratio

$$\frac{|J_x^{-,in}|}{|J_x^{+,in}|},$$

recalling that $+,-$ are related to the modes that we labeled with $1, 2$ respectively. Then, noticing that $|k_1| = |k_2|$, in the ratio $|J_x^{-,in}|/|J_x^{+,in}|$, the wavevectors of the two modes do not give any contribution. The thermality depends indeed only on the square of $D_{1,2}(\omega)$ and $E_{1,2}(\omega)$. Since we have that $c_{1,\phi} = c_{1,\varphi}$ and $c_{2,\phi} = c_{2,\varphi}$, we can rewrite the two currents as

$$\begin{aligned} |J_x^{-,in}| &= \frac{\hbar}{m}\left[|D(\omega;k_1)|^2 + |E(\omega;k_1)|^2\right] |c_1|^2 |k_1|, \\ |J_x^{+,in}| &= \frac{\hbar}{m}\left[|D(\omega;k_2)|^2 + |E(\omega;k_2)|^2\right] |c_2|^2 |k_2|. \end{aligned}$$

If we compute the sum of the squares of the coeffiecients $D(\omega)$ and $E(\omega)$ in the two cases, we can see that, both for $|J_x^{-,in}|$ and $|J_x^{+,in}|$, the sum is

$$|D(\omega;k_{1,2})|^2 + |E(\omega;k_{1,2})|^2 \propto \frac{1}{\eta^2}\frac{2}{\bar{c}^2}(v_*^2 - c_*^2)\left(v_*^2 + \frac{c_*^2}{2\bar{c}^2}(v_*^2 - c_*^2)\right),$$

where we have considered the leading term, depending on η, in the two wavevectors.
Since in the ratio we have the same constants at numerator and denominator, it simplifies to

$$\frac{|J_x^{-,in}|}{|J_x^{+,in}|} = \frac{|c_2|^2}{|c_1|^2} = \frac{e^{-\frac{\pi\omega}{\kappa}}}{e^{\frac{\pi\omega}{\kappa}}} = e^{-\frac{2\pi\omega}{\kappa}} = e^{-\beta_H \omega}$$

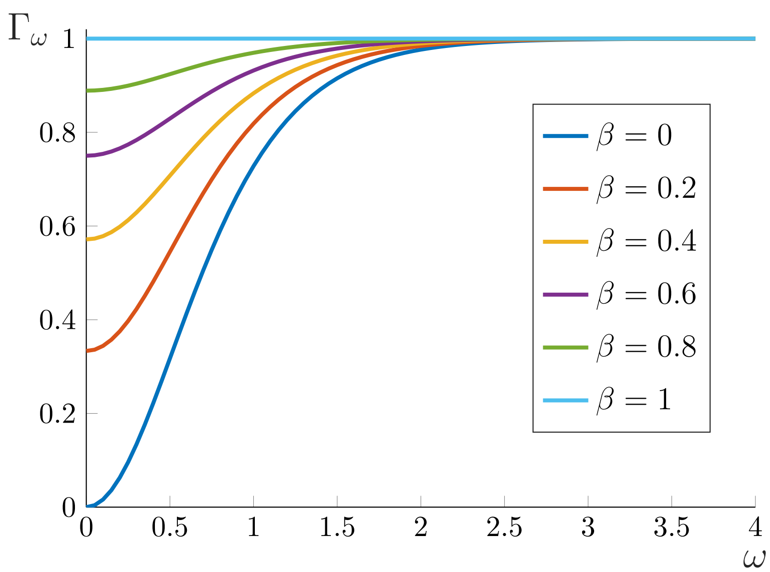

Figure 4.6: Greybody factor as function of the frequency. The surface gravity is $\kappa = 1$ and $\beta = |v|/|c|$, where v, c are the asymptotic constant velocities in the far horizon region.

where

$$T_H = \frac{1}{\beta_H} = \frac{\kappa}{2\pi}$$

is the Hawking temperature [9].
Now we move to evaluate the greybody factor, which is defined as

$$\Gamma_\omega = 1 - \frac{|J_x^{v,in}|}{|J_x^{u,out}|}.$$

Also in this case, the two currents can be written as

$$|J_x^{v,in}| = \frac{\hbar}{m}\left[|D(\omega;k_3)|^2 + |E(\omega;k_3)|^2\right]|c_3|^2|k_3|,$$
$$|J_x^{u,out}| = \frac{\hbar}{m}\left[|D(\omega;k_4)|^2 + |E(\omega;k_4)|^2\right]|c_4|^2|k_4|,$$

since $c_{3,\phi} = c_{3,\varphi}$ and the same for c_4. As before, in the ratio $|J_x^{v,in}|/|J_x^{u,out}|$, recalling the definitions of k_3 and k_4 in (4.3.21) and using the dispersion relation (4.3.23), we have that

$$\frac{|D(\omega;k_3)|^2 + |E(\omega;k_3)|^2}{|D(\omega;k_4)|^2 + |E(\omega;k_4)|^2} = 1.$$

[9] In this formula $\hbar = k_B = 1$

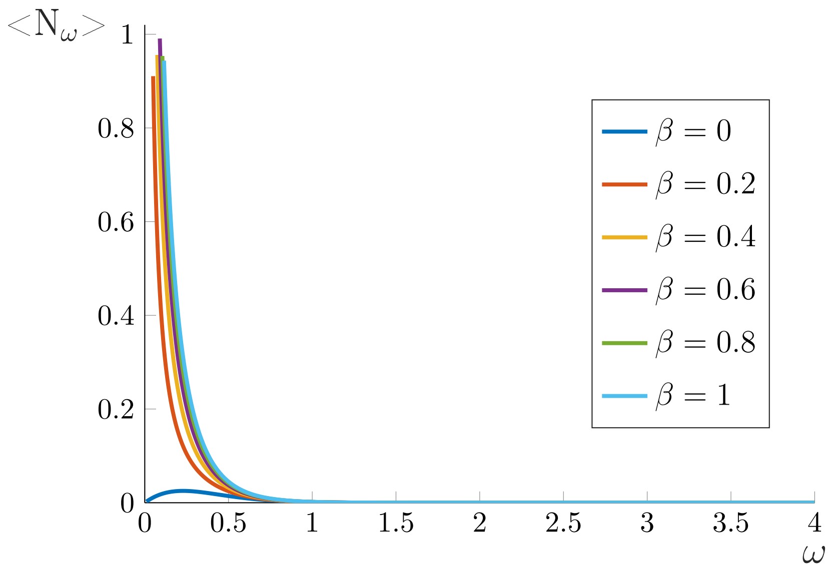

Figure 4.7: Number of particles created as function of ω.

At this point, evaluating the squared modulus of the coefficient c_4, we get:

$$|c_4|^2 = \frac{1}{\pi^2}\left|\Gamma\left(-\frac{i\omega}{\kappa}\right)\right|^2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|^2.$$

Using the property of the Gamma function [GR07]

$$|\Gamma(iy)|^2 = \frac{\pi}{y\sinh(\pi\, y)}$$

we get that

$$|c_4|^2 = \frac{\kappa}{\pi\omega}\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|.$$

Since $c_{3,\phi} = c_{3,\varphi} = 1$, we are able to obtain the greybody factor as

$$\Gamma_\omega = 1 - \frac{|J_x^{v,in}|}{|J_x^{u,out}|} = 1 - \frac{\pi\omega}{\kappa}\frac{1}{\sinh\left(\frac{\pi\omega}{\kappa}\right)}\frac{1-\frac{|v|}{|c|}}{1+\frac{|v|}{|c|}}.$$

A graphycal representation of the greybody factor can be found, for different values of the ratio $|v|/|c|$, in Figure 4.6.
This is a foundamental result that allows us to compute the number of created particles.
Taking the current balance (4.4.2) and dividing by the left-hand side, we get

$$1 = \frac{|J_x^{v,in}|}{|J_x^{u,out}|} + \frac{|J_x^{+,in}|}{|J_x^{u,out}|} - \frac{|J_x^{-,in}|}{|J_x^{u,out}|}.$$

By definition of greybody factor, we can rewrite the previous equation as

$$\Gamma_\omega = \frac{|J_x^{+,in}|}{|J_x^{u,out}|} - \frac{|J_x^{-,in}|}{|J_x^{u,out}|} \tag{4.4.4}$$

and, using the relationship between temperature and currents

$$|J_x^{-,in}| = e^{\frac{2\pi\omega}{\kappa}} |J_x^{+,in}|,$$

we can subsitute this expression into (4.4.4), to obtain

$$\Gamma_\omega = \left(e^{\frac{2\pi\omega}{\kappa}} - 1\right) \frac{|J_x^{+,in}|}{|J_x^{u,out}|}.$$

We can find out the number of created particles as

$$< N_\omega > := \frac{|J_x^{+,in}|}{|J_x^{u,out}|} = \frac{\Gamma_\omega}{\left(e^{\frac{2\pi\omega}{\kappa}} - 1\right)},$$

which exactly represents the Planckian distribution with a correction, given by the greybody factor. A graphycal representation of the number of particles created can be found in Figure 4.7.

Chapter 5

Conclusions

After recalling some fundamental notions about quantum field theory and black hole geometry, we have introduced the BEC model. In this framework, we were able to compute the WKB approximations, far from the turning point, and the near horizon ones, for the solutions.
Thanks to matched asymptotic expansion method, we have sewn the two approximations. Exploiting current conservation, we found an analytical expression for the greybody factor and for the thermality.
To extend this work, expansions beyond the leading order in WKB and near horizon approximations could be studied, evaluating also the ratio between the leading and the next-to leading order. I expect a more complete study to generate a more precise black hole spectroscopy.

Appendix A

Coefficients

A.1 Initial equation

A.1.1 ϕ Coefficients

$$\bar{c}_4 = \frac{\hbar^2}{4m^2c^2(x)}, \tag{A.1.1}$$

$$\bar{c}_3 = -\frac{\hbar^2 c'(x)}{m^2c^3(x)} - \frac{\hbar^2 v'(x)}{2m^2c^2(x)v(x)}, \tag{A.1.2}$$

$$\begin{aligned}\bar{c}_2 = &-\frac{\hbar^2 c''(x)}{2m^2c^3(x)} + \frac{3\hbar^2 c'(x)v'(x)}{2m^2c^3(x)v(x)} + \frac{3\hbar^2 c'(x)^2}{2m^2c^4(x)} - \frac{i\hbar v(x)c'(x)}{mc^3(x)} \\ &-\frac{\hbar^2 v''(x)}{2m^2c^2(x)v(x)} + \frac{3\hbar^2 v'(x)^2}{4m^2c^2(x)v^2(x)} + \frac{i\hbar v'(x)}{mc^2(x)} + \frac{v^2(x)}{c^2(x)} - 1\end{aligned}$$

$$\begin{aligned}\bar{c}_1 = &\frac{\hbar^2 c''(x)v'(x)}{2m^2c^3(x)v(x)} - \frac{i\hbar v(x)c''(x)}{mc^3(x)} + \frac{\hbar^2 c'(x)v''(x)}{m^2c^3(x)v(x)} \\ &-\frac{3\hbar^2 c'(x)v'(x)^2}{2m^2c^3(x)v^2(x)} - \frac{3\hbar^2 c'(x)^2 v'(x)}{2m^2c^4(x)v(x)} - \frac{2i\hbar c'(x)v'(x)}{mc^3(x)} \\ &+\frac{3i\hbar v(x)c'(x)^2}{mc^4(x)} - \frac{2\omega\hbar c'(x)}{mc^3(x)} - \frac{2v^2(x)c'(x)}{c^3(x)} \\ &-\frac{\hbar^2 v^{(3)}(x)}{4m^2c^2(x)v(x)} - \frac{3\hbar^2 v'(x)^3}{4m^2c^2(x)v^3(x)} + \frac{\hbar^2 v'(x)v''(x)}{m^2c^2(x)v^2(x)} \\ &+\frac{i\hbar v''(x)}{mc^2(x)} - \frac{i\hbar v'(x)^2}{mc^2(x)v(x)} + \frac{v(x)v'(x)}{c^2(x)} \\ &-\frac{2i\omega v(x)}{c^2(x)} + \frac{v'(x)}{v(x)},\end{aligned}$$

$$\bar{c}_0 = -\frac{\omega\hbar c''(x)}{mc^3(x)} + \frac{\omega\hbar c'(x)v'(x)}{mc^3(x)v(x)} + \frac{3\omega\hbar c'(x)^2}{mc^4(x)} + \frac{2i\omega v(x)c'(x)}{c^3(x)} - \frac{\omega^2}{c^2(x)}.$$

A.1.2 φ Coefficients

$$\bar{c}_4 = \frac{\hbar^2}{4m^2c^2(x)},$$

$$\bar{c}_3 = -\frac{\hbar^2 c'(x)}{m^2c^3(x)} - \frac{\hbar^2 v'(x)}{2m^2c^2(x)v(x)},$$

$$\bar{c}_2 = -\frac{\hbar^2 c''(x)}{2m^2c^3(x)} + \frac{3\hbar^2 c'(x)v'(x)}{2m^2c^3(x)v(x)} + \frac{3\hbar^2 c'(x)^2}{2m^2c^4(x)} + \frac{i\hbar v(x)c'(x)}{mc^3(x)} - \frac{\hbar^2 v''(x)}{2m^2c^2(x)v(x)} + \frac{3\hbar^2 v'(x)^2}{4m^2c^2(x)v^2(x)} - \frac{i\hbar v'(x)}{mc^2(x)} + \frac{v^2(x)}{c^2(x)} - 1,$$

$$\begin{aligned}\bar{c}_1 =& \frac{\hbar^2 c''(x)v'(x)}{2m^2c^3(x)v(x)} + \frac{i\hbar v(x)c''(x)}{mc^3(x)} + \frac{\hbar^2 c'(x)v''(x)}{m^2c^3(x)v(x)} \\ &- \frac{3\hbar^2 c'(x)v'(x)^2}{2m^2c^3(x)v^2(x)} - \frac{3\hbar^2 c'(x)^2 v'(x)}{2m^2c^4(x)v(x)} + \frac{2i\hbar c'(x)v'(x)}{mc^3(x)} \\ &- \frac{3i\hbar v(x)c'(x)^2}{mc^4(x)} + \frac{2\omega\hbar c'(x)}{mc^3(x)} - \frac{2v^2(x)c'(x)}{c^3(x)} \\ &- \frac{\hbar^2 v^{(3)}(x)}{4m^2c^2(x)v(x)} - \frac{3\hbar^2 v'(x)^3}{4m^2c^2(x)v^3(x)} + \frac{\hbar^2 v'(x)v''(x)}{m^2c^2(x)v^2(x)} \\ &- \frac{i\hbar v''(x)}{mc^2(x)} + \frac{i\hbar v'(x)^2}{mc^2(x)v(x)} + \frac{v(x)v'(x)}{c^2(x)} \\ &- \frac{2i\omega v(x)}{c^2(x)} + \frac{v'(x)}{v(x)},\end{aligned}$$

$$\bar{c}_0 = -\frac{\omega\hbar c''(x)}{mc^3(x)} - \frac{\omega\hbar c'(x)v'(x)}{mc^3(x)v(x)} - \frac{3\omega\hbar c'(x)^2}{mc^4(x)} + \frac{2i\omega v(x)c'(x)}{c^3(x)} - \frac{\omega^2}{c^2(x)}.$$

A.2 Reduced equation

A.2.1 ϕ Coefficients

$$\widetilde{c}_4 = \frac{\hbar^2}{4m^2c^2(x)}, \tag{A.2.1}$$

$$\begin{aligned}\widetilde{c}_2 =& \frac{\hbar^2 c''(x)}{m^2 c^3(x)} - \frac{3\hbar^2 c'(x)^2}{2m^2 c^4(x)} - \frac{i\hbar v(x) c'(x)}{m c^3(x)} + \frac{\hbar^2 v''(x)}{4m^2 c^2(x) v(x)} \\ &- \frac{3\hbar^2 v'(x)^2}{8m^2 c^2(x) v^2(x)} + \frac{i\hbar v'(x)}{m c^2(x)} + \frac{v^2(x)}{c^2(x)} - 1,\end{aligned}$$

$$\begin{aligned}\widetilde{c}_1 =& \frac{\hbar^2 c^{(3)}(x)}{m^2 c^3(x)} - \frac{i\hbar v(x) c''(x)}{m c^3(x)} + \frac{3\hbar^2 c'(x)^3}{m^2 c(x)^5} - \frac{i\hbar c'(x) v'(x)}{m c^3(x)} \\ &+ \frac{i\hbar v(x) c'(x)^2}{m c^4(x)} - \frac{2\omega\hbar c'(x)}{m c^3(x)} - \frac{2c'(x)}{c(x)} - \frac{4\hbar^2 c'(x) c''(x)}{m^2 c^4(x)} \\ &+ \frac{\hbar^2 v^{(3)}(x)}{4m^2 c^2(x) v(x)} + \frac{3\hbar^2 v'(x)^3}{4m^2 c^2(x) v^3(x)} - \frac{\hbar^2 v'(x) v''(x)}{m^2 c^2(x) v^2(x)} \\ &+ \frac{i\hbar v''(x)}{m c^2(x)} + \frac{2v(x) v'(x)}{c^2(x)} - \frac{2i\omega v(x)}{c^2(x)},\end{aligned}$$

$$\begin{aligned}\widetilde{c}_0 =& \frac{\hbar^2 c^{(4)}(x)}{4m^2 c^3(x)} - \frac{\hbar^2 c''(x)^2}{2m^2 c^4(x)} + \frac{i\hbar c''(x) v'(x)}{2m c^3(x)} - \frac{\omega\hbar c''(x)}{m c^3(x)} \\ &+ \frac{v^2(x) c''(x)}{c^3(x)} - \frac{c''(x)}{c(x)} - \frac{\hbar^2 v^{(3)}(x) c'(x)}{4m^2 c^3(x) v(x)} + \frac{\hbar^2 c'(x)^2 v''(x)}{4m^2 c^4(x) v(x)} \\ &- \frac{3\hbar^2 c'(x) v'(x)^3}{4m^2 c^3(x) v^3(x)} - \frac{3\hbar^2 c'(x)^2 v'(x)^2}{8m^2 c^4(x) v^2(x)} + \frac{\hbar^2 c'(x) v'(x) v''(x)}{m^2 c^3(x) v^2(x)} \\ &+ \frac{i\hbar c'(x) v''(x)}{2m c^3(x)} - \frac{3i\hbar c'(x) v'(x)^2}{4m c^3(x) v(x)} - \frac{3i\hbar c'(x)^2 v'(x)}{2m c^4(x)} \\ &+ \frac{3i\hbar v(x) c'(x)^3}{m c(x)^5} + \frac{\omega\hbar c'(x)^2}{m c^4(x)} + \frac{v(x) c'(x) v'(x)}{c^3(x)} - \frac{2v^2(x) c'(x)^2}{c^4(x)} \\ &- \frac{\hbar^2 c^{(3)}(x) c'(x)}{m^2 c^4(x)} + \frac{3\hbar^2 c'(x)^2 c''(x)}{2m^2 c(x)^5} - \frac{2i\hbar v(x) c'(x) c''(x)}{m c^4(x)} \\ &+ \frac{\hbar^2 v^{(4)}(x)}{8m^2 c^2(x) v(x)} - \frac{7\hbar^2 v''(x)^2}{16m^2 c^2(x) v^2(x)} - \frac{63\hbar^2 v'(x)^4}{64m^2 c^2(x) v^4(x)} \\ &- \frac{5\hbar^2 v^{(3)}(x) v'(x)}{8m^2 c^2(x) v^2(x)} + \frac{31\hbar^2 v'(x)^2 v''(x)}{16m^2 c^2(x) v^3(x)} - \frac{3i\hbar v'(x)^3}{4m c^2(x) v^2(x)} \\ &+ \frac{i\hbar v'(x) v''(x)}{m c^2(x) v(x)} + \frac{v(x) v''(x)}{2c^2(x)} - \frac{i\omega v'(x)}{c^2(x)} + \frac{v'(x)^2}{4c^2(x)} \\ &- \frac{\omega^2}{c^2(x)} - \frac{v''(x)}{2v(x)} + \frac{3v'(x)^2}{4v^2(x)}.\end{aligned}$$

A.2.2 φ Coefficients

$$\widetilde{c}_4 = \frac{\hbar^2}{4m^2 c^2(x)},$$

$$
\begin{aligned}
\widetilde{c}_2 =& \frac{\hbar^2 c''(x)}{m^2 c^3(x)} - \frac{3\hbar^2 c'(x)^2}{2m^2 c^4(x)} + \frac{i\hbar v(x) c'(x)}{mc^3(x)} + \frac{\hbar^2 v''(x)}{4m^2 c^2(x) v(x)} \\
&- \frac{3\hbar^2 v'(x)^2}{8m^2 c^2(x) v^2(x)} - \frac{i\hbar v'(x)}{mc^2(x)} + \frac{v^2(x)}{c^2(x)} - 1,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{c}_1 =& \frac{\hbar^2 c^{(3)}(x)}{m^2 c^3(x)} + \frac{i\hbar v(x) c''(x)}{mc^3(x)} + \frac{3\hbar^2 c'(x)^3}{m^2 c(x)^5} + \frac{i\hbar c'(x) v'(x)}{mc^3(x)} \\
&- \frac{i\hbar v(x) c'(x)^2}{mc^4(x)} + \frac{2\omega\hbar c'(x)}{mc^3(x)} - \frac{2c'(x)}{c(x)} - \frac{4\hbar^2 c'(x) c''(x)}{m^2 c^4(x)} \\
&+ \frac{\hbar^2 v^{(3)}(x)}{4m^2 c^2(x) v(x)} + \frac{3\hbar^2 v'(x)^3}{4m^2 c^2(x) v^3(x)} - \frac{\hbar^2 v'(x) v''(x)}{m^2 c^2(x) v^2(x)} \\
&- \frac{i\hbar v''(x)}{mc^2(x)} + \frac{2v(x) v'(x)}{c^2(x)} - \frac{2i\omega v(x)}{c^2(x)},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{c}_0 =& \frac{\hbar^2 c^{(4)}(x)}{4m^2 c^3(x)} - \frac{\hbar^2 c''(x)^2}{2m^2 c^4(x)} - \frac{i\hbar c''(x) v'(x)}{2mc^3(x)} + \frac{\omega\hbar c''(x)}{mc^3(x)} \\
&+ \frac{v^2(x) c''(x)}{c^3(x)} - \frac{c''(x)}{c(x)} - \frac{\hbar^2 v^{(3)}(x) c'(x)}{4m^2 c^3(x) v(x)} + \frac{\hbar^2 c'(x)^2 v''(x)}{4m^2 c^4(x) v(x)} \\
&- \frac{3\hbar^2 c'(x) v'(x)^3}{4m^2 c^3(x) v^3(x)} - \frac{3\hbar^2 c'(x)^2 v'(x)^2}{8m^2 c^4(x) v^2(x)} + \frac{\hbar^2 c'(x) v'(x) v''(x)}{m^2 c^3(x) v^2(x)} \\
&- \frac{i\hbar c'(x) v''(x)}{2mc^3(x)} + \frac{3i\hbar c'(x) v'(x)^2}{4mc^3(x) v(x)} + \frac{3i\hbar c'(x)^2 v'(x)}{2mc^4(x)} \\
&- \frac{3i\hbar v(x) c'(x)^3}{mc(x)^5} - \frac{\omega\hbar c'(x)^2}{mc^4(x)} + \frac{v(x) c'(x) v'(x)}{c^3(x)} - \frac{2v^2(x) c'(x)^2}{c^4(x)} \\
&- \frac{\hbar^2 c^{(3)}(x) c'(x)}{m^2 c^4(x)} + \frac{3\hbar^2 c'(x)^2 c''(x)}{2m^2 c(x)^5} + \frac{2i\hbar v(x) c'(x) c''(x)}{mc^4(x)} \\
&+ \frac{\hbar^2 v^{(4)}(x)}{8m^2 c^2(x) v(x)} - \frac{7\hbar^2 v''(x)^2}{16m^2 c^2(x) v^2(x)} - \frac{63\hbar^2 v'(x)^4}{64m^2 c^2(x) v^4(x)} \\
&- \frac{5\hbar^2 v^{(3)}(x) v'(x)}{8m^2 c^2(x) v^2(x)} + \frac{31\hbar^2 v'(x)^2 v''(x)}{16m^2 c^2(x) v^3(x)} + \frac{3i\hbar v'(x)^3}{4mc^2(x) v^2(x)} \\
&- \frac{i\hbar v'(x) v''(x)}{mc^2(x) v(x)} + \frac{v(x) v''(x)}{2c^2(x)} - \frac{i\omega v'(x)}{c^2(x)} + \frac{v'(x)^2}{4c^2(x)} \\
&- \frac{\omega^2}{c^2(x)} - \frac{v''(x)}{2v(x)} + \frac{3v'(x)^2}{4v^2(x)}.
\end{aligned}
$$

A.3 Healing length equation

A.3.1 ϕ Coefficients

$$c_4 = \frac{\bar{c}^2\eta^2}{2c^2(x)},$$

$$\begin{aligned} c_2 =& \frac{\bar{c}^2\eta^2 v''(x)}{2c^2(x)v(x)} - \frac{3\bar{c}^2\eta^2 v'(x)^2}{4c^2(x)v^2(x)} + \frac{2\bar{c}^2\eta^2 c''(x)}{c^3(x)} - \frac{i\sqrt{2}\bar{c}\eta v(x)c'(x)}{c^3(x)} \\ & - \frac{3\bar{c}^2\eta^2 c'(x)^2}{c^4(x)} + \frac{i\sqrt{2}\bar{c}\eta v'(x)}{c^2(x)} + \frac{v^2(x)}{c^2(x)} - 1, \end{aligned}$$

$$\begin{aligned} c_1 =& \frac{\bar{c}^2\eta^2 v^{(3)}(x)}{2c^2(x)v(x)} + \frac{3\bar{c}^2\eta^2 v'(x)^3}{2c^2(x)v^3(x)} - \frac{2\bar{c}^2\eta^2 v'(x)v''(x)}{c^2(x)v^2(x)} \\ & - \frac{i\sqrt{2}\bar{c}\eta v(x)c''(x)}{c^3(x)} - \frac{i\sqrt{2}\bar{c}\eta c'(x)v'(x)}{c^3(x)} + \frac{i\sqrt{2}\bar{c}\eta v(x)c'(x)^2}{c^4(x)} \\ & - \frac{2\sqrt{2}\bar{c}\eta\omega c'(x)}{c^3(x)} - \frac{2c'(x)}{c(x)} + \frac{2\bar{c}^2\eta^2 c^{(3)}(x)}{c^3(x)} - \frac{8\bar{c}^2\eta^2 c'(x)c''(x)}{c^4(x)} \\ & + \frac{6\bar{c}^2\eta^2 c'(x)^3}{c(x)^5} + \frac{i\sqrt{2}\bar{c}\eta v''(x)}{c^2(x)} + \frac{2v(x)v'(x)}{c^2(x)} - \frac{2i\omega v(x)}{c^2(x)}, \end{aligned}$$

$$
\begin{aligned}
c_0 =& \frac{\bar{c}^2\eta^2 v^{(4)}(x)}{4c^2(x)v(x)} - \frac{7\bar{c}^2\eta^2 v''(x)^2}{8c^2(x)v^2(x)} - \frac{63\bar{c}^2\eta^2 v'(x)^4}{32c^2(x)v^4(x)} \\
&- \frac{5\bar{c}^2\eta^2 v^{(3)}(x)v'(x)}{4c^2(x)v^2(x)} + \frac{31\bar{c}^2\eta^2 v'(x)^2 v''(x)}{8c^2(x)v^3(x)} - \frac{\bar{c}^2\eta^2 c''(x)^2}{c^4(x)} \\
&+ \frac{i\bar{c}\eta c''(x)v'(x)}{\sqrt{2}c^3(x)} - \frac{\sqrt{2}\bar{c}\eta\omega c''(x)}{c^3(x)} + \frac{v^2(x)c''(x)}{c^3(x)} - \frac{c''(x)}{c(x)} \\
&+ \frac{i\bar{c}\eta c'(x)v''(x)}{\sqrt{2}c^3(x)} - \frac{3i\bar{c}\eta c'(x)v'(x)^2}{2\sqrt{2}c^3(x)v(x)} - \frac{3i\bar{c}\eta c'(x)^2 v'(x)}{\sqrt{2}c^4(x)} \\
&+ \frac{3i\sqrt{2}\bar{c}\eta v(x)c'(x)^3}{c(x)^5} + \frac{\sqrt{2}\bar{c}\eta\omega c'(x)^2}{c^4(x)} + \frac{v(x)c'(x)v'(x)}{c^3(x)} \\
&- \frac{2v^2(x)c'(x)^2}{c^4(x)} + \frac{\bar{c}^2\eta^2 c^{(4)}(x)}{2c^3(x)} - \frac{\bar{c}^2\eta^2 v^{(3)}(x)c'(x)}{2c^3(x)v(x)} \\
&+ \frac{\bar{c}^2\eta^2 c'(x)^2 v''(x)}{2c^4(x)v(x)} - \frac{3\bar{c}^2\eta^2 c'(x)v'(x)^3}{2c^3(x)v^3(x)} - \frac{3\bar{c}^2\eta^2 c'(x)^2 v'(x)^2}{4c^4(x)v^2(x)} \\
&+ \frac{2\bar{c}^2\eta^2 c'(x)v'(x)v''(x)}{c^3(x)v^2(x)} + \frac{3\bar{c}^2\eta^2 c'(x)^2 c''(x)}{c(x)^5} - \frac{2i\sqrt{2}\bar{c}\eta v(x)c'(x)c''(x)}{c^4(x)} \\
&- \frac{2\bar{c}^2\eta^2 c^{(3)}(x)c'(x)}{c^4(x)} - \frac{3i\bar{c}\eta v'(x)^3}{2\sqrt{2}c^2(x)v^2(x)} + \frac{i\sqrt{2}\bar{c}\eta v'(x)v''(x)}{c^2(x)v(x)} \\
&+ \frac{v(x)v''(x)}{2c^2(x)} - \frac{i\omega v'(x)}{c^2(x)} + \frac{v'(x)^2}{4c^2(x)} - \frac{\omega^2}{c^2(x)} - \frac{v''(x)}{2v(x)} + \frac{3v'(x)^2}{4v^2(x)}.
\end{aligned}
$$

A.3.2 φ Coefficients

$$
c_4 = \frac{\bar{c}^2\eta^2}{2c^2(x)},
$$

$$
\begin{aligned}
c_2 =& \frac{\bar{c}^2\eta^2 v''(x)}{2c^2(x)v(x)} - \frac{3\bar{c}^2\eta^2 v'(x)^2}{4c^2(x)v^2(x)} + \frac{2\bar{c}^2\eta^2 c''(x)}{c^3(x)} + \frac{i\sqrt{2}\bar{c}\eta v(x)c'(x)}{c^3(x)} \\
&- \frac{3\bar{c}^2\eta^2 c'(x)^2}{c^4(x)} - \frac{i\sqrt{2}\bar{c}\eta v'(x)}{c^2(x)} + \frac{v^2(x)}{c^2(x)} - 1,
\end{aligned}
$$

$$
\begin{aligned}
c_1 =& \frac{\bar{c}^2\eta^2 v^{(3)}(x)}{2c^2(x)v(x)} + \frac{3\bar{c}^2\eta^2 v'(x)^3}{2c^2(x)v^3(x)} - \frac{2\bar{c}^2\eta^2 v'(x)v''(x)}{c^2(x)v^2(x)} \\
&+ \frac{i\sqrt{2}\bar{c}\eta v(x)c''(x)}{c^3(x)} + \frac{i\sqrt{2}\bar{c}\eta c'(x)v'(x)}{c^3(x)} - \frac{i\sqrt{2}\bar{c}\eta v(x)c'(x)^2}{c^4(x)} \\
&+ \frac{2\sqrt{2}\bar{c}\eta\omega c'(x)}{c^3(x)} - \frac{2c'(x)}{c(x)} + \frac{2\bar{c}^2\eta^2\bar{c}^{(3)}(x)}{c^3(x)} - \frac{8\bar{c}^2\eta^2 c'(x)c''(x)}{c^4(x)} \\
&+ \frac{6\bar{c}^2\eta^2 c'(x)^3}{c(x)^5} - \frac{i\sqrt{2}\bar{c}\eta v''(x)}{c^2(x)} + \frac{2v(x)v'(x)}{c^2(x)} - \frac{2i\omega v(x)}{c^2(x)},
\end{aligned}
$$

$$
\begin{aligned}
c_0 =& \frac{\bar{c}^2\eta^2 v^{(4)}(x)}{4c^2(x)v(x)} - \frac{7\bar{c}^2\eta^2 v''(x)^2}{8c^2(x)v^2(x)} - \frac{63\bar{c}^2\eta^2 v'(x)^4}{32c^2(x)v^4(x)} \\
&- \frac{5\bar{c}^2\eta^2 v^{(3)}(x)v'(x)}{4c^2(x)v^2(x)} + \frac{31\bar{c}^2\eta^2 v'(x)^2 v''(x)}{8c^2(x)v^3(x)} - \frac{\bar{c}^2\eta^2 c''(x)^2}{c^4(x)} \\
&- \frac{i\bar{c}\eta c''(x)v'(x)}{\sqrt{2}c^3(x)} + \frac{\sqrt{2}\bar{c}\eta\omega c''(x)}{c^3(x)} + \frac{v^2(x)c''(x)}{c^3(x)} - \frac{c''(x)}{c(x)} \\
&- \frac{i\bar{c}\eta c'(x)v''(x)}{\sqrt{2}c^3(x)} + \frac{3i\bar{c}\eta c'(x)v'(x)^2}{2\sqrt{2}c^3(x)v(x)} + \frac{3i\bar{c}\eta c'(x)^2 v'(x)}{\sqrt{2}c^4(x)} \\
&- \frac{3i\sqrt{2}\bar{c}\eta v(x)c'(x)^3}{c(x)^5} - \frac{\sqrt{2}\bar{c}\eta\omega c'(x)^2}{c^4(x)} + \frac{v(x)c'(x)v'(x)}{c^3(x)} \\
&- \frac{2v^2(x)c'(x)^2}{c^4(x)} + \frac{\bar{c}^2\eta^2\bar{c}^{(4)}(x)}{2c^3(x)} - \frac{\bar{c}^2\eta^2 v^{(3)}(x)c'(x)}{2c^3(x)v(x)} \\
&+ \frac{\bar{c}^2\eta^2 c'(x)^2 v''(x)}{2c^4(x)v(x)} - \frac{3\bar{c}^2\eta^2 c'(x)v'(x)^3}{2c^3(x)v^3(x)} - \frac{3\bar{c}^2\eta^2 c'(x)^2 v'(x)^2}{4c^4(x)v^2(x)} \\
&+ \frac{2\bar{c}^2\eta^2 c'(x)v'(x)v''(x)}{c^3(x)v^2(x)} + \frac{3\bar{c}^2\eta^2 c'(x)^2 c''(x)}{c(x)^5} + \frac{2i\sqrt{2}\bar{c}\eta v(x)c'(x)c''(x)}{c^4(x)} \\
&- \frac{2\bar{c}^2\eta^2\bar{c}^{(3)}(x)c'(x)}{c^4(x)} + \frac{3i\bar{c}\eta v'(x)^3}{2\sqrt{2}c^2(x)v^2(x)} - \frac{i\sqrt{2}\bar{c}\eta v'(x)v''(x)}{c^2(x)v(x)} \\
&+ \frac{v(x)v''(x)}{2c^2(x)} - \frac{i\omega v'(x)}{c^2(x)} + \frac{v'(x)^2}{4c^2(x)} - \frac{\omega^2}{c^2(x)} - \frac{v''(x)}{2v(x)} + \frac{3v'(x)^2}{4v^2(x)}
\end{aligned}
$$

A.3.3 φ Solutions and coefficients

WKB solutions

$$
\begin{aligned}
\varphi_1^{(WKB)}(x) =& \exp\left(\frac{1}{\eta}\frac{\sqrt{2}i}{\bar{c}}\int^x \mathrm{d}x' \sqrt{v^2(x')-c^2(x')}\right)\frac{1}{\left(v^2(x)-c^2(x)\right)^{3/4}} \\
& \exp\left(-i\omega\int^x \mathrm{d}x' \frac{v(x')}{v^2(x')-c^2(x')}\right)\left(\sqrt{\frac{v^2(x)}{c^2(x)}-1}+\frac{v(x)}{c(x)}\right) \\
\varphi_2^{(WKB)}(x) =& \exp\left(-\frac{1}{\eta}\frac{\sqrt{2}i}{\bar{c}}\int^x \mathrm{d}x' \sqrt{v^2(x')-c^2(x')}\right)\frac{1}{\left(v^2(x)-c^2(x)\right)^{3/4}} \\
& \exp\left(-i\omega\int^x \mathrm{d}x' \frac{v(x')}{v^2(x')-c^2(x')}\right)\left(\sqrt{\frac{v^2(x)}{c^2(x)}-1}+\frac{v(x)}{c(x)}\right)^{-1} \\
\varphi_3^{(WKB)}(x) =& \exp\left(-i\omega\int^x \mathrm{d}x \frac{1}{c-v}x\right) \\
\varphi_4^{(WKB)}(x) =& \exp\left(i\omega\int^x \mathrm{d}x \frac{1}{c+v}x\right).
\end{aligned}
\tag{A.3.1}
$$

NHE solutions

$$
\begin{aligned}
\varphi_{out}^{(NHE)}(z) &= \frac{\sqrt{\pi}}{2\pi i}|z|^{-3/4}\Bigg[\exp\left(i\theta(u_{max,out})\right)\exp\left(|z|^{3/2}(\pm\frac{2}{3})\right) \\
&\qquad \exp\left(-\frac{i\omega}{2\kappa}\log|z|\right)(\pm 1)^{(-1-i\omega/\kappa)}\Bigg] \\
\varphi_{in}^{(NHE)}(z) &= \frac{\sqrt{\pi}}{2\pi i}|z|^{-3/4}\exp\left(\frac{\pi\omega}{2\kappa}\right)\Bigg[\exp\left(i\theta(u_{max,in})\right)\exp\left(|z|^{3/2}(\mp\frac{2}{3}i)\right) \\
&\qquad \exp\left(-\frac{i\omega}{2\kappa}\log|z|\right)\exp\left(\mp\frac{\pi}{2}i\right)\Bigg] \\
\varphi_3^{NHE}(z) &= 1, \\
\varphi_4^{(NHE)}(z) &= -\frac{1}{\pi i}\,\Gamma\left(-\frac{i\omega}{\kappa}\right)\;\exp\left(\frac{i\omega}{\kappa}\log|z|\right)\;\sinh\Big(\frac{\pi\omega}{\kappa}\Big).
\end{aligned}
\tag{A.3.2}
$$

Matching coefficients

$$
\begin{aligned}
c_1 &= +\,2^{-\frac{3}{4}}\sqrt{\frac{c_0\kappa\bar{c}}{\pi}}\,\eta^{\frac{1}{2}}\left(\left(\frac{4c_0\kappa}{\bar{c}^2}\right)^{\frac{1}{3}}\eta^{-\frac{2}{3}}\right)^{-\frac{i\omega}{2\kappa}}\exp\left(+\frac{\pi}{2}\frac{\omega}{\kappa}+\frac{3}{4}\pi i\right) \\
c_2 &= -\,2^{-\frac{3}{4}}\sqrt{\frac{c_0\kappa\bar{c}}{\pi}}\,\eta^{\frac{1}{2}}\left(\left(\frac{4c_0\kappa}{\bar{c}^2}\right)^{\frac{1}{3}}\eta^{-\frac{2}{3}}\right)^{-\frac{i\omega}{2\kappa}}\exp\left(-\frac{\pi}{2}\frac{\omega}{\kappa}+\frac{1}{4}\pi i\right) \\
c_3 &= 1 \\
c_4 &= -\frac{1}{\pi i}\left(\left(\frac{4c_0\kappa}{\bar{c}^2}\right)^{1/3}\eta^{-2/3}\right)^{i\omega/\kappa}\Gamma\left(-\frac{i\omega}{\kappa}\right)\sinh\Big(\frac{\pi\omega}{\kappa}\Big)
\end{aligned}
$$

Appendix B

WKB approximation

The WKB (Wentzel, Kramers, and Brillouin) method is a useful tool to solve second order linear differential equation, capable to capture fast oscillating phenomena. It is based on the assumption that the fast time scale dependence is exponential and its a reasonable assumption since a lot of the physical problem have this dependence at the end [Hol12].

B.1 Introductory example

To present the ideas underlying the WKB method, we will use an introductory example. Let us consider the equation

$$\epsilon^2 y'' - q(x)y = 0. \tag{B.1.1}$$

For the moment we will only assume that $q(x)$ is sufficiently regular. If we assume $q(x)$ to be constant, the general solution of the equation (B.1.1) is

$$y(x) = a_0\, e^{\frac{-x\sqrt{q}}{\epsilon}} + b_0\, e^{\frac{x\sqrt{q}}{\epsilon}}. \tag{B.1.2}$$

The main hypotesis of WKB method is that a solution to (B.1.1), with non constant $q(x)$, can be found starting from(B.1.2). The specific assumption made using the WKB method is that the asymptotic expansion of a solution is

$$y(x) \sim e^{\frac{\theta(x)}{\epsilon^\alpha}} \left(y_0(x) + \epsilon^\alpha y_1(x) + \dots\right). \tag{B.1.3}$$

From (B.1.3)

$$y' \sim \left(\epsilon^{-\alpha}\theta' y_0 + y_0' + \theta' y_1 + \dots\right) e^{\frac{\theta(x)}{\epsilon^\alpha}}$$

and

$$y'' \sim \left[\epsilon^{-2\alpha}\theta_x^2 y_0 + \epsilon^{-\alpha}(\theta_{xx} y_0 + 2\theta_x y_0' + \theta_x^2 y_1) + \dots\right] e^{\frac{\theta(x)}{\epsilon^\alpha}}. \tag{B.1.4}$$

For the problem under consideration, it is sufficient to stop the expansion at second order. Nonetheless, for higer-order problems, higher-order expansions

are needed.
The next step is to substitute (B.1.3) and (B.1.4) into (B.1.1). Doing this, one finds that

$$\epsilon^2 \left\{ \frac{1}{\epsilon^{2\alpha}}(\theta_x)^2 y_0 + \frac{1}{\epsilon^{\alpha}}[\theta'' y_0 + 2\theta' y_0' + (\theta_x)^2 y_1 + \dots] \right\} \\ -q(x)(y_0 + \epsilon^{\alpha} y_1 + \dots) = 0. \tag{B.1.5}$$

Doing so, the exponential simplifies, thanks to the linearity of the equation. Now, balancing the terms in (B.1.5), one finds that $\alpha = 1$, leading to the following equation:

$$O(1) \qquad (\theta_x)^2 = q(x). \tag{B.1.6}$$

This is called *eikonal equation* and its solutions are

$$\theta(x) = \pm \int^x \mathrm{d}s \sqrt{q(s)}. \tag{B.1.7}$$

To obtain the first term of the expansion, we need also to find $y_0(x)$. It is then necessary to look also at the $O(\epsilon)$ problem, also known as *transport equation*.

$$O(\epsilon) \qquad \theta'' y_0 + 2\theta' y_0' + (\theta_x)^2 y_1 = q(x) y_1.$$

Appendix C

Cut-integral

C.1 Integral

$$
\begin{aligned}
& -\frac{1}{2\pi i}\;|z|^{-\frac{i\omega}{2\kappa}}\int_{\mathbb{R}}\mathrm{d}t\;t^{-\frac{i\omega}{\kappa}-1}\;\exp\!\left(|z|^{3/2}\left(t-\frac{t^3}{3}\right)\right)\\
&= -\frac{1}{2\pi i}\;|z|^{-\frac{i\omega}{2\kappa}}\frac{1}{(-\frac{i\omega}{\kappa})}\left[t^{-i\omega/\kappa}\;\exp\!\left(|z|^{3/2}\left(t-\frac{t^3}{3}\right)\right)\right]_{-\infty}^{+\infty}\\
&\quad+\frac{1}{2\pi i}\;|z|^{-\frac{i\omega}{2\kappa}}\int_{\mathbb{R}}\mathrm{d}t\left(\frac{1}{-\frac{i\omega}{\kappa}}\right)t^{-\frac{i\omega}{\kappa}}|z|^{3/2}\left(1-t^2\right)\;\exp\!\left(|z|^{3/2}\left(t-\frac{t^3}{3}\right)\right)=
\end{aligned}
$$

where, by construction, since the extrema of the circuit around the cut, starts and ends at $t=+\infty$, the boundary term vanishes thanks to $e^{-t^3/3}$,

$$
\begin{aligned}
&= +\frac{1}{2\pi i}\frac{\Gamma(-i\omega/\kappa)}{\Gamma(1-i\omega/\kappa)}|z|^{-\frac{i\omega}{2\kappa}+\frac{3}{2}}\int_{\mathbb{R}}\mathrm{d}t\left(1-t^2\right)t^{-\frac{i\omega}{\kappa}}\;\left(1-t^2\right)\;\exp\!\left(|z|^{3/2}\left(t-\frac{t^3}{3}\right)\right)\\
&= +\frac{1}{2\pi i}\frac{\Gamma(-i\omega/\kappa)}{\Gamma(1-i\omega/\kappa)}|z|^{-\frac{i\omega}{2\kappa}+\frac{3}{2}}\left(1-e^{-2\pi i(-i\omega/\kappa)}\right)\frac{\Gamma(1-i\omega/\kappa)}{(|z|^{3/2})^{1-i\omega/\kappa}}(-1)^{-i\omega/\kappa}(-1)\\
&= -\frac{1}{2\pi i}\Gamma\left(-\frac{i\omega}{\kappa}\right)|z|^{i\omega/\kappa}\left(1-e^{-2\pi\omega/\kappa)}\right)e^{\pi\omega/\kappa}\\
&= -\frac{1}{2\pi i}\,\Gamma\left(-\frac{i\omega}{\kappa}\right)|z|^{i\omega/\kappa}\;2\sinh\!\left(\frac{\pi\omega}{\kappa}\right).
\end{aligned}
$$

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