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THE MERTON PROBLEM WITH DERIVATIVES

Supervisors: Prof. Emilio BARUCCI
Prof. Daniele MARAZZINA

Author: Paola MILANESI
Matr: 905130

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Abstract

This thesis considers the problem of optimizing a continuous time portfolio of an investor with a constant risk aversion coefficient, which maximizes the expected utility of his final wealth. We study optimal investment strategies given investor access not only to bond and stock markets but also to the derivatives market. The problem is solved in a closed form through the stochastic control approach.
In the first analysis, the equivalence between Merton's classic problem and a *buy-and-hold* strategy that includes a possible investment in the options market compared to the other problem is presented. Then we introduce the solution of the problem in dynamic way when the derivatives are written on Stock that introduces stochastic volatility and it takes into consideration the fact that the price of the Stock can undergo great variations in short temporal instants (jump risks in the stock market). Finally a solution is proposed in the case in which the incomplete markets are completed by derivatives.

Keywords: Asset allocation; Portfolio optimization; Derivatives; HJB equation

Sommario

In questa tesi si considera il problema di ottimizzazione di un portafoglio a tempo continuo di un investitore con coefficiente di avversione al rischio costante, che massimizza l'utilità attesa della sua ricchezza finale. Studiamo strategie d'investimento ottimali dando l'accesso agli investitori non solo ai mercati obbligazionari e azionari, ma anche al mercato dei derivati. Il problema viene risolto in forma chiusa attraverso l'approccio di controllo stocastico.
In prima analisi viene presentata l'equivalenza tra il problema classico di Merton e una strategia *buy-and-hold* che include rispetto all'altro problema anche un possibile investimento sul mercato delle opzioni. Viene poi presentata la soluzione del problema in modo dinamico quando i derivati sono scritti su stock che presentano volatilità stocastica e tengono in considerazione il fatto che il prezzo del titolo possa subire grandi variazioni in brevi istanti temporali (il rischio di salto del titolo). Infine viene proposta una soluzione nel caso in cui mercati incompleti vengano completati da strumenti derivati.

Parole chiave: Asset allocation; Portfolio optimization; Derivati; Equazione di HJB

Contents

Introduction

Banks, investment funds and insurance companies are examples of investors that invest money in the financial markets. They want to make as much money as possible on their investments, but any serious investor also need to consider the risk involved. An investor is to a certain degree risk-averse, i.e the investor is reluctant to invest in an asset with high potential if this means that the risk of losing money is also high. The aim of such investors is to maximize the expected returns on their investments while at same time limiting the risk involved. One way of modelling such behaviour is through the theory of stochastic control and the maximization of expected utility.
The objects that are considered for a potential investment are divided into two categories: *risky assets*, for example stock, real estate, commodities, derivatives, which are assets with an uncertain future return and *risk-free assets*, for example bond, which are considered safe because they have a known future return at the time the investment starts. An investor can compose an investment portfolio by choosing the weights to be assigned to risky and risk-free assets, to match the level of risk the investor is comfortable with. In the problem dealt with in the thesis, the degree of aversion is represented by the investor's utility function. For such a risk-averse investor, it is natural to ask: which allocation strategy or investment strategy will maximize the expected utility of the portfolio? This is the question that awarded the Nobel Prize to the American economist Robert C. Merton, who addressed and solved it mathematically in 1969 using stochastic control. The problem is popularly known as "Merton's portfolio problem", which has become a well researched problem in articles and literature.
The most basic version of the problem gives an investor the limited choice of investing her wealth in a risky asset (stock) and a risk-free asset (bond). Given some additional assumptions, Merton found that the optimal allocation strategy is to keep a constant fraction of the wealth in the risky asset and hence, a constant fraction in the risk-free asset. This can be generalized to a situation with several risky assets and one risk-free asset and the conclusion is basically the same. This strategy is indeed a frequently used strategy among investors.
From a realistic point of view, the conclusion of "Merton's portfolio problem" is based on rather stylized mathematics as well as stylized assumptions. For example, one such assumption is that the dynamics of the risky assets are geometric Brownian motions, implying normally distributed log returns.
Another assumption that simplifies the problem is that the conclusion is based on

a continuous mathematical framework. It is also a fact that in today's extremely liquid financial markets, stocks and other risky assets change value almost continuously in time. This means that to follow the optimal strategy an investor has to rebalance her portfolio at the same rate as the prices changes. This is obviously not very realistic seen from a practical point of view. Also, transaction costs would make such a behaviour extremely expensive.

In Merton's portfolio problem the investor has the possibility to invest only in stock assets. In the discussion of this thesis, we have tried to understand what changes should be made to the solution in the event that the investor has the opportunity to access both the stock market and the derivatives market.
As mentioned before, the investor's aim is not only to gain as much as possible but also to have a degree of risk in line with his nature. Derivatives were created precisely with risk hedging objectives, and within the portfolio they provide an additional investment opportunity. A derivative instrument is a financial instrument that does not behave autonomously, since its performance is always linked to that of another instrument, called the underlying instrument, and can be of a different nature: it can be stock, bonds, financial indices, commodities such as oil or even another derivative, but there are derivatives based on the most diverse variables such as the amount of snow fallen in a given area. So, for example, an oil derivative will follow a similar trend to that of physical oil, although with some small differences. New derivatives are born every day, with different financial profiles and different degrees of sophistication. Standard types are called *plain vanilla*, while more complex types are called *exotic*. The most well-known and widespread types are: futures, forward rate agreement, swap, and options. In our discussion we primarily consider options and in the last part we define futures; both these types of derivatives are defined as regulated derivatives because they are traded on the regulated financial market.
An investor might ask: why include derivative instruments in the investment portfolio if they are always linked to the underlying asset and not buy the latter directly? The reasons for the attractiveness of derivatives are different and of considerable importance: reduced capital use in the immediate future, there is no need to store the products in the case of physical underlying assets and they have the ability to hedge risks without cancelling out potential benefits at the same time.

As mentioned above, the aim of this thesis is to understand how the solution to Merton's problem changes when in addition to the stock market we have the opportunity to invest in the derivatives market.
The first result found is based on the equivalence between Merton's problem and a static problem in which the risky assets that can be invested in are stocks and derivatives. For this result we refer to *The equivalence of the static and dynamic asset allocation problems* of R.V.Kohn, O.M.Papazoglu-Statescu [22] and to *Asset allocation and derivatives* of M.B.Haugh, A.W.Lo [15].

The second important result found is based on J.Liu and J.Pan's article *Dynamic derivative stategies* [11]. We find the solution in closed form of the optimal dynamic investment strategy when the portfolio consists of bonds, stocks and derivatives. In the analysis of this problem we considered that the stock dynamics do not follow Geometric Brownian motion but consider stochastic volatility following the Heston model and the risk of jumps.
The equivalence demonstrated above is based on the fact that any derivative in a complete market can be replicated by a trading strategy involving just stock and risk-free bond. So we wondered how the solution changes from Merton's problem if we consider an incomplete market that is complemented by derivatives. We achieve two great results with this framework. The first result of an optimal allocation strategy is obtained from a market that considers a stock and a derivative written on an underlying asset that cannot be traded, both governed by two random resources. Secondly, we find that there are infinite solutions to achieve an optimal portfolio consisting of a stock driven by two Brownian motions and a derivative written on the stock itself.

In particular, the thesis work is divided into 6 chapters:

- Chapter 1. Some theoretical notations necessary to understand the treatment are introduced. In particular the concepts and the theorems of stochastic analysis that are used during the discussion to succeed in resolving the problems are presented, the theory of the stochastic optimal control is presented and in particular the equation of Hamilton Jacobi Bellman (HJB) that allows to find the solution in closed form of the proposed problems. Finally, the concept of utility function and risk aversion is introduced.

- Chapter 2. The concept of portfolio optimization, which was studied by Merton, is introduced. In particular, the Black-Scholes model is presented, which is the main assumption of Merton's problem, and the solution of the latter is provided in the simplest case where only a risky asset (stock) and a risk-free assets are considered.

- Chapter 3. It is shown that the equivalence between the simple Merton model presented in the previous chapter and a portfolio optimization in which a buy-and-hold strategy is presented in which the investor also has access to the options market. This equivalence is proved through a theorem and illustrated by numerical example.

- Chapter 4. The solution is found with a dynamic strategy when the portfolio consists of a stock, two derivatives and a bond. When the stock follows the dynamics of B&S, investing in the derivatives market is similar to having Merton's problem, since in the case of a complete market each option is redundant. In this chapter we present two solutions: in the first we relax the assumption of the Black and Scholes model: we consider the stochastic volatility. The second result, in addition to the stochastic volatility, we consider that there is the risk of jumps, i.e. that the risky security on which the derivatives are written does not only have continuous trajectories.

- Chapter 5. The solution is found in a closed form in an incomplete market that is completed by a derivative. First, we analyse the case where the derivative is written on an underlying asset which is not traded; therefore both the European index and a derivative on the American index are held without trading the American index. We then consider the case where the option is written on a stock on the market.

- Chapter 6. All the results found during the treatment are summarized.

Chapter 1

Theoretical notions

Before starting the treatment of the problem of portfolio optimization, theoretical notions is introduced that are useful for understanding the results of the different applications of portfolio optimization.
In section 1.1 the concepts of stochastic process, martingale and stochastic integral are defined to provide a series of results that are used during the treatment. In section 1.2 we present the classical theory of stochastic optimal control, i.e. we introduce the problem of optimal control in its generality and we describe the solving approach characterized by the dynamic programming principle. In particular, we show how the solution of the optimization problem is linked to the resolution of a non-linear partial derivative equation, known as Hamilton Jacobi Bellman equation. Section 1.3 introduces the concept of mathematical utility and risk aversion of an individual and presents the main utility functions that are used in economics.

1.1 Elements of stochastic analysis

1.1.1 Stochastic Process

A stochastic process is a mathematical object that is intended to model the evolution of a random phenomenon.

Definition 1.1.1 (Stochastic process)**.** A stochastic process is an object of the form

$$\mathrm{X} = (\Omega,\, \mathscr{F},\, (\mathscr{F}_t)_{t\in T},\, (X_t)_{t\in T},\, \mathrm{P})$$

where

- $(\Omega,\, \mathscr{F},\, \mathrm{P})$ is a probability space.
- T (the times) is a subset of $\mathbb{R}^+$.
- $(\mathscr{F}_t)_{t\in T}$ is a filtration, i.e. an increasing family of sub σ-algebras of $\mathscr{F}$: $\mathscr{F}_s \subset \mathscr{F}_t$ whenever s $\leq$ t.

- $(X_t)_{t \in T}$ is a family of random variables on (Ω, $\mathscr{F}$) taking values into a measurable space (E, $\mathscr{E}$) such that, for every t, X_t is $\mathscr{F}_t$-measurable. This fact is also expressed by saying that $(X_t)_t$ is *adapted* to the filtration $(\mathscr{F}_t)_t$.

A stochastic process can therefore be seen as a family of random variables $(X_t)_{t \in [0,T]}$ indexed over time. Alternatively, if we consider a ω realization of the process, the trajectory X(ω) : t $\rightarrow X_t(\omega)$ defines a function of time with values in E. Finally, if we consider both t and ω as variables, a stochastic process can also be seen as a function of the process X : [0,T] $\times$ Ω $\rightarrow$ (E,$\mathscr{E}$).

Definition 1.1.2 (Brownian motion)**.** A $\mathbb{R}^m$-valued process X = (Ω, $\mathscr{F}$, $(\mathscr{F}_t)_t$, $(X_t)_t$, P) is a m-dimensional Brownian motion if

- $X_0 = 0$ a.s.;
- for every 0 $\leq$ s $\leq$ t the random variable X_t - X_s is independent of $\mathscr{F}_s$;
- for every 0 s $\leq$ t, $X_t - X_s$ is N(0, (t-s)I)-distributed (I is the m$\times$m identity matrix).

Definition 1.1.3 (Stopping time)**.** Let $(\mathscr{F}_t)_{t \in T}$ be a filtration. A random variable $\tau : \Omega \rightarrow T \cup \{+\infty\}$ is said to be *stopping time* if, for every $t \in T$, $\{\tau \leq t\} = \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathscr{F}_t$.

Martingales are stochastic processes that enjoy many important properties. When studying a process X, it is always a good idea to look for martingales "associated" to X, in order to take advantage of these properties.

Definition 1.1.4 (Martingale)**.** A real valued process M = (Ω, $\mathscr{F}$, $(\mathscr{F}_t)_t$, $(M_t)_t$, P) is a martingale if M_t is integrable for every t $\in$ T and

$$\mathbb{E}[M_t|\mathscr{F}_s] = M_s$$

for every s $\leq$ t.

We have that a process M is a *supermartingale* if $\mathbb{E}[M_t|\mathscr{F}_s] \leq M_s$, and respectively a *submartingale* when $\mathbb{E}[M_t|\mathscr{F}_s] \geq M_s$.

1.1.2 Stochastic Integral

Let B = $(B_t^1, \cdots B_t^m)_{t \in T}$ be a continuous standard m-dimensional Brownian motion fixed one for all, and X = $(X_t^1, \cdots X_t^m)_{t \in T}$ an adapted d-dimensional process, then

$$\int_0^T X_s(\omega)\, dB_s(\omega) \tag{1.1}$$

will be called a *stochastic integral.*
For instance, once the stochastic integral is defined, it is possible to consider a *stochastic differential equation*: it will be something of the kind

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \tag{1.2}$$

where b and σ are suitable function. To solve it will mean to find a process $(X_t)_t$ such that for every t ≥ 0

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s.$$

Definition 1.1.5. Let us consider $M^p_{loc}([a,b])$ as a space of equivalence classes of real stochastic processes X = (Ω, $\mathscr{F}$, $\mathscr{F}_t$, X_t, P) such that X is progressive (i.e adapted and right continuous) and $\int_a^b |X_t|^p dt < \infty$ a.s.
Instead, we define $M^p([a,b])$ space of equivalence classes of real stochastic processes X such that X is progressive and $\mathbb{E}[\int_a^b |X_t|^p dt] < \infty$.

From the definition of stochastic integral we can define a multitude of processes that admit a particular differential representation, the processes of Ito.
Let X be a process such that, for every $0 \leq t_1 < t_2 \leq T$,

$$X_{t_2} - X_{t_1} = \int_{t_1}^{t_2} F_t dt + \int_{t_1}^{t_2} G_t dB_t$$

where F $\in M^1_{loc}([0,T])$ and G $\in M^2_{loc}([0,T])$. We say then that X admits the *stochastic differential*

$$dX_t = F_t dt + G_t dB_t.$$

It is clear that such X is continuous and therefore X $\in M^p_{loc}([0,T])$ for every p $\geq$ 0. A process admitting a stochastic differential is also called an *Ito process.*

Theorem 1.1.1 (Ito's formula)**.** Let X be a process with stochastic differential

$$dX_t = F_t\ dt + G_t\ dB_t$$

and let f: $\mathbb{R}^m{}_x \times \mathbb{R}^+{}_t \to \mathbb{R}$ be a continuous function in (x,t), with continuous derivatives $u_t, u_{x_i}, u_{x_i x_j}$, i,j = 1,...m.
Then the process $(f(X_t,t))_t$ has stochastic differential

$$df(X_t,t) = \frac{\partial f}{\partial t}(X_t,t)\ dt + \sum_{i=1}^m \frac{\partial f}{\partial x_i}(X_t,t)\ dX_i(t) + \frac{1}{2}\sum_{i,j=1}^m \frac{\partial f}{\partial x_i \partial x_j}(X_t,t)\ A_{ij}(t)\ dt \tag{1.3}$$

where $A = GG^T$.

Proof. The proof can be found in chapter 7 *Stochastic Calculus* of the book by P.Baldi [17]. □

1.1.3 Stochastic Differential Equations

Let $b(x,t) = (b_i(x,t))_{1\leq i\leq m}$ and $\sigma(x,$t$) = (\sigma_{ij}(x,t))_{1\leq i\leq m,\ 1\leq j\leq d}$ be measurable functions defined on $\mathbb{R}^m \times [0,T]$ and $\mathbb{R}^m$- and M(m,d)-valued respectively.

Definition 1.1.6 (SDE)**.** The couple of processes (X,B) = (Ω, $\mathscr{F}$, $(\mathscr{F}_t)_{t\in[0,T]}$, $(X_t)_{t\in[u,T]}$, $(B_t)_{t\in[0,T]}$, P) is said to be a solution of the Stochastic Differential Equation (SDE)

$$\begin{aligned} dX_t &= b(X_t,t)\ dt + \sigma(X_t,t)\ dB_t, \\ X_u &= x \qquad\qquad\qquad\qquad x\in\mathbb{R}^m \end{aligned}$$

if

- (Ω, $\mathscr{F}$, $(\mathscr{F}_t)_t$,$(B_t)_t$, P) is a d-dimensional standard Brownian motion
- (Ω, $\mathscr{F}$, $(\mathscr{F}_t)_t$,$(X_t)_t$, P) is an Ito process with respect to B such that
  - $F_t = b(X_t,t) \ \in M^1_{loc}([u,T])$
  - $G_t = \sigma(X_t,t) \ \in M^2_{loc}([u,T])$
  - for every t $\in$ [u,T] we have

$$X_t = x + \int_u^t b(X_s,s)\ ds + \int_u^t \sigma(X_s,s)\ dB_s. \tag{1.4}$$

We remark that the solution of a SDE as above is necessarily a continuous process. In case we want to model some real life phenomenon with such a SDE, it is important to realize that the model presented above does not fit a model with jumps: discontinuous behaviors must be modeled using different SDE's in which the Brownian motion replaced by some more suitable stochastic process.
Terminology: b is a *drift*, σ is the *diffusion coefficient*.

Definition 1.1.7 (Strong solution)**.** We say that SDE defined in definition (1.1.6) has strong solutions if for every standard Brownian motion (Ω, $\mathscr{F}$, $(\mathscr{F}_t)_t$, $(B_t)_t$, P) there exists a process X that satisfies equation (1.4).

Theorem 1.1.2 (Existence and uniqueness of strong solution)**.** Let SDE defined in (1.1.6) with $b : \mathbb{R}^m \times [0,T] \to \mathbb{R}^m$ and $\sigma : \mathbb{R}^m \times [0,T] \to M(m\times d)$ such that

- measurable in (x, t);
- $\exists M > 0 : \begin{cases} |b(x,t)| & \leq M(1+|x|) \\ |\sigma(x,t)| & \leq M(1+|x|) \end{cases} \quad \forall x\in\mathbb{R}^m, \forall t\in[0,T];$ (Linearity)
- $\exists N > 0, \exists L_N > 0 : \begin{cases} |b(x,t)-b(y,t)| & \leq L_N\,|x-y| \\ |\sigma(x,t)-\sigma(y,t)| & \leq L_N\,|x-y| \end{cases} \ \forall\,|x|\,,|y| \leq N, \forall t\in [0,T]$. (Locally lipschitz)

And let B = $(\Omega,\ \mathscr{F},\ (\mathscr{F}_t),\ (B_t),\ \mathrm{P})$ be a d-dimensional continuous Brownian motion and let X $\in L^2(\mathscr{F}_u)$ $\mathbb{R}^m$-valued, $u \in [0, T]$, then exists a strong solution of SDE defined in (1.1.6) and we we will have uniqueness of the solution X from x in t, i.e $X_t = x$. Moreover, exists a constant C_T such that

$$\mathbb{E}[\sup_{t \leq s \leq T} |X_s|^p] \leq C_T(1 + \mathbb{E}[|x_s|^p]).$$

Proof. The proof can be found in chapter 8 *Stochastic Calculus* of the book by P.Baldi [17]. □

1.2 Stochastic Optimization Problems

In this section we describe the classic approach to a generic optimal control problem. For the totality of the theoretical results that are presented we refer to H.Pham [7].

We consider a dynamic system characterized by its state at any time, and evolving in an uncertain environment formalized by a probability space (Ω, $\mathscr{F}$, P). We denote by $X_t(\omega)$ the state of the system at time t in a world scenario $\omega \in \Omega$.
The dynamics of the system is typically influenced by a control modeled as a process $\alpha = (\alpha_t)_t$ whose value is decided at any time t in function of the available information. The control α should satisfy some constraints, and is called *admissible control*. We denote by $\mathscr{A}$ the set of admissible controls.
The objective is to maximize (or minimize) over all admissible controls a functional $J(X, \alpha)$. We shall consider objective functions in the form

$$J(X, t; \alpha) = \mathbb{E}\bigg[\int_t^T f(X_s, \omega, \alpha_s)ds + g(X_T, \omega)\bigg] \tag{1.5}$$

on a finite horizon T$< \infty$, and

$$J(X; \alpha) = \mathbb{E}\bigg[\int_0^\infty e^{-\beta t}\ f(X_t, \omega, \alpha_t)dt\bigg] \tag{1.6}$$

on an infinite horizon. The function f is a running profit function, g is a terminal reward function, and $\beta > 0$ is a discount factor. In some situation the controller may also decide directly the horizon or ending time of this objective. The corresponding optimization problem is called *optimal stopping time*. The control can be mixed, composed of a pair control/stopping time (α, τ), and the objective functional is in the form

$$J(X, \alpha, \tau) = \mathbb{E}\bigg[\int_0^\tau f(X_t, \alpha_t)dt + g(X_\tau)\bigg].$$

The optimization problem can be formulated as a search for the v, called value function, defined respectively in the three cases as

$$v(X, t) = \sup_\alpha J(X, t; \alpha);$$

$$v(X) = \sup_\alpha J(X, \alpha);$$

$$v(X) = \sup_{\alpha,\tau} J(X, \alpha, \tau).$$

The main goal is a stochastic optimization problem is to find the maximizing control process and/or stopping time attaining the value function to be determined.

1.2.1 Formulation

We proceed with the formulation of the problem described above in a more rigorous way. $(\Omega, \mathscr{F}, \mathrm{P})$ is a complete probability space on which we defined a d-dimensional Brownian $W = (W_t^1, ..., W_t^d)_{t \in T}$. Suppose now that our evolving system is described through the stochastic differential equation with values in $\mathbb{R}^n$

$$dX_t = b(X_t, \alpha_t)\ dt + \sigma(X_t, \alpha_t)\ dW_t \tag{1.7}$$

where the control $\alpha = (\alpha_t)_{t \in T}$ is progressively measurable and it has value in $A \subseteq \mathbb{R}^m$. The measurable functions b: $\mathbb{R}^n \times A \to \mathbb{R}^n$ and σ: $\mathbb{R}^n \times A \to \mathbb{R}^{n \times d}$ satisfy a uniform Lipschitz condition in A, i.e. $\exists K \geq 0$ such that $\forall x, y \in \mathbb{R}^n$ and $\forall a \in A$

$$|b(x,a) - b(y,a)| + |\sigma(x,a) - \sigma(y,a)| \leq K\,|x-y|\,.$$

We assume also that the control α is a progressive measurable process such that

- $\mathbb{E}\left[\int_0^T |b(0,\alpha_t)|^2 + |\sigma(0,\alpha_t)|^2\,dt\right] < \infty$ for problem with finite horizon
- $\mathbb{E}\left[\int_0^T |b(0,\alpha_t)|^2 + |\sigma(0,\alpha_t)|^2\,dt\right] < \infty$ $\forall T > 0$ for problem with infinite horizon.

Under these hypotheses we can write that $\alpha \in \mathscr{A}$ and in particular we will have that for each initial condition (t,x) $\in \mathbb{T} \times \mathbb{R}^n$ the SDE (1.7) admits one and only one strong solution X to continuous trajectories.
We describe separately the problems of optimal control over finite and infinite horizon.

Problem with finite horizon: In a problem with finite horizon, we define the running profit function and terminal reward function respectively as the functions f : [0,T]$\times\mathbb{R}^n \times A \to \mathbb{R}$ and g: $\mathbb{R}^n \to \mathbb{R}$ measurable such that:

- g is lower-bounded and satisfies a quadratic growth condition $|g(x)| \leq C\big(1+|x|^2\big)$, $\forall x \in \mathbb{R}^n$, for some constant C independent of x.
- f is such that $\forall (t,x) \in [0,T] \times \mathbb{R}^n$ exists a subset of controls $\mathscr{A}(t,x) \subset \mathscr{A}$ called *admissible controls* such that

$$\mathbb{E}\left[\int_t^T |f(s, X_s, \alpha_s)|\,ds\right] < \infty.$$

We can define the objective functional like in (1.5) and value function like

$$v(t,x) = \sup_{\alpha \in \mathscr{A}(t,x)} J(t,x;\alpha) = \sup_{\alpha \in \mathscr{A}(t,x)} \mathbb{E}\left[\int_t^T f(s, X_s, \alpha_s) ds + g(X_T)\right]. \tag{1.8}$$

Given an initial condition (t,x) we say that $\hat{\alpha} \in \mathscr{A}(t,x)$ is an optimal control if v(t,x) = J(t,x;$\hat{\alpha}$).

Problem with infinite horizon: We define the running profit function f for problem with infinite horizon like f: $\mathbb{R}^n \times A \to \mathbb{R}$ measurable such that, given $\beta > 0$, exists a subset of controls $\mathscr{A}(x) \subset \mathscr{A}$ called *admissible controls* such that

$$\mathbb{E}\bigg[\int_0^\infty e^{-\beta s}\,|f(X_s, \alpha_s)|\,ds\bigg] < \infty.$$

We can define the objective functional like in (1.6) and value function like

$$v(t,x) = \sup_{\alpha \in \mathscr{A}(x)} J(x;\alpha) = \sup_{\alpha \in \mathscr{A}(x)} \mathbb{E}\bigg[\int_0^\infty e^{-\beta s} f(X_s, \alpha_s) ds\bigg]. \tag{1.9}$$

Given an initial condition x we say that $\hat{\alpha} \in \mathscr{A}(x)$ is optimal control if $v(x)$ = J(x, $\hat{\alpha}$).

1.2.2 Dynamic Programming Principle

The dynamic programming principle (DPP) is a fundamental principle in the theory of optimal stochastic control and allows to reconstruct the value function at the initial instant from the value function to a subsequent instant. In practice, the optimal solution to the sub problem that starts at a later time can be used to find the optimal solution to the entire problem. This allows, note the final condition to reconstruct backwards the optimal control instant by instant.
Below are the finite and infinite horizon versions of the dynamic programming principle.

Finite horizon problem

Theorem 1.2.1 (DPP with finite horizon)**.** Let $\tau_{t,T}$ family of stopping time on finite horizon [t,T] and v(t,x) is a value function like (1.8), then fixed (t,x)$\in$ [0,T]$\times\mathbb{R}^n$

$$v(t,x) \;=\; \sup_{\alpha \in \mathscr{A}(t,x)} \mathbb{E}\bigg[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)ds + v(\theta, X_\theta^{t,x})\bigg] \quad \forall \theta \in \tau_{t,T}. \tag{1.10}$$

Proof. The proof can be found in chapter 3 of *Continuous-time stochastic control and optimization with financial application* by H.Pham [7]. □

We can observe that $\forall \theta \in \tau_{t,T}$ and $\forall \alpha \in \mathscr{A}(t,x)$, we have

$$v(t,x) \geq \mathbb{E}\bigg[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)ds + v(\theta, X_\theta^{t,x})\bigg].$$

The dynamic programming principle allows us to split the finite horizon optimization problem into two sub problems:

- Search for an optimal control starting from $\theta \in [t, T]$ and the corresponding value function $v(\theta, X_\theta^{x,t})$,
- Search for an optimal control for a finite horizon problem described through (1.10).

Infinite horizon problem

Theorem 1.2.2 (DDP with infinite horizon)**.** Let τ family of stopping time on infinite horizon and $v(x)$ is a value function like (1.9), then fixed $x \in \mathbb{R}^n$

$$v(x) \;=\; \sup_{\alpha \in \mathscr{A}(x)} \mathbb{E}\bigg[\int_0^\theta e^{-\beta s} f(X_s^{t,x}, \alpha_s) ds + e^{-\beta\theta} v(X_\theta^{t,x})\bigg] \quad \forall\theta \in \tau \tag{1.11}$$

with $e^{-\beta\ \theta(\omega)} = 0$ when $\theta(\omega) = \infty$.

Proof. The proof can be found in chapter 3 of *Continuous-time stochastic control and optimization with financial application* by H.Pham [7]. □

We can also observe that $\forall\theta \in \tau$ and $\forall\alpha \in \mathscr{A}(x)$ we have

$$v(x) \geq \mathbb{E}\big[\int_t^\theta e^{-\beta s} f(X_s^{t,x}, \alpha_s) ds + e^{-\beta\theta} v(X_\theta^x)\big].$$

The dynamic programming principle allow us to split the infinite horizon optimization problem into two sub problems:

- Search for an optimal control starting form $\theta \geq 0$ and the corresponding value function $v(X_\theta^{t,x})$,
- Search for an optimal control for a finite horizon problem described thought (1.11).

1.2.3 Hamilton Jacoby Bellman equation

The Hamilton Jacoby Bellman equation (HJB) is the infinitesimal version of the dynamic programming principle and describes the local behavior of the value function when we send $\theta \to t$ in (1.10) and $\theta \to 0$ in (1.11). The HJB equation is also called *dynamic programming equation*.
A crucial step in the dynamic programming approach is to prove that given a regular solution of the Hamilton Jacoby Bellman equation, it coincides with the value function of the optimization problem. This result is known as the *verification theorem*.
Below we heuristically derive the equation of Hamilton Jacoby Bellman in the cases with finite and infinite horizon and we enunciate respectively the theorems of verification.

Finite horizon problem: Let h > 0, we consider the time $\theta = t + h$ and a constant control $\alpha_s = a$ in the dynamic programming principle, we have

$$v(t,x) \geq \mathbb{E}\bigg[\int_t^{t+h} f(s, X_s^{t,x}, a)ds + v(t+h, X_{t+h}^{t,x})\bigg].$$

By assuming that v is smooth enough, we may apply Ito's formula between t and $t+h$:

$$v(t+h, X_{t+h}^{t,x}) = v(t,x) + \int_t^{t+h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x})ds + \\ + \int_t^{t+h} \sum_{i=1}^n \frac{\partial v}{\partial x}(s, X_s^{t,x})(\sigma(X_s^{t,x}, a))_i^T dW_s,$$

where the infinitesimal diffusion generator $\mathcal{L}^a$ is defined as follows

$$\mathcal{L}^a v(t,x) = b(x,a)\ D_x v(t,x) + \frac{1}{2} tr(\sigma(x,a)\ \sigma^T(x,a)\ D_{xx}^2 v(t,x)). \tag{1.12}$$

Sending $h \to 0$, if the stochastic integral is a martingale, we will have that

$$v(t,x) \geq f(t,x,a) + v(x,t) + \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(t,x),$$

thus simplifying for v(t,x) and reordering the terms, we obtain

$$-\frac{\partial v}{\partial t}(t,x) - \mathcal{L}^a v(t,x) - f(t,x,a) \geq 0 \quad \forall a \in A.$$

Since this holds true for any $a \in A$, we obtain the following inequality

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A}(\mathcal{L}^a v(t,x) - f(t,x,a)) \geq 0. \tag{1.13}$$

On the other hand, suppose that α^* is an optimal control then from the principle of dynamic programming we have

$$v(t,x) = \mathbb{E}\bigg[\int_t^{t+h} f(s, X_s^{t,x}, \alpha_s^*)ds + v(t+h, X_{t+h}^{t,x})\bigg].$$

By similar arguments as above, we then get

$$-\frac{\partial v}{\partial t}(t,x) - \mathcal{L}^{\alpha_t^*} v(t,x) - f(t,x,\alpha_t^*) = 0.$$

Which combined with (1.13), we have that v should satisfy the equality and we can formulate the *Hamilton Jacobi Bellman equation*:

$$\begin{cases} -\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A}[\mathcal{L}^a v(t,x) + f(t,x,a)] = 0 & \forall (t,x) \in [0,T) \times \mathbb{R}^n \\ v(T,x) = g(x) & \forall x \in \mathbb{R}^n \end{cases} \tag{1.14}$$

Theorem 1.2.3 (Verification theorem in finite horizon)**.** Let ω be a function in $C^{1,2}([0,T)\times\mathbb{R}^n)\cap C^0([0,T]\times\mathbb{R}^n)$, and satisfying a quadratic growth condition, i.e. there exists a constant C such that

$$|\omega(t,x)| \;\leq\; C(1+|x|^2), \quad \forall (t,x)\in[0,T]\times\mathbb{R}^n.$$

- Suppose that

  $$\begin{cases} -\frac{\partial\omega}{\partial t}(t,x) - \sup_{a\in A}[\mathcal{L}^a\omega(t,x)+f(t,x,a)] \geq 0 & \forall (t,x)\in[0,T)\times\mathbb{R}^n \\ \omega(T,x)\geq g(x) & \forall x\in\mathbb{R}^n \end{cases}$$

  Then $\omega \geq v$ on $[0,T]\times\mathbb{R}^n$.

- Suppose further that ω(T,x) = g(x) and there is a measurable function $\hat{\alpha}(t,x)$ with values in A and with (t,x) $\in[0,T)\times\mathbb{R}^n$ such that

$$-\frac{\partial\omega}{\partial t}(t,x)-\sup_{a\in A}[\mathcal{L}^a\omega(t,x)+f(t,x,a)] = -\frac{\partial\omega}{\partial t}(t,x)-\mathcal{L}^{\hat{\alpha}(t,x)}\omega(t,x)-f(t,x,\hat{\alpha}(t,x)) = 0$$

and such that the SDE

$$dX_s = b(X_s,\hat{\alpha}(s,X_s))\;ds + \sigma(X_s,\hat{\alpha}(s,X_s))\;dW_s \qquad X_t = x$$

admits a unique solution, denoted by $\hat{X}_s^{t,x}$ for which the process $\left\{\hat{\alpha}(s,\hat{X}_s^{t,x})\right\}\in A(t,x)$, then

$$\omega = v \quad \text{on [0,T]} \times\mathbb{R}^n$$

is a value function and $\hat{\alpha}$ is an optimal Markovian control.

Proof. The proof can be found in chapter 3 of *Continuous-time stochastic control and optimization with financial application* by H.Pham [7]. □

Infinite horizon problem: By using similar arguments as in the finite horizon case, we consider $\theta = h$ and a constant control $\alpha_s = a$ in the principle of dynamic programming, we therefore have

$$v(x)\geq \mathbb{E}\left[\int_0^h e^{-\beta s}f(X_s^x,a)\;ds + e^{-\beta h}\;v(X_h^x)\right].$$

We suppose that v is smooth enough, we may apply Ito's formula to the term $e^{-\beta t}v(x)$ we have

$$e^{-\beta h}v(X_h^x) = v(x)\;+\int_0^h(-\beta v+\mathcal{L}^a v)(X_s^x)ds\;+\int_t^{t+h}\sum_{i=1}^n\frac{\partial v}{\partial x}(X_s^x)\;(\sigma(X_s^x,a))_i^T\;dW_s.$$

Sending h $\to$ 0 like before, if the stochastic integral is a martingale, we will have that

$$\beta v(x)-\mathcal{L}^a v(x)-f(x,a)\geq 0 \quad \forall a\in A,$$

and then we arrive at this inequality

$$\beta v(x) - \sup_{a \in A} \left[\mathcal{L}^a v(x) + f(x,a)\right] \geq 0. \tag{1.15}$$

Taking α^* optimal control then we can show that

$$\beta v(x) - \mathcal{L}^{\alpha^*} v(x) - f(x, \alpha^*) = 0.$$

We can formulate the Hamilton Jacobi Bellman equation in case of infinite horizon

$$\beta v(x) - \sup_{a \in A} \left[\mathcal{L}^a v(x) + f(x,a)\right] = 0 \quad \forall x \in \mathbb{R}^n. \tag{1.16}$$

Theorem 1.2.4 (Verification theorem in infinite horizon)**.** Let ω be a function in $C^2(\mathbb{R}^n)$ and satisfies a quadratic growth condition, i.e. exists a constant C such that

$$|\omega(t,x)| \;\leq\; C(1 + |x|^2), \quad \forall x \in \mathbb{R}^n.$$

- Suppose that

  $$\beta\omega(x) - \sup_{a \in A}[\mathcal{L}^a \omega(x) + f(x,a)] \geq 0 \quad \forall x \in \mathbb{R}^n$$

  and holds

  $$\lim_{T \to \infty} sup\{e^{-\beta T}\ \mathbb{E}[\omega(X_T^x)]\} \geq 0 \qquad \forall x \in \mathbb{R}^n,\ \forall \alpha \in A(x)$$

  then $\omega \geq v$ on $\mathbb{R}^n$.

- Suppose further that $\forall x \in \mathbb{R}^n$, there exists a measurable function $\hat{\alpha}(x)$ with values in A such that

  $$\beta\omega(x) - \sup_{a \in A}[\mathcal{L}^a \omega(x) + f(x,a)] = \beta\omega(x) - \mathcal{L}^{\hat{\alpha}(x)}\omega(x) - f(x, \hat{\alpha}(x)) = 0$$

and such that the SDE

$$dX_s = b(X_s, \hat{\alpha}(s, X_s))\ ds + \sigma(X_s, \hat{\alpha}(s, X_s))\ dW_s \qquad X_0 = x$$

admits a unique solution, denoted by $\hat{X}_s^x$ for which the process $\left\{\hat{\alpha}(s, \hat{X}_s^x)\right\} \in A(x)$, and satisfying

$$\lim_{T \to \infty} inf\{e^{-\beta T}\ \mathbb{E}[\omega(\hat{X}_T^x)]\} \leq 0.$$

Then

$$\omega = v \quad \forall x \in \mathbb{R}^n$$

is a value function and $\hat{\alpha}$ is an optimal Markovian control.

Proof. The proof can be found in chapter 3 of *Continuous-time stochastic control and optimization with financial application* by H.Pham [7]. □

1.3 Utility Function

The utility function represents a consumer's preference ordering over a choice set.

1.3.1 Preference ordering

In the optimization problems that we will face in the treatment, the functions f and g of (1.5)-(1.6) will have to be functions that represent the preferences of the individual. Let (Ω, P) a probability space and X $\in \Omega$ a random variable. The level of satisfaction of an individual is based on a preference ordering $\mathcal{R}$, i.e. given two lotteries X,Y $\in \Omega$ we will say that the lottery X is at least as good as Y for the individual if it is that X$\mathcal{R}$Y, that X and Y are indifferent if X$\mathcal{I}$Y, i.e. X$\mathcal{R}$Y and Y$\mathcal{R}$X, and finally that X is strictly preferred to Y if X$\mathcal{R}$Y, but not Y$\mathcal{R}$X.
Let u be a utility function which represents the preference ordering if we are in a risk-free environment. It is shown that if the preference ordering respects some assumptions then it can be represented by the *expected utility*

$$U(X) = \mathbb{E}[u(X)],$$

i.e. the expected value of the utility function. We set out below the assumptions that must be satisfied for this result to be valid.

Assumption 1 (Rationality) The preference ordering $\mathcal{R}$ is rational if it satisfies the following properties:

- Reflexivity: $\forall X \in \Omega$, X$\mathcal{R}$X;
- Completeness: $\forall X, Y \in \Omega$, X$\mathcal{R}$Y or Y$\mathcal{R}$X;
- Transitivity: $\forall X, Y, Z \in \Omega$, if X$\mathcal{R}$Y and Y$\mathcal{R}$Z then X$\mathcal{R}$Z.
- Continuity: $\forall Y \in \Omega$, {X $\in \Omega$ | X$\mathcal{R}$Y } and {X $\in \Omega$ | Y$\mathcal{R}$X } are closed set.

Assumption 2 (Continuity) The preference ordering $\mathcal{R}$ is continuous if $\forall\, X, Y, Z \in \Omega$ such that X$\mathcal{R}$Y and Y$\mathcal{R}$Z, there exists $\alpha \in$ [0,1] such that $(\alpha X + (1-\alpha)Z)\ \mathcal{I}$Y.

Assumption 3 (Independence) The preference ordering $\mathcal{R}$ is independent if $\forall\, X, Y, Z \in \Omega$ and $\forall \alpha \in [0, 1]$, X$\mathcal{R}$Y if and only if $(\alpha X + (1-\alpha)Z)\ \mathcal{R}\ (\alpha Y + (1-\alpha)Z)$.

Finally, we observe that the assumption of independence is a rather strong hypothesis because it assumes that the individual is always able to identify the common part of two lotteries and evaluate them only for what they differ.

1.3.2 Risk Aversion

The form of the utility function must be able to reflect not only the preferences of the individual in terms of expected returns, but also in terms of risk appetite. A lottery is said to be *currently fair* if it has an expected value of zero. Given a level of wealth obtained with certainty, an individual is said to be:

- *Risk averse* if he does not accept or is indifferent to any currently fair lottery;
- *Risk neutral* if every currently fair lottery is indifferent to him;
- *Risk loving* if he accepts every currently fair lottery;
- *Strictly risk averse* if he rejects any currently fair lottery.

We can link an individual's risk aversion to the characteristics of the utility function, i.e. an individual is (strictly) risk averse if and only if u is (strictly) convex. In fact from the inequality of Jensen for $X \in \Omega$ and u concave we will have that

$$\mathbb{E}[u(X)] \leq u(\mathbb{E}[X])$$

i.e the risk averse individual prefers to reject any lottery and to have with certainty his expected value.

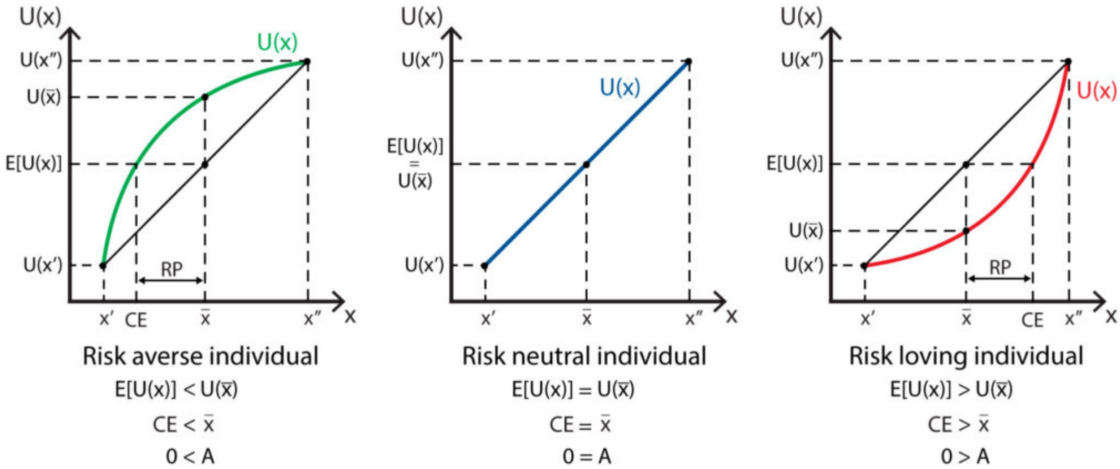

Figure 1.1: Behaviour of utility function in different case of risk aversion

The difference between the expected value and the certainty equivalent is called the *risk premium* ρ_u (RP in the Figure 1.1). For risk averse individuals with increasing $u(\cdot)$, the risk premium is positive, for risk neutral persons it is zero, and for risk loving individuals their risk premium is negative.

$$u(\mathbb{E}[X] - \rho_u(X)) = \mathbb{E}[u(X)] \tag{1.17}$$

where the quantity $\mathbb{E}[X] - \rho_u(X)$ is called certainty equivalent. Assuming that $X = x + \epsilon$ with $x \in \mathbb{R}$, $\epsilon \in \Omega$ random variable with zero mean and variance σ^2 and u is twice differentiable, then we can write the Taylor series with respect to ϵ

$$u(x+\epsilon) = u(x) + \epsilon u'(x) + \frac{\epsilon^2}{2} u''(x) + o(\epsilon^2).$$

Taking the expected value we obtain

$$\mathbb{E}[u(X)] = \mathbb{E}[u(x+\epsilon)] = u(x) + \frac{\sigma^2}{2} u''(x). \tag{1.18}$$

We develop now at the first order u in correspondence of the certain equivalent

$$u(x - \rho_u(X)) = u(x) - \rho_u(X) u'(x) + o(\rho_u(X)) \tag{1.19}$$

from (1.17) putting equal the right terms of (1.18) and (1.19), we obtain an expression for the risk premium

$$\rho_u(X) \approx -\frac{1}{2} \frac{u''(x)}{u'(x)} \sigma^2.$$

Thanks to this characterisation the risk premium can be broken down into an objective factor given by the lottery variance and a subjective factor given by the relationship between the second derivative and the first derivative of the utility function. The latter is called the *coefficient of absolute risk aversion*

$$r_u^A(x) = -\frac{u''(x)}{u'(x)}$$

and characterizes different classes of utility functions. Another indicator for classifying utility functions is the *coefficient of relative risk aversion*,

$$r_u^R(x) = -x \frac{u''(x)}{u'(x)}$$

which depends directly on wealth x.

1.3.3 Classification of utility functions

An initial classification of utility functions can be made from the behaviour of the absolute risk aversion coefficient as wealth changes x. We will have the following classes of utility functions:

- $u \in$ DARA (Decreasing Absolute Risk Aversion) if $\frac{dr_u^A(x)}{dx} < 0 \ \forall x \in \mathbb{R}^+$, $r_u^A(x)$ decreasing as wealth grows;
- $u \in$ CARA (Constant Absolute Risk Aversion) if $\frac{dr_u^A(x)}{dx} = 0 \ \forall x \in \mathbb{R}^+$, $r_u^A(x)$ constant as wealth grows;

- $u \in$ IARA (Increasing Absolute Risk Aversion) if $\frac{dr_u^A(x)}{dx} > 0\ \forall x \in \mathbb{R}^+$, $r_u^A(x)$ increasing as wealth grows.

Let us list the main utility functions that are encountered in economic problems, specifying the class of each of them.

Exponential utility function

$$u(x) = -\frac{1}{a}e^{-ax}$$

with $a > 0$, it belongs to CARA utility function, in fact its coefficient of absolute risk aversion is

$$r_u^A(x) = -\frac{-ae^{-ax}}{e^{-ax}} = a$$

while the coefficient of relative risk aversion is increasing $r_u^R(x) = ax$.

Quadratic utility function

$$u(x) = x - \frac{b}{2}x^2$$

with $b > 0$, it belongs to IARA utility function, in fact its coefficient of absolute risk aversion is

$$r_u^A(x) = -\frac{b}{1 - bx}.$$

We notice that for $0 \leq x < \frac{1}{b}$ is positive in x while for $\frac{1}{b} < x$ is negative but always increasing. The coefficient of relative risk aversion is increasing in x $r_u^R(x) = \frac{bx}{1-bx}$.

Power utility function

$$u(x) = \frac{1}{1-\gamma}x^{1-\gamma}$$

with $\gamma > 0$ and $\gamma \neq 1$, it belongs to DARA utility function, in fact its coefficient of absolute risk aversion is

$$r_u^A(x) = \frac{\gamma}{x}.$$

We notice that the coefficient of absolute risk aversion is decreasing with respect to x. The coefficient of relative risk aversion is constant $r_u^R(x) = \gamma$, this property allows that often the power utility is also identified in the CRRA class (Constant Relative Risk Aversion).

Logarithmic utility function

$$u(x) = ln(x)$$

it belongs to DARA utility function, in fact its coefficient of absolute risk aversion is

$$r_u^A(x) = \frac{1}{x}.$$

The coefficient of relative risk aversion is $r_u^R(x) = 1$, this property allows that often the power utility is also identified in the CRRA class (Constant Relative Risk Aversion).
We also note that the logarithmic utility function can be obtained from the power utility function by sending $\gamma \to 1$.

The mentioned utility functions belong to the broader category of HARA functions (Hyperbolic Absolute Risk Aversion).
The HARA utility function is presented in the form

$$u(x) = \frac{1-\gamma}{\gamma}\left[\frac{ax}{1-\gamma} + b\right]^\gamma$$

with $b > 0$, $\gamma \neq 1$ (sending $\gamma \to 1$ we obtain a linear utility function). The coefficient of absolute risk aversion is

$$r_u^A(x) = a\left(\frac{ax}{1-\gamma} + b\right)^{-1}.$$

We note that for a HARA function, risk tolerance defined as

$$t_u(x) = \frac{1}{r_u^A(x)}$$

is a linear function of wealth: $t_u(x) = A + Bx$ with A $= \frac{b}{a}$ and B $= \frac{1}{1-\gamma}$.

Chapter 2

Portfolio Optimization

The optimization problem of a portfolio mainly consists in describing the investment choices of an individual whose degree of risk aversion is known, described by the utility function. The model in its first version, assumes a market with a certain number of risky securities and a security without risk. The individual has an initial wealth to invest freely in the available *assets* and the objective function to maximize is given by the function of expected utility of the wealth to a certain future moment that represents the temporal horizon of the optimization problem. This problem has been tackled by the American economist Robert C. Merton.
The main assumption of the Merton portfolio allocation problem is to be in the framework of the Black-Scholes model.
In section 2.1 we will present the Black-Scholes model deriving the formula and the equation of the dynamics of an option. In section 2.2 we provide the solution of portfolio optimization studied by Merton, in which only a risk-free and a risky assets are considered.

2.1 Black-Scholes model

The Black-Scholes is a pricing model used to evaluate option prices. The derivation of Black-Scholes formula does not use stochastic calculus, it is essential to understand significance of Black-Scholes equation which is one of the most famous applications of Itò lemma.

In finance, what is an option? An option is a contract which gives the buyer (the owner or holder of the option) the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price prior to or on a specified date, depending on the form of the option. The strike price may be set by reference to the spot price (market price) of the underlying security or commodity on the day an option is taken out, or it may be fixed at a discount or at a premium. The seller has the corresponding obligation to fulfill the transaction, to sell or buy, if the buyer (owner) exercises the option.
The basic option are: the *call option* that conveys to the owner the right to buy

at a specific price and the options that conveys the right of the owner to sell at a specific price is referred to as a *put option.* There are numerous variants to the basic definition of option, also called *plain vanilla option*, and they are represented by *exotic options.*
We can define the *European option* which is a type of option that can be exercised only on a specified future date.

In the following sub-sections we find the evolution and the formula that provides the price of a Call option. The payoff of the Call option C_t written on an underlying S_t with strike price K is

$$C_t = max(S_t - K, 0).$$

2.1.1 Black-Scholes equation

The investors are often interested in predicting the future price of an option to build a profitable portfolio.
Black-Scholes partial differential equation does the work by describing the price of option over time.

Theorem 2.1.1 (Black-Scholes Equation)**.** Let the value of an option be $f(t, S_t)$, standard deviation of stock be stock's returns be σ, and risk-free interest rate be r. Then the price of an option over time can be expressed by the following partial differential equation:

$$\frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S_t}S_t + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial S_t^2} = rf. \tag{2.1}$$

Proof. The proof can be found in section 5 of *Stochastic Calculus and Black-Scholes model* by Younggeun Yoo [26]. □

By hedging away all randomness, we make sure that the portfolio has no risk, and that allows us to use the assumption of risk-neutrality. If we did not do this, then it seems very natural that the price of an option must take on the perceived risk with which the investor views the stock.

In the proof of the Black-Scholes equation there is an intermediate step that is useful to us in the discussion.
The evolution of the stock (risky asset) is described by a Geometric Brownian motion, i.e.

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

and

$$dS_t^2 = \mu^2 S_t^2 dt^2 + \mu\sigma S_t^2 dt \, dW_t + \sigma^2 S_t^2 dW_t^2 = \sigma^2 S_t^2 dt.$$

Thanks to Ito's formula (1.3) and the above properties of the stock, we are able to find the dynamics of the Call option C_t in Black-Scholes framework.

$$dC_t = \mu_c \, dt + \sigma_c \, dW_t \tag{2.2}$$

where

$$\mu_c = \frac{\partial C_t}{\partial t} + \mu S_t \frac{\partial C_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2},$$

$$\sigma_c = \sigma S_t \frac{\partial C_t}{\partial S_t}.$$

2.1.2 Black-Scholes Formula

Theorem 2.1.2 (Black-Scholes Formula)**.** The value of an European call option (C_0) can be calculated given its stock price (S_0), exercise price (K), time to expiration (T), standard deviation of log returns (σ), and risk free interest rate (r). Assume that the option satisfied the following conditions:

- The short-term interest rate is known and is constant thought time;
- The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal. The variance rate of the return on the stock is constant;
- The stock pays no dividends or other distributions;
- The option is "European", it can only be exercised at maturity;
- There are no transaction costs in buying or selling the stock or the option;
- It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate;
- There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

Then, the price can be calculated by

$$C_0 = S_0 N(d_1) - K e^{-rT} N(d_2), \tag{2.3}$$

where

$$d_1 = \frac{ln(\frac{S_0}{k}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = \frac{ln(\frac{S_0}{k}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and N(x) represents a cumulative distribution function for normally distributed random variable x.

Proof. The proof can be found in section 4 of *Stochastic Calculus and Black-Scholes model* by Younggeun Yoo [26]. □

The formula gave a good approximation, they found that the option buyers pay prices consistently higher than those predicted by the formula.
In real market, real interest rates are not constant as assumed in Black-Scholes model. Most stocks pay some form of distributions including dividends. Due to such factors, volatility (σ) in Black-Scholes formula may be underestimated. Since the price of an option (C_0) is a monotonically increasing function of the volatility, such a difference in volatility could be one for underestimation of option prices.

2.2 Merton problem

The Merton problem is the first continuous time optimal investment model that assumes a market with N risky and a risk-free assets. The dynamic of each risky asset is described by a log-normal process.
Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, $W = (W_t^1, ..., W_t^N)_{t\in\mathbb{T}}$ N-dimensional Brownian motion with $\mathbb{T} = [0, T]$ which represents the time interval over which the problem is defined, and $(\mathscr{F})_{t\in\mathbb{T}}$ is complete natural filtration.
The dynamic of the risky asset (stock) is the following

$$dS_t^i = \mu^i S_t^i \ dt \ + \sum_{j=1}^{N} \sigma^{ij} \ S_t^i \ dW_t^j \tag{2.4}$$

with the initial condition $S_0^i = s_0^i$.
i=1,..N, $\mu = (\mu^1, ..., \mu^N)$ constant vector in $\mathbb{R}^N$ and $\sigma = (\sigma^{ij})_{i,j=1,..N}$ a constant matrix in $\mathbb{R}^{N\times N}$. Let $r > 0$ be the interest rate that guarantees a non-risk investment, for example a bank deposit, then the risk-free asset (bond) will have dynamics

$$dB_t = r \ B_t \ dt, \quad B_0 = 1. \tag{2.5}$$

We introduce the vector process $\pi_t = (\pi_t^1, ...\pi_t^N)_{t\in\mathbb{T}}$ that represent an amount of wealth invested in risky securities, and $(1 - \sum_{i=1}^N \pi_t^i)$ amount of wealth invested in the bond. We notice that the process vector is closely connected with the process that describes the dynamics of wealth as we will see later and so it will often be constructed through a measurable function $\pi : [0, T] \times \mathbb{R} \to \mathbb{R}^n$ of time and wealth. More precisely we have that if $X = (X_t)_{t\in\mathbb{T}}$ is the process that describes the dynamics of portfolio wealth starting from an initial wealth $X_0 = 1$, then the process π is defined as $\pi_t = \pi(t, X_t)$.
Let us now formulate the optimization problem: given an utility function $u : \mathbb{R}^+ \to \mathbb{R}$ which describes the level of utility of the individual, the aim is to find the optimal control given by the vector $\hat{\pi}$, which maximizes the expected utility

$$v(t, x) = \sup_{\pi} \mathbb{E}[u(X_T^{t,x})]$$

where $v(t, x)$ is the value function of the problem. We study the case in which we consider the power utility function, i.e. fixed $\gamma > 0$ and $\gamma \neq 1$, we have

$$u(x) = \frac{1}{1-\gamma} x^{1-\gamma}.$$

2.2.1 Budget equation

We consider a portfolio in which, given an initial wealth, it is no longer possible to deposit or withdraw money, and whose dynamics are therefore linked only to fluctuations in the value of the assets that compose it. A portfolio of this type is called *self-financing.* The dynamics of a self-financing portfolio is summarised in the following proposition:

Proposition 2.2.1. Let X be a stochastic process that described the wealth of a self-financing portfolio. Given N risky assets $\{S_t^i\}_{i=1,..N}$ and a risk-free asset B_t, the dynamic at time t of a portfolio investing a wealth π_t^i on S_t^i risky asset with $i = 1, ...N$ and a wealth $(1 - \sum_{i=1}^N \pi_t^i)$ on risk-free asset B_t, is given by

$$dX_t = \sum_{i=1}^{N} \frac{dS_t^i}{S_t^i} \pi_t^i + \frac{dB_t}{B_t}\left(X_t - \sum_{i=1}^{N} \pi_t^i\right). \tag{2.6}$$

Proof. For the proof we refer to *Arbitrage Theory in Continuous Time* by *T.Björk* [23]. Let $h^i(t)$ the number of shares held by the i-th asset, i.e.

$$h^i(t) = \frac{\pi_t^i}{S_t^i} \quad i = 1, ...N,$$

$$h^0(t) = \frac{\left(X_t - \sum_{i=1}^N \pi_t^i\right)}{B_t}.$$

Let us assume that we can only re-balance the portfolio after an interval Δt, then the wealth at time t can be seen as

$$X_t = \sum_{i=1}^{N} h^i(t - \Delta t)S_t^i + h^0(t - \Delta t)B_t.$$

The budget equation at time t for a self-financing portfolio requires that the value of the re-balanced portfolio is equal to the value of the portfolio before re-balancing, i.e.

$$\sum_{i=1}^{N} h^i(t - \Delta t)S_t^i + h^0(t - \Delta t)B_t = \sum_{i=1}^{N} h^i(t)S_t^i + h^0(t)B_t.$$

Thus, introducing the notation $\Delta f(t) = f(t) - f(t - \Delta t)$ we obtain

$$\sum_{i=1}^{N} \Delta h^i(t)\ S_t^i + \Delta h^0(t)\ B_t = 0.$$

Adding and subtracting the term $\sum_{i=1}^N \Delta h^i(t)\ S_{t-\Delta t}^i + \Delta h^0(t)\ B_{t-\Delta t}$ in order to have forward increments

$$\sum_{i=1}^{N} \Delta h^i(t)\ S_{t-\Delta t}^i + \Delta h^0(t)\ B_{t-\Delta t} + \sum_{i=1}^{N} \Delta h^i(t)\ \Delta S_t^i + \Delta h^0(t)\ \Delta B_t = 0.$$

Sending $\Delta t \to 0$, we obtain

$$\sum_{i=1}^{N} dh^i(t)\ S_t^i + dh^0(t)\ B_t + \sum_{i=1}^{N} dh^i(t)\ dS_t^i + dh^0(t)\ dB_t = 0. \tag{2.7}$$

Let us consider the Ito differential in wealth

$$dX_t = \sum_{i=1}^{N} h^i(t)dS_t^i + h^0(t)dB_t + \sum_{i=1}^{N} dh^i(t)S_t^i + dh^0(t)B_t + \sum_{i=1}^{N} dh^i(t)dS_t + dh^0(t)dB_t,$$

using (2.7) we obtain the Budget equation for a self-financing portfolio

$$dX_t = \sum_{i=1}^{N} h^i(t)\ dS_t^i + h^0(t)\ dB_t$$

which rewritten in terms of the percentage of wealth invested in assets, we obtain

$$dX_t = \sum_{i=1}^{N} \frac{dS_t^i}{S_t^i}\ \pi_t^i + \frac{dB_t}{B_t}\Big(X_t - \sum_{i=1}^{N} \pi_t^i\Big).$$

□

2.2.2 Hamilton Jacobi Bellman

Let us examine the particular case where there is only one risky and one risk-free assets to simplify the handling. We consider $W = (W_t, t \in \mathbb{T})$ a one-dimensional Brownian motion, S risky asset and B a risk-free asset with the following dynamics:

$$dS_t = \mu S_t\ dt + \sigma S_t\ dW_t, \quad S_0 = s_0$$
$$dB_t = rB_t\ dt, \quad B_0 = 1$$

with $\mu, \sigma \in \mathbb{R}$ and $r \geq 0$ constant. An agent invests a portion π_t of this wealth in a stock of price S and (1-π_t) in a bond of price B. His wealth process evolves according to

$$dX_t = \frac{X_t\pi_t}{S_t}\ dS_t + \frac{X_t(1-\pi_t)}{B_t}\ dB_t = X_t(\pi_t\mu + (1-\pi_t)r)\ dt + X_t\pi_t\sigma\ dW_t. \quad (2.8)$$

We denote by $\mathscr{A}$ the set of progressively measurable process π valued in A, and such that $\int_0^T |\pi_s|^2 ds < \infty$ a.s. This integrability condition ensures the existence and uniqueness of a strong solution to the SDE governing the wealth process controlled by $\alpha \in \mathscr{A}$. Given a portfolio strategy $\alpha \in \mathscr{A}$, we denote by $X^{t,x}$ the corresponding wealth process starting from an initial capital $X_t = x > 0$ at time t. The agent wants to maximize the expected utility from terminal wealth at horizon T. The value function of the utility maximization problem is then defined by

$$v(t,x) = \sup_{\alpha \in \mathscr{A}} \mathbb{E}[u(X_T^{t,x})], \quad (t,x) \in [0,T] \times \mathbb{R}_+.$$

The utility function u is increasing and concave on $\mathbb{R}_+$. Let us check that for all $t \in [0,T]$, $v(t,\cdot)$ is also increasing and concave in x.

We define the infinitesimal diffusion $\mathcal{L}^\pi$ generator associated with equation (2.8) as

$$\mathcal{L}^\pi v(t,x) = (\pi(\mu - r) + rx)\frac{\partial v}{\partial x}(t,x) + \frac{1}{2}\pi^2\sigma^2\frac{\partial^2 v}{\partial x^2}(t,x). \tag{2.9}$$

Following the principle of dynamic programming, given $h > 0$ we have that

$$v(t,x) \geq \mathbb{E}[v(t+h, X_{t+h}^{t,x})]$$

since $v \in C^0([0,T]\times\mathbb{R}) \cap C^{1,2}([0,T]\times\mathbb{R})$, for hypothesis we can apply the Ito formula

$$v(t,x) \geq \mathbb{E}\left[v(t,x) + \int_t^{t+h}\left(\frac{\partial v}{\partial t} + \mathcal{L}^{\pi_s} v\right)(s, X_s)ds + \int_t^{t+h}\pi_s\sigma X_s^{t,x}\frac{\partial v}{\partial x}(s, X_s^{t,x})dW_s\right]$$

where $\mathcal{L}^\pi$ is defined like in (2.9). Sending $h \to 0$ and if the last term of inequality is a martingale, we have that

$$0 \geq \left(\frac{\partial v}{\partial t} + \mathcal{L}^\pi v\right)(t,x).$$

In particular, the inequality above becomes an equality if I consider the control $\hat{\pi}$ such that it achieved by $\sup_\pi$. Now, we are able to formulate the Hamilton Jacobi Bellman equation

$$-\frac{\partial v}{\partial t}(t,x) - \sup_\pi \mathcal{L}^\pi v(t,x) = 0 \qquad (t,x) \in [0,T)\times\mathbb{R},$$

$$v(T,x) = u(x) \qquad x \in \mathbb{R}.$$

This is analogous to (1.14) in the case where the running profit function f is identically zero and the terminal reward function is u.
Let's assume that the solution of Hamilton Jacobi Bellman equation has the form $v(t,x) = \varphi(t)\ u(x) = \varphi\ \frac{1}{1-\gamma}x^{1-\gamma}$ with φ such that $\varphi(T) = 1$ so that the final condition is verified. We have that

$$\frac{\partial v}{\partial x}(t,x) = \varphi(t)x^{-\gamma}, \qquad \frac{\partial^2 v}{\partial x^2}(t,x) = -\gamma\ \varphi(t)\ x^{-\gamma-1}.$$

By replacing the derivatives in the Hamilton Jacobi Bellman equation and explicitly rewriting the infinitesimal diffusion generator we obtain

$$-\varphi'(t)\frac{x^{1-\gamma}}{1-\gamma} - \sup_\pi\left\{[\pi(\mu-r) + rx]\varphi(t)\ x^{-\gamma} - \frac{1}{2}\pi^2\sigma^2\varphi(t)\gamma x^{-\gamma-1}\right\} = 0$$

with $(t,x) \in [0,T)\times\mathbb{R}$.
From which, by collecting and simplifying for $\frac{x^{1-\gamma}}{1-\gamma}$, we derive the ordinary differential equation

$$-\varphi'(t) - \varphi(t)(1-\gamma)\sup_\pi\left\{[\pi(\mu-r) + rx]\frac{1}{x} - \frac{1}{2}\pi^2\sigma^2\gamma\frac{1}{x^2}\right\} = 0 \qquad t \in [0,T),$$

$$\varphi(T) = 1.$$

Searching now for the portfolio π that realizes the $\sup_\pi$, setting to zero the derivative with respect to π of the term between the curly brackets, we get

$$\hat{\pi} = \frac{(\mu - r)}{\sigma^2 \ \gamma} x.$$

We note that the portfolio at instant t is a function of wealth at the same time or rather the optimal portfolio represents a Markovian feedback control in the sense that the portfolio is automatically re-balanced by intervening on the dynamics of wealth starting from the value of wealth at the last recorded moment. In these cases we speak of closed-loop control. Replacing $\hat{\pi}$ in the differential equation for φ, we obtain

$$-\varphi'(t) - \rho\varphi(t) = 0 \qquad t \in [0, T),$$
$$\varphi(T) = 1$$

where

$$\rho = (1 - \gamma)\left(r + \frac{(\mu - r)^2}{2\sigma^2\gamma}\right).$$

The solution of the differential equation will be $\varphi(t) = e^{\rho(T-t)}$ and therefore the solution of Hamilton Jacobi Bellman equation can be written explicitly as

$$v(t, x) = e^{\rho(T-t)} \frac{x^{1-\gamma}}{1 - \gamma}.$$

Chapter 3

Equivalence of the static and dynamic asset allocation problems

Our goal in this discussion is to extend Merton's problem to the case where we consider: an risky asset (stock), a risk-free asset (bond) and options written on stock. For now we will consider the Black and Scholes framework (The assumptions are explained in section 2.1 of chapter 2).

A classic *dynamic asset allocation problem* optimizes the expected final-time utility of wealth for an individual who can invest in a risky stock and a risk-free bond, trading continuously in time. We want to consider the corresponding *static asset allocation problem* in which the individual cannot trade but can invest in options as well as the underlying. Surprisingly, however, for some market models the two approaches are equivalent.
In section 3.1 we provide the necessary and sufficient conditions in order for the static and dynamic approach to be equivalent, we refer to *On the equivalence of the static and dynamic asset allocation problems* by R.V.Kohn and O.M. Papazoglu-Statescu [22]. In section 3.2 we present the equivalence through a numerical example.

3.1 Theoretical results of equivalence

The classic model is
Dynamic Asset Allocation Problem: Consider an individual who can invest in a risky stock and a risk-free bond, trading continuously, in time. Suppose the stock price follows a known diffusion process: $dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t$, and assume there is no consumption.
Merton found the answer in 1969 using the method of dynamic programming and we presented the solution in section 2.2 of chapter 2.

Static Asset Allocation Problem: Consider an individual whose investment opportunities include not only a risk-free bond and a risky stock, but also (European style) derivatives on the stock. However this individual cannot trade, he is a *buy-and-hold* investor, who assumes an initial portfolio then holds it to maturity.

The optimal static asset allocation can do no better than the optimal dynamic one, since the market is complete every option can be replicated by a trading strategy involving just stock and the risk-free bond. Thus the static problem maximizes the expected final-time utility over a restricted class of trading strategies, those that replicate options.
When the two problems are equivalent or nearly so, the static strategy is clearly preferable, indeed, it avoids exposure to market frictions such as transaction cost and limited liquidity. Such frictions were ignored in formulating and solving the dynamic asset allocation problem, but their effect can be significant in practise.
The static and dynamic approaches are equivalent when the underlying stock process is log-normal.
The static investor can buy any option, i.e. he can buy an option with any payoff. We consider portfolios with many call or put options since it is convenient because permit us to use continuous methods.

The main assumption is that the market is **complete** and as noted above, the static problem maximizes the expected final-time utility over a restricted class of trading strategies. One can of course consider the static asset allocation problem even when the market is incomplete but in this case there is no relation between the static and dynamic problems. Options cannot be replicated, so they are redundant for the dynamic investor, and it is natural to include them among the admissible investments.

3.1.1 Conditions for completeness of the market

In this part of the treatment we require that the market be complete. This places certain conditions on μ and σ, r. The main conditions involve the market price of risk, defined by

$$\theta_t = \frac{\mu(t, S_t) - r}{\sigma(t, S_t)}.$$

It must satisfy

$$\int_0^T \theta_t^2 dt < \infty$$

almost surely; moreover the associated local martingale

$$Z_0(t) = exp\Big[- \int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds \Big]$$

must be a martingale. A well-known sufficient condition is the Novikov criterion:

$$\mathbb{E}\left[exp\left(\frac{1}{2}\int_0^T \theta_t^2 dt\right)\right] < \infty.$$

The martingale Z_0 is the density of the risk-neutral measure with respect to the subjective measure. In particular, the initial value of an option with payoff f at time T is $e^{-rT}\mathbb{E}[Z_0(t)f(S_T)]$.
For a more detailed discussion, and a proof that the preceding conditions imply completeness, see chapter 1 of *Methods of Mathematical Finance* by I.Karatzas and S.Shreve [9].

3.1.2 Martingale approach to Merton's problem

In order to solve the Merton problem we can use alternative approach: *Martingale method*, studied by Pliska, Cox and Huang in the 1980s, that provides the same solution seen in chapter 2.
Consider, for any utility function u, the problem

$$\max_{\pi} \mathbb{E}[u(X_{\pi,T})] \tag{3.1}$$

where π ranges over all admissible trading strategies with fixed initial wealth X_0 and $X_{\pi,T}$ is the time T wealth achieved by π. The martingale approach splits the problem into two sub-problems:

- Find the optimal final-time wealth by solving

  $$\max_{\mathbb{E}[H_T X]=X_0} \mathbb{E}[u(X)] \tag{3.2}$$

  over all time T measurable random variable X. We can define the state price density by

  $$H_t = exp\left(-rt - \int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right).$$

- Find an admissible trading strategy π that achieves the optimal X identified in step 1.

The first sub-problem derives from a reformulation of the objective function (3.1) as a static optimization problem, with a constraint that the discounted portfolio has to be equal to X_0 to make it feasible.

$$\max_{\pi} \mathbb{E}[u(X_{\pi,T})]$$

$$s.t.\ \mathbb{E}[H_T X] = X_0 \tag{3.3}$$

where the constraint (3.3) is widely referred to as the budget constraint in academic literature.

Since the market is complete, the second sub-problem always has a solution. Therefore to find the final optimal final-time wealth we need only consider the first sub-problem. In order to find the solution of (3.2) we use the method of Lagrange multipliers. The Lagrangian corresponding to (3.2) is

$$L(X,\lambda) = \mathbb{E}[u(X) - \lambda(H_T X - X_0)].$$

The optimal X and the Lagrange multiplier λ are characterized by the first-order optimality conditions of the Lagrangian. Taking the first variation with respect to λ gives, as usual, the constraint we started with:

$$L_\lambda(X,\lambda) = X_0 - \mathbb{E}[H_T X] = 0. \tag{3.4}$$

Taking the first variation with respect to X gives

$$< L_X(X,\lambda), \delta X > = \ \mathbb{E}\left[\Big(u'(X) - \lambda H_T\Big)\delta X\right] = 0 \tag{3.5}$$

for every perturbation δX; this implies $u'(X) = \lambda H_T$. There exists a unique solution of $u'(X) = \lambda H_T$, namely, $X = (u')^{-1}(\lambda H_T)$. This formula gives the optimal X_T. It remains only to specify the Lagrangian multiplier λ; it is determined by (3.4) since

$$X_0 = \mathbb{E}[H_T(u')^{-1}(\lambda H_T)] := F(\lambda) \ \Rightarrow \ \lambda = F^{-1}(X_0).$$

In solving for λ, we have used the fact that the inverse of F exists. One can see that this is true by taking the derivative of F with respect to λ, using the fact that H_T is positive and the hypothesis that u (being a utility function) is strictly concave. In conclusion: the optimal final time wealth is given by

$$X_T = (u')^{-1}(F^{-1}(X_0)H_T). \tag{3.6}$$

3.1.3 Necessary and sufficient condition for equivalence

Our starting point is the observation that the static and dynamic problems are equivalent if and only if Merton's final time wealth X_T is **path-independent**, i.e. X_T is path-independent if and only if there exists a function f such that $X_t = f(t, S_t)$, for every $t \in [0,T]$. Our goal is thus to understand when the right-hand side of (3.6) is a path-independent function of the final-time stock price S_T for every T. By inspection, this amounts to asking when the state price density H_t is a path-independent function of S_t for all t.
It will be convenient to work with the logarithm of the stock price

$$P_t = ln\ S_t.$$

Clearly H_t is a path-independent function of S_t if and only if there exists a deterministic function $g(t, P_t)$ such that $H_t = g(t, P_t)$ for all t. The following theorem gives a necessary and sufficient condition by identifying g (if it exists) as the solution of a suitable PDE, allowing us to say that the static and dynamic problem are equivalent (Theorem 1 of [22]).

Theorem 3.1.1. Assume the market model satisfied the condition summarized in subsection 3.1.1. Then the static and dynamic problems are equivalent if and only if there exists a function $g(t, P_t)$ with $g(0, P_0) = 0$ such that the following relations hold:

$$\frac{\mu - r}{\sigma^2} = g_P, \tag{3.7}$$

$$\frac{\mu - r}{\sigma^2}\left(\frac{-\mu - r + \sigma^2}{2}\right) = \frac{1}{2} g_{PP}\sigma^2 + g_t, \tag{3.8}$$

where $g_P = \partial g/\partial P$, $g_{PP} = \partial^2 g/\partial P^2$ and $g_t = \partial g/\partial t$.

Proof. Consider

$$h_t = \int_0^t \frac{\mu - r}{\sigma} dW_s + \frac{1}{2}\int_0^t \left(\frac{\mu - r}{\sigma}\right)^2 ds.$$

Note that $h_0 = 0$. From the definition of h we have

$$dh_t = \frac{\mu - r}{\sigma}\, dW_t + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 dt. \tag{3.9}$$

As explained above, the static and dynamic problems are equivalent if and only if

$$h_t = g(t, P_t) \tag{3.10}$$

for some function g.

Suppose there is such a g. Then we can find an SDE for h by applying Ito's lemma to the right-hand side of (3.10). The SDE for $P_t = lnS_t$ is

$$dP_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t$$

so Ito's lemma applied to $g(t, P_t)$ gives

$$dh_t = \left[g_P\left(\mu - \frac{\sigma^2}{2}\right) + \frac{1}{2} g_{PP}\, \sigma^2 + g_t\right] dt + g_P\, \sigma dW_t. \tag{3.11}$$

The SDE associated with the diffusion process is unique, so the corresponding terms in (3.9) and (3.11) must be identical. The condition that the coefficients of dW_t match is precisely (3.7). The condition that the dt terms match is

$$g_P\left(\mu - \frac{\sigma^2}{2}\right) + \frac{1}{2} g_{PP}\, \sigma^2 + g_t = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2.$$

With the aid of (3.7), we can rewrite this as

$$\frac{\mu - r}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right) + \frac{1}{2} g_{PP}\, \sigma^2 + g_t = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2$$

or equivalently as

$$\frac{1}{2}g_{PP}\ \sigma^2 + g_t = \frac{1}{2}\bigg(\frac{\mu - r}{\sigma^2}\bigg)\bigg((\mu - r) - 2\Big(\mu - \frac{\sigma^2}{2}\Big)\bigg).$$

Thus (3.8) holds too.
The preceding calculation is reversible. If (3.7) and (3.8) hold then the SDE characterizing h is the same as the one solved by $g(t, P_t)$. If in addition $g(0, P_0) = 0$ then the initial conditions match as well, and it follows that $h_t = g(t, P_t)$ for all t. □

Remark: When the static and dynamic problems are equivalent, the proof of Theorem 3.1.1 gives a formula for the optimal final-time wealth. Indeed, when $h_T = g(T, P_T)$ we have $H_T = e^{-rT}e^{g(T,P_T)}$, so

$$X_T = (u')^{-1}(\lambda e^{-rT}e^{g(T,P_T)}).$$

Simplified conditions when μ and σ depend only on t
Our necessary and sufficient condition simplifies dramatically when μ and σ depend on time alone. This leads to a simple, explicit condition for path independence of X_T.

Theorem 3.1.2. Suppose the stock price process is $dS_t = \mu_t S_t\ dt + \sigma_t S_t\ dW_t$ where μ and σ are deterministic functions of time alone. Assume the "*usual conditions*" hold. Then the static and dynamic problems are equivalent if and only if

$$\frac{\mu_t - r}{\sigma_t^2} = constant.$$

Proof. Assume $(\mu_t - r)/\sigma_t^2$ is constant, and call its value a. We shall find the associated solution of (3.7)-(3.8) explicitly. Remember that in (3.7) and (3.8), the derivatives of g are taken with respect to $P_t = ln(S_t)$.
From (3.7) we have that $g_P = constant = a$ which implies $g(t, P_t) = aP_t + c_t$. Then from (3.8) we have

$$a\bigg[-\frac{\mu}{2} + \frac{-r + \sigma^2}{2}\bigg] = c_t' \ \Rightarrow\ a\bigg[-\frac{a\sigma^2 + r}{2} + \frac{-r + \sigma^2}{2}\bigg] = c_t'$$

$$\Rightarrow c_t' = a\bigg(\frac{\sigma^2}{2}(1 - a) - r\bigg) \ \Rightarrow\ c_t = a\bigg(\frac{\sigma^2}{2}(1 - a) - r\bigg)t + constant.$$

Using this, we get $g(t, P_t) = aP_t + a((\sigma^2/2)\ (1 - a) - r)t + constant$. Using the initial condition $g(0, P_0) = 0$, we find that the *constant* is $-aP_0$. Thus finally

$$g(t, P_t) = \frac{\mu - r}{\sigma^2}(P_t - P_0) + \frac{\mu - r}{\sigma^2}\bigg(\frac{-\mu - r + \sigma^2}{2}\bigg)t.$$

One verifies by inspection that if $(\mu_t - r)/\sigma_t^2 = a$ then this g does indeed satisfy $g(0, P_0) = 0$ and our condition (3.7) and (3.8). Thus the static and dynamic problems are equivalent in this case.
Conversely, if the static and dynamic problems are equivalent there must be a function g which satisfies (3.7) and (3.8). Because μ and σ are functions of t alone, (3.7) implies

$$g = \frac{\mu - r}{\sigma^2} P_t + A_t.$$

But then $g_{PP} = 0$ and $g_t = ((\mu - r)/\sigma^2)' P + A_t'$. Since the left side of (3.8) depend only on t this implies that $((\mu - r)/\sigma^2)' = 0$. Thus $(\mu - r)/\sigma^2$ is constant, as asserted. □

3.2 Numerical results of equivalence

In this section we show the equivalence of the Dynamic Asset Allocation Problem and Static Asset Allocation Problem based on numerical result; to do this we follow the *Asset allocation and derivatives* by M.B.Haugh and A.W.Lo [15].

3.2.1 The model

The simplest formulation that we just presented in section (2.2.2), one without intermediate consumption, consists of an investor's objective to maximize the expected utility $\mathbb{E}[u(X_T)]$ of end-of-period wealth X_T by allocating his wealth between two assets, a risky security (the stock) and a riskless security (the bond), over some investment horizon $[0, T]$. The standard asset allocation problem is then:

$$\max_{\pi_t} \ \mathbb{E}[u(X_T)] \tag{3.12}$$

subject to

$$dX_t = (r + \pi_t(\mu - r))X_t dt + \pi_t \sigma X_t dW_t \tag{3.13}$$

where π_t is the fraction of the investor's portfolio invested in the stock at time t and (3.13) is a budget constraint that wealth X_t must satisfy at all time $t \in [0, T]$.

The Static Asset Allocation Problem is a reformulation of the standard asset allocation problem with two modifications: we allow the investor to include up to n European Call Option in his portfolio at time $t = 0$ and we do not allow the investor to trade after setting up his initial portfolio of stocks, bond and options. Specifically, denote by D_i the payoff of a European Call Option with strike price equal to k_i, hence:

$$D_i = (S_T - k_i)^+.$$

Then the *buy-and-hold* asset allocation problem for the investor is given by:

$$\max_{\alpha,\beta,\gamma} \ \mathbb{E}[u(X_T)] \tag{3.14}$$

subject to

$$X_T = \alpha + \beta S_T + \gamma_1 D_1 + \gamma_2 D_2 + ... + \gamma_n D_n, \tag{3.15}$$

$$X_0 = exp(-rT) \ \mathbb{E}[X_T] \tag{3.16}$$

where α and β denote the investor's position in bonds and stocks, and $\gamma_1, \gamma_2, ...\gamma_n$ the number of options with strike price prices $k_1, k_2, ...k_n$ respectively.
The budget constraint (3.16) is highly non linear in the option strikes k_i. Moreover, for certain utility functions, it is necessary to impose solvency constraints to avoid bankruptcy, and such constraints add to the computational complexity of the problem.
Assuming that the strike price k_i are fixed, we reduce the problem to maximizing

a concave objective function subject to linear constraints and we are able to find a unique global optimum. This is done by discretizing the distribution of S_T and solving the Karush-Kush-Tucker conditions which, in this case, are sufficient.

We notice that CRRA utility function is not defined for negative wealth. In such cases, the following $n+2$ solvency constraints must be imposed along with the budget constraint to ensure non negative wealth:

$$0 \leq \alpha$$

$$\begin{gathered} 0 \leq \alpha + \beta k_1 \\ 0 \leq \alpha + \ (\beta + \gamma_1) k_2 - \ \gamma_1 k_1 \\ \vdots \\ 0 \leq \alpha + \ (\beta + \gamma_1 + \cdots + \gamma_{n-1}) k_n - \ (\gamma_1 k_1 + \cdots + \gamma_{n-1} k_{n-1}) \\ 0 \leq \beta + \gamma_1 + \cdots + \gamma_n \\ 0 \leq k_1 \leq k_2 \leq \cdots \leq k_n \end{gathered}$$

In addition to this condition, we impose that the sum of the weights of all assets in the portfolio must be equal to one.

$$\alpha + \beta + \gamma_1 + \cdots + \gamma_n = 1.$$

3.2.2 Numerical example

We provide a numerical example to illustrate the practical relevance of our optimal buy-and-hold portfolio. [1]
The Stock price follows Geometric Brownian motion. We set the following parameters:

$$X_0 = 1 \text{ €}, \qquad S_0 = 1 \text{ €}, \qquad \text{T} = 5 \text{ years},$$

$$\text{r} = 0.05, \qquad \mu = 0.06, \qquad \sigma = 0.30.$$

We use CRRA preferences under each of the three stochastic processes, so we have the following utility function:

$$u(X_T) = \frac{X_T^p}{p}$$

using the relative risk-aversion coefficient of 0.7 (i.e. p = 0.3).
The parameters have been chosen in a personal way, changing them you can notice the same behavior of the portfolio.

[1] The codes written on Matlab that solve the Static Asset Allocation problem with different number of options are reported in the appendix.

A portfolio weight of 15.87% for the stock in the optimal dynamic asset allocation policy, and a certainty equivalent of 3.60 € for X_T^*. The solution to this problem has been found in section 2.2.2 of this discussion.
Now we consider the problem of constructing an optimal buy-and-hold portfolio containing stock, bond and options. In particular in the example we consider only a Call options but considering a Put options we can notice that we obtain the same behaviour of portfolio.

Number of call options	1	2	3
w_0^* for stock	*44.02 %*	*51.07 %*	*51.45 %*
w_0^* for $call_1$ (k_1=0.7)	*- 44.02 %*	*-74.75 %*	*-81.06 %*
w_0^* for $call_2$ (k_2=1)		*23.69 %*	*36.18 %*
w_0^* for $call_3$ (k_3=1.3)			*-6.53 %*
Value of optimal portfolio (€)	*3.59*	*3.59*	*3.59*

Remark. The strike price of the options are placed in the table and their unit of measurement is €. Also this type of parameter is also chosen at pleasure.

With options, the optimal buy and hold portfolio yields 3.59 €, which is 99.87% of the optimal dynamic asset allocation strategy, a strategy that requires continuous trading over a 5-year period. The estimation error is due to the discretization of the distribution of S_T.

Chapter 4

Dynamic derivative strategies with stochastic volatility and price jumps

In this chapter we find the optimal investment strategy including bonds, stocks and derivatives. The problem is solved in a closed form. There are two differences with respect to the problem analyzed in the previous chapter: the portfolio found previously was through a static strategy, we build a *buy and hold* portfolio, i.e the weights that are attributed to the different elements of the portfolio are chosen at time $t = 0$ and are no longer traded, now we analyze the portfolio through a dynamic strategy in which the weights are optimal at any time t. The second change is that derivatives extend the risk and return trade-offs associated with stochastic volatility and price jumps. So we adopt an empirically realistic model for the stock market that incorporate three types of risk factors: diffusive price shocks, volatility risks and jump risks.
In particular in section 4.1 we consider the case where the stock is subject only to volatility risk. In section 4.2 we add the jump risk to the model just analysed. We refer to *Dynamic derivative strategies* by J.Liu and J.Pan [11].

4.1 Derivatives with stochastic volatility

Why do we introduce stochastic volatility and not remain in the Black and Scholes framework?
In the traditional theory of derivative pricing (B&S framework), derivative assets like options are viewed as redundant securities, the payoffs of which can be replicated by portfolios of primary assets. The market is generally assumed to be complete without options, thus in a complete market setting, an exclusion of derivatives is justified by the fact that derivatives are redundant.
When the completeness of the market breaks down, either because of infrequent trading or the presence of additional sources of uncertainly, it becomes sub-optimal to exclude derivatives.

The non linear nature of derivative securities can serve to complete the market. In particular the stock returns are not instantaneously perfectly correlated with time-varying volatility. In this chapter we set the derivative securities written on the stock as non-redundant asset, in this way they can provide differential exposure to the imperfect instantaneous correlation and make the market complete.

4.1.1 The model

We assume that wealth comprises investments in traded assets only: a risk less bond that pays a constant rate of interest r, its instantaneous return is

$$dB_t = r\ B_t\ dt.$$

A risky stock that represent the aggregate equity market. Its instantaneous total return dynamics are given by

$$dS_t = S_t\ (rdt + \sqrt{V_t}\ dZ_s)$$

where S_t denotes the price of the risky asset at time t, $\sqrt{V_t}$ is the time-varying instantaneous standard deviation of the return on the risky asset, and dZ_s is a Brownian motion. We assume that the short rate is constant in order to focus on the stochastic volatility of the risky asset.
From the following setting, the investment opportunity is time-varying. We assume that the instantaneous variance process is

$$dV_t = k(\bar{v} - V_t)dt + \sigma\sqrt{V_t}\ (\rho\ dZ_s + \sqrt{1-\rho^2}\ dZ_v).$$

The instantaneous variance process V is a stochastic process with long-run mean $\bar{v} > 0$, mean-reversion rate $k > 0$, and volatility coefficient $\sigma \geq 0$. This formulation of stochastic volatility (Heston, 1993), allows the diffusive price shock Z_s to enter the volatility dynamics via the constant coefficient $\rho \in (-1, 1)$, introducing correlation between the price and volatility shocks; notice that Z_s and Z_v are assumed to be independent.

We consider the class of derivatives whose time-t price O_t depends on the underlying stock price S_t and the stock volatility V_t thought $O_t = g(S_t, V_t)$ for some function g. At time t will have the following price dynamics

$$dO_t = rO_tdt + (g_sS_t + \sigma\rho g_v)\sqrt{V_t}\ dZ_s + \sigma\sqrt{1-\rho^2}\ g_v(\lambda V_tdt + \sqrt{V_t}dZ_v) \quad (4.1)$$

where λ determines the stochastic volatility risk premium, i.e it controls the additional volatility risk Z_v; g_s and g_v measure the sensitivity of the derivative price to infinitesimal changes in the stock price and volatility, specifically:

$$g_s = \left.\frac{\partial g(s,v)}{\partial s}\right|_{(S_t,V_t)} \qquad g_v = \left.\frac{\partial g(s,v)}{\partial v}\right|_{(S_t,V_t)}.$$

A derivative with non-zero g_s provides exposure to the diffusive price shock Z_s, and a derivative with non-zero g_v provides exposure to additional volatility risk Z_v.

Partial Differential Equation for Option price

In this subsection we explain how we are able to obtain the dynamics of a derivative presented in the formula (4.1). Indeed we have an alternative characterisation of the option price when the stochastic volatility is also an Ito process, namely as the solution of a parabolic partial differential equation similar to the Black-Scholes pricing PDE, but with an extra dimension representing the dependence on the volatility process. We try to construct a hedge portfolio of assets which can be priced by the no-arbitrage principle. Unlike the Black-Scholes case, it is not sufficient to hedge solely with the underlying asset, since the dZ_s term can be balanced but the dZ_v term cannot. Then we try and hedge with the underlying asset and another option which has a different expiration date.
Let $O^{(1)}(s, v, t)$ be the price of an option with expiration date T_1, and try to find processes (a_t, b_t, c_t) such that

$$O^{(1)}_{T_1} = a_{T_1} S_{T_1} + b_{T_1} B_{T_1} + c_{T_1} O^{(2)}_{T_1}. \tag{4.2}$$

$O^{(1)}_t, O^{(2)}_t$ are the price of a European options with the same properties but different expiration date $T_2 > T_1 > t$ and B_t is the price of a riskless bond under the prevailing short-term constant interest rate r. The right-hand side of (4.2) is a portfolio whose payoff at time T_1 equals almost surely the payoff of $O^{(1)}$. In addition, the portfolio is to be self-financing so that

$$dO^{(1)}_t = a_t dS_t + b_t dB_t + c_t dO^{(2)}_t. \tag{4.3}$$

If such a portfolio can be found, for there to be no arbitrage opportunities, it must be that

$$O^{(1)}_t = a_t S_t + b_t B_t + c_t O^{(2)}_t$$

for all $t < T_1$. Expanding (4.3) by Ito's formula,

$$\left(\frac{\partial O^{(1)}}{\partial t} + \mathcal{L}_1 O^{(1)}\right) dt + \frac{\partial O^{(1)}}{\partial s} dS_t + \frac{\partial O^{(1)}}{\partial v} dV_t = \tag{4.4}$$

$$\left(a_t + c_t \frac{\partial O^{(2)}}{\partial s}\right) dS_t + c_t \frac{\partial O^{(2)}}{\partial v} dV_t + \left(c_t \mathcal{L}_1 O^{(2)} + b_t r B_t + \frac{\partial O^{(2)}}{\partial t}\right) dt$$

where

$$\mathcal{L}_1 = \frac{1}{2} v s^2 \frac{\partial^2}{\partial s^2} + \rho \sigma v s \frac{\partial^2}{\partial v \partial s} + \frac{1}{2} \sigma^2 v \frac{\partial^2}{\partial v^2},$$

and $O^{(1)}, O^{(2)}$ are evaluated at (S_t, V_t, t). The risk from the dZ_v terms can be eliminated by balancing the dV_t terms, which gives

$$c_t = \frac{\partial O^{(1)}/\partial v}{\partial O^{(2)}/\partial v},$$

and to eliminate the dZ_s terms associated with S_t, we must have

$$a_t = \frac{\partial O^{(1)}}{\partial s} - c_t \frac{\partial O^{(2)}}{\partial s}.$$

Substituting for a_t, c_t and $b_t B_t = O_t^{(1)} - a_t S_t - c_t O_t^{(2)}$ and comparing dt terms in (4.4) gives

$$\left(\frac{\partial O^{(1)}}{\partial v}\right)^{-1} \mathcal{L}_2 O^{(1)}(S_t, V_t, t) = \left(\frac{\partial O^{(2)}}{\partial v}\right)^{-1} \mathcal{L}_2 O^{(2)}(S_t, V_t, t) \tag{4.5}$$

where

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \mathcal{L}_1 + r\big(s\frac{\partial}{\partial s} - \cdot\big).$$

That is, $\mathcal{L}_2$ is a classical Black-Scholes differential operator with volatility parameter $\sqrt{v}$, plus second-order terms from the V_t diffusion process.
Now, the left-hand side of (4.5) contains terms depending on T_1 but not T_2 and vice versa for the right-hand side. Thus both sides must be equal to a function that does not depend on expiration date. Denoting this function $\lambda V\sqrt{1-\rho^2}\sigma - k(\bar{v} - v)$, the pricing function $O(s, v, t)$, with the dependence on expiry date suppressed, must satisfy the PDE

$$\frac{\partial O}{\partial t} + \frac{1}{2}vs^2\frac{\partial^2 O}{\partial s^2} + \rho\sigma vs\frac{\partial^2 O}{\partial v \partial s} + \frac{1}{2}\sigma^2 v\frac{\partial^2 O}{\partial v^2} + (k(\bar{v} - v) - \lambda V\sqrt{1-\rho^2}\sigma)\frac{\partial O}{\partial v} + \\ + r\big(s\frac{\partial O}{\partial s} - O\big) = 0.$$

To find the dynamics of the derivative we need to replace the PDE in the following formula found through Ito formula

$$\begin{aligned} dO &= \left(\frac{\partial O}{\partial t} + \frac{1}{2}vs^2\frac{\partial^2 O}{\partial s^2} + \rho\sigma vs\frac{\partial^2 O}{\partial v \partial s} + \frac{1}{2}\sigma^2 v\frac{\partial^2 O}{\partial v^2}\right)dt + \frac{\partial O}{\partial s}dS_t + \frac{\partial O}{\partial v}dV_t = \\ &= \left(\frac{\partial O}{\partial t} + \frac{1}{2}vs^2\frac{\partial^2 O}{\partial s^2} + \rho\sigma vs\frac{\partial^2 O}{\partial v \partial s} + \frac{1}{2}\sigma^2 v\frac{\partial^2 O}{\partial v^2}\right)dt + \frac{\partial O}{\partial s}s(rdt + \sqrt{v}dZ_s) \\ &\quad + \frac{\partial O}{\partial v}\big(k(\bar{v} - v)dt + \sigma\sqrt{v}(\rho dZ_s + \sqrt{1-\rho^2}\, dZ_v)\big). \end{aligned}$$

Now let's replace the term $\frac{\partial O}{\partial t}$ with the PDE found above

$$\begin{aligned} dO = \bigg(&-\frac{1}{2}vs^2\frac{\partial^2 O}{\partial s^2} - \rho\sigma vs\frac{\partial^2 O}{\partial v \partial s} - \frac{1}{2}\sigma^2 v\frac{\partial^2 O}{\partial v^2} - (k(\bar{v} - v) - \lambda V\sqrt{1-\rho^2}\sigma)\frac{\partial O}{\partial v} + \\ &- r\big(s\frac{\partial O}{\partial s} - O\big) + \frac{1}{2}vs^2\frac{\partial^2 O}{\partial s^2} + \rho\sigma vs\frac{\partial^2 O}{\partial v \partial s} + \frac{1}{2}\sigma^2 v\frac{\partial^2 O}{\partial v^2}\bigg)dt + \frac{\partial O}{\partial s}s(rdt + \sqrt{v}dZ_s) + \\ &+ \frac{\partial O}{\partial v}\big(k(\bar{v} - v)dt + \sigma\sqrt{v}(\rho dZ_s + \sqrt{1-\rho^2}\, dZ_v)\big). \end{aligned}$$

Adding up all the terms we get

$$dO = rOdt + \left(\frac{\partial O}{\partial s}S_t + \sigma\rho\frac{\partial O}{\partial v}\right)\sqrt{V_t}\, dZ_s + \sigma\sqrt{1-\rho^2}\,\frac{\partial O}{\partial v}(\lambda V_t dt + \sqrt{V_t}dZ_v). \tag{4.6}$$

We found the dynamics of a derivative in case the stochastic volatility follows Heston's model.

4.1.2 The investment problem

The investor starts with positive wealth X_0. Given the opportunity to invest in the riskless asset, the risky stock and the derivative securities, he chooses, at each time t, $0 \le t \le T$, to invest a fraction ϕ_t of his wealth in the stock S_t, and fractions $\psi_t^{(1)}$ and $\psi_t^{(2)}$ in the two derivative securities $O_t^{(1)}$ and $O_t^{(2)}$, respectively. The investment objective is to maximize the expected utility of his terminal wealth X_T,

$$\max_{\phi_t, \psi_t, 0 \le t \le T} \mathbb{E}\left(\frac{X_T^{1-\gamma}}{1-\gamma}\right), \tag{4.7}$$

where $\gamma > 0$ is the relative risk-aversion coefficient of the investor, and where the wealth process satisfies the self-financing condition

$$dX_t = rX_t dt + \theta_t^s X_t \sqrt{V_t} dZ_s + \theta_t^v X_t (\lambda V_t dt + \sqrt{V_t} dZ_v)$$

where θ_t^s, θ_t^v are defined, for given portfolio weights ϕ_t and ψ_t on the stock and the derivatives, by

$$\theta_t^s = \phi_t + \sum_{i=1}^{2} \psi_t^{(i)} \left(\frac{g_s^{(i)} S_t}{O_t^{(i)}} + \rho\sigma \frac{g_v^{(i)}}{O_t^{(i)}}\right); \qquad \theta_t^v = \sigma\sqrt{1-\rho^2} \sum_{i=1}^{2} \psi_t^{(i)} \frac{g_v^{(i)}}{O_t^{(i)}}.$$

Effectively, by taking position ϕ_t and ψ_t on the risky assets, the investor invests θ^s in the diffusive price shock Z_s and θ_v in the additional volatility risk Z_v. For example, a portfolio position θ_t in the risky stock provides exposures only on the diffusive price shock. Similarity, a portfolio position ψ_t in the derivative security provides exposure both to the volatility risk Z_v via a non-zero g_v and to the diffusive price shock Z_s via a non-zero g_s.

Before solving for this problem, we should point out that the maturities of the chosen derivatives do not have to match the investment horizon T. For example, it might be hard for an investor with ten-year investment horizon to find an option with a matching maturity. He might choose to invest in options with a much shorter time to expiration, which typically expire in one or two years, and switch or roll over to the other derivatives in the future. For the purpose of choosing the optimal portfolio weights at time t, what matters is the choice of derivative securities O_t at that time, not the future choice of derivatives. At each point in the future, there exist non-redundant derivative to complete the market.

To solve the investment problem in (4.7) we use the stochastic control approach. We define the indirect utility function by

$$J(t, x, v) = \max_{\phi_s, \psi_s, t \le s \le T} \mathbb{E}\left(\frac{X_T^{1-\gamma}}{1-\gamma}\middle| X_t = x, V_t = v\right), \tag{4.8}$$

which, by the principle of optimal stochastic control, satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\max_{\phi_t, \psi_t} \left\{ J_t + X_t J_X (r + \theta_t^v \lambda V_t) + \frac{1}{2} X_t^2 J_{XX} V_t ((\theta_t^s)^2 + (\theta_t^v)^2) \right. \tag{4.9}$$

$$+k(\bar{v}-V_t)J_V + \frac{1}{2}\sigma V_t J_{VV} + \sigma V_t X_t J_{XV}(\rho\theta_t^s + \sqrt{1-\rho^2}\theta_t^v)\Big\} = 0$$

where J_t, J_X, J_V denote the derivatives of $J(t, X, V)$ with respect to t, X and V, and similar notations for higher derivatives.
To solve the HJB equation, we notice that it depends explicitly on the portfolio weights θ^s, θ^v which, as defined before, are linear transformations of the portfolio weights ϕ_t, ψ_t on the risky assets. Taking advantage of this structure, we first solve the optimal positions on the risk factors Z_s, Z_v, and then transform them back via the linear relation to the optimal positions on the risky assets.
This transformation is feasible as long as the chosen derivatives are non-redundant in the following sense:

Definition 4.1.1. At any time t, the derivative $O_t^{(1)}$ and $O_t^{(2)}$ are non-redundant if

$$\mathscr{D}_t \neq 0 \quad where \quad \mathscr{D}_t = \frac{g_s^{(2)} S_t}{O_t^{(2)}} \frac{g_v^{(1)}}{O_t^{(1)}} - \frac{g_s^{(1)} S_t}{O_t^{(1)}} \frac{g_v^{(2)}}{O_t^{(2)}}.$$

The non-redundant condition guarantees market completeness with respect to the chosen derivative securities, the risky stock, and the riskless bond since it does not allow the two options to be linearly dependent. Without access to derivatives, linear position in risky stock provide exposure only to diffusive risk, and none to volatility risk. To complete the market with respect to volatility risk, we need to bring in a risky asset that is sensitive to changes in volatility: $g_v \neq 0$.

The solution

Proposition 4.1.1. Assume that there are non-redundant derivatives available for trade at any time $t < T$. Then, for given wealth X_t and volatility V_t, the solution to the HJB equation is given by

$$J(t, X_t, V_t) = \frac{X_t^{1-\gamma}}{1-\gamma} exp(\gamma h(T-t) + \gamma H(T-t)V_t), \tag{4.10}$$

where h(·) and H(·) are time dependent coefficients that are independent of the state variables. That is, for any $0 \leq \tau \leq T$,

$$h(\tau) = \frac{2k\bar{v}}{\sigma^2} ln\left(\frac{2k_2 exp((k_1+k_2)\tau/2)}{2k_2 + (k_1+k_2)(exp(k_2\tau)-1)}\right) + \frac{1-\gamma}{\gamma} r\tau,$$

$$H(\tau) = \frac{exp(k_2\tau)-1}{2k_2 + (k_1+k_2)(exp(k_2\tau)-1)}\delta$$

where

$$\delta = \frac{1-\gamma}{\gamma^2}\lambda^2;$$

$$k_1 = k - \frac{1-\gamma}{\gamma}\lambda\sigma\sqrt{1-\rho^2}; \qquad k_2 = \sqrt{k_1^2 - \delta\sigma^2}.$$

The optimal portfolio weights on the risk factor Z_s, Z_v are given by

$$\theta_t^{*s} = \sigma\rho H(T-t) \qquad \theta_t^{*v} = \frac{\lambda}{\gamma} + \sigma\sqrt{1-\rho^2}H(T-t).$$

Transforming the θ^*'s to the optimal portfolio weights on the risky assets, ϕ_t^* for the stock and ψ_t^* for derivatives, we have

$$\phi_t^* = \theta_t^{*s} - \sum_{i=1}^{2} \psi_t^{*(i)} \left(\frac{g_s^{(i)} S_t}{O_t^{(i)}} + \sigma\rho \frac{g_v^{(i)}}{O_t^{(i)}} \right);$$

$$\psi_t^{*(1)} = \frac{1}{\mathscr{D}_t} \left[\frac{g_v^{(2)}}{O_t^{(2)}} \left(-\theta_t^{*s} - \frac{\theta_t^{*v}\rho}{\sqrt{1-\rho^2}} \right) + \frac{g_s^{(2)} S_t}{O_t^{(2)}} \frac{\theta_t^{*v}}{\sigma\sqrt{1-\rho^2}} \right];$$

$$\psi_t^{*(2)} = \frac{1}{\mathscr{D}_t} \left[\frac{g_v^{(1)}}{O_t^{(1)}} \left(+\theta_t^{*s} + \frac{\theta_t^{*v}\rho}{\sqrt{1-\rho^2}} \right) - \frac{g_s^{(1)} S_t}{O_t^{(1)}} \frac{\theta_t^{*v}}{\sigma\sqrt{1-\rho^2}} \right].$$

Proof. The proof is an application of the stochastic control method. Suppose that the indirect utility function J exists, and is of the conjectured form in (4.10).

$$J(t, X_t, V_t) = \frac{X_t^{1-\gamma}}{1-\gamma} exp(\gamma h(T-t) + \gamma H(T-t)V_t).$$

Then the first-order condition of the HJB equation (4.9) implies the following optimal portfolio weights.

$$X_t^{1-\gamma}\ exp(\gamma h(T-t) + \gamma H(T-t)V_t)\ \max_{\phi_t,\psi_t} \Bigg(\frac{1}{1-\gamma}(-\gamma h'(T-t) - \gamma H'(T-t)V_t)$$
$$+ (r + \theta_t^v \lambda V_t) - \frac{\gamma}{2} V_t((\theta_t^s)^2 + (\theta_t^v)^2) + k(\bar{v} - V_t)\frac{\gamma}{1-\gamma} H(T-t) + \frac{1}{2}\sigma V_t \frac{\gamma^2}{1-\gamma} H(T-t)^2$$
$$+ \sigma V_t \gamma H(T-t)(\rho\theta_t^s + \sqrt{1-\rho^2}\theta_t^v) \Bigg) = 0.$$

Now I impose that the derivative with respect to θ_t^s and θ_t^v is null, in this way I get the optimal weights.

$$\frac{\partial}{\partial\theta_t^s} = 0 \ \Rightarrow\ -\gamma V_t \theta_t^{s*} + \sigma V_t \gamma H(T-t)\rho = 0 \ \Rightarrow\ \theta_t^{s*} = \sigma\rho H(T-t).$$

$$\frac{\partial}{\partial\theta_t^v} = 0 \ \Rightarrow\ \lambda V_t - \gamma V_t \theta_t^{v*} + \sigma V_t \gamma H(T-t)\sqrt{1-\rho^2} = 0 \ \Rightarrow\ \theta_t^{v*} = \frac{\lambda}{\gamma} + \sigma\sqrt{1-\rho^2}H(T-t).$$

Through a linear transformation of θ^*, we obtain the optimal portfolio weights on the risky assets, ϕ_t^* for stock and ψ_t^* for derivatives.

Then substituting this weights into the HJB equation, one can show that the conjectured form for the indirect utility function J indeed satisfies the HJB equation if the following ordinary differential equations are satisfied:

$$\frac{dh(t)}{dt} = k\bar{v}H(t) + \frac{1-\gamma}{\gamma}r,$$
$$\frac{dH(t)}{dt} = \Big(-k + \frac{1-\gamma}{\gamma}\lambda\sigma\sqrt{1-\rho^2}\Big)H(t) + \frac{\sigma^2}{2}H(t)^2 + \frac{1-\gamma}{2\gamma^2}\lambda^2.$$

The solution of these differential equations has the form of H, h proposed in the proposition. [1] □

[1] All the calculations that are necessary to find the solution are reported in the appendix.

4.2 Derivatives with stochastic volatility and price jumps

In this subsection we relax another critical assumption. In Black-Scholes model the underlying stock dynamics can be described by a stochastic process with a continuous sample path. Now we consider the underlying stock returns are generated by a mixture of both continuous and jump processes.
In the presence of large, negative price jumps, the investor is reluctant to hold too much jump risk regardless of the premium assigned to it. This is because in contrast to diffusive risk, which can be controlled via continuous trading, the sudden, high-impact nature of jump risk takes away the investor's ability to continuously trade out of a leveraged position to avoid negative wealth. For investors with a reasonable range of risk aversion, jump risk is compensated more highly than diffusive risk.
For investors with a reasonable range of risk aversion, jump risk is compensated more highly than diffusive risk.

4.2.1 The model

When we consider both volatility risk and jump risk, we assume the following dynamics for the price process S of the risky stock:

$$dS_t = (r + \mu(\xi - \xi^Q)V_t)S_t dt + \sqrt{V_t} S_t dZ_s + \mu S_{t-}(dN_t - \xi V_t dt),$$

$$dV_t = k(\bar{v} - V_t)dt + \sigma\sqrt{V_t}\ (\rho\ dZ_s + \sqrt{1-\rho^2}\ dZ_v).$$

where Z_s and Z_v are standard Brownian motions, and N is a pure-jump process. All the three random shocks are assumed to be independent.

The random arrival of jump of jump events is dictated by the pure-jump process N with stochastic arrival intensity $\{\xi V_t : t \geq 0\}$ for constant $\xi \geq 0$. The conditional probability at time t of another jump before $t + \Delta t$ is, for some small Δt, approximately $\xi V_t \Delta t$.
ξ^Q is a constant coefficient capturing the component of the equity premium for jump risk N.
We impose a condition to the jump amplitudes: the stock price jumps is multiplied by a constant $\mu > -1$, with the limiting case of -1 representing the situation of total ruin. This specification of deterministic jump amplitude simplifies our analysis in the sense that only one additional derivative security is needed to complete the market with respect to the jump component.

The price of derivatives O_t depends on the underlying stock price S_t and the stock volatility V_t thought $O_t = g(S_t, V_t)$ for some function g. Compared to the dynamics of O_t found in the previous section where we used only the stochastic volatility, we must add the term related to the risk of jumps. In fact, based on the theory of jumps, the part of continuous dynamics remains unchanged and is analogous to the equation (4.6) of the dynamics of the option with stochastic volatility. For the part of the jumps we refer to the results that are obtained in J.Liu and J.Pan's article [11]. At time t will have the following dynamics

$$dO_t = rO_t dt + (g_s S_t + \sigma\rho g_v)\sqrt{V_t}\ dZ_s + \sigma\sqrt{1-\rho^2}\ g_v(\lambda V_t dt + \sqrt{V_t} dZ_v)$$
$$+ \Delta g((\xi - \xi^Q)V_t dt + dN_t - \xi V_t dt)$$

where g_s and g_v measure the sensitivity of the derivative price to infinitesimal changes in the stock price and volatility, respectively, and where Δg measures the change in the derivative price of each jump in the underlying stock price.

$$g_s = \left.\frac{\partial g(s,v)}{\partial s}\right|_{(S_t,V_t)} \qquad g_v = \left.\frac{\partial g(s,v)}{\partial v}\right|_{(S_t,V_t)}$$

$$\Delta g = g((1+\mu)S_t, V_t) - g(S_t, V_t).$$

Letting γ be the relative risk aversion coefficient of the representative agent, the coefficient for the jump risk premium is $\xi^Q/\xi = (1+\mu)^{-\gamma}$. In the presence of adverse jump risk ($\mu < 0$), the investor fears that jumps are more likely to occur ($\xi^Q > \xi$), consequently requiring the positive premium for holding jump risk.

4.2.2 The investment problem

Our goal is like the previous section: to maximize the expected utility of terminal wealth X_T. The investor starts with an initial wealth X_0 and has the opportunity to invest in the riskless asset, risky stock and derivatives.
At each time t, $0 \le t \le T$, he chooses to invest a fraction ϕ_t of this wealth in the stock S_t, and fractions $\psi_t^{(1)}$ and $\psi_t^{(2)}$ in the two derivative securities $O_t^{(1)}$ and $O_t^{(2)}$, respectively.

$$\max_{\phi_t,\psi_t,0\le t\le T} \mathbb{E}\left(\frac{X_T^{1-\gamma}}{1-\gamma}\right)$$

where $\gamma > 0$ is the relative risk-aversion coefficient of the investor. The wealth process satisfies the following dynamics

$$dX_t = rX_t\ dt + \theta_t^s X_t\sqrt{V_t}\ dZ_s + \theta_t^v X_t\ (\lambda V_t dt + \sqrt{V_t} dZ_v)$$
$$+ \theta_{t\_}^N X_{t\_}\mu\ ((\xi - \xi^Q)V_t dt + dN_t - \xi V_t dt)$$

where θ_t^s, θ_t^v, and θ_t^N are defined, foe given portfolio weights ϕ_t and ψ_t on the stock and the derivatives by,

$$\theta_t^s = \phi_t + \sum_{i=1}^{2}\psi_t^{(i)}\left(\frac{g_s^{(i)} S_t}{O_t^{(i)}} + \rho\sigma\frac{g_v^{(i)}}{O_t^{(i)}}\right); \qquad \theta_t^v = \sigma\sqrt{1-\rho^2}\sum_{i=1}^{2}\psi_t^{(i)}\frac{g_v^{(i)}}{O_t^{(i)}};$$

$$\theta_t^N = \phi_t + \sum_{i=1}^{2} \psi_t^{(i)} \frac{\Delta g^{(i)}}{\mu O_t^{(i)}}.$$

The investor invest θ^s in the diffusive price shock Z_s, θ^v in the additional volatility risk Z_v, and θ^N in the jump risk N.So, now if we consider a portfolio position ψ_t in the risky stock provides equal exposure to both diffusive and jump risks in stock price; similarity, a portfolio position ψ_t in the derivative provides exposure to the volatility risk Z_v via a non-zero g_v, exposure to the diffusive price shock Z_s via a non-zero g_s and exposure to the jump risk via a non-zero Δg.

We now proceed to solve the investment problem using the stochastic control approach like before. The indirect utility function $J(t, x, v)$ (5.4) now satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\max_{\phi_t, \psi_t} \Bigg(J_t + X_t J_X (r + \theta_t^v \lambda V_t - \theta_t^N \mu \xi^Q V_t) + \frac{1}{2} X_t^2 J_{XX} V_t ((\theta_t^s)^2 + (\theta_t^v)^2) + \xi V_t \Delta J$$
$$+ k(\bar{v} - V_t) J_V + \frac{1}{2} \sigma V_t J_{VV} + \sigma V_t X_t J_{XV} (\rho \theta_t^s + \sqrt{1 - \rho^2} \theta_t^v) \Bigg) = 0$$

where $\Delta J = J(t, X_t(1 + \theta^N \mu), V_t) - J(t, X_t, V_t)$ denotes the jump in the indirect utility function J for given jumps in the stock price, and where J_t, J_X and J_V denote the derivative of $J(t, X, V)$ with respect to t, X and V respectively.

Even now we need the derivatives that we consider in our portfolio to be non-redundant, which is why we have the following definition.

Definition 4.2.1. At any time t, the derivative securities $O_t^{(1)}$ and $O_t^{(2)}$ are non-redundant if

$$\mathscr{D}_t \neq 0 \quad where \quad \mathscr{D}_t = \left(\frac{\Delta g^{(1)}}{\mu O_t^{(1)}} - \frac{g_s^{(1)} S_t}{O_t^{(1)}} \right) \frac{g_v^{(2)}}{O_t^{(2)}} - \left(\frac{\Delta g^{(2)}}{\mu O_t^{(2)}} - \frac{g_s^{(2)} S_t}{O_t^{(2)}} \right) \frac{g_v^{(1)}}{O_t^{(1)}}.$$

Compared to the considerations made in the previous section of the non-redundant condition, we have that without access to derivatives, linear positions in the risky stock provide equal exposures to diffusive and jump risks, and none to volatility risk. To complete the market with respect to jump risk, we need a risky asset with different sensitivities to infinitesimal and large changes in stock prices: $g_s S_t / O_t \neq \Delta g / \mu O_t$. Moreover, this condition also ensures that the two chosen derivative securities are not identical in covering the two risk factor.

The solution

Proposition 4.2.1. Assume that there are non-redundant derivatives available for trade at any time $t < T$. Then, for given wealth X_t and volatility V_t, the solution to the HJB equation is given by

$$J(t, X_t, V_t) = \frac{X_t^{1-\gamma}}{1-\gamma} exp(\gamma h(T - t) + \gamma H(T - t) V_t), \tag{4.11}$$

where h(·) and H(·) are time dependent coefficients that are independent of the state variables. That is, for any $0 \leq \tau \leq T$,

$$h(\tau) = \frac{2k\bar{v}}{\sigma^2} ln\left(\frac{2k_2 exp((k_1 + k_2)\tau/2)}{2k_2 + (k_1 + k_2)(exp(k_2\tau) - 1)}\right) + \frac{1-\gamma}{\gamma} r\tau,$$

$$H(\tau) = \frac{exp(k_2\tau) - 1}{2k_2 + (k_1 + k_2)(exp(k_2\tau) - 1)}\delta.$$

where

$$\delta = \frac{1-\gamma}{\gamma^2}\lambda^2 + 2\xi^Q\left[\left(\frac{\xi}{\xi^Q}\right)^{1/\gamma} + \frac{1}{\gamma}\left(1 - \frac{\xi}{\xi^Q}\right) - 1\right];$$

$$k_1 = k - \frac{1-\gamma}{\gamma}\lambda\sigma\sqrt{1-\rho^2}; \qquad k_2 = \sqrt{k_1^2 - \delta\sigma^2}.$$

The optimal portfolio weights on the risk factor Z_s, Z_v and N are given by

$$\theta_t^{*s} = \sigma\rho H(T-t), \qquad \theta_t^{*v} = \frac{\lambda}{\gamma} + \sigma\sqrt{1-\rho^2} H(T-t),$$

$$\theta_t^{*N} = \frac{1}{\mu}\left(\left(\frac{\xi}{\xi^Q}\right)^{1/\gamma} - 1\right).$$

Transforming the θ^*'s to the optimal portfolio weights on the risky assets, ϕ_t^* for the stock and ψ_t^* for derivatives, we have

$$\phi_t^* = \theta_t^{*s} - \sum_{i=1}^{2} \psi_t^{*(i)}\left(\frac{g_s^{(i)} S_t}{O_t^{(i)}} + \sigma\rho\frac{g_v^{(i)}}{O_t^{(i)}}\right);$$

$$\psi_t^{*(1)} = \frac{1}{\mathscr{D}_t}\left[\frac{g_v^{(2)}}{O_t^{(2)}}\left(\theta_t^{*N} - \theta_t^{*s} - \frac{\theta_t^{*v}\rho}{\sqrt{1-\rho^2}}\right) - \left(\frac{\Delta g^{(2)}}{\mu O_t^{(2)}} - \frac{g_s^{(2)} S_t}{O_t^{(2)}}\right)\frac{\theta_t^{*v}}{\sigma\sqrt{1-\rho^2}}\right];$$

$$\psi_t^{*(2)} = \frac{1}{\mathscr{D}_t}\left[\frac{g_v^{(1)}}{O_t^{(1)}}\left(\theta_t^{*s} - \theta_t^{*N} + \frac{\theta_t^{*v}\rho}{\sqrt{1-\rho^2}}\right) + \left(\frac{\Delta g^{(1)}}{\mu O_t^{(1)}} - \frac{g_s^{(1)} S_t}{O_t^{(1)}}\right)\frac{\theta_t^{*v}}{\sigma\sqrt{1-\rho^2}}\right].$$

Proof. The demonstration is similar to that made in the previous section with the addition of the term on jump risk.
θ_t^{*s} and θ_t^{*v} values remain the same. The term θ_t^N appears in the two terms we have added in the rewrites in the HJB equation. I impose that the derivative with respect to θ_t^N is null

$$\frac{\partial}{\partial\theta_t^N} = 0 \Rightarrow \frac{\partial}{\partial\theta_t^N}(-X_t J_X \theta_t^N \mu\xi^Q V_t + \xi V_t \Delta J) = 0.$$

Knowing that

$$\Delta J = J(t, X_t(1 + \theta_t^N\mu), V_t) - J(t, X_t, V_t) =$$
$$= \left(\frac{X_t^{1-\gamma}(1 + \theta_t^N\mu)^{1-\gamma}}{1-\gamma} - \frac{X_t^{1-\gamma}}{1-\gamma}\right) exp(\gamma h(T-t) + \gamma H(T-t)V_t).$$

And so

$$
\begin{aligned}
& - X_t^{1-\gamma}\mu\xi^Q V_t + \xi V_t X_t^{1-\gamma}\mu(1+\theta_t^{*N}\mu)^{-\gamma} = 0 \;\Rightarrow\; -\xi^Q + \xi(1+\theta_t^{*N})^{-\gamma} = 0 \\
& \Rightarrow\; \theta_t^{*N} = \frac{1}{\mu}\Big(\Big(\frac{\xi}{\xi^Q}\Big)^{1/\gamma} - 1\Big).
\end{aligned}
$$

Then substituting this weights into the HJB equation, one can show that the conjectured form for the indirect utility function J indeed satisfies the HJB equation if the following ordinary differential equations are satisfied:

$$
\begin{aligned}
\frac{dh(t)}{dt} &= k\bar{v}H(t) + \frac{1-\gamma}{\gamma}r, \\
\frac{dH(t)}{dt} &= \Big(- k + \frac{1-\gamma}{\gamma}\sigma\sqrt{1-\rho^2}\Big)H(t) + \frac{\sigma^2}{2}H(t)^2 + \frac{1-\gamma}{2\gamma^2}\lambda^2 + \\
&\quad + \xi^Q\Big[\Big(\frac{\xi}{\xi^Q}\Big)^{1/\gamma} + \frac{1}{\gamma}\Big(1 - \frac{\xi}{\xi^Q}\Big) - 1\Big].
\end{aligned}
$$

The solution of these differential equations has the form of H, h proposed in the proposition. The proof that shows how to obtain the values of H, h is shown in the appendix and is analogous to the one proposed in the previous section where we make the change that

$$
\delta = \frac{1-\gamma}{\gamma^2}\lambda^2 + 2\xi^Q\Big[\Big(\frac{\xi}{\xi^Q}\Big)^{1/\gamma} + \frac{1}{\gamma}\Big(1 - \frac{\xi}{\xi^Q}\Big) - 1\Big].
$$

□

Chapter 5

Completion of the incomplete market through derivatives

In the previous chapter of the treatment the solution in dynamic strategy was found when the underlying of the options was not in the Black and Scholes framework. As mentioned above, adding an option in a complete market is not convenient in fact every derivative is redundant and we can replicate the same strategy with the use of assets only.
In this chapter we find the optimal investment strategy including bond, stocks and derivatives in an incomplete market. Now the dynamic of stock remains within the framework of Black and Scholes, and we break the completion of the market by introducing more number of random sources than the number of stocks.
In particular in the first section 5.1 we present the model that we use in this chapter of the discussion, for reasons of simplicity we consider at most two risky assets.
In section 5.2 the classic Merton problem is quickly presented in which in a complete market are considered two stocks and two Brownian motion. In section 5.3 we present a problem of optimal allocation in which we have a risky asset traded, a derivative written on another stock that cannot be traded on the market and two Brownian motions. In section 5.4 we find the optimal allocation solution when a option written on a traded stock is introduced in an incomplete market.

5.1 Model

As we mentioned in the section 2.2, the Merton problem is the first continuous time optimal investment model.
Let's start by defining the market framework that we consider in this section.
Assumed a market with K risky assets and a risk-free asset. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, $W = (W_t^1, ...W_t^N)_{t \in \mathbb{T}}$ N-dimensional Brownian motion with $\mathbb{T} = [0, T]$ which represents the time interval over which the problem is defined.

When the market is complete and with absence of arbitrage? A complete market is a market with two conditions: negligible transaction costs and therefore also perfect information, and there is a price for every asset in every possible state of the world. Derivatives are priced using the no-arbitrage or arbitrage-free principle: the price of the derivative is set at the same level as the value of the replicating portfolio, so that no trader can make a risk-free profit by buying one and selling the other.
In chapter 8 of*Arbitrage Theory in Continuous Time* by *T.Björk* [23] we have a general rule of thumb for quickly determining whether a certain model is complete and/or free of arbitrage: "meta theorem".

Theorem 5.1.1. Let K denote the number of underlying *traded* assets in the model excluding the risk-free asset, and let N denote the number of random sources. Generally we then have the following relations:

- The model is arbitrage free if and only if $K \leq N$.
- The model is complete if and only if $K \geq N$.
- The model is complete and arbitrage free if and only if $K = N$.

For reasons of simplicity in our treatment we consider at most $K = 2$ stocks and $N = 2$ Brownian motion which represented the random sources in the time horizon $[0, T]$.
The dynamics of each risky asset is described by log-normal process and follows the assumption of the Black and Scholes model.

$$dS_t^{(1)} = \mu_1 \ S_t^{(1)} dt + \sigma_{11} \ S_t^{(1)} dW_t^{(1)} + \sigma_{12} \ S_t^{(1)} dW_t^{(2)} \tag{5.1}$$

$$dS_t^{(2)} = \mu_2 \ S_t^{(2)} dt + \sigma_{21} \ S_t^{(2)} dW_t^{(1)} + \sigma_{22} \ S_t^{(2)} dW_t^{(2)} \tag{5.2}$$

where $\mu = (\mu_1, \mu_2)$ is constant vector in $\mathbb{R}$ and $\sigma = (\sigma_{ij})_{i,j=1,2}$ a constant matrix in $\mathbb{R}^{2\times 2}$. We also define, in order to simplify the calculations later on, with

$$\alpha_1 = \sigma_{11}^2 + \sigma_{12}^2, \qquad \alpha_2 = \sigma_{21}^2 + \sigma_{22}^2, \qquad \beta = \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}. \tag{5.3}$$

Let $r > 0$ be the interest rate that guarantees a non-risk investment, for example a bank deposit, then the risk-free asset (bond) will have the following dynamics

$$dB_t = rB_t \ dt.$$

We want to find an amount of wealth invested in stocks, bond and after in derivatives. We are able to find these quantities by solving a Stochastic Optimal Problem, the theory of which is proposed in section 1.2 of the discussion.
More precisely we have that $(X_t)_{t\in\mathbb{T}}$ is the process that describes the dynamics of portfolio wealth. The investment objective is to maximize the expected utility of terminal wealth X_T,

$$\max_{\phi_t, \psi_t, 0\leq t\leq T} \mathbb{E}\left(\frac{X_T^{1-\gamma}}{1-\gamma}\right)$$

where $\gamma > 0$ and $\gamma \neq 1$ is the relative risk-aversion coefficient of the investor.
To solve the investment problem we use the stochastic control approach and we define the indirect utility function by

$$J(t, x) = \max_{\phi_s, \psi_s, t \leq s \leq T} \mathbb{E}\left(\frac{X_T^{1-\gamma}}{1-\gamma} \middle| X_t = x \right). \tag{5.4}$$

For each of the problems that we have dealt with in the below sections we have that the J function has the following form:

$$J(t, X_t) = \frac{X_t^{1-\gamma}}{1-\gamma}\, e^{h(T-t)}$$

where $h(\cdot)$ is time dependent coefficient that is independent on the state variable.

5.2 Complete market with two stocks

In this section we quickly see the solution to Merton's classic problem when considering two stocks. In this case we consider that our portfolio is composed of $S_t^{(1)}, S_t^{(2)}$ and the bond B_t. The dynamics of the two risky assets is the one reported in equation (5.1)-(5.2); in this case we have that the market is **complete** since the number of stocks ($K = 2$) equals the number of random sources ($N = 2$). The dynamics of the portfolio wealth is

$$
\begin{aligned}
dX_t = & \sum_{i=1}^{2} \phi_t^{(i)} X_t \, \frac{dS_t^{(i)}}{S_t^{(i)}} + \left(1 - \sum_{i=1}^{2} \phi_t^{(i)}\right) X_t \, \frac{dB_t}{B_t} = \\
& = rX_t \, dt + \sum_{i=1}^{2} X_t \left(\phi_t^{(i)} (\mu_i - r) dt + \sigma_{i1} dW_t^{(1)} + \sigma_{i2} dW_t^{(2)} \right) = \\
& = rX_t \, dt + \left(\phi_t^{(1)} (\mu_1 - r) + \phi_t^{(2)} (\mu_2 - r) \right) X_t \, dt + \left(\phi_t^{(1)} \sigma_{11} + \phi_t^{(2)} \sigma_{21} \right) X_t \, dW_t^{(1)} + \\
& \quad + \left(\phi_t^{(1)} \sigma_{12} + \phi_t^{(2)} \sigma_{22} \right) X_t \, dW_t^{(2)}.
\end{aligned}
$$

The solution

To find the solution to the Merton problem in the two-dimensional case we followed the same steps as in section 2.2. Obtaining the following results.
For given wealth X_t the solution to the HJB equation is given by

$$
J(t, X_t) = \frac{X_t^{1-\gamma}}{1-\gamma} \, e^{h(T-t)}
$$

where, for any $0 \leq \tau \leq T$,

$$
h(\tau) = \tau(1-\gamma) \left(r + \frac{\alpha_2(\mu_1 - r)^2 - 2\beta(\mu_1 - r)(\mu_2 - r) + \alpha_1(\mu_2 - r)^2}{2\gamma(\alpha_1\alpha_2 - \beta^2)} \right).
$$

The optimal portfolio weights on the risky assets $S_t^{(1)}$ and $S_t^{(2)}$ are given by

$$
\phi_t^{*(1)} = \frac{\alpha_2(\mu_1 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)} - \frac{\beta(\mu_2 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)};
$$

$$
\phi_t^{*(2)} = \frac{\alpha_1(\mu_2 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)} - \frac{\beta(\mu_1 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)};
$$

where the value of $\alpha_{1,2}$ and β are defined in (5.3).

5.3 Incomplete market with derivative written on non-traded stock

In this section, starting from a complete market like the one analyzed in the previous section, we add a derivative written on one of the two stocks. We require, however, that the stock on which the derivative is written is non-negotiable, so we are in a situation where the investor takes the derivative written on it instead of the stock.
We are therefore considering an incomplete market, because we have $K = 1$ traded stock and $N = 2$ random sources. **This market incomplete is completed by the introduction of the derivative** written on the other stock.

Our portfolio consists of a risky asset $S_t^{(2)}$, a bond B_t and a derivative written on the other stock $S_t^{(1)}$. The dynamics of the two risky assets is the one reported respectively in equation (5.1)-(5.2).

We consider that the derivative instrument is an option. The price of option O_t depends only on the underlying stock price $S_t^{(1)}$, i.e $O_t = g(t, S_t^{(1)})$ for some function g. We can find the dynamics of the option through Ito's formula.

$$dO_t = \left(\frac{\partial O_t}{\partial t} + \frac{1}{2} S_t^{(1)^2} \alpha_1 \frac{\partial^2 O_t}{\partial S_t^{(1)^2}}\right) dt + \frac{\partial O_t}{\partial S_t^{(1)}}\, dS_t^{(1)}.$$

Since the stock has a dynamic that respects the assumptions of the Black and Scholes model, we can use the Black-Scholes equation (2.1) for a derivative defined in chapter 2 to simplify the dynamics.

$$\begin{aligned} dO_t = & \left(- rS_t^{(1)} \frac{\partial O_t}{\partial S_t^{(1)}} - \frac{1}{2} S_t^{(1)^2} \alpha_1 \frac{\partial^2 O_t}{\partial S_t^{(1)^2}} + rO_t + \frac{1}{2} S_t^{(1)^2} \alpha_1 \frac{\partial^2 O_t}{\partial S_t^{(1)^2}} \right) dt + \\ & + \frac{\partial O_t}{\partial S_t^{(1)}} S_t^{(1)} \left(\mu_1 dt + \sigma_{11} dW_t^{(1)} + \sigma_{12} dW_t^{(2)} \right) = \\ & = rO_t\, dt + (\mu_1 - r) S_t^{(1)} \frac{\partial O_t}{\partial S_t^{(1)}} dt + S_t^{(1)} \frac{\partial O_t}{\partial S_t^{(1)}} (\sigma_{11} dW_t^{(1)} + \sigma_{12} dW_t^{(2)}). \end{aligned} \tag{5.5}$$

Our aim is to maximize the expected utility of terminal wealth X_T starting with an initial wealth X_0. At each time t, $0 \le t \le T$, the investor chooses to invest a fraction $\phi_t^{(2)}$ of this wealth in the stock $S_t^{(2)}$, a fraction ψ_t in the derivative security O_t written on the stock $S_t^{(1)}$ that is no traded asset, i.e $\phi_t^{(1)} = 0$ and a fraction $(1 - \phi_t^{(2)} - \psi_t)$ in the bond.

The wealth process is the following

$$\begin{aligned}\frac{dX_t}{X_t} &= \phi_t^{(2)}\ \frac{dS_t^{(2)}}{S_t^{(2)}} + \psi_t\ \frac{dO_t}{O_t} + (1-\phi_t^{(2)}-\psi_t)\frac{dB_t}{B_t} = \\ &= \phi_t^{(2)}(\mu_2 dt + \sigma_{21}dW_t^{(1)} + \sigma_{22}dW_t^{(2)}) + \psi_t r\ dt + \\ &\quad + \psi_t \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\Big((\mu_1 - r)dt + \sigma_{11}dW_t^{(1)} + \sigma_{12}dW_t^{(2)}\Big) + \\ &\quad + (1-\phi_t^{(2)}-\psi_t)r dt.\end{aligned}$$

So we obtain

$$\begin{aligned}\frac{dX_t}{X_t} &= r\ dt + \psi_t \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}(\mu_1 - r)dt + \phi_t^{(2)}(\mu_2 - r)\ dt + \\ &\quad + \left(\phi_t^{(2)}\sigma_{21} + \psi_t \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\sigma_{11}\right)dW_t^{(1)} + \\ &\quad + \left(\phi_t^{(2)}\sigma_{22} + \psi_t \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\sigma_{12}\right)dW_t^{(2)}.\end{aligned}$$

When we describe the model, we define the indirect utility function by (5.4). This function $J(t,x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\max_{\phi_t,\psi_t}\left\{J_t + X_t J_X\left(r + \psi_t \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}(\mu_1 - r) + \phi_t^{(2)}(\mu_2 - r)\right) + \frac{1}{2}X_t^2 J_{XX}A^2\right\} = 0 \quad (5.6)$$

where

$$A^2 = \phi_t^{(2)^2}\alpha_2 + \psi_t^2\left(\frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\right)^2\alpha_1 + 2\ \phi_t^{(2)}\psi_t\ \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\ \beta \quad (5.7)$$

where the value of $\alpha_{1,2}$ and β are defined in (5.3).

The solution

Assume that there are option and stock $S_t^{(2)}$ available for trade at any time $t < T$. Then, for a given wealth X_t the solution to the HJB equation defined in (5.6) is given by

$$J(t, X_t) = \frac{X_t^{1-\gamma}}{1-\gamma}\ e^{h(T-t)}$$

where, for any $0 \le \tau \le T$,

$$h(\tau) = \tau(1-\gamma)\left(r + \frac{\alpha_2(\mu_1 - r)^2 - 2\beta(\mu_1 - r)(\mu_2 - r) + \alpha_1(\mu_2 - r)^2}{2\gamma(\alpha_1\alpha_2 - \beta^2)}\right).$$

The optimal portfolio weights on the risky asset $S_t^{(2)}$ and on the option O_t, imposing that $\phi_t^{(1)} = 0$ are given by

$$\phi_t^{*(2)} = \frac{\alpha_1(\mu_2 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)} - \frac{\beta(\mu_1 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)};$$

$$\psi_t^* = \left(\frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\right)^{-1} \frac{\alpha_2(\mu_1 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)} - \frac{\beta(\mu_2 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)};$$

where the value of $\alpha_{1,2}$ and β are defined in (5.3).

Proof. The proof is an application of the stochastic control method. Suppose that the indirect function $J(t, X_t)$ exists, and we suppose that it has the following form

$$J(t, X_t) = \frac{X_t^{1-\gamma}}{1-\gamma}\, e^{h(T-t)}.$$

Then the first-order condition of the HJB equation (5.6) implies the following form.

$$X_t^{1-\gamma} e^{h(T-t)} \max_{\phi_t, \psi_t} \left\{ -\frac{h'(T-t)}{1-\gamma} + r + \psi_t \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}(\mu_1 - r) + \phi_t^{(2)}(\mu_2 - r) - \frac{\gamma}{2}A^2 \right\} = 0$$

where A^2 is defined in (5.7).
Now we impose that the derivative with respect to $\phi_t^{(2)}$ and ψ_t is null, in this way we obtain a system with two equations in two unknown variables.

$$\frac{\partial}{\partial \phi_t^{(2)}} = 0 \;\Rightarrow\; (\mu_2 - r) - \frac{\gamma}{2}\left(2\phi_t^{*(2)}\alpha_2 + 2\psi_t^* \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\beta\right) = 0.$$

$$\frac{\partial}{\partial \psi_t} = 0 \;\Rightarrow\; (\mu_1 - r)\frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}} - \frac{\gamma}{2}\left(2\psi_t^* \left(\frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\right)^2 \alpha_1 + 2\phi_t^{*(2)} \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\beta\right) = 0.$$

Simplifying, we get the following the system

$$\begin{cases} \phi_t^{*(2)}\beta + \psi_t^* \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\alpha_1 & = \frac{\mu_1 - r}{\gamma} \\ \phi_t^{*(2)}\alpha_2 + \psi_t^* \frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\beta & = \frac{\mu_2 - r}{\gamma} \end{cases}$$

By solving the system we get the values found above

$$\phi_t^{*(2)} = \frac{\alpha_1(\mu_2 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)} - \frac{\beta(\mu_1 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)};$$

$$\psi_t^* = \left(\frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\right)^{-1} \left(\frac{\alpha_2(\mu_1 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)} - \frac{\beta(\mu_2 - r)}{\gamma(\alpha_1\alpha_2 - \beta^2)}\right).$$

Then substituting this weights into the HJB equation, one can show that the conjectured form for the indirect utility function $J(t, X_t)$ satisfies the HJB equation if the following ordinary differential equation is satisfied:

$$h'(\tau) = \frac{dh(\tau)}{d\tau} = (1-\gamma)\left(r + \frac{\alpha_2(\mu_1 - r)^2 - 2\beta(\mu_1 - r)(\mu_2 - r) + \alpha_1(\mu_2 - r)^2}{2\gamma(\alpha_1\alpha_2 - \beta^2)}\right).$$

The solution of this differential equation is very simple:

$$h(\tau) = \tau(1-\gamma)\left(r + \frac{\alpha_2(\mu_1 - r)^2 - 2\beta(\mu_1 - r)(\mu_2 - r) + \alpha_1(\mu_2 - r)^2}{2\gamma(\alpha_1\alpha_2 - \beta^2)}\right).$$

□

Comment: We can see that the solution to this problem is similar to the solution found in the previous section. The HJB equation in the two problems has the same form and also the function h(·) which depends only on time is the same and we obtain that the set of attainable wealth profiles is the same.
The optimal weight to invest in the $S_t^{(2)}$ stock is the same in the two problems, while the weight assigned to the option O_t differs from that assigned to the $S_t^{(1)}$ stock (when it was negotiable and the derivative was not considered) for the term $\left(\frac{S_t^{(1)}}{O_t}\frac{\partial O_t}{\partial S_t^{(1)}}\right)^{-1}$.
We can see that in the Black and Scholes framework the term $\frac{\partial O_t}{\partial S_t}$ corresponds to the delta of the option. This term is equal to the slope of the curve that links the option price to the price of the underlying asset. The delta of a call in B&S model written on a security that does not pay dividends is $\Delta_C = N(d_1)$, while for a put option it is $\Delta_P = N(d_1) - 1$, where N(·) is the standard normal cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-x^2/2}dz.$$

This similarity of the results is due to the fact that when we go looking for the solution for this last problem, we start from a market that is complete in a similar way to the problem we were dealing with in the previous section. The derivative written on the $S_t^{(1)}$ stock when added to an already complete market provides only one more trade possibility.
In the solution of the Merton problem we find that the optimal allocation strategy is to keep a constant fraction of the wealth in the various assets that the investor the investor considers to build the portfolio, in fact the optimal weights depend exclusively on the parameters of the dynamics of the assets. In this case the fraction of wealth that is assigned to the option is composed of a constant part, that coincides with the classic Merton problem, and a term that depends on time t, on the value at time t of the stock on which the option is written and on the price at time t of the option itself.

5.4 Incomplete market with one stock

In this section, we start by considering an incomplete market composed of a single risky asset governed by two Brownian motion and a risk-free asset. To complete the market we introduce a derivative instrument that is written on the traded stock.
We find endless optimal allocation strategies that allow us to build an optimal portfolio.

In *Financial Economics* by P.Ireland [19] we have a rule that allows us to continue with the discussion: the following preposition ensures that in our framework with the introduction of the derivative the market is completed.

Proposition 5.4.1. A necessary and sufficient condition for a creation of a complete set of Arrow-Debreu securities is that there exists a single portfolio with the property that options can be purchased and written on it and such that its payoff pattern distinguishes among all future states.

Our portfolio consists of a stock S_t, a bond B_t and an option written on this stock, so the price of option O_t depends only on the underlying stock price S_t. The dynamics of the risky assets are the following

$$dS_t = \mu S_t dt + \sigma_1 S_t dW_t^{(1)} + \sigma_2 S_t dW_t^{(2)},$$

$$dO_t = rO_t\ dt + (\mu - r) S_t \frac{\partial O_t}{\partial S_t}\ dt + S_t \frac{\partial O_t}{\partial S_t}\ (\sigma_1 dW_t^{(1)} + \sigma_2 dW_t^{(2)}).$$

The dynamics of the option is obtained with the same steps made in the equation (5.5).

Our aim is to maximize the expected utility of terminal wealth X_T. At each time t, $0 \leq t \leq T$, the investor chooses to invest a fraction ϕ_t of this wealth in the stock S_t, a fraction ψ_t in the derivative instrument O_t and a fraction $(1 - \phi_t - \psi_t)$ in the bond B_t.
The wealth process X_t has the following dynamics:

$$\begin{aligned}
\frac{dX_t}{X_t} &= \phi_t \frac{dS_t}{S_t} + \psi_t \frac{dO_t}{O_t} + (1 - \phi_t - \psi_t) \frac{dB_t}{B_t} = \\
&= \phi_t(\mu dt + \sigma_1 dW_t^{(1)} + \sigma_2 dW_t^{(2)}) + \psi_t r\ dt + \psi_t \frac{S_t}{O_t} \frac{\partial O_t}{\partial S_t} \Bigg((\mu - r) dt + \sigma_1 dW_t^{(1)} \\
&\quad + \sigma_2 dW_t^{(2)} \Bigg) + (1 - \phi_t - \psi_t) r dt = \\
&= r dt + \phi_t(\mu - r) dt + \psi_t \frac{S_t}{O_t} \frac{\partial O_t}{\partial S_t} (\mu - r) dt + \left(\phi_t \sigma_1 + \psi_t \frac{S_t}{O_t} \frac{\partial O_t}{\partial S_t} \sigma_1 \right) dW_t^{(1)} + \\
&\quad + \left(\phi_t \sigma_2 + \psi_t \frac{S_t}{O_t} \frac{\partial O_t}{\partial S_t} \sigma_2 \right) dW_t^{(2)}.
\end{aligned}$$

When we describe the model, we define the indirect utility function by (5.4) in the following way

$$J(t,x) = \max_{\phi_s,\psi_s,t\le s\le T} \mathbb{E}\left(\frac{X_T^{1-\gamma}}{1-\gamma}\middle| X_t = x\right).$$

This function $J(t,x)$ satisfies the following Hamilton-Jacobi-Bellman equation:

$$\max_{\phi_t,\psi_t}\left\{J_t + X_t J_X\left(r + \left(\phi_t + \psi_t \frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t}\right)(\mu - r)\right) + \frac{1}{2}X_t^2 J_{XX} A^2\right\} = 0 \qquad (5.8)$$

where

$$A^2 = (\sigma_1^2 + \sigma_2^2)\left(\phi_t^2 + \psi_t^2\left(\frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t}\right)^2 + 2\phi_t\psi_t \frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t}\right). \qquad (5.9)$$

The solution

Assume that there are option and stock available for trade at any time $t < T$. Then, for a given wealth X_t the solution of the HJB equation defined in (5.8) is given by

$$J(t,X_t) = \frac{X_t^{1-\gamma}}{1-\gamma}\, e^{h(T-t)}$$

where, for any $0 \le \tau \le T$,

$$h(\tau) = \tau\, \frac{1-\gamma}{\gamma}\, \frac{(\mu-r)^2}{2(\sigma_1^2+\sigma_2^2)}.$$

We obtain infinite optimal allocation strategies such that the weights that are assigned to the stock S_t and option O_t respect the following equation

$$\phi_t^* + \psi_t^*\, \frac{S_t}{O_t}\, \frac{\partial O_t}{\partial S_t} = \frac{\mu - r}{\gamma(\sigma_1^2+\sigma_2^2)}. \qquad (5.10)$$

Proof. The proof is an application of the stochastic control method. Suppose that the indirect function $J(t,X_t)$ exists, and we suppose that it has the following form

$$J(t,X_t) = \frac{X_t^{1-\gamma}}{1-\gamma}\, e^{h(T-t)}.$$

Then the first-order condition of the HJB equation (**??**) implies the following form

$$X_t^{1-\gamma} e^{h(T-t)} \max_{\phi_t,\psi_t}\left\{-\frac{h'(T-t)}{1-\gamma} + r + \phi_t(\mu-r) + \psi_t \frac{S_t}{O_t}\, \frac{\partial O_t}{\partial S_t}(\mu - r) - \frac{\gamma}{2}A^2\right\} = 0$$

where A^2 is defined in (5.9).
Now we impose that the derivative with respect to ϕ_t and ψ_t is null, in this way we obtain a system with two equations in two unknown variables.

$$\frac{\partial}{\partial \phi_t} = 0 \;\Rightarrow\; (\mu - r) - \frac{\gamma}{2}\left(2\phi_t^*\alpha + 2\psi_t^* \frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t}\alpha\right) = 0.$$

$$\frac{\partial}{\partial \psi_t} = 0 \;\Rightarrow\; \frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t}(\mu - r) - \frac{\gamma}{2}\left(2\psi_t^*\left(\frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t}\right)^2 \alpha + 2\phi_t^* \frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t}\alpha\right) = 0.$$

We define with $\alpha = \sigma_1^2 + \sigma_2^2$.
So we obtain the following system:

$$\begin{cases} \phi_t^* + \psi_t^* \frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t} & = \frac{\mu - r}{\gamma\alpha} \\ \phi_t^* + \psi_t^* \frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t} & = \frac{\mu - r}{\gamma\alpha} \end{cases}$$

We can notice that these two equations of the system is equal, and we are not able to solve it, so we have an infinite optimal solution of ϕ_t^* and ψ_t^*. However, the optimal weights must satisfy the following equation

$$\phi_t^* + \psi_t^* \frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t} = \frac{\mu - r}{\gamma\alpha}.$$

Replacing ϕ_t^* as a function of ψ_t^* within the HJB equation, one can show that the form for the indirect utility function $J(t, X_t)$ satisfies the HJB equation if the following ordinary equation is satisfied:

$$h'(\tau) = \frac{dh(\tau)}{d\tau} = \frac{1-\gamma}{\gamma}\,\frac{(\mu - r)^2}{2\alpha}.$$

The solution of this differential equation is very simple:

$$h(\tau) = \tau\frac{1-\gamma}{\gamma}\,\frac{(\mu - r)^2}{2(\sigma_1^2 + \sigma_2^2)}.$$

□

Comment: We can note that the solution that we obtain takes up the solution that is obtained in Merton's problem where markets are complete and is considered a single stock, the solution is given in section 2.2.2 of chapter 2 of the discussion. In the problem just proposed, we find infinite optimal solutions such that the weights satisfy the equation (5.10). The right-side of this equation is equal to the fraction of wealth invested in the risky asset of the classic Merton problem, so the sum that is invested in the risky securities in the two problems less than the term $\frac{S_t}{O_t}\frac{\partial O_t}{\partial S_t}$ is the same.

Chapter 6

Conclusion

Let's retrace the steps taken in this discussion. The objective of the thesis was to understand how the optimal allocation strategy changes when derivatives are added to Merton's problem, i.e. when the investor has the possibility to choose between risky assets not only equities but also options contracts. We want to find the optimal strategy in order to maximize the expected utility of an investor with constant risk aversion.

We start by considering the same framework of Merton's problem where the market is composed of a risky stock described by Geometric Brownian motion and the non-risk asset characterized by its risk free interest rate r. We find a necessary and sufficient condition for the Merton problem to be equivalent to a *buy-and-hold* strategy in which derivatives are added to the market. A *buy-and-hold* strategy means that the weights to be assigned to the different assets are chosen at time $t = 0$ and cannot be changed. In the Merton problem we have instead a dynamic strategy, even if the fraction of wealth to invest in the assets is constant. We have that the optimal static asset allocation can do no better that the optimal dynamic one since the market is complete every option can be replicated by a trading strategy involving just stock and the risk-free bond. Since this two problems are equivalent the static strategy is preferable since it avoid exposure to market frictions such as transaction cost (e.g. settlement cost for the equities) and limited liquidity.

Since remaining in the Black and Scholes framework each derivative is redundant, we find a solution to the stochastic optimization problem by relaxing a model hypothesis. We therefore consider that the volatility is not constant but the dynamics of the stock is characterized by stochastic volatility following the Heston model. The portfolio includes one stock, one bond and two derivatives. We find in closed form the optimal fraction of wealth that the investor allocates to each of the assets.

The dynamics considered so far to describe the risky security is unrealistic, in fact, it presents only continuous trajectories excluding the possibility of jumps in the price of the risky security. In addition to stochastic volatility, we have added the risk of jumps to the dynamics of the stock, finding an optimal allocation strategy

for a portfolio composed of a risky stock, a bond and two derivatives.
In the view of the results seen in the first part, we try to ask ourselves what would happen if derivatives were added to an incomplete market when the stocks follow the Geometric Brownian motion.
The first result found is the case where we start from a complete market composed of two stocks governed by two random resources. We consider a derivative instrument written on one of the two stocks. In this case we would have an infinite solution because we have an additional investment possibility represented by the derivative, as we find in chapter 2. We place a greater restriction: the stock that serves as the underlying of the derivative cannot be traded. In this way we started from an incomplete market which we complete with the addition of a derivative. We find an optimal allocation solution when the derivative is an option contract. We get interesting results because in both cases they are comparable with the solution of Merton's problem when considering two stocks.
The latest result found is when we consider an incomplete market composed of one stock and two random resources. To complete the market we add an option written on the stock, we find infinite solutions of optimal allocation strategy.

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Appendix A

Proof

A.1 Proof of Proposition 4.1.1

In Chapter 4 we have found in closed form an optimal allocation strategy. The portfolio consists of a stock, a bond and two derivatives written on the stock.
In section 4.1 the stock has stochastic volatility. After having found the ideal weights to associate to the different assets, in order to verify that the indirect utility function $J(t,x)$ hypothesized is correct it is necessary to find the expression of the functions $H(\cdot)$ and $h(\cdot)$.
We suppose in particular that for given wealth X_t and volatility V_t, the solution to the HJB equation is given by

$$J(t, X_t, V_t) = \frac{X_t^{1-\gamma}}{1-\gamma} exp(\gamma h(T-t) + \gamma H(T-t)V_t).$$

Here below we will illustrate all the steps to find the expression of these two functions for Proposition 4.1.1.
We can do the same steps again for proposition 4.2.1, where the difference is that the stock has both stochastic volatility and jump risk. The difference lies in the definition of the term δ.

Starting from the equation of HJB (4.9), replacing the derivatives of the utility function J, we obtain the following equation:

$$W_t^{1-\gamma}\ exp(\gamma h(T-t) + \gamma H(T-t)V_t)\ \max_{\phi_t,\psi_t} \left(\frac{1}{1-\gamma}(-\gamma h'(T-t) - \gamma H'(T-t)V_t) \right.$$
$$+ (r + \theta_t^v \lambda V_t) - \frac{\gamma}{2} V_t((\theta_t^s)^2 + (\theta_t^v)^2) + k(\bar{v} - V_t)\frac{\gamma}{1-\gamma} H(T-t) + \frac{1}{2}\sigma V_t \frac{\gamma^2}{1-\gamma} H(T-t)^2$$
$$\left. + \sigma V_t \gamma H(T-t)(\rho\theta_t^s + \sqrt{1-\rho^2}\theta_t^v) \right) = 0.$$

We collect the terms in which the volatility V_t appears and replace the optimal θ^*

values found previously. We obtain:

$$
\begin{aligned}
&-\frac{\gamma}{1-\gamma}H'(\tau)+\lambda\bigg(\frac{\lambda}{\gamma}+\sigma\sqrt{1-\rho^2}H(\tau)\bigg)-\frac{\gamma}{2}\bigg(\sigma^2\rho^2H^2(\tau)+\frac{\lambda^2}{\gamma^2}+\sigma^2(1-\rho^2)H^2(\tau)\\
&+2\sigma\frac{\lambda}{\gamma}\sqrt{1-\rho^2}H(\tau)\bigg)-\frac{\gamma}{1-\gamma}kH(\tau)+\sigma\gamma H(\tau)\bigg(\sigma\rho^2H(\tau)+\frac{\lambda}{\gamma}\sqrt{1-\rho^2}+\\
&+\sigma(1-\rho^2)H(\tau)\bigg)+\frac{\gamma^2}{2(1-\gamma)}\sigma H^2(\tau)=0.
\end{aligned}
$$

Simplifying and adding up the terms we find

$$
-\frac{\gamma}{1-\gamma}H'(\tau)+\bigg(-k\frac{\gamma}{1-\gamma}+\lambda\sigma\sqrt{1-\rho^2}\bigg)H(\tau)+\frac{\gamma}{1-\gamma}\frac{\sigma^2}{2}H^2(\tau)+\frac{\lambda^2}{2\gamma}=0,
$$

$$
H'(\tau)=\bigg(-k+\frac{\gamma}{1-\gamma}\lambda\sigma\sqrt{1-\rho^2}\bigg)H(\tau)+\frac{\sigma^2}{2}H^2(\tau)+\frac{\lambda^2}{2\gamma^2}(1-\gamma).
$$

Defining

$$
k_1=k-\frac{\gamma}{1-\gamma}\lambda\sigma\sqrt{1-\rho^2},\qquad\qquad \delta=\frac{\lambda^2}{\gamma^2}(1-\gamma).
$$

We obtain

$$
H'(\tau)=-k_1H(\tau)+\frac{\sigma^2}{2}H^2(\tau)+\frac{\delta}{2}. \tag{A.1}
$$

Now we collect from the HJB the terms in which the volatility V_t does not appear. We obtain:

$$
-\frac{\gamma}{1-\gamma}h'(\tau)+r+k\bar{v}\frac{\gamma}{1-\gamma}H(\tau)=0,
$$

$$
h'(\tau)=r\frac{\gamma}{1-\gamma}+k\bar{v}H(\tau). \tag{A.2}
$$

Let's start by solving equation (A.1).
This is an ordinary non-homogeneous differential equation that is quadratic in the unknown function. It is known as **Riccardi's equation**.
We make the following replacement

$$
H=-\frac{y'}{y\ \sigma^2/2}.
$$

The derivative of H is

$$
H'=-\frac{y''(y\ \sigma^2/2)-y'(y'\ \sigma^2/2)}{(y\ \sigma^2/2)^2}=\frac{(y')^2\sigma^2/2-y''(y\ \sigma^2/2)}{(y\ \sigma^2/2)^2}.
$$

By inserting this into the starting equation, a homogeneous equation of the second order is obtained

$$
y''+k_1y'+\frac{\sigma^2}{4}\delta y=0. \tag{A.3}
$$

The characteristic polynomial associated to equation (A.3) is

$$x^2 + k_1 x + \frac{\sigma^2}{4}\delta = 0,$$
$$\Delta = k_1^2 - 4\frac{\sigma^2}{4}\delta = k_1^2 - \sigma^2\delta := k_2^2$$
$$x_{1,2} = \frac{-k_1 \pm k_2}{2}.$$

The solution of (A.3) is

$$y = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$

where

$$\lambda_1 = \frac{-k_1 - k_2}{2}, \qquad \lambda_2 = \frac{-k_1 + k_2}{2}.$$

Returning to the initial replacement where now we know the y function, H has the following form

$$H(\tau) = -\frac{c_1\lambda_1 e^{\lambda_1\tau} + c_2\lambda_2 e^{\lambda_2\tau}}{\frac{\sigma^2}{2}(c_1 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau})}.$$

We now impose the initial condition $H(0) = 0$

$$H(0) = 0 = -\frac{c_1\lambda_1 + c_2\lambda_2}{\frac{\sigma^2}{2}(c_1 + c_2)} \quad \Rightarrow \quad c_1 = -\frac{\lambda_2}{\lambda_1}c_2.$$

So we have:

$$H(\tau) = -\frac{-c_2\lambda_2 e^{\lambda_1\tau} + c_2\lambda_2 e^{\lambda_2\tau}}{\frac{\sigma^2}{2}(-\frac{\lambda_2}{\lambda_1}c_2 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau})} = -\frac{\lambda_2(-e^{\lambda_1\tau} + e^{\lambda_2\tau})}{\frac{\sigma^2}{2}(-\frac{\lambda_2}{\lambda_1}e^{\lambda_1\tau} + e^{\lambda_2\tau})}$$

$$-e^{\lambda_1\tau} + e^{\lambda_2\tau} = -e^{\frac{-k_1-k_2}{2}\tau} + e^{\frac{-k_1+k_2}{2}\tau} = e^{\frac{-k_1}{2}\tau}(-e^{\frac{-k_2}{2}\tau} + e^{\frac{+k_2}{2}\tau}) = e^{\frac{-(k_1+k_2)}{2}\tau}(e^{k_2\tau} - 1)$$

$$-\frac{\lambda_2}{\lambda_1}e^{\lambda_1\tau} + e^{\lambda_2\tau} = -\frac{\lambda_2}{\lambda_1}e^{\frac{-k_1-k_2}{2}\tau} + e^{\frac{-k_1+k_2}{2}\tau} = e^{\frac{-k_1}{2}\tau}\left(-\frac{\lambda_2}{\lambda_1}e^{\frac{-k_2}{2}\tau} + e^{\frac{k_2}{2}\tau}\right) =$$
$$= e^{\frac{-(k_1+k_2)}{2}\tau}\left(-\frac{\lambda_2}{\lambda_1} + e^{k_2\tau}\right)$$

$$-\frac{\lambda_2}{\lambda_1} + e^{k_2\tau} = \frac{-k_1 + k_2}{k_1 + k_2} + e^{k_2\tau} = \frac{-k_1 + k_2 + k_1 e^{k_2\tau} + k_2 e^{k_2\tau}}{k_1 + k_2} =$$
$$= \frac{2k_2 + (k_1 + k_2)(e^{k_2\tau} - 1)}{k_1 + k_2}$$

$$-\lambda_2(k_1 + k_2) = \frac{k_1 - k_2}{2}(k_1 + k_2) = \frac{k_1^2 - k_2^2}{2} = \frac{k_1^2 - k_1^2 + \delta\sigma^2}{2} = \frac{\delta\sigma^2}{2}$$

$$H(\tau) = -\frac{\lambda_2 e^{\frac{-(k_1+k_2)}{2}\tau}(e^{k_2\tau} - 1)}{\frac{\sigma^2}{2}e^{\frac{-(k_1+k_2)}{2}\tau}\left(-\frac{\lambda_2}{\lambda_1} + e^{k_2\tau}\right)} = -\frac{\lambda_2(e^{k_2\tau} - 1)}{\frac{\sigma^2}{2}\left(-\frac{\lambda_2}{\lambda_1} + e^{k_2\tau}\right)}$$
$$= -\frac{\lambda_2(e^{k_2\tau} - 1)}{\frac{\sigma^2}{2}(2k_2 + (k_1 + k_2)(e^{k_2\tau} - 1))}(k_1 + k_2) = \frac{e^{k_2\tau} - 1}{2k_2 + (k_1 + k_2)(e^{k_2\tau} - 1)}\delta.$$

Now that we have found the formula for $H(\tau)$, we can find the formula for $h(\tau)$ by solving the equation (A.1).

$$h'(\tau) = r\frac{\gamma}{1-\gamma} + k\bar{v}H(\tau).$$

This is a differential equation with separable variables.

$$dh(t) = \left(r\frac{\gamma}{1-\gamma} + k\bar{v}H(t)\right)dt \quad \Rightarrow \quad \int_0^\tau dh(t) = \int_0^\tau \left(r\frac{\gamma}{1-\gamma} + k\bar{v}H(t)\right)dt$$
$$\Rightarrow \quad h(\tau) = r\tau\frac{\gamma}{1-\gamma} + k\bar{v}\int_0^\tau H(t)dt.$$

Now we focus on solving the integral

$$\int_0^\tau H(t)dt = \int_0^\tau \frac{e^{k_2\tau}-1}{2k_2+(k_1+k_2)(e^{k_2\tau}-1)}\delta dt =$$
$$= \delta\left[\frac{t(k_1+k_2) - 2ln(k_1e^{k_2t} - k_1 + k_2e^{k_2t} + k2}{k_1^2 - k_2^2}\right]_0^\tau =$$
$$= \frac{\delta}{k_1^2 - k_1^2 + \delta\sigma^2}\left[\tau(k_1+k_2) - 2ln(k_1e^{k_2\tau} - k_1 + k_2e^{k_2\tau} + k2) + 2ln(2k_2)\right] =$$
$$= \frac{2}{\sigma^2}\left[ln\left(exp\left(\frac{\tau(k_1+k_2)}{2}\right)\right) - ln(2k_2 + (k_1+k_2)(e^{k_2\tau}-1)) + ln(2k_2)\right] =$$
$$= \frac{2}{\sigma^2}ln\left(\frac{2k_2exp\left(\frac{\tau(k_1+k_2)}{2}\right)}{2k_2+(k_1+k_2)(e^{k_2\tau}-1)}\right).$$

So the formula of $h(\tau)$ is

$$h(\tau) = r\ \tau\frac{\gamma}{1-\gamma} + \frac{2k\bar{v}}{\sigma^2}ln\left(\frac{2k_2\ exp(\tau(k_1+k_2)/2)}{2k_2+(k_1+k_2)(e^{k_2\tau}-1)}\right).$$

Appendix B

Matlab

B.1 Numerical results of equivalence

In Chapter 3 we have defined a necessary and sufficient condition for the dynamic problem in which a stock and a bond is considered to be equivalent to the static problem in which the model is enriched with options. In particular in section 3.2 we show this equivalence through a numerical example.
Here below is reported the Matlab code that refers to the example reported in the treatment.

```
clc
clear
close all

%% Parameters
r = 0.05;    mu = 0.06;   sigma = 0.3;     T = 5;
Nsim = 1e6; W0 = 1;        S0 = 1;
p = 0.3;

%% Simulation
% Simulate ST
ST = S0 * exp((mu-sigma^2/2)*T + sigma*sqrt(T)*randn(Nsim
    ,1));
[price_S, n, I] = normfit(exp(-r*T)*ST);

% Simulate value of the first Call option
CT= ones(Nsim,1);          K = 0.7;
for i = 1:Nsim
    CT(i) = max(0, ST(i)-K);
end
[price_C, n, I] = normfit(exp(-r*T)*CT);

```

```
%% ONE Call option

%Function that we want to maximise
f = @(x) mean((x(1)*ST + x(2)*CT + (1-x(1)-x(2))).^ p/ p,1);

% Starting point of the optimization algorithm
weights0 = [0.5; 0.5];

% Matrices for indicating the linear constraints of inequality Ax <= b
A = [-1 -1 ; (-K+1) 1 ; 1 1 ];
b = [0; 1; 1];

% Equality non linear constraint
eq = @(x) x(1)*price_S + x(2)*price_C + (1-x(1)-x(2))*exp(-r*T) - W0;

% Minimization options
options = optimset ('Largescale', 'off', 'MaxFunEvals', 1000, 'GradObj', 'off', ...
                    'Tolfun', 1e-9, 'TolX', 1e-9, 'TolCon', 1e-6, 'Display', 'iter');

[xopt_1, fval_1, exitFlag] = fmincon(@(x) -f(x), weights0, A, b, [], [], ...
                                     [], [], @(x) nonlin_con(x,eq) , options);
%% BUY-and-HOLD Portfolio (Static Asset Allocation Problem)
%% TWO Call options

% Simulate value of the second Call option
CT1= ones(Nsim,1);        K1 = 1;
for i = 1:Nsim
    CT1(i) = max(0, ST(i)-K1);
end
[price_C1, n, I] = normfit(exp(-r*T)*CT1);

%Function that we want to maximise
f = @(x) mean((x(1)*ST + x(2)*CT + x(3)*CT1 + (1-x(1)-x(2)-x(3))).^ p/ p,1);

% Starting point of the optimization algorithm
```

```
weights0 = [0.25; 0.25; 0.5];

% Matrices for indicating the linear constraints of inequality Ax <= b
A = [1 1 1; 1-K 1 1; 1-K1 1-K1+K 1; -1 -1 -1];
b = [1; 1; 1; 0];

% Equality non linear constraint
eq = @(x) x(1)*price_S + x(2)*price_C + x(3)*price_C1 + (1-x(1)-x(2)-x(3))*exp(-r*T) - W0;

% Minimization options
options = optimset ('Largescale', 'off', 'MaxFunEvals', 1000, 'GradObj', 'off', ...
                    'Tolfun', 1e-9, 'TolX', 1e-9, 'TolCon', 1e-6, 'Display', 'iter');

[xopt_2, fval_2, exitFlag] = fmincon(@(x) -f(x), weights0, A, b, [], [], ...
                    [], [], @(x) nonlin_con(x,eq) , options);

%% THREE Call options

% Simulate value of the third Call option
CT2= ones(Nsim,1);        K2 = 1.3;
for i = 1:Nsim
    CT2(i) = max(0, ST(i)-K2);
end
[price_C2, n, I] = normfit(exp(-r*T)*CT2);

%Function that we want to maximise
f = @(x) mean((x(1)*ST + x(2)*CT + x(3)*CT1 + x(4)*CT2 + (1-x(1)-x(2)-x(3)-x(4))).^ p/ p,1);

% Starting point of the optimization algorithm
weights0 = [0.25; 0.25; 0.25; 0.25];

% Matrices for indicating the linear constraints of inequality Ax <= b
A = [1 1 1 1; 1-K 1 1 1; 1-K1 1-K1+K 1 1; -1 -1 -1 -1; 1-K2 1-K2+K 1-K2+K1 1];
b = [1; 1; 1; 0; 1];

```

```
% Equality non linear constraint
eq = @(x) x(1)*price_S + x(2)*price_C + x(3)*price_C1 + x
    (4)*price_C2 + ...
          (1-x(1)-x(2)-x(3)-x(4))*exp(-r*T) - W0;

% Minimization options
options = optimset ('Largescale', 'off', 'MaxFunEvals',
    1000, 'GradObj', 'off', ...
                    'Tolfun', 1e-9, 'TolX', 1e-9, 'TolCon'
                      , 1e-6, 'Display', 'iter');

[xopt_3, fval_3, exitFlag] = fmincon(@(x) -f(x), weights0,
    A, b, [], [], ...
                               [], [], @(x) nonlin_con(x
                                 ,eq) , options);

%% MERTON Problem (Dynamic Asset Allocation Problem)

x_opt = (mu -r) / ((1-p)*sigma^2);

rho = p*(x_opt*(mu-r) + r - 0.5*x_opt^2*(1-p)*sigma^2);
fval_Dyn = exp(rho*T)*W0.^p/p;

%% Print the result

disp('Optimal problem with ONE call option');
disp('Weights');            disp(xopt_1);
disp('Value of ptf');       disp(-fval_1);

disp('Optimal weights with TWO call options');
disp('Weights');            disp(xopt_2);
disp('Value of ptf');       disp(-fval_2);

disp('Optimal weights with THREE call options');
disp('Weights');            disp(xopt_3);
disp('Value of ptf');       disp(-fval_3);

disp('Merton optimal problem');
disp('Weight');            disp(x_opt);
disp('Value of ptf');      disp(fval_Dyn);
```