POLITECNICO

MILANO 1863

SCUOLA DI INGEGNERIA INDUSTRIALE
E DELL'INFORMAZIONE

Continuous-time system identification with functional basis expansions

Author: José Antonio Rebollo

Advisor: Prof. Marco Lovera

Co-advisor: Enrico Barbiero

Academic year: 2024-2025

1. Introduction

System identification seeks to build accurate mathematical models from experimental data. This field is of great interest for applications such as rotorcraft flight dynamics, where a detailed knowledge of the linear and non-linear behavior is critical for safe and effective operation, control design, and performance analysis. This presents a series of challenges, as the collected data are often noisy and of limited duration or bandwidth, and usually collected in closed-loop. One of the most widely used identification techniques in closed-loop applications is the Predictor Based Subspace IDentification (PBSID) algorithm and its variants [1], which combines subspace projections with prediction error minimization. It identifies the system by first estimating prediction errors using oblique projections of the data and then solving a least-squares problem to determine the state-space matrices. This approach improves model accuracy, especially in closed-loop settings, by explicitly accounting for the feedback structure during the estimation process.

While most identification methods focus on discrete-time systems, there is a need for directly identifying continuous-time models, especially when dealing with stiff systems or non-uniformly sampled data. Several approaches for continuous-time identification have been proposed, often relying on basis functions to avoid the numerical issues related to high order derivative computation. This includes transformations based on Laguerre basis expansions or generalized orthonormal bases [2, 3].

Although the PBSID technique offers a powerful approach to system identification, without the need of an initial estimate nor an iterative approach, and while providing good statistical guarantees, the extension to continuous time systems is an open area of research. The analysis performed in [4] led to the Continuous-Time PBSID (CT-PBSID) algorithm, based on two different state-of-the-art strategies: the application of Laguerre filters and the projection into a basis of Laguerre functions, with the posterior providing better results.

2. Main contributions

This work develops a novel framework for transforming linear continuous-time (CT) systems into discrete-time (DT) systems using orthonormal bases in a functional analysis setting. By projecting system signals and operators onto these bases, a direct mathematical link between CT and DT systems is established, addressing

the gap in the literature concerning the CT-DT transformation from a basis transformation viewpoint. A specific case using Hermite functions leads to the Hermite Domain PBSID (HD-PBSID) method, contributing to the field of system identification and control.

The framework is general and not limited to a specific basis. Leveraging Sturm-Liouville theory, the orthonormal bases arising from eigenproblems of second-order linear operators can be systematically determined, as well as the associated projection of the differentiation operator, crucial for the identification problem. Practical aspects are explored through simulations and experimental quadrotor dynamics identification, validating theoretical results.

This work aims to answer three key research questions: the relationship between Laguerre filtering and projection methods in continuous-time system identification, the applicability of prior methods to other function families, and the possibility of a unified framework for continuous-time system identification across different bases.

3. CT-DT transformation via basis expansions

The first part of this work delves into the formal characterization of CT-DT transformations. This involves first laying out the foundations on functional spaces, basis transformations, and orthogonality. Then, leveraging results from quantum mechanics, a novel system identification framework is proposed.

3.1. Formal characterization

A Hilbert space is a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$ satisfying some properties of bilinearity, conjugate symmetry and positive definiteness. CT linear systems are accurately modeled as operators $T : \mathcal{L}^2 \to \mathcal{L}^2$ acting on Hilbert spaces (e.g., $\mathcal{L}^2(\mathbb{R})$, the space of square integrable functions on the real axis), with the objective of finding a bijective transformation to equivalent DT operators $T_d : \ell^2 \to \ell^2$ in discrete Hilbert spaces (e.g., $\ell^2(\mathbb{N})$, the space of square summable sequences). A key requirement for system identification is preserving the mean and covariance of Wiener processes, this is, the behavior of white noise.

Schauder or countable bases, like Laguerre functions $\{l_n(t)\}$ and Hermite functions $\{h_n(t)\}$, allow representing continuous signals as sequences. These orthonormal bases facilitate CT-DT transformation via projections

$$v = \sum_{i \in \mathbb{N}} \alpha_i b_i, \quad \forall v \in \mathcal{L}^2, \tag{1}$$

where $\{b_i\}$ is the basis and $\alpha_i = \langle b_i, v \rangle$. Applying this to a linear map $v' = Tv$,

$$\langle b_j, v' \rangle = \sum_{i \in \mathbb{N}} \langle b_i, v \rangle \langle b_j, Tb_i \rangle \implies \tilde{v}'_j = \sum_{i \in \mathbb{N}} \tilde{T}_{ji} \tilde{v}_i, \tag{2}$$

where $\tilde{v}_i = \langle b_i, v \rangle$ and $\tilde{T}_{ji} = \langle b_j, Tb_i \rangle$, thus transforming the CT operator into a DT representation in ℓ^2.

A revision of CT-PBSID using Laguerre functions proposed in [2, 4] highlights the connection between projection and convolution methods, revealing that the latter is an approximation, valid only for $t \to \infty$, failing to fully preserve noise statistics, and leading to an inaccurate transformation. This justifies the fact that the projection method performs better than its convolution counterpart. The projection method introduced in [3] coincides with a Schauder basis expansion, particularized for Laguerre functions.

3.2. The HD-PBSID method

One core innovation of this work is a novel Hermite function-based CT-DT transformation inspired by the quantum harmonic oscillator. Using the creation $(a^\dagger)$ and annihilation (a) operators, the derivative operator is expressed as

$$\frac{d}{dt} h_n(t) = \frac{1}{\sqrt{2}} \left(a - a^\dagger \right) h_n(t), \tag{3}$$

leading to a matrix-like derivative operator $\underline{D}$ for the Hermite coefficients

$$\frac{d}{dt} \underline{f} = \underline{f}\underline{D}, \tag{4}$$

where $\underline{f}$ is the vector of Hermite coefficients of a general function $f(t)$. This allows transforming CT dynamics $\frac{d}{dt} x = Ax + Bu$ into $\underline{x}\underline{D} = A\underline{x} + B\underline{u}$. A reconstruction formula completes the CT-DT process

$$\begin{pmatrix} f(t_0) & f(t_1) & \cdots & f(t_N) \end{pmatrix} = \underline{f}\underline{H}, \tag{5}$$

where $\underline{H}$ is the matrix of sampled Hermite functions (from $h_0(t)$ to $h_N(t)$) at times

$t_0, t_1, \ldots, t_N$. Numerical examples demonstrate the effectiveness of this transformation, including accurate differentiation and integration in the Hermite domain, confirming that, for Wiener processes, the covariance of random processes is conserved.

Using this new formulation in tandem with the DT PBSID algorithm, the Hermite Domain PBSID (HD-PBSID) method is developed, addressing practical issues like effective support of the Hermite basis, time scaling, and the introduction of a modified differentiation operator $\underline{D}' = \left(\frac{\underline{D}}{\alpha} + \beta I\right)\frac{1}{\gamma}$ to ensure DT stability, and resulting in an integrator form for improved noise robustness. Parameters α, β and γ address time scaling, eigenvalue shifting and eigenvalue scaling, respectively, and allow adjusting the derivative operator for each identification problem. These results provide a ready-to-use algorithm with practical guidance, addressing limitations and enhancing the existing body of CT-DT transformation knowledge.

4. A general theory of CT-DT transformation

The previous results motivate generalizing the CT-DT transformation framework, encompassing both Laguerre-based and Hermite-based methods (HD-PBSID), by leveraging the Sturm-Liouville theory for differential equations. The goal is to establish a systematic methodology for constructing CT-DT transformations based on different orthogonal bases.

The Sturm-Liouville problem is defined by the equation

$$Lf(x) = \lambda w(x)f(x), \qquad (6)$$

where $L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$ is the Sturm-Liouville operator, λ is the eigenvalue, $w(x)$ is the weight function, and $f(x)$ are the eigenfunctions. The operator L is formally self-adjoint under suitable boundary conditions. This ensures real eigenvalues and orthogonal eigenfunctions, namely, for f_n, f_m solutions of the Sturm-Liouville problem with eigenvalues λ_n, λ_m,

$$\int_a^b f_n(x)f_m(x)w(x)dx = d_n^2\delta_{nm}, \qquad (7)$$

where d_n^2 is a normalization constant.

Any second-order eigenvalue problem of the form

$$p_0(x)f''(x) + p_1(x)f'(x) - q_0(x)f(x) = -\lambda f(x), \qquad (8)$$

can be recast into the Sturm-Liouville form by multiplying by a weighting factor $w(x)$ satisfying

$$w(x) = \frac{1}{p_0(x)}\exp\left(\int \frac{p_1(y)}{p_0(y)}dy\right). \qquad (9)$$

If $p_0(x), p_1(x)$, and $q_0(x)$ are polynomials, the eigenfunctions $f_n(x)$ can be represented by Rodrigues' formula

$$f_n(x) = \frac{c_n}{w(x)}\frac{d^n}{dx^n}(w(x)p_0^n(x)), \qquad (10)$$

where c_n is a normalization constant and where $\lambda_n = -n\left(p_1' + \frac{n-1}{2}p_0''\right)$.

A key result is the derivation of generalized index rising ($\hat{a}^\dagger$) and lowering ($\hat{a}$) operators and a general derivative operator for orthogonal functions

$$\begin{aligned}
2p_0(x)\frac{d}{dx} = \\
\frac{c_n d_{n+1}}{c_{n+1} d_n}\frac{\lambda_n}{n\pi_n'}\hat{a}^\dagger - \frac{c_{n-1}d_n}{c_n d_{n-1}}\frac{\lambda_{n-1}}{(n-1)\pi_{n-1}'}\hat{a} \\
-\frac{\lambda_n \pi_n(x)}{n\pi_n'} + \frac{\lambda_{n-1}\pi_{n-1}(x)}{(n-1)\pi_{n-1}'} - p_0'(x),
\end{aligned} \qquad (11)$$

where $\pi_n(x) = p_1(x) + np_0'(x)$. This framework is applied to specific standard cases, computationally validating the results with well-known literature. Equation (11) provides a systematic approach for obtaining the derivative operator systematically, extending the HD-PBSID to all bases arising from (8).

By inspecting the relationship between the maximum degree of $p_0(x)$ and the structure of the bases obtained from (8), solutions can be classified into Hermite-like (order zero), Laguerre-like (order one), Legendre and Jacobi-like (order two). For each case, respectively, the derivative is obtained multiplying a constant, x and x^2 term. Each power of x acts like an inverse derivative, implying that its presence in the system behaves like a filter. This provides a foundation for understanding the filtering properties of different orthogonal bases in CT-DT transformations.

5.   Simulation analysis

This work presents a computational verification and validation of the proposed Hermite Domain PBSID (HD-PBSID) algorithm through a series of detailed simulation studies. The primary goal is to assess the performance of the HD-PBSID under varying operational conditions and to directly compare its capabilities with those of the existing CT-PBSID algorithm.

The initial computational verification employs a well-defined second-order linear time-invariant (LTI) system, characterized by specific system matrices. To emulate real-world scenarios, a noise component with varying Signal-to-Noise Ratio (SNR) is systematically introduced into the simulated data. The system is excited using a linear sine sweep signal. Furthermore, to enhance the quality of the system representation within the Hermite domain, the excitation signal undergoes projection onto a truncated Hermite basis, up to a certain maximum order $n_{max,u}$, yielding a better suited excitation signal.

The results of this verification phase reveal several key insights. For scenarios with relatively low noise, the HD-PBSID algorithm demonstrates successful convergence towards accurate representations of the real system. As the SNR decreases, deviations emerge, particularly at the extremes of the frequency spectrum, attributable to the sine sweep's limited excitation range. The study then explores HD-PBSID parameters through simulations across various configurations, maintaining a moderate-to-high noise level (SNR = 20 dB). The impact of the number of samples is examined, ranging from 100 to 10000. The findings confirm that an increasing number of samples leads to improved accuracy in system identification. The effect of varying Hermite basis orders, spanning from 10 to 1000, is also investigated. The simulations indicate that expansion orders below 30 result in poor identification, while increasing the order beyond 50 yields negligible improvements in accuracy. Lastly, differentiation operator parameters α and β have minimal impact, provided they stay within stable, numerically sound ranges.

To provide a robust comparative analysis, the HD-PBSID and CT-PBSID algorithms are applied to the identification of a closed-loop controlled linear dynamics model of a quadrotor. The system is excited using a carefully con-

structed control signal, and both algorithms are used to identify the system for different SNR levels. The comparative results, summarized in Table 1, reveal a clear trend. While both algorithms perform comparably well under low-noise conditions, the HD-PBSID exhibits significantly more robust performance in highly noisy environments. The HD-PBSID consistently provides accurate and unbiased estimates, even at low SNR (e.g., SNR = 10 dB). In contrast, the CT-PBSID encounters difficulties, often resulting in divergent behavior and inaccurate estimations, especially when SNR levels drop. For instance, Figure 1 depicts the results of a Monte Carlo simulation for SNR = 6 dB. The HD-PBSID exhibits fewer outliers and more tightly clustered eigenvalues than the CT-PBSID, highlighting the improved robustness of the new framework in high noise scenarios. This better performance stems from its rigorous, non-redundant approach to signal and operator projection, along with the lower-order Hermite expansion-based input setup.
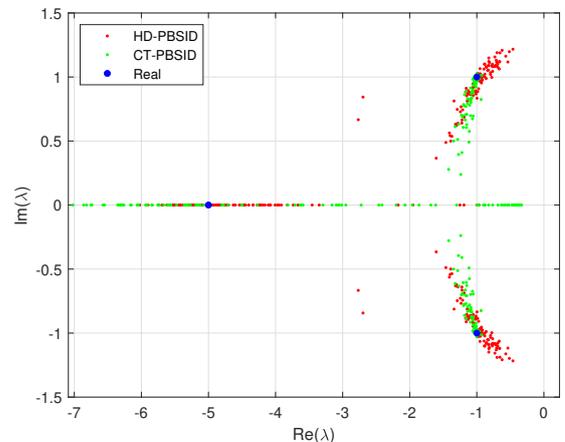

Figure 1: Eigenvalues obtained by the CT-PBSID and HD-PBSID for SNR = 6 dB, zoomed for detail.

6.   Experimental implementation

An experimental validation of the HD-PBSID algorithm was performed on a quadrotor UAV within the ANT-X Drone Lab environment. The Drone Lab provides a quadrotor, a ground control station, and a motion capture system for high-frequency kinematic data acquisition. The

SNR (dB)	Method	Bias		STD	
		Dim 1	Dim 2	Dim 1	Dim 2
50	Hermite	0.005854	0.00071335	0.012384	0.0038998
	Laguerre	0.00047555	$6.6732 \cdot 10^{-5}$	0.00095678	0.00017196
20	Hermite	0.00046385	0.0048368	0.077576	0.035362
	Laguerre	0.10017	0.0061251	0.1052	0.0083464
10	Hermite	0.05526	0.027607	0.23503	0.11277
	Laguerre	2.5129	0.18066	4.6628	0.30477
6	Hermite	0.15335	1.8918	0.45364	0.19165
	Laguerre	45.89	1.8254	227.25	1.0726

Table 1: Bias and standard deviation (STD) results for different SNR values for the CT-PBSID and HD-PBSID.

Experiment	Eigenvalues			
1	-28.1279	-10.4272	$1.2072 + 5.8205i$	$1.2072 - 5.8205i$
2	-29.6969	-12.2414	$1.5850 + 4.9288i$	$1.5850 - 4.9288i$
3	-37.5718	-6.1102	$0.5082 + 4.3069i$	$0.5082 - 4.3069i$

Table 2: Eigenvalues of the identified plant for the longitudinal dynamics.

focus is on identifying the longitudinal and lateral angular dynamics of the drone, utilizing measurements of angular velocity and acceleration.

The angular dynamics are modeled as those of a rigid body, and are linearized around steady-state conditions. A fourth order system is theorized in consequence of the coupling between translational and rotational dynamics.

System control and excitation are achieved using a combination of feedback control and an LMNF injection, that is, a perturbation of the moments generated by the propellers. Three experiments were performed for each axis, exciting frequencies between 1 and 4 Hz. The excitation signal was constructed via Hermite projections of a sine sweep, ensuring both frequency content and Hermite domain representation requirements are met.

Using the sum of an LMNF injection and the closed loop control as input, the HD-PBSID algorithm, with a 400-term Hermite basis expansion, was used to identify a fourth-order system, motivated by singular value analysis. The eigenvalues for the longitudinal and lateral dynamics are shown in Tables 2 and 3, respectively.

The obtained system matrices were verified by comparing the experimental outputs to the simulated ones for the used excitation inputs. As an example, the results obtained for the longitudinal dynamics are included in Figure 2. In general, all cases provide rather similar results, mainly in the frequency range excited by the input signal, that is, for $\omega \in [2\pi, 8\pi]$ rad. A good agreement of the three identified systems was observed for higher frequencies. To obtain a better fitting for a wider frequency range, and in particular for a better characterization of the slow modes, a longer experiment is necessary. An unstable oscillating mode was observed around $0 \pm 5i$. The remaining eigenvalues were obtained as slow damping modes, which could be associated to dissipation effects.

The obtained results provide a promising experimental validation of the HD-PBSID algorithm, demonstrating its potential for practical CT system identification applications.

7. Conclusions

This thesis introduced a novel framework for identifying CT systems using orthogonal function bases, with a specific focus on the HD-PBSID method. The proposed approach offers an alternative to existing CT identification

Experiment	Eigenvalues			
1	-28.9986	-11.1375	$2.4045 + 4.9270i$	$2.4045 - 4.9270i$
2	-27.4011	-12.2941	$1.1678 + 5.3542i$	$1.1678 - 5.3542i$
3	-31.7832	-9.7939	$0.8442 + 5.3266i$	$0.8442 - 5.3266i$

Table 3: Eigenvalues of the identified plant for the lateral dynamics.
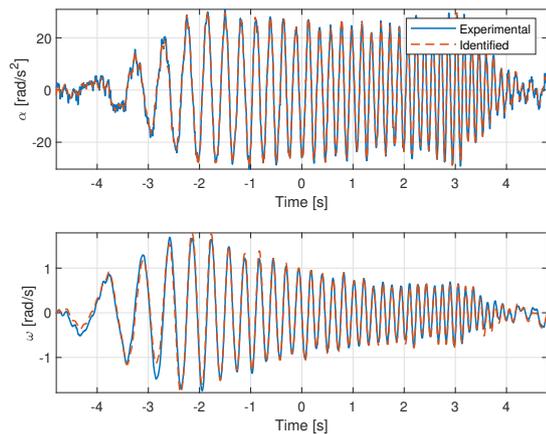

Figure 2: Simulated and experimental output for the longitudinal dynamics.

methods like CT-PBSID, particularly excelling in noisy environments. A key contribution of this work is its rigorous mathematical formulation, which links the choice of orthonormal function bases to filtering properties and numerical stability. While previous research has extensively explored Laguerre-based transformations, this thesis expands the framework by incorporating Hermite functions, leveraging their compact support and well-defined spectral characteristics to enhance CT system identification.

The effectiveness of the HD-PBSID method was evaluated through both numerical simulations and experimental validation. Simulation results demonstrated its robustness against noise and model uncertainties, while real-world testing on a quadrotor UAV confirmed its practical applicability. The method successfully identified both longitudinal and lateral dynamics, showing strong agreement between estimated and measured responses. These findings suggest that the use of orthonormal function bases, as proposed in this thesis, can significantly improve the accuracy and reliability of CT system identification, particularly in applications requiring high-fidelity models. However, challenges re-

main, including increased computational complexity with higher-order basis functions and the need for further validation across different systems and conditions.

Future research should explore alternative orthonormal function bases, such as Legendre or Jacobi functions, to optimize basis selection for specific system dynamics. Another promising direction is extending the proposed CT-DT transformation framework to other system identification algorithms. Additionally, exploring and improving computational efficiency through adaptive order selection or sparsity-based representations could make the method more scalable and suitable for real-time applications. Finally, testing the approach on more complex aerospace, robotic, or industrial systems would further validate its effectiveness and expand its potential applications in system identification and control design.

References

[1] A. Chiuso and G. Picci, "Consistency analysis of certain closed-loop subspace identification methods," *Automatica*, vol. 41, no. 3, pp. 377–391, 2005.

[2] B. Haverkamp, C. Chou, M. Verhaegen, and R. Johansson, "Identification of continuous-time mimo state space models from sampled data, in the presence of process and measurement noise," in *Proceedings of 35th IEEE Conference on Decision and Control*, vol. 2, pp. 1539–1544, IEEE, 1996.

[3] Y. Ohta, "Realization of input-output maps using generalized orthonormal basis functions," *Syst Control Lett*, vol. 22, no. 6, pp. 437–444, 2005.

[4] M. Bergamasco, "Continuous-time model identification with applications to rotorcraft dynamics," in *PhD thesis, Politecnico di Milano*, 2012.