POLITECNICO DI MILANO

SCUOLA DI INGEGNERIA INDUSTRIALE E DELL'INFORMAZIONE

Corso di Laurea Magistrale in Ingegneria Matematica

POLITECNICO
MILANO 1863

On the evolution of Aharonov-Berry superoscillations

Candidata:
Nadia Alloppi
Matricola 892146

Relatore:
Prof.ssa I. Sabadini

Anno Accademico 2018 - 2019

Abstract

The studies conducted by Aharonov, Berry and their collaborators in the last 50 years led to the discovery of a new phenomenon in quantum physics field called *superoscillations*. The surprising fact is that it seems to violate the well known principle of harmonic analysis, indeed superoscillations can be thought as functions that can oscillate faster than their fastest Fourier component.

One of the main purposes of this dissertation is to give a unified overview of the various definitions of superoscillating functions, which have been evolving, in the past few years in [6], [9], [8] and [19], in order to adapt to the increasing level of generality required by the problem at hand.

We complement the discussion with plots to illustrate the phenomenon of superoscillations and to display the superoscillation region and the region of fast superoscillation.

Furthermore, I discuss the persistence of the superoscillatory behavior when superoscillating sequences are taken as initial values of Schrödinger type equations. This leads to the study of a general strategy to approach those problems, which is mainly focused on proving the continuity of operators acting on entire functions.

Finally, I analyse in details the case of the quantum harmonic oscillator and, in order to take into account singularities of the evolved datum, I consider a more general notion of superoscillation, termed *supershift*. Thanks to this new definition it is possible to enlarge the superoscillatory notion also to distributions and hyperfunctions.

Sommario

Gli studi condotti da Aharonov, Berry e i loro collaboratori negli ultimi 50 anni hanno portato alla scoperta di un nuovo fenomeno nel campo della fisica quantistica chiamato*superoscillazioni*. Il fatto sorprendente è che sembra violare il noto principio dell'analisi armonica, infatti le superoscillazioni possono essere pensate come funzioni che possono oscillare più velocemente del loro componente di Fourier più veloce.

Uno degli scopi principali di questa tesi è fornire una panoramica unificata delle varie definizioni di funzioni superoscillanti, che si sono evolute, negli ultimi anni in [6], [9], [8] e [19], al fine di adattarsi al crescente livello di generalità richiesto dal problema in questione.

Completiamo la discussione con grafici volti ad illustrare il fenomeno delle superoscillazioni e a visualizzare la regione di superoscillazione e la regione di superoscillazione rapida.

Inoltre, ho discusso la persistenza del comportamento superoscillante quando sequenze superoscillanti sono prese come valori iniziali di equazioni della tipologia di Schrödinger. Questo porta allo studio di una strategia generale per affrontare questi problemi, che si concentra principalmente sulla dimostrazione della continuità di operatori che agiscono su funzioni intere.

Infine, analizzo in dettaglio il caso dell'oscillatore armonico quantico e, per tenere conto delle singolarità del dato evoluto, considero una nozione più generale di superoscillazione, indicata con *supershift*. Grazie a questa nuova definizione è possibile estendere la nozione di superoscillazione anche a distribuzioni e iperfunzioni.

Ringraziamenti

I miei ringraziamenti vanno innanzitutto alla professoressa Sabadini: l'attenzione e la disponibilità che mi ha dimostrato in questi mesi, anche da oltre oceano, sono del tutto rare e preziose.
In secondo luogo non posso che ringraziare i miei genitori, per avermi supportato con ogni mezzo possibile e per il loro sguardo discreto e orgoglioso che mi ha sempre spinto a fare il meglio.
Grazie alla mia sorellina Deborah, per la complicità e l'aiuto (non solo morale!) che mi ha dedicato sempre e in particolar modo durante questi mesi di stesura della tesi.
Grazie ai miei zii e ai miei cugini perché sia da lontano che da più vicino hanno sempre avuto un pensiero per me.
Grazie a Sabrina, Chiara F e Chiara M per essere rimaste al mio fianco nonostante i miei tagli di capelli improbabili e la mia indecisione cronica e, soprattuto, grazie per aver condiviso con me circa qualsiasi cosa degna di nota mi sia successa nella vita.
Grazie a chi come Silvia, Valentina e il Vincio, non ha permesso che tempo, spazio e circostanze varie potessero allontanarci.
Grazie anche agli amici che si sono aggiunti durante questi meravigliosi cinque anni: Eva, Robi, Aline con cui ho condiviso chiacchiere, appunti e biro colorate; le mie (in)decision makers Chiara Mz e Lucia, Vanessa, Chiara L; e infine i miei compagni di viaggio (che sia su un regionale o su un aereo) Ale, Marci, Martina, G. Botti e J. Inzani, Mattia e Ludovica per non avermi mai permesso di annoiarmi e per aver badato a Giacomo quando non c'ero.
Grazie ai miei colleghi di Deloitte per aver rispettato i miei bisogni di studentessa e per avermi insegnato tanto.
Grazie infine a Giacomo, per avermi sempre ascoltata anche quando non avevo niente di nuovo da dire, per la pazienza e la criticità con cui ha sempre saputo consigliarmi e per la sua capacità di vedere sempre qualcosa di bello in me.

Contents

Chapter 1

Introduction

The interest for superoscillation phenomenon arose in the quantum physics field, thanks to the studies of Aharonov and his collaborators on weak values, a notion that provides a different way to regard measurements in quantum physics.

One of the purposes of this introduction is to offer a quick and basic overview of the background from quantum physics which generated the notion of superoscillations.

In quantum mechanics any system is described by its quantum state, that is a (usually infinite-dimensional) vector in a Hilbert space. It is a postulate of quantum mechanics that all measurements have an associated operator (called an observable) such that it is a Hermitian (self-adjoint) operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$, where $D(A)$ is the domain of A and $\mathcal{H}$ is a Hilbert space. Furthermore, the observable's eigenvectors, also called eigenstates, form an orthonormal basis spanning $D(A)$, that means that any quantum state can be represented as a superposition of the eigenstates of an observable.

The measurement is usually assumed to be ideally accurate, so the state of a system after measurement is assumed to collapse into an eigenstate of the associated operator, mathematically speaking we call *measurement of a quantum state* its projection on the eigenspace generated by the eigenvectors of A. If the system was prepared in a specific eigenstate, then the measurement will coincide with probability one with the associated eigenvalue, while if the system is in a generic quantum state then one needs to repeat the measurements more times and the final result will be a set of different eigenvalues, each one with its probability.

Now, let us consider a large set of particles, then we can impose an initial condition and obtain the so-called pre-selected ensembles, after that we evolve the system according to Schrödinger equation and observe the obtained final state.

What I explained so far is a time asymmetric view of quantum mechanics. Let us notice that the process of preparation is actually a kind of filtering of results: only one quantum state of many possible ones is chosen to begin with. Then, by introducing the concept of post-selection, i.e. filtering also the final results, the theory can be made time-symmetric.

In 1964 in a paper by Aharonov, Bergman, and Lebowitz [3], the authors showed that the initial conditions of a quantum mechanical system can be selected independently of the final conditions. Subsequently, despite of what it is traditionally believed, Aharonov, Albert and Vaidman [1] showed that information can be obtained even if the system was not disturbed, i.e. even if the measurement interaction is weakened. The outcomes of these weak measurements, denoted in [1] as *weak values*, depend on both the pre- and the post-selection and can have values outside the allowed eigenvalue spectrum. In this way Aharonov and his collaborators showed that the weak values lead to a new phenomenon called superoscillations [2].

Let us consider a pre-selected initial state $\mid \psi_{in}\rangle$ and post-selected final state $\mid \psi_{fin}\rangle$, with $\langle\psi_{fin} \mid \psi_{in}\rangle \neq 0$. Assume that, between the two pre- and post-selected states, we measure a non-degenerate Hermitian operator A. We assume also that, during the measurement, the time evolution operator for the measured system and measuring device is $e^{-(i/\hbar)AP_d}$, where P_d is the observable associated with the measuring device. Moreover, let us denote by $\phi_{in}(Q_d)$ the initial state of the measuring device, then, after the post-selection, the final state of the measuring device when we consider a small P_d is (proceeding formally)

$$\begin{aligned}\langle\psi_{fin} \mid e^{-(i/\hbar)AP_d} \mid \psi_{in}\rangle\phi_{in}(Q_d) &\approx \langle\psi_{fin} \mid 1-(i/\hbar)AP_d \mid \psi_{in}\rangle\phi_{in}(Q_d)\\ &= \langle\psi_{fin} \mid \psi_{in}\rangle - (i/\hbar)\langle\psi_{fin} \mid A \mid \psi_{in}\rangle P_d]\phi_{in}(Q_d)\\ &= \langle\psi_{fin} \mid \psi_{in}\rangle[1-(i/\hbar)\langle A\rangle_w P_d]\phi_{in}(Q_d)\\ &\approx \langle\psi_{fin} \mid \psi_{in}\rangle e^{-(i/\hbar)\langle A\rangle_w P_d}\phi_{in}(Q_d),\end{aligned}$$

where

$$\langle A\rangle_w = \frac{\langle\psi_{\rm fin}|A|\psi_{\rm in}\rangle}{\langle\psi_{\rm fin}|\psi_{\rm in}\rangle}, \tag{1.0.1}$$

is the formal definition of what Aharonov, Albert, and Vaidman called weak value of A. Let us notice that the result of weakly measuring the operator A is not its expectation value, but rather

$\langle A\rangle_w$, that can exceed the eigenvalue range (and even assume complex values).

Such behavior has important applications in several areas, including metrology, signal processing, antenna theory and theory of super resolution in optics as it is discussed in [17]. Moreover, in papers like [16], [14] and [13] Berry and his coauthors deeply investigate the phenomenon from a more physical perspective.

From a mathematical point of view, we think to superoscillatory functions as a superposition of small Fourier components with a bounded Fourier spectrum, that result in a shift by an arbitrarily large a. In order to have a concrete example of a superoscillatory function let us consider, the sequence of functions:

$$F_n(x,a) = \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n \tag{1.0.2}$$

where $a \in \mathbb{R}$, $a > 1$. If we perform a binomial expansion, this sequence can be written as

$$\sum_{j=0}^{n} C_j(n,a)e^{i(1-2j/n)x}$$

for suitable coefficients and thus we see that the wavelength in the expansion is always smaller or equal to 1. However, it can be proven that $F_n(x,a)$ can be approximated as e^{iax} in a interval of $\mathbb{R}$ whose width grows as n grows. The puzzling fact is that the wavelength of e^{iax} can be any $a \in \mathbb{R}$, in particular it can be much larger than one.

This phenomenon is very general and holds for a wide range of functions and coefficients. As it is shown throughout this work, we can obtain a very large class of superoscillating function just considering the evolution of $F_n(x,a)$ through Schrödinger type equations.

As avenues for further research one may consider evolution using different type of equations, for example Klein-Gordon, and the interest is always to establish the persistence and robustness of the superoscillating behaviour in time.

As we have already pointed out the possible impact of this research for applications has tremendous potential, as it has discussed in [17], and various laboratories are performing experiments in order to validate the theoretical predictions of the theory which is in continuous development. Of particular interest are the applications to the construction of super resolution microscopes for the impact in biology and medicine.

The thesis in organized as follows: in Chapter 2 I define what a superoscillatory function is, I present the prototypical superoscillation function and I finally introduce the framework where superoscillations are studied from a mathematical point of view; in Chapter 3 I analyse the evolution of a superoscillatory initial datum through Schrödinger type equations, wondering if it is possible to obtain newly a superoscillatory function as result; finally in Chapter 4 I present the case of the Schrödinger equation for the harmonic oscillator, this problem is of particular interest due to the singularities appearing in the solution.

Chapter 2

Some basic results on superoscillating sequences

This chapter is devoted to the study of the basic mathematical properties of superoscillating sequences.

I start presenting, in Section 2.1, the development of the definition of superoscillating sequences, then in Section 2.2 I will focus on the prototypical superoscillating sequence $F_n(x,a)$ presenting the main preliminary results and discussing the size of the superoscillating region. Finally, in Section 2.3, I will present the mathematical environment where superoscillating sequences are studied.

2.1 Superoscillating sequences

Let us define the objects that will be the main characters through all the thesis.

Definition 2.1.1. *(Generalized Fourier sequence)*
We call generalized Fourier sequence a sequence $Y_n(x) : \mathbb{R} \longmapsto \mathbb{C}$, such that, for all $n \in \mathbb{N}$, is of the form

$$Y_n(x) := \sum_{j=0}^{n} C_j(n)\, e^{ik_j(n)x} \tag{2.1.1}$$

where $C_j(n)$ and $k_j(n)$ are real valued functions.

Remark 2.1.1. The sequence of partial sums of a Fourier expansion is a particular case of this notion where $C_j(n) = C_j \in \mathbb{R}$ and $k_j(n) = k_j \in \mathbb{R}$ are multiples of a real number.

We can also consider generalized Fourier sequences dependent on a parameter $a \in \mathbb{R}^+$, formally we have

$$Y_n(x,a) := \sum_{j=0}^{n} C_j(n,a)\, e^{ik_j(n)x}. \tag{2.1.2}$$

Definition 2.1.2. *(Superoscillating sequence 1)*
Let $a, \alpha \in \mathbb{R}^+$. A generalized Fourier sequence

$$Y_n(x,a) = \sum_{j=0}^{n} C_j(n,a)\, e^{ik_j(n)x}$$

is said to be a superoscillating sequence if:

i) $|k_j(n)| < \alpha$ for all n and $j \in \mathbb{N} \cup \{0\}$;

ii) there exists a compact subset of $\mathbb{R}$, which will be called a ***superoscillation set***, on which Y_n converges uniformly to $e^{ig(a)x}$ where g is a continuous real-valued function such that $|g(a)| > \alpha$.

Remark 2.1.2. The usual Fourier sequence of a function is obviously not superoscillating because it violates i).

Remark 2.1.3. *(Superoscillating sequence 2)*
More in general, one could substitute clause $ii)$ in Definition 2.1.2 with

$ii)'$ there exists an open subset $U \subseteq \mathbb{R}$, which will be called a *superoscillation domain*, on which Y_n converges uniformly on any compact subset of U to the restriction to U of a trigonometric polynomial function $Y = P(e^{ig(a)x})$, where $P \in \mathbb{C}[X, X^{-1}]$ is a Laurent polynomial (i.e a polynomial that may have terms of negative degree) with no constant term and $|g(a)| > \alpha$.

Remark 2.1.4. It is clear that Remark 2.1.3 gives a notion slightly more general than Definition 2.1.2. Indeed if $Y_n(x,a)$ is a superoscillating sequence in the sense of Remark 2.1.3 with superoscillation domain U, then it is also a superoscillating sequence in the sense of Definition 2.1.2 with superoscillation set any segment $[a,b] \in U$.

Moreover the second definition allows us to consider polynomial of trigonometric functions instead of simply $e^{ig(a)x}$.

It is possible to further extend the definition of superoscillating sequence to the case of several variables. First of all we should define a generalized Fourier sequence in several variables and then we can define what it means that it superoscillates.

Definition 2.1.3. *(Generalized Fourier sequence in several variables)*
We call generalized Fourier sequence a sequence $Y_n(x) : \mathbb{R}^m \longmapsto \mathbb{C}$, such that, for all $n \in \mathbb{N}$, is of the form

$$Y_n(x_1, \dots, x_m) := \sum_{j=0}^{n} C_j(n) P(e^{ik_{j,1}(n)x_1}, \dots, e^{ik_{j,m}(n)x_m}) \tag{2.1.3}$$

where $P \in \mathbb{C}[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$ is a Laurent polynomial, $C_j(n)$ is a real valued function and $k_j(n)$ is a map from $\mathbb{N}^*$ to $\mathbb{R}^m$.

Definition 2.1.4. *(Superoscillating sequence in several variables)*
Let $a, \alpha \in \mathbb{R}^+$. A generalized Fourier sequence in several variables is said to be a superoscillating sequence if:

i) $|k_{j,l}(n)| < \alpha$ for all n, $j \in \mathbb{N} \cup \{0\}$ and $l = 1, \dots, m$;

ii) there exists an open subset $U \subseteq \mathbb{R}^m$, which will be called a *superoscillation domain*, on which Y_n converges uniformly on any compact subset of U to the restriction to U of a trigonometric polynomial function $Y = P_\infty(e^{ig_1(a)x_1}, \dots, e^{ig_m(a)x_m})$, where $P_\infty \in \mathbb{C}[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$ is a Laurent polynomial with no constant term and $|g_l(a)| > \alpha$ for all $l = 1, \dots, m$.

Remark 2.1.5. Definition 2.1.1 and Definition 2.1.3 can be enhanced considering $C_j(n) \in \mathbb{C}$ instead of $C_j(n) \in \mathbb{R}$.

Once again, the definition of superoscillating sequence can be extended, in this case, to the definition of super-shift. The main reason for which this definition is needed is the possibility to generalize it from functions to distributions and hyperfunctions, but this purpose will be discussed in details in the next chapters of the thesis (see in particular Section 4.3). Moreover, the definition of supershift takes into account general functions, instead of only functions with exponential form.

Definition 2.1.5. *(Supershift for a family of functions)*

Let $\mathcal{T}$ be a locally compact topological space and $\mathcal{F} = \{\varphi_\lambda : \mathcal{T} \to \mathbb{C}\,;\, \lambda \in \mathbb{R}\}$ be a family of $\mathbb{C}$-valued functions on $\mathcal{T}$ indexed by $\mathbb{R}$. A sequence $\boldsymbol{\psi} = \{\psi_n(\tau)\}_{n\geq 1}$ of $\mathbb{C}$-valued functions on $\mathcal{T}$ is called supershift for the family $\mathcal{F}$ (or $\mathcal{F}$ admits $\boldsymbol{\psi}$ as supershift) if:

i) any entry ψ_n is of the form $\psi_n = \sum_{j=0}^{n} C_j(n)\,\varphi_{k_j(n)}$ with $k_j(n) \leq 1 \;\; \forall n \in \mathbb{N}^*$ and $0 \leq j \leq n$;

ii) there exists an open subset U^{ssh} of $\mathcal{T}$, called a $\mathcal{F}$-supershift domain, such that the sequence $\{\psi_n(\tau)\}_{n\geq 1}$, when $\tau \in U^{ssh}$, converges locally uniformly towards the restriction to U^{ssh} of a function ψ_∞ which is a $\mathbb{C}$-finite linear combination of elements in $\mathcal{F}$ of the form $\varphi_{\nu k(\infty)}$ with $\nu \in \mathbb{Z}^*$, where $k(\infty) \in \mathbb{R} \setminus [-1, 1]$.

Remark 2.1.6. Definition 2.1.5 would be equivalent if in requirement *i)* one asks for $k_j(N) \leq \alpha \;\; \forall N \in \mathbb{N}^*$, $\alpha \in \mathbb{R}^+$, and, consequently, in requirement *ii)* $k(\infty) \in \mathbb{R} \setminus [-\alpha, \alpha]$.

Remark 2.1.7. Remark 2.1.3 and Definition 2.1.5 seems to be more different than they actually are. Let us notice that a Laurent trigonometric polynomial $Y = P(e^{ig(a)x})$ is a $\mathbb{C}$-finite linear combination of elements of the form $e^{i\nu g(a)x}$ with $\nu \in \mathbb{Z}^*$. So if in Definition 2.1.5 we have $\mathcal{F} = \{e^{i\lambda x}\,;\, \lambda \in \mathbb{R}\}$, asking that ψ_∞ is a $\mathbb{C}$-finite linear combination of elements in $\mathcal{F}$ of the form $\varphi_{\nu k(\infty)}$, is the same to ask that ψ_∞ is a Laurent trigonometric polynomial $Y = P(e^{i\nu k(\infty)x})$.

As it is previously stated, this definition could be enlarged in order to include distributions and hyperfunctions.

Definition 2.1.6. *(Supershift for a family of distributions (resp. hyperfunctions))*

Let $\mathcal{T}$ be of the form $[0, T) \times U$, where U is an open subset in $\mathbb{R}^{m-1}_x$ $(m \geq 2)$ and $T \in (0, \infty]$. Let $\mathcal{F} = \{\varphi_\lambda : \mathcal{T} \to \mathbb{C}\,;\, \lambda \in \mathbb{R}\}$ be a family of $\mathbb{C}$-valued distributions (resp. hyperfunctions) in $\mathbb{R} \times U$ with support in $\mathcal{T}$. A sequence $\boldsymbol{\psi} = \{\psi_n(\tau)\}_{n\geq 1}$ of $\mathbb{C}$-valued functions on $\mathcal{T}$ is called supershift for the family $\mathcal{F}$ (or $\mathcal{F}$ admits $\boldsymbol{\psi}$ as supershift) if:

i) any entry ψ_n is of the form $\psi_n = \sum_{j=0}^{n} C_j(n)\,\varphi_{k_j(n)}$ with $k_j(n) \leq 1 \;\; \forall n \in \mathbb{N}^*$ and $0 \leq j \leq n$;

ii) there exists an open subset $U^{ssh} = V \cap \mathcal{T}$ (where V is an open subset in $\mathbb{R} \times U$), called a $\mathcal{F}$-supershift domain, such that the sequence $\{\psi_n(\tau)_{|V}\}_{n\geq 1}$ converges weakly in the sense of distributions (resp. hyperfunctions) in V towards the restriction to V of a distribution (resp. hyperfunction) ψ_∞ which is a $\mathbb{C}$-finite linear combination of elements in $\mathcal{F}$ of the form $\varphi_{\nu k(\infty)}$ with $\nu \in \mathbb{Z}^*$, where $k(\infty) \in \mathbb{R} \setminus [-1,1]$.

2.2 The archetypical superoscillating sequence $F_n(x,a)$

2.2.1 Properties of $F_n(x,a)$

This section is devoted to the presentation of the properties of the archetypical superoscillating sequence, already mentioned in the introduction, whose expression is:

$$F_n(x,a) = \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n, \tag{2.2.1}$$

where $a > 1$, $n \in \mathbb{N}$, and $x \in \mathbb{R}$.

Proposition 2.2.1. *Consider the sequence (2.2.1). Then we have*

(1) For every $x_0 \in \mathbb{R}$

$$\lim_{n\to\infty} F_n(x_0,a) = e^{iax_0}.$$

(2) The functions $F_n(x,a)$ can be written in terms of their Fourier coefficients $C_j(n,a)$ as

$$F_n(x,a) = \sum_{j=0}^{n} C_j(n,a)e^{i(1-2j/n)x}, \tag{2.2.2}$$

where

$$C_j(n,a) := \frac{(-1)^j}{2^n}\binom{n}{j}(a+1)^{n-j}(a-1)^j.$$

(3) For every $p \in \mathbb{N}$ the following relation

$$F_n^{(p)}(0,a) = \sum_{j=0}^{n} C_j(n,a)\left[i\left(1-\frac{2j}{n}\right)\right]^p$$

between the Taylor and the Fourier coefficients of $F_n(x,a)$ holds.

Proof. Point (1) follows from:

$$\lim_{n\to\infty} F_n(x_0,a) = \lim_{n\to\infty}\left(\cos\left(\frac{x_0}{n}\right) + ia\sin\left(\frac{x_0}{n}\right)\right)^n = \lim_{n\to\infty}\left(1+\frac{iax_0}{n}\right)^n = e^{iax_0} \qquad \forall x_0 \in \mathbb{R} \tag{2.2.3}$$

Point (2) is a consequence of the Newton binomial formula, as it is showed below:

$$\begin{aligned}F_n(x,a) &= \Big(\frac{e^{ix/n}+e^{-ix/n}}{2} + a\,\frac{e^{ix/n}-e^{-ix/n}}{2}\Big)^n \\ &= \Big(\frac{1+a}{2}\,e^{ix/n} + \frac{1-a}{2}\,e^{-ix/n}\Big)^n = \\ &= \sum_{j=0}^{n}\binom{n}{j}\left(\frac{1+a}{2}\,e^{ix/n}\right)^{n-j}\left(\frac{1-a}{2}\,e^{-ix/n}\right)^{j} \\ &= \sum_{j=0}^{n}\binom{n}{j}\left(\frac{1+a}{2}\right)^{n-j}\left(\frac{1-a}{2}\right)^{j} e^{i(1-2j/n)x} \\ &= \sum_{j=0}^{n} C_j(n,a)\, e^{i(1-2j/n)x}.\end{aligned} \tag{2.2.4}$$

Let us notice that since $(1-2j/n) \leq 1\ \forall n \in \mathbb{N}$ we have that the sequence (2.2.2) satisfies the first requirement of definition Definition 2.1.2 with $\alpha = 1$.

Point (3) follows by taking the derivatives of

$$\sum_{j=0}^{n} C_j(n,a)\, e^{i(1-2j/n)x}$$

and computing them at the origin. □

Let observe that we have pointwise convergence of $F_n(x,a) \to e^{iax}$ as $n \to \infty$ on all $\mathbb{R}$. Now the focus will be on uniform convergence in order to exploit also the second requirement of Definition 2.1.2 and prove that (2.2.2) is a superoscillating sequence.

Theorem 2.2.2. *Let $M > 0$ be a fixed real number and $a \in \mathbb{R}$. Then the sequence $F_n(x,a)$ converges uniformly to e^{iax} on $[-M,M]$. Thus $F_n(x,a)$ is a superoscillating sequence.*

Proof. We have to show that for every interval $[-M;M]$ we have

$$\sup_{|x|\leq M} |F_n(x,a) - e^{iax}| \to 0 \quad \text{as} \quad n \to \infty.$$

To this purpose, we will compute an estimate for the modulus of the function $F_n(x,a) - e^{iax}$. Let us set

$$w = F_n(x,a) \qquad \text{and} \qquad z = e^{iax},$$

and observe that the modulus and the angles associated with w and z are, respectively,

$$p_w = \left(\cos^2\left(\frac{x}{n}\right) + a^2\sin^2\left(\frac{x}{n}\right)\right)^{n/2}, \qquad \theta_w = n\arctan\left(a\tan\left(\frac{x}{n}\right)\right)$$

and

$$p_z = 1, \qquad \theta_z = ax.$$

The Carnot theorem for triangles gives

$$|w - z|^2 = 1 + p_w^2 - 2p_w \cos(\theta_w - \theta_z)$$

so that we obtain

$$\begin{aligned} |F_n(x,a) - e^{iax}|^2 &= 1 + \Big(\cos^2\Big(\frac{x}{n}\Big) + a^2 \sin^2\Big(\frac{x}{n}\Big)\Big)^n \\ &- 2\Big(\cos^2\Big(\frac{x}{n}\Big) + a^2 \sin^2\Big(\frac{x}{n}\Big)\Big)^{n/2} \cos\Big[n \arctan\Big(a \tan\Big(\frac{x}{n}\Big)\Big) - ax\Big]. \end{aligned} \tag{2.2.5}$$

Let us set

$$\mathcal{E}_n^2(x,a) := |F_n(x,a) - e^{iax}|^2 \tag{2.2.6}$$

and observe that for any x such that $|x| \leq M$ we have

$$\Big(\cos^2\Big(\frac{x}{n}\Big) + a^2 \sin^2\Big(\frac{x}{n}\Big)\Big)^n \to 1 \quad \text{as} \quad n \to \infty$$

and

$$\cos\Big[n \arctan\Big(a \tan\Big(\frac{x}{n}\Big)\Big) - ax\Big] \to 1 \quad \text{as} \quad n \to \infty.$$

Using (2.2.5), we deduce that $\mathcal{E}_n^2(x,a) \to 0$. Note that $\mathcal{E}_n^2(x,a)$, as a function of x, is continuous on the compact set $[-M, M]$ for any $n > \dfrac{2M}{\pi}$ so $\mathcal{E}_n^2(x,a)$ has maximum. Set

$$\varepsilon(n,a) = \max_{x \in [-M,M]} \mathcal{E}_n(x,a).$$

It is now easy to see that $\varepsilon(n,a) \to 0$ as $n \to \infty$ and since

$$\sup_{|x| \leq M} |F_n(x,a) - e^{iax}| = \varepsilon(n,a)$$

the convergence is uniform in $[-M, M]$. □

The next result however shows that on all $\mathbb{R}$ the sequence $F_n(x,a)$ does not converge uniformly.

Proposition 2.2.3. *The sequence $F_n(x,a)$ does not converge uniformly to e^{iax} on $\mathbb{R}$.*

Proof. Uniform convergence on $\mathbb{R}$ would be equivalent to

$$\sup_{x\in\mathbb{R}} |F_n(x,a) - e^{iax}| \to 0 \quad \text{as} \quad n\to\infty.$$

If $x=0$ obviously $F_n(0,a) - e^0 = 0$, however, if we consider the points $x_n = j\pi n$ for $j \in \mathbb{Z}\setminus\{0\}$ we have

$$\begin{aligned}\sup_{x\in\mathbb{R}} |F_n(x,a) - e^{iax}| &\geq |F_n(x_n,a) - e^{iax_n}| = |(\pm 1)^n - e^{iax_n}| \\ &= |(\pm 1)^n - e^{i(aj\pi)n}| \not\to 0 \qquad \text{as} \quad n\to\infty\end{aligned}$$

if $a \in \mathbb{R}\setminus(\mathbb{Z}\setminus\{0\})$, indeed the angle $aj\pi n$ keeps rotating of $aj\pi$ as $n\to\infty$, so the convergence cannot be uniform.

If $a\in\mathbb{Z}\setminus\{0\}$, we reason in the same way by taking $x_n = n\dfrac{\pi}{2}$. □

Thanks to this last results we proved not only that the sequence $F_n(x)$ is a superoscillating sequence, but also that it actually is a supershift as remarked below.

Remark 2.2.1. If $\mathcal{T} = \mathbb{R}$ and $\mathcal{F}$ denotes the family of characters $x\in\mathbb{R}\mapsto\varphi_\lambda = e^{i\lambda x}$ indexed by $\lambda\in\mathbb{R}$, then, one says that $\lambda \longmapsto \{x\mapsto F_n(x,\lambda)\}_{n\geq 1}$ realizes a supershift for $\lambda\mapsto\varphi_\lambda$, or also that $\lambda\mapsto\varphi_\lambda$ admits $\lambda\mapsto\{x\mapsto F_n(x,\lambda)\}_{n\geq 1}$ as a supershift.

Remark 2.2.2. In Theorem 2.2.2 the term

$$|F_n(x,a) - e^{iax}|^2 = \mathcal{E}_n^2(x,a)$$

is given by formula (2.2.5). We can give a first approximation of $\mathcal{E}_n^2(x,a)$ by considering the principal part of the infinitesimum $\mathcal{E}_n^2(x,a)$ for $|x|\leq M$. If we can find two constants j and $\beta\in\mathbb{R}^+$ such that

$$\mathcal{E}_n^2(x,a) = \beta\left(\frac{x}{n}\right)^j + o\left(\frac{x}{n}\right)^j, \quad \text{as} \quad n\to\infty,$$

we can choose

$$\mathcal{E}_n^2(x,a) \approx \beta\left(\frac{x}{n}\right)^j \quad \text{as} \quad n\to\infty,$$

as first approximation of $\mathcal{E}_n^2(x,a)$.

With some computations we have

$$(\mathcal{E}_n^2(0,a))' = 0 \qquad \text{and} \qquad (\mathcal{E}_n^2(0,a))'' = \frac{\beta}{2} = \frac{3}{2}(a^2-1),$$

and so

$$\mathcal{E}_n(x,a) \approx \frac{x}{n}\sqrt{\frac{3}{2}\Big(a^2-1\Big)}.$$

This proves that also $\mathcal{E}_n^2(x,a) \to 0$ uniformly over compact sets.

Let us consider now, the superoscillating sequence $F_n(z,a)$ where the real variable x and the real parameter a have been replaced with the corresponding complex counterparts. What we have claimed so far in Theorem 2.2.2 and Remark 2.2.2 does not hold in this complex setting and so we need the following lemma.

Lemma 2.2.4. *Let $a \in \mathbb{C}$, set $\alpha := \max(1, |a|)$ and, for any $z \in \mathbb{C}$, consider $F_n(z,a)$ defined as in (2.2.2) but with $z, a \in \mathbb{C}$ instead of $x, a \in \mathbb{R}$.*
Then, for any $N \in \mathbb{N}^*$ and any $z \in \mathbb{C}$, the following inequalities hold

$$\begin{aligned} &|F_n(z,a)| \le e^{(|a|+1)|z|} \\ &|F_n(z,a) - e^{iaz}| \le \frac{2}{3}\,\frac{a^2-1}{n}\,|z|^2 e^{(\alpha+1)|z|}. \end{aligned} \tag{2.2.7}$$

Proof. In order to prove the first inequality (2.2.7), let

$$\text{sinc}\ : z \in \mathbb{C} \mapsto \frac{\sin z}{z} = \int_0^1 t\,\cos(tz)dt$$

be the sinus cardinal function and recall that it satisfies $|\text{sinc}(z)| \le e^{|\text{Im}(z)|}\ \forall z \in \mathbb{C}$. One has then the uniform estimates valid $\forall N \in \mathbb{N}^*$, $\forall z \in \mathbb{C}$

$$\begin{aligned} |F_n(z,a)| &= \Big|\cos\Big(\frac{z}{n}\Big) + ia\sin\Big(\frac{z}{n}\Big)\Big|^n = \Big|\cos\Big(\frac{z}{n}\Big) + iaz\,\text{sinc}\Big(\frac{z}{n}\Big)\Big|^n \\ &\le e^{\text{Im}(z)}\Big(1 + \frac{|az|}{n}\Big)^n \le \exp(|a||z| + |\text{Im}(z)|) \le e^{(|a|+1)|z|}. \end{aligned} \tag{2.2.8}$$

Let us now prove the second inequality in (2.2.7), that gives us an estimate on the error $|F_n(z,a) - e^{iaz}|$ when z is complex.

Let us first state the following, thanks to Werner formula and Eulero identity, for any $n \in \mathbb{N}^*$

$$\begin{aligned} \Big|\cos\Big(\frac{z}{n}\Big) - \cos\Big(\frac{az}{n}\Big)\Big| &= 2\,\Big|\sin\Big(\frac{(a-1)z}{2n}\Big)\sin\Big(\frac{(a+1)z}{2n}\Big)\Big| \\ &\le \frac{|a^2-1|}{2n^2}|z|^2\exp\Big(\frac{|a-1|+|a+1|}{2n}|z|\Big) \\ &\le \frac{|a^2-1|}{2n^2}|z|^2\exp\Big(\frac{\alpha+1}{n}|z|\Big), \end{aligned} \tag{2.2.9}$$

and, recalling that $\sum_{l=0}^{k-1} a^{2l+1} = a\frac{1-a^{2k}}{1-a^2}$, we have

$$
\begin{aligned}
\Big|a\sin\Big(\frac{z}{n}\Big) - \sin\Big(\frac{az}{n}\Big)\Big| &= \Big|\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(a-a^{2k+1})\Big(\frac{z}{n}\Big)^{2k+1}\Big| \\
&= \frac{|a^2-1|}{n^2}|z|^2\Big|\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\Big(\sum_{l=0}^{k-1} a^{2l+1}\Big)\Big(\frac{z}{n}\Big)^{2k-1}\Big| \\
&\leq \frac{|a^2-1|}{2n^2}|z|^2 \sum_{k=0}^{\infty} \frac{\alpha^{2k-1}}{(2k-1)!(2k+1)}\Big(\frac{|z|}{n}\Big)^{2k-1} \qquad (2.2.10)\\
&\leq \frac{|a^2-1|}{6n^2}|z|^2 \sum_{k=0}^{\infty} \frac{1}{(2k-1)!}\Big(\frac{\alpha|z|}{n}\Big)^{2k-1} \\
&\leq \frac{|a^2-1|}{6n^2}|z|^2 \exp\Big(\frac{\alpha}{n}|z|\Big).
\end{aligned}
$$

The identity $A^n + B^n = (A+B)\sum_{k=0}^{N-1} A^k B^{N-1-k}$ entails

$$
\begin{aligned}
|F_n(z,a) - e^{iaz}| = &\Big|\cos\Big(\frac{z}{n}\Big) - \cos\Big(\frac{az}{n}\Big) + i\Big(a\sin\Big(\frac{z}{n}\Big) - \sin\Big(\frac{az}{n}\Big)\Big)\Big| \\
&\times \sum_{k=0}^{N-1} |F_n(z,a)|^k \Big|\exp\Big(\frac{iaz}{N}\Big)^{N-1-k}\Big|.
\end{aligned}
$$

Then using (2.2.8), (2.2.9) and (2.2.10) we can prove the second inequality in (2.2.7)

$$
\begin{aligned}
|F_n(z,a) - e^{iaz}| &\leq \frac{2}{3}\frac{|a^2-1|}{n^2}|z|^2 \exp\Big(\frac{\alpha+1}{n}|z|\Big)\sum_{k=0}^{N-1} \exp\Big(k(|a|+1)|z| + \frac{N-1-k}{N}|a||z|\Big)\Big) \\
&\leq \frac{2}{3}\frac{|a^2-1|}{n^2}|z|^2 \exp\Big(\alpha+1)|z|\Big).
\end{aligned}
$$

□

The following remark further clarifies the behaviour of the sequence F_n.

Remark 2.2.3. Consider a point x_0 and an increment δx. The superoscillating sequence

$$
F_n(x,a) := \Big(\cos\Big(\frac{x}{n}\Big) + ia\sin\Big(\frac{x}{n}\Big)\Big)^n
$$

is such that $F_n(x_0, a) \to e^{iax_0}$ and $F_n(x_0 + \delta x, a) \to e^{ia(x_0+\delta x)}$. This can also be seen in a different way, which sheds some light on the behaviour of the superoscillatory sequence. Indeed,

$$\begin{aligned}
F_n(x_0+\delta x, a) &= \left(\cos\left(\frac{x_0+\delta x}{n}\right) + ia\sin\left(\frac{x_0+\delta x}{n}\right)\right)^n \\
&= \left(\cos\left(\frac{x_0}{n}\right)\cos\left(\frac{\delta x}{n}\right) - \sin\left(\frac{x_0}{n}\right)\sin\left(\frac{\delta x}{n}\right) + ia\left[\sin\left(\frac{x_0}{n}\right)\cos\left(\frac{\delta x}{n}\right) + \cos\left(\frac{x_0}{n}\right)\sin\left(\frac{\delta x}{n}\right)\right]\right)^n \\
&= \left\{\cos\left(\frac{\delta x}{n}\right)\left[\cos\left(\frac{x_0}{n}\right) + ia\sin\left(\frac{x_0}{n}\right)\right] + ia\sin\left(\frac{\delta x}{n}\right)\left[\frac{i}{a}\sin\left(\frac{x_0}{n}\right) + \cos\left(\frac{x_0}{n}\right)\right]\right\}^n \\
&= \left\{\cos\left(\frac{\delta x}{n}\right) + ia\sin\left(\frac{\delta x}{n}\right)\left(\frac{\cos\left(\frac{x_0}{n}\right) + \frac{i}{a}\sin\left(\frac{x_0}{n}\right)}{\cos\left(\frac{x_0}{n}\right) + ia\sin\left(\frac{x_0}{n}\right)}\right)\right\}^n\left[\cos\left(\frac{x_0}{n}\right) + ia\sin\left(\frac{x_0}{n}\right)\right]^n \\
&= \left\{\cos\left(\frac{\delta x}{n}\right) + i\tilde{a}_n\sin\left(\frac{\delta x}{n}\right)\right\}^n\left[\cos\left(\frac{x_0}{n}\right) + ia\sin\left(\frac{x_0}{n}\right)\right]^n
\end{aligned}$$

where

$$\tilde{a}_n = a\frac{\left[\cos\left(\frac{x_0}{n}\right) + \frac{i}{a}\sin\left(\frac{x_0}{n}\right)\right]\left[\cos\left(\frac{x_0}{n}\right) - ia\sin\left(\frac{x_0}{n}\right)\right]}{\cos^2\left(\frac{x_0}{n}\right) + a^2\sin^2\left(\frac{x_0}{n}\right)}$$

which can also be written as

$$\tilde{a}_n = a\frac{1 - \frac{i}{2}\frac{a^2-1}{a}\sin\left(2\frac{x_0}{n}\right)}{\cos^2\left(\frac{x_0}{n}\right) + a^2\sin^2\left(\frac{x_0}{n}\right)};$$

since $\tilde{a}_n \to a$ as $n \to \infty$, we re-obtain the previous result and we see that the modulus of the limit function now grows as a grows. The amplitude of the superoscillations decreases when a increases. We see that $a = 1$ is a fixed point while if a is large then we obtain large variations.

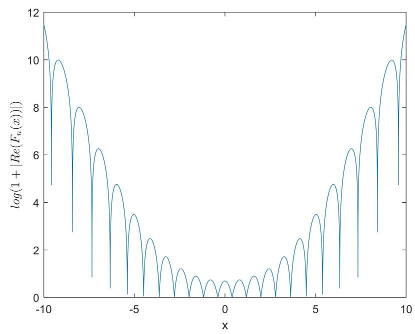

Figure 2.1: $a = 4$ and $n = 50$

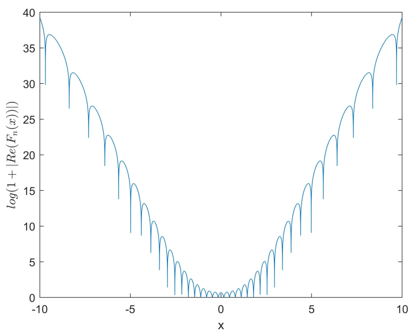

Figure 2.2: $a = 10$ and $n = 50$

In order to better understand the properties of $F_n(x, a)$, I compared in the figure above the cases in which $a = 4$ and $a = 10$, fixing $n = 50$.

In Fig. 2.1 and Fig. 2.2 I plot $F_n(x)$ using a representation that will be employed extensively in what follows. To display the oscillations, and to accommodate the exponentially large variation in the modulus of $F_n(x)$, I plot $\log(|1+\mathrm{Re}(F_n(x))|)$, so that the oscillations are visible as downward spikes at the zeros of $\mathrm{Re}(F_n)$, that coincide with the zeros of $\log(|1+\mathrm{Re}(F_n(x))|)$. Let us notice that pictures are very similar with $\mathrm{Im}(F_n(x))$ rather than $\mathrm{Re}(F_n(x))$.

2.2.2 On the superoscillating region of $F_n(x, a)$

To understand the superoscillating phenomenon of the prototypical sequence (expressed in formula (2.2.2)) in greater detail it will be worth to define and study the size of its superoscillatory region.

Before starting, it is useful to point out that the material in this work is based on a precise definition of superoscillation phenomenon in terms of the uniform convergence of sequences of functions. Here I follow a different approach with respect to the one in [15] where superoscillations are not studied in terms of the uniform convergence of functions. In [15] the authors treat a different case by describing superoscillations with wavenumbers different from a, in the region away from the origin when n is large but finite. Consequently, they consider the sequence

$$G_n(x,a) := (\cos x + ia\sin x)^n$$

that converges only at $x = 0$, and therefore does not fit in the current setting.

Let us start now the analysis on the size of the region where $F_n(x)$ is a good approximation of e^{iax}. In order to do that I write $F_n(x, a)$ in exponential form, involving the local wavenumber.

First of all, let us notice that (I will omit the dependence on a unless necessary)

$$F_n(x) = |g_n(x)|^{n/2} e^{in\arg(g_n(x))}.$$

Where

$$\begin{aligned} &g_n(x) = \cos(x/n) + ia\sin(x/n), \\ &|g_n(x)|^2 = \cos^2(x/n) + a^2\sin^2(x/n), \\ &\arg(g_n(x)) = x/n. \end{aligned}$$

Then let us recall that the local wavenumber is, roughly speaking, the number of waves per unit distance and it is defined by:

$$k(x) = Im\left[\frac{\partial}{\partial x}(log(F_n(x))\right].$$

It is possible to obtain a more explicit formula for $k(x)$, as showed in the following computations:

$$\begin{aligned}\frac{\partial}{\partial x}(log(F_n(x)) &= \frac{1}{\cos(x/n)+ia\sin(x/n)}(ia\cos(x/n)-\sin(x/n))\frac{\cos(x/n)-ia\sin(x/n)}{\cos(x/n)-ia\sin(x/n)}\\ &= \frac{ia+\frac{1}{2}(a^2-1)\sin(2x/n)}{\cos^2(x/n)+a^2\sin^2(x/n)}.\end{aligned}$$

Thus

$$k(x) = \frac{a}{\cos^2(x/n)+a^2\sin^2(x/n)}. \tag{2.2.11}$$

We can now rewrite $F_n(x)$ using (2.2.11) as:

$$\begin{aligned}F_n(x,a) = |g_n(x)|^{n/2}e^{in\arg(g_n(x))} &= \left(\cos^2\left(\frac{x}{n}\right)+a^2\sin^2\left(\frac{x}{n}\right)\right)^{n/2}\exp\left(in\int_0^x k(x')\,dx'\right) =\\ &= \left(\frac{a}{k(x)}\right)^{n/2}\exp\left(in\int_0^x k(x')\,dx'\right).\end{aligned} \tag{2.2.12}$$

We call *superoscillatory region*, the region where the local wavenumber of $F_n(x)$ is smaller than 1.

As it is showed in Fig. 2.3, the wavenumber varies from the superoscillatory $k(0)=a$ to the slowest variation $k(n\pi)=1/a$. So, the superoscillatory region, within which $|k(x)|>1$, delimited in Fig. 2.3 by red lines, explicitly is:

$$|x| = x_s < n\,\mathrm{arccot}(\sqrt{a}).$$

Equation (2.2.12) shows that in the superoscillatory region $F_n(x)$ is exponentially smaller (in n) than in the region where $|k(x)|\leq 1$.

However, the region where $F_n(x)$ is a good approximation of e^{iax}, that is, where the amplitude of the superoscillations is approximately constant it is defined to be the region of *fast superoscillations*. Let us point out that this region is smaller than the region in which $|k(x)|<1$.

To explore it, we expand $F_n(x)$ near $\frac{x}{n}=0$ (i.e. $n\to\infty$) to get an approximation slightly more accurate than $F_n(x)\approx e^{iax}$.

In particular we consider:

$$\begin{aligned}F_n(x) = e^{n\log((g_n(x))} &\approx \exp\left(n(\log(g(0))+nx\frac{d}{dx}\log(g(0))+nx^2\frac{d^2}{dx^2}\log(g(0))\right)\\ &\approx \exp(iax+\frac{1}{2n}(a^2+1)x^2) = e^{\frac{1}{2n}(a^2+1)x^2}e^{iax} \qquad as \quad x/n\to 0.\end{aligned} \tag{2.2.13}$$

So, the region of fast superoscillations, delimited by green vertical lines in Fig. 2.3, is where the coefficient $\frac{1}{n}(a^2-1)x^2$ is smaller than 1, i.e. where:

$$|x| < x_{fs} = \sqrt{\frac{n}{a^2-1}}.$$

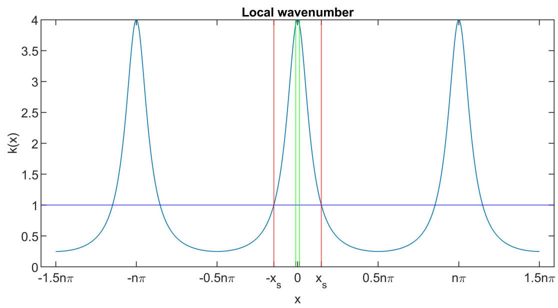

Figure 2.3: Local wavenumber $k(x)$ (2.2.11) when $a = 4$ and $n = 50$

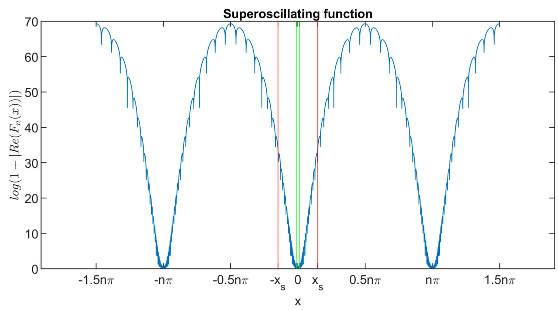

Figure 2.4: Representation of $F_n(x)$ when $a = 4$ and $n = 50$

It is clear both from the mathematical expression and from Fig. 2.3 and Fig. 2.4 that we are handling $n\pi$-periodic functions.

It is immediate to notice that both the superoscillating and fast superoscillating regions become larger as n increases until they invade the entire real axis, indeed we proved in Proposition 2.2.1 the pointwise convergence of $F_n(x) \to e^{iax}$ on all $\mathbb{R}$.

In order to better visualize the approximation phenomenon, I zoomed in $x \in [-5, 5]$. Here, I consider n taking values from $n = 20$ to $n = 100$, and so the fast superoscillation region enlarge from $[-\sqrt{20}, \sqrt{20}] = [-4.5, 4.5]$ to $[-10, 10]$. I plot on the same graph $F_n(x)$ for different values of n and its asymptotic function, namely e^{iax}, and the max norm error $\mathcal{E}_n^2(x, a)$ defined in (2.2.6) (Fig. 2.8).

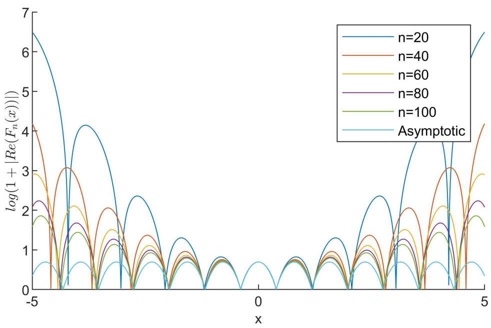

Figure 2.5: Representation of $F_n(x)$ when $a = 4$ and n increases

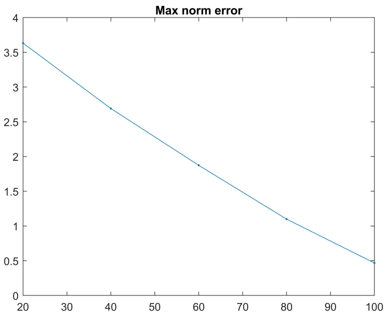

Figure 2.6: Representation of the logarithm of $\max(\mathcal{E}_n^2(x, a))$

Let us notice that if we consider a larger portion of the real axis including points well outside the fast superoscillatory region, the convergence is no more assured. For example, let us consider the largest superoscillatory region, that is $[-100 \operatorname{arccot}(\sqrt{4}), 100 \operatorname{arccot}(\sqrt{4})] = [-46.26, 46.26]$, then we have an exponential growth of the error.

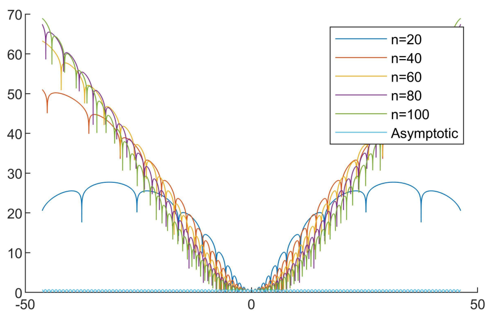

Figure 2.7: Representation of $F_n(x)$ when $a = 4$ and n increases

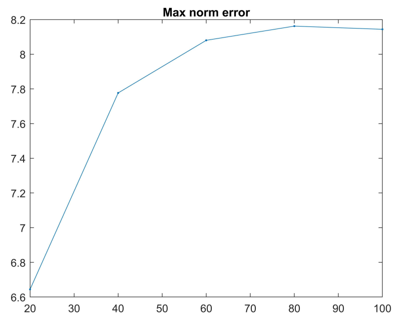

Figure 2.8: Representation of the logarithm of $\max(\mathcal{E}_n^2(x, a))$

Naturally, a key component to the superoscillatory phenomenon is the extremely rapid oscillation in the coefficients C_j and since the regions of superoscillations are created at the expense of having the function grow exponentially in other regions, it would be natural to conclude that the superoscillations would be quickly "over-take" by tails coming from the exponential regions and would thus be short-lived as far as n is finite.

2.3 $\mathcal{A}_p$ spaces

The superoscillating sequences we are studying are finite sums of exponentials, and their complex extensions is formally obtained by replacing the real variable x with the complex variable z. It holds that the extensions are not simply holomorphic in a neighborhood of $\mathbb{R}$, but they are actually entire, so from now on the powerful theory of holomorphic functions can be used.

Before going ahead with that, I need to point out that since the exponentials that appear in the superoscillating sequences are of the form $e^{i\lambda x}$, with $|\lambda| \leq 1$, and x real, they have frequencies bounded by 1, but we need to interpret this in different way when one considers the entire extension of such functions. Indeed, functions of the form $e^{i\lambda z}$ are entire and of exponential type and order 1 (see Definition 2.3.1 and Definition 2.3.2). Thus, we are naturally led to the study of entire functions with growth conditions at infinity.

The scope of this section is to present, without aiming of completeness, the spaces of entire functions with growth conditions, the topology with which they are endowed (Section 2.3.1) and some operators acting on these spaces (Section 2.3.2).

2.3.1 Basics on $\mathcal{A}_p$ spaces

I restrict the analysis to two special kinds of space of entire functions with growth conditions $\mathcal{A}_p(\mathbb{C})$ and $\mathcal{A}_{p,0}(\mathbb{C})$, such spaces are classical, see e.g. [11], and their introduction goes back to Hörmander.

Let f be a non-constant entire function of a complex variable z. We define

$$M_f(r) = \max_{|z|=r} |f(z)|, \qquad \text{for } r \geq 0.$$

Definition 2.3.1. *(Order)*

The non-negative real number p defined by

$$p = \limsup_{r\to\infty} \frac{\ln \ln M_f(r)}{\ln r}$$

is called the order of f.

If p is finite then f is said to be of *finite order* and if $p = \infty$ the function f is said to be of *infinite order*.

Definition 2.3.2. *(Type)*

In the case f is of finite order we define the non negative real number

$$\sigma = \limsup_{r \to \infty} \frac{\ln M_f(r)}{r^p}$$

which is called the type of f.

If $\sigma \in (0, \infty)$ we call f of *normal type*, while we say that f is of *minimal type* if $\sigma = 0$ and of *maximal type* if $\sigma = \infty$.

Constant entire functions are considered of minimal type and order zero.

Definition 2.3.3. *($\mathcal{A}_p(\mathbb{C})$ space)*

Let p be a positive number. The space $\mathcal{A}_p(\mathbb{C})$ is defined by

$$\mathcal{A}_p(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) \ : \ \exists\ A > 0,\ B > 0\ :\ |f(z)| \leq A \exp(B|z|^p)\}$$

and it is called the space of entire functions of order less or equal to p and of finite type.

Remark 2.3.1. An $\mathcal{A}_p$ space of particular interest is $\mathcal{A}_1$, namely the set of entire functions such that there exists $A > 0$ and $B > 0$ for which

$$|f(z)| \leq A\, e^{B|z|} \qquad \forall z \in \mathbb{C}.$$

This space is called the space of entire functions of exponential type and it is denoted also with $\mathrm{Exp}(\mathbb{C})$.

Definition 2.3.4. *($\mathcal{A}_{p,0}(\mathbb{C})$ space)*

Let p be a positive number. The space $\mathcal{A}_{p,0}(\mathbb{C})$ is defined by

$$\mathcal{A}_{p,0}(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) \ : \ \forall \varepsilon > 0,\ \exists A_\varepsilon > 0\ :\ |f(z)| \leq A_\varepsilon \exp(\varepsilon|z|^p)\},$$

and it is called the space of entire functions of order less or equal p and of minimal type.

Remark 2.3.2. Also in this case, a space of particular interest turns out when we consider $p = 1$ and it is called the space of entire functions of infra-exponential type and it is denoted with $\mathcal{A}_{1,0}$.

Let us point out some important relations between the spaces I presented so far.

Proposition 2.3.5. *The following property holds*

$$\mathcal{A}_{p,0} \subseteq \mathcal{A}_p \subseteq \mathcal{A}_{q,0} \subseteq \mathcal{A}_q \qquad \forall\ p \leq q.$$

Proof. First of all, let us prove that, for all p, $\mathcal{A}_{p,0}(\mathbb{C}) \subseteq \mathcal{A}_p(\mathbb{C})$. If $\forall \varepsilon > 0$, $\exists A_\varepsilon > 0$ such that $|f(z)| \leq A_\varepsilon \exp(\varepsilon|z|^p)$ (i.e. $f \in \mathcal{A}_{p,0}$) then, in particular, if we fix $\varepsilon = B$ we can find an A_ε that we denote with A, such that $|f(z)| \leq A\exp(B|z|^p)$ (i.e. $f \in \mathcal{A}_p$).

Furthermore, we have that $\mathcal{A}_p \subseteq \mathcal{A}_{q,0}$ for every $p \leq q$. Indeed, given a $f \in \mathcal{A}_p(\mathbb{C})$, for any $\varepsilon > 0$ consider a $R_\varepsilon > 0$ such that $B|z|^p < \varepsilon|z|^q$ in the open set $|z| > R_\varepsilon$ and then

$$|f(z)| \leq A\exp(B|z|^p) \leq A\exp(\varepsilon|z|^q).$$

On the compact set $|z| \leq R_\varepsilon$, we reason as before and we claim that f is bounded in that set, so it admits a maximum C. Thus we have

$$|f(z)| \leq C \leq C\exp(\varepsilon|z|^q).$$

Considering $A_\varepsilon = \max\{A, C\}$ we get the result.

Finally, this discussion leads to state: $\mathcal{A}_{p,0} \subseteq \mathcal{A}_p \subseteq \mathcal{A}_{q,0} \subseteq \mathcal{A}_q \ldots$ and so on. □

To define a topology on these spaces I need to introduce some tools from functional analysis and topology, for a detailed treatment refer to [11][Section 2.1] and [12][Section 4.5].

Definition 2.3.6. *(Fréchet space)*

A topological vector space is called a Fréchet space, when X is metrizable with a translation invariant metric, complete and locally convex.

Intuitively, the combination of a vectorial structure with a topological one allows us to define a local topology near 0 and to translate it in every other point of the space. Since, here, the aim is to generalize the concept of Fréchet space in the way that follows, this property turns out to be useful.

Theorem 2.3.7. *Let*

$$X_1 \subset X_2 \subset \cdots \subset X_j \subset \cdots$$

be a sequence of Fréchet spaces such that for every $j \in \mathbb{N}$, X_j is a subspace of X_{j+1} and the topology on X_j is the topology induced from X_{j+1}, namely if $W \subset X_j$ is an open set in X_j iff *there exists $Y \subset X_{j+1}$ open in X_{j+1} such that $W = Y \cap X_j$. Let*

$$X = \bigcup_{j=1}^{\infty} X_j$$

and consider the sets W that satisfy: $W \cap X_j$ is an open, convex, balanced neighborhood of 0 in X_j for all j. Then X has a unique locally convex vector space topology whose open, convex, balanced neighborhoods of 0 are precisely the sets W.

The topology determined in this way is called the inductive limit topology, and spaces of this kind are called LF spaces (short for: inductive limits of Fréchet spaces). Let us point out that an LF space is almost never a Fréchet space.

The fundamental property of the inductive limit topology that we use, is that important concepts such as convergence and continuity in connection with the topology on X can be referred to the corresponding concepts for one of the simpler spaces X_j.

Now, let us use this concepts in our framework, and let us define the spaces that will play the role of X_j spaces.

Definition 2.3.8. *($\mathcal{A}_p^B$ space)*
Let p and B be positive real numbers. The space $\mathcal{A}_p^B(\mathbb{C})$ is defined by

$$\mathcal{A}_p^B(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) \ : \ \exists \ A > 0 \ : \ |f(z)| \leq A \exp(B|z|^p)\}.$$

Then we can induce a topology on $\mathcal{A}_p^B(\mathbb{C})$ through the definition of the following norm:

$$\|f\|_B := \sup_{z \in \mathbb{C}} \{|f(z)| \exp(-B|z|^p)\}.$$

$\|\cdot\|_B$ defines a norm on $\mathcal{A}_p^B(\mathbb{C})$ so that $(\mathcal{A}_p^B(\mathbb{C}), \|\cdot\|_B)$ is a Banach space, and in particular a Fréchet space.

For any sequence $\{B_n\}_{n\geq 1}$ of positive numbers, strictly increasing to infinity, we can notice that $\mathcal{A}_p^{B_n}(\mathbb{C}) \subseteq \mathcal{A}_p^{B_{n+1}}(\mathbb{C})$ and that $\mathcal{A}_p(\mathbb{C})$ is given by the inductive limit

$$\mathcal{A}_p(\mathbb{C}) := \lim_{\rightarrow} \mathcal{A}_p^{B_n}(\mathbb{C}).$$

So we can induce in a canonical way an LF-topology on $\mathcal{A}_p(\mathbb{C})$, as it is described in Theorem 2.3.7.

In this case, it can be proven that this topology is independent of the choice of the sequence $\{B_n\}_{n\geq 1}$. Thus given f and a sequence $\{f_N\}_{N\geq 1}$ in $\mathcal{A}_p(\mathbb{C})$, we say that $f_N \to f$ in $\mathcal{A}_p(\mathbb{C})$ if and only if there exists $n \in \mathbb{N}^*$ such that $f, f_N \in \mathcal{A}_p^{B_n}(\mathbb{C})$ for all $N \in \mathbb{N}^*$, and $\|f_N - f\|_{B_n} \to 0$ for $N \to \infty$.

On the other hand, the topology on $\mathcal{A}_{p,0}(\mathbb{C})$ is given by the projective limit

$$\mathcal{A}_{p,0}(\mathbb{C}) := \lim_{\leftarrow} \mathcal{A}_p^{\varepsilon_n}(\mathbb{C}).$$

where $\{\varepsilon_n\}_{n\geq 1}$ is a strictly decreasing sequence of positive numbers converging to 0.

The following result is an immediate consequence of the definition of the topology in the spaces $\mathcal{A}_p(\mathbb{C})$ for $p > 0$.

Proposition 2.3.9. *Let $\boldsymbol{f} = \{f_N\}_{N\geq 1}$ be a sequence of elements in $\mathcal{A}_p(\mathbb{C})$. The two following assertions are equivalent:*

- *the sequence $\boldsymbol{f}$ converges towards 0 in $\mathcal{A}_p(\mathbb{C})$;*
- *the sequence $\boldsymbol{f}$ converges towards 0 in $\mathcal{H}(\mathbb{C})$ and there exists $A_{\boldsymbol{f}} \geq 0$ and $B_{\boldsymbol{f}} \geq 0$ such that*

$$\forall N \in \mathbb{N}^*, \forall z \in \mathbb{C} \quad |f_N(z)| \leq A_{\boldsymbol{f}}\, e^{B_{\boldsymbol{f}}|z|^p}. \tag{2.3.1}$$

Remark 2.3.3. Here $\mathcal{H}(\mathbb{C})$ is equipped with its usual topology of uniform convergence on any compact subset.

Proof. The first assertion means that there exists $B > 0$ with $\lim_{N\to\infty} \|f_N\|_B = 0$ which implies that the sequence $\boldsymbol{f}$ converges to 0 in $\mathcal{H}(\mathbb{C})$.

Furthermore, since $\lim_{N\to\infty} \|f_N\|_B = 0$, there exists $N_1 > 0$ such that $\|f_N\|_B \leq 1$ for $N \geq N_1$, and then $|f_N(z)| \leq Ae^{B|z|^p}$ with B and $A = \sup(\tilde{A}_1, \ldots, \tilde{A}_{N_1}, 1)$ independent of N ($\tilde{A}_j = \|f_j\|_B = \sup_{\mathbb{C}}(|f_j(z)|e^{-B|z|^p})$ for $j = 1, \ldots, N_1$.

Conversely, assume that the second assertion holds and take $B > B_{\boldsymbol{f}}$, so that, given $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\forall N \in \mathbb{N}^*, \quad \sup_{|z|\geq R_\varepsilon} |f_N(z)|e^{-B|z|^p} \leq A_{\boldsymbol{f}}\, e^{(B_{\boldsymbol{f}}-B)R_\varepsilon^p} < \varepsilon.$$

On the other hand, since $\boldsymbol{f}$ converges to 0 uniformly on any compact subset of $\mathbb{C}$, in particular on $\overline{D(0, R_\varepsilon)}$, there exists $N_\varepsilon \in \mathbb{N}^*$ such that for $N \geq N_\varepsilon$

$$\sup_{|z|\geq R_\varepsilon} |f_N(z)|e^{-B|z|^p} \leq \sup_{|z|\geq R_\varepsilon} |f_N(z)| < \varepsilon.$$

Therefore $\sup_{N\geq N_\varepsilon} \|f_N(z)\|_B < \varepsilon$ and the sequence $\boldsymbol{f}$ converges to 0 in $\mathcal{A}_p(\mathbb{C})$. $\square$

This result has two important consequences stated in the theorem and in the lemma below. The first is a convergence result of $F_n(z,a)$ that can be compared with its real counterpart stated in Section 2.2, whereas the latter is a characterization of the coefficients of entire functions with growth conditions.

Theorem 2.3.10. *For any $a \in \mathbb{C}$, the sequence $\{z \mapsto F_N(z,a)\}_{N\geq 1}$ converges to $z \mapsto e^{iaz}$ in $\mathcal{A}_1(\mathbb{C})$.*

Proof. It follows from estimates (2.2.8) that the sequence $\boldsymbol{f} = \{z \mapsto F_N(z,a)\}_{N\geq 1}$ satisfies the estimates (2.3.1) with $p = 1, B_{\boldsymbol{f}} = |a| + 1$ and $C_{\boldsymbol{f}} = 1$. Lemma 2.2.4 implies on the other hand that the sequence $\boldsymbol{f}$ converges towards $z \mapsto e^{iaz}$ in $\mathcal{H}(\mathbb{C})$. The result is then a consequence of Proposition 2.3.9. □

Lemma 2.3.11. *The function*

$$f(z) = \sum_{j=0}^{\infty} f_j z^j$$

belongs to $\mathcal{A}_p$ if and only if there exists $C_f > 0$ and $b > 0$ such that

$$|f_j| \leq C_f \frac{b^j}{\Gamma(\frac{j}{p}+1)}.$$

Proof. First suppose that $f(z) \in \mathcal{A}_p$ and let us prove that the estimate on the coefficients f_j follows by the Cauchy formula. In fact, we have

$$f^{(j)}(z) = \frac{j!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{j+1}}\, dw,$$

where the path of integration γ is the circle $|w - z| = s|z|$, where s is a positive real number and $z \neq 0$. Then we have

$$|f^{(j)}(z)| \leq \frac{j!}{(s|z|)^j} \max_{|w-z|=s|z|} |f(w)| \leq \frac{C_f j!}{(s|z|)^j} \exp B(1+s)^p |z|^p$$

for all $s > 0$, where we have used the fact that $f \in \mathcal{A}_p$ and $|w| \leq (1+s)|z|$. The well known estimate

$$(a+b)^p \leq 2^p(a^p + b^p), \quad a > 0,\, b > 0,\, p > 0$$

gives

$$(1+s)^p \leq 2^p(s^p + 1)$$

for all $s > 0$. Hence we have

$$|f^{(j)}(z)| \leq C_f \frac{j!}{(s|z|)^j} \exp(B\, 2^p s^p |z|^p) \exp(B\, 2^p |z|^p)$$

for all $z \in C$ and $s > 0$. We now take the minimum of the right-hand side of the above estimate with respect to s, i.e. the minimum of the function

$$g(s) := \frac{1}{(s|z|)^j} \exp(B\, 2^p s^p |z|^p)$$

in $(0, \infty)$. The minimum is at the point

$$s_{min} = \Big(\frac{j}{2^p B^p}\Big)^{1/p} \frac{1}{|z|}$$

so that we obtain

$$|f^{(j)}(z)| \leq C_f j! \Big(\frac{2^p B^p}{j}\Big)^{j/p} e^{j/p} \exp(B\, 2^p |z|^p)$$

So if we set

$$b := (2^p B\, pe)^{1/p}$$

we obtain

$$|f^{(j)}(z)| \leq C_f j! \frac{b^j}{j^{j/p}} \exp(B\, 2^p |z|^p)$$

for all $z \in \mathbb{C}$. Since $f_j = \dfrac{f^{(j)}(0)}{j!}$ we have, by the maximum modules principle applied in a disc centered at the origin and with radius $\epsilon > 0$ sufficiently small,

$$\begin{aligned} |f_j| &\leq C_f \frac{b^j}{j^{j/p}} \exp(B\, 2^p \epsilon^p) \leq 2\, C_f \frac{b^j}{j^{j/p}} \\ &= C'_f \frac{b^j}{(j!)^{1/p}} \leq C'_f \frac{b^j}{\Gamma(\frac{j}{p}+1)} \end{aligned}$$

where the last inequality follows from $(j!)^{1/p} \leq \Gamma(\frac{j}{p}+1)$.

The other direction follows from the properties of the Mittag-Leffler function (see Appendix A). In fact

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

is an entire function of order $1/\alpha$ (and of type 1) for $\alpha > 0$ and $Re(\beta) > 0$. So, in our case, f is entire of order p. □

Thanks to this important characterization and recalling that $\Gamma(j+1) = j!$, we can define:

Definition 2.3.12. *($\mathcal{A}_1^{\gamma,\beta}$ spaces)*

We say that $f \in \mathcal{A}_1^{\gamma,\beta}(\mathbb{C})$ iff $f \in \mathcal{A}_1$ such that $f : W \in \mathbb{C} \longmapsto \sum_{j=0}^{\infty} a_j W^j$ and

$$|a_j| \leq \frac{\gamma}{j!}\beta^j \qquad \forall j \in \mathbb{N}$$

2.3.2 Operators on $\mathcal{A}_p(\mathbb{C})$

Let us now focus our attention on the operators acting on $\mathcal{A}_p(\mathbb{C})$ and $\mathcal{A}_{p,0}(\mathbb{C})$ and on the properties of the dual spaces.

Definition 2.3.13. *(Analytic functional space)*

Let Ω be an open subset of $\mathbb{C}$. An analytic functional $T : \mathcal{H}(\Omega) \to \mathbb{C}$ is a continuous linear functional on the space of the holomorphic functions $\mathcal{H}(\Omega)$.

The space of analytic functional on Ω is denoted with $\mathcal{H}'(\Omega)$.

Definition 2.3.14. *(Fourier-Borel transform)*

Let Ω be an open subset of $\mathbb{C}$. The Borel-Fourier transform of an analytic functional $\mu \in \mathcal{H}'(\Omega)$ is the function

$$\hat{\mu}(w) = \mu(\exp(-z \cdot w)), \qquad w \in \mathbb{C}.$$

Remark 2.3.4. It can be proven that the Fourier-Borel transform is an entire function of exponential type, that is $\hat{\mu} \in \mathcal{A}_1(\mathbb{C})$. Thus, thanks to Proposition 2.3.5, $\hat{\mu} \in \mathcal{A}_p(\mathbb{C})$ $\forall p$.

Let us denote by $\mathcal{A}_p'(\mathbb{C})$ the strong dual of $\mathcal{A}_p(\mathbb{C})$, namely the space of continuous linear functional on $\mathcal{A}_p(\mathbb{C})$ endowed with the strong topology, then one can extend the definition of Fourier-Borel transform to $\mu \in \mathcal{A}_p'(\mathbb{C})$.

I can now state the following duality results which we will be useful in the sequel:

Theorem 2.3.15. *Let $p, p' \in \mathbb{R}$, $p > 1$, $p' > 1$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then the following isomorphisms*

$$\widehat{\mathcal{A}_p(\mathbb{C})'} \cong \mathcal{A}_{p',0}(\mathbb{C})$$

and

$$\widehat{\mathcal{A}_{p,0}(\mathbb{C})'} \cong \mathcal{A}_{p'}(\mathbb{C}),$$

are algebraic and topological as well.

The duality is realized as

$$\mu \in \mathcal{A}'_{p'}(\mathbb{C}) \longmapsto \hat{\mu} \in \mathcal{A}_{p,0}(\mathbb{C})$$

That means that every holomorphic function with growth conditions, i.e. $f \in \mathcal{A}_{p,0}(\mathbb{C})$, is the Fourier-Borel transform of an analytic functional on $\mathcal{A}_{p'}(\mathbb{C})$, namely $\mu \in \mathcal{A}'_{p'}(\mathbb{C})$.

In the extreme case $p = 1$, $\mathcal{A}_1(\mathbb{C})$ is isomorphic to the space $\mathcal{H}(\mathbb{C})$ of analytic functionals, and, similarly as above, the duality is realized as

$$\mu \in \mathcal{H}'(\mathbb{C}) \longmapsto \hat{\mu} \in \mathcal{A}_1(\mathbb{C})$$

That is a rephrasing of Remark 2.3.4.

The last concept I want to introduce in this section is the definition of convolution operators on $\mathcal{A}_p(\mathbb{C})$ spaces. This objects will turn out to be very used in the study of superoscillating sequences.

Definition 2.3.16. *(Convolution between an analytic functional and an entire function)* Let $\mu \in \mathcal{H}'(\mathbb{C})$ and $f \in \mathcal{H}(\mathbb{C})$, then their convolution is the function defined by

$$(\mu * f)(z) = \mu(f)(z) := \mu(\zeta \mapsto f(\zeta + z)) \qquad z \in \mathbb{C}.$$

There are some special cases of interest, for example the convolution of any analytic functional with an exponential. In this case one can see that

$$(\mu * e^{ia\zeta})(z) = \mu(\zeta \mapsto e^{ia(z-\zeta)}) = e^{iaz}\hat{\mu}(a)$$

where $\hat{\mu}(a)$ is the Fourier transform of μ at the point a. This fact is important because it has a direct impact on superoscillating sequences, indeed if one considers any convolutor $\mu*$ acting on $\mathcal{H}(\mathbb{C})$, we see that

$$\mu * F_n(z) = \sum_{j=0}^{n} C_j(n,a) e^{i(1-2j/n)z} \hat{\mu}(1-2j/n) = \sum_{j=0}^{n} \tilde{C}_j(n,a) e^{i(1-2j/n)z},$$

where for all j

$$\tilde{C}_j(n,a) = C_j(n,a)\hat{\mu}(1-2j/n).$$

Before introducing another special class of convolutors on the space of entire functions, let us state this preliminary definition.

Definition 2.3.17. *(Carrier)*
Let $\mu \in \mathcal{H}'(\mathbb{C})$. A compact set $K \subseteq \mathbb{C}$ is a carrier of μ if for every neighborhood w of K, there exists $C_w \geq 0$ such that

$$|\mu(h)| \leq C_w \sup_{z \in w} |h(z)| \qquad \forall h \in \mathcal{H}(\mathbb{C}).$$

Now we are ready to introduce the class of infinite order differential operators, which arise when we consider an analytic functional μ carried by the origin. The reason for the nomenclature of *infinite order differential operator* stems from the fact that the Fourier-Borel transform of an analytic functional carried by the origin belongs to the space $\mathcal{A}_{p,0}$, and therefore its Taylor expansion converges everywhere on $\mathbb{C}$ and its action can truly be considered as the action of a differential operator of infinite order. Specifically, we can give the following definition, see [18].

Definition 2.3.18. *(Infinite-order differential operator)*
An operator of the form

$$\sum_{m=0}^{\infty} b_m(z) \frac{d^m}{dz^m} \tag{2.3.2}$$

is an infinite-order differential operator, which acts continuously on $\mathcal{H}(\mathbb{C})$ if and only if, for every compact set $K \subset \mathbb{C}$,

$$\lim_{k \to \infty} \sqrt[k]{\sup_{z \in K} |b_k(z)| \, k!} = 0. \tag{2.3.3}$$

There are many instances when we can write a differential operator in the form (2.3.2), while they do not satisfy condition (2.3.3). This is, classically, the case for the translation operator, which can be defined as

$$\tau f(x) = f(x+1) = \exp(d/dx) f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dx^m} f(x). \tag{2.3.4}$$

While the operator appears to be expressed as an infinite sum of derivatives, the function on the right hand side of (2.3.4) does not converge, in general, and in fact makes no sense, except in an intuitive way. To be precise, the translation is only a convolutor on the space, say, of entire functions. Operators of this kind are quite important and therefore one may ask whether they can be treated in a general way. This is guaranteed by Theorem 2.3.15, which can now be rephrased as follows:

Theorem 2.3.19. *Let p, p' be real numbers such that $p \geq 1$, $p' \geq 1$ and*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The space of convolutors on $\mathcal{A}_p$ is isomorphic (via Fourier-Borel transform) to the space $\mathcal{A}_{p',0}$ and conversely, the space of convolutors of $\mathcal{A}_{p,0}$ is isomorphic to $\mathcal{A}_{p'}$.

Remark 2.3.5. Let us consider the differential operator defined as

$$U\left(t, \frac{d}{dz}\right) = \sum_{m=0}^{\infty} \frac{\lambda(t)^m}{m!} \frac{d^m}{dz^m},$$

which may act on holomorphic functions and let us denote by $h(t, \zeta)$ its symbol. By symbol of a linear differential operator we mean, roughly speaking, its Fourier transform. It is the series (or the polynomial) formally obtained replacing each partial derivative by a new variable. Then $h(t, \zeta)$ is such that

$$h(t, \zeta) = \sum_{m=0}^{\infty} \frac{\lambda(t)^m}{m!} \zeta^m.$$

Depending on the value of $\lambda(t)$ this symbol may or may not define an infinite order differential operator in the sense of Definition 2.3.18. However, $h(t, \zeta)$ can be thought as the symbol of a convolution operator for suitable choices of $\lambda(t)$.

For instance, if $\lambda(t) \equiv 1$, then $h(t, \zeta) = e^{\zeta}$ and therefore is the symbol of the translation of the unit operator which is nothing but the convolution with the Dirac delta centered at $z = -1$, see (2.3.4).

Moreover, the function e^{ζ} is clearly a multiplier operator on $\mathcal{A}_1(\mathbb{C})$, where by multiplier operator I mean an operator for which this property holds

$$\widehat{Tf}(\xi) = m_T(\xi)\hat{f}(\xi),$$

where $m(\xi)$ is a $\mathbb{C}$-valued function. In other words, the Fourier transform of Tf at a frequency ξ is given by the Fourier transform of f at that frequency, multiplied by the value of the multiplier at ξ. The convolutor T that e^{ζ} defines is the translation as indicated above.

If we now consider the symbol

$$h(t, \zeta) = \sum_{m=0}^{\infty} \frac{\lambda(t)^m}{m!} \zeta^m$$

it is easy to see that such operator defines, for suitable choices of $\lambda(t)$ and t, a convolutor on $\mathcal{H}(\mathbb{C})$, in fact the translation by $\lambda(t)$. This is easily seen because $h(t,\zeta)$ is actually nothing but $\exp(\lambda(t)\zeta)$.

The previous reasoning can be summarized in the following scheme:

$$\underbrace{\sum_{m=0}^{\infty} \frac{\lambda(t)^m}{m!}\frac{d^m}{dz^m}}_{\text{differential operator}} \longrightarrow \underbrace{\sum_{m=0}^{\infty} \frac{\lambda(t)^m}{m!}\zeta^m}_{\text{symbol}} \longrightarrow \underbrace{e^{\lambda(t)\zeta}}_{\text{multiplier}} \longrightarrow \underbrace{\delta_{\lambda(t)\zeta} \text{ or translation by } \lambda(t)}_{\text{convolutor}}.$$

Remark 2.3.6. Theorem 2.3.19 shows that any continuous multiplier on $\mathcal{A}_p$ defines a continuous convolutor on the space $\mathcal{A}_{p',0}$, and then a continuous differential operator on $\mathcal{A}_{p',0}$. This result turns out to be very useful in the sequel.

Chapter 3

Evolution through Schrödinger type equations

Since superoscillations arise naturally in the context of quantum mechanics, it is important to study the evolution of superoscillatory functions under Schrödinger equation, which general expression is stated below:

$$i\frac{\partial}{\partial t}\psi(t,x) = \mathscr{H}\psi(t,x) \qquad \psi(x,0) = \psi_0(x).$$

The main question to address concerns the persistence of the superoscillatory behaviour when superoscillating sequencess are taken as initial values of a particular Schrödinger equation. In other words, we wonder if the solution is again superoscillating according to definitions presented in Section 2.1.

Depending on the expression of the Hamiltonian operator $\mathscr{H}$ the physical meaning of the equation changes. In particular in this thesis I will treat the following cases:

- in Section 3.2 I consider $\mathscr{H} = -\frac{\partial^2}{\partial x^2}$, that is the Schrödinger equation for the free particle;
- in Section 3.3, I analyse the modified Schrödinger equation considering $\mathscr{H} = -\frac{\partial^p}{\partial x^p}$;
- in Section 3.4 I present a generalization of the modified Schrödinger equation, examining $\mathscr{H} = -\sum_{p=0}^{\infty} a_p \frac{\partial^p}{\partial x^p}$;
- in Section 3.5 I consider $\mathscr{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} - x$, that is the Schrödinger equation for the electric field. Let us point out that in this case we introduce a potential $V(x) = -x$;

- finally, Chapter 4 is totally devoted to the study of the Schrödinger equation for the harmonic oscillator, so I consider $\mathscr{H} = -\frac{1}{2}\Big(\frac{\partial^2}{\partial x^2} - x^2\Big)$, where $V(x) = \frac{1}{2}x^2$ is the potential.

It is important to note that the persistence of superoscillations only occurs when one takes the limit for $n \to \infty$. If one fixes the value of n, persistence is only for a finite time and superoscillations are, for large n, exponentially weak, see Section 2.2 and [15].

In writing this chapter I refer to [10], [8], [9], [6], [4], [7], [5] and [13].

3.1 General strategy and continuity results

Before analysing each case in details in the following sections, I explain the general strategy that it has been followed to address the question of persistence of superoscillation and I present the common results that will be used later on.

General strategy:

1. We start considering the Schrödinger equation in the real setting with initial datum $\psi(0, x) = F_n(x)$ (or any superoscillatory function). We denote the solution with $\psi_n(t, x)$.

2. We can write the solution in the form

$$\psi_n(t, x) = U\Big(t, \frac{\partial}{\partial x}\Big)F_n(x)$$

   where $U(t, \frac{\partial}{\partial x})$ is an operator which is formally defined by a series of derivatives.

   It can be formally seen as an infinite order differential operator (recall Definition 2.3.18), in fact, it is a convolution operator when considered on a suitable space of holomorphic functions.

3. We complexify the variable x both in the operators and in the functions on which the operators act. The key point here is that the real analytic functions on which the operators act can be extended not simply to holomorphic functions in a neighborhood of the real axis (as it would customarily happen) but in fact to entire functions satisfying specific growth conditions on the whole $\mathbb{C}$.

   Once we are in the complex setting, in order to prove that these operators act continuously on the space of the entire functions containing $F_n(z)$, we can decide whether to apply the methods from the theory of $\mathcal{A}_p$ spaces and Fourier transform or to use the more limited theory of holomorphic functions.

4. Continuity implies that we can calculate the limit $\lim_{n\to\infty} \psi_n(z,t)$. One can then restrict this result to the real axis and thus demonstrate that the superoscillatory nature of the initial value F_n is preserved in the evolved solution ψ_n.

Remark 3.1.1. Let us explain what is the relationship between the continuity of $U\Big(t, \frac{\partial}{\partial x}\Big)$ and the parameter t.

We could prove the operator continuity in x both for every t fixed and uniformly with respect to t. This leads, in the first case, to prove that $\psi_n(t,x)$ superoscillates for each t in a subset of $\mathbb{R}$, whereas in the latter this leads to prove that $\psi_n(t,x)$ superoscillates in a subset of $\mathbb{R}^2$.

In general we have two different approaches to prove continuity. One was already presented in Theorem 2.3.19, as remarked in 2.3.6.

The second possibility, presented in the following, is a direct method that avoids the use of the Fourier transform and uses just the theory of holomorphic functions.

Theorem 3.1.1. *Let $\lambda(t)$ be a bounded function for $t \in [0,T]$ for some $T \in (0,\infty)$, $f \in \mathcal{A}_1$ and let $U_\lambda(t,\partial_z)$ be the following operator*

$$U_\lambda(t,\partial_z) := \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \partial_z^{pn}.$$

Then, for $p \in \mathbb{N}$, we have $U_\lambda(t,\partial_z)f \in \mathcal{A}_1$ and $U_\lambda(t,\partial_z)$ is continuous on $\mathcal{A}_1$, that is $U_\lambda(t,\partial_z)f \to 0$ as $f \to 0$.

Proof. Let us consider the action of $U_\lambda(t,\partial_z)$ on $f(z)$

$$\begin{aligned} U_\lambda(t,\partial_z)f(z) &= \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \partial_z^{pn} f(z) \\ &= \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \partial_z^{pn} \sum_{j=0}^{\infty} f_j z^j \\ &= \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \sum_{j=pn}^{\infty} f_j \frac{j!}{(j-pn)!} z^{j-pn} \\ &= \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \sum_{k=0}^{\infty} f_{pn+k} \frac{(pn+k)!}{k!} z^k \end{aligned} \tag{3.1.1}$$

and now let us take the modulus

$$|U_\lambda(t,\partial_z)f(z)| \le \sum_{n=0}^{\infty} \frac{|\lambda(t)|^n}{n!} \sum_{k=0}^{\infty} |f_{pn+k}| \frac{(pn+k)!}{k!} |z|^k.$$

Using Lemma 2.3.11, on the coefficients f_{pn+k} we have the estimate

$$|f_{pn+k}| \le C_f \frac{b^{pn+k}}{\Gamma(pn+k+1)}$$

and the Gamma function estimate $(a+b)! \le 2^{a+b} a! b!$ gives

$$(pn+k)! \le 2^{pn+k}(pn)!k!$$

so we get

$$|U_\lambda(t,\partial_z)f(z)| \le \sum_{n=0}^{\infty} \frac{|\lambda(t)|^n}{n!} \sum_{k=0}^{\infty} C_f \frac{b^{pn+k}}{\Gamma(pn+k+1)} \frac{2^{pn+k}(pn)!k!}{k!} |z|^k.$$

Let us now use the estimate $\Gamma(a+b+2) \ge \Gamma(a+1)\Gamma(b+1)$ to separate the two series, so we have

$$\frac{1}{\Gamma(pn - \frac{1}{2} + k - \frac{1}{2} + 2)} \le \frac{1}{\Gamma(pn + \frac{1}{2})} \frac{1}{\Gamma(k + \frac{1}{2})}$$

and so

$$|U_\lambda(t,\partial_z)f(z)| \le C_f \sum_{n=0}^{\infty} \underbrace{\frac{((2b)^p |\lambda(t)|)^n}{n!} \frac{(pn)!}{\Gamma(pn+\frac{1}{2})}}_{A_n} \sum_{k=0}^{\infty} \underbrace{\frac{(2b|z|)^k}{\Gamma(k+\frac{1}{2})}}_{B_k}.$$

Now observe that, due to the properties of the Mittag-Leffler function, the series in k, whose terms are denoted with B_k, is smaller of $Ce^{2b|z|}$, for some constant $C > 0$, as described in the Appendix A. Now we have to show that the series in n is convergent. In fact, we have that it has positive terms, denoted with A_n, so we study the asymptotic behaviour.

Let us recall the duplication formula for the Gamma function and its functional equation

$$\Gamma(pn)\Gamma(pn + \tfrac{1}{2}) = 2^{1-2pn}\sqrt{\pi}\,\Gamma(2pn)$$

$$z\Gamma(z) = \Gamma(z+1)$$

so we have

$$\begin{aligned} A_n &= \frac{((2b)^p|\lambda(t)|)^n}{n!} \frac{(pn)!\Gamma(pn)}{2^{1-2pn}\sqrt{\pi}\Gamma(2pn)} \\ &= \frac{((2b)^p|\lambda(t)|)^n}{n!} \frac{(pn)!}{2^{1-2pn}\sqrt{\pi}} \frac{\frac{\Gamma(pn+1)}{pn}}{\frac{\Gamma(2pn+1)}{2pn}} \\ &= \frac{((8b)^p|\lambda(t)|)^n}{n!} \frac{(pn)!}{\sqrt{\pi}} \frac{\Gamma(pn+1)}{\Gamma(2pn+1)} \\ &= \frac{((8b)^p|\lambda(t)|)^n}{n!} \frac{(pn)!}{\sqrt{\pi}} \frac{(pn)!}{(2pn)!}. \end{aligned}$$

Using the Stirling formula $m! \sim \sqrt{2\pi m}(m/e)^m$ we get

$$\begin{aligned} A_n &\sim \frac{1}{\sqrt{\pi}} \frac{((8b)^p|\lambda(t)|)^n}{n!} \frac{[\sqrt{2\pi pn}(pn/e)^{pn}]^2}{\sqrt{2\pi 2pn}(2pn/e)^{2pn}} \\ &\sim \sqrt{p}\frac{((8b)^p|\lambda(t)|)^n}{n!}\sqrt{n}, \end{aligned}$$

so the series is convergent. Setting

$$G_\lambda(t) := \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \frac{((2b)^p|\lambda(t)|)^n}{n!} \frac{(pn)!}{\Gamma(pn+\frac{1}{2})}$$

we obtain the estimate

$$|U_\lambda(t,\partial_z)f(z)| \le C_f\, G_\lambda(t)\, C\, e^{2b|z|}. \tag{3.1.2}$$

This tells that $U_\lambda(t,\partial_z)$ takes $\mathcal{A}_1$ into $\mathcal{A}_1$, indeed it is enough to take, for each $t \in [0,T]$, $A = C_f G_\lambda(t) C$ and $B = 2b$ in Definition 2.3.3.

The continuity follows from the fact that for $C_f \to 0$ we have $U_\lambda(t,\partial_z)f(z) \to 0$. □

Let us now notice that the last estimate (3.1.2) varies with t. In order to enhance those results it is possible to prove also the uniformity with respect to t of the continuity of $U(t,\partial_z)$.

Lemma 3.1.2. *Let $\mathcal{T}$ be a set of parameters and $t \in \mathcal{T} \mapsto U(t, \frac{d}{dW})$ be the differential operator-valued map*

$$t \in \mathcal{T} \longmapsto U(t, \frac{d}{dW}) = \sum_{j=0}^{\infty} b_j(t)\Big(\frac{d}{dW}\Big)^j$$

(with $b_j : \mathcal{T} \to \mathbb{C}$ for $j \in \mathbb{N}$) whose formal symbol

$$h : (t,W) \in \mathcal{T} \times \mathbb{C} \longmapsto \sum_{j=0}^{\infty} b_j(t) W^j$$

realizes for each $t \in \mathcal{T}$ an entire function of W such that

$$\sup_{t\in\mathcal{T}, W\in\mathbb{C}} (|h(t,W)|e^{-B|W|^p}) = A < +\infty \tag{3.1.3}$$

for some $p \ge 1$ and $B \ge 0$. Then $U(t, \frac{d}{dW})$ acts as a continuous operator from $\mathcal{A}_1(\mathbb{C})$ into itself uniformly with respect to the parameter $t \in \mathcal{T}$.

Proof. It follows from Lemma 2.3.11 that the coefficient functions $t \mapsto b_j(t)$ satisfy then uniform (i.e. independent on t) estimates

$$\forall j \in \mathbb{N}, \quad \forall t \in \mathcal{T}, \quad |b_j(t)| \le C\frac{b^j}{\Gamma(j/p+1)} = C\frac{b^j}{j!}$$

for some positive constants $C = C(U)$ and $b = b(U)$ depending only on the finite quantity A and B in (3.1.3).

Let us consider $f : W \mapsto \sum_{\ell=0}^{\infty} a_\ell W^\ell \in \mathcal{A}_1(\mathbb{C})$, then there are (see again Lemma 2.3.11) positive constants γ and β such that

$$\ell \in \mathbb{N} \qquad |a_\ell| \leq (\gamma/\ell!)\beta^\ell.$$

Let us now focus on the action of U on such f. Following the same formal steps as in (3.1.1) one has

$$\forall\, t \in \mathcal{T}, \qquad U(t, \frac{d}{dW})(f) = \sum_{\ell=0}^{\infty} \underbrace{\Big(\sum_{j=0}^{\infty} \frac{(j+\ell)!}{\ell!} b_j(t) a_{\ell+j} \Big)}_{\alpha_\ell} W^\ell \tag{3.1.4}$$

with coefficients α_ℓ such that

$$|\alpha_\ell| = \sum_{j=0}^{\infty} \frac{(j+\ell)!}{\ell!} |b_j(t)||a_{\ell+j}| \leq \gamma\, C \frac{\beta^\ell}{\ell!} \sum_{j=0}^{\infty} \frac{(b\,\beta)^j}{j!} = K(b, C, \beta, \gamma)\, \frac{\beta^\ell}{\ell!}. \tag{3.1.5}$$

Therefore the formal identity (3.1.4) is in fact a true one for any $W \in \mathbb{C}$, which shows that $U(t, \frac{d}{dW})[f] \in \mathcal{A}_1(\mathbb{C})$ for any $t \in \mathcal{T}$, with

$$\forall\, t \in \mathcal{T}, \quad \forall\, W \in \mathbb{C}, \qquad |U(t, \frac{d}{dW})(f)| \leq \sum_{\ell=0}^{\infty} |\alpha_\ell| W^\ell \leq K(b, C, \beta, \gamma)\, e^{\beta |W|}.$$

We have proved that $U(t, \frac{d}{dW})$ acts from $\mathcal{A}_1(\mathbb{C})$ to $\mathcal{A}_1(\mathbb{C})$. Let us now focus on the continuity.

Let $\boldsymbol{f} = \{f_N\}_{N\geq 1}$ be a sequence converging to 0 in $\mathcal{A}_1(\mathbb{C})$ which is equivalent to say that $\sup(b_{f_N} + C_{f_N}) < +\infty$ and that $\boldsymbol{f}$ converges to 0 in $\mathcal{H}(\mathbb{C})$, see Proposition 2.3.9. Then we have

$$\forall N \in \mathbb{N}^*, \ \forall t \in \mathcal{T}, \ \forall W \in \mathbb{C} \qquad\qquad |U(t, \frac{d}{dW})(f_N)(W)| \leq A_{\boldsymbol{f}}\, e^{B_{\boldsymbol{f}} |W|}$$

for some positive constants $A_{\boldsymbol{f}}$ and $B_{\boldsymbol{f}}$ depending only on U and $\boldsymbol{f}$. Let $B_1 > B_{\boldsymbol{f}}$ and $\varepsilon > 0$. Let $R = R_\varepsilon$ large enough such that

$$\forall N \in \mathbb{N}^*, \ \forall t \in \mathcal{T}, \ \forall W \in \mathbb{C} \text{ with } |W| > R \qquad\qquad |U(t, \frac{d}{dW})(f_N)(W)| e^{-B_1 |W|} \leq \varepsilon.$$

Since $U(t, \frac{d}{dW})(f_N)(W) = \sum_{\ell=0}^{\infty} a_{N,\ell} W^\ell$ with $|a_{N,\ell}| \leq (C_{\boldsymbol{f}}/\ell!) b_{\boldsymbol{f}}^\ell$ for some constants $C_{\boldsymbol{f}}$ and $b_{\boldsymbol{f}}$ independent on $t \in \mathcal{T}$ and on N and the sequence $\boldsymbol{f}$ converges to 0 in $\mathcal{H}(\mathbb{C})$, one can find $N = N_\varepsilon$ such that

$$|U(t, \frac{d}{dW})(f_N)(W)| \leq \varepsilon \qquad\qquad \forall N \leq N_\varepsilon, \ \forall t \in \mathcal{T}, \ \forall W \in \mathbb{C} \text{ with } |W| \leq R$$

Hence the sequence $\{U(t, \frac{d}{dW})(f_N)(W)\}_{N\geq 1}$ converges to 0 in $\mathcal{A}_1(\mathbb{C})$, uniformly with respect to the parameter t. □

3.2 Schrödinger equation for the free particle

In physics, a free particle is a particle that, in some sense, is not bound by an external force, or equivalently not in a region where its potential energy varies. In quantum mechanics, it means a region of uniform potential, usually set to zero.

In this section I consider the Schrödinger equation that describes the behaviour of a free particle, when the initial value is the superoscillating sequence $F_n(x)$ fully presented in Section 2.2:

$$i\frac{\partial}{\partial t}\psi(t,x) = -\frac{\partial^2}{\partial x^2}\psi(t,x) \qquad \psi(x,0) = F_n(x).$$

In order to find out if the solution of this equation keeps to be superoscillating, I follow the steps described in Section 3.1.

1. Let us starting finding the solution of the differential equation.

Theorem 3.2.1. *The time evolution through Schrödinger equation for the free particle of the spatial superoscillating sequence $F_n(x)$, is given by*

$$\psi_n(t,x) = \sum_{j=0}^{n} C_j(n,a)e^{i(1-2j/n)x - i(1-2j/n)^2 t}.$$

Proof. To solve the Schrödinger equation with $F_n(x)$ as initial condition, we will work in the space of the tempered distributions $\mathcal{S}'(\mathbb{R})$ and use a standard Fourier transform argument. Let us denote by $\hat{\psi}(t,\lambda)$ the Fourier transform of ψ. Taking the Fourier transform of the Schrödinger equation we obtain

$$i\frac{\partial}{\partial t}\hat{\psi}(t,\lambda) = \lambda^2\hat{\psi}(\lambda,t)$$

and integrating we get

$$\hat{\psi}(\lambda,t) = C(\lambda)e^{-i\lambda^2 t},$$

where the arbitrary function $C(\lambda)$ can be determined by the initial condition and therefore, using the fact that $\mathcal{F}(e^{imx}) \overset{\mathcal{S}'}{=} 2\pi\delta(x-m)$

$$\begin{aligned} C(\lambda) = \hat{\psi}(\lambda,0) &= \int_{\mathbb{R}} \Big[\sum_{j=0}^{n} C_j(n,a)e^{i(1-2j/n)x}\Big] e^{i\lambda x} dx \\ &= \sum_{j=0}^{n} C_j(n,a) \int_{\mathbb{R}} e^{i(1-2j/n)x} e^{i\lambda x} dx \\ &= \sum_{j=0}^{n} C_j(n,a)\delta(\lambda-(1-2j/n)). \end{aligned}$$

So we obtain

$$\hat{\psi}(\lambda,t) = \sum_{j=0}^{n} C_j(n,a)\delta(\lambda-(1-2j/n))e^{-i\lambda^2 t}.$$

Taking now the inverse Fourier transform we have

$$\begin{aligned} \psi(t,x) &= \int_{\mathbb{R}} \Big[\sum_{j=0}^{n} C_j(n,a)\delta(\lambda-(1-2j/n))e^{-i\lambda^2 t}\Big] e^{i\lambda x} d\lambda \\ &= \sum_{j=0}^{n} C_j(n,a) \int_{\mathbb{R}} \Big[\delta(\lambda-(1-2j/n))e^{-i\lambda^2 t}\Big] e^{i\lambda x} d\lambda \\ &= \sum_{j=0}^{n} C_j(n,a)e^{-i(1-2j/n)^2 t} e^{i(1-2j/n)x}. \end{aligned}$$

□

2. In this step we give an equivalent representation of the time evolution of ψ_n in terms of the derivatives of the functions $F_n(x)$.

   Using the well known expansion of the exponential, the function $\psi_n(t,x)$ can be rewritten as

$$\begin{aligned} \psi_n(t,x) &= \sum_{j=0}^{n} C_j(n,a)e^{-i(1-2j/n)^2 t} e^{i(1-2j/n)x} \\ &= \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \sum_{j=0}^{n} C_j(n,a)(1-2j/n)^{2m} e^{i(1-2j/n)x} \\ &= \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \sum_{j=0}^{n} C_j(n,a) \frac{d^{2m}}{dx^{2m}} e^{i(1-2j/n)x} \\ &= \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dx^{2m}} \sum_{j=0}^{n} C_j(n,a)e^{i(1-2j/n)x} \\ &= \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dx^{2m}} F_n(x). \end{aligned}$$

3. We are now led to study the operator formally defined by

$$U_2\left(\frac{d}{dx}, t\right) := \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dx^{2m}}$$

and the spaces of functions on which it acts continuously. To this purpose, we extend $U_2\left(\frac{d}{dx}, t\right)$ to an operator which may act on holomorphic functions, i.e. we consider operators of the form

$$U_2\left(\frac{d}{dz}, t\right) = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dz^{2m}}$$

whose symbol is, recalling the notation used in Remark 2.3.5, $h(\zeta^2, t)$.

It is obvious to see that $h(\zeta^2, t)$ does not define a multiplication operator on $\mathcal{A}_1(\mathbb{C})$ because it grows at infinity too fast. The appropriate space for which h would be a multiplier and therefore the appropriate space for which h would induce a convolution operator is $\mathcal{A}_2(\mathbb{C})$. In particular, as an immediate application of Theorem 2.3.19 and Remark 2.3.6, we have that for any value of t, the operator $U_2\left(\frac{d}{dz}, t\right)$ acts continuously on the space $\mathcal{A}_{2,0}(\mathbb{C})$.

Moreover if we recall Theorem 3.1.1 and we consider $\lambda(t) = it$ and $p = 2$ we have that $U_2(\frac{d}{dz}, t)$ acts continuously, for any t, on the space $\mathcal{A}_1(\mathbb{C})$. This result is weaker than the previous one, since we should consider as input for the operator entire functions with stronger growth conditions at infinity.

Furthermore, we can enhance this result using Lemma 3.1.2, where we consider $b_j(t) = \dfrac{(it)^m}{m!}$ and $\mathcal{T} = \mathbb{R}$. We notice that the symbol h realizes for each $t \in \mathbb{R}$ an entire function of z such that

$$\sup_{t\in\mathbb{R}, z\in\mathbb{C}} (|h(t,z)|e^{-B|z|^p}) = A < +\infty \tag{3.2.1}$$

for $p = 2$ and for some $B \geq 0$. Then all the hypotheses of Lemma 3.1.2 are satisfied and it holds that $U_2(t, \frac{d}{dz})$ acts as a continuous operator from $\mathcal{A}_1(\mathbb{C})$ into itself uniformly with respect to the parameter $t \in \mathbb{R}$.

4. As a consequence of the previous step we can show that the superoscillatory phenomenon persists for $n \to \infty$ according to each of the four definitions presented in Section 2.1.

Indeed, we can prove that $\psi_n(t, x)$ superoscillates for all values of the time t (see Definition 2.1.2 and Remark 2.1.3), but also that $\{\psi_n(t, x)\}_{n\geq 1}$ is a superoscillating sequence in two variables (t,x) (see Definition 2.1.4) and finally we could prove that the sequence $\{\psi_n(t,x)\}_{n\geq 1}$ is a $\mathcal{F}$-supershift for the family $\mathcal{F} = \{\varphi_a(t,x) = e^{iax-ia^2t}\}$ (see Definition 2.1.5).

In order to do that we need to state the theorem below.

Theorem 3.2.2. *For $a > 1$, and for every $x, t \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} \psi_n(t,x) = e^{iax-ia^2t}. \tag{3.2.2}$$

Proof. The functions $F_n(x)$ can be extended to an entire function in $\mathcal{A}_1(\mathbb{C})$, and this space is clearly contained in $\mathcal{A}_{2,0}$. Therefore it is enough to take the limit and recall that

$$F_n(x) \to e^{iax}. \tag{3.2.3}$$

So, since the operator is continuous, we obtain

$$\begin{aligned}
\psi(t,x) &= \lim_{n\to\infty} U_2\left(\frac{d}{dx}, t\right) F_n(x) \\
&= U_2\left(\frac{d}{dx}, t\right) \lim_{n\to\infty} F_n(x) \\
&= \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dx^{2m}} e^{iax} \\
&= \sum_{m=0}^{\infty} \frac{(-ia^2t)^m}{m!} e^{iax} \\
&= e^{iax-ia^2t}.
\end{aligned}$$

□

This theorem can be read in different ways: we can interpret the convergence of $F_n(x)$ to e^{iax} simply as uniform convergence on any compact set of $\mathbb{R}$ (see Theorem 2.2.2) and conclude that the sequence $\psi_n(t,x)$ superoscillates for each t with superoscillation domain $\mathbb{R}$ and superoscillation sets any compact set in $\mathbb{R}$.

On the other hand, we can interpret (3.2.3) in $\mathcal{A}_1(\mathbb{C})$ (see Theorem 2.3.10) and recall that, thanks to Lemma 3.1.2, $U_2(t, \frac{d}{dz})$ acts continuously from $\mathcal{A}_1(\mathbb{C})$ to $\mathcal{A}_1(\mathbb{C})$ uniformly with respect to t. Then, according to Definition 2.1.4, we have that $\psi_n(t,x)$ is a superoscillating sequence with superoscillation domain $\mathbb{R}^2$, $P_\infty(T,X) = TX$, $g_1(a) = a^2$ and $g_2(a) = a$.

Finally, in the same way, we have that $\psi_n(t,x)$ is a $\mathcal{F}$-supershift for the family $\mathcal{F} = \{\varphi_a(t,x) = e^{iax-ia^2t}\}$, with super-shift domain $\mathbb{R}^2$.

3.3 Modified Schrödinger equation

We now consider the Cauchy problem associated with a modified version of the Schrödinger equation. The reason why this kind of problem is studied lies in the following observation.

If in (3.2.2) we fix $x = 0$, we obtain

$$\lim_{n\to\infty} \psi_n(t,0) = \lim_{n\to\infty} \sum_{k=0}^{\infty} C_k(n,a) e^{-i(1-2k/n)^2 t} = e^{-ia^2 t}.$$

This observation opens the way to construct a larger class of superoscillatory functions using differential equations. In particular, one is naturally led to wonder if it is possible to compute the limit

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} C_k(n,a) e^{i(1-2k/n)^p x}$$

when $(1-2k/n)^2$ has been replaced by $(1-2k/n)^p$, where p is an arbitrarily natural number.

This is indeed possible by replacing the Schrödinger equation with a modified version. In order to state the results, it is convenient to distinguish the cases p even and p odd, because different differential equations are involved. In particular we have:

a) $i\dfrac{\partial \psi(t,x)}{\partial t} = -\dfrac{\partial^p \psi(t,x)}{\partial x^p}, \qquad \psi(x,0) = F_n(x,a) \qquad \text{if } p \text{ is even.}$

b) $\dfrac{\partial \psi(t,x)}{\partial t} = \dfrac{\partial^p \psi(t,x)}{\partial x^p}, \qquad \psi(x,0) = F_n(x,a) \qquad \text{if } p \text{ is odd.}$

c) $-i \cdot i^{p \mod 2} \dfrac{\partial \psi(t,x)}{\partial t} = \dfrac{\partial^p \psi(t,x)}{\partial x^p}, \qquad \psi(x,0) = F_n(x,a) \qquad p \in \mathbb{N}.$

d) $i^{p-1} \dfrac{\partial \phi(t,x)}{\partial t} = \dfrac{\partial^p \phi(t,x)}{\partial x^p}, \qquad \phi(x,0) = F_n(x,a) \qquad p \in \mathbb{N}.$

Let us notice that the case $a)$ and $b)$ can be summarized in the expression $c)$, while equation $d)$ represents a slightly different case.

As I did in Section 3.2 for the Schrödinger equation for the free particle, I will follow the four steps presented at the beginning of the chapter in order to prove that the solution of all the modified Schrödinger equations listed above is again a superoscillating sequence.

1. Let us solve the equations stated above.

Theorem 3.3.1. *The time evolution through a modified Schrödinger equation of the spatial superoscillating sequence $F_n(x)$, is given by*

a) $\psi_n(t,x) = \sum_{k=0}^n C_k(n,a) e^{i(1-2k/n)x} e^{it(-i(1-2k/n))^p}$ *if p is even.*

b) $\psi_n(t,x) = \sum_{k=0}^n C_k(n,a) e^{i(1-2k/n)x} e^{t(-i(1-2k/n))^p}$ *if p is odd.*

c) $\psi_n(t,x) = \sum_{k=0}^n C_k(n,a) e^{i(1-2k/n)x} e^{i\cdot i^{p \mod 2} t(1-2k/n)^p}$ $p \in \mathbb{N}$.

d) $\phi_n(t,x) = \sum_{k=0}^n C_k(n,a) e^{i(1-2k/n)x} e^{it(1-2k/n)^p}$ $p \in \mathbb{N}$.

Proof. We will prove the result only for the case p even, the others are analogous.

The proof is totally similar to the one of Theorem 3.2.1, so I will skip some computations. As before, we work in the space of the tempered distributions $\mathcal{S}'(\mathbb{R})$. We start with the equation

$$i\frac{d\hat{\psi}(\lambda,t)}{dt} = -(-i\lambda)^p \ \hat{\psi}(\lambda,t)$$

and, integrating, we obtain

$$\hat{\psi}_n(\lambda,t) = 2\pi \sum_{k=0}^n C_k(n,a)\delta(\lambda-(1-2k/n))e^{i(-i\lambda)^p t},$$

and taking the inverse Fourier transform we have

$$\begin{aligned}\psi_n(t,x) &= \int_{\mathbb{R}} \Big[\sum_{k=0}^n C_k(n,a)\delta(\lambda-(1-2k/n))e^{it(-i\lambda)^p}\Big] e^{i\lambda x} d\lambda \\ &= \sum_{k=0}^n C_k(n,a) e^{i(1-2k/n)x} e^{it(-i(1-2k/n))^p}.\end{aligned}$$

□

2. This step is devoted to the rewriting of the solution $\psi_n(t,x)$ and, again, it follows the same passages as in the previous case. In particular we get

$$\psi_n(t,x) = \sum_{m=0}^\infty \frac{(it)^m}{m!}\frac{d^{mp}}{dx^{mp}} F_n(x,a).$$

Similar computations for p odd and p general lead to state that

a) $U_p\Big(\frac{d}{dx},t\Big) = \sum_{m=0}^\infty \frac{(it)^m}{m!}\frac{d^{mp}}{dx^{mp}}$ if p is even.

b) $U_p\left(\frac{d}{dx}, t\right) = \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \frac{d^{mp}}{dx^{mp}}$ if p is odd.

c) $U_p\left(\frac{d}{dx}, t\right) = \sum_{m=0}^{\infty} \frac{(i \cdot i^{p \mod 2} t)^m}{m!} \frac{d^{mp}}{dx^{mp}}$ $p \in \mathbb{N}$.

d) $\tilde{U}_p\left(\frac{d}{dx}, t\right) = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{1}{i^{mp}} \frac{d^{mp}}{dx^{mp}}$ $p \in \mathbb{N}$.

3. We now notice that the operators listed above differ only in terms of powers of the imaginary unit i, this does not influence the continuity of the differential operator and the growth at infinity of its symbol. So, in this step, I will refer to a generic $U_p\big(\frac{d}{dx}, t\big)$.

   As a consequence of Theorem 2.3.19 we have that $U_p\big(\frac{d}{dx}, t\big)$ is continuous, when we replace x by z, in the space $\mathcal{A}_p(\mathbb{C})$ for every $t \in \mathbb{R}$.

   Moreover, if we recall Theorem 3.1.1 and we consider a suitable $\lambda(t)$ according to the equation one is studying, we have that $U_p(\frac{d}{dz}, t)$ acts continuously, for any t, also on the space $\mathcal{A}_1(\mathbb{C})$. Let us notice that the previous result is a bit stronger than this, since it allows us to consider as input for the operator also entire functions that belong to $\mathcal{A}_p(\mathbb{C})$ and not only to $\mathcal{A}_1(\mathbb{C})$.

   Furthermore, as before, we can enhance this result using Lemma 3.1.2, all its hypotheses are satisfied and it holds that $U_p(t, \frac{d}{dz})$ acts as a continuous operator from $\mathcal{A}_1(\mathbb{C})$ into itself uniformly with respect to the parameter $t \in \mathbb{R}$.

4. Thanks to step 3) we can now pass to the limit. Following the computation as in the step 4) of Section 3.2 and considering p even we have:

   Theorem 3.3.2. *For $a > 1$, and for every $x, t \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} \psi_n(x,t) = e^{it(-ia)^p} e^{iax}. \tag{3.3.1}$$

   Proof. Similarly to Theorem 3.2.2:

$$\begin{aligned}
\psi(t,x) = \lim_{n\to\infty} \psi_n(t,x) &= \lim_{n\to\infty} U_p\left(\frac{d}{dx}, t\right) F_n(x) \\
&= \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{mp}}{dx^{mp}} e^{iax} \\
&= \sum_{m=0}^{\infty} \frac{((ia)^p it)^m}{m!} e^{iax} = e^{it(ia)^p} e^{iax}.
\end{aligned}$$

□

Summing up for all cases:

a) $\psi(t,x) = e^{it(-ia)^p + iax}$ if p is even.

b) $\psi(t,x) = e^{t(-ia)^p + iax}$ if p is odd.

c) $\psi(t,x) = e^{i \cdot i^{p \bmod 2} t(ia)^p + iax}$ $p \in \mathbb{N}$.

d) $\phi(t,x) = e^{ita^p + iax}$ $p \in \mathbb{N}$.

We can deduce exactly the same conclusions as before (substituting 2 with a general p). So if the superoscillating sequence $F_n(x)$ evolves through the modified Schrödinger equation, the solution is again a superoscillating sequence, accordingly to each of the definitions in Section 2.2.

We are now able to address the problem we presented at the beginning of this section, in particular, fixing $x = 0$ in (3.3.1), we have that the sequence

$$\sum_{k=0}^{n} C_k(n,a) e^{it(-i(1-2k/n))^p}$$

is $e^{it(-ia)^p}$-superoscillating, for p even.

One may ask what happens if we use a different initial datum for the modified Schrödinger equation, specifically, if we use a datum of the form

$$\sum_{k=0}^{n} C_k(n,a)\, e^{-i(1-2k/n)^\ell x}.$$

It can be proved that in this case, the solution is given by

$$\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a)\, e^{-ix(1-2k/n)^\ell} e^{it(-i(1-2k/n)^\ell)^p}.$$

Moreover, for all t, we have

$$\lim_{n\to\infty} \psi_n(t,x) = e^{it(-i)^p a^{p\,\ell}} e^{iax}.$$

The limit is uniform for x in the compact sets of $\mathbb{R}$.

Remark 3.3.1. This result shows that in this way we do not further enlarge the class of superoscillating sequences. If we want to find a more general limit function $e^{ig(a)x}$, then we have to leave the kingdom of the differential equations as I present in the next section.

3.4 Modified Schrödinger equation with series of derivatives

We will now consider a much more general situation in which the right hand side of the differential equation to solve is an infinite series of derivatives.

Let $\{a_p\}$ be a sequence of complex numbers and consider the convolution equation formally defined by

$$i\frac{\partial\psi(x,t)}{\partial t} = -\sum_{p=0}^{\infty} a_p \frac{\partial^p \psi(x,t)}{\partial x^p}. \tag{3.4.1}$$

Let us notice that if $a_p \equiv 0$ for every $p > \bar{p}$ we obtain that $\sum_{p=0}^{\bar{p}} a_p \frac{\partial^p}{\partial x^p}$ is simply a polynomial of degree $\bar{p}$ in $\frac{\partial}{\partial x}$.

1. The following theorem gives us the solution of (3.4.1) when we take $F_n(x)$ as initial datum (the hypotheses will be clearer afterwards).

Theorem 3.4.1. *Let $a \in \mathbb{R}$, $a > 1$. Consider a sequence of complex numbers $\{a_p\}$ such that the function $\sum_{p=0}^{\infty} a_p z^p$ is holomorphic in $\Delta_\rho = \{z \in \mathbb{C} : |z| < \rho\}$ for $\rho > a$. Consider the Cauchy problem for the generalized Schrödinger equation*

$$i\frac{\partial\psi(t,x)}{\partial t} = -G\left(\frac{d}{dx}\right)\psi(t,x), \qquad \psi(z,0) = F_n(z,a), \tag{3.4.2}$$

where

$$G\left(\frac{d}{dx}\right) = \sum_{p=0}^{\infty} a_p \frac{d^p}{dx^p}.$$

Then the solution $\psi_n(t,x)$, is given by

$$\psi_n(t,x) = \sum_{k=0}^{n} C_k(n,a)\, e^{-ix(1-2k/n)}\, e^{itG(-i(1-2k/n))}.$$

Proof. Using the usual method we solve the equation in the space of the tempered distributions $\mathcal{S}'(\mathbb{R})$ and, with standard Fourier transform argument, we obtain the result. □

2. Through the following computation we are able to write $\psi_n(t,x)$ as a result of the action

of a convolution operator on $F_n(x)$.

$$\begin{aligned}
\psi_n(t,x) &= \sum_{k=0}^{n} C_k(n,a)\, e^{-ix(1-2k/n)} e^{it\sum_{p=0}^{\infty} a_p(-i(1-2k/n))^p} \\
&= \sum_{k=0}^{n} C_k(n,a)\, e^{-ix(1-2k/n)} \prod_{p=0}^{\infty} e^{ita_p(-i(1-2k/n))^p} \\
&= \sum_{k=0}^{n} C_k(n,a)\, e^{-ix(1-2k/n)} \prod_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{(ita_p)^m}{m!} (-i(1-2k/n))^{mp} \\
&= \prod_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{(ita_p)^m}{m!} \sum_{k=0}^{n} C_k(n,a) \frac{d^{mp}}{d^{mp}} e^{-ix(1-2k/n)} \\
&= \prod_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{(ita_p)^m}{m!} \frac{d^{mp}}{d^{mp}} F_n(x,a).
\end{aligned}$$

3. In this step, we need to study the continuity of $U_\infty\left(\frac{d}{dz},t\right)$ that can formally be written as the infinite product of the operators we have just considered in Section 3.3, i.e.

$$U_\infty\left(\frac{d}{dz},t\right) = \prod_{p=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(ita_p)^m}{m!}\frac{d^{pm}}{dz^{pm}}\right) = \prod_{p=0}^{\infty} U_p\left(\frac{d}{dz},a_p t\right).$$

In order to understand if it is continuous, we first need to understand on which space it actually operates.

This operator is associated with the multiplier given by the function

$$h_\infty(\zeta,t) := \prod_{p=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(ita_p)^m}{m!}\zeta^{pm}\right).$$

That can be written in the form

$$h_\infty(\zeta,t) = \prod_{p=0}^{\infty}\exp(ita_p\zeta^p) = \exp\left(it\sum_{p=0}^{\infty}a_p\zeta^p\right).$$

When $\{a_p\}$ is a sequence of complex numbers such that the function $\sum_{n=0}^{\infty} a_p\zeta^p$ is analytic in the disc $|\zeta|<\rho$, then, h_∞ realizes, for each value of $t\in\mathbb{R}$, an holomorphic function on $\Delta_\rho\subset\mathbb{C}$.

More precisely, one has

$$\forall(t,\zeta)\in\mathbb{R}\times\Delta_\rho,\qquad h_\infty(\zeta,t) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}b_{j,k}t^k\right)\zeta^j = \sum_{j=0}^{\infty}b_j(t)\zeta^j$$

where, for $R>0$, the radius of convergence of the power series $\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}|b_{j,k}|R^k\right)\zeta^j$ is at least equal to ρ.

Therefore, this framework does not fit with the one needed in Theorem 2.3.19 or Lemma 3.1.2, so in order to prove the continuity of $U_\infty\left(\frac{d}{dz},t\right)$, we need to expand these results.

First of all, let us consider the space

$$\mathcal{A}_{1,\rho}(\mathbb{C}) := \lim_{\leftarrow} \mathcal{A}_1^{\rho_n}(\mathbb{C}) = \{f \in \mathcal{A}_1(\mathbb{C}) \ : \ \forall\varepsilon > 0, \ \exists A_\varepsilon > 0 \ : \ |f(z)| \leq A_\varepsilon e^{(\rho-\varepsilon)|z|}\},$$

where $\{\rho_n\}_{n\geq 1}$ is a strictly increasing sequence converging to ρ.

This space is called space of entire functions of exponential type less than ρ.

Now, we are ready to state the analogous in this framework of Theorem 2.3.19.

Proposition 3.4.2. *Let $\rho > 0$ and let X be the space of entire functions of exponential type less than ρ. Then X is isomorphic via Fourier-Borel transform to the space of functions holomorphic in the disc Δ_ρ.*

As a consequence of this we have that:

Theorem 3.4.3. *Let ρ the radius of convergence of $\sum_{n=0}^\infty a_p \zeta^p$. Then the function $h_\infty(\zeta,t)$ is a continuous multiplier on the space of functions analytic in the disc Δ_ρ and the associated operator $U_\infty\left(\frac{d}{dz},t\right)$ acts continuously on the space $\mathcal{A}_{1,\rho}(\mathbb{C})$.*

We can also prove that the continuity is uniform with respect to t, as it was done in Lemma 3.1.2 for the previous cases.

Theorem 3.4.4. *We distinguish two cases based on ρ.*

i) When $\rho = +\infty$, the convolutor operator $U_\infty\left(\frac{d}{dz},t\right)$ acts continuously locally uniformly with respect to $t \in \mathbb{R}$ from $\mathcal{A}_1(\mathbb{C})$ into itself.

ii) When $\rho \in (0,+\infty)$ it acts continuously locally uniformly with respect to $t \in \mathbb{R}$ from the space $\mathcal{A}_{1,\rho}(\mathbb{C})$ into itself.

Proof. The proofs of both the assertions follow the lines of Lemma 3.1.2.

i) Let $R > 0$ and $K \subset [-R,R] \subset$ be a compact subset of $\mathbb{R}$. Let $\gamma > 0$, $\beta > 0$ and $f \in \mathcal{A}_1^{\gamma,\beta}$ (in the sense of Definition 2.3.12).

Since, for any $R > 0$, $h_\infty(\zeta, t)$ is holomorphic in all $\mathbb{R}$, one can check as in the proof of Lemma 3.1.2 (compare to (3.1.5)) that, for any $t \in K$ and $j \in \mathbb{N}$,

$$\sum_{j=0}^{\infty} \frac{(j+\ell)!}{\ell!} |b_j(t)| \, |f_{\ell+j}| \leq \gamma \frac{\beta^\ell}{\ell!} \sum_{j=0}^{\infty} \Big(\sum_{k=0}^{\infty} |b_{j,k}| R^k \Big) \beta^j = K_U(\beta, \gamma) \frac{\beta^\ell}{\ell!}.$$

This is indeed enough to conclude as in the proof of Lemma 3.1.2 that U_∞ acts continuously locally uniformly in t from $\mathcal{A}_1(\mathbb{C})$ into itself.

ii) Consider now the case where $\rho \in (0, +\infty)$. For any $R > 0$, the radius of convergence of the power series $\sum_{j=0}^{\infty} \Big(\sum_{k=0}^{\infty} |b_{j,k}| R^k \Big) \zeta^j$ is now at least equal to ρ. Repeating the preceding argument (but taking now $\beta \leq \rho - \varepsilon$ for some $\varepsilon > 0$ arbitrarily small), one concludes that U_∞ acts continuously locally uniformly in t from $\mathcal{A}_{1,\rho}(\mathbb{C})$ into itself.

□

4. Thanks to the continuity results obtained in step 3), we can compute the limit as $n \to \infty$.

Theorem 3.4.5. *Let $a \in \mathbb{R}$, $1 < |a| < \rho$. For all fixed t we have*

$$\lim_{n\to\infty} \psi_n(t, x) = e^{itG(ia)} e^{iax},$$

and the convergence is uniform on all compact sets of $\mathbb{R}$.

Proof. As usual, we recall that $F_n(x) \to e^{iax}$ and we pass to the limit for $n \to \infty$

$$\begin{aligned}
\psi(t, x) &= \prod_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{(ita_p)^m}{m!} \frac{d^{mp}}{d^{mp}} e^{iax} \\
&= \prod_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{(ita_p)^m}{m!} (ia)^{mp} e^{iax} \\
&= \prod_{p=0}^{\infty} e^{(ita_p (ia)^p)} e^{iax} \\
&= e^{it \sum_{p=0}^{\infty} (a_p (ia)^p)} e^{iax}.
\end{aligned}$$

Finally, we can write

$$\lim_{n\to\infty} \psi_n(t, x) = e^{itG(ia)} e^{iax}.$$

□

Once again we can claim that the superoscillatory behaviour is preserved.

Indeed, we can interpret the convergence of $F_n(x)$ to e^{iax} as uniform convergence on any compact set of $\mathbb{R}$ and conclude that the sequence $\psi_n(t,x)$ superoscillates for each t with superoscillation domain $\mathbb{R}$ and superoscillation sets any compact set in $\mathbb{R}$, in agreement with Definition 2.1.2 and Remark 2.1.3.

On the other hand, we can interpret the convergence in $\mathcal{A}_1(\mathbb{C})$ (see Theorem 2.3.10). It is clear that $\mathcal{A}_1(\mathbb{C}) \subseteq \mathcal{A}_{1,\rho}(\mathbb{C})$ for $\rho > a$, so recall that, thanks to Theorem 3.4.4, $U_\infty\left(t, \dfrac{d}{dz}\right)$ acts continuously from $\mathcal{A}_{1,\rho}(\mathbb{C})$ to $\mathcal{A}_{1,\rho}(\mathbb{C})$ uniformly with respect to t.

Adding the requests that G is a real polynomial such that $|G(ia)| \geq 1$ and $\sup_{x\in[-1,1]} G(x) < 1$, we can conclude that $\psi_n(t,x)$ superoscillates according to Definition 2.1.4. Indeed, we have that $\psi_n(t,x)$ is a superoscillating sequence with superoscillation domain $\mathbb{R}^2$, $P_\infty(T,X) = TX$, $g_1(a) = G(ia)$ and $g_2(a) = a$.

Finally, we can also prove that $\psi_n(t,x)$ is a $\mathcal{F}$-supershift for the family $\mathcal{F} = \{\varphi_a(t,x) = e^{iax-iG(ia)t}\}$, with super-shift domain $\mathbb{R}^2$.

Remark 3.4.1. By setting $g(a) = G(ia)$ and by suitably choosing the coefficients a_p of the series expressing G, we can obtain a very large class of superoscillating functions.
This is an impressive result: if so far we can draw superoscillatory sequence using only powers of a, thanks to this we are now able to use many more functions.

3.5 Schrödinger equation for the electric field

In this section we try to adapt the previous results also to the case of the Schrödinger equation for the electric field expressed below:

$$i\frac{\partial\psi(t,x)}{\partial t} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(t,x) - x\psi(t,x), \qquad \psi(0,x) = F_n(x).$$

As one can immediately notice, this equation present an addition with respect to the previous ones. In fact, this equation present a non-zero potential $V(x) = -x$.

1. Let $a > 1$. Then the solution is given by

$$\psi_n(t,x) = \sum_{k=0}^{n} C_k(n,a)\, e^{-it^3/6} e^{-i(1-2k/n)t((1-2k/n)+t)/2} e^{i((1-2k/n)+t)x}. \tag{3.5.1}$$

I will not provide the proof of this result, since it is not interesting within the scope of this thesis.

2. In this step I write the solution (3.5.1) in terms of convolution operators. Indeed, considering the series expansion

$$e^{-i(1-2k/n)t((1-2k/n)+t)/2} = \sum_{m=0}^{\infty} \frac{1}{m!}(-i(1-2k/n)t((1-2k/n)+t)/2)^m,$$

we observe that the functions

$$\psi_n(t,x) = \quad e^{-it^3/6} e^{itx} \sum_{k=0}^{n} C_k(n,a)\, e^{-i(1-2k/n)t((1-2k/n)+t)/2} e^{ix(1-2k/n)}$$

can be written in the following way (when passing to the complex variable z)

$$\psi_n(t,z) = e^{-it^3/6} e^{itz} U(t,\partial_z) F_n(z,a),$$

where

$$U(t,\partial_z) = \sum_{m=0}^{\infty} \frac{(-it/2)^m}{m!}(t+\partial_z)^m \partial_z^m.$$

3. Due to the peculiarity of the form of $U(t,\partial_z)$, any of the results stated in Section 3.1 can be used to prove that $U(t,\partial_z)$ is a continuous operator, so the following tailored theorem is needed.

Theorem 3.5.1. *The operator*

$$U(t,\partial_z) = \sum_{m=0}^{\infty} \frac{(-it/2)^m}{m!}(t+\partial_z)^m \partial_z^m$$

acts continuously from $\mathcal{A}_1$ into itself.

Proof. We have

$$U(t,\partial_z) f(z) = \sum_{m=0}^{\infty} \frac{(-it/2)^m}{m!}(t+\partial_z)^m \partial_z^m \sum_{j=0}^{\infty} f_j z^j$$

and

$$\begin{aligned}
U(t,\partial_z)f(z) &= \sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}(t+\partial_z)^m\partial_z^m\sum_{j=0}^{\infty}f_jz^j\\
&= \sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}t^{m-\ell}\partial_z^{\ell+m}\sum_{j=0}^{\infty}f_jz^j\\
&= \sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}t^{m-\ell}\sum_{j=\ell+m}^{\infty}f_j\frac{j!}{(j-\ell-m)!}z^{j-\ell-m}\\
&= \sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}t^{m-\ell}\sum_{k=0}^{\infty}f_{m+\ell+k}\frac{(m+\ell+k)!}{k!}z^k.
\end{aligned}$$

With similar computations, as we did in Theorem 3.1.1, we get

$$|U(t,\partial_z)f(z)|\leq C_f\sum_{m=0}^{\infty}\frac{(|t|/2)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}|t|^{m-\ell}\sum_{k=0}^{\infty}\frac{b^{m+\ell+k}}{\Gamma(m+\ell+k+1)}\frac{2^{m+\ell+k}(m+\ell)!k!}{k!}|z|^k,$$

and therefore

$$|U(t,\partial_z)f(z)|\leq C_f\sum_{m=0}^{\infty}\frac{(b|t|)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}|t|^{m-\ell}(2b)^{\ell}\frac{(m+\ell)!}{\Gamma(m+\ell+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{(2b|z|)^k}{\Gamma(k+\frac{1}{2})}.$$

Now observe that, thank to the duplication formula and to the functional equation of the Gamma function $z\Gamma(z)=\Gamma(z+1)$ we have

$$\begin{aligned}
\frac{(m+\ell)!}{\Gamma(m+\ell+\frac{1}{2})} &= 4^{m+\ell}\frac{(m+\ell)!}{2\sqrt{\pi}}\frac{\Gamma(m+\ell)}{\Gamma(2(m+\ell))}\\
&= 4^{m+\ell}\frac{(m+\ell)!}{2\sqrt{\pi}}\frac{\frac{\Gamma(m+\ell+1)}{m+\ell}}{\frac{\Gamma(2(m+\ell)+1)}{2(m+\ell)}}\\
&= 4^{m+\ell}\frac{(m+\ell)!}{\sqrt{\pi}}\frac{(m+\ell)!}{(2(m+\ell))!},
\end{aligned}$$

but since $\frac{(n!)^2}{(2n)!}\leq 1$ we get

$$\frac{(m+\ell)!}{\Gamma(m+\ell+\frac{1}{2})}\leq 4^{m+\ell}.$$

So the estimate of the operator becomes

$$|U(t,\partial_z)f(z)|\leq C_f\sum_{m=0}^{\infty}\frac{(b|t|)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}|t|^{m-\ell}(2b)^{\ell}4^{m+\ell}\sum_{k=0}^{\infty}\frac{(2b|z|)^k}{\Gamma(k+\frac{1}{2})}$$

and, observing that, due to the properties of the Mittag-Leffer function (see Appendix A), the series in k is smaller of $Ce^{2b|z|}$, for some constant $C>0$, we have

$$|U(t,\partial_z)f(z)|\leq C_f\sum_{m=0}^{\infty}\frac{(4b|t|)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}|t|^{m-\ell}(8b)^{\ell}Ce^{2b|z|}$$

but, noting that the series in ℓ is nothing that a binomial expansion of $(|t|+8b)^m$, we finally get

$$|U(t,\partial_z)f(z)| \leq C\ C_f\, e^{4b|t|(|t|+8b)}e^{2b|z|}$$

and so we get the statement. □

Let us point out that the last estimate is not uniform with respect to t.

4. Finally, we can compute the limit:

$$\lim_{n\to\infty} \psi_n(t,x) = e^{-it^3/6}e^{-iat(a+t)/2}e^{i(a+t)x}.$$

Let us notice that in this case we can prove that the evolved solution $\psi_n(t,x)$ superoscillates only accordingly to Definition 2.1.2 and Remark 2.1.3, so it is a superoscillation in the unique variable x, with $g(a) = (a+t)$ and, as usual, superoscillation domain $\mathbb{R}$ and superoscillation set any compact subset of $\mathbb{R}$.

Since we do not have any uniform continuity result we cannot say that $\psi_n(t,x)$ is a superoscillating sequence in the two variables (t,x) (it can also be foreseen looking at the algebraic expression of the solution).

Despite this, we can claim that $\psi_n(t,x)$ is a $\mathcal{F}$-supershift for the family

$$\mathcal{F} = \{\varphi_a(x) = e^{-it^3/6}e^{-iat(a+t)/2}e^{i(a+t)x}\},$$

with super-shift domain $\mathbb{R}$.

Chapter 4

The case of the harmonic oscillator

In this chapter I will study the evolution of superoscillations under Schrödinger equation for the quantum harmonic oscillator.

The mathematical strategy I will apply is the same presented in Section 3.1, but in this case we need more tools than the ones used before, due to the presence of a non-null potential as in the case of electric field, see Section 3.5.

In particular, the Hamiltonian considered to describe an harmonic oscillator is:

$$\mathscr{H}(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2(t)x^2 - f(t)x.$$

From a physical point of view, this Hamiltonian represents a harmonic oscillator of mass m and time-dependent frequency $\omega(t)$ under the influence of the external time-dependent force $f(t)$.

For the sake of simplicity, in the sequel we will re-scale the variables in order to solve the Cauchy problem for the quantum harmonic oscillator, stated below:

$$i\frac{\partial\psi(t,x)}{\partial t} = \frac{1}{2}\Big(-\frac{\partial^2}{\partial x^2} + x^2\Big)\psi(t,x), \qquad \psi(0,x) = F_n(x,a). \tag{4.0.1}$$

This chapter is organized as follows: in Section 4.1 I present (and then use) the tools needed to solve the equation (exploiting step 1 of the general strategy); after that, in Section 4.2, I study the continuity of the operator arising in the solution, in order to prove that the superoscillatory behaviour of the initial datum persists (step 2, 3 and 4); I finally conclude this chapter with a particular focus on the singularities in the quantum harmonic oscillator evolution (see Section 4.3).

4.1 Solve the equation

In this case it is no more effective to solve the differential equation in the space of tempered distributions $\mathcal{S}'$, so we need to invoke the Green's function.

The Green's function $G(t,x,0,x')$ is such that we can write the solution of a Cauchy problem with initial datum $\psi_0(x)$ as

$$\psi(t,x) = \int_{\mathbb{R}} G(t,x,0,x')\psi_0(x')dx'.$$

In our case, where the Cauchy problem is (4.0.1) the Green's function is the locally integrable function in $(0,\infty)\times\mathbb{R}\times\mathbb{R}$ expressed by

$$G(t,x,0,x') := (2\pi i \sin t)^{-1/2} e^{(2xx' - (x^2+x'^2)\cos t)/(2i\sin t)}. \tag{4.1.1}$$

So, formally, we can already write the solution, but we are allowed to make this step effective only if the integral converges.

In order to settle the approach to non-absolutely convergent integrals on the half-line $\mathbb{R}^{+*}$ ($\mathbb{R}^{+*}u$ represents an half-line in $\mathbb{C}$ passing through u) or $\mathbb{R}$ through the principle of regularization, we need to explain what regularization of formal Fresnel-type integrals on $\mathbb{R}^{+*}$ or $\mathbb{R}$ means.

Let us introduce the framework.

Suppose that $\mathcal{T}$ is a set of parameters. Let $G:(t,Z)\in\mathcal{T}\times\mathbb{C}\longmapsto G(t,Z)$ be a function which is entire as a function of Z for each $t\in\mathcal{T}$ fixed.

Let also ϕ be a non-vanishing real function on $\mathcal{T}$ that will play the role of a phase function.

Let finally χ be a real number such that $\chi > -1$.

In order to give a meaning to the formal integral

$$\int_0^\infty (x)^\chi e^{-i\phi(t)x^2}\, G(t,x)\, dx \tag{4.1.2}$$

we distinguish the cases where $\phi(t) > 0$ and $\phi(t) < 0$.

In the first case ($\phi(t) > 0$), we substitute $x \leftrightarrow e^{-i\pi/4}Z$ and we obtain this (for the moment formal) expression:

$$\begin{aligned}\int_0^\infty (x)^\chi e^{-i\phi(t)x^2} G(t,x)\, dx \;&=\; e^{-i(\chi+1)\pi/4}\int_{\mathbb{R}^{+*}e^{i\pi/4}} Z^\chi e^{-\phi(t)Z^2}\; G(t,e^{-i\pi/4}Z)\, dZ\\ = \int_{\mathbb{R}^{+*}e^{i\pi/4}} Z^\chi e^{-\phi(t)Z^2}\; F_+(t,Z)\, dZ\end{aligned} \tag{4.1.3}$$

where $\mathbb{R}^{+*}e^{i\pi/4}$ is the bisector of the first quadrant and $F_+(t,Z) := e^{-i(\chi+1)\pi/4}G(t,e^{-i\pi/4}Z)$ for any $t \in \mathcal{T}$ and $Z \in \mathbb{C}$.

In the second case ($\phi(t) < 0$), we use the substitution $x \leftrightarrow e^{i\pi/4}Z$ and we rewrite (4.1.3) as

$$\begin{aligned}\int_0^{\infty} (x)^{\chi} e^{-i\phi(t)x^2}\, G(t,x)\, dx \;&=\; e^{i(\chi+1)\pi/4} \int_{\mathbb{R}^{+*}e^{-i\pi/4}} Z^{\chi} e^{\phi(t)Z^2}\; G(t,e^{i\pi/4}Z)\, dZ \\ = \int_{\mathbb{R}^{+*}e^{-i\pi/4}} Z^{\chi} e^{\phi(t)Z^2}\; F_-(t,Z)\, dZ\end{aligned} \tag{4.1.4}$$

where $\mathbb{R}^{+*}e^{-i\pi/4}$ is the bisector of the fourth quadrant and $F_-(t,Z) := e^{i(\chi+1)\pi/4}G(t,e^{i\pi/4}Z)$ for any $t \in \mathcal{T}$ and $Z \in \mathbb{C}$.

The following lemma gives the conditions to satisfy in order to make the Fresnel integral convergent.

Lemma 4.1.1. *Let $\mathcal{T}$, ϕ, χ as above and $F : \mathcal{T} \times \mathbb{C} \to \mathbb{C}$ such that $F(t,\cdot) \in \mathcal{A}_{q,0}(\mathbb{C})$ uniformly in t, for some $q \in (1,2]$. Then, for any $u = e^{i\theta}$ with $\theta \in (-\pi/4, \pi/4)$, the integral*

$$\int_{\mathbb{R}^{+*}u} Z^{\chi} e^{-|\phi(t)|Z^2} F(t,Z)\, dZ \tag{4.1.5}$$

is absolutely convergent and remains independent of u. It equals in particular its value for $u = 1$.

Proof. The absolute convergence follows from the estimates

$$\forall \varepsilon > 0, \qquad \sup_{t\in\mathcal{T}, Z\in\mathbb{C}} (|F(t,Z)|e^{-\varepsilon|Z|^q}) < +\infty, \tag{4.1.6}$$

together with the fact that if $u = e^{i\theta}$, $\mathrm{Re}((tu)^2) = t^2 cos(2\theta) > 0$ for $t > 0$.

The fact that the integrals do not depend of u follows from residue theorem. □

In view of this lemma, the regularization of an integral of the Fresnel-type such as (4.1.2) consists in the successive two operations:

i) first transform the formal expression (4.1.2) into one of the representations (4.1.3) or (4.1.4) according to $\mathrm{sign}(\phi(t))$;

ii) then invoke Lemma 4.1.1 (provided the required hypothesis are satisfied) and consider the regularization of (4.1.2).

Remark 4.1.1. In order to give a meaning (if possible of course) to the formal integral expression

$$\int_{\mathbb{R}} x^{\chi}\, e^{-i\phi(t)x^2} G(t,x)\, dx$$

one splits it as

$$\int_0^\infty |x|^\chi \, e^{-i\phi(t)x^2} G(t,x)\, dx + \int_0^\infty |x|^\chi \, e^{-i\phi(t)x^2} G(t,-x)\, dx$$

and proceed as above for the two formal expressions involved into this formal decomposition.

Proposition 4.1.2. *Let $G \in \mathcal{A}_{2,0}(\mathbb{C})$ and $\chi > -1$. Then, for all $\omega \in \mathbb{R}^*$*

$$\lim_{\varepsilon \to 0} \int_0^\infty |x|^\chi \, e^{i\omega x^2} e^{-\varepsilon x^2} G(x) dx$$

exists and coincides with the integral regularized under the approach described above.

Proof. For the proof I refer the reader to [19]. □

We are now ready to use the Green's function to solve the following Schrödinger equation, where the initial datum is simply $\psi(0,x) = e^{iax}$.

Proposition 4.1.3. *Let $a \in \mathbb{R}$. Then the solution of the Cauchy problem*

$$i\frac{\partial \psi(t,x)}{\partial t} = \frac{1}{2}\Big(-\frac{\partial^2}{\partial x^2} + x^2 \Big)\psi(t,x), \qquad \psi(0,x) = e^{iax} \tag{4.1.7}$$

is

$$\psi_a(t,x) = (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t} e^{-(i/2)a^2 \tan t} e^{iax/\cos t}. \tag{4.1.8}$$

Proof. Using the Green function expressed in (4.1.1) we get

$$\begin{aligned}
\psi_a(t,x) &= (2\pi i \sin t)^{-1/2} \int_{\mathbb{R}} \exp\Big(\frac{2xx' - (x^2 + x'^2)\cos t}{2i\sin t} \Big) \, e^{iax'} \, dx' \\
&= (2\pi i \sin t)^{-1/2} \, \exp\Big(-\frac{x^2 \cos t}{2i \sin t} \Big) \int_{\mathbb{R}} \exp\Big(\frac{2xx' - x'^2 \cos t + iax'(2i\sin t)}{2i\sin t} \Big) \, dx',
\end{aligned}$$

and rewriting

$$\frac{2xx' - x'^2\cos t - 2ax'\sin t}{2i\sin t} = -i\frac{(x - a\sin t)^2}{2\sin t\cos t} + i\frac{(x'\cos t - (x - a\sin t))^2}{2\sin t \cos t},$$

we obtain

$$\begin{aligned}
\psi_a(t,x) = (2\pi i \sin t)^{-1/2} & \\
&\times \exp\Big(\frac{i\, x^2 \text{cotan } t}{2} \Big) \, \exp\Big(-\frac{i(x - a\sin t)^2}{2\sin t\cos t} \Big) \\
&\times \int_{\mathbb{R}} \exp\Big(\frac{i(x'\cos t - (x - a\sin t))^2}{2\sin t\cos t} \Big) \, dx'.
\end{aligned}$$

We now perform a change of variable and, applying Lemma 4.1.1 and Proposition 4.1.2 with $\chi = 0$ and $F(t, Z) \equiv 1$, we can use the regularized integral

$$\int_{\mathbb{R}} e^{i\alpha x^2}\, dx = \lim_{\beta \to 0^+} \int_{\mathbb{R}} e^{-x^2(\beta - i\alpha)}\, dx = \left(\frac{i\pi}{\alpha}\right)^{1/2}. \tag{4.1.9}$$

Performing a change of variables $y = x' \cos t - (x - a \sin t)$ and considering $\alpha = (2 \sin t \cos t)^{-1}$

$$\begin{aligned}
&\int_{\mathbb{R}} \exp\left(\frac{i(x'\cos t - (x - a\sin t))^2}{2\sin t\cos t}\right) dx' \\
&= \int_{\mathbb{R}} \frac{1}{\cos t} \exp\left(\frac{iy^2}{2\sin t\cos t}\right) dy \\
&= \frac{1}{\cos t}(2\pi i \sin t\cos t)^{1/2},
\end{aligned}$$

from which one finally obtains

$$\begin{aligned}
\psi_a(t, x) = (2\pi i \sin t)^{-1/2}\ &\exp\left(\frac{i\, x^2 \mathrm{cotan}\ t}{2}\right) \\
\times \exp\left(-\frac{i(x - a\sin t)^2}{2\sin t\cos t}\right) &\times \frac{1}{\cos t}(2\pi i \sin t\cos t)^{1/2}.
\end{aligned}$$

And then with standard computations we have

$$\psi_a(t, x) = (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t} e^{-(i/2)a^2 \tan t} e^{iax/\cos t}.$$

□

For any $a \in \mathbb{R}$, we can interpret the evolution of the initial datum $x \in \mathbb{R} \mapsto e^{iax}$ through the Cauchy Schrödinger equation (4.1.7) also in the sense of distributions, that means that Proposition 4.1.3 can be enhanced as follows.

Corollary 4.1.4. *Let $\mathcal{T} = (0, +\infty) \times \mathbb{R}$. The solution of the Cauchy problem (4.1.7) is a $\mathbb{C}$-valued distribution $\mu_a \in \mathcal{D}'(\mathcal{T}, \mathbb{C})$ with singular support $\dfrac{\pi(2\mathbb{N}+1)}{2} \times \mathbb{R}$.*

Remark 4.1.2. Let us recall that the singular support of a distribution μ is the set of points where μ cannot be accurately expressed as a function in relation to test functions with support including that points.

Proof. The proof is a simple consequence of Proposition 4.1.3.

Indeed, since $(t, x) \in (0, +\infty) \times \mathbb{R} \longmapsto (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t} e^{-(i/2)a^2 \tan t}$ is a locally integrable function, the initial datum $x \in \mathbb{R} \mapsto e^{iax}$ evolves through the Schrödinger equation (4.1.7) as a distribution φ_a (in fact defined by a locally integrable function).

The singular support of μ_a is actually $\dfrac{\pi(2\mathbb{N}+1)}{2} \times \mathbb{R}$, due to the fact that $\tan t$ is not defined in this set of points. □

Now, using the previous results, we can solve the original Cauchy problem (the one with initial datum $\psi_0(x) = F_n(x)$), both in classical sense and in the sense of distributions.

Theorem 4.1.5. *The solution of the Cauchy problem*

$$i\frac{\partial \psi(t,x)}{\partial t} = \frac{1}{2}\Big(-\frac{\partial^2}{\partial x^2} + x^2\Big)\psi(t,x), \qquad \psi(0,x) = F_n(x,a) \tag{4.1.10}$$

is

$$\begin{aligned}\psi_n(t,x) &= (\cos t)^{-1/2}\exp(-(i/2)x^2\tan t)\\ &\times \sum_{k=0}^{n} C_k(n,a)\exp\Big(\frac{ix(1-2k/n)}{\cos t}\Big)\exp(-(i/2)(1-2k/n)^2\tan t).\end{aligned} \tag{4.1.11}$$

Proof. We observe that the initial datum $F_n(x,a)$ is a linear combination of the exponentials $e^{ix(1-2k/n)}$, then formula (4.1.11) follows from Proposition 4.1.3. □

Corollary 4.1.6. *Let $\mathcal{T} = (0,+\infty)\times\mathbb{R}$. The solution of the Cauchy problem (4.1.10) is a $\mathbb{C}$-valued distribution $\mu_n \in \mathcal{D}'(\mathcal{T},\mathbb{C})$ with singular support $\dfrac{\pi(2\mathbb{N}+1)}{2}\times\mathbb{R}$.*
In particular, $\mu_{(n)} \overset{\mathcal{D}'}{=} \sum_{k=0}^{n} C_k(n,a)\mu_{1-2k/n}$.

As always we have a solution in two variables, that means that we need a 3-dimensional plot to represent it, otherwise we can fix a variable and represent slices of function.

Below I follow the second approach and in Fig. 4.1 I fix different values of x and I plot $\log(1+|\mathrm{Re}(\psi_n(x,t)|)$ for $a = 2$ and $n = 20$, with a focus on the singularity point $\dfrac{\pi}{2}$, while in Fig. 4.2 I consider x varying in the interval $[-5,5]$ and I fix the value of t both far and close to $\dfrac{\pi}{2}$.

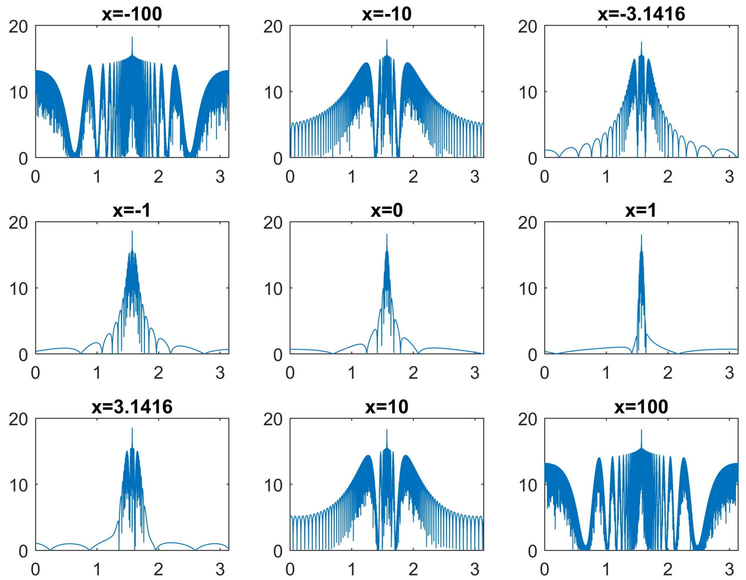

Figure 4.1: $\log(1 + |\mathrm{Re}(\psi_n(x,t)|)$, when $a = 2$, $n = 20$ and x fixed.

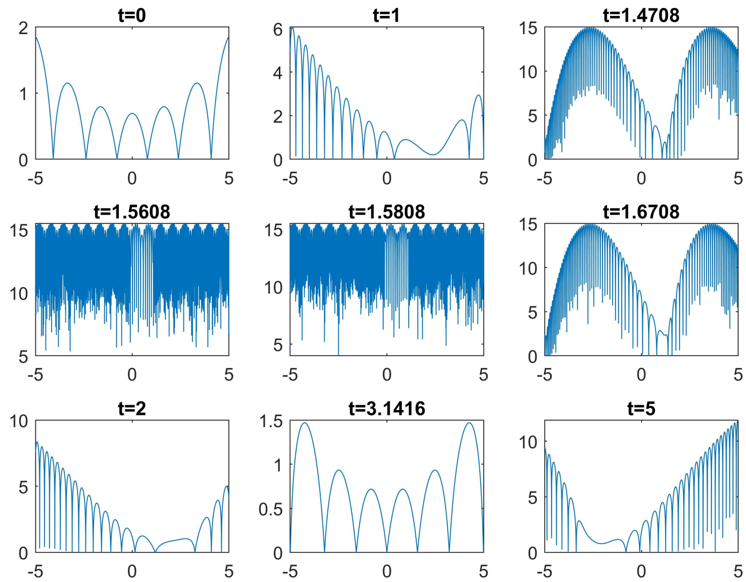

Figure 4.2: $\log(1 + |\mathrm{Re}(\psi_n(x,t)|)$, when $a = 2$, $n = 20$ and t fixed.

4.2 Continuity of the operator

We start performing the second step of the general strategy.

In order to write $\psi_n(t,x)$ in terms of a differential operator applied to $F_n(x)$, we first observe that the exponential $e^{-(i/2)(1-2k/n)^2 \tan t}$ can be expanded in series as

$$e^{-(i/2)(1-2k/n)^2 \tan t} = \sum_{m=0}^{\infty} \frac{[-(i/2)(1-2k/n)^2 \tan t]^m}{m!}$$

and that the following identity holds

$$(-\cos^2 t)^m \frac{\partial^{2m}}{\partial x^{2m}} e^{ix(1-2k/n)/\cos t} = (1-2k/n)^{2m} e^{ix(1-2k/n)/\cos t}.$$

Those observations and some additional computations give that formula (4.1.11) is indeed

$$\psi_n(t,x) = (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{i}{2}\sin t \cos t\right)^m \frac{\partial^{2m}}{\partial x^{2m}} F_n(x/\cos t).$$

So we can define:

$$U\left(\frac{d}{dx}, t\right) := \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{i}{2}\sin t \cos t\right)^m \frac{d^{2m}}{dx^{2m}} \odot H_{1/\cos t}\ , \tag{4.2.1}$$

where the symbol $\odot$ stands for the composition of operators and $H_{1/\cos t}$ is the dilation operator acting on entire functions defined, for $\alpha \in \mathbb{C}$, as $H_\alpha : f \mapsto f(\alpha(\cdot))$.

Now we are ready to perform step 3, that is the study of the continuity of the operator U we have just found.

In this section we consider this problem under two different points of view: in Section 4.2.1 we follow the lines of the previous cases, while in Section 4.2.2 we look at the operator like a Fresnel-type integral operator and we prove its continuity in this framework. The aim of all these discussions is to show, in Section 4.2.3, the persistence of superoscillatory behaviour under the Schrödinger equation for the harmonic oscillator.

4.2.1 Classical results

First of all, let us notice that U is a composition of two operators:

$$\tilde{U}\left(\frac{d}{dx}, t\right) := \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{i}{2}\sin t \cos t\right)^m \frac{d^{2m}}{dx^{2m}} \qquad \text{and} \qquad H_{1/\cos t}\ . \tag{4.2.2}$$

$H_{1/\cos t}$ is clearly a continuous operators from $\mathcal{A}_p(\mathbb{C})$ to $\mathcal{A}_p(\mathbb{C})$, for any p. So we can focus only on the continuity of $\tilde{U}$, since the composition of continuous operators is again a continuous operator.

We start using the theory of $\mathcal{A}_p$ spaces and Fourier transform, in particular we invoke Theorem 2.3.19. Thanks to this result one can prove that the complexified version of $\tilde{U}\left(\frac{d}{dx}, t\right)$, i.e. $\tilde{U}\left(\frac{d}{dz}, t\right)$, is continuous on the space $A_{2,0}$ for any value of t (for a more detailed treatment I refer the reader to step 3 of Section 3.2).

Moreover, we can reach similar conclusions also restricting our tools and using only the theory of holomorphic functions. In particular, we can apply Theorem 3.1.1 with $\lambda(t) = \left(\frac{i}{2}\sin t\cos t\right)$ and $p = 2$ and conclude that $\tilde{U}\left(\frac{d}{dx}, t\right)$ acts continuously also from $\mathcal{A}_1(\mathbb{C})$ to itself.

Both the above continuity results hold for every t, but they do not treat the issue of the uniform continuity with respect to t. In order to take into account also this feature we apply Lemma 3.1.2 with $b_j(t) = \frac{1}{m!}\left(\frac{i}{2}\sin t\cos t\right)^m$ and $\mathcal{T} = \mathbb{R}$.

4.2.2 Continuity of Fresnel-type integral operators

We could talk also about the continuity of Fresnel-type integral operators involved in the process to find a solution for the equation.

In order to do that we need a preliminary lemma (Lemma 4.2.1), that is an extension of Lemma 3.1.2 and an ad-hoc continuity theorem for the Fresnel-type integral (Theorem 4.2.2).

Lemma 4.2.1. *Let $\mathscr{T}$ be a set of parameters and $t \in \mathscr{T} \mapsto U(t, Z)$ be a differential operator-valued map*

$$t \in \mathscr{T} \longmapsto U(t, Z) = \sum_{j=0}^{\infty} b_j(t, Z)\left(\frac{d}{dZ}\right)^j$$

(with $b_j : \mathscr{T} \times \mathbb{C} \to \mathbb{C}$ holomorphic in Z for $j \in \mathbb{N}$) whose formal symbol

$$h : (t, Z, W) \in \mathcal{T} \times \mathbb{C} \times \mathbb{C} \longmapsto \sum_{j=0}^{\infty} b_j(t, Z) W^j$$

is such that

$$\forall\, \varepsilon > 0, \quad \sup_{t \in \mathscr{T}, (Z,W) \in \mathbb{C}^2} \left(\left(\sum_{j=0}^{\infty} |b_j(t, Z)|\, |W|^j\right) \exp(-\varepsilon\, |Z|^q - B\, |W|^p)\right) = A^{(\varepsilon)} < +\infty \tag{4.2.3}$$

for some $q > 1$, $p > 1$ and $B \geq 0$. Then $U(t, Z)$ acts as a continuous operator from $\mathcal{A}_1(\mathbb{C})$ into $\mathcal{A}_{q,0}(\mathbb{C})$ uniformly with respect to the parameter $t \in \mathscr{T}$.

Proof. As in Lemma 3.1.2 the proof is structured in two steps: before one proves that $U(t,Z)$ acts from $\mathcal{A}_1(\mathbb{C})$ into $\mathcal{A}_{q,0}(\mathbb{C})$, and then one treats the continuity.

The function

$$h\,:(t,Z,W)\longmapsto \sum_{j=0}^{\infty}\underbrace{\Big(\sum_{k=0}^{\infty}b_{j,k}(t)Z^k\Big)}_{b_j(t,Z)}W^j=\sum_{k=0}^{\infty}Z^k\Big(\sum_{j=0}^{\infty}b_{j,k}(t)W^j\Big) \tag{4.2.4}$$

is well defined since $b_j(t,Z)$ is holomorphic in Z, and it is an entire function of two variables (Z and W), this also justifies in (4.2.4) the application of Fubini theorem.
Cauchy formulae in $\mathbb{C}\times\mathbb{C}$ shows that for any $t\in\mathscr{T}$, for any $j,\,k\in\mathbb{N}$,

$$\begin{aligned}|b_{j,k}(t)| &= \frac{1}{4\pi^2}\Big|\int_{|Z|=r_z,|W|=r_w}h(t,Z,W)\frac{dZ}{Z^{k+1}}\wedge\frac{dW}{W^{j+1}}\Big|\\ &\le A^{(\varepsilon)}\inf_{r_z>0}\frac{e^{\varepsilon r_z^q}}{r_z^k}\times\inf_{r_w>0}\frac{e^{Br_w^p}}{r_w^j}\\ &= A^{(\varepsilon)}\Big(\frac{1}{k}\Big)^{k/q}\times\Big(\frac{1}{j}\Big)^{j/p}(\varepsilon qe)^{k/q}(Bpe)^{j/p}\\ &\le C_\eta\frac{1}{\Gamma(k/q+1)\Gamma(j/p+1)}(\eta d)^k b^j\end{aligned} \tag{4.2.5}$$

for each $\eta>0$, with constants C_η, β and b independent on the parameter t.

Let now $\boldsymbol{f}=\{f_N\}_{N\ge1}$ be a sequence of elements in $\mathcal{A}_1(\mathbb{C})$ which converges to 0 in $\mathcal{A}_1(\mathbb{C})$. All differential operators

$$U_k(t):=\sum_{j=0}^{\infty}b_{j,k}(t)\Big(\frac{d}{dW}\Big)^j \qquad (k\in\mathbb{N})$$

act continuously on $\mathcal{A}_1(\mathbb{C})$, as seen Lemma 3.1.2. Moreover, recalling (3.1.4) and (3.1.5), one has that for any $f\in\mathcal{A}_1^{\gamma,\beta}(\mathbb{C})$ (see Definition 2.3.12), we can plug in (3.1.5) the estimates (4.2.5) and obtain $\forall t\in\mathcal{T},\ \forall k\in\mathbb{N},\ \forall\ell\in\mathbb{N}$ that the coefficients $\tilde{\alpha}_\ell$ of $U_k(t)(f)(W)$ satisfy:

$$\begin{aligned}\tilde{\alpha}_\ell &\le \gamma\,C\sum_{j=0}^{\infty}C_\eta\frac{1}{\Gamma(k/q+1)\Gamma(j/p+1)}(\eta d)^k\,b^j\,\frac{\beta^{\ell+j}}{\ell!}\\ &\le \gamma\,\tilde{C}_\eta\,\frac{(\eta d)^k}{\Gamma(k/q+1)}E_{1/p,1}(\beta\,b)\frac{\beta^\ell}{\ell!},\end{aligned}$$

where

$$E_{1/p,1}(\xi):=\sum_{j=0}^{\infty}\frac{\xi^j}{\Gamma(j/p+1)}$$

is the entire (with order $1/p$ and type 1) Mittag-Leffler function (see Appendix A). One has therefore for such $f \in \mathcal{A}^1_{\gamma,\beta}(\mathbb{C})$ that $\forall t \in \mathcal{T},\ \forall k \in \mathbb{N},\ \forall W \in \mathbb{C}$,

$$|U_k(t)(f)(W)| \le \gamma\, \tilde{C}_\eta\, E_{1/p,1}(\beta\, b)\, \frac{e^{\beta|W|}}{\Gamma(k/q+1)}.$$

Taking now $W = Z$ we have that $\forall t \in \mathcal{T},\ \forall W \in \mathbb{C}$,

$$\sum_{k=0}^{\infty} |U_k(t)(f)(Z)|\,|Z|^k \le \gamma\, \tilde{C}_\eta\, E_{1/p,1}(\beta\, b)\, e^{\beta|Z|} \underbrace{\sum_{k=0}^{\infty} \frac{(\eta\, b\,|Z|)^k}{\Gamma(k/q+1)}}_{E_{1/q,1}(\eta\, d\,|Z|)}.$$

Since the Mittag-Leffer function $E_{1/q,1}(\eta\, d\,|Z|)$ has order $q > 1$, the estimates above (uniform in the parameter t as well as in the function $f \in \mathcal{A}^1_{\gamma,\beta}(\mathbb{C})$) show that the differential operator acts from $\mathcal{A}_1(\mathbb{C})$ into $\mathcal{A}_{q,0}(\mathbb{C})$, for any $t \in \mathcal{T}$.

In order to prove that $U(t,Z)$ acts continuously one just needs to repeat here the end of the proof of Lemma 3.1.2. □

Now we have all the tools needed to prove the continuity of Fresnel-type integral operators on our spaces.

Let $\mathcal{T}$ be a set of parameters and $t \in \mathcal{T} \longmapsto U(t,Z)$ be a differential operator-valued map that satisfies all the hypotheses in Lemma 4.2.1. Let also ϕ be a non-vanishing real function on $\mathcal{T}$ and $\chi > -1$.

It follows from the estimates (4.2.3), together with Lemma 4.1.1, that the regularization approach described in Section 4.1 allows to define the operator

$$t \quad\longmapsto\quad \int_0^{\infty} Z^{\chi}\, e^{-i\phi(t)Z^2} \sum_{j=0}^{\infty} b_j(t,Z)\Big(\frac{d}{dZ}\Big)^j (\cdot)\, dZ.$$

One needs to consider for the moment these operators as acting on entire functions of the complex variable Z. The discussion is with respect to the sign of $\phi(t)$.

- When $\phi(t) > 0$,

$$\begin{aligned}&\int_0^{\infty} Z^{\chi} e^{-i\phi(t)Z^2} \sum_{j=0}^{\infty} b_j(t,Z)\Big(\frac{d}{dZ}\Big)^j (\cdot) dZ \\ &= e^{-i(1+\chi)\pi/4} \int_0^{\infty} y^{\chi} e^{-\phi(t)y^2} \Big(\sum_{j=0}^{\infty} b_j(t, e^{-i\pi/4}Z)\Big(e^{ij\pi/4}\Big(\frac{d}{dZ}\Big)^j \odot H_{e^{-i\pi/4}}\Big)(\cdot)\Big)(y) dZ.\end{aligned} \tag{4.2.6}$$

- When $\phi(t) < 0$,

$$\begin{aligned}&\int_0^\infty Z^\chi e^{-i\phi(t)Z^2}\sum_{j=0}^\infty b_j(t,Z)\Big(\frac{d}{dZ}\Big)^j(\cdot)dZ\\&= e^{i(1+\chi)\pi/4}\int_0^\infty y^\chi e^{\phi(t)y^2}\Big(\sum_{j=0}^\infty b_j(t,e^{i\pi/4}Z)\Big(e^{-ij\pi/4}\Big(\frac{d}{dZ}\Big)^j\odot H_{e^{i\pi/4}}\Big)(\cdot)\Big)(y)dZ.\end{aligned}\tag{4.2.7}$$

Theorem 4.2.2. *Suppose that the parameter space $\mathcal{T}$ is a topological space and that ϕ is continuous. Consider functions $B_j : \mathcal{T}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ ($j\in\mathbb{N}$) which are entire in the two complex entries and such that*

$$\begin{aligned}&\forall\varepsilon>0,\ \exists A^{(\varepsilon)},\ \exists B_\varepsilon\geq 0\ \text{such that}\ \forall t\in\mathcal{T},\ \forall Z\in\mathbb{C},\ \forall Y\in\mathbb{C},\ \forall W\in\mathbb{C},\\&\qquad\sum_{j=0}^\infty |B_j(t,Z,Y)|\,|W|^j\leq A^{(\varepsilon)}\,e^{\varepsilon|Z|^q+B_\varepsilon|Y|^q+B|W|^p},\end{aligned}\tag{4.2.8}$$

for some $p>1$, $q\in(1,2]$ and $B>0$. Then the operator

$$\int_0^\infty Z^\chi\,e^{i\phi(t)Z^2}\Big(\sum_{j=0}^\infty |B_j(t,Z,Y)|\Big(\frac{d}{dZ}\Big)^j(\cdot)\Big)dZ$$

(understood through the process of regularization) acts continuously locally uniformly in t from $\mathcal{A}_1(\mathbb{C})$ into $\mathcal{A}_{q,0}(\mathbb{C})$.

Proof. It is enough to consider $\mathcal{T}$ as a neighborhood of a point t_0 in which $\phi(t)\geq\varepsilon_0>0$ (since ϕ is continuous).

Let $\boldsymbol{f}=\{Z\mapsto f_n(Z)\}_{n\geq1}$ be a sequence of elements in $\mathcal{A}_1(\mathbb{C})$ that converges towards 0 in $\mathcal{A}_1(\mathbb{C})$, which means (see Proposition 2.3.9) that all f_n belong to some $\mathcal{A}_1^{C,b}(\mathbb{C})$ for some constants $C,b>0$ independent on N (namely $f_n=\sum_\ell a_{n,\ell}Z^\ell$ with $|a_{n,\ell}|\leq Cb^\ell/\ell!$). It is clear that the operator

$$H=\sum_{j=0}^\infty B_j(t,e^{-i\pi/4}Z,Y)\Big(e^{ij\pi/4}\Big(\frac{d}{dZ}\Big)^j\odot H_{e^{-i\pi/4}}\Big)$$

involved in the integrand of (4.2.6) is governed by estimates of the form (4.2.8).

For each $n\in\mathbb{N}^*$ the function that we obtain applying this operator to f_n is

$$\begin{aligned}&H(f_n)\,(t,Z,Y)\in\mathcal{T}\times\mathbb{C}\times\mathbb{C}\\&\qquad\longmapsto\sum_{j=0}^\infty B_j(t,e^{-i\pi/4}Z,Y)\Big(e^{ij\pi/4}\Big(\frac{d}{dZ}\Big)^j\odot H_{e^{-i\pi/4}}\Big)(f_n)(Z).\end{aligned}$$

It follows then from Lemma 4.2.1, taking into account estimates (4.2.8), that $H(f_n)\in\mathcal{A}_{q,0}(\mathbb{C})$ whether it is considered as an entire function in the complex variable Z or it is considered entire in Y. This means that for each $\varepsilon > 0$, there exists $\tilde{A}^{(\varepsilon)} \geq 0$ (depending on $\mathcal{T}$, A_ε, the B_j, b and C, but not on the n) such that

$$\forall (t,Z,Y) \in \mathcal{T} \times \mathbb{C} \times \mathbb{C}, \qquad |H(f_n)(t,Z,Y)| \leq \tilde{A}^{(\varepsilon)}\, e^{\varepsilon|Z|^q + B_\varepsilon |Y|^q}.$$

Take in particular $\varepsilon < \varepsilon_0$, so it holds $\phi(t) \geq \varepsilon_0 > \varepsilon$. Then the function

$$Y \in \mathbb{C} \longmapsto \int_0^\infty y^\chi\, e^{-\phi(t)y^2} H(f_n)(t,y,Y)\, dy$$

is in $\mathcal{A}_{q,0}(\mathbb{C})$ since it is estimated as

$$\Big| \int_0^\infty y^\chi\, e^{-\phi(t)y^2} H(f_n)(t,y,Y)\, dy \Big| \leq \tilde{A}^{(\varepsilon)} \Big(\int_0^\infty y^\chi\, e^{-\varepsilon_0 y^2}\, e^{\varepsilon y^q}\, dy \Big)\, e^{B_\varepsilon |Y|^q} \qquad \forall\, Y \in \mathbb{C}$$

(remember that $q \in (1,2]$ and then the integral is convergent). It remains to show that the sequence

$$\Big\{ Y \longmapsto \int_0^\infty y^\chi\, e^{-\phi(t)y^2} H(f_n)(t,y,Y)\, dy \Big\}_{n \geq 1}$$

converges to 0 in $\mathcal{A}_{q,0}(\mathbb{C})$. It is enough (see Proposition 2.3.9) to prove that it converges to 0 in $\mathcal{H}(\mathbb{C})$, in other words we should prove the convergence uniformly on any closed disk $\overline{D(0,r)}$ in $\mathbb{C}$.

Fix $\varepsilon < \varepsilon_0$ and $\eta > 0$. Choose then $R_\eta >> 1$ such that

$$\begin{aligned} \forall\, n \in \mathbb{N}, \qquad & \Big| \int_{R_\eta}^\infty y^\chi\, e^{-\phi(t)y^2} H(f_n)(t,y,Y)\, dy \Big| \\ & \leq \tilde{A}^{(\varepsilon)} \Big(\int_0^\infty y^\chi\, e^{-\varepsilon_0 y^2}\, e^{\varepsilon y^q}\, dy \Big)\, e^{B_\varepsilon |Y|^q} \\ & \leq \eta\, e^{-B_\varepsilon r^q}\, e^{B_\varepsilon |Y|^q} \leq \eta \qquad \forall\, Y \in \overline{D(0,r)}. \end{aligned}$$

On $[0, R_\eta]$, one uses the uniform convergence of $\boldsymbol{f}$ towards 0 on any compact set, hence of $H[\boldsymbol{f}]$ on any compact set, to conclude that for $n \geq N_\eta >> 1$, one has

$$\Big| \int_0^{R_\eta} y^\chi\, e^{-\phi(t)y^2} H(f_n)(t,y,Y)\, dy \Big| \leq \eta \qquad \forall\, Y \in \overline{D(0,r)}.$$

Note that our estimates show that the convergence towards 0 in $\mathcal{A}_{q,0}(\mathbb{C})$ thus obtained is uniform in $t \in \mathcal{T}$. □

Let us apply this theorem in our setting, where the Fresnel integral operator is defined through the Green's function. In particular we have that the solution of the Cauchy problem we are examining in this chapter (4.1.7) is

$$\begin{aligned}\psi_a(t,x) &= (2\pi i \sin t)^{-1/2} \int_{\mathbb{R}} \exp\Big(\frac{2xx' - (x^2 + x'^2)\cos t}{2i\sin t}\Big)\, e^{iax'}\, dx' \\ &= (2\pi i \sin t)^{-1/2}\, \exp\Big(i\frac{\text{cotan } t}{2}x^2\Big) \\ &\quad \times \int_{\mathbb{R}} \exp\Big(i\frac{\text{cotan } t}{2}(x')^2\Big)\exp\Big(-i\frac{x\,x'}{\sin t}\Big)\exp(iax')\, dx'.\end{aligned}$$

So the Fresnel integral operator we want to study is:

$$t \in (0,+\infty)\setminus \pi\mathbb{N}^*/2 \quad \longmapsto \quad \int_{\mathbb{R}} \exp\Big(i\frac{\text{cotan } t}{2}Z^2\Big)\exp\Big(-i\frac{Y\,Z}{\sin t}\Big)(\cdot)\, dZ.$$

In order to write it in a suitable form for Theorem 4.2.2, we perform a change of variables $Z \leftrightarrow |\sin t| Z$ on $[0,+\infty)$ and split accordingly to $\omega = \pm 1$, so we obtain

$$t \in (0,+\infty)\setminus \pi\mathbb{N}^*/2 \;\longmapsto\; |\sin t| \int_{\mathbb{R}} \exp\Big(i\frac{\sin 2t}{4}Z^2\Big)\exp(-i\,\omega\ \text{sign}(\sin t))YZ) \odot H_{\omega|\sin t|}(\cdot)\, dZ. \tag{4.2.9}$$

Set now

$$\begin{aligned}&\mathcal{T} = (0,+\infty)\setminus \pi\mathbb{N}^*/2\,,\\ &\phi\,: t \in \mathcal{T} \longmapsto -\frac{\sin(2t)}{4}\,,\\ &\chi = 0\,,\\ &B_j\,: (t,Z,Y) \in \mathcal{T}\times\mathbb{C}\times\mathbb{C} \longmapsto \begin{cases}\exp(-i\,\omega\ \text{sign}(\sin t))ZY) \odot H_{\omega|\sin t|} & \text{if } j=0\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{4.2.10}$$

Theorem 4.2.2 applies here with $p = 1$ and $q = 2$ and the two operators (4.2.9) act continuously from $\mathcal{A}_1(\mathbb{C})$ to $\mathcal{A}_2(\mathbb{C})$ (locally uniformly with respect to the parameter $t \in \mathcal{T}$).

Note again that where the Fresnel-type integral operators (4.2.9) are applied to $Z \mapsto e^{iaZ}$ ($a \in \mathbb{R}$), are semi-convergent and their values as semi-convergent integrals coincide with the values that are obtained by regularization in Section 4.1.

4.2.3 Persistence of superoscillations

We have presented so far four methods one can apply in order to prove the continuity of the operator involved in the solution: three of these are the classical ones presented in Section 4.2.1 and the fourth is the one discussed in Section 4.2.2.

Now we are ready to perform the final step of the strategy summarized in Section 3.1: the computation of the limit when n approaches $+\infty$.

Proposition 4.2.3. *Let* $\mathcal{T} = (0,+\infty) \setminus \dfrac{\pi(2\mathbb{N}+1)}{2}$. *For any* $n \in \mathbb{N}$, *let* $\psi_n(t,x)$ *be the solution of* (4.1.10), *then*

$$\lim_{n\to\infty} \psi_n(t,x) = (\cos t)^{-1/2} \exp(-(i/2)(x^2+a^2)\tan t + iax/\cos t),$$

for any t *in* $\mathcal{T}$.

Proof. Thanks to our continuity results we can compute the limit below as follows

$$\psi(t,x) := \lim_{n\to\infty} \psi_n(t,x) = (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t}\, \tilde{U}\left(t, \frac{d}{dx}\right) \lim_{n\to\infty} F_n(x/\cos t)$$

Let us define $F(x) := e^{iax}$. Then, by Theorem 2.2.2, $F_n(x/\cos t)$ converges uniformly to $F(x/\cos t)$ on the compact set $|x| \le M$, where $M > 0$, for every fixed t in $\mathcal{T}$, so we have

$$\begin{aligned}
\psi(t,x) &= (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t}\, \tilde{U}\left(t, \frac{d}{dx}\right) F(x/\cos t) \\
&= (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{i}{2}\sin t \cos t\right)^m \frac{d^{2m}}{dx^{2m}}\, e^{iax/\cos t} \\
&= (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{i}{2}\sin t \cos t\right)^m \left(ia/\cos t\right)^{2m} e^{iax/\cos t} \\
&= (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t} \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{i}{2}a^2 \tan t\right)^m e^{iax/\cos t}, \\
&= (\cos t)^{-1/2} \exp(-(i/2)(x^2+a^2)\tan t + iax/\cos t).
\end{aligned}$$

□

As always, we can interpret this theorem in different ways and, as consequence of that, we can prove that ψ_n superoscillates according to all the versions of the definition of superoscillation presented in Section 2.1.

For instance, if we consider the convergence of $F_n(x/\cos t)$ to $e^{iax/\cos t}$ simply as uniform convergence on any compact set of $\mathbb{R}$, as we did in the proof, we conclude that the sequence $\psi_n(t,x)$ superoscillates for each $t \in \mathcal{T} = (0,+\infty) \setminus \dfrac{\pi(2\mathbb{N}+1)}{2}$ with superoscillation domain $\mathbb{R}$ and superoscillation sets any compact set in $\mathbb{R}$.

On the other hand, we can interpret $F_n(x/\cos t) \to e^{iax/\cos t}$ in $\mathcal{A}_1(\mathbb{C})$ (see Theorem 2.3.10) and recall that, thanks to Lemma 3.1.2, $\tilde{U}(t, \frac{d}{dz})$ acts continuously from $\mathcal{A}_1(\mathbb{C})$ to $\mathcal{A}_1(\mathbb{C})$ uniformly with respect to $t \in \mathcal{T}$. Then, according to Definition 2.1.4, we have that $\psi_n(t,x)$ is a superoscillating sequence in two variables with superoscillation domain $\mathcal{U} := \mathcal{T} \times \mathbb{R}$, $P_\infty(T,X) = TX$, where $T = e^{ig_1(a)\frac{\tan t}{2}}$ and $X = e^{ig_2(a)\frac{x}{\cos t}}$, $g_1(a) = a^2$ and $g_2(a) = a$.

Finally, we can also prove that the sequence of functions $\left\{\sum_{j=0}^{n} C_j(n,a)(\psi_{1-2j/n})_{|\mathcal{U}}\right\}_{n\geq 1}$ is, for any $a \in \mathbb{R}\setminus[-1,1]$, a supershift for the family $\mathcal{F} = \{(\psi_a)_{|\mathcal{U}}\,;\, a\in\mathbb{R}\}$, see (4.1.8), with $\mathcal{F}$-supershift domain $\mathcal{U}\subset\mathbb{R}^2$.

Since we have considered the solutions also as $\mathbb{C}$-valued distributions $\mu_n \in \mathcal{D}'(\mathcal{T},\mathbb{C})$ (see Corollary 4.1.6), we can claim that the superoscillatory behaviour persists also in the sense of Definition 2.1.6. Indeed, let us recall that $\mu_a \in \mathcal{D}'(\mathcal{T},\mathbb{C})$ is the distribution expressed by the locally integrable function $(\cos t)^{-1/2}e^{-(i/2)x^2\tan t}e^{-(i/2)a^2\tan t}$, with singular support $\dfrac{\pi(2\mathbb{N}+1)}{2}\times\mathbb{R}$ (see Corollary 4.1.4).

Now, let $\mathcal{F} = \{(\mu_a)_{|\mathcal{U}}\,;\, a\in\mathbb{R}\}$ be a family of distributions and let $\{(\sum_{j=0}^{n} C_j(n,a)\mu_{1-2j/n})_{|\mathcal{U}}\}_{n\geq 1}$ be a sequence. Then, thanks to the continuity of the Fresnel integral proved in Theorem 4.2.2 we can conclude that the sequence above converges weakly in the sense of distributions in $\mathcal{U}$ to the restriction of $\mathcal{U}$ of the distribution $\mu_a \in \mathcal{D}'(\mathcal{T},\mathbb{C})$.

Remark 4.2.1. Let us point out that the superoscillations are amplified by the potential and the analytic solution blows up for $t=\pi/2$. Moreover, even when $a\in(-1,1)$, the harmonic oscillator displays a superoscillatory phenomenon since the solution contains the term $\exp(-(i/2)(x^2+a^2)\tan t + iax/\cos t)$, which increases arbitrarily as t approaches $\pi/2$. This is a peculiarity of the case of the harmonic oscillator.

In Fig. 4.3 I plot $\log(1+|\mathrm{Re}(\psi_n(5,t)|)$ fixing $a=2$ and I zoomed near $\frac{\pi}{2}$, while in Fig. 4.4 I plot $\log(1+|\mathrm{Re}(\psi_n(5,t)|)$ fixing $a=0.2$. In both cases I try to display the convergence phenomenon towards the asymptotic function, starting from small $n=20$ and reaching $n=45$. Due to computational limitations, I cannot test greater values of n.

Looking at the plots we can notice that the convergence happens in both cases, when $a>1$ $\psi_n(x)$ move close $\psi_a(x)$ from above, instead when $a<1$ they move close $\psi_a(x)$ from below.

This is not the only effect due to the presence of singularities in the solution, indeed given $k\in\mathbb{N}$ and $x_0\in\mathbb{R}$, it is impossible to interpret

$$\left\{\alpha \mapsto \Big(\sum_{j=0}^{n} C_j(n,a)\mu_{1-2j/n}\Big)_{\text{about }((2k+1)\pi/2,\,x_0)}\right\}_{n\geq 1} \tag{4.2.11}$$

(when $a\in\mathbb{R}\setminus[-1,1]$) as supershift for $a\mapsto(\mu_{1-2j/n})_{\text{about }((2k+1)\pi/2,\,x_0)}$ (all maps being considered here as distribution-valued about $((2k+1)\pi/2,\,x_0)$). In order to interpret (4.2.11) as a super-

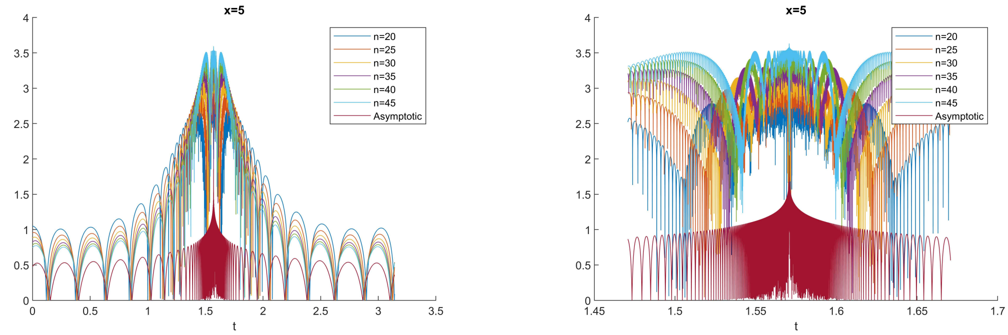

Figure 4.3: $a = 2$.

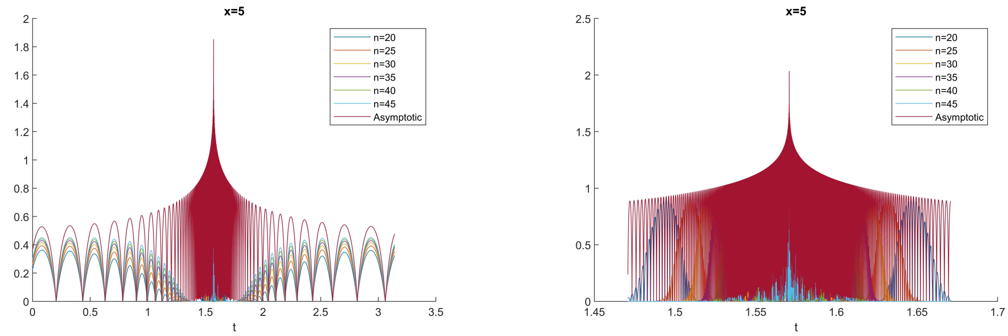

Figure 4.4: $a = 0.2$.

shift for $a \mapsto (\mu_a)_{\text{about }((2k+1)\pi/2,\, x_0)}$, one needs to consider the map $a \mapsto (\mu_a)_{|\text{about }((2k+1)\pi/2,\, x_0)}$ as hyperfunction (in t) times distribution (in x)-valued instead of distribution-valued in (t, x).

4.3 Singularities in the quantum harmonic oscillator evolution

Before starting with the analysis of the singularities affecting the solution founded so far, it is worth to present a quick overview on the theory of hyperfunctions, since these objects turns out to be useful in Section 4.3.2.

4.3.1 Elements of hyperfunction theory

The aim of this section is to give a basic understanding of the hyperfunction theory, I refer the reader who is willing to go deeper inside the topic to see [20].

The birth of the hyperfunction theory goes back to the late 1950s, when the Japanese mathematician M. Sato needed to construct a space of generalized functions which would be the "analytic" equivalent of Schwartz's distributions. His inspiration came from some work in theoretical physics which showed the necessity of dealing with boundary values of holomorphic functions.

Let us recall that with boundary value we mean the following: f has a boundary value, denoted with $f(x+i0)$, in the sense of distributions if $\forall \varphi \in \mathcal{D}((a,b))$ the limit

$$\lim_{\varepsilon \to 0^+} \int_a^b f(x+i\varepsilon)\varphi(x)\,dx = \langle f(x+i0),\, \varphi \rangle$$

exists.

The aim is to generalize the concept of distribution so that this limit always exists.

Let us consider an open set $\Omega \subset \mathbb{R}$; an open set $U \subset \mathbb{C}$ such that Ω is a closed subset of U, is said to be a complex neighborhood of Ω. Let us consider the complex vector space $\mathcal{H}(U \setminus \Omega)$, its subspace $\mathcal{H}(U)$ and their quotient $\mathcal{H}(U \setminus \Omega)/\mathcal{H}(U)$.

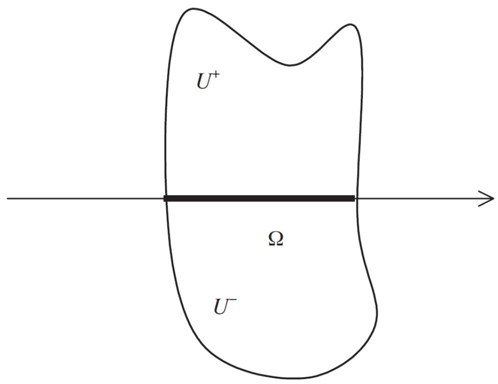

Figure 4.5: Representation of U and Ω in Gauss plane.

We will define a hyperfunction $f(x)$ as an equivalence class $f(x) = [F(z)]$ in this quotient. Any function $F(z)$ in the equivalence class is said to be a defining function for $f(x)$. More precisely, if $f(x) = [F(z)] = [G(z)]$ then $F, G \in \mathcal{H}(U \setminus \Omega)$ and $F - G \in \mathcal{H}(U)$.

Let us notice that the quotient $\mathcal{H}(U \setminus \Omega)/\mathcal{H}(U)$ depends a priori on the choice of the open set $U \subset \mathbb{C}$, but it can be shown that this is not the case., in fact for any U, $V \subset \mathbb{C}$

$$\frac{\mathcal{H}(U \setminus \Omega)}{\mathcal{H}(U)} \simeq \frac{\mathcal{H}(V \setminus \Omega)}{\mathcal{H}(V)}.$$

Definition 4.3.1. *(Space of hyperfunctions)*

Let Ω be an open set in $\mathbb{R}$. The vector space of hyperfunctions on Ω is defined as

$$\mathcal{B}(\Omega) = \frac{\mathcal{H}(U \setminus \Omega)}{\mathcal{H}(U)}, \tag{4.3.1}$$

where U is any complex neighborhood of Ω.

Remark 4.3.1. Let $F \in \mathcal{H}(U \setminus \Omega)$ and denote by $f = [F]$ the hyperfunction f defined by the quotient (4.3.1). If the function F is holomorphic at every point of Ω, then f is the zero hyperfunction (due to Liuville's theorem). Note, however, that it is not possible to speak about the value of a hyperfunction at a given point, so it is not correct to think of $f(x) = 0$ as a numerical value.

Before giving some examples of hyperfunctions, it is convenient to define some elementary operations on them, besides the operation of sum and multiplication by a complex number that are naturally implied by the vector space structure.

Note that, since a hyperfunction is determined by the equivalence class of a function $F \in \mathcal{H}(U \setminus \Omega)$, we can set $U \setminus \Omega = U^+ \cup U^-$ with $U^\pm = U \cup \{\pm \mathrm{Im} z > 0\}$ and $F = (F^+, F^-)$, $F^\pm \in \mathcal{H}(U^\pm)$ so that the hyperfunction f can be represented by the pair (F^+, F^-).

Definition 4.3.2. *(Multiplication of a hyperfunction by a real analytic function)*

Let $\phi(x)$ be a real analytic function on Ω and let $f = [F] \in \mathcal{B}(\Omega)$.

We define $\phi(x) f(x) = [\phi(z) F(z)]$ where $\phi(z)$ is an analytic continuation of $\phi(x)$.

Definition 4.3.3. *(Differentation)*

We define

$$\frac{d^n}{dx^n} f(x) = \left[\frac{d^n}{dz^n} F(z)\right].$$

It is possible to define the notion of definite integral for hyperfunctions: let f be a hyperfunction defined on a neighborhood of the interval $[a, b]$ and let f be real analytic in a neighborhood of each of the two endpoints of the interval. Let $F = (F^+, F^-)$, be a defining function for f: by definition both F^+ and F^- can be analytically continued to some neighborhoods of the points a and b. Let $\gamma^\pm \subset U^\pm$ be piece-wise smooth arcs connecting the points a, b, then (see Fig. 4.6)

$$\int_a^b f(x)\, dx = \int_{\gamma^+} F^+(z)\, dz - \int_{\gamma^-} F^-(z)\, dz.$$

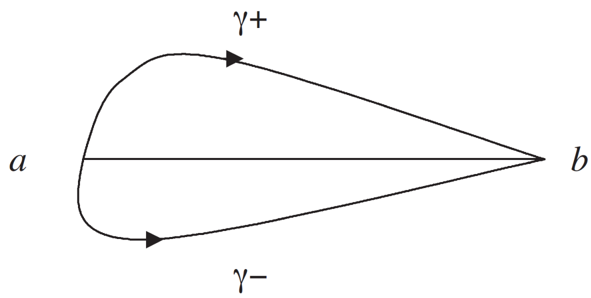

Figure 4.6

The definition is not affected by the choices we made.

Definition 4.3.4. *(Integration)*
Let $F(z) \in \mathcal{H}(U \setminus \Omega)$ be a defining function for a hyperfunction $f(x)$ supported in $K \subset \mathbb{R}$, where U is any complex neighborhood of K. Let $\gamma \subset U$ be a closed, piece-wise smooth curve encircling once K. We will assume γ oriented counterclockwise. We define

$$\int_{\mathbb{R}} f(x)\, dx = -\oint_{\gamma} F(z)\, dz.$$

Using Cauchy's integral theorem, it is immediate to verify that the notion is independent of the choices of F, U and γ.

Remark 4.3.2. Let us recall that the Cauchy principal value is a method for assigning values to certain improper integrals which would otherwise be undefined. In particular, if integrand f has a singularity at the finite number b, the Cauchy principal value is defined as follows

$$\lim_{\varepsilon \to 0^+} \left[\int_a^{b-\varepsilon} f(x)\, dx + \int_{b+\varepsilon}^{c} f(x)\, dx \right]$$

where b is a point at which the behaviour of the function f is such that the integral from a to b or from b to c diverges and $a, c \in [-\infty, +\infty]$. In particular let us notice that the principal value of $\frac{1}{x}$ can be consider as a map acting on functions on $\mathcal{D}(\mathbb{R})$, then thanks to the Cauchy principal value method we can describe the effect of $p.v. \left(\frac{1}{x}\right)$ on $u \in \mathcal{D}(\mathbb{R})$ as

$$p.v. \left(\frac{1}{x}\right) u(x) = \lim_{\varepsilon \to 0^+} \left[\int_{-\infty}^{-\varepsilon} \frac{u(x)}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{u(x)}{x}\, dx \right].$$

It can be proven that $p.v. \left(\frac{1}{x}\right)$ belongs to $\mathcal{D}'(\mathbb{R})$, i.e. it is a distribution.

Let us present some examples of hyperfunction.

Remark 4.3.3. *Dirac delta hyperfunction*

The Dirac delta can be defined as the hyperfunction

$$\delta(x) = \left[-\frac{1}{2\pi i}\frac{1}{z} \right]$$

Since $\frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, $\delta(x) = 0$ on $\mathbb{R} \setminus \{0\}$ so its support is the origin. Let $\phi(x)$ be a real analytic function, then the chain of equalities

$$\int_{\mathbb{R}} \phi(x)\delta(x)\,dx = \frac{1}{2\pi i}\int_{\gamma} \frac{\phi(x)}{z}\,dz = \phi(0)$$

proves, thanks to Cauchy formula, that the δ hyperfunction behaves like the δ-distribution. In particular, considering $\phi(x) \equiv 1$, we have

$$\int_{\mathbb{R}} \delta(x)\,dx = 1.$$

Moreover, we also have

$$\frac{d}{dx}\delta(x) = \left[\frac{1}{2\pi i}\frac{1}{z^2} \right]$$

and, in general,

$$\frac{d^n}{dx^n}\delta(x) = \left[-\frac{(-1)^n\, n!}{2\pi i}\frac{1}{z^{n+1}} \right].$$

Remark 4.3.4. *Unit hyperfunction*

Consider now the following hyperfunctions:

$$\varepsilon(z) = \begin{cases} 0 & \text{if } \mathrm{Im} z > 0 \\ 1 & \text{if } \mathrm{Im} z < 0 \end{cases} \qquad \bar{\varepsilon}(z) = \begin{cases} 0 & \text{if } \mathrm{Im} z > 0 \\ -1 & \text{if } \mathrm{Im} z < 0. \end{cases}$$

The hyperfunction associated to ε is defined on $\mathbb{R}$ and it can be seen as the unit hyperfunction 1 if we think of a hyperfunction as the difference of boundary values of holomorphic functions. We obviously have $[\varepsilon] = [\bar{\varepsilon}]$.

Remark 4.3.5. A function F holomorphic on $U \setminus \Omega$ defines a hyperfunction $f = [F]$ that can be realized as the boundary value of F as follows. Let us set

$$F(x + i0) = [F \cdot \varepsilon], \qquad F(x - i0) = [F \cdot \bar{\varepsilon}].$$

We have

$$f(x) = F(x + i0) - F(x - i0).$$

Remark 4.3.6. If F is any function that is holomorphic everywhere except for an essential singularity at 0 (for example $e^{1/z}$), then $f = [F]$ is a hyperfunction with support 0 that is not a distribution.

Instead, if f has a pole of finite order at 0 then $f = [F]$ is a distribution, so when f has an essential singularity then f looks like a "distribution of infinite order" at 0. (Note that distributions always have finite order at any point).

The first is indeed the case we are analysing, recalling Corollary 4.1.4, $(t, x) \in (0, +\infty) \times \mathbb{R} \longmapsto (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t} e^{-(i/2)a^2 \tan t}$ is holomorphic everywhere in $\mathbb{R}^2$ except for the countable set of essential singularities at $\frac{2\mathbb{N}+1}{2}\pi \times \mathbb{R}$.

Since we have a definition of differentiation on the space of hyperfunctions, we can talk about differential equations and differential operators.

Lemma 4.3.5. *Let Ω be an open interval in $\mathbb{R}$ and let $f \in \mathcal{B}(\Omega)$. Then $\frac{d^n}{dx^n} f = 0$ if and only if f is a polynomial of degree less than n.*

Proposition 4.3.6. *Let Ω be an open interval in $\mathbb{R}$ and let $g \in \mathcal{B}(\Omega)$. Then the ordinary differential equation*

$$\frac{d^n}{dx^n} f = g$$

has a hyperfunction solution f on Ω unique up to polynomials of degree less than n.

Proof. Let us consider a representative $G \in \mathcal{H}(U \setminus \Omega)$ of g; it is not reductive to suppose that U, U^+ and U^- are simply connected. By the monodromy principle (let us recall that the monodromy theorem makes possible to extend an analytic function to a larger set via curves connecting a point in the original domain of the function to points in the larger set) there exists $F \in \mathcal{H}(U \setminus \Omega)$ such that $\frac{d^n}{dx^n} F = G$ on $U \setminus \Omega$). It is clear that $f = [F]$ is a solution to the differential equation. The uniqueness up to polynomials of degree less than n follows by Lemma 4.3.5. □

More generally, let us consider the operator

$$P\left(x, \frac{d}{dx}\right) = \sum_{i=0}^{m} a_i(x) \frac{d^i}{dx^i} \tag{4.3.2}$$

where $a_i \in \mathcal{A}(\Omega)$ (space of real analytic functions) for $i = 0, \ldots, m$, $a_m(x) \not\equiv 0$ and let $f = [F] \in \mathcal{B}(\Omega)$. Let U be an open set to which all the a_i can be holomorphically extended, and let

$F \in \mathcal{H}(U \setminus \Omega)$. We define

$$P\left(x, \frac{d}{dx}\right) f = \left[P\left(z, \frac{d}{dz}\right) F\right]$$

where $P\left(z, \frac{d}{dz}\right)$ is the complexified version of $P\left(x, \frac{d}{dx}\right)$. It is easy to check that this definition does not depend on the choice of the open set U, the defining function F and the extensions $a_i(z)$ of $a_i(x)$.

We have the following result about the existence of hyperfunction solutions to the differential equation

$$P\left(x, \frac{d}{dx}\right) f = g. \tag{4.3.3}$$

Theorem 4.3.7. *For any $g \in \mathcal{B}(\Omega)$ there exists a solution $f \in \mathcal{B}(\Omega)$ to the differential equation* (4.3.3).

Proof. The proof is similar to the one of Proposition 4.3.6. □

Now we wish to consider differential operators more general than (4.3.2). As it is well known, every differential operator on distributions is a finite sum of convolutions with the Dirac delta and its derivatives. On the other hand, one of the great advantages of the hyperfunctions is that they allow the use of a larger class of differential operators in which infinitely many derivatives can be considered. More generally, we can consider the operator

$$P\left(x, \frac{d}{dx}\right) = \sum_{i=0}^{\infty} a_i(x) \frac{d^i}{dx^i} \tag{4.3.4}$$

where $\{a_i(x)\}_{i \in \mathbb{N}}$ is a sequence of functions. In order for this operator to act on a hyperfunction $f = [F]$ supported at the origin, we need to ensure that when $F(z)$ is holomorphic in a neighborhood of $\{0\}$, then

$$P\left(z, \frac{d}{dz}\right) F = \sum_{i=0}^{\infty} a_i(z) \frac{d^i}{dz^i} F(z) \tag{4.3.5}$$

is holomorphic in U, so that we can define

$$P\left(x, \frac{d}{dx}\right) f = \left[P\left(z, \frac{d}{dz}\right) F(z)\right].$$

The series on the right-hand side of (4.3.5) converges under suitable growth conditions on the coefficients $a_i(z)$, that turn out to be the ones such that make the operator an infinite order differential operator (recall Definition 2.3.18). That means that for every compact set $K \subset \mathbb{C}$,

$$\lim_{i\to\infty} \sqrt[i]{\sup_{z\in K} |a_i(z)|\, i!} = 0. \tag{4.3.6}$$

So, it turns out that a hyperfunction on a compact set K is nothing but a locally analytic functional on K. It is then natural to compare distributions and hyperfunctions. Since there is a canonical injection $i : \mathcal{A}(\mathbb{R}) \hookrightarrow \mathcal{C}^\infty(\mathbb{R})$ with dense image, there is also an injective dual map $i^* : (\mathcal{C}^\infty)'(\mathbb{R}) \hookrightarrow \mathcal{A}'(\mathbb{R})$ preserving the supports (where $(\mathcal{C}^\infty)'(\mathbb{R})$ represents the set of compactly supported distributions, usually denoted with $\mathcal{E}'(\mathbb{R})$).

Remark 4.3.7. Let K be a compact set in $\mathbb{R}$ and let U be any complex neighborhood of K. We have the following isomorphism

$$\mathcal{B}[K] \cong \frac{\mathcal{H}(U \setminus K)}{\mathcal{H}(U)}.$$

Indeed, let $\Omega \in \mathbb{R}$ be any open set containing K, then f can be represented by an element in $\mathcal{B}(\Omega)$ by extending it to zero outside K.

Thanks to the previous remark, we can claim the following:

Theorem 4.3.8. *Let K be a compact set in $\mathbb{R}$. Then we have*

$$\mathcal{A}'(K) \cong \mathcal{B}[K].$$

In particular, it is possible to show that the following formal inclusions

$$\mathcal{A} \hookrightarrow \mathcal{L}^1_{loc} \hookrightarrow \mathcal{D}' \hookrightarrow \mathcal{B}$$

hold.

4.3.2 Application to the case on harmonic oscillator

Let $\theta(t,x) \in \mathcal{D}(\mathcal{T},\mathbb{C})$ be a test function with support in a small neighborhood of $\left(\frac{\pi}{2}, x_0\right)$. We want now to study the effect of the distribution μ_a on such a test-function. Since the support of $\theta(t,x)$ contains the singular point $\left(\frac{\pi}{2}, x_0\right)$, we cannot describe the effect of μ_a through a function in L^1_{loc}.

For this purpose, let us then consider instead $\theta\left(\frac{\pi}{2} - t, x_0\right)$ that has support in a small neighborhood of $(0, x_0)$.

We start by writing formally

$$\langle \mu_a, \theta \rangle =$$
$$= \int_{\mathbb{R}^2} \left[\frac{1}{\sqrt{2\pi i \sin t}} \exp\Big(\frac{i\hat{x}^2 \text{cotan } t}{2}\Big) \int_{\mathbb{R}} \exp\Big(\frac{iz^2 \text{cotan } t}{2}\Big) \exp\Big(-\frac{i\hat{x}z}{\sin t}\Big)(e^{ia(\cdot)})\, dz \right]_{\hat{x}=x} \theta(t,x)\, dt\, dx,$$

translating by $\frac{\pi}{2}$ in t and recalling that $\sin(\pi/2 - t) = \cos t$ and $\cos(\pi/2 - t) = \sin t$, we obtain

$$\langle \mu_a, \theta \rangle =$$
$$= -\int_{\mathbb{R}^2} \left[\frac{1}{\sqrt{2\pi i \cos t}} \exp\Big(\frac{i\hat{x}^2 \text{tan} t}{2}\Big) \int_{\mathbb{R}} \exp\Big(\frac{iz^2 \text{tan} t}{2}\Big) \exp\Big(-\frac{i\hat{x}z}{\cos t}\Big)(e^{ia(\cdot)})\, dz \right]_{\hat{x}=x} \theta(\pi/2 - t, x)\, dt\, dx.$$

Let us notice that this expression makes sense, indeed we can express the effect of μ_a on a test-function with support in $(0, x_0)$ by the function in L^1_{loc} founded before and represented between squared brackets above.

Now, let us define $\xi(t,x) = \theta(t,x)\frac{1}{\sqrt{2\pi i}} \exp\Big(\frac{ix^2 \text{cotan } t}{2}\Big)$, then we have

$$\langle \mu_a, \theta \rangle = -\int_{\mathbb{R}^2} \left[\frac{1}{\sqrt{\cos t}} \int_{\mathbb{R}} \exp\Big(\frac{iz^2 \text{tan} t}{2}\Big) \exp\Big(\frac{-i\hat{x}z}{\cos t}\Big)(e^{ia(\cdot)})\, dz \right]_{\hat{x}=x} \xi(\pi/2 - t, x)\, dt\, dx.$$

Substituting $z \leftrightarrow \sqrt{\cos t}\; z$ we obtain

$$\langle \mu_a, \theta \rangle = -\int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} \exp\Big(\frac{iz^2 \sin t}{2}\Big) \exp\Big(\frac{-i\hat{x}z}{\sqrt{\cos t}}\Big)(e^{ia(\cdot)})\, dz \right]_{\hat{x}=x} \xi(\pi/2 - t, x)\, dt\, dx.$$

And performing another substitution, i.e. $x \leftrightarrow \sqrt{\cos t}\; x$, we have

$$\langle \mu_a, \theta \rangle = -\int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} \exp\Big(\frac{iz^2 \sin t}{2}\Big)\; e^{-i\hat{x}z}\; (e^{ia(\cdot)})\, dz \right]_{\hat{x}=x} \xi(\pi/2 - t, x)\sqrt{\cos t}\, dt\, dx.$$

Finally, we can define $\tilde{\xi}(u,x) = -\sqrt{\cos t}\; \xi(\pi/2 - t, \sqrt{\cos t} x_0)$ and rewrite the previous expression as

$$\langle \mu_a, \theta \rangle = \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} \exp\Big(\frac{iz^2 \sin t}{2}\Big)\; e^{-i\hat{x}z}\; (e^{ia(\cdot)})\, dz \right]_{\hat{x}=x} \tilde{\xi}(u,x)\, du\, dx. \tag{4.3.7}$$

In order to compute the value of the Fresnel integral between squared brackets we use the same trick we used in Proposition 4.1.3, that means that we rewrite the exponent as

$$iz^2 \frac{\sin u}{2} - i\hat{x}z + iaz = 2\frac{i(z \sin u/2 - (\hat{x} - a)/2)^2}{\sin u} - 2\frac{i((\hat{x} - a)/2)^2}{\sin u}.$$

We can now rewrite (4.3.7) in the following way

$$\begin{aligned}
\langle \mu_a, \theta\rangle &= \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} e^{\varepsilon z^2} \exp\left(\frac{iz^2 \sin u}{2}\right) e^{-i\hat{x}z}\, e^{iaz}\, dz \right]_{\hat{x}=x} \tilde{\xi}(u,x)\, du\, dx \\
&= \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} e^{\varepsilon z^2} \exp\left(iz^2 \frac{\sin u}{2} - i\hat{x}z + iaz\right) dz \right]_{\hat{x}=x} \tilde{\xi}(u,x)\, du\, dx \\
&= \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} e^{\varepsilon z^2} \exp\left(2\frac{i(z\sin u/2 - (\hat{x}-a)/2)^2}{\sin u} - 2\frac{i((\hat{x}-a)/2)^2}{\sin u}\right) dz \right]_{\hat{x}=x} \tilde{\xi}(u,x)\, du\, dx \\
&= \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}^2} \exp\left(\frac{i(x-a)^2}{2\sin u}\right) \left[\int_{\mathbb{R}} e^{\varepsilon z^2} \exp\left(i\left(z\frac{\sin u}{2} - \frac{\hat{x}-a}{2}\right)^2 \frac{2}{\sin u}\right) dz \right]_{\hat{x}=x} \tilde{\xi}(u,x)\, du\, dx \\
&= \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}^2} \exp\left(\frac{i(x-a)^2}{2\sin u}\right) \frac{2}{\sin t} \left[\int_{\mathbb{R}} e^{\varepsilon z^2} \exp\left(iy^2 \frac{2}{\sin u}\right) dz \right] \tilde{\xi}(u,x)\, du\, dx,
\end{aligned}$$

where in the last equality we performed the change of variables $y \leftrightarrow \dfrac{z\sin u}{2} - \dfrac{(x-a)}{2}$.

Then using Lebesgue dominated convergence theorem we can exchange the limit with the first integral and, then, we can apply the well know regularization (4.1.9) and finally obtain

$$\langle \mu_a, \theta\rangle = \int_{\mathbb{R}^2} \exp\left(\frac{i(x-a)^2}{2\sin u}\right) \sqrt{\frac{2i\pi}{\sin u}}\, \tilde{\xi}(u,x)\, du\, dx.$$

Let us use now the change of variables $v \leftrightarrow 2\sin u$ about $u = 0$, here $v \approx 2u$ and we can write

$$\langle \mu_a, \theta\rangle = \int_{\mathbb{R}^2} \exp\left(\frac{i(x-a)^2}{v}\right) \sqrt{\frac{1}{v}}\, \tilde{\theta}(v,x)\, dv\, dx,$$

where $\tilde{\theta}(v,x) := \sqrt{i\pi}\, \tilde{\xi}(2u,x)$ is a test-function supported in $(0, x_0)$.

Though such expression makes sense when $a \in \mathbb{R}$ (since $\left|\exp(i(x-a)^2/v)\right| = 1$ for any point $(v,x) \in \mathrm{Supp}(\tilde{\xi})$), it does not make sense anymore when $a \in \mathbb{C}$.

Lemma 4.3.9. *Let $U(Y)$ $(Y \in \mathbb{C})$ be a differential operator of the form*

$$\sum_{k=0}^{\infty} \left[\frac{A_k(Y, d/dZ)}{k!}(\cdot)\right]_{Z=0} (d/dv)^k, \tag{4.3.8}$$

(where $A_k \in \mathbb{C}[[Y, d/dZ]]$ for any $k \in \mathbb{N}$), considered as acting from the space of entire functions of the variable Z to the space $\mathbb{C}[Y][[d/dv]]$.

Suppose that there exist $p \geq 1$ and $q \geq 1$ and $B, D \geq 0$ such that

$$\sup_{k\in\mathbb{N}, Y\in\mathbb{C}} \left(|A_k(Y,W)| \exp(-B|W|^p - D|Y|^q)\right) < +\infty. \tag{4.3.9}$$

Then, for any $b \geq 0$, there exists $A^{(b)} \geq 0$ such that

$$\forall C \geq 0, \quad \forall f \in \mathcal{A}_1^{C,b}(\mathbb{C}), \quad \sup_{k\in\mathbb{N}} |A_k(Y, (d/dZ))(f)(0)| \leq C\, A^{(b)}\, e^{D|Y|^q}. \tag{4.3.10}$$

In particular, for any $f \in \mathcal{A}_1^{C,b}(\mathbb{C})$, $U(Y)(f)$ remains an infinite order differential operator of the form

$$\sum_{k\geq 0} \alpha_k(Y)(f)(d/dv)^k$$

with coefficients satisfying (independently of $f \in \mathcal{A}_1^{C,b}(\mathbb{C})$)

$$\sum_{k\in\mathbb{N}} k!\, |\alpha_k(Y)(f)|\, \exp(-B|Y|^q) = C\, A^{(b)} < +\infty.$$

Remark 4.3.8. Before proving this result, let us make some clarification about the notation and the statement itself.

With the notation $\mathbb{C}[[Y, d/dZ]]$ I denote the set of formal differential operator of infinite order both in Y and $\frac{d}{dZ}$, instead $\mathbb{C}[Y][[d/dv]]$ stands for the set of formal infinite order differential operators in $\frac{d}{dv}$ depending on the parameter Y.

Moreover, let us notice that condition (4.3.9) means that the formal symbol of A_k belongs to $\mathcal{A}_p(\mathbb{C})$ as a function of W and to $\mathcal{A}_q(\mathbb{C})$ as a function of Y. Analogously, (4.3.10) assures that the function $Y \in \mathbb{C} \mapsto A_k(Y, d/dZ)(f)(0)$ is in $\mathcal{A}_q(\mathbb{C})$.

Proof. The coefficients of A_k as a polynomial in d/dZ satisfy

$$\sum_{k,j\in\mathbb{N},\, Y\in\mathbb{C}} |a_{k,j}(Y)| \leq C_0 \frac{b_0^j}{\Gamma(j/p)+1} e^{D|Y|^q}$$

for some absolute constants C_0 and b_0 (see Lemma 2.3.11).

As in the proof of Lemma 3.1.2, one concludes that for any $f \in \mathcal{A}_1^{C,b}(\mathbb{C})$ and any $k \in \mathbb{N}$, one has uniform estimates

$$|A_k(Y, (d/dZ))(f)| \leq C\, A^{(b)} e^{b_0 b|Z|}\, e^{D|Y|^q}$$

for some positive constant $A^{(b)}$. Evaluating at $Z = 0$, one gets the required estimates.

In particular we obtain that $|\alpha_k(Y)(f)| = |A_k(Y, d/dZ)(f)|$ are the coefficients of the resulting differential operator that turns to satisfy the condition (2.3.3) and then to be an infinite order differential operator. □

One can then complete the discussion in Section 4.1 and Section 4.2 into the following companion proposition.

Proposition 4.3.10. *Let $\mathcal{T} = (0, +\infty) \times \mathbb{R}$. For any $a \in \mathbb{R}$, let $\varphi_a \in \mathcal{D}'(\mathcal{T}, \mathbb{C})$ be the evolved distribution from the initial datum $x \in \mathbb{R} \mapsto e^{iax}$ through the Cauchy Schrödinger equation (4.1.10). Let $\mathcal{F} = \{\varphi_a\,;\, a \in \mathbb{R}\}$, where each φ_a is considered as a hyperfunction in $\mathcal{T}$.*
Then, for any $a \in \mathbb{R} \setminus [-1, 1]$, the sequence $\{\sum_{j=0}^{n} C_j(n)\varphi_{1-2j/n}\}_{n\geq 1}$ is a $\mathcal{F}$-supershift of hyperfunctions over the $\mathcal{F}$-supershift domain $\mathcal{T}$.

Proof. Let $\theta \in \mathcal{D}(\mathbb{R}^2_{t,x}, \mathbb{C})$ with support a small neighborhood V of the point $(\pi/2, x_0)$, $x_0 \in \mathbb{R}$, and $\tilde{\theta}$ the test-function with support $(0, x_0) \in V \setminus (\pi/2, 0)$ that corresponds to it through the successive transformations exhibited at the beginning of the section. One has for any $a \in \mathbb{R}$,

$$\begin{aligned}
&\langle \varphi_a, \theta \rangle = \\
&= \int_{\mathbb{R}} \int_0^{\infty} \Big[\exp\Big(\frac{i}{v}(Y-a)^2\Big)\Big]_{Y=x} \frac{\tilde{\theta}(v,x)}{\sqrt{v}}\, dv\, dx \\
&\qquad\qquad - i \int_{\mathbb{R}} \int_0^{\infty} \Big[\exp\Big(-\frac{i}{v}(Y-a)^2\Big)\Big]_{Y=x} \frac{\tilde{\theta}(-v,x)}{\sqrt{v}}\, dv\, dx \\
&= \int_{\mathbb{R}} \int_0^{\infty} \Big(\Big[\sum_{k=0}^{\infty} \frac{i^k}{k!} \frac{(Y-a)^{2k}}{v^{1/2+k}}\Big]_{Y=x} \tilde{\theta}(v,x) - i\Big[\sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \frac{(Y-a)^{2k}}{v^{1/2+k}}\Big]_{Y=x} \tilde{\theta}(-v,x)\Big) dv\, dx.
\end{aligned} \tag{4.3.11}$$

For any $k \in \mathbb{N}$, the distribution $v_+^{-1/2-k} \in \mathcal{D}'([0, +\infty), \mathbb{R})$ can be expressed as

$$v_+^{-1/2-k} = \frac{2^k}{\prod_{\ell=1}^{k}(2(k-\ell)+1)} \Big(-\frac{d}{dv}\Big)^k (v_+^{-1/2})$$

in the sense of distributions in $\mathcal{D}'([0, +\infty), \mathbb{R})$.

Furthermore, for any $k \in \mathbb{N}$ we have

$$(Y-a)^{2k}\, e^{iaZ} = \Big(Y + i\frac{d}{dZ}\Big)^{2k} (e^{iaZ}).$$

Let us denote with M the number resulting from the product $\prod_{\ell=1}^{k}(2(k-\ell)+1)$. Then, one can reformulate formally (4.3.11) as

$$\begin{aligned}
\langle \varphi_a, \theta \rangle = \sum_{k=0}^{\infty} \frac{(2i)^k}{k!\, M} \\
\int_{\mathbb{R}} \Big\langle \Big[\Big(Y + i\frac{d}{dZ}\Big)^{2k} (e^{ia(\cdot)})\Big]_{Y=x} (0) \Big(\frac{d}{dv}\Big)^k (v_+^{-1/2}), \tilde{\theta}(\cdot, x) - i(-1)^k \tilde{\theta}(-\cdot, x) \Big\rangle\, dx.
\end{aligned} \tag{4.3.12}$$

Lemma 4.3.9 applies to the two operators

$$\begin{aligned} U_1(Y) &= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{M}\Big[(2i)^k\Big(Y+i\frac{d}{dZ}\Big)^{2k}(\cdot)\Big]_{Z=0}(d/dv)^k \\ U_2(Y) &= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{M}\Big[-i(2i)^k\Big(Y+i\frac{d}{dZ}\Big)^{2k}(\cdot)\Big]_{Z=0}(d/dv)^k \end{aligned} \tag{4.3.13}$$

with $p = q = 2$.

These two operators act then continuously (locally uniformly with respect to the parameter Y) from $\mathcal{A}_1(\mathbb{C})$ into the space of infinite order differential operators in d/dv (depending on the parameter $Y \in \mathbb{C}$).

Such differential operators can be considered as hyperfunctions on $\mathbb{R}_v$, indeed it can be proven that the coefficients respect condition (2.3.3). Since $v_+^{-1/2}$ is a hyperfunction in the real line $\mathbb{R}$, the two $\mathcal{H}(\mathbb{R})$-valued operators $f \in \mathcal{A}_1(\mathbb{C}) \longmapsto U_1(Y)(f) \odot v_+^{-1/2}$ and $f \in \mathcal{A}_1(\mathbb{C}) \longmapsto U_2(Y)(f) \odot v_+^{-1/2}$ are well defined and depend continuously (locally uniformly with respect to Y) on the entry $f \in \mathcal{A}_1(\mathbb{C})$.

The conclusion follows then from the fact that $F_n(x) \to e^{iax}$ in $\mathcal{A}_1(\mathbb{C})$ (as showed in Theorem 2.3.10) and from the evaluation of $\langle \varphi_a, \theta \rangle$ when $a \in \mathbb{R}$ and φ_a is considered as an element in $\mathcal{D}'(\mathcal{T}, \mathbb{C})$ (acting on $\theta \in \mathcal{D}(\mathcal{T}, \mathbb{C})$) which can be also interpreted a hyperfunction on $\mathcal{T}$. □

Appendix A

The Mittag-Leffler function

The Mittag-Leffler function is defined by the power series:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha \in \mathbb{C}, \quad \mathrm{Re}(\alpha) > 0.$$

The series converges in the whole complex plane for all $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$. For all $\mathrm{Re}(\alpha) < 0$ it diverges everywhere on $\mathbb{C} \setminus \{0\}$. For $\mathrm{Re}(\alpha) = 0$ the radius of convergence is $\mathrm{R} = e^{\pi |Im(\alpha)|/2}$. The most interesting fact is that for $\mathrm{Re}(\alpha) > 0$ the Mittag-Leffler function is an entire function of finite order. Indeed using Stirling's asymptotic formula

$$\Gamma(\alpha k + 1) = \sqrt{2\pi}(\alpha k)^{\alpha k + 1/2} e^{-\alpha k}(1 + o(1)), \quad for \quad k \to \infty$$

so that for

$$c_k = \frac{1}{\Gamma(\alpha k + 1)}$$

for $\alpha > 0$, according to the definition of order stated in Definition 2.3.1, we have

$$\limsup_{k\to\infty} \frac{k \ln k}{\ln \frac{1}{|c_k|}} = \limsup_{k\to\infty} \frac{k \ln k}{\ln |\Gamma(\alpha k + 1)|} = \frac{1}{\alpha}$$

and, this time according to the definition of type (recall Definition 2.3.2),

$$\limsup_{k\to\infty} \left(k^{1/\rho} \sqrt[k]{|c_k|} \right) = \limsup_{k\to\infty} \left(k^{1/\rho} \sqrt[k]{\frac{1}{|\Gamma(\alpha k + 1)|}} \right) = (e/\alpha)^\alpha.$$

This means that for each $\alpha \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$ the Mittag-Leffler function is an entire function of order $\rho = 1/\mathrm{Re}(\alpha)$ and of type $\sigma = 1$.

This function provides a generalization of the exponential function because we replace $k! = \Gamma(k+1)$ by $(\alpha k)! = \Gamma(\alpha k + 1)$ in the denominator of the power terms of the exponential series.

A useful generalization that was widely used in the computations of this thesis is the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha,\, \beta \in \mathbb{C}, \quad \operatorname{Re}(\alpha) > 0.$$

The function $E_{\alpha,\beta}(z)$ for $\alpha,\, \beta \in \mathbb{C}$ and $\operatorname{Re}(\alpha) > 0$ is an entire function of $\rho = 1/\operatorname{Re}(\alpha)$ and of type $\sigma = 1$ for every $\beta \in \mathbb{C}$. In other words, we have the following estimates

$$E_{\alpha,\beta}(z) \leq C \exp(|z|^{1/\operatorname{Re}(\alpha)}) \qquad \forall z \in \mathbb{C}.$$

Bibliography

[1] Y. Aharonov, D. Z. Albert, and L. Vaidman. "How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100". In: *Physical Review Letters* 60.14 (Apr. 1988), pp. 1351–1354. DOI: 10.1103/physrevlett.60.1351.

[2] Y. Aharonov, E. Ben-Reuven, S. Popescu, and D. Rohrlich. "Perturbative induction of vector potentials". In: *Physical Review Letters* 65.25 (Dec. 1990), pp. 3065–3067. DOI: 10.1103/physrevlett.65.3065.

[3] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz. "Time Symmetry in the Quantum Process of Measurement". In: *Physical Review* 134.6B (June 1964), B1410–B1416. DOI: 10.1103/physrev.134.b1410.

[4] Y. Aharonov, F. Colombo, I. Sabadini, D. C. Struppa, and J. Tollaksen. "Superoscillating Sequences in Several Variables". In: *Journal of Fourier Analysis and Applications* 22.4 (Oct. 2015), pp. 751–767. DOI: 10.1007/s00041-015-9436-8.

[5] Y. Aharonov, F. Colombo, I. Sabadini, D. C. Struppa, and J. Tollaksen. "Evolution of superoscillatory initial data in several variables in uniform electric field". In: *Journal of Physics A: Mathematical and Theoretical* 50.18 (Mar. 2017), p. 185201. DOI: 10.1088/1751-8121/aa66d9.

[6] Y. Aharonov, F. Colombo, I. Sabadini, D. Struppa, and J. Tollaksen. "The Mathematics of Superoscillations". In: *Memoirs of the American Mathematical Society* 247.1174 (May 2017), pp. 0–0. DOI: 10.1090/memo/1174.

[7] Y. Aharonov, F. Colombo, I. Sabadini, D.C. Struppa, and J. Tollaksen. "Superoscillating sequences as solutions of generalized Schrödinger equations". In: *Journal de Mathématiques Pures et Appliquées* 103.2 (Feb. 2015), pp. 522–534. DOI: 10.1016/j.matpur.2014.07.001.

[8] Y. Aharonov, I. Sabadini, J. Tollaksen, and A. Yger. "Classes of superoscillating functions". In: *Quantum Studies: Mathematics and Foundations* 5.3 (Feb. 2018), pp. 439–454. DOI: 10.1007/s40509-018-0156-z.

[9] T. Aoki, F. Colombo, I. Sabadini, and D. C. Struppa. "Continuity of some operators arising in the theory of superoscillations". In: *Quantum Studies: Mathematics and Foundations* 5.3 (Feb. 2018), pp. 463–476. DOI: 10.1007/s40509-018-0159-9.

[10] T. Aoki, F. Colombo, I. Sabadini, and D. C. Struppa. "Continuity theorems for a class of convolution operators and applications to superoscillations". In: *Annali di Matematica Pura ed Applicata (1923 -)* 197.5 (Mar. 2018), pp. 1533–1545. DOI: 10.1007/s10231-018-0736-x.

[11] C. A. Berenstein and R. Gay. *Complex Analysis and Special Topics in Harmonic Analysis*. Springer New York, 1995. DOI: 10.1007/978-1-4613-8445-8.

[12] C. A. Berenstein and R. Gay. *Complex Variables*. Springer New York, Oct. 3, 1997. 668 pp. ISBN: 0387973494. URL: https://www.ebook.de/de/product/5168862/carlos_a_berenstein_roger_gay_complex_variables.html.

[13] M. V. Berry. "Faster than Fourier". In: *Quantum Coherence and Reality; in celebration of the 60th Birthday of Yakir Aharonov* (1994), pp. 55–65.

[14] M. V. Berry. "Exact nonparaxial transmission of subwavelength detail using superoscillations". In: *Journal of Physics A: Mathematical and Theoretical* 46.20 (May 2013), p. 205203. DOI: 10.1088/1751-8113/46/20/205203.

[15] M. V. Berry and S. Popescu. "Evolution of quantum superoscillations and optical superresolution without evanescent waves". In: *Journal of Physics A: Mathematical and General* 39.22 (May 2006), pp. 6965–6977. DOI: 10.1088/0305-4470/39/22/011.

[16] M. V. Berry and P. Shukla. "Pointer supershifts and superoscillations in weak measurements". In: *Journal of Physics A: Mathematical and Theoretical* 45.1 (Nov. 2011), p. 015301. DOI: 10.1088/1751-8113/45/1/015301.

[17] M. Berry et al. "Roadmap on superoscillations". In: *Journal of Optics* 21.5 (Apr. 2019), p. 053002. DOI: 10.1088/2040-8986/ab0191.

[18] F. Colombo, I. Sabadini, F. Sommen, and D. C. Struppa. *Analysis of Dirac Systems and Computational Algebra*. Birkhäuser Boston, 2004. DOI: 10.1007/978-0-8176-8166-1.

[19] F. Colombo, I. Sabadini, D.C. Struppa, and A. Yger. "Superoscillating sequences and supershifts for families of generalized functions". In: *Preprint* (Apr. 2019).

[20] A. Kaneko. *Introduction to the Theory of Hyperfunctions*. Springer Netherlands, Feb. 28, 1989. 476 pp. ISBN: 9027728372. URL: https://www.ebook.de/de/product/4060292/a_kaneko_introduction_to_the_theory_of_hyperfunctions.html.