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Kantian evolutionary game theory

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Abstract

Evolutionary game theory has long been an essential tool for understanding the dynamics of natural selection and competition among individuals in various populations. However, classical evolutionary game theory primarily focuses on fitness and often falls short in capturing the complexities of cooperation exhibited by the players involved. This thesis introduces a novel framework, named as "Evolutionary Kantian Theory," which addresses this limitation by incorporating elements of moral and cooperative behavior into classical settings.

The objectives of the research are, first to provide a comprehensive review and synthesis of non-Nashian game theory, shedding light on alternative approaches to modeling strategic interactions; then to enable us to develop a framework that not only accommodates the traditional notions of fitness and competition but also effectively captures the inherent cooperation observed in real-world scenarios. Our approach involves adapting classical evolutionary game theory by introducing various parameters, borrowed from Nashian game theory, that enable us to capture the complex tradeoff between selfishness and morality. By doing so, we create a more nuanced and comprehensive model that better aligns with the complex decision-making processes of individuals.

Finally, we simulated these new equations on various games, where we observed the emergence of novel features that defy classical expectations. These simulations serve as empirical evidence of the utility and relevance of the Evolutionary Kantian Theory in capturing the subtleties of human and animal behaviors that transcend the simplistic assumptions of classical models. This framework might provide valuable insights into the dynamics of cooperation, the emergence of moral norms, and the evolution of altruistic behaviors.

In conclusion, the Evolutionary Kantian Theory represents a significant advancement in the field of evolutionary game theory. By bridging the gap between traditional fitness-based models and the complexities of cooperation, this research offers a more holistic understanding of the strategic interactions that shape the evolutionary landscape.

Keywords: Game theory, Cooperation, Evolution, Morality, Nashian optimization

Abstract in lingua italiana

La teoria evolutiva dei giochi è da tempo uno strumento essenziale per comprendere le dinamiche della selezione naturale e della competizione tra individui di varie popolazioni. Tuttavia, la teoria evolutiva classica si concentra principalmente sulla fitness e spesso non riesce a catturare le complessità della cooperazione esibita dai giocatori coinvolti. Questa tesi introduce un nuovo schema, denominato "Teoria kantiana evolutiva", che affronta questa limitazione incorporando elementi di comportamento morale e cooperativo in contesti classic.

Gli obiettivi della ricerca sono, in primo luogo, fornire una revisione completa e una sintesi della teoria dei giochi non Nashiana, facendo luce su approcci alternativi alla modellazione delle interazioni strategiche; poi sviluppare uno schema che non solo accoglie le nozioni tradizionali di fitness e competizione, ma anche la cooperazione intrinseca osservata negli scenari del mondo reale. Il nostro approccio prevede l'adattamento della teoria dei giochi evolutiva classica introducendo vari parametri, presi in prestito dalla teoria dei giochi classica, che ci consentono di catturare il complesso compromesso tra egoismo e moralità. In questo modo, creiamo un modello più sfumato e completo che si allinea meglio ai complessi processi decisionali dei singoli individui.

Infine, abbiamo simulato queste nuove equazioni su vari giochi, dove abbiamo osservato l'emergere di nuove caratteristiche. Queste simulazioni servono come prova empirica dell'utilità e della rilevanza della teoria evolutiva kantiana nel catturare le sottigliezze dei comportamenti umani e animali che trascendono i presupposti semplicistici dei modelli classici. Questo quadro potrebbe fornire preziose informazioni sulle dinamiche della cooperazione, sull'emergere di norme morali e sull'evoluzione dei comportamenti altruistic.

In conclusione, crediamo che la Teoria Evolutiva Kantiana rappresenti un avanzamento significativo nel campo della teoria evolutiva dei giochi. Colmando il divario tra i modelli tradizionali basati sul fitness e le complessità della cooperazione, questa ricerca offre una comprensione più olistica delle interazioni strategiche che modellano il panorama evolutivo.

Parole chiave: Teoria dei giochi, Cooperazione, Evoluzione, Moralità, Ottimizzazione Nashiana

Contents

Introduction

In the ever-evolving landscape of decision-making, the interplay between morality, cooperation, and competition stands as a central challenge. Immanuel Kant's moral philosophy, grounded in the notion of treating others as ends in themselves, offers a profound ethical framework. At the same time, evolutionary game theory provides a robust platform for exploring how strategies and behaviors emerge and persist in dynamic populations. Combining these two seemingly disparate realms, "Kantian Evolutionary Game Theory" seeks to illuminate the intricate relationship between moral principles and the evolution of cooperative behaviors within a competitive environment.

This thesis is structured to explore and bridge the gap between classical game theory, moral-based game theory, and evolutionary game theory. The following chapters lay out the roadmap for our journey of inquiry:

In the first chapter, we delve into the foundational concepts and results of classical game theory. Here, we establish the fundamental framework of rational decision-making, the Nash equilibrium concepts, and the dynamics of competitive interactions. We will also glimpse the limitations of such a manner of optimizing, in terms of optimality and its predictive power in real-case scenarios.

It leads to the second chapter, dedicated to a comprehensive review of the existing literature in collaborative and moral-based game theory. We survey the various approaches and models that incorporate moral considerations into strategic interactions. This literature review forms the bedrock for our exploration of morality within the realm of game theory and is denoted by the generic term of Kantian optimization in opposition to Nashian optimization.

In the third chapter, we introduce the classical evolutionary game theory framework. Here, we delve into the principles of evolutionary dynamics, replicator dynamics, and the study of equilibrium in such an environment. This chapter lays the groundwork for understanding how strategies propagate and evolve over time, within the classical game theory framework. Then, we built a model that aims to estimate the resilience of Kantian optimizers against Nashian ones in a coordination game, using a double dynamics game that we numerically simulate.

Finally, the last chapter of our thesis navigates the delicate balance between pure competitive evolutionary game theory and a moral-based approach. We explore how Kantian ethics can be incorporated into the evolutionary dynamics of strategy selection, by defining the so-called replimorator equation that merges the selfish and moral incentives of players. Through numerical simulations and analysis of benchmark games, we seek to shed light on what extent moral considerations impact the emergence and sustainability of cooperative behaviors within competitive environments.

As we embark on this intellectual journey, we aim to unravel the intricate landscape of human decision-making between self-interest and cooperation. Through these exploration and reasoning, we aspire to positively contribute to our understanding of the dynamics of moral reasoning and cooperation in complex social systems.

1 | Game theory: A quick introduction

This section is a quick introduction to *Game Theory* and introduces the essential notations of the document. We recall the main result of the so-called classical game theory, which we will consider as the base of our journey. Suggests for more precise and complete results on the classical theory the following references [8] [12].

As its name suggests, *Game Theory* aims to investigate the interaction between different players in a given environment. Given the choice of each of these players, they will obtain a different reward which we can quantify. In order to perform mathematical analysis, we will denote the finite set of players of a given game by $\mathcal{P} = \{1, ..., I\}$. Then, each player i has a pure strategy space S_i, which represents the possible choices that each player can take. Finally, a utility function u_i is associated with each player i, that associates to each possible common choice $(s_1, ..., s_I) \in S_1 \times ... \times S_I$ a reward. This framework, known as *normal form* or *strategic form*, can capture a wide range of interactions between players and is widely used in social science, evolution theory, or economics to name just a few. However, it is worth noticing that this representation does not integrate all possible games. For instance, special interactions such as sequential games just to name one, where players take decisions not in simultaneous time, require another framework.

Once the rule of a game has been settled, in a strategic form in our case, a straightforward question that follows is which configuration the players will end up. If we consider a trivial game with only one player, the answer is rather simple and is equal to the strategy $s_1 \in S_1$ that maximizes its utility u_1. We don't actually need another theory in this case, it is already a well-known mathematical field called optimization. However, starting with a situation where we have at least 2 players, the problem becomes rather more complicated.

To illustrate this fact, let's consider the famous prisoner dilemma. This name describes more precisely a category of games that share special features and can be found in different daily life situations, in several versions. To add some storytelling to the situation, let's say we have two prisoners who have committed a crime and get arrested later on as some suspicions but no proof arises on them. As soon as they got arrested, they were both interrogated separately without any way of communication. Moreover, we suppose each of them only cares about their fate, no matter the outcome for his previous partner in crime. In other terms, they are completely *selfish* and have no *altruism* or *morality*, we will come back to these concepts later.

To come back to the story, the justice in charge doesn't have any proof of their culpability therefore they will try their best in order to obtain confessions. Let's say that at first, with a naïve approach, the court will inflict 10 years of prison on both players if they obtain any confession and they will be released without any punishment in the case of no confession. Therefore, each player has the choice of confessing or denying, denoted respectively by C and D. As such, their utility function will be both equal to $u(C,C) = u(C,D) = u(D,C) = -10$ and $u(D,D) = 0$, situation summarise thanks to the *payoff matrix* 1.1a. However, this setting is rather problematic for the court, indeed we understand quite rapidly that none of the convicts has the interest to confess, and they will both deny the crime. More precisely, whatever has chosen the other convict, the alternative denying is always one (and in the case of the other one denying the only one) of the best solution in order to maximize, in a selfish manner, his own utility. The outcome (D,D) in the naïve case corresponds to a famously called *Nash equilibrium*, the main concept of the classical game theory according to definition 2 introduced later. As we will see, under the assumption of the selfishness of the players, no coordination, and rational players fully informed of the outcome, the output of the game will necessarily end up in one of the Nash Equilibrium. As in the naïve setting, (D,D) is the only Nash Equilibrium, therefore the situation is rather problematic for the tribunal...

An approach that could be used by the tribunal is to make the confession option more appealing and the denying option less attractive, in a technical way to perform game designing. In order to do so, they can decide to send them to one year in jail if they both deny instead of nothing, in the end, they have already several pieces of proof against them. Moreover, they decide to punish even more a convict who has been caught lying, so if the other criminal talks while the first one as denied, then he will do 10 years of jail plus 5 for lying to the court. All this enables the D option to be less considered, but is it enough to make the option (C,C) the outcome, in other words the Nash Equilibrium? Knowing that the other one chooses to confess, then in this new setting the right option is to confess as well. However, if we know that the other one denies, then following his move in denying is still the best option for us as well because we will take 1 year of jail instead of 10. Therefore we would need to reward a betrayer, and for instance not pursue him if he talks while the other stays muted. The modified game can be seen in the matrix representation 1.1b.

Convict 1 \ Convict 2	C	D
C	$(-10,-10)$	$(-10,-10)$
D	$(-10,-10)$	$(0,0)$

(a) Naïve court setting

Convict 1 \ Convict 2	C	D
C	$(-10,-10)$	$(0,-15)$
D	$(-15,0)$	$(-1,-1)$

(b) Setting whit the modified sentence

Figure 1.1: Sequences in the two different justice setting.

Therefore, by modifying the penalty in a smart way, the tribunal has been able to provide the right incentive in order to make the convicts betray each other. Examples of this game design are numerous, such as the *Programme de clémence* [11] [18] just to name one, that aims to fight against corporate cartels.

1.1. Foundations of game theory

As we have seen in the introduction, being able to design a theoretical framework for the interaction between players is essential in order to determine the possible rational outcomes. The objective of this section is, therefore, to provide the necessary definitions as well as the main result of the discipline, taking as a reference the book "Game Theory" [8] of Jean Tirole and Drew Fudenberg.

1.1.1. Strategic form game

A game in strategic (or normal) form is composed of three elements: a finite set of player $\mathcal{P} = \{1, ..., I\}$, for each player i a *pure-strategy space* S_i, a *utility function* u_i associated to player i which associates to each possible common choice $s = (s_1, ..., s_I) \in S_1 \times ... \times S_I$ the i's von Neumann-Morgenstern utility $u_i(s)$.

The utility functions u_i associated with each player, are by assumption in the classical game theory equal to the *material payoff* obtained, as the players are assumed to be *selfish* and care only about their personal gain. Moreover, it is also supposed that players know perfectly the structure of the game and they are aware that the other players know the structure as well. To be complete, it is assumed that every player knows that all the other ones know that they know that they know, and so on *ad infinitum*. In addition to this common knowledge structure, players cannot communicate and play their strategy in simultaneous time.

For simplicity, we suppose the game to be finite i.e. $S = \prod_i S_i$ to be finite, which simplifies the theoretical justification for what follows. Indeed, players are also authorized to play in a stochastic manner, by playing their strategies in a random way. More specifically, each player selects a probability distribution σ_i over his finite strategic space S_i. The utility of the player therefore, according to the *mixed strategy profile* $\sigma = \prod_i \sigma_i$, therefore corresponds to the expected value of his pure strategic payoff. We also remark that this extended framework still does include the case where players play deterministic. We denote by Δ_i the set of mixed strategies of player i, and by $\Delta = \prod_i \Delta_i$ the set of mixed strategies.

As the players cannot communicate and play at the same time, the different random variables that correspond to the pure strategy selected by each player i, which belong to S_i, are independent. Then, the player's payoff associated with profile $\sigma \in \Delta$ is given by

$$u_i(\sigma) = \sum_{s=(s_1,\dots,s_I)\in S} [\prod_{j=1}^{I} \sigma_j(s_j)] u_i(s) \tag{1.1}$$

With these foundations and assumptions, we are now looking at the potential solution where such a game can end up.

1.1.2. Rational outcomes and Nash equilibrium

To investigate the outcome of a game, we will start to make a very intuitive definition and reasoning out of it. Indeed, it can be deduced that for some players i, some pure strategies in S_i are not relevant at all to be played.

Definition 1 (strictly dominated strategy)**.** *A pure strategy $s_i \in S_i$ is strictly dominated for player i if there exist $\sigma_i \in \Delta_i$ such that*

$$\forall s_{-i} \in S_{-i}, u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \wedge \quad \exists s'_{-i} \in S_{-i}, u_i(\sigma_i, s'_{-i}) > u_i(s_i, s'_{-i}) \tag{1.2}$$

We have introduced the very useful notation S_{-i} which corresponds to $\prod_{j\neq i} S_j$, which is the set of all the combinations of the pure strategies without these of player i. According to this definition of dominated strategy and thanks to the player rationality, if s_i is dominated at least with a strict inequality, then it would never be played. By repeatedly canceling the (strictly) dominated strategies, we can considerably reduce the game and in some lucky cases even end up in the unique rational solution of the game. For instance, one can verify that, by performing this strategy in the case of the prisoner dilemma, the only rational strategy is to betray.

Unfortunately, many if not most games of economic interest are not completely solvable by iterated strict dominance. In contrast, the concept of a Nash-equilibrium solution has the advantage of existing in a broad class of games.

Definition 2 (Nash-equilibrium). *A mixed-strategy profile σ^* is a Nash-equilibrium if*

$$\forall i \in \mathcal{P}, \forall \sigma_i \in \Delta_i : \quad u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \tag{1.3}$$

Therefore, a Nash equilibrium is a strategy profile such that each player's strategy is an optimal response to the other's strategies. Nash equilibriums are "consistent" predictions of how the game will be played as if such a solution is played, no one has the interest in deviating from this position by their own initiative. If a strategy that has been played is not a Nash equilibrium, then we can argue that at least one of the players has made a mistake as he could have gotten a better payoff. We will not discuss further the other deep reasons that make Nash equilibrium, the strategies that focalize the attention in the classic game theory, as well as in its application in other fields such as economics. However, one of the reasons is that there exists at least one Nash equilibrium for a wide range of games, as the following theorem stands.

Theorem 1.1 (Exitence of a Mixed-Strategy Equilibrium). *Every finite strategic-form game has a mixed-strategy equilibrium*

This result is well-known in the discipline and was provided by John Nash in 1950, who first introduced the previous equilibrium concept. The proof of this result is out of the scope of this summarized work but can be found in [8] for the interested reader. Similar results can be demonstrated, for an infinite game with a continuous payoff for instance, but for our needs the Theorem 1.1 is enough.

However, if the theorem stands, then it provides at least one mixed-strategy equilibrium, but that could be not unique. Indeed, many games present several Nash equilibriums, such as the Stag and Hare game. Another main issue with the Nash equilibrium is that it often corresponds to a sub-optimal strategy.

Definition 3 (Pareto dominancy). *A strategy profile $\sigma \in \Delta$ is said to be dominated by $\nu \in \Delta$ in the sense of Pareto if*

$$(\forall i \in \mathcal{P}: \quad u_i(\sigma) \leq u_i(\nu)) \quad \wedge \quad (\exists i^+ \in \mathcal{P}: \quad u_{i^+}(\sigma) < u_{i^+}(\nu)) \tag{1.4}$$

As it turns out, there exist numerous games where Nash equilibriums are not optimal in the sense of Pareto, i.e. it exists another profile that will do better or equal for all players and strictly better for at least one of them. The prisoner dilemma 1.1b is a perfect example of that fact, as the Nash equilibrium provides a penalty of 10 years to each player, whereas they would have taken only 1 if they had cooperated.

2 | Non-Nashian game theory: an overview

"Mutual Aid: A Factor of Evolution"[9] by Peter Kropotkin, published in 1902, is a groundbreaking book that challenges the prevailing social and scientific theories of its time. It presents a compelling argument for the significance of mutual aid and cooperation in the natural world, particularly in the evolution of species.

Nowadays, we have built complex societies that are also based on the principle of collaboration as the different tasks, even essential for our survival such as food production with agriculture, are done by different economic actors. Indeed, through time we managed to develop societies where economic actors are more and more specialising which increases the overall economic efficiency. We are able to run complex projects, create states with taxation to provide public goods and commons. However, it required an efficient way of exchanging different goods, then the emergence of money which is a formidable cooperation. Indeed, a currency must have three basic properties which are a medium of exchange, a scale of value, and finally, a reserve of value [1]. Its two first properties empathize with the utility of a currency in order to run such complex economic interaction, the latter underlines the trust in the community and future cooperation. Phenomenons such as hyperinflation are characteristic of the loss of trust of people in the economy and currency management associated with the monetary zone.

Unlikely, the reverse processes concerning cooperation can also be observed in human society, where a progressive lack of trust in the common project or exogenous destabilization can progressively lead in the worth cases to a society collapse, with severe consequences for the population, due to a rapid switch to less efficient and more chaotic structure[4].

In order to determine what sustains the collaboration in such a structure, many factors are responsible for it. From our developed elocution system to talk, our big brain to perform complex reasoning, and even our eyebrows to understand better aggressivity level to the cultural learning process of cooperate, our species is deeply cooperative as it is investigated in [14]. In such a complex environment, we often collaborate and do the right things without noticing it, by following what others do and moral norms. That way of optimizing is then, in essence, morally based.

Such a player that optimizes according to what is best for the community can be denoted as *Homo moralis* in opposition to the classical *Homo oeconomicus* of the game theory. Here we make the first, but not the last, distinction between morality and altruism as they are often confused. Indeed, a fully altruistic player would only care about the happiness of the other player, whereas the morale player cares about all the players including himself, in other words, the community. This is, in my personal comprehension of those concepts, the main difference that is not easy to grasp as they are both assimilated into cooperative ways to optimize. However, I reassure the readers that those concepts will be fully discussed during this paper, and I hope they will become crystal clear to them.

2.1. Homo moralis and homo altruistic: an overview

In this section, we summarize the different propositions of the article [2] in order to properly define a non-Nashian comportment.

In the classical game theory framework, agents are assumed to be *Homo oeconomicus* who are rational agents that only care about their payoff. In other words, its utility function is strictly equal to its payoff function. Two other kinds of players, that do not care only about their individual payoff are introduced, namely *Homo moralis* and *Homo altruis*.

In this framework, we consider an n-player normal-form game where all players have the same set of strategy $\mathcal{X}$. The material payoff function π is common to all the players, which implies that the game is symmetric. If a player is playing the strategy $x \in \mathcal{X}$, whereas the others are playing the strategies $\boldsymbol{y} \in \mathcal{X}^{n-1}$, he would get the *material payoff* $\pi(x, \boldsymbol{y}) \in \mathbb{R}$. In the classical game theory framework, the material payoff is what a player would look to maximize at all costs because it is assimilated to his utility. Indeed, what players are looking to optimize would be their utility.

As such, a player is the *Homo oeconomicus* of the classical game theory if its utility is given by

$$\forall (x_i, \boldsymbol{x}_{-i}) \in \mathcal{X}^n : \quad u(x_i, \boldsymbol{x}_{-i}) = \pi(x_i, \boldsymbol{x}_{-i}) \tag{2.1}$$

An individual is said to be *altruist*, with a *degree of altruism* $\alpha \in [0, 1]$, if he cares as well about the materials payoff of the other to some degree. The associated utility is then given by

$$\forall (x_i, \boldsymbol{x}_{-i}) \in \mathcal{X}^n : \quad v(x_i, \boldsymbol{x}_{-i}) = \pi(x_i, \boldsymbol{x}_{-i}) + \alpha \sum_{j \neq i} \pi(x_j, \boldsymbol{x}_{-j}) \tag{2.2}$$

Finally, a player is said to be a *Homo moralis* if it attaches, also to some degree, what would be his personal outcome if the other play like him. Formally, the utility of a Homo moralis with a degree of morality $k \in [0, 1]$ is

$$\forall (x_i, \boldsymbol{x}_{-i}) \in \mathcal{X}^n : \quad w(x_i, \boldsymbol{x}_{-i}) = \mathbb{E}[\pi(x_i, \tilde{\boldsymbol{x}}_{-i})] \tag{2.3}$$

where $\tilde{\boldsymbol{x}}_{-i}$ is a random vector belonging to $\mathcal{X}^{n-1}$, which derive from $\boldsymbol{x}_{-i}$ by replacing each of its $n-1$ components by x_i with a probability equal to the degree of morality k.

We then remark that, in the case where X represents pure strategy, with a morality coefficient $k = 1$ players are going to be focalized on the SKE 4 of the game. It means that the definition of *Homo k-moralis* is coherent with the definition of Kantian optimizer, defined in the following section when they are fully moral i.e. $k = 1$.

2.2. How we cooperate: Main concepts of Kantian equilibrium

How we cooperate is a book [6] that was published in 2019, and written by John E. Roemer who is a professor at Yale University. However, before the release of the book, several papers by the same author already exposed the so-called Kantian optimization spirit, named in honor of the German philosopher Immanuel Kant. Indeed Kant's categorical imperative stipulates that "Act only according to that maxim whereby you can at the same time will that it should become a universal law.", which is the core idea of Kantian optimization. This new approach to game theory has inspired numerous research on non-Nashian optimization game theory, at least this is where I started my journey on non-Nashian game theory. The main content of the book is to investigate cooperation through the concept of Kantian optimization.

As opposed to a Nash optimizer, who optimizes by wondering what would be his new outcome if only he deviates from the current position, a Kantian player wonders what would be his new outcome if only he deviated from the current state, with the other players deviating in a "similar way" as well. In other terms, a Kantian player asks himself "What would happen if all of us take that kind of decision". This way of optimizing is then, in essence, morally based, and such a player can be denoted as *Homo moralis* in opposition to the classical *Homo oeconomicus* of the game theory.

Moreover, in the special case of symmetric games, where players can deeply feel that they are in the same boat, the author supposes the different players to be *superrational*. This assumption means that if a player, rationally chooses to play a strategy x, then the exact same reasoning would compel other all agents to do the same. The consequence of this reasoning is when considering the normal form of the game, only the strategy giving the greatest reward on the diagonal should be played. In this framework of a symmetric game, this is where we can define the most intuitive kind of Kantian equilibrium that we will introduce later, the so-called Simple Kantian Equilibrium (SKE) given by definition 4.

In this section, we denote by S the set of common strategies which is assumed to be a real interval. In general, S represents the effort or the quantity of work that a player provides, hence the fact that it is defined in the positive real value line. We will consider games with n players, and for all player $i \in [|1, n|]$ we denote by $V^i : S^n \to \mathbb{R}$ its payoff function. This environment composes the game that we denote by $\mathbf{V}$. Up to now, we will introduce the new concept in the study of Kantian Theory as is it done in [6].

Indeed, there exist different kinds of Kantian equilibrium.'We start to introduce the one which is the simplest to understand conceptually, the (SKE) standing for Simple Kantian Equilibrium.

Definition 4 (SKE)**.** *In a symmetric game* $\mathbf{V}$*, a SKE is a strategy profile* $\mathbf{s} = (s, ..., s) \in S^n$ *such that* $\forall i \in [|1, n|], V^i(\mathbf{s}) = V(\mathbf{s})$ *is maximized*

Remark 1. *We underline the fact that this definition can be extended to common diagonal games, by the same definition.*

For the following definitions, we suppose $S = [0, +\infty)$, which make the definition of the following concepts easier.

Definition 5 (multiplicative Kantian equilibrium)**.** *A multiplicative Kantian equilibrium in the game* $\mathbf{V}$ *is a strategy vector* $(s_1, ..., s_n)$ *such that nobody would prefer to rescale the strategy of everyone by a nonnegative common factor. Formally,*

$$\forall i, \forall r \geq 0, V^i(s_1, ..., s_n) \geq V^i(rs_1, ..., rs_n).$$

We denote such an allocation a $K^\times$ *equilibrium.*

Following the same reasoning, we can define a *additive Kantian equilibrium* where in this case the players would consider translation instead of rescaling.

Definition 6 (additive Kantian equilibrium)**.** *An additive Kantian equilibrium (K^+) is an allocation such that nobody would prefer to translate its strategy by a constant. Formally,*

$$\forall i \forall r \geq -s_i, V^i(s_1, ..., s_n) \geq V^i(s_1 + r, ..., s_n + r)$$

where $s_j + r = 0$ if $s_j + r < 0$.

The last two definitions embedded the morality roots of "what would happen if everyone deviates as I do" by considering the impacts of common similar deviation. In the book, many examples inspired by real-world scenarios, and the relevancy of using whether the K^+ or $K^\times$ are deeply discussed. Moreover, a special category of games that are said to be monotone presents very interesting features towards those concepts.

Definition 7 (Monotone increasing games)**.** *A Game* $\mathbf{V}$ *is said to be increasing (respectively strictly) if, for all player i, V^i is increasing (respectively strictly) with respect to the strategy of the player different that i. Formally,* $\forall i \in [|1, n|], \forall (S_1, ..., S_n), \forall \{r_j\}_{j \in [|1,n|] \setminus \{i\}} \geq 0, V^i(S_1 + r_1, ..., S_i, ..., S_n + r_n) \geq V^i(S_1, ..., S_i, ..., S_n)$

In a similar way, we define decreasing games. A game is said to be (strictly) *monotone* if it is either (strictly) monotone increasing or decreasing. This kind of game includes games where the tragedy the of common (for strictly decreasing games) or free riders problem (respectively for strictly increasing games) occur, which are very frequent when dealing with common policy management. This concept of strictly monotone games turns out to be an essential feature. In fact, in such a game if a Kantian equilibrium does exist, then it is Pareto efficient, and this is for all the different kinds of Kantian optimizations described above, which is not the case at all concerning Nash equilibriums.

Theorem 2.1 (Pareto efficiency in strictly increasing games)**.** *Let* $\mathbf{V}$ *be a strictly monotone game. There any SKE, K^+ and $K^\times$ equilibrium are Pareto optimal.*

2.3. Alternative tentative to define cooperative equilibrium

The concept that are presented in this section is taken from [7]. The objective, which is clearly stated in the paper is to extend the notion of Kantian optimization to a wider range of games, as the notions defined in the previous section are quite restrictive.

2.3.1. Kantian equilibrium generalize in terms of a fair common variation function

We start by defining the notion of Kantian Hofstadter equilibrium, which in particular generalizes the notion of K^{+} and $K^{\times}$ to in terms of strategy space and deviation possibilities.

Definition 8 (Kantian Hofstadter equilibrium). *Let G be a game with a common set of actions A. We define a variation function as $\phi : \Theta \times A \to A$, for some set of parameter Θ. A Kantian (Hofstadter) equilibrium is a pure strategy profile $x^{opt} = (x_1^{opt}, ..., x_n^{opt})$ that maximizes the material payoff of each agent, in the case that players move in a similar way with respect to the variation function. Formally, for all agent i and parameter $\theta \in \Theta$,*

$$V_i(x_1^{opt}, ..., x_n^{opt}) \geq V_i(\phi(\theta, x_1^{opt}), ..., \phi(\theta, x_n^{opt}))$$

Remark 2. *This is a generalization that extends the previous definition given in Roemer's book, and this is for the various kind of equilibrium. Note that a similar idea of variation function is also introduced in [6] in order to generalize the concept of $K^{\times}$ and K^{+} equilibrium. However, the definition 8 introduced in [7] is an even wider generalization.*

The main concept of the previous definition is the so-called *variation function* ϕ that represents a "moral rule". Indeed, this function associates, for each initial action $x_{initial}$ and a manner deviation θ, the unique right action x_{moral}. In a formal way, for each player i currently playing the action $x^i_{intial} \in A$ and deviation parameter $\theta \in \Theta$, we have that the moral move to switch denotes by x^i_{moral} is such that $x^i_{moral} = \phi(x^i_{intial}, \theta)$. As all the players consider the same deviation function ϕ, this partially embedded the idea of "fairness" and "morality" that we aim to represent. We said partially because the payoff matrix of the game in the general framework could be very different given the player, and the idea of fairness could be very far in various games. However, the important point is that all the players agreed on a common way to move from their personal strategy, which embedded the principle of reciprocity.

2.3.2. Attempt of Generalizing Kantian equilibrium for mixed strategies

One of the major issues of the Simple Kantian Equilibrium (SKE) concept is that it concerns pure strategies, which is a narrow view of the potential possibilities. Here are presented different attempts to generalize this notion into the classical stochastic choice among pure strategies.

Definition 9 (Mixed Kantian agent). *Given a game G with identical actions, a mixed Kantian agent will choose a mixed strategy $X^{OPT} \in \Delta_G$ that maximizes its expected utility, should everyone play X^{OPT}.*

This is a natural way to expand the notion of Kantian strategy in the case of mixed strategies. However, this concept might not be very relevant to the fact that determining the right X^{opt} is not a simple task in general. Therefore, the irrelevancy rises when in real scenarios, players would have to find this common optimal action and thus should be spotted in a limited time frame. The following theorem is in that sense a piece of bad news.

Theorem 2.2. *The following problem, called "mixed kantian equilibrium", is NP-hard:*

Given a two-player symmetric game G, and an aspiration level $r \in \mathbb{Q}$, is there a mixed strategy profile $x = (x_1, ..., x_N)$ such that the utility of every player under the common mixed action $x_1a_1 + x_2a_2 + ... + x_ma_m$ is at least r?

A very elegant concept introduced in [7] is the concept of *Kantian program equilibrium.* it provides a nice way to generalize the Kantian optimization procedure in cases where mixed strategy or even dis-coordination is necessary in order to obtain optimal solutions.

Definition 10 (Kantian program equilibrium). *Given a (Pareto symmetric) game G with identical actions set for all players, a Kantian program equilibrium in G is a probability distribution p on the action profiles of G such that:*

a) p has its support on the set of Pareto optimal strategies.

b) p is implemented by agents playing a common program P in the extended game.

c) there exists no probability distribution q with the two previous properties such that the vector of expected utilities $(\mathbb{E}[u_i(q)])_{i\in[|1,n|]}$ strictly dominates the vector $(\mathbb{E}[u_i(p)])_{i\in[|1,n|]}$.

A crucial point of this definition is indeed to properly formalize the concept of *program.* We invite an interested reader to take a look at the original article, where proper definitions are given in Section 15. This last concept seems to us promising, as in several games a third entity might be necessary in order to reach coordination and cooperation.

3 | Classical Evolutionary Game theory: The Replicator equation

The aim of this chapter is, at first, to present the basic notion of the so-called Evolutionary Game Theory (EGT). We will see in particular, the core notion of this theory, which is called the Replicator equation.

In a second time, we will present a model that uses the previous equation in a non-standard way in order to simulate the stability and resilience of Kantian players toward Nasher's invasion in a coordination game. The former model is a two-strategic game that has been developed in the following paper [10], and will be extended in a three-strategic one.

3.1. A brief introduction of EGT

Evolutionary game theory is a powerful framework that merges the principles of evolution and strategic interaction to explore how various behaviors and strategies can emerge, persist, and evolve within populations over time. Originating at the intersection of biology and economics, evolutionary game theory provides a captivating lens through which researchers can better understand the dynamics of competition, cooperation, and decision-making in a wide range of contexts.

Evolutionary game theory has far-reaching applications across disciplines. In biology, it helps explain the emergence of cooperation in scenarios where traditional game theory predicts selfishness [5]. It sheds light on the development of social behaviors, such as group living, altruism, and the formation of alliances among animals. In economics, evolutionary game theory offers insights into the evolution of norms, conventions, and market dynamics, where the persistence of certain behaviors is not solely determined by rational considerations [19].

At its core, evolutionary game theory builds upon the foundation of classical game theory by incorporating elements from evolutionary biology. Where game theory traditionally studies the strategic choices made by rational individuals in one-shot interactive situations where each person's outcome depends not only on their actions but also on the choices of others. Evolutionary game theory extends this framework by considering how these strategic choices are subject to change over multiple generations, driven by processes akin to biological evolution.

In the natural world, organisms adopt certain strategies or behaviors to enhance their chances of survival and reproduction. These strategies might involve cooperation, competition, altruism, or a combination of these traits. Evolutionary game theory captures this dynamic by introducing the concept of *fitness*, which represents an individual's reproductive success. Strategies that result in higher fitness are more likely to be passed on to the next generation, influencing the composition of the population over time. Evolutionary game theory, in a similar framework as the classical game theory, aims to investigate games. Players make decisions based on their chosen strategies, and these decisions lead to certain payoffs. These payoffs, in an evolutionary setting, are assimilated to reproductive success, and determine an individual's *fitness*. As strategies propagate or decline within a population, the relative frequencies of different strategies can shift, creating a dynamic interplay of behaviors that mirrors the complex interactions observed in nature.

In the following subsections, we are going to expose the most basic model of the evolutionary game, known as the one-population dynamic. Moreover, we will introduce the *Replicator equation* that explicitly drives the different strategies frequencies in time. Moreover, we will introduce the concept of *evolutionarily stable strategy* (ESS), which can be considered as the counterpart of the Nash equilibrium in a revolutionary setting. We rely on the book [15] of Jörgen W. Weibull as a reference for the following results.

3.1.1. Mathematical environment setting of the evolutionary game theory setting

In the classical evolutionary game theory setting, we have a large population of players and each time the game is played, two players are picked randomly within this unique massive population in order to interact with each other. As the population is large enough, we can focus on the evolution of the different strategies' proportion in a continuous way.

Mathematically speaking, we are looking for an Ordinary Differential Equation (ODE) that would describe the evolution of those proportions through time. The set of pure strategies considered is finite and of size n, and the players can only play those n pure strategies that we denote by $\{e_i\}_{i\in[|1,n|]}$. Therefore, contrary to classical game theory, players cannot play mixed strategies. However, we can add a finite amount of them by adding them to the set of pure strategies if we need to.

Moreover, both players are picked randomly from the same population, therefore the game must necessarily be symmetric as they would be indistinguishable. As the set of pure strategies is finite, we can represent the payoff function of both players by a pair of $n\times n$ matrices $(\mathbf{A},\mathbf{B})$. The matrix $\mathbf{A}$ represents the payoff function of player 1, where $\mathbf{A}_{ij}$ is the payoff of player one when he plays strategy e_i against strategy e_j. The mandatory symmetry of the game implies that $\mathbf{B}=\mathbf{A}'$. Therefore, the intrinsic characteristic of the game in this setting is fully characterized by the matrix $\mathbf{A}$, whereas the state of the game will be represented by the vector of proportion $\underline{x}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Hence, we must have always that $\sum_{i=1}^n x_i = 1 \wedge \forall 1\leq i\leq n, x_i\geq 0$, where we denote by Σ_n the set of the vector fulfilling those previous conditions.

Remark 3. *This proportion setting with $\underline{x}$ is very similar to the mixed strategy framework with σ, and in the book [15] the same notation is used for both concepts. Even though they are both different and do not represent the same object, they are kind of related. Indeed, if a proportion $\underline{x}$ is an ESS, which stands for evolutionarily stable strategy in the evolutionary game, then it is necessarily a mixed strategy Nash equilibrium in the classical game theory equivalent game.*

3.1.2. The Replicator dynamic

As we have properly set up the system under study, we are now describing how it can evolve. As described in the introduction of this section, fitness that captures the ability of a behavior to reproduce to the following generations is associated with each strategy. For all pure strategy e_i, its fitness is indeed related to the average payoff and is therefore only a function of the current state of the system $\underline{x}$. As such, if we denote by f_i the fitness associated with each pure strategy e_i we then obtain the following relation:

$$\forall i\in[|1,n|], \forall \underline{x}\in\Sigma_n, f_i(\underline{x}) = \underline{\delta_i'}\mathbf{A}\underline{x} \tag{3.1}$$

Here, $\underline{\delta_i'}$ corresponds to the case of a population playing only the strategy e_i, or in the game theory setting the pure strategy e_i. As such, the value $\underline{\delta_i'}\mathbf{A}\underline{x} = \sum_{j=1}^n x_j A_{ij}$ represent the payoff of an agent who plays the pure strategy e_i against of population playing the pure strategies with the proportion $\underline{x}$. In a more general case, taking two vector $(\underline{x}, \underline{y}) \in \Sigma_n^2$, then $\underline{y}'\mathbf{A}\underline{x} = \sum_{i=1}^n \sum_{j=1}^n x_j y_i A_{ij}$. This value represents the output of the mixed strategy $\underline{y}$ against the strategy $\underline{x}$ in a classical game theory setting, or in the population game setting, the previous quantity represents the average payoff of a population with the strategy proportions $\underline{y}$ that plays against one described by $\underline{x}$. In particular, $\forall \underline{x} \in \Sigma, \underline{x}'\mathbf{A}\underline{x}$ represent the *population average payoff*.

It is in that framework that the famous replicator equation is built, by comparing the average payoff of each pure strategy with the previously defined population average payoff. The change in the proportion of each strategy would be proportional to the difference between the average payoff of that strategy and the average payoff of the entire population. Therefore, for a given strategy e_i, if its fitness is higher than the population average payoff, it means that it does better than the average and naturally it will be more represented within the population. Naturally, there is the opposite incentive when the fitness is less than the average.

Moreover, the evolution of a given proportion will be proportional to the current proportion. This enables an evolutionary setting to capture the reproductive behavior, and in social behavior environment represents the mimicking comportment. It implies the limitation of the model, as it captures only the selection mechanism, as mutation or behavioral initiative are simply not considered. More refined models such as *Replicator-Mutator Equation* or *Stochastic Replicator Dynamics* have been developed in order to include this aspect, but are out of the scope of this work.

Definition 11 (The Replicator equation)**.** *For all pure strategies $e_i \in \{e_j\}_{j \in [|1,n|]}$, and all strategy proportions $\underline{x} \in \Sigma_n$, the evolution of the strategy within the population is given by:*

$$\dot{x}_i(\underline{x}) = x_i(\underline{\delta_i'}\mathbf{A}\underline{x} - \underline{x}'\mathbf{A}\underline{x}) \tag{3.2}$$

By associating the vector fields $\phi : \mathbb{R}^n \to \mathbb{R}^n$ that is defined by $\forall i \in [|1,n|], \phi_i(\underline{x}) = x_i(\underline{\delta_i'}\mathbf{A}\underline{x} - \underline{x}'\mathbf{A}\underline{x})$, we aim to apply the Picard-Lindelöf theorem as the vector field previously defined is polynomial, the local existence and uniqueness of the solution is then well established.

Moreover, the solution of such an equation must have its trajectory fully in Σ_n as must always remain a strategy proportion. This necessarily implies that for all $\underline{x}$, we have that $\sum_{i=1}^n x_i(\underline{x}) = 0$. Moreover, if a proportion x_i is either equal to 0 or 1, then its evolution must be constant, in order to keep proportions in the interval [0,1].

Those basic requirements must be indeed satisfied by a system of (ODE) that describes the evolution of proportions. In the following chapter, when we will introduce the so-called moralisator, one of our first concerns will be to respect those constraints. First, let's prove that the replicator equation does satisfy them.

Proof. The fact that the derivative vanishes when the proportion is equal to 0 is clear. When it is equal to 1, then $\underline{x} = \underline{\delta_i}$, hence the result.

In order to show that for any state $\underline{x}, \sum_{i=1}^{n} x_i(\underline{x}) = 0$, it is enough to observe that $\sum_{i=1}^{n} x_i \underline{\delta_i} = \underline{x}$. Therefore, $\sum_{i=1}^{n} x_i \underline{\delta_i'} \mathbf{A} \underline{x} = \underline{x}' \mathbf{A} \underline{x} = \sum_{i=1}^{n} x_i \underline{x}' \mathbf{A} \underline{x}$ as $\sum_{i=1}^{n} x_i = 1$ for justifying the last equality, Hence the fact that the sum is null. □

We now want to study the potential convergence of the population, and we are introducing the concept of *evolutionary stable strategy* (ESS), which as we will see is strictly related to the concept of Nash equilibrium in the classical setting

3.1.3. Potential convergence of the population

Evolutionary Stable Strategy (ESS) is a concept within the realm of evolutionary game theory that provides a framework for analyzing the stability and persistence of strategies in the population. An ESS is a strategy that, once established as the dominant strategy in a population, is resistant to the intrusion of alternative strategies, preventing them from displacing the ESS. This concept plays a crucial role in understanding the dynamics of strategy evolution and stability in natural and social systems.

Definition 12. *Formally, a strategy $\underline{x}^*$ in a symmetric n-player game represented by the matrix $\mathbf{A}$ is an ESS if, for every other strategy $\underline{y} \neq \underline{x}^*$ it exists and small enough $\varepsilon_y > 0$ such that for all $\varepsilon \in (0, \varepsilon_y)$, $\underline{x}^{*\prime} \mathbf{A}(1-\varepsilon)\underline{x}^* + \varepsilon \underline{y} > \underline{y}' \mathbf{A}(1-\varepsilon)\underline{x}^* + \varepsilon \underline{y}$.*

However, as with Nash equilibrium, the ESS does not explain how a population arrives at such a strategy. Instead, it asks whether once reached, a strategy is robust to evolutionary pressures. In that sense, ESS can be seen are the counterpart of Nash equilibrium in the evolutionary setting, and therefore the set of ESS denoted by Δ_{ESS} is of high interest. Moreover, the following theorem furthers the analogy between both concepts.

Theorem 3.1 ($\Delta_{ESS} \subseteq \Delta_{NE}$). *Let $\mathbf{A}$ a $n \times n$ matrix that represents a symmetric finite game. Denoting by Δ_{ESS} the set of ESS in the evolutionary setting of the game and Δ_{NE} the set of mixed Nash equilibrium in the two-player game setting, we have that $\Delta_{ESS} \subseteq \Delta_{NE}$*

Proof. Let $n > 0$ and $\mathbf{A}$ a $n \times n$ matrix. Taking $\underline{x} \in \Sigma_n$, it can be seen as the mixed strategy among $\{e_i\}_{i\in[|1,n|]}$ in the two-player game setting, or the proportions of pure strategy play in a population one. In the two-player setting, $\underline{x}^* \in \Delta_{NE}$ if and only it is one of the best responses to himself i.e. $\underline{x}^* \in \text{argmax}_{\underline{x}\in\Sigma_n} \underline{x}\mathbf{A}\underline{x}^*$.

An ESS $\underline{x}^*$ in the evolutionary setting is necessarily optimal against itself, if not there exists another strategy $\underline{y} \in \Sigma_n$ that, if played within a small amount, does better than $\underline{x}^*$ i.e $\underline{y}\mathbf{A}\underline{x}^* = \sum_{i=1}^{n} y_i\underline{\delta_i}\mathbf{A}\underline{x}^* > \underline{x}^*\mathbf{A}\underline{x}^*$. This implies that there exists a pure strategy e_i that is doing better than the average fitness and will be increasingly played within the population, and then will move $\underline{x}^*$.

As the property of being optimal against itself fully characterizes a Nash equilibrium, therefore $\Delta_{ESS} \subseteq \Delta_{NE}$. □

The inclusion is however not an equality, as some Nash equilibrium in the classical setting do not correspond to an ESS in the evolutionary setting. Indeed, being an ESS also requires to be a better reply to all its best replies different from itself. See the dedicated section in the reference book of Jörgen W. Weibull [15] for more details.

3.2. Application on the Stag and Hare hunters game

The well known French philosopher Jean-Jacques Rousseau published a book named "Discours sur l'origine et les fondements de l'inégalité parmi les hommes", translate in English as "Discourse on the Origin and Basis of Inequality Among Men," published in 1755. In this text, Rousseau uses a metaphor involving hare and deer hunters to explain his vision of the evolution of human society and inequality. He discusses the transition from a state of nature, where humans lived as solitary individuals or in small groups, to a more complex society. Within this context, Rousseau introduces the concept of stag and hare hunters who, at first, act independently and later begin to collaborate.

In the state of nature, humans were primarily concerned with meeting their basic needs for survival, such as food, shelter, and protection from danger. Initially, individuals hunted small game like hares or other easily attainable resources independently. They used simple tools and had little need for cooperation.

However, as the human population grew and resources became scarcer or more difficult to obtain, there arose a need for collaboration and cooperation. Rousseau suggests that humans realized the benefits of working together, especially when hunting larger and more challenging prey, such as stags. Stags were more substantial sources of food and other resources, and hunting them required coordinated efforts.

As a result, individuals began to come together in groups to hunt stags effectively. This collaboration led to the emergence of social bonds and cooperation among humans. Rousseau highlights this transition from solitary or independent hunting to collaborative hunting as a crucial step in the development of human society.

The act of working together towards common goals like hunting stags signified the beginning of social organization and the formation of early communities. This collaborative effort not only allowed humans to acquire more significant resources but also fostered a sense of interdependence and mutual reliance. This shift from individualistic behavior to collective action laid the foundation for the development of more complex societies and social structures.

In Rousseau's view, this collaborative phase was still relatively egalitarian, as people cooperated for mutual benefit rather than competing for individual gain. However, as societies continued to evolve and the concept of private property emerged, it introduced new dynamics of inequality and competition, ultimately leading to the more significant inequalities that Rousseau criticized in his discourse.

Overall, the collaboration of stag and hare hunters in Rousseau's discourse symbolizes the initial steps towards social organization and the transition from a state of nature to more complex forms of human society.

3.2.1. The stag and hare game

In Rousseau'book, the interaction between hunters as such:

> In this manner, men may have insensibly acquired some gross ideas of mutual undertakings, and of the advantages of fulfilling them:
>
> that is, just so far as their present and apparent interest was concerned: for they were perfect strangers to foresight, and were so far
>
> from troubling themselves about the distant future, that they hardly thought of the morrow. If a deer was to be taken, every one saw that, in order to succeed, he must abide faithfully by his post: but if a hare happened to come within the reach of any one of them, it is not to be doubted that he pursued it without scruple, and, having seized his prey, cared very little, if by so doing he caused his companions to miss theirs. It is easy to understand that such intercourse would not require a language much more refined than that of rooks or monkeys, who associate together for much the same purpose.[1]

[1]Translation by G. D. H. Cole. See http://www.constitution.org/jjr/ineq.htm. The exact quote is: "Voilà comment les hommes purent insensiblement acquérir quelque idée grossière des engagements

As a consequence, the 2-player game where each of them must choose between collaborating and hunting the stag (S), or being an opportunist and going for the hare (H) is represented in figure 3.1.

If both players decide to hunt the stag, they will both take the maximum payoff of 1 as such capture provides a huge amount of food and precious resources. If they both go for the hare, they will get just enough to eat and survive, with the associated material payoff of 0. Finally, if one decides to keep hunting the stag in every circumstance whereas the other is more opportunistic by betraying and going for the hare, the first one would get -1 as he is going have nothing to eat, and the second would get 0.5 as he will have more hare to catch.

		Hunter 2	
		Stag	*Hare*
Hunter 1	*Stag*	$(1, 1)$	$(-1, 0.5)$
	Hare	$(0.5, -1)$	$(0, 0)$

Figure 3.1: The stag and hare game

We can observe that the game is symmetric, and it has two Nash equilibriums which are (S, S) and (H, H). Indeed, if we know that the other hunter is going for the stag, then we should also go for it and take the best payoff. However, if the other is not worth trusting and we know that he will go for hare, we should do the same as we have no chance to catch stag alone. We call it coordination as it has two symmetric Nash-equilibrium of different "quality", where players must succeed to coordinate themselves to the best equilibrium.

On the other hand, the SKE in the game is unique, corresponds to the best possible equilibrium and is as we would have guessed equal to (S, S). Indeed, we all agree that a situation where both players cooperate and go hunting the stag is the best. However, enough trust is necessary in order to achieve this equilibrium as it is more risky, in the sense that in the worth case, the payoff -1 can be taken instead of 0 if he goes for the hare.

Concretely, if $\underline{x} = (x_S, x_H)' \in \Sigma_2$ represents the proportion of stag and hare hunters in

mutuels, et de l'avantage de les remplir, mais seulement autant que pouvait l'exiger l'intérêt présent et sensible; car la prévoyance n'était rien pour eux, et loin de s'occuper d'un avenir éloigné, ils ne songeaient pas même au lendemain. S'agissait-il de prendre un cerf, chacun sentait bien qu'il devait pour cela garder fidèlement son poste; mais si un lièvre venait à passer à la portée de l'un d'eux, il ne faut pas douter qu'il ne le poursuivît sans scrupule, et qu'ayant atteint sa proie il ne se souciât fort peu de faire manquer la leur à ses compagnons. Il est aisé de comprendre qu'un pareil commerce n'éxigeait pas un language beaucoup plus raffiné que celui des corneilles ou des singes, qui s'attroupent à peu près de même."

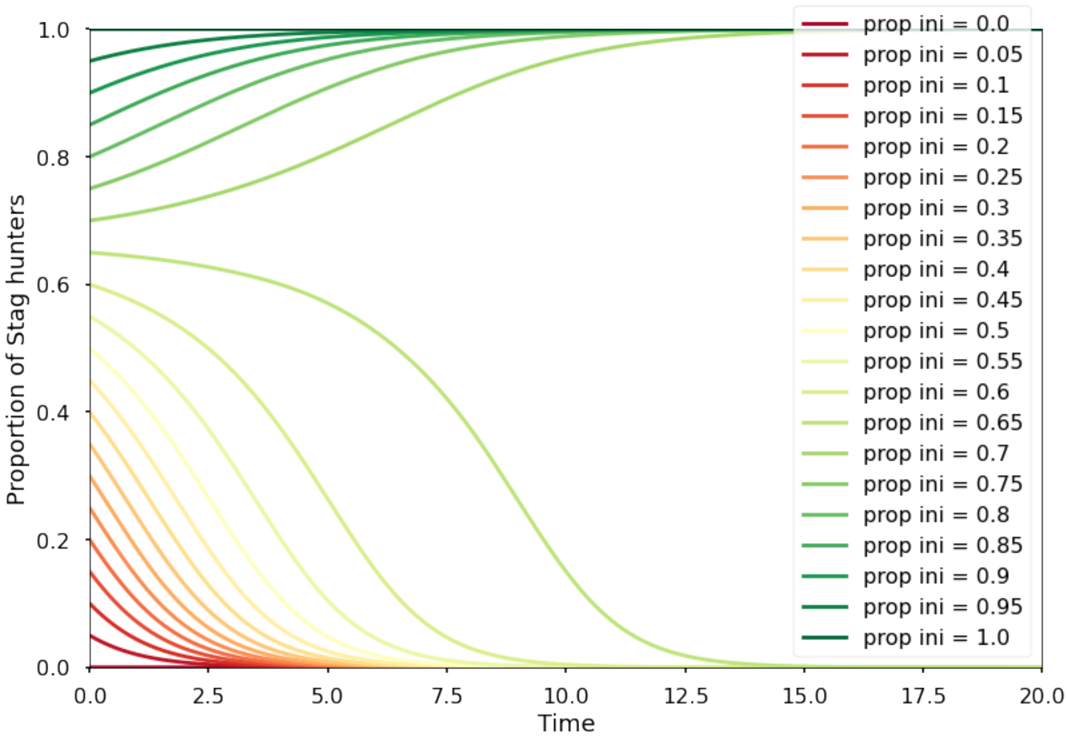


Figure 3.2: Evolution of the proportion of Stag hunter in time according to different initial proportions

the population, then it becomes convenient to hunt stages instead of hare if and only if $2x_S - 1 > \frac{x_S}{2}$ i.e. $x_S > 2/3$. Indeed, above this threshold, a population of Nasher who only maximize their individual benefits, simulated through the replicator equation will only cooperate if $x_H(0) > k_tip = 2/3$ as shown by the figure 4.7 and 3.2.

3.2.2. Application on a double dynamics game (Nashers vs Kantian optimizers)

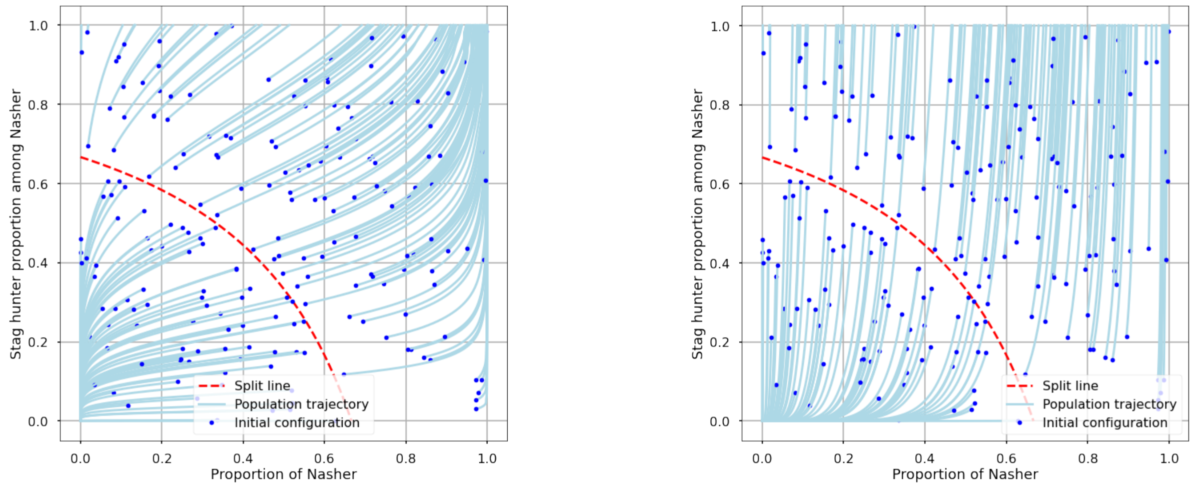
In "How we cooperate", a full chapter is dedicated to the resistance of a *kantian equilibrium* to the invasion of Nashers. However, this approach has reserved some criticism on how it represents the underlying interaction [13]. In this optic, the paper of J.-F. Laslier[10] addresses the challenge of studying the stability of the SKE in the stag and hare game.

More precisely, in Roemer'book Kantian optimizers are assumed to play the strategy (S) whereas the Nashers all play (H). However, this approach does not represent well the situation, as the Nashers would play what is best for them. For that purpose, J.-F. Laslier introduced a so-called double-dynamic game, where the first dynamic is the proportion of Nashian versus Kantian players, and the second is the frequency of playing Stag versus Hare of the mixed Nashe-optimizer's strategy. Both dynamics i.e. the switch from Nashian to Kantian optimizer and the evolution of Nashier's strategy are evolving according to a replicator dynamic.

The simulation is performed as such, with two groups $i = 1, 2$, the evolution of the size of the population i is given by $\dot{n_i} = f_i n_i$ where f_i is the fitness associated with group i. If we only consider the evolution of the proportion within the population, if we denote $n = n_1 + n_2$ and $x_i = n_i/n$, we have the following computation $\dot{x_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n_i}{n}\right) = \frac{\dot{n_i}n_j - \dot{n_j}n_i}{n^2} = \frac{n_i n_j f_i}{n^2} - \frac{n_i n_j f_j}{n^2} = x_i x_j (f_i - f_j)$. Finally, the evolution of the proportion $i = 1, 2$ is given by $\dot{x_i} = x_i(1 - x_i)\delta_{i-j}$ where δ_{i-j} is the fitness differential between the population i and j.

If we denote by ν the proportion of Kantian players, and by x the frequency of stag hunting among Nashers, the whole proportion of stag hunters within the population is therefore given by $y = \nu + (1-\nu)x$. The objective of the simulation would be to determine the basins of attraction of the system, i.e. to determine the partition of the initial values $(\nu, x) \in [0, 1]^2$ such that the system converges to different associated equilibriums.

As such, the difference in fitness between Kantian and Nasher is given by $\delta_{K-N} = [y - (1 - y)] - [x(2y - 1) + (1 - x)y/2] = (1 - x)(3y/2 - 1)$, and between stag vs hare hunters by $\delta_{S-H} = [y - (1 - y)] - [y/2] = 3y/2 - 1$. So finally we obtain that $\dot{\nu} = \nu(1 - \nu)\delta_{K-N}$, and that $\dot{x} = s.x(1 - x)\delta_{S-H}$. The coefficient s just introduced in the last equation is to modulate the pace at which Nashian changes their common mixed strategy, with respect to the one of changing from Nasher to Kantian player and vice versa. Therefore, as it is precise by J.-F. Laslier, this s coefficient is expected to be higher than 1, as switching its mixed strategy as a Nashe optimizer is done in a shorter time frame with respect to changing to become a Kantian optimizer. result of the simulation can be found in figure 3.3.



(a) Simulated with s =0.2

(b) Simulated with s = 5

Figure 3.3: Double dynamic Stag and Hare hunters simulations

In all cases, we observe that the partitions are separated by the parametric equation given by $x = \frac{2-3\nu}{3(1-\nu)}$, and this is independent of the value of s. Indeed, if we take consider δ_{K-N}, we observe by the calculus that $\delta_{K-N} > 0 \equiv x > \frac{2-3\nu}{3(1-\nu)}$, and we observe the exactly the same equivalence for δ_{S-H}. Therefore, we can intuitively explain that the separating border as such shape, as if it is above a snowball effect is launched, and will make growth stag hunter's proportion, that will become even more appealing to be played by the hunter community. If the initial configuration is such that $x < \frac{2-3\nu}{3(1-\nu)}$, then the same snowball effect will be initiated, but this time for the hare hunter's strategy.

We invite the interested reader to consult the original article [10], where the motivation and assumptions of the model are deeply exposed. Now we are going to extend the previous scheme in a coordination game composed of three strategies.

3.2.3. The cow, stag and hare game

We consider an environment in which in addition to hunting stags or hares, players have the possibility of farming and creating cow herds. The way of preceding does bring stability and well-being to the people choosing this way of getting food if they agree all together at respecting the property of the other and do not betray themselves. Indeed, stag and hare hunters can take big advantage of the situation, and loot the production of farmers. The situation can be represented for instance by the matrix 3.4, and we are going to take this game for the following simulation. We can observe that the game has three pure Nash equilibriums, which are $(C, C), (S, S)$ and (H, H) and is, therefore, a

		Player 2		
		Cow	Stag	Hare
Player 1	Cow	(10, 10)	(3, 6)	(0, 8)
	Stag	(6, 3)	(5, 5)	(1, 4)
	Hare	(8, 0)	(4, 1)	(2, 2)

Figure 3.4: The cow, stag and hare payoff matrix

coordination game.

We aim to apply this extended framework to the previous double dynamic, Kantian vs. Nashian optimizers, and to the mixed strategy of the Nashers. As the SKE is unique, Kantian optimizers are unconditionally playing cow (C), whereas we have to simulate the evolution of the Nashian mixed strategy which is composed of 3 possible choices.

Lemma 3.2. *In the case of 3 possible strategies, denoted by i, j, k, we have that*

$$\dot{x_i} = x_i[x_j(f_i - f_j) + x_k(f_i - f_k)]$$

Proof. We recall that to each strategy i is associated with a fitness f_i and $\dot{n_i} = n_i.f_i$, moreover in this case$n = n_i + n_j + n_k$. Therefore, $\dot{x_i} = \frac{\dot{n_i}n - \dot{n}n_i}{n^2} = \frac{\dot{n_i}(n_i+n_j+n_k)-(\dot{n_i}+\dot{n_j}+\dot{n_k})n_i}{n^2} = \frac{n_i.f_i(n_j+n_k)-n_i(n_jf_j+n_kf_k)}{n^2} = x_i[f_i(x_j + x_k) - (x_jf_j + x_kf_k)]$. So finally $\dot{x_i} = x_i[x_j(f_i - f_j) + x_k(f_i - f_k)]$. □

Therefore, in order to simulate the game, it is necessary to compute as before δ_{K-N}, but in addition, three other fitness differences denoted as δ_{C-S}, δ_{S-H}, and δ_{C-H}, which all depend of the current configuration. Similarly as before, we denote by $y = \nu + (1-\nu).c$ the proportion of the most cooperative option i.e. of cow players. For the mixed strategy, we need to introduce a double parameterization as we have three parameters with an equality constraint. We decide to introduce c, h which are the respective proportions of cow and hare players within the Nashian optimizers, which are constrained to be positive and such that $c + h \leq 1$. As a consequence, the proportion of hare hunters within the overall population is $\mu = (1-\nu).(1-c-h)$, and the proportion of hare hunters is given by $x = (1-\nu).s$. Of course, all quantities evolve and therefore depend on time, for simplicity of the notation, we don't right this dependency. Here the computation of the differential payoff according to the previously introduced parameters.

- $\delta_{K-N} = 10y + 3\mu - [c(10y + 3\mu) + h(8y + 4\mu + 2h) + (1 - h - c)(6h + 5\mu + x)]$
- $\delta_{C-S} = 10y + 3\mu - [6y + 5\mu + x] = 4y - 2\mu - x$

- $\delta_{H-S} = 8y + 4\mu + 2x - [6y + 5\mu + x] = 2y - \mu + x$
- $\delta_{C-H} = 10y + 3\mu - [8y + 4\mu + 2x] = 2y - \mu - 2x$

We then perform, out of this built framework, the associated simulations. They were performed in Python, and we used the library *scipy.integrate* in order to solve the ODE. The associated code is available in Annex A, as well as the code of the previous subsection that performed the simulation of the paper [10]. We performed the simulation on a regular grid in $[0, 1]^3$ with 11 points by dimension, which we can observe in Figure 3.5 and has been truncated in order to compile with the condition $c + h \leq 1$.

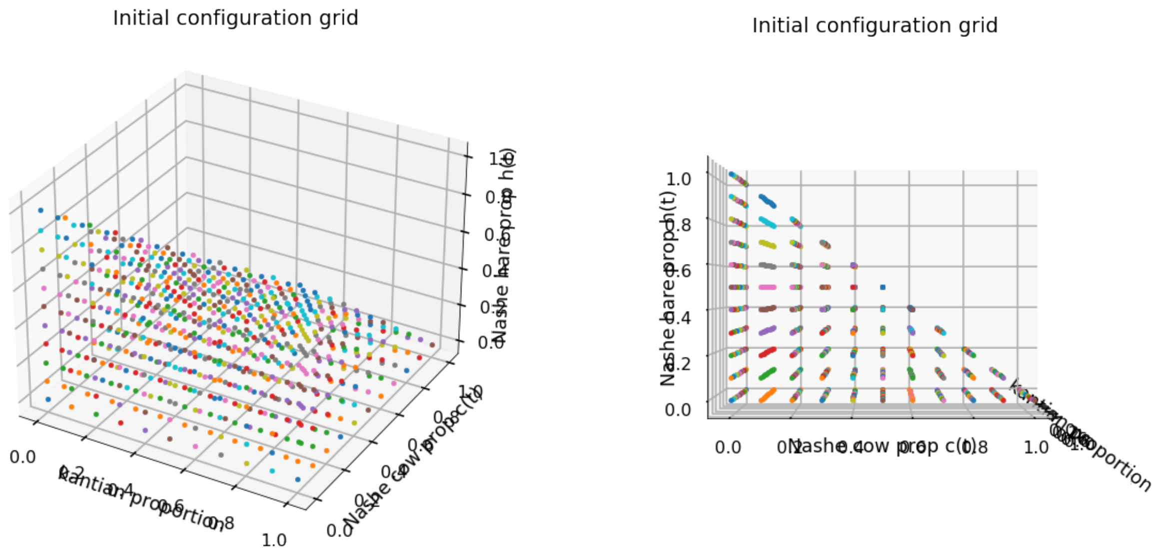


Figure 3.5: Initial configuration for the Simulation of the Cow, Stag, Hare game

Then, we performed the different simulations starting from each initial point, which are represented in Figure 3.6. As before, we also use a coefficient $s = 5$, which implies that Nashers are changing their mixed strategy 5 times quicker with respect to changing to become a Kantian dude.

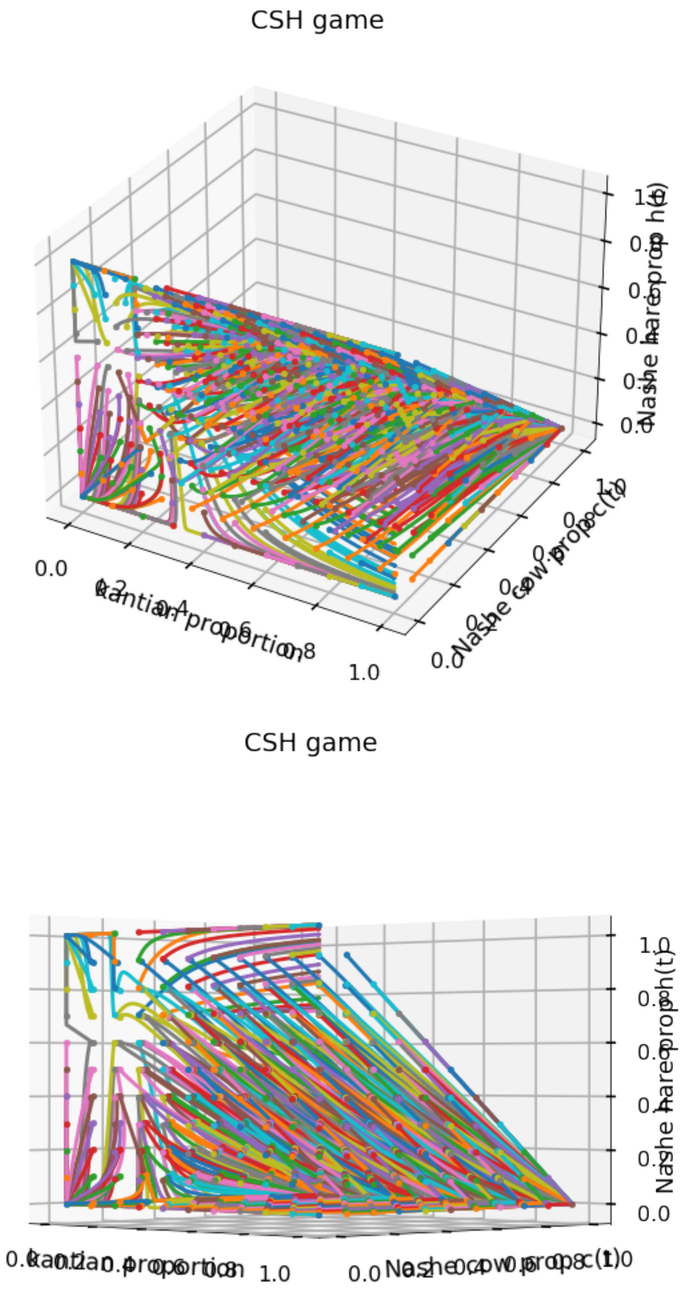


Figure 3.6: Simulation of the Cow, Stag, Hare game

Out of the simulations, we can observe three zones of attraction. First, the convergence to the *Hare hunting society*, which is located in the top left corner of the triangular prism, and corresponds to the ordinates $(\nu = 0, h = 1, c = 0)$.

The second one corresponds to the case where society evolves into a *Stag hunting society*, where the point of attraction is located in the bottom left corner of the Figure 3.6, and has associated coordinates ($\nu = 0, h = 0, c = 0$).

Finally, the last zone of attraction corresponds to the case where the society becomes a *farming society*. In this case, it is not only represented by a point but a much larger zone due to our parametrization. The zone of attraction is composed of all the points with the proportion of Kantian optimizer ν equal to one, as well as all the points that entail the fact that all Nash optimizers are playing the pure strategy cow i.e. all the points with c equal to 1.

We see clearly in Figure 3.6 the emergence of the 3 basins of attraction. As the configuration space is of dimension 3, the basins of attraction will be separated by parametric surfaces. It would be an important result to determine the parametric equation of those surfaces, but it is a rather more complex approach than in the classical stag and hare game. A starting point to determine those equations could be, as we did in the case of its little brother game, to study the sign of the different δ_{i-j} according to the configuration. With our initial points setting, we obtain that 90.7% of the area corresponds to the basin of attraction of the *farming society*, 6.7% for the *Stag hunting society* and 2.2% to the *Hare hunting society*, the remaining $1,4\%$ are not attributed probably to numerical issues.

4 | Kantian Evolutionary Game theory: the Replimorator equation

In the annals of game theory, a contest emerged in the early 1980s that would forever alter our understanding of cooperation, strategy, and the very essence of human behavior. It was the Iterated Prisoner's Dilemma Tournament, orchestrated by the visionary Robert Axelrod. This tournament served as the crucible in which the simple yet powerful *tit-for-tat* strategy came to prominence, revealing a remarkable facet of human behavior—the capacity for cooperation even in the face of self-interest.

The brilliance of Axelrod's tournament lay in its simplicity and elegance. He invited researchers from various fields to submit computer programs representing strategies for playing the Iterated Prisoner's Dilemma. Unlike the single-round dilemma, the Iterated version allowed participants to engage in repeated interactions over numerous rounds. This setup introduced a crucial element: the memory of past interactions.

Among the strategies submitted, tit-for-tat stood out as both simple and effective. This strategy, proposed by Anatol Rapoport, was straightforward in its approach. It began by cooperating in the first round, setting a cooperative tone. In subsequent rounds, it mirrored the opponent's previous move. If the opponent had cooperated in the last round, tit-for-tat cooperated in return; if the opponent had defected, tit-for-tat defected as well. What surprised everyone was that this seemingly simplistic strategy outperformed its more complex counterparts. In a tournament filled with intricate, sophisticated algorithms, tit-for-tat emerged as the victor.

Axelrod's book, "The Evolution of Cooperation,"[3] which followed the tournament, emphasized the profound implications of tit-for-tat's success. It demonstrated that in repeated interactions, simple, reciprocal, and forgiving strategies could lead to cooperative outcomes, even when faced with initial self-interested defections. This finding challenged the prevailing assumption that self-interest should always lead to defection.

Out of this competition, it appears that even more developed strategies emerged, taking into account reputation and forgiveness. As such, we would like to study how a population of players acting morally would evolve, using assumptions and a similar setting of the classical evolutionary game theory.

4.1. The Moralisator equation: the Kantian optimization in EGT

In this section, we try to describe how a population of players that optimizes in a Kantian way would evolve in time. More precisely, we are looking for such optimization in the case of a symmetric game, i.e. where the matrix $(\mathbf{A}, \mathbf{B})$ of the game are such that $\mathbf{B} = \mathbf{A}'$. We are looking for a solution in this framework because the players of the game should feel that the other ones are facing a similar situation, and are able to optimize as Kantian players. As it is well underlined in Roemer's book[6], players should feel the Benjamin Franklin motto "We should hang all together, or hang all of us alone".

Therefore, the game represented by the matrix $\mathbf{A}$ is symmetric and possesses n possible pure strategies $\{e_i\}_{i\in[|1,n|]}$. As in the replicator equation framework, the state of the game is represented by the vector $\underline{x} = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}' \in \Sigma_n$, where each component x_i is the proportion of the population playing the strategy i. It naturally follows that the vector $\underline{x}$ has each of its components positive and that sum up to 1. As such, the ODE (Ordinary Differential Equation) that follows $\underline{x}$ must fulfill the following constraints.

1. $\sum_{i=1}^{n} x_i = 0$ as the sum up of the component must remain 1 through time.
2. $\forall i \in [|1, n|]: \quad x_i = 0 \implies \dot{x}_i = 0$

As it is discussed in Chapter 3, we recall that the replicator equation defines as $\forall i \in [|1, n|], \dot{x}_i = x_i[\underline{\delta_i}'\mathbf{A}\underline{x} - \underline{x}'\mathbf{A}\underline{x}]$, the previous conditions are fulfilled. Let's look at different possibilities to capture how a moralist population will evolve

4.1.1. Reflection around potential extensions

We have begun our journey by first of all getting inspiration from the different lectures on cooperative game theory, mainly the concepts of SKE of Roemer and as well the concept of *Homo moralis* [2] of Alger and Weibull. In those approaches, a fully cooperative or moral player will consider common, or more precisely group strategies.

Therefore, the positive growth incentive associated with strategy e_i, which is equal to

$\underline{\delta_i}'\mathbf{A}\underline{x}$ in the replicator equation should be replaced in an intuitive way by $\underline{\delta_i}'\mathbf{A}\underline{\delta_i}$. Indeed, the first equation represents the average output of a player that plays always strategy e_i in a population composed of the strategy distribution $\underline{x}$, which perfectly fits the positive incentive of self-interest behavior. In the second case, $\underline{\delta_i}'\mathbf{A}\underline{\delta_i} = \mathbf{A}_{ii}$ represents what we commonly get as an individual if we all play strategy e_i, which correspond exactly the positive incentive associated to a player that acts in a moral way.

However, the previous positive incentive associated with the strategy e_i must be balanced by a negative incentive as well. Indeed, we recall that the ODE must fulfill the following constraint $\forall \underline{x} \in \Sigma_n, \sum_{i=1}^{n} x_i = 0$. Therefore if some strategies grow, others should decrease on the same scale. At first, the idea selected in order to make this negative incentive was to subtract the arithmetic mean of the pure strategy. Therefore, if we denote by $\mathbf{D} = diag(\mathbf{A})$ then in this first approach, the guess moralis equation would be proportional to $\dot{x}_i = x_i[\underline{\delta_i}'\mathbf{D}\underline{\delta_i} - \text{Tr}(\mathbf{D})/n]$.

In the classical framework, such as in the replicator equation, $\dot{x}_i$ is proportional to x_i. Whether we add such a term in the equation, then the required constraints 4.1 are not fulfilled. In particular, it appeared on the simulations that some proportion x_i overshot 1. As such, an idea was for a 2-strategy game, i.e $n = 2$, to multiply the equation by the other proportion in order to make the derivative vanish when $x_i = 1$. Extending this reasoning to a larger game, we proposed the following equation, which quite nicely fulfills the required statements:

Lemma 4.1. *The following equation fulfill conditions 4.1:*

$$\forall i \in [|1, n|], \dot{x}_i(\underline{x}) = \prod_{j=1}^{n}(x_j)[\underline{\delta_i}'\mathbf{D}\underline{\delta_i} - \text{Tr}(\mathbf{D})/n] \tag{4.1}$$

Proof. First, we prove that the derivatives component-wisely sum up to 0 for all $\underline{x}$. Given any state $\underline{x}$, we have $\sum_{i=1}^{n} \dot{x}_i(\underline{x}) = \prod_{j=1}^{n}(x_j) \sum_{i=1}^{n}[\underline{\delta_i}'\mathbf{D}\underline{\delta_i} - \text{Tr}(\mathbf{D})/n]$ as the full product is independent of the i-th strategy. Moreover, recalling that $\underline{\delta_i}'\mathbf{D}\underline{\delta_i}$ is the i-th element of the diagonal of $\mathbf{D}$, $\sum_{i=1}^{n} \underline{\delta_i}'\mathbf{D}\underline{\delta_i} = \text{Tr}(\mathbf{D})$, hence the result.

The second point to prove, concerning the cancellation of the i-th derivative when the i-th proportion is equal to 1 or 0 is even more straightforward. Indeed, if one of the proportions reaches 0 or 1, it vanishes the coefficient $\prod_{j=1}^{n}(x_j)$, which automatically cancels all the derivatives. □

Even though this equation fulfills the mathematics requirements, as well as integrating the "moralis spirit", some issues remain. Indeed, integrating the factor $\prod_{j=1}^{n}(x_j)$ enable stabilizing the equation when one of the proportion becomes close to 1 or 0, but have very little interpretability. Even more problematic, when one of the proportions becomes 0, the system gets locked and doesn't evolve anymore.

Moreover, the factor containing the "moralis spirit" compares the i-th common strategy with all possible common strategies means, but doesn't depend on the current situation of the system. This property seems to be rather unrealistic in practice, as the preference of a player for an option i should depend on the current situation $\underline{x}$. Indeed, we human beings have the tendency to compare ourselves to the other. Unlikely, in this first proposition the current state only plays the role of a common pace evolution factor among the different strategies, with the previously mentioned degenerated case where one of the proportions vanished.

4.1.2. The Moralisator equation

Out of the weakness of the equation 4.1, which is the first attempt to characterize the moral comportment, we built another one that nicely fits the equation requirements 4.1, that we named *moralisator equation*.

Definition 13 (The moralisator equation)**.**

$$\forall i \in [|1,n|], \dot{x}_i(\underline{x}) = x_i[\underline{\delta_i}'\mathbf{D}\underline{\delta_i} - \underline{x}'\mathbf{D}\underline{1}] \tag{4.2}$$

As in the case of the replicator equation, by associating the vector fields $\phi : \mathbb{R}^n \to \mathbb{R}^n$ that is componently defined by $\forall i \in [|1,n|], x_i[\underline{\delta_i}'\mathbf{D}\underline{\delta_i} - \underline{x}'\mathbf{D}\underline{1}]$, we aim to apply the Picard-Lindelöf theorem as the vector field previously defined is polynomial, the local existence and uniqueness of the solution is then well established.

Remark 4. *We have chosen to write the Moralisator equation in a matrix form, which shortens the notation but might overcomplicate what it represents. If we expand the equation, we obtain* $\dot{x}_i(\underline{x}) = x_i[\mathbf{D}_{ii} - \sum_{j=1}^{n} x_j \mathbf{D}_{jj}] = x_i[\mathbf{A}_{ii} - \sum_{j=1}^{n} x_j \mathbf{A}_{jj}]$*. In particular, we observe that the term* $\underline{x}'\mathbf{D}\underline{1}$ *represents the weighted mean of the common strategies.*

Theorem 4.2. *The moralisator equation does fulfill conditions 4.1.*

Proof. As in the previous equation, we prove at first that the derivatives componentwisely sum up to 0 for all $\underline{x}$. Given any state $\underline{x}$, we have $\sum_{i=1}^{n} \dot{x}_i(\underline{x}) = \sum_{i=1}^{n} x_i[\mathbf{A}_{ii} - \sum_{j=1}^{n} x_j \mathbf{A}_{jj}] = \sum_{i=1}^{n} x_i \mathbf{A}_{ii} - \sum_{i=1}^{n}\sum_{j=1}^{n} x_j x_i \mathbf{A}_{jj}$. Then it follows that both terms are equal, indeed $\sum_{i=1}^{n}\sum_{j=1}^{n} x_j x_i \mathbf{A}_{jj} = \sum_{j=1}^{n}\sum_{i=1}^{n} x_j x_i \mathbf{A}_{jj} = \sum_{j=1}^{n} x_j \mathbf{A}_{jj} \sum_{i=1}^{n} x_i = \sum_{j=1}^{n} x_j \mathbf{A}_{jj}$ as the x_i sum up to 1, which implies that the sum of the derivatives does component-wisely sum up to 0.

The point that remains to prove, concerns the cancellation of the i-th derivative when the i-th proportion is equal to 1 or 0. When the i-th proportion is 0, it is straightforward that $\dot{x}_i(\underline{x}) = 0$ as x_i is in factor. In the other case, when $x_i = 1$ then the weight mean $\underline{x}'\mathbf{D}\underline{1} = D_{ii}$ which cancels the other factor of the derivative, hence the result. □

The interpretation of the equation can be done as follows. As before, the rise of a proportion i is proportional to the difference between the i-th common strategy and the mean of all possible common, but not the arithmetic mean this time. Indeed, the considered mean in this case is the weighted mean, where the weight associated with each common strategy is equal to the proportion of players currently playing this strategy. This new way of comparing the potential strategies, from the point of view of a player, makes more sense. In fact, in this framework, the comparison between strategies does depend on the current state $\underline{x}$ in that case. It also makes sense to weigh the mean with the current proportion of the strategy played, as players would compare themselves to the others, and take as a moral reference the common output of the mixed strategy profile played in a coordinated way.

Moreover, the variation of a given strategy is also proportional to the current proportion of players in this strategy, as in the replicator equation. This term was indeed necessary in order to fulfill the required equation properties, but it could be as well nicely interpreted. As it is mentioned in [15], this term was originally placed in order to represent when the EGT studied reproduction in terms of populations, of different species of animal for instance. When we deal with strategy choices, the more this strategy is played, the quicker it would evolve, which represents the mimicking comportment within the population, which implies sadly a limit in our modelization. Indeed, if at the beginning, or at some point, a strategy i has vanished then it cannot appear again even if it becomes later on advantageous to play it again. Therewith, the human capacity to undertake and improve is neglected, which is somehow understandable as those characteristics are by nature stochastic and we are working in a deterministic framework. A stochastic approach might be a further step in order to integrate this aspect, through a Stochastic Differential Equation (SDE). For the moment, as the integration of that undertaken behavior is out of the scope of our studies, we have to be very attentive to the initial state of our system and not let proportions get too close to 0 as we would get unrealistic simulation behaviors.

4.1.3. A few properties of the Moralisator equation

We know that in the case of the replicator equation, the mean reward of the population through time, which can be computed as $\underline{x}'(t)\mathbf{A}\underline{x}(t)$ is an increasing quantity. In the case of the moralisator equation, we have a similar behavior for a similar quantity.

Theorem 4.3 (Increase of $M_W(\underline{x}(t))$ under the moralisator equation)**.** *The quantity $M_W(\underline{x}(t)) = \sum_{j=1}^{n} x_j(t)A_{jj}$ is an increasing quantity through time when $\underline{x}$ follow a moralisator dynamic.*

Proof. Let $t_1 > t_0 \geq 0$, let's prove that $\sum_{j=1}^{n} x_j(t_1)A_{jj} \geq \sum_{j=1}^{n} x_j(t_0)A_{jj}$.

$\sum_{j=1}^{n} x_j(t_1)A_{jj} - \sum_{j=1}^{n} x_j(t_0)A_{jj} = \sum_{j=1}^{n} A_{jj}[x_j(t_1) - x_j(t_0)] = \sum_{j=1}^{n} A_{jj} \int_{t_0}^{t_1} \dot{x_j}(u)\, \mathrm{d}u$ according to the fundamental theorem of integration. As $\underline{x}$ follows the moralisator equation, we have that $\forall j \in [|1, n|], \forall u \geq 0, \dot{x_j}(u) = x_j(u)[A_{jj} - \sum_{j=i}^{n} x_j(u)A_{ii}]$.

Therefore, we finally have the equality $\sum_{j=1}^{n} x_j(t_1)A_{jj} - \sum_{j=1}^{n} x_j(t_0)A_{jj} = \int_{t_0}^{t_1} \sum_{j=1}^{n} A_{jj}x_j(u)[A_{jj} - \sum_{j=i}^{n} x_j(u)A_{ii}]$. Then by the positivity of the integral, it is enough to conclude the proof to show that $\forall u \geq 0, \sum_{j=1}^{n} A_{jj}x_j(u)[A_{jj} - \sum_{j=i}^{n} x_j(u)A_{ii}] = \sum_{j=1}^{n} x_j(u)A_{jj}^2 - \sum_{j=1}^{n} \sum_{i=1}^{n} x_i(u)x_j(u)A_{ii}A_{jj} \geq 0$.

This is the case as the last equality corresponds to the variance of a specific random variable. Indeed, for a fix $u \geq 0$, considering the random variable $R(u)$ that to all A_{jj} is associated the probability $x_j(u)$ to be played, then $\sum_{j=1}^{n} x_j(u)A_{jj}^2 - \sum_{j=1}^{n} \sum_{i=1}^{n} x_i(u)x_j(u)A_{ii}A_{jj} = \mathbb{E}[R(u)^2] - \mathbb{E}[R(u)]^2 = Var[R(u)] \geq 0$. □

Moreover, studying the long-term convergence of the population under the dynamic of such an equation is a critical question. As the game is symmetric, there exists at least one SKE which corresponds to the strategy achieving the maximum in the diagonal of the matrix $\mathbf{A}$. Without loss of generality, we reorder the strategies in order to get a decreasing diagonal, i.e. $\mathbf{A}_{11} \geq ... \geq \mathbf{A}_{nn}$. In the case of a unique SKE, if the initial proportion associated with this strategy is non-null, then the population will converge to the case where they fully play this strategy.

It might also happen, in some degenerate cases that there are several strategies achieving the maximum on the diagonal, so we are in the case of $2 \leq r \leq n$, $\mathbf{A}_{11} = ... = \mathbf{A}_{rr}$. In that case, the equation would converge to a mixed strategy, with support on the strategies that enable achieving maximum payoff on the diagonal. I call this situation degenerate because in real-world games, in opposition to artificial ones, two outputs cannot be equal and they at least differ slightly. Moreover, a population acting morally and ending up playing different moral strategies is falling, as they are playing different moral actions and will get payoffs out of the diagonal of $\mathbf{A}$. This might be due to a limit of the model, that simulates only one population playing with itself. However, it is necessary to explain that this situation is degenerate, as in a real-world scenario, using different arguments of habits or customs, the population will focalize on one of the SKE.

Theorem 4.4. *Let* $\mathbf{A}$ *be a symmetric game where we have, without loss of generality reordered the strategies in order to get a decreased diagonal. We also denote by* $r \in [|1, n|]$ *the last coefficient such that* $\mathbf{A}_{11} = \mathbf{A}_{rr}$*.*

Then we have that $\sum_{i=1}^{r} x_i(0) > 0 \implies \sum_{i=1}^{r} x_i(t) \xrightarrow{t\to\infty} 1$.

Proof. If $r = n$, then there is nothing to prove.

If $r < n$, let suppose that $\sum_{i=1}^{r} x_i(0) > 0$.We define $x_{1-r} \equiv \sum_{i=1}^{r} x_i$, therefore $x_{1-r}(0) > 0$.

The first step of the demonstration is to show that $\exists T_1 > 0 : \sum_{j=1}^{n} x_j(T_1)A_{jj} > A_{r+1,r+1}$. In order to do so, let's suppose the opposite that is $\forall t > 0 : \sum_{j=1}^{n} x_j(t)A_{jj} \leq A_{r+1,r+1}$. Then we deduce $\forall t > 0 : \dot{x}_{1-r}(t) = x_{1-r}(A_{11} - \sum_{j=1}^{n} x_j(t)A_{jj}) \geq x_{1-r}(A_{11} - A_{r+1,r+1})$. According to the inequality, we have that $x_{1-r}(t) \geq x_{1-r}(0)\exp[(A_{11} - A_{r+1,r+1})t]$, which is an obvious contradiction as it will get quickly greater than 1.

Therefore, $\exists T_1 > 0 : \sum_{j=1}^{n} x_j(T_1)A_{jj} > A_{r+1,r+1}$. Therefore, using theorem 4.3 we deduce that the inequality also stands for all the values greater than T_1. We will conclude the proof by showing that $1 - x_{1-r} \xrightarrow{t\to\infty} 0$ which implies that $x_{1-r} \xrightarrow{t\to\infty} 1$. Indeed, $\forall t \geq T_1$: $1 - \dot{x}_{1-r}(t) = x_{r+1}(t)[A_{r+1,r+1} - \sum_{j=1}^{n} x_j(T_1)A_{jj}] + ... + x_n(t)[A_{nn} - \sum_{j=1}^{n} x_j(T_1)A_{jj}] \leq (1 - x_{1-r})(t)[A_{r+1,r+1} - \sum_{j=1}^{n} x_j(T_1)A_{jj}]$. The last inequality that implies $\forall t \geq T_1$: $0 \leq (1 - x_{1-r})(t) \leq (1 - x_{1-r})(T_1)\exp[(A_{r+1,r+1} - \sum_{j=1}^{n} x_j(T_1)A_{jj})t] \xrightarrow{t\to\infty} 0$. Hence the result. □

4.2. The k-Replimorator equation: a compromise between Nasher and Kantian optimization

It is fairly known that in many aspects of life, "things" are not simply black or white but slightly more complex, like a scale of grey. Human interaction is part of those "things", whether it be in the economy, politics, geopolitics, or even at the base scale in business, family, or friendships. In all those fields, there are countless situations where an individual extra-benefit would be in opposition to a solution more suitable for the other parties. In those situations, the outcome in general never ends up in one of both extremes, but in a complex compromise that has been driven by a mix of selfish and moral forces.

It also happens, in some really close and specific interactions that players are driven by altruism, which means they only aim to maximize the payoff of the other players. Such an example can be a mother towards is children, who only cares about the well-being of her progeny. Overall, altruism does exist in very specific interactions, but it is completely negligible in the majority of them. We will also argue, that, in other types of interactions, the opposite driver known as malignancy can rise out of rivalry or hate.

Overall, in the symmetric games that are under our studies, players can deeply feel the reciprocity and have the sentiment of being part of the same boat. In this framework, the moral force should be as well considered in addition to the self-interest intrinsic driver. In order to set what extent a player would be influenced by his morality, we denote by $k \in [0,1]$ the degree of morality. Under those assumptions, proportions of strategies played in the game are then driven by the *k-Replimorator* equation, defined as the convex combination between the Replicator and Moralisator equation.

$$\forall i \in [|1,n|], \dot{x}_i(\underline{x}) = x_i \left\{ k[\underline{\delta_i}'\mathbf{D}\underline{\delta_i} - \underline{x}'\mathbf{D}\underline{1}] + (1-k)[\underline{\delta_i}'\mathbf{A}\underline{x} - \underline{x}'\mathbf{A}\underline{x}] \right\} \tag{4.3}$$

As the Replicator and the Moralisator equation did fulfill the basic requirement 4.1 of such equation, it directly follows that the convex combination of any kind will fulfill them as well. In fact, for any positive linear combinations, the requirements would have to stand, which might open the way to the integration of other behavior-driving forces. The new equation would enable us to study a wide range of population behavior according to its degree of morality. The degree of morality is kept constant in the equation to keep things as simple as possible, even though making it time and state path depend would be possible in order to make it more refined. We are now going to simulate numerically the k-Replimorator equation in different classical games.

The different simulations are performed numerically with Python, using the library Scipy in order to simulate the ODEs. All codes concerning those simulations are available in Appendix B.

4.2.1. The prisoner dilemma

The first game we would like to study is the prisoner dilemma. Indeed, despite the fact that it is the most famous game of the discipline, some very interesting behaviors toward the Replimorator equation can be expected. In fact, this game has an SKE equilibrium as well as a Nash equilibrium that corresponds to the two possible common strategies. The

Kantian equilibrium corresponds to both convicts that collaborate with each other and deny (D), whereas the Nash equilibrium corresponds to the situation where they both confess (C).

According to the theory, if the population converges, it would converge to the Nash equilibrium if it evolves according to the replicator equation, and to the SKE in the case of the Moralisator equation. Therefore, in the case of the k-replimorator equation, we can expect a different comportment according to the morality coefficient k, with an increasing incentive to collaborate between prisoners as k increases.

According to the dedicated Wikipedia page [17], the general form of the matrix is given by 4.1a, at little details that in the web page the *Cooperate* strategy correspond to our *Deny* one strategy. Indeed, it can be easily understood that Denying the crime is associated with the players' cooperative strategy, where a confession with justice corresponds to a defection toward the other player. As the game is symmetric, it can be represented by 4 coefficients of the matrix **A**. According to the Wikipedia page, coefficients can be denoted by R,P,T and S that are associated with Reward for cooperation, Punishment, Temptation to betray, and the Sucker payoff. Morever, the inequality $T > R > P > S$ must hold in order get the properties of the prisoner dilemma.

We then perform simulations and several analyses over two game settings. The first one, represented in the matrix 4.1b, corresponds to the game as the introduction. The second one 4.1c, represents the previous one where we have increased the incentives to defect by increasing T and lower S.

		Convict 2	
		Cooperate	*Defect*
Convict 1	*Cooperate*	(R, R)	(S, T)
	Defect	(T, S)	(P, P)

(a) The general **A** matrix for the prisoner dilemma

		P_2	
		Confess	*Deny*
P_1	*Confess*	-10	0
	Deny	-15	-1

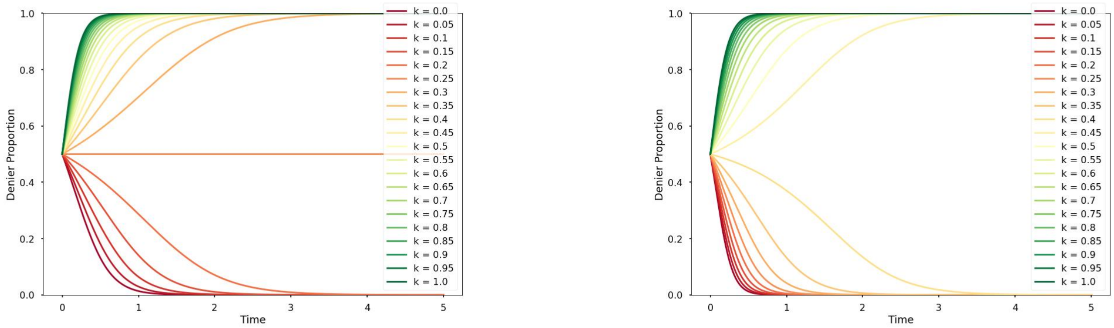
(b) **A** matrix of Chapter one

		P_2	
		Confess	*Deny*
P_1	*Confess*	-10	2
	Deny	-20	-1

(c) **A** matrix modified prisoner

Figure 4.1: **A** matrix int the prisoners dilemma setting

We have therefore performed the simulations in both cases with an even proportion of cooperator and defector, the evolution through time and according to the coefficient of morality can be seen in Figure 4.2. We can observe, in both cases, that without morality the population evolves quickly to full defectors. When the morality coefficient increase, it still goes to this equilibrium but at a slower pace, until a tipping point of morality k_{tip}, where above the opposite behavior is observed and the population converges to a fully cooperative one.

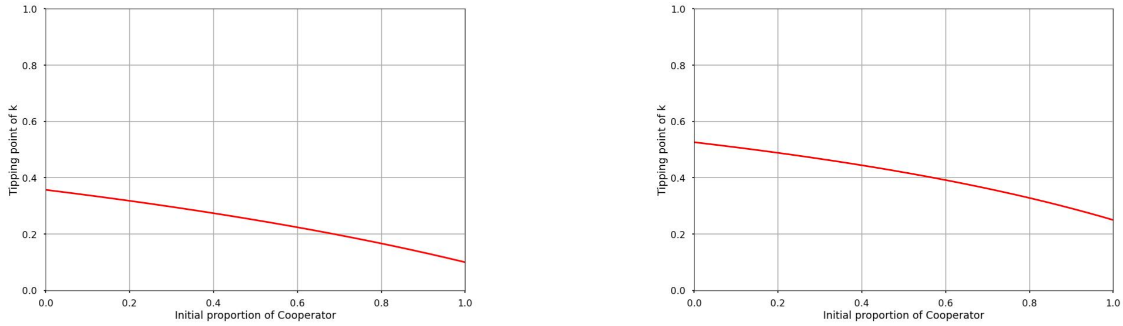


(a) Simulation with the basic **A** matrix

(b) Simulation with the modified **A** matrix

Figure 4.2: Both simulation in prisoners dilemma different setting

We can also observe that in the second game, the value of k_{tip} is higher than in the first case, which was an expected result as we made the second game more appealing for defectors, then the level of morality necessary in order to cooperate must be higher. The value of k_{tip} should also depend on the initial proportion of the different strategies within the population. Indeed, if the proportion of defectors is high, the level of morality in order to achieve cooperation is expected to be higher with respect to the case where the number of defectors is low. In order to investigate this intuition, we have computed k_{tip} with respect to the initial proportion of cooperators within the population, and we do observe the expected behavior.



(a) k_{tip} w.r.t the initial proportion of cooperator in the basic $\mathbf{A}$ matrix

(b) k_{tip} w.r.t the initial proportion of cooperator in the modified $\mathbf{A}$ matrix

Figure 4.3: k_{tip} w.r.t the initial proportion of cooperator in both prisoners dilemma setting

4.2.2. The Hawk and Dove

The Hawk and Dove is another famous and well-studied game. It can represent many different situations, for example, two drivers crossing a narrow road or even nuclear deterrence, even if the usual storytelling is to study two possible behaviors of a population of birds. One of them is aggressive and denoted by Hawk, whereas the other one is docile and characterized by Dove [16].

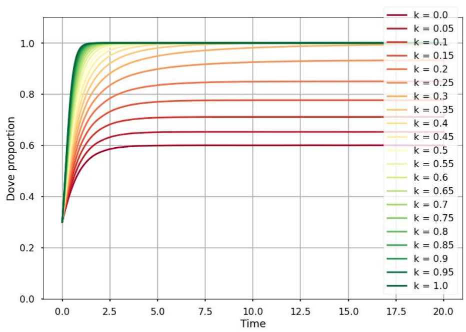
Within this population, it often happens that birds find a rival resource that would bring them the payoff V. If they are both doves, they will share the resource and get the payoff $V/2$, but a hawk against a dove would take all the resources. Finally, if two hawks meet in those circumstances, they will escalate the conflict and fight which would cost $C > V$ to the loser. The average payoff, in that final case, is then equal to $(V - C)/2 < 0$. The game is summarized in Figure 4.4 and the $\mathbf{A}$ is deduced from it by taking the payoff matrix of the first bird.

		Bird 2: *Hawk*	Bird 2: *Dove*
Bird 1	*Hawk*	$(\frac{V-C}{2}, \frac{V-C}{2})$	$(V, 0)$
	Dove	$(0, V)$	$(\frac{V}{2}, \frac{V}{2})$

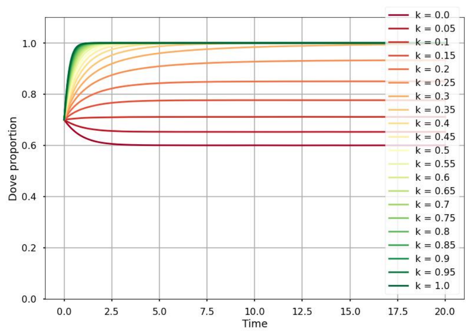
Figure 4.4: The general hawk and dove game

Moreover, it can be observed that neither the Hawk nor the Dove strategy is Nash equilibrium. Indeed, a full population of hawks is always fighting and taking negative payoffs on average, whereas in this setting a dove would take a zero payoff which is an evolutionary advantage. In the opposite case, if all birds are doves, which corresponds to the Kantian equilibrium, then an invasive hawk will always take the full resources instead of half of it for the doves. Therefore, even if the population composed exclusively of Dove is most optimal in terms of common payoff, as no resources are lost during conflicts, this equilibrium is not reached in a classical evolutionary setting where the Nash equilibrium corresponds to a proportion of hawk of V/C.

We have performed the following simulation with the values $V = 4$ and $C = 10$, and started them for two different proportion settings of $(30\%, 70\%)$ and $(70\%, 30\%)$, we can observe them in Figure 4.5. We observe, in this game, a convergence of the strategy proportions after a short time for each value of the morality coefficient k. More precisely, the game converges to the Nash equilibrium when $k = 0$, which corresponds to a proportion of $\frac{C-V}{C}$ of doves, and as long as k increases the proportion of dove corresponding to the equilibrium increase as well. We then observe a value critic of k, between 0.25 and 0.3 for the values of V and C previously selected, where the equilibrium is exclusively composed of Doves for values of k higher.



(a) Starting proportion of Dove $= 0.3$



(b) Starting proportion of Dove $= 0.7$

Figure 4.5: Hawk and Dove simulations throw time and according to k

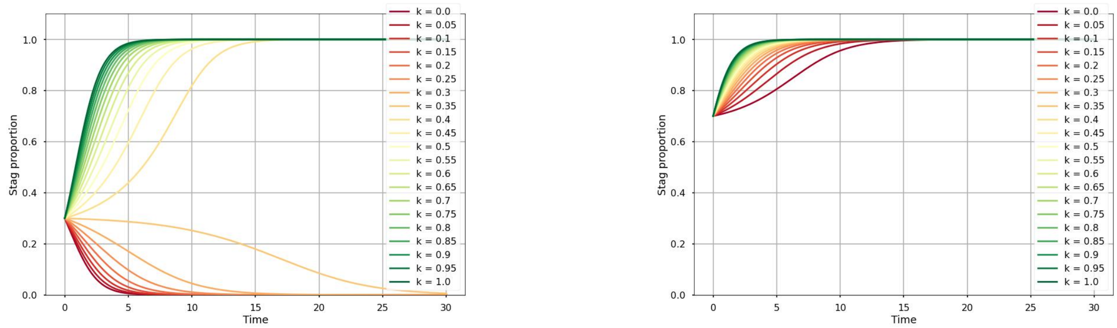
In particular, we empirically observe that the equilibrium reached is independent of the initial proportion of birds, except for the degenerated case where only one kind of bird is initially present.

As we previously discussed, the higher the proportion of Dove, the higher the average payoff within the population, as fewer resources are lost in the fights. Indeed, the average payoff of the population is given by $\underline{x}'\mathbf{A}\underline{x} = x_H^2(\frac{V-C}{2} + x_H x_D V + x_D^2$. As $x_D = 1 = x_H$, we obtain that the derivative of the population average payoff is given by $-Cx_H < 0$ which implies that it is a decreasing function of x_1.

As an increasing moral coefficient k induces a higher proportion of doves at the equilibrium, we can conclude that an increase in morality would imply an increase in the common well-being of the population at the equilibrium. This is a common feature with the previous game studied, except that in that game, the behavior is achieved in a continuous way until the k_{tip} is reached. Whereas, in the prisoner dilemma the switch of behavior binary, in the sense that it switches from one extreme equilibrium to the other whether k is under or above k_{tip}.

4.2.3. The stag and hare hunters

The stag and hare hunters game is introduced in section 3.2.1, and the payoff matrix associated with the game is given by Figure 3.1.



(a) Initial proportion of stag hunter $= 0.3$

(b) Initial proportion of stag hunter $= 0.7$

Figure 4.6: Stag and Hare simulations throw time and according to k

In Figure 4.6, we have reported the simulation of the game for different initial conditions and different coefficients of morality k. We observe that in the long run, the game ends up either in a *stag hunter* or in an *hare hunter* society.

We also observe that whatever the initial proportion in the game, there exists a morality coefficient k_{tip} that enables the game to converge to the cooperative equilibrium where players are hunting the stag. This is a similar feature that we have observed in the prisoner dilemma, except that in the case the k_{tip} is rather different as we can observe in Figure 4.7. Indeed, there exists a threshold of cooperator proportion where above, even a morality coefficient $k = 0$ makes the game converge to a fully cooperating population, which is not the case in the prisoner dilemma.

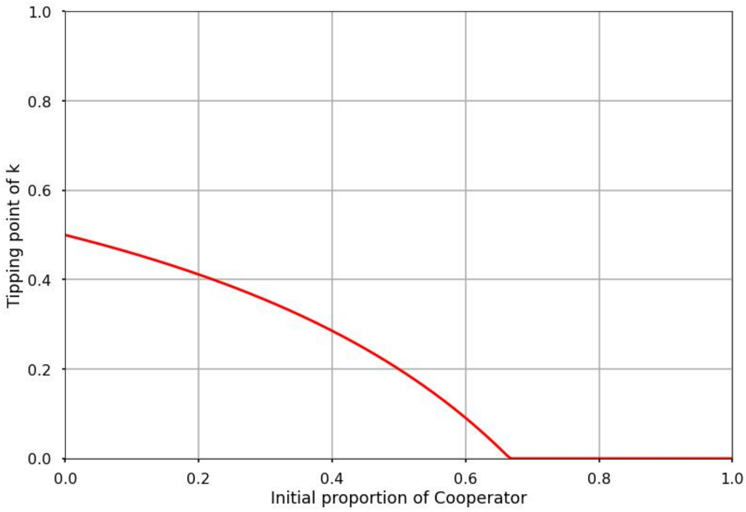
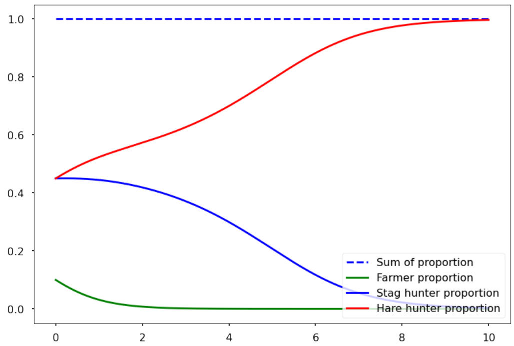


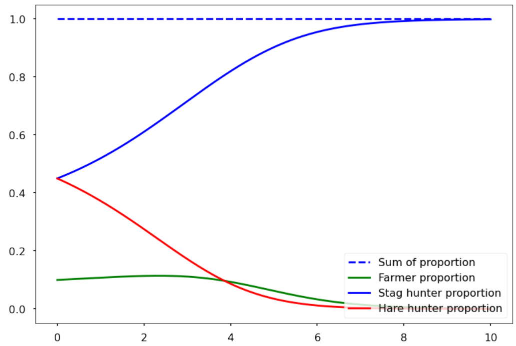
Figure 4.7: Value of k_{tip} in the Stag and Hare game according to the initial proportion of Hare hunters

4.2.4. The cow, stag and hare game

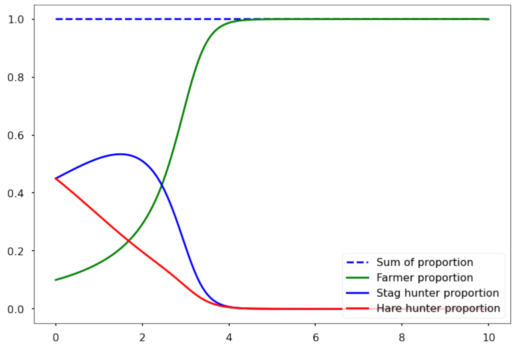
The cow, stag and hare game is previously introduced in section 3.2.3 and has for payoff matrix the one represented by 3.4. It is, as the stag and hare hunters a coordination game and we aim to investigate the influence of the morality coefficient on the capacity of the players in order to coordinate themselves.



(a) morality coefficient $k = 0$



(b) morality coefficient $k = 0.15$



(c) morality coefficient $k = 0.2$

Figure 4.8: Cow, Stag and Hare simulations with initial proportions $(0.1, 0.45, 0.45)$

As we can expect, the higher the morality coefficient k, the better the capacity of the players to coordinate on a higher-quality equilibrium. As justified by the simulation summarised in Figure 4.8, which are performed with the same initial proportions but with increasing k. In all cases, the system ends up in one of the three possible societies described in section 3.2.3, with a bigger incentive for cooperation as long as k increases.

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A | Appendix A

A.1. The stag and hare game double dynamic

```
###packages###
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt
from functools import partial
plt.style.use("seaborn-poster")

###give the differents derivatives according to (v,x)###
 #S is the vector of the system i.e; (v,x)
 #t the time and s the parameter in the equation
def F_2_derivative(t, S, s):
    y = S[0] + (1-S[0]) * S[1]
    ret = np.zeros(2)
    ret[0] = S[0] * (1 - S[0]) * (1 - S[1]) * (1.5 * y -1)
    ret[1] = s * S[1] * (1 - S[1]) * (1.5 * y -1)
    return ret

###Solver for an initial point S0###
def my_solver_2(t_span, S0, s):
    f = partial(F_2_derivative, s=s)
    sol = solve_ivp(f, [t_span[0],t_span[-1]] , S0, t_eval=t_span)
    return sol

### Solver for multiple initial points in S0_gris###
def my_solver_2_multistart(t_span, S0_grid, s):
    n = len(S0_grid[1])
    l = []
```

```
    f = partial(F_2_derivative, s=s)
    for i in range(n):
        sol = solve_ivp(f, [t_span[0],t_span[-1]] ,...
            [S0_grid[0][i], S0_grid[1][i]], t_eval=t_span)
        l.append(sol)
    return l

### example of initial points grid, time line, and s coefficient###
grid = [np.random.uniform(size=200),np.random.uniform(size=200)]
t_span = np.linspace(0,100,10001)
s=5
list_sol = my_solver_2_multistart(t_span, grid, s)

### plot example###
def parabole(x):
    n = len(x)
    y = [0]*n
    for i in range(n):
        y[i] = (2-3*x[i])/(3*(1-x[i]))
    return y

plt.figure(figsize = (10, 10))
x_space = np.linspace(0,2/3,100)
y_space = parabole(x_space)
plt.plot(x_space,y_space,'--r', label= "split line")
for i in range(len(list_sol)):
    plt.plot(list_sol[i].y[0], list_sol[i].y[1],'#add8e6')
    plt.plot(list_sol[i].y[0][0], list_sol[i].y[1][0],'.b')
plt.ylabel("Stag hunter proportion among Nasher")
plt.xlabel("Proportion of Nasher")
plt.legend(["Split line","Population trajectory","Initial configuration"])
plt.grid(True)
plt.show()
```

A.2. The cow, stag and hare double dynamic

```
###packages importation###
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt
from functools import partial
plt.style.use("seaborn-poster")

#creation of the function that gives the derivative wrt (time, state_of_system, s)
#S is the vector of the system, t the time and s the parameter in the equation
def F_3_derivative(t, S, s):
    y = S[0] + (1-S[0]) * S[1] #proportion of cooperative players
    neut = (1 - S[0]) * (1 - S[1] - S[2]) #proportion of neutral player
    x = (1 - S[0]) * S[2] #proportion of selfish players
    ret = np.zeros(3)
    delta_cn = 4*y -2*neut # value of the differential fitness btw cow and stag
    delta_sn = x -2*neut
    delta_cs = 4*y - x
    delta_kn = (1-S[1]) * (8*y + 2*neut + x) -(1-S[1]-S[2])*(4*y + 4*neut +x) - ...
        S[2]*(4*y + 4*neut +2*x)
    ret[0] = S[0] * (1 - S[0]) * delta_kn
    ret[1] = s * S[1] * ( (1 - S[1]-S[2]) * delta_cn + S[2] * delta_cs)
    ret[2] = s * S[2] * ( (1 - S[1]-S[2]) * delta_sn - S[1] *delta_cs)
    return ret

#function that solve the ODE given (time_span, list_initial_states, s_parameter)
def my_solver_2_multistart(t_span, S0_grid, s):
    n = len(S0_grid[1])
    l = []
    f = partial(F_3_derivative, s=s)
    for i in range(n):
        sol = solve_ivp(f, [t_span[0],t_span[-1]] ,...
            [S0_grid[0][i], S0_grid[1][i], S0_grid[2][i]], t_eval=t_span)
        l.append(sol)
    return l
```

```
#simpler one version
def my_solver_2(t_span, S0, s):
    f = partial(F_3_derivative, s=s)
    sol = solve_ivp(f, [t_span[0],t_span[-1]] , S0, t_eval=t_span)
    return sol

###we need to delete the points of the grid that have y + z > 1###
def make_grid(n_rafined):
    x = np.linspace(0, 1, n_rafined)
    y = np.linspace(0, 1, n_rafined)
    z = np.linspace(0, 1, n_rafined)
    xv, yv, zv = np.meshgrid(x, y, z)
    x = list(xv.flatten())
    y = list(yv.flatten())
    z = list(zv.flatten())
    indexes = []

    n = len(x)
    for i in range(n):
        if y[i] + z[i] > 1:
            indexes.append(i)


    for index in sorted(indexes, reverse=True):
        del x[index]
        del y[index]
        del z[index]

    grid=[x,y,z]
    return grid

###setting use for simulation in the document###
grid = make_grid(11)
t_span = np.linspace(0,50,2001)
s = 5
list_sol = my_solver_2_multistart(t_span, grid, s)
```

```
###plot###
ax = plt.figure().add_subplot(projection='3d')
n = len(list_sol)
for i in range(0,n):
    #ax.scatter(list_sol[i].y[2][0], list_sol[i].y[1][0],list_sol[i].y[0][0])
    ax.scatter(grid[0][i], grid[1][i],grid[2][i])
    ax.plot3D(list_sol[i].y[0], list_sol[i].y[1],list_sol[i].y[2])
ax.set_xlabel("kantian proportion")
ax.set_ylabel("Nashe cow prop c(t)")
ax.set_zlabel("Nashe hare prop h(t)")
ax.set_title("CSH game")
plt.grid(True)
plt.show()
```

B | Appendix B

B.1. Basics common code of k-replimorator equation

```
#packages importation
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib as mpl
import matplotlib.pyplot as plt
from functools import partial
plt.style.use("seaborn-poster")
from matplotlib.axis import Axis

###creation of the function that gives the derivative wrt
#for the k-replimorator equation (time, state_of_system, s_parameter)
#S is the vector of the system, t the time and A is the matrix of the game
def kantion_mean_derivative(t, X_0, A, k):
    X = np.matrix(X_0)                          # k is the degree of morality
    n = X.shape[1]
    ret = np.zeros(n)
    diag = A.diagonal()
    for i in range(n):
        delta_i = np.zeros((1,n))
        delta_i[0,i] = 1
        ret[i] =(1-k)*(((delta_i @ A @ X.transpose())-(X @ A @ X.transpose()))[0,0]
        ret[i] = ret[i] * X[0,i]
        ret[i] += X[0,i] *k * ...
            ( (delta_i @ A @ delta_i.transpose())[0,0] - X @ diag.transpose() )
    return ret

#for the arithmetic mean and product instead of the weighted mean
```

```
def kantion_mean_derivative_prod(t, X_0, A, k):
    X = np.matrix(X_0)
    n = X.shape[1]
    ret = np.zeros(n)
    mean_common_strategy = np.trace(A)/n #mean of the common strategies
    diag = A.diagonal()
    for i in range(n):
        delta_i = np.zeros((1,n))
        delta_i[0,i] = 1
        ret[i] =(1-k) * (( (delta_i @ A @ X.transpose())-(X @ A @ X.transpose()))[0,0])
        ret[i] = ret[i] * X[0,i]
        ret[i] += np.prod(X) *k *( (delta_i @ A @ delta_i.transpose())[0,0] - ...
            mean_common_strategy )
    return ret

def solver_kantian_mean_replicator(t_span, X0, A, k):
    f = partial(kantion_mean_derivative, A=A, k=k)
    sol = solve_ivp(f, [t_span[0],t_span[-1]] , X0, t_eval=t_span)
    return sol
```

B.1.1. The hawk and dove

```
#let's try for howk and dove game, with parameter C and G
# with the classical contrain C>G
C,G = 10,4
A = np.matrix([[(G-C)/2, G],[0,G/2]])

t_span = np.linspace(0,10,1000)
X_0 = np.array([0.9, 0.1])
k = 0.2


sim = solver_kantian_mean_replicator(t_span, X_0, A, k)

fig, ax = plt.subplots()
```

```
plt.plot(t_span,sim.y[1], c='green', label='Dove') #propotion of dove
plt.plot(t_span,sim.y[0]+sim.y[1],'--b', label='Sum of proportion')
plt.plot(t_span,sim.y[0], c = 'red', label= 'Hawk')
leg = ax.legend
```

B.1.2. Stag and hare

```
#let's try on Stag and Hare, taken from Laslier paper
A = np.matrix([[1, -1],[0.5,0]])

t_span = np.linspace(0,30,10000)
X_0 = np.array([0.5,0.5])
k = 0

sim = solver_kantian_mean_replicator(t_span, X_0, A, k)

fig, ax = plt.subplots()

plt.plot(t_span,sim.y[0]+sim.y[1],'--b', label='Sum of proportion')
plt.plot(t_span,sim.y[0], c='green', label='Stag hunter') #proportion of stag
plt.plot(t_span,sim.y[1], c='red', label='Hare hunter')
leg = ax.legend(loc ="lower right")
```

B.1.3. Prisonner dilemma

```
#let's try on prisoner dilemma, with parameter T>R>P>S according to wikipedia page
T,R,P,S = 0,-1,-5,-10
A = np.matrix([[R, S],[T,P]])

t_span = np.linspace(0,7.5,1000000)
X_0 = np.array([0.5, 0.5]) #classical setting, 0.5 0.5
k = 0.8 #limite


sim = solver_kantian_mean_replicator(t_span, X_0, A, k)
print(len(t_span),len(sim.y[0]))
fig, ax = plt.subplots()
```

```
plt.plot(t_span,sim.y[0]+sim.y[1], '--b', label='Sum of proportion')
plt.plot(t_span,sim.y[0], c='green', label='Cooperate proportion')
plt.plot(t_span,sim.y[1], c='red', label='Betrayer proportion')

leg = ax.legend(loc ="lower right")
```

Here is the code for the multi-visualisation according to the value of k:

```
#let's try on prisoner dilemma, with parameter T>R>P>S according to wikipedia page
# but it doesn't make sense in a population game...
T,R,P,S = 0,-1,-5,-10
A = np.matrix([[R, S],[T,P]])
#A = np.matrix([[-1,-15],[0,-10]])

t_span = np.linspace(0,15,100000)
X_0 = np.array([0.5, 0.5]) #classical setting, 0.5 0.5
k = 0 #limite

fig, ax = plt.subplots()
n_curve = 20
for i in range(n_curve+1):
    k = i/n_curve
    lab = 'k = '+ str(k)
    sim = solver_kantian_mean_replicator(t_span, X_0, A, k)
    plt.plot(sim.t, sim.y[0],  c=mpl.colormaps['RdYlGn'](k), label=lab)


leg = ax.legend(loc ="lower right")
```

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Acknowledgements

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