POLITECNICO
MILANO 1863

SCUOLA DI INGEGNERIA INDUSTRIALE
E DELL'INFORMAZIONE

Sustainable Investing and Climate Change

Tesi di Laurea Magistrale in
Mathematical Engineering - Ingegneria Matematica

Author: **Filippo Lombardi**

Student ID: 10529961
Advisor: Prof. Emilio Barucci
Academic Year: 2021-22

Abstract

In this work, we aim to integrate the aspects given by the ESG factors on stock returns and pricing, and investigate how an investor's portfolio management may vary with the presence of CO_2 in the atmosphere. To do so, we first examine the studies by Pedersen [25] and Pastor [24] on the use of ESG factors, supported by empirical analysis, for stock premium calculations, and then develop the dynamics of the variable that describes pollution inspired by the work of Hambel [15]. In the modified model, therefore, the investor makes decisions based on both the ESG factors of the company and the more conventional returns, as well as the state of atmospheric pollution. Based on these factors, an optimal portfolio is constructed that is distinguished by being composed of 4 different funds: the riskless fund, the myopic fund, the externality fund, and the hedge fund. Pollution can be controlled through a strategy of reducing the concentration of CO_2 in the air, as a control variable. In this way, investments can also be controlled and guided towards a more green policy or, on the contrary, less focused on sustainability. Finally, the value of the Social Cost of Carbon is studied and compared with the literature as this measure is very important from an economic point of view, and its dependence on the damage elasticity links it to climate risk.

Keywords: Sustainable finance, Climate change, Climate risk, ESG, Portfolio management, Social Cost of Carbon

Sommario

Nel presente lavoro vogliamo andare ad integrare gli aspetti dati dai fattori ESG sui return e il pricing dello sotck, inoltre vogliamo indagare come il managing di portafoglio di un investitore possa variare rispetto alla presenza di CO_2 nell'atmosfera. Per far ciò, dapprima prendiamo in esame gli studi di Pedersen [25] e Pastor [24] sull'utilizzo dei fattori ESG, supportati da analisi empiriche, per il calcolo dei premi degli stock, per poi andare a svuluppare la dinamica della variabile che descrive l'inquinamento ispirandoci al lavoro di Hambel [15]. Nel modello modificato, quindi, l'investitore prende decisioni in base sia ai fattori ESG dell'azienda, oltre ai più canonici return, che in base allo stato dell'inquinamento atmosferico. Seguendo tali fattori viene costruito un portafoglio ottimo che si distingue per essere composto da 4 fondi differenti: il riskless fund, il myopic fund, l'externality fund e l'hedge fund. L'inquinamento può essere controllato tramite una strategia di abbattimento, o variabile di controllo, della concentrazione di CO_2 nell'aria. In questo modo anche gli investimenti possono essere controllati e guidati verso o una politica più green o, al contrario, meno indirizzata alla sotenibilità. Da ultimo viene studiato il valore del Social Cost of Carbon e comparato con la letteratura in quanto tale misura risulta essere molto importante da un punto di vista economico e la sua dipendenza dalla damage elasticity la lega al rischio climatico.

Parole chiave: Finanza sostenibile, Cambiamento climatico, Rischio climatico, ESG, Gestione di portafoglio, Social Cost of Carbon

Contents

Introduction

Climate change refers to the long-term changes in the Earth's climate, including changes in temperature, precipitation patterns, and weather events. These changes are largely driven by human activities, such as the burning of fossil fuels and deforestation, which release large amounts of greenhouse gases into the atmosphere. These greenhouse gases, including carbon dioxide, methane, and nitrous oxide, trap heat in the atmosphere, leading to global warming.

The consequences of climate change are far-reaching and include rising sea levels, more frequent and severe weather events, and changes to ecosystems and the natural world. These impacts can have significant social and economic consequences, such as displacement of people, damage to infrastructure and property, and food and water shortages.

In recent years, efforts have been made to mitigate climate change and its consequences, with many national and international bodies working to reduce emissions or to counter rising temperatures.

The Paris Agreement, which was signed in 2015 and went into effect in 2016, marked a significant step forward in the global effort to address climate change. The Agreement aims to limit global temperature increases to well below 2 degrees Celsius above pre-industrial levels, and to pursue efforts to limit the increase to 1.5 degrees Celsius. This has had a major impact on the sustainable finance landscape, as investors are increasingly seeking to align their portfolios with the goals of the Paris Agreement. The G20 Green Finance Study Group in 2016, where the G20 established the Green Finance Study Group to promote the development of green finance and to support the transition to a low-carbon, sustainable economy.

The Task Force on Climate-related Financial Disclosures (TCFD) which is a global organization that was established in 2015 in response to the growing need for better climate-related financial disclosures.

There have been done a number of regulatory developments in recent years that have influenced the sustainable finance landscape, including the EU's Action Plan on Financing Sustainable Growth and the US Securities and Exchange Commission's (SEC) recent guidance on climate change disclosure.

In 2021, the European Union (EU) introduced the Taxonomy Regulation, which provides a common framework for classifying and identifying environmentally sustainable economic activities.

Following this growing interest in climate change, attention has also been drawn to sustainable investing from investors who seek to achieve both financial returns and positive social and environmental impact. The rising engagement of the finance industry with climate change is a result of the call to non-governmental actors to join the fight against climate change.

This trend is driven by a growing awareness of the environmental and social issues facing the world, as well as by the recognition that companies with sustainable practices are often better positioned to create long-term value.

Investors are increasingly looking to allocate capital to companies that are committed to sustainability and environmental, social and governance (ESG) principles. This trend is supported by a growing body of evidence that suggests that companies with strong ESG performance tend to outperform their peers over the long term. Moreover, investors are increasingly concerned about the risks associated with climate change and are looking for ways to manage these risks in their portfolios.

As a result, there has been a surge in demand for sustainable investment strategies and a corresponding increase in the number of investment products that incorporate ESG

criteria into their investment processes. These products range from ESG-screened mutual funds and exchange-traded funds (ETFs) to impact investment funds that seek to generate social and environmental benefits alongside financial returns.
The growth of sustainable investing reflects a broader shift in investor attitudes towards sustainability and social responsibility. As investors seek to align their portfolios with their values and address the risks associated with climate change, sustainable investing is likely to become an increasingly important part of the investment landscape.

Since there is considerable uncertainty regarding the climate, it is hard to gauge the climate impact of a particular portfolio and, indeed, there are many other factors that affect the climate apart from investments.

The literature in sustainable finance includes a wide range of strategies and initiatives, including sustainable investing, green bonds, corporate social responsibility, temperature increasing and pollution. Sustainable investing involves considering environmental, social, and governance (ESG) factors in the investment decision-making process. This can include selecting companies with strong sustainability practices, or avoiding those with poor ESG records. Green bonds are debt instruments issued to finance environmentally sustainable projects, while corporate social responsibility involves companies taking responsibility for their impact on society and the environment.

The financial world so needs models that use to price, in equilibrium, stocks taking into account the ESG factor of the companies. The traditional pricing models have not taken into account the impact of environmental, social, and governance (ESG) factors on financial markets. In recent years, researchers have developed new models that incorporate ESG considerations, providing a more comprehensive understanding of the relationship between ESG and asset pricing.
Some examples are the models of Pastor and Pedersen, which take into account the impact of ESG factors on asset prices [24, 25]. The models incorporate measures of ESG performance into the pricing equation, allowing investors to evaluate the impact of ESG

on asset returns. This approach provides a more complete understanding of the impact of ESG on market dynamics, and helps investors to make more informed decisions.
Empirical evidence has also demonstrated the importance of considering ESG factors in financial analysis. A recent study [3, 4] analysed the relationship between ESG and stock returns. The study demonstrate that firms with higher ESG scores has higher stock returns than those with lower ESG scores, suggesting that ESG considerations are becoming increasingly important in financial markets.
Taken together, these developments highlight the importance of considering ESG factors in financial analysis and modelling. As the impact of ESG on financial markets continues to grow, it is essential that pricing models evolve to reflect this changing landscape.

The continuous increase in temperatures and atmospheric pollution caused by human presence have been the subject of study in recent times. These factors, in fact, represent a risk component due to climate change that an investor cannot ignore. The dynamics of temperature and pollution is become very significative prom a financial prospective. Its importance is linked to the fact that an increasing in pollution could be dangerous not only for the quality of life but also for the investor's portfolio. The tool used to study the economic loss of pollution concentration is the damage function.
Many authors tried to model a damage function. But the most famous is the one with exponential dependence on $C0_2$ concentration [13].
This potential loss linked to the carbon emissions introduces the problem of carbon price.

Carbon price is a policy tool used to incentivize companies to reduce their carbon emissions by assigning a cost to carbon. One example of carbon pricing is the carbon tax. Carbon tax is a policy mechanism aimed at reducing carbon emissions by assigning a cost to carbon. One example of a carbon tax is a direct fee imposed on the carbon content of fossil fuels, such as coal, oil, and gas. The goal of a carbon tax is to incentivize companies to reduce their carbon footprint by creating a financial cost for carbon emissions. A study by [15, 16] found that a carbon tax can effectively reduce carbon emissions in the electricity sector. Their research highlights the potential for a carbon tax to encourage companies to transition towards a more sustainable future by reducing their reliance on carbon-intensive energy sources.

So the questions that we pose ourselves are: how can someone invest optimally, taking into account the climate? How does uncertainty regarding the climate impact of various investments affect optimal portfolios and equilibrium returns?

To answer this questions we take into account the literature adding some major importance to the pollution in the dynamics of the stocks [9].
The original model consists in the combination of two frameworks: The Integrated Assessment Model (IAM) [22] and the Intertemporal Capital Asset Pricing Model (ICAPM) [20, 21].
The Integrated Assessment Model (IAM) developed by William Nordhaus is a widely used tool in climate change economics. The IAM is a model that attempts to simulate the interactions between the economy and the climate system, and to project the impacts of climate change on the economy and society [22].
The ICAPM argues that investors consider not only the expected return and risk of an investment, as in the traditional CAPM, but also the effect of their investment decisions on their future consumption and investment opportunities. The ICAPM has been applied to various areas of finance, including sustainable finance, where it has been used to analyse the relationship between sustainability factors and asset pricing.

The presence of carbon dioxide in the atmosphere has two main effects that are relevant in the context of finance and climate change.
First, it contributes to the overall climate risk that comes with the potential damage to the environment, human health, and the economy. Climate risk is usually measured by using a damage function that estimates the economic cost of climate change. The damage function can be influenced by the concentration of CO_2 in the atmosphere, among other factors. Therefore, reducing CO_2 emissions can help mitigate climate risk and potentially

limit the damage caused by climate change.
Secondly, the concentration of CO_2 in the atmosphere can also have an impact on the pricing of financial assets, such as stocks. In particular, some investors and analysts are concerned that companies that are highly dependent on fossil fuels, such as oil and gas, may become less profitable in the future if governments around the world adopt policies that aim to reduce carbon emissions.

The model proposed has the peculiarity of an effect of the carbon concentration, not only on the non-financial incomes, but also in the prices of the stocks. We consider that the pollution negatively affect the prices and increases the volatility of the firms.
As mentioned, the concentration of CO_2 drives the non-financial income that is characterized by a damage function that depends on the pollution.
There is a vast literature with the challenging aim of quantifying the impact of climate change on economic output. For the expression of the damage function, we consider an exponential dependence on the CO_2 concentration [13].
The model is also extended, introducing a Stochastic Differential Utility (SDU) [8].
The SDU framework incorporates the idea that individuals have preferences not only for the expected value of different outcomes, but also for the variability or uncertainty of those outcomes. This utility has been applied to various areas of finance, including sustainable finance, where it has been used to analyse the trade-offs between risk and sustainability considerations. For the stochasticity of economic variables and the uncertainty of future events, the SDU framework can help investors make more informed decisions about their investments in the face of climate change and other sustainability challenges.

To maximize the expected utility, we solve the Hamilton-Jacobi-Bellman (HJB). The HJB equation is a mathematical tool used in finance to find the optimal portfolio strategy that maximizes expected utility over time, subject to the constraints of the market dynamics and the investor's preferences. The solution to the HJB equation provides the investor with the optimal allocation of wealth across different assets, such as stocks and bonds, taking into account the uncertainty and risk associated with each asset. The optimal portfolio strategy found through the HJB equation is an essential concept in modern finance and is widely used in portfolio management, asset pricing, and risk management.

The optimal portfolio found solving the HJB equation consists of four different funds:

1. a risk-free asset
2. a myopic fund
3. a fund accounting for pollution externalities
4. a fund hedging the impact of CO_2 emissions on future production

That's a result in line with the literature of portfolio management.

1 | Environmental Social and Governance

Environmental, Social, and Governance (ESG) criteria have become increasingly important in the world of investing and business as they are critical to sustainable development and mitigating climate change. ESG factors are important because they help businesses and investors make informed decisions that promote sustainable growth and reduce the negative impacts of human activities on the environment and society.

ESG factors focus on three key areas: environmental, social, and governance. Environmental factors are crucial for climate change mitigation and adaptation. The degradation of natural resources, deforestation, pollution, and climate change all have adverse impacts on the environment, including water scarcity, soil erosion, and biodiversity loss. These environmental factors have negative impacts on human health, food security, and the quality of life of communities. Therefore, businesses and investors must consider environmental factors such as carbon emissions, energy efficiency, renewable energy, waste management, and sustainable sourcing of raw materials.

Social factors are equally essential as they consider the social impacts of business activities. Social factors include labor standards, human rights, and community engagement. Socially responsible businesses and investors prioritize creating safe and healthy working environments, promoting diversity and inclusion, respecting human rights, and engaging with local communities to improve the quality of life. These factors ensure that businesses contribute positively to society and improve the well-being of communities.

Governance is also an important aspect of ESG criteria, which evaluates the internal management and oversight of a company. Companies that demonstrate transparency, accountability, and ethical business practices have better long-term performance and are less likely to engage in activities that harm the environment or society. Governance factors include board diversity, executive compensation, audit quality, and risk management. These factors ensure that companies are well-managed and operate with integrity, creating trust and confidence among investors and stakeholders.

ESG factors are crucial for climate change mitigation and adaptation. Climate change has significant impacts on the environment, society, and the economy. Rising sea levels, extreme weather events, and changes in temperature and precipitation patterns are threatening food security, human health, and the livelihoods of millions of people. Businesses and investors have a role to play in mitigating climate change by reducing greenhouse gas emissions and promoting sustainable practices. ESG criteria help businesses and investors identify companies that are taking action to reduce their environmental impact and are committed to creating a more sustainable future.
ESG criteria are essential for sustainable development and mitigating climate change. Businesses and investors have a responsibility to consider the environmental, social, and governance factors in their decision-making processes. By doing so, they can contribute positively to society, improve the well-being of communities, and promote sustainable growth. ESG factors are important because they ensure that companies are well-managed, transparent, and operate with integrity, creating trust and confidence among investors and stakeholders.

ESG investing offers numerous benefits to investors. For one, an ESG-optimized portfolio can offer a hedge against traditional risks such as negative environmental or social impacts, reputational damage, regulatory fines, and legal liabilities. Moreover, investing in ESG companies helps build investor confidence that their investments are working to achieve greater social, environmental, and governance goals. Finally, ESG investing tends

to be a long-term investment, allowing investors to align themselves with companies that are also focused on long-term, sustainable growth.
As ESG investing continues to gain momentum, it also presents new challenges. The impact of ESG metrics on the financial performance of a company is difficult to measure, and there are no clear, universal standards to gauge ESG performance. Moreover, there are debates about whether companies that perform well from an ESG perspective can, in fact, generate better long-term returns compared to those without strong ESG scores.

Despite these challenges, investors, corporations, and policymakers have all begun to recognize the importance of a broad-based approach to environmental, social, and governance factors. In recent years, the popularity of ESG investments has led to the creation of an entire ecosystem of ESG-focused funds, including passive index funds, actively managed mutual funds, and ETFs.

1.1. ESG as investment strategy

The integration of environmental, social, and governance (ESG) factors into investment decision-making is becoming increasingly important in the financial industry. In addition to contributing to sustainable and responsible investing, there is growing evidence that ESG factors can have a material impact on the expected returns of investments.
The inclusion of ESG factors in investment decisions can be seen as a response to global challenges such as climate change, social inequality, and lack of accountability by corporations. Investors are increasingly recognizing that corporations with strong ESG credentials are more likely to generate sustainable long-term returns than those without. Companies that are taking serious steps towards mitigating their environmental impact, promoting social issues like diversity and inclusion, and have strong governance practices are more likely to gain investor favour and therefore access to capital.

ESG investing considers a broad range of environmental factors including a company's carbon footprint, energy efficiency, waste reduction initiatives, water conservation, and resource efficiency. These considerations are designed to support the global transition to a greener economy and mitigate the adverse effects of climate change. ESG investing also takes into account the social impact of a company's operations, including working conditions in its supply chain, employee welfare, human rights policies, social inclusion, and diversity, among other factors.

It's evaluated the company's governance practices, such as executive pay, board diversity, and the effectiveness of the management team. These considerations reflect shareholders' desire for greater accountability and transparency in corporate decision-making.

As a result, investors are increasingly incorporating ESG considerations into their investment decision-making processes to identify opportunities for long-term value creation. By integrating ESG factors into their analysis, investors can identify companies with a sustainable business model, strong risk management practices, and a focus on long-term value creation. ESG factors can also help investors to identify risks that may not be captured by traditional financial analysis, such as reputational risk or regulatory risk. Furthermore, the demand for ESG investments is on the rise, with institutional investors and individual investors alike seeking to align their investments with their values and social responsibility.

The growing importance of ESG factors in the expected returns of assets reflects a broader shift in the investment landscape towards a more holistic, sustainable approach to investing. As investors increasingly recognize the material impact of ESG factors on expected returns, they are seeking out new tools and strategies to incorporate these factors into their investment decision-making processes. This shift not only benefits investors by improving risk-adjusted returns, but it also encourages companies to prioritize sustainability and responsible business practices, leading to a more sustainable and prosperous future for all.

1.1.1. ESG portfolio frontier

Pedersen proposes an interesting theory where each stock's environmental, social and governance (ESG) score is important to provide information about firm fundamentals, and it also affects investor preferences [25]. They introduce an ESG efficient frontier, in alternative to the usual portfolio frontier, that, for each ESG level, provides the highest attainable Sharpe Ratio.

The portfolio frontier is a concept in modern portfolio theory that represents the set of all possible portfolios that an investor can construct given a set of available investments [2]. The portfolio frontier shows the trade-off between risk and return, where the risk is measured by the portfolio's volatility, and the return is measured by the portfolio's expected return. On the other hand, the ESG portfolio frontier extends the traditional portfolio frontier by including ESG considerations [25].

The main difference between the traditional portfolio frontier and the ESG portfolio frontier is that the ESG portfolio frontier adds another constraint on the portfolio optimization problem. Specifically, it requires that the selected portfolios meet a certain level of ESG criteria. This constraint reduces the set of feasible portfolios compared to the traditional portfolio frontier, as some portfolios may not meet the required ESG standards.

The ESG portfolio frontier also allows investors to compare the risk-return trade-off of portfolios that meet different levels of ESG criteria. For example, an investor may be able to construct a portfolio with a higher expected return but lower ESG score, or a portfolio with a lower expected return but higher ESG score. The ESG portfolio frontier can help investors make informed decisions about their investments by considering both financial performance and ESG considerations. The ESG portfolio frontier is a useful tool for investors who want to incorporate ESG considerations into their portfolio construction and make investment decisions that align with their values and objectives. It extends the traditional portfolio frontier by considering ESG factors and provides a way to visualize the trade-off between risk and return while meeting specific ESG criteria. The portfolio created by an investor that follows such frontier satisfy four-fund separation [25].

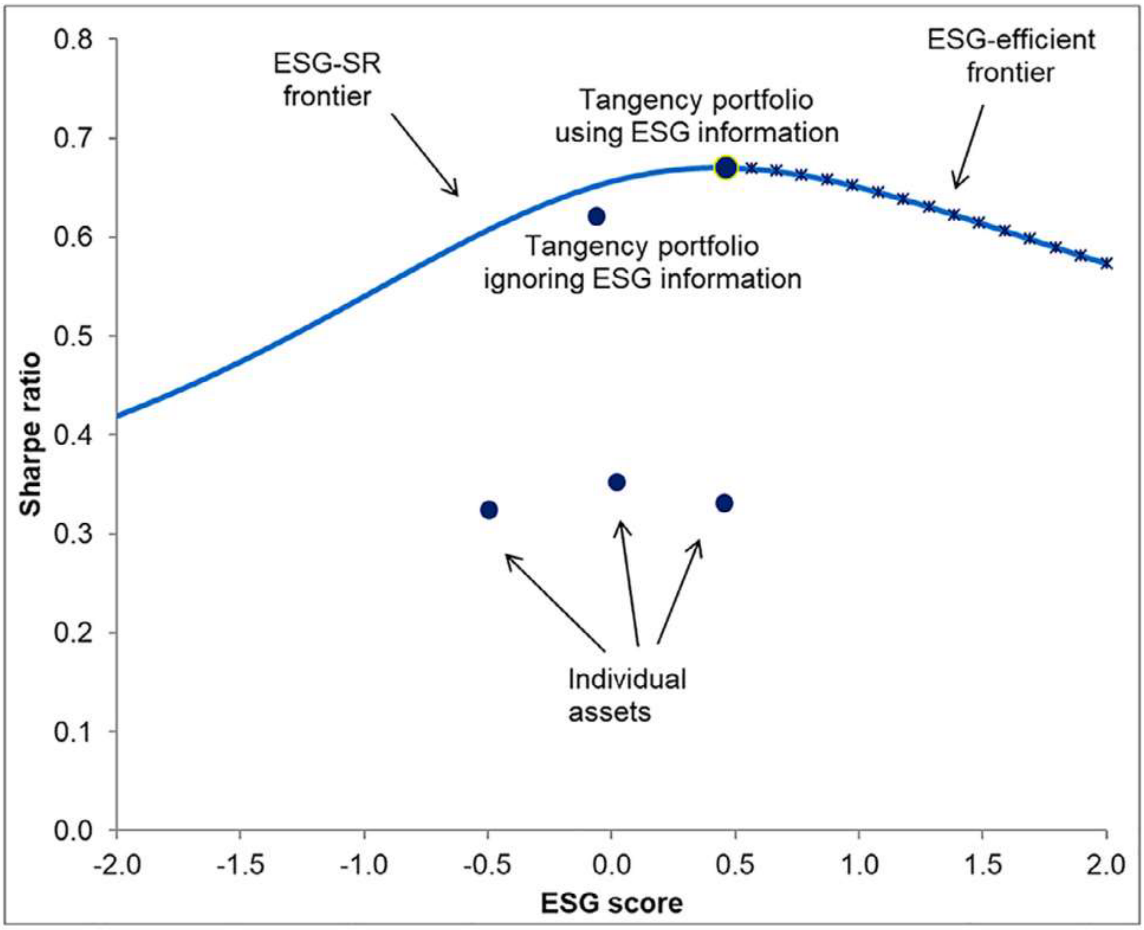

Figure 1.1: ESG frontier [25]

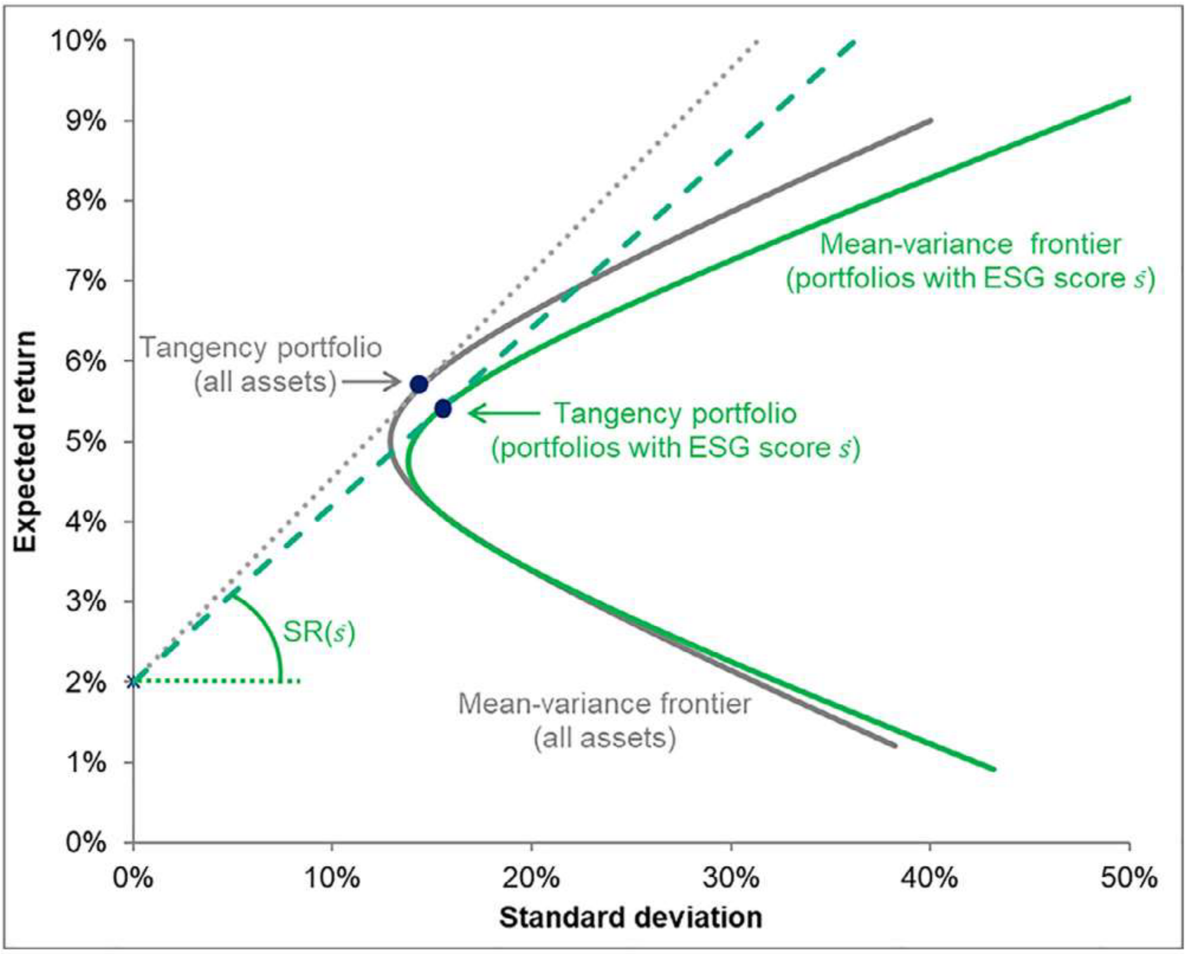

Figure 1.2: Given a specific ESG score, portfolio and assets mean-variance [25]

1.2. Climate risk

Climate risk refers to the potential negative impacts of climate change on the natural and built environment, as well as on social and economic systems. These risks can manifest in different ways, such as more frequent extreme weather events, rising sea levels, and changes in precipitation patterns, all of which can cause physical damage to infrastructure, property, and natural resources. Climate risks can also result in social and economic costs, such as lost income and employment opportunities, health impacts, and increased demand for emergency services.

To understand the economic impacts of climate change, economists rely on a mathematical tool called the damage function. This function attempts to quantify the relationship between changes in temperature or other climate variables and the economic losses that would result from these changes. The damage function considers both the direct and indirect impacts of climate change on the economy. Direct impacts include physical damage to infrastructure, buildings, and other assets. Indirect impacts can include the impacts of climate change on trade, productivity, and competition. The damage function is a critical tool for policymakers and investors looking to estimate the potential economic losses associated with different levels of climate change. It can help identify the most significant risks and costs associated with climate change and inform strategies to mitigate these risks. By taking into account the direct and indirect impacts of climate change, the damage function can help ensure that efforts to address climate change are grounded in sound economic analysis. In section 3.1 different damage functions are presented, with a particular focus on exponential formulation [13].

Climate risk can be seen as an additional source of risk for investors, which is non-traded and only partially insurable. It can be treated as a background risk which the investors want to mitigate.

Investors need to consider climate risk and take steps to hedge their portfolios, investing in both green and brown firms.
Green firms have a smaller carbon footprint and are, therefore, less exposed to climate risks such as legislation changes or natural disasters. By investing in green firms, investors can mitigate climate risk by supporting companies that are actively working to address climate change.
On the other hand, brown firms are traditional fossil fuel companies. Investing in brown firms may be perceived as riskier due to their carbon-intensive operations, they can be an important part of a diversified portfolio that hedges against climate risk. Brown firms also have the potential for significantly higher returns if they improve their sustainability practices or if governments take a less stringent stance on carbon emissions.

Investing in both green and brown firms can help investors hedge their portfolios against potential climate risks while also providing opportunities for growth. A well-diversified portfolio that includes both green and brown firms can reduce overall risk and volatility, protecting investors from sudden market changes or disruptions. Moreover, investing in both green and brown firms can create a positive feedback loop. By supporting green firms and pushing brown firms to develop sustainable practices, investors can help drive change in the industry. This can lead to more sustainable practices and technologies being developed, which can further reduce the risks of climate change.

While it is true that investing in brown firms can be seen as counterproductive to efforts to combat climate change, investors can take a more constructive approach. One way is to engage with these companies to encourage them to improve their sustainability practices. Investors can use their position as shareholders to vote on key issues and influence decision-making to push for more sustainable operations.

In conclusion, investors need to hedge their portfolios against climate risk, and one approach is to invest in both green and brown firms. Investing in green firms can mitigate risk while supporting sustainability, while investing in brown firms can hedge against the potential economic impacts of divestment while also providing opportunities for growth. By engaging with brown firms and supporting sustainable practices, investors can create a positive feedback loop that drives change in the industry. Ultimately, investors must balance the risks and opportunities of investing in green and brown firms to build a diversified portfolio that can weather the challenges of climate change while supporting sustainable growth.

The dichotomy of green and brown stocks is a matter of study and discussion in literature. As we will see in Section 2.2, Pastor creates a hedging portfolio and comes to the conclusion that the better type of stock that mitigate against climate risk is the green one [24]. Even Fischer studies the situation in which a hypothetical investor can invest in 2 risky assets, brown and green [9].

So climate risk is related to both direct and indirect losses, one can be predicted through a damage function, the other one is characterized by many factors, the strategy socially adopted to decrease the pollution and other, and the investor can try to hedge his portfolio against this risk.

2 | Equilibrium models

Equilibrium asset prices are determined by an ESG adjusted capital asset pricing model, instead of the traditional CAPM, showing when ESG raises or lowers the required return.

The Capital Asset Pricing Model (CAPM) is a widely used financial model that estimates the expected return on an asset by taking into account its risk and the market risk premium. The model assumes that investors are rational and only consider the expected return and risk of an asset when making investment decisions. According to the CAPM, the expected return of a security or a portfolio is equal to the risk-free rate plus a risk premium, where the risk premium is proportional to the asset's beta, which measures the asset's sensitivity to changes in the market as a whole. In the following analysis, the authors take into account the excess returns of the stocks instead of the returns, which means that the formulation is:

$$\mathbb{E}\left(r^i\right) = \beta\left[\mathbb{E}\left(r^m\right)\right] + \alpha$$

where $\mathbb{E}\left(r^i\right)$ is the expected excess return on the asset, $\mathbb{E}\left(r^m\right)$ is the expected return on the market portfolio, β is the systematic risk and α is the abnormal rate of return, used mainly for hedging purpose.

However, the CAPM model does not consider ESG (environmental, social, and governance) factors, which can have a significant impact on the performance and risk of an asset. The evolution from the standard CAPM is the inclusion of ESG factors in the asset pricing process. The ESG-adjusted CAPM recognizes that ESG factors can impact the expected return and risk of an asset, and therefore adjusts the pricing model to incorporate these considerations. In addition, the ESG-adjusted CAPM provides investors with a more comprehensive view of an asset's risk and return profile. By incorporating ESG considerations, the model can better capture the long-term risks and opportunities associated with an asset. For example, an asset with a high ESG score may be expected to have a lower risk profile over the long term, leading to a higher expected return.

2.1. Market scenarios

Pedersen suggests the evolution of the market starting from an ESG unaware situation to a more sustainable driven one in which the investments take into account the ESG scores of the firms. These two different scenarios bring not only the prices and the expected values to change, but also the firms start to make investments to increase their ESG score [25].

Consider an economy which, at time t, has security prices $p_t = (p_t^1, p_t^2, ..., p_t^n)'$ and excess return from time $t-1$ to t, $r_t = (r_t^1, r_t^2, ..., r_t^n)'$. There are also the exogenous variables as ESG score s^i, risk-free rate r^f and security dividend payoffs $v_t = (v_t^1, v_t^2, ..., v_t^n)'$.
The total market dividend is $v_t^m = v_t^1 + v_t^2 + ... + v_t^n$, assuming that dividends are independent and identically distributed over time.
The informational value of ESG scores is $\mathbb{E}(v_t|s) = \hat{\mu} + \lambda(s - s^m)$ with $s^m = \sum_i m^i s^i$ that is the weighted -average ESG score of market portfolio, $m^i = p^i / \sum_j p^j$ is the weight of the market portfolio in stock i, and the parameter $\lambda \in \mathbb{R}$ determines how informative ESG scores are for future profits. A positive λ means that more ESG friendly firms are also more profitable on average, and a negative λ has the opposite interpretation.

The aim of this work is to find out equilibrium prices p_t and excess returns r_t which are related by

$$r_t^i = \frac{v_t^i + p_t^i}{p_{t-1}^i} - 1 - r^f.$$

In steady-state equilibrium, both the prices and the expected returns are constant in time: $p_t = p$. With this hypothesis, the relationship becomes the more simplified form $r_t^i = \frac{v_t^i}{p^i} - r^f$. The steady-state equilibrium described exists because dividends are i.i.d., ESG scores are constant and the wealth of different investor types is constant (ESG unaware, ESG aware with no ESG utility and ESG aware that have preference for high average ESG score).

2.1.1. Market ESG unaware

The first scenario is when all investors are ESG unaware. They ignore ESG, so the model is a standard CAPM in equilibrium, where the beta of each security expected excess return is $\beta^i = \frac{cov\left(r_t^i, r_t^m\right)}{var(r_t^m)}$.
Following this hypothesis, any security i has steady-state equilibrium price

$$p^i = \frac{\mu^i - \frac{\gamma}{W} cov(v^i, v^m|s)}{r^f}$$

and expected return

$$\mathbb{E}(r_t^i) = \beta^i \mathbb{E}(r_t^m)$$

$$\mathbb{E}(r_t^i|s) = \beta^i \mathbb{E}(r_t^m) + \lambda \frac{s^i - s^m}{p^i}.$$

The price is given by its expected cash flow payoff (μ^i) less a risk premium ($\frac{\gamma}{W}\sigma_{im}$), discounted by a risk-free rate. In this formulation, if an investor ignores ESG scores (s), the expected excess return are driven by market betas. Otherwise, if an investor uses the

ESG scores, stocks returns have alphas relative to the CAPM that depend linearly on ESG. If $\lambda > 0$, which means that a high ESG score is indicative of a high future profit, stocks with ESG scores (s^i) above average (s^m) have higher conditional expected returns than those with below average ESG scores.

2.1.2. Market ESG aware

After that market participants gradually learn about the usefulness of governance and impound it into prices, in this way the scenario becomes that all investors are ESG motivated.

In this situation, the returns can be written as

$$r_t = diag\left(\frac{1}{p^i}\right) v_t - r^f$$

where $diag\left(\frac{1}{p^i}\right)$ is the diagonal matrix with elements $\left(\frac{1}{p^1}, \frac{1}{p^2}, ..., \frac{1}{p^n}\right)$.

In equilibrium, the investors must choose the market portfolio which maximize for Sharpe Ratio among all portfolios with an ESG equal to that of the market, s^m.

To create the optimal portfolio, the investor follows the proposition.

Proposition 2.1. *Given an average ESG score $\hat{s}$, the optimal portfolio is*

$$x = \frac{1}{\gamma}\Sigma^{-1}\left(\mu + \pi(s - 1\hat{s})\right)$$

as long as $x'1 > 0$, where

$$\pi = \frac{c_{1\mu}\hat{s} - c_{s\mu}}{c_{ss} - 2c_{1s}\hat{s} + c_{11}\hat{s}^2}$$

The optimal portfolio is therefore a combination of the risk-free asset, the tangency portfolio, $\Sigma^{-1}\mu$, the minimum-variance portfolio, $\Sigma^{-1}1$, and the ESG-tangency portfolio, $\Sigma^{-1}s$

Where the notation is $c_{ab} = a'\Sigma^{-1}b$ and $\Sigma = var(r|s)$ is the conditional variance-covariance matrix of excess returns.

So the portfolio that the investor wants to buy is

$$x = \frac{1}{\gamma} diag\left(p^i\right)\Sigma^{-1} diag\left(p^i\right) \times \left(diag\left(\frac{1}{p^i}\right)\hat{\mu} - r^f + \pi\left(s - 1s^m\right)\right)$$

with γ risk aversion.

The total wealth invested in each stock is Wx, hence the equilibrium condition is $p = Wx$. In a market where all the investors are ESG friendly, the conditional market beta is $\hat{\beta}^i = \frac{cov\left(r_t^i, r_t^m|s\right)}{var(r_t^m|s)}$. Then any security i has equilibrium price

$$p^i = \frac{\mu^i + \lambda(s^i - s^m) - \frac{\gamma}{W}cov(v^i, v^m|s)}{r^f - \pi(s^i - s^m)}$$

and expected return

$$\mathbb{E}(r_t^i|s) = \beta^i \mathbb{E}(r_t^m|s) - \pi(s^i - s^m)$$

The price of any firm's equity is influenced by its ESG scores in two ways because of the presence both in numerator and denominator of the price expression. The result is that the firm's cost of capital is lower if its ESG score is higher, this implies that high ESG firms make real investments because of their low discount rate.

2.1.3. Mixed market

Another possible scenario in which Pedersen focuses is the one in which there exist all types of investors. In this possibility, many things can happen. If a security has a higher

ESG score, its expected return can be higher. A higher ESG score increases the demand for the stock from the investors that consider the ESG scores in their choices, leading to a higher price and a lower required return. On the other hand, a low ESG score means a lower demand from these investors and so a lower price.

2.2. Portfolio weights based on ESG

A different approach is proposed by Pastor [24]. They argue that owning green stocks can lead to lower expected returns. Pastor's argument highlights the challenges that investors face when trying to balance their desire for sustainability with their need for returns. While investing in green stocks can align with investors' values and social responsibilities, it may not always result in the highest returns. It is important for investors to carefully consider the trade-offs between sustainability and returns, and to develop investment strategies that meet their financial and ethical objectives.

The model is developed considering a single time period, from 0 to 1. $\tilde{r}_n$ denote the return of the nth firm's shares in excess respect to the riskless one r_f. The market is assumed to be composed by N firms.
The excess return is

$$\tilde{r} = \mu + \tilde{\epsilon}.$$

This formulation means that $\tilde{r}$, that is the $N \times 1$ vector of $\tilde{r}_i$, is normally distributed. μ is the equilibrium expected excess return and $\tilde{\epsilon} \sim N(0, \Sigma)$. In addition, each firm n has an "ESG characteristic" g_n which can be positive for green firms and negative for the brown ones.
Let X_i denote a $N \times 1$ vector whose nth element is the fraction of agent i's wealth invested in stock n. Agent i's wealth at time 1 is $\tilde{W}_{1i} = W_{0i}(1 + r_f + X_i'\tilde{r})$, where W_{0i} is the initial wealth of agent i. The utility is considered exponential

$$V(\tilde{W}_{1i}, X_i) = -e^{-A_i \tilde{W}_{1i} - b_i' X_i}$$

where A_i is the agent's absolute risk aversion and $b_i = d_i g$ is a $N \times 1$ vector of nonpecuniary benefits that the agent derives from his stock holdings, g is the $N \times 1$ vector which contains the ESG characteristics of the stock and $d_i \geq 0$ measures the ESG taste of the agent.

He takes into account a four funds portfolio divided in a risk-free portfolio, a market portfolio, an ESG portfolio and, in conclusion, a hedging portfolio. He starts with a two funds portfolio composed by the risk-free and the market ones, then, adding one by one, he composes the final portfolio.

- To individuate the market portfolio, form the utility explained before, the portfolio weights of agent i are computed as

$$X_i = \frac{1}{a} \Sigma^{-1} \left(\mu + \frac{1}{a} b_i \right)$$

where a_i is the risk aversion which is assumed to be the same for each agent, so it's denoted with a. The market portfolio requires that the vector of weights ω_m satisfies:

$$\omega_m = \int_i \omega_i X_i di = \frac{1}{a} \Sigma^{-1} \mu + \frac{\bar{d}}{a^2} \Sigma^{-1} g$$

where $\bar{d} = \int_i \omega_i d_i di \geq 0$ is the average of ESG tastes d_i. Solving for μ, we obtain

$$\mu = a \Sigma \omega_m - \frac{\bar{d}}{a} g$$

Premultiplying by ω_m' gives the market equity premium, $\mu_m = \omega_m' \mu$:

$$\mu_m = a\sigma_m^2 - \frac{\bar{d}}{a}\omega_m' g$$

As we can see, the equity premium depends on $\bar{d}$ through $\omega_m' g$ which is the overall greenness of the market portfolio. Then on the hypothesis of a market portfolio that is ESG neutral which means that $\omega_m' g = 0$ the expected excess return in equilibrium should be calculated as

$$\mu = \mu_m \beta_m - \frac{\bar{d}}{a} g$$

where $\beta_m = \frac{1}{\sigma_m^2}\Sigma\omega_m$ and $a = \frac{\mu_m}{\sigma_m^2}$. This result is important because it means that if $\bar{d} > 0$, then the expected return on stock n is decreasing in g_n. Agents are willing to pay more for greener firms, thereby lowering the firms' expected returns.

- The ESG portfolio is the third portfolio, with the riskless and the market ones. In fact, substituting the value of μ into the expression of X_i, we can find the following equilibrium portfolio weights

$$X_i = \omega_m + \frac{\delta_i}{a^2}\Sigma^{-1} g.$$

The agent i allocates a fraction ϕ_i in the ESG portfolio and a fraction $1 - \phi_i$ is invested in the market portfolio.

The vector of weights is

$$\omega_i = (1 - \phi_i)\omega_m + \phi_i \omega_g$$

where

$$\phi_i = \frac{\frac{\delta_i}{a^2} i' \Sigma^{-1} g}{1 + \frac{\delta_i}{a^2} i' \Sigma^{-1} g}$$

$$\omega_g = \frac{1}{i'\Sigma^{-1}g}\Sigma^{-1}g$$

$\phi_i = 0$ for agents that have average ESG concerns and so $d_i = \bar{d}$ and thus $\gamma_i = d_i - \bar{d} = 0$. In fact, the market portfolio is optimal for the agents with average concerns about ESG but not for those indifferent to ESG, which should tilt away from the market portfolio. If there is no dispersion in ESG tastes in the agents, then all of them hold the market portfolio.

- The hedging portfolio is the last type of portfolio introduced by Pastor because it is important to note that investing in green stocks alone may not be enough to achieve long-term financial goals. Investors may also consider hedging their portfolios against climate related risks, such as extreme weather events, natural disasters, and regulatory changes. Hedging can help investors to mitigate the risks associated with their investments and to protect their portfolios against potential losses.

  Agents whose climate sensitivity is above average go short on hedging portfolio, whereas agents who are below average go long.

  In order to modify the utility are introduced $\tilde{C}$ as the climate at time 1 which is unknown at time 0 and $\bar{c} = \int_i \omega_i c_i di$ as the wealth weighted mean of climate sensitivity across agents.

$$V\left(\tilde{W}_{1i}, X_i, \tilde{C}\right) = -e^{-A_i\tilde{W}_{1i} - b_i'X_i - c_i\tilde{C}}$$

  From the new expression of the utility and considering the assumption of $\bar{c} > 0$, agents dislike low realizations of $\tilde{C}$. In addition, is taken the hypothesis that $\tilde{C}$ is normally distributed

$$\tilde{C} \sim N(0,1)$$

With the introduction of the climate hedging portfolio, the expected excess returns in equilibrium become

$$\mu = \mu_m \beta_m - \frac{\bar{d}}{a} g + \bar{c}(1 - \rho_{mC}^2)\psi$$

where ψ is the vector of climate betas and ρ_{mC} is the correlation between the unexpected market return $\tilde{\epsilon_m}$ and $\tilde{C}$. Expected returns depend on climate betas which represent firms' exposures to nonmarket climate risk and on $\tilde{\epsilon}_m$ which is the unexpected market return. Climate betas are likely to be negative correlated to g_n.

The weights of the four fund portfolio in equilibrium are given by

$$X_i = \omega_m + \frac{\delta_i}{a^2}(\Sigma^{-1} g) - \frac{\gamma_i}{a}(\Sigma^{-1}\sigma_{\epsilon C})$$

with $\gamma_i = c_i - \bar{c}$ and $\sigma_{\epsilon C}$ vector of covariances between $\tilde{\epsilon}_n$ and $\tilde{C}$. As we can see from the equation, the weights of the climate hedging portfolio are proportional to $\Sigma^{-1}\sigma_{\epsilon C}$

But in which kind of stocks someone should invest more for hedging purpose? Determining whether green stocks or brown stocks are better climate hedges is not trivial, since there are sensible economic arguments to support both sides. The argument that green stocks should be used as a hedge against climate risk can be justified through two different channels.

First, the customer channel. If the climate unexpectedly worsens, consumers may become more concerned about climate issues and may start demanding goods and services from greener providers. This demand can be driven by both consumer preferences and government regulations. Negative climate shocks can lead to regulations that favour green providers and penalize brown ones, such as subsidies for green products and taxes or even bans on brown products.

Second, the investor channel. If the climate unexpectedly worsens, investors may have a stronger preference for green holdings. This could be due to stronger public pressure on institutional investors to divest from brown assets.

The evidence suggests the better climate hedges are green stocks. Green firms, as measured by low carbon emissions, outperform brown firms during months with abnormally warm weather, which alerts investors to climate change.

Taking the special case where the climate betas ψ_n are perfectly negative correlated to g_n

$$\psi_n = -\xi g_n$$

where $\xi > 0$ is a constant, so

$$\mu = \mu_m \beta_m - \left[\frac{\bar{d}}{a} + \bar{c}(1 - \rho_{mC}^2)\xi \right] g.$$

.

The alpha CAPM of stock n is thus given by

$$\alpha_n = - \left[\frac{\bar{d}}{a} + \bar{c}(1 - \rho_{mC}^2)\xi \right] g.$$

Green stocks are not only favoured by investors due to their preference for eco-friendly investments, but also because of their potential to mitigate climate risk more effectively. This is one of the reasons why greener stocks have lower CAPM alphas compared to brown stocks, as climate risk presents a compelling argument for green stocks to underperform over the long term.

2.3. What type of stock has the higher return?

There appears to be a paradox between the two claims presented [24, 25]. The former asserts that green stocks have a higher expected return than their less eco-friendly counterparts, while the latter contends that the expected excess return of green stocks is lower than that of brown stocks. However, this aspect requires a more depth analysis, as the underlying assumptions differ.

Pedersen postulates two different scenarios: one where only investors who do not care about the ESG factor are present, and another where everyone is concerned with this parameter. In the former case, the market does not account for the possibility of an economic loss resulting from climate risk. Consequently, green stocks have hidden value that is not detected by the market, enabling investors who are aware of this to buy a stock that will yield more than the market can predict. Conversely, in a market where all investors take the ESG factor into account, the expected excess return of the greenest firms will be lower than that of the browner ones. This is because in a market solely focused on green stocks, demand for brown stocks is greatly reduced, necessitating a higher expected excess return for those stocks. Pedersen also introduces a mixed market, with investors who consider the ESG factor and those who do not, but he does not continue his analysis, as more possibilities arise [25].

Pastor, on the other hand, considers the investor's ESG taste by using it as a variable, thereby not excluding any possible type of investor, as each is driven by their interest in the ESG factor. This analysis also introduces a non-pecuniary return for the investor, who receives a type of premium in their investment through the environmental factor. This premium does not generate a cash flow for the investor but is purely linked to how much an investor prefers to invest in green rather than brown stocks. By hypothesizing this additional premium, the return of green stocks is further reduced, since a portion is due to a non-pecuniary premium [24].

To deepen our understanding of the expected returns associated with ESG factors, we can turn to empirical studies to confirm that firms with higher level of CO_2 emissions have higher returns [3, 4]. Bolton present a regression model defined as

$$RET_{i,t} = a_0 + a_1 LOG(TOTEmissions)_{i,t} + a_2 Controls_{i,t-1} + \mu_t + \epsilon_{i,t}$$

where $RET_{i,t}$ is the stock return of a firm i at time t and the vector of $Controls$ includes some known variables to predict returns but not useful for our analysis and not relevant in the results. The parameter $Emissions$ is a term standing for $SCOPE1$, $SCOPE2$, or $SCOPE3$ emissions, which are the three different sources of emissions used by the Greenhouse Gas Protocol. $SCOPE1$ emissions are direct emissions over one year from establishments that are owned or controlled by the company; these include all emissions from fossil fuel used in production. $SCOPE2$ emissions come from the generation of purchased heat, steam, and electricity consumed by the company. $SCOPE3$ emissions are caused by the operations and products of the company but occur from sources not owned or controlled by the company [4].

Consider that the kind of $SCOPE$ is taken one at a time, so they don't exist in the same model at the same time, but in every model their values have always high value of significance. So the variable of interest, for our aim, is a_1 which takes the following values

VARIABLES	(1)	(2)	(3)	(4)	(5)	(6)
LOG (SCOPE 1 TOT)	0.051**			0.181***		
	(0.022)			(0.044)		
LOG (SCOPE 2 TOT)		0.103**			0.179***	
		(0.039)			(0.052)	
LOG (SCOPE 3 TOT)			0.148***			0.327***
			(0.040)			(0.082)

Table 2.1: Results of the pooled regression with standard errors in parentheses. In columns (4) through (6), industry-fixed effects are additionally included. Here, the results for the natural logarithm of total firm-level emissions are reported.
$***$ 1% significance; $**$ 5% significance; $*$ 10% significance.

A similar analysis with similar results is conducted on the percentage change in carbon total emissions with the regression model

$$RET_{i,t} = a_0 + a_1 \Delta(Emissions)_{i,t} + a_2 Controls_{i,t-1} + \mu_t + \epsilon_{i,t}$$

With the following results

VARIABLES	(1)	(2)	(3)	(4)	(5)	(6)
$\Delta SCOPE1$	0.0718***			0.706***		
	(0.181)			(0.164)		
$\Delta SCOPE2$		0.400**			0.379**	
		(0.150)			(0.143)	
$\Delta SCOPE3$			1.311***			1.303***
			(0.388)			(0.383)

Table 2.2: Results of the pooled regression with standard errors in parentheses. In columns (4) through (6), industry-fixed effects are additionally included. Here, the results for the percentage change in carbon total emissions are reported.
$***$ 1% significance; $**$ 5% significance; $*$ 10% significance.

A positive and statistically significant effect both of total firm-level emissions and of the growth in emissions on stock returns is founded. This analysis support the thesis that stocks of firms with higher total CO_2 emissions and growth in emissions, earn higher returns.

There is, instead, a vast literature that wants to prove that higher ESG factor firms tends to over perform in a long term horizon, but there is no evidence that a portfolio with only high scoring ESG firms maximize Sharpe Ratios [10, 11].

3 | Temperature and pollution

A large asset-pricing literature seeks to explain the dependencies of stock returns based on exposures to aggregate risk factors such as size and book-to-market ratios, or firm-specific risk linked to its own characteristics. One variable that has so far been missing from the analysis is corporate carbon emissions. The concerns over global warming linked to CO_2 emissions from human activity have only recently become a matter of concern. Both the evidence of rising temperature and the policy efforts to curb CO_2 emissions, raise whether carbon emissions represent a material risk today for investors.
Major curbs in CO_2 emissions are likely to be introduced over the next decade. Primarily affected by these curbs are the companies with operations generating high CO_2 emissions, or with activities linked to companies in the value chain that have high CO_2 emissions. Fossil fuels are a critical input to production, so economic growth increases greenhouse gas emissions. Those emissions induce climate change, and climate change has a potentially large negative feedback effect on future economic activity.

The literature tries to approximate a climate model in order to include the impact that increasing temperature has on the economy. A simplified model which links temperature and cumulative emissions is the one where the increase of temperature linearly depends on the cumulative emissions [18].

$$T_t - T_0 \approx \beta \eta_t$$

Where η_t are the cumulative emissions and β is a parameter

This model is clearly oversimplified and we might lose some information. A more sophisticated climate model is the logarithmic relationship between global warming and atmospheric carbon dioxide concentrations [15]:

$$T_t = \eta_t \log\left(\frac{M_t^{\Sigma}}{M^{PI}}\right)$$

where M_t^{Σ} is the total current concentration of carbon dioxide in the atmosphere and M^{PI} is the average pre-industrial concentration. The total concentration is the sum of the pre-industrial average and the amount of atmospheric $C0_2$ caused by human activities, $M_T^{\Sigma} = M^{PI} + M_t$. The pre-industrial average is constant, so it doesn't need a more in-depth study. Different the choice of M_t which dynamics could be expressed in a lot of different ways. Hambel assumes an expression of it as

$$dM_t = M_t[(g_m(t) - \alpha_t)dt + \sigma_m dW_t^m]$$

$W^m = (W_t^m)_{t\geq 0}$ is a standard Brownian motion that models unexpected shocks on the CO_2 concentration that can be caused from volcano eruptions or earthquakes, for example, but they can be the results of the actions of man. σ_m is the volatility of these shocks and is assumed to be constant. The expected increasing of pollution in the expression is expressed by $(g_m(t) - \alpha_t)$ which represents the growth rate if society doesn't take additional actions to reduce emissions, or the business-as-usual drift g_m, minus the new policies to reduce those emissions, or the abatement strategy $\alpha = (\alpha_t)_{t\geq 0}$ [15].

It's important to write the dynamics of pollution because the model aims to find the optimal level of carbon abatement in a stochastic environment where future outcomes are uncertain. In order to accurately estimate the optimal level of abatement, it is necessary to incorporate the dynamics of pollution as it accumulates over time and affects future outcomes.

The model takes into account the fact that pollution has long-lasting effects on the environment, and its level at any given time depends on the historical level of carbon emissions. By explicitly modelling the dynamics of pollution, the model proposed is able to account for the effects of past carbon emissions on future outcomes, and thus provide a more realistic assessment of the optimal level of carbon abatement. This information is essential for policymakers and investors who are making decisions about how to address climate change and reduce carbon emissions.

3.1. Damage function

The damage function is used to capture the relationship between carbon emissions and the resulting economic damage. The damage function plays a crucial role in determining the optimal level of carbon abatement, as it provides a measure of the cost of climate change. By estimating the damage function, we are able to quantify the impact of carbon emissions on economic welfare and use this information to guide the optimal policy response. Specifically, we use the damage function to determine the marginal cost of carbon abatement, which reflects the additional cost of reducing emissions by a small amount. The optimal level of carbon abatement is then determined by equating this marginal cost to the marginal benefit of reduced emissions, which depends on the impact of emissions on economic welfare as captured by the damage function.

The environmental damage function is a strictly increasing function, which takes on values between zero and one [1]. There is a vast literature with the challenging aim of quantifying the impact of climate change on economic output. This function can be expressed both in dependence of the pollution or the temperature, thanks to the link between the two variables. Some examples of damage functions could be the quadratic damage function that is commonly used in climate models and assumes that damages from climate change increase quadratically with temperature. However, some studies have extended

this function to include a dependence on pollution.
A function proposed by Martin Weitzman assumes that there is a small but finite probability of catastrophic damage from climate change [29]. This model incorporates dependence on pollution by assuming that the probability of catastrophic damage increases with the level of atmospheric carbon concentration. Another function used is the non-linear damage function, where damages from climate change increase non-linearly with temperature. This function also includes dependence on pollution by considering the effects of different levels of greenhouse gas emissions on the economic output.

A different damage function is first proposed by Golosov [13], the exponential damage function. He formulates the output of non-financial incomes as

$$F_{1t}(K_{1t}, N_{1t}, E_{1t}, S_t) = (1 - D(\gamma_t S_t))\tilde{F}_{1t}(K_{1t}, N_{1t}, E_{1t})$$

S_t is the stock of CO_2 in the atmosphere and D is the damage function that translate such parameter in economic loss. captures the non-financial income in a scenario with no pollution, in fact it does not depend on S_t but only on the parameters K_{1t}, N_{1t} and E_{1t} which are respectively the capital, labour and energy inputs which influence the output. The costs of climate change are not determined precisely. This uncertainty is summarized by a stochastic variable γ which parameterizes the dependence of output damages on atmospheric carbon concentration.

$$D_t(S) = 1 - \exp\left(-\gamma_t\left(S - \bar{S}\right)\right)$$

with $\bar{S}$ which is the pre-industrial CO_2 concentration in the atmosphere.

This function assumes that damages increase exponentially with carbon dioxide concentration in the atmosphere. This function predicts that the cost of carbon emissions will increase over time as atmospheric carbon concentration increases.

The damage function can be written as

$$D(\eta_t) = 1 - \exp(-\psi\eta_t)$$

where η_t is the amount of carbon in the atmosphere in excess of prehistoric ages and ψ is a coefficient. Due to the fact that this coefficient is estimated to be near to zero, $\psi = 5.3 \cdot 10^{-5}$ [13], we can approximate the damage function at the Taylor's first order, and it becomes linear in the pollution

$$D(\eta_t) \approx \psi\eta_t$$

In our work, we take into account this kind of damage function because it is able to capture the non-linear and potentially catastrophic effects of climate change. This is especially important when considering the potential long-term and irreversible damage that could occur due to climate change.

3.2. Social Cost of Carbon

The Social Cost of Carbon (SCC) is an important concept in environmental economics that measures the economic cost associated with each additional ton of carbon dioxide emitted into the atmosphere. It is the estimate of the dollar value of the long-term damage caused by one additional ton of CO_2 emissions, including impacts on human health, agriculture, infrastructure, and the environment. The SCC is used to inform public policy decisions related to climate change mitigation and adaptation.
Calculating the SCC is a complex task that involves modelling the costs and benefits of carbon emissions over long time horizons. Several agencies in the US government, such

as the Environmental Protection Agency (EPA) and the Department of Energy (DOE), have developed models to estimate the SCC. These models incorporate factors such as the discount rate, the time horizon, the economic impact of damages, and the uncertainty of future climate change impacts. The SCC is typically expressed in terms of a dollar value per metric ton of CO_2 emissions, and it varies depending on the specific assumptions and inputs used in the modelling process.

The SCC has important implications for climate policy, as it provides a way to compare the costs and benefits of different policy options. For example, a carbon tax or cap-and-trade system that charges emitters for each ton of CO_2 they release into the atmosphere can help reduce emissions and generate revenue that can be used to fund climate mitigation and adaptation efforts. The SCC can be used to estimate the economic benefits of such policies, such as the avoided damages from climate change.

One of the challenges in using the SCC to inform policy decisions is the uncertainty associated with modelling the impacts of climate change over long time horizons. There is also debate over the appropriate discount rate to use in calculating the SCC. A higher discount rate, which reflects the preference for present consumption over future consumption, would result in a lower SCC estimate, while a lower discount rate would result in a higher SCC estimate.

Another challenge is the potential for the SCC to vary depending on the geographic region and sector being analysed. For example, the SCC estimate may be higher for regions that are more vulnerable to the impacts of climate change, such as low-lying coastal areas, or for sectors that are more carbon-intensive, such as the energy or transportation sectors. This variability can complicate policy decisions, as policymakers may need to tailor policies to specific regions or sectors to achieve their desired outcomes [6, 23, 26].

The SCC is an important tool for policymakers and analysts working to address climate change. It provides a way to estimate the economic costs associated with carbon emissions and to compare the costs and benefits of different policy options. As the global community continues to grapple with the challenges of climate change, the SCC will likely play an increasingly important role in shaping policy decisions and driving progress towards a more sustainable future.

The social cost of carbon could be defined as the marginal rate of substitution between carbon dioxide emission and GDP [16]

$$SCC_t = -\frac{\frac{\delta V_t}{\delta E_t}}{\frac{\delta V_t}{\delta Y_t}} \tag{3.1}$$

where V_t is the society's utility index at time t, Y_t is the output and E_t is the anthropological emissions of carbon dioxide.
This formulation explains that the SCC measures the increase in current GDP that is required to compensate for economic damages caused by a marginal increase of emissions at time t. In literature, many authors follow the same idea (Traeger 2014).
Gross Domestic Product (GDP) is a measure of the total value of all goods and services produced in a country within a specific period. It's commonly used to measure a country's economic performance and growth. GDP is calculated by summing up the final value of all goods and services produced in a country, including consumer goods, investments, government spending, and exports.

This form of SCC is a simplified expression, but that clearly explain the role of the SCC. The Social Cost of Carbon can serve as a carbon tax, as it provides a monetary value for the damages caused by each additional ton of carbon dioxide emitted into the atmosphere. This value can be used to inform policy decisions and market mechanisms, such as cap-and-trade programs, that aim to reduce greenhouse gas emissions. In other words, the Social Cost of Carbon serves as a proxy for the costs imposed on society by carbon emissions, and can be incorporated into decision-making processes to account for the negative externalities of climate change. As such, it represents an important tool for policymakers and businesses looking to mitigate the impacts of climate change and transition to a low-carbon economy.

More sophisticated formulations of SCC exist in the literature [27]

$$SCC(0) = -\int_0^\infty e^{-\delta_S t} R(t) \frac{\partial \psi}{\partial S}(t) \int_t^\infty \epsilon e^{\epsilon(t-s)} R(s;t) \frac{\partial F}{\partial T}(s) ds dt. \tag{3.2}$$

where $R(s;t) = \frac{R(s)}{R(t)}$ is the consumption discount factor between time t and s, δ_S is the depreciation rate with S that is the CO_2 stock over the pre-industrial level, $\psi(S)$ is the physical long-run equilibrium level, ϵ is the rate at which temperatures adjust, F depends on capital K, emissions E, and the global average surface temperature T, and on time t that may capture technological development.

The main problem in calculating the SCC is that the current range of values of the Social Cost of Carbon depends on the country or organization that calculates it and the methodology used. For example, in the United States, the Social Cost of Carbon was recently reviewed and increased by the Biden administration from $43 to $51 per ton of CO_2 with some studies that estimate it to $185 per ton of CO_2[1], while Trump's administration considered a range of $3-$5 in the USA. Other countries, such as the United Kingdom, have similar assessments of the social cost of carbon, while other countries may have very different estimates.
The value of the Social Cost of Carbon can vary depending on the methodology used for calculation [14]. Some authors also say that SCC's variance grows faster than its mean [5].

[1] www.brookings.edu

4 | Model Asset Pricing and Portfolio Management

4.1. Starting model

In this section, we introduce the model developed by Fischer that is the starting point of our work [9]. The model combines Merton's portfolio analysis and derivation of equilibrium return with Nordhaus' model of economic damages due to climate change[20–22]. In this model, the expected returns of stocks are influenced by both traditional risk factors and a climate risk factor, which is related to the expected damage from climate change.

4.1.1. Model setup

The model represent an investor who has access to n risky assets, a riskless asset with constant rate of return r and an infinite time horizon. Under these conditions, the price of the ith risky financial asset is a stochastic process of the form

$$\frac{dS_i}{S_i} = R_i dt + \sigma_i' d\mathbf{Z}_t \tag{4.1}$$

where $\mathbf{Z}_t$ is a $n \times 1$ vector of Brownian motions, $\mathbf{R}$ is the $n \times 1$ vector of instantaneously expected returns and $\mathbf{\Omega}^{\frac{1}{2}} = [\gamma_1', \gamma_2', ..., \gamma_n']'$. $\mathbf{\Omega} = \mathbf{\Omega}^{\frac{1}{2}} \left(\mathbf{\Omega}^{\frac{1}{2}}\right)'$ is the instantaneous variance-covariance matrix of returns.

While Merton [21] considers expected utility, in his work, Fischer [9], uses the more general Stochastic Utility framework [7, 8] with a normalized aggregator function f(C, V) [17].

$$V_t = \mathbf{E}\left[\int_t^{\infty} f(C_s, V_s)ds\right] \tag{4.2}$$

with $f(C_s, V_s) = U(C) - \tilde{\rho}J$, and $U(C)$ is the utility function [8].

With the dynamics of the prices described before, the investor's wealth is:

$$dW_t = (rW_t + \mathbf{w}_t'(\mathbf{R} - r\mathbf{1})W_t + Y_t - C_t)dt + \mathbf{w}_t'\mathbf{\Omega}^{\frac{1}{2}}W_t d\mathbf{Z}_t \tag{4.3}$$

where $\mathbf{1}$ is a $n \times 1$ unit vector, $\mathbf{w}_t$ is the vector of weights and Y_t is the non-financial income. This term is crucial in all the literature of climate change because it captures the background risk associated with the climate change induced by natural disasters or climate related events that bring to an economic loss.
For every investment in a specific firm i, there is an output of CO_2 equivalent to the portion of wealth invested in that firm, $\omega_{i,t}W_t$, multiplied by $k_i \geq 0$. If a firm reduces the stock of carbon in the atmosphere, theoretically this value could even be negative, but it's an unrealistic scenario. For that reason, the parameter k_i couldn't be negative.

For these reasons we can write the costs, in terms of CO_2 emissions, of an investor portfolio as

$$\mu_t = \mathbf{w}_t'\mathbf{k}W_t \tag{4.4}$$

where $\mathbf{k}$ is the vector of k_i referred to each firm in the portfolio.

This model does not consider the climate change as an exogenous process, but the decisions of the investors act directly on the amount of CO_2 in the atmosphere.

The amount of carbon dioxide in the world is represented by the variable η_t, which capture the amount of carbon in the atmosphere in excess of pre-industrial ages.
Many authors in literature present the amount of CO_2 considering the beginning of industrial ages as starting point with null value, on the other hands there are others that consider also the amount in the pre-industrial era. Fisher takes in addition only the pollution caused by industrial evolution and describes its dynamic as

$$d\eta_t = (\mu_t - \gamma\eta_t)dt + b_1(\eta)'dZ_t + b_2(\eta)'dB_t \tag{4.5}$$

where B_t is a $m \times 1$ vector of independent Brownian motions mutually independent of the Brownian motions in Z_t. The vectors $b_1(\eta)$ and $b_2(\eta)$ determine stochastic fluctuations in the stock of CO_2. The process Z_t captures components that affect both the return dynamics and the stock of CO_2. If there is no human activities ($\mu_t = 0$) the carbon concentration will converge back to its prehistoric level ($\eta = 0$).

The investor earns both financial and non-financial incomes. Non-financial income (Y_t) is the link between climate and financial. In fact, a certain amount of $C0_2$ means a corresponding loss through the damage function. Increasing η_t determines decreasing, Y_t which depends on the damage function D

$$Y_t = A_0(1 - D(\eta_t)) \tag{4.6}$$

where A_0 is the non-financial income when $\eta_t = 0$.

The damage function $D\,(\eta)$ is a strictly increasing function which takes on values between zero and one. In this paper the authors take into account the exponential damage function [13], but in literature is a very discussed matter the expression of the damage function and which quantifies the effects of climate change as economic output.

$$D(\eta_t) = 1 - exp(-\psi\eta_t) \tag{4.7}$$

where $\psi = 5.3 \cdot 10^{-5}$.

Because the small value of ψ the damage function can be approximated at the first order Taylor as

$$D(\eta_t) \approx \psi\eta_t \tag{4.8}$$

Combining these model elements gives rise to the Hamilton-Jacobi-Bellman (HJB) equation [7, 8]

$$\begin{aligned} 0 = \max_{C,w}\{f(C,J) + [rW + w'(R - r1)W - A_0\psi\eta + (A_0 - C)]J_W + [w'kW - \gamma\eta]J_\eta \\ + \frac{1}{2}J_{WW}w'\Omega wW^2 + J_{W\eta}w'\Omega^{\frac{1}{2}}b_1(\eta)W + \frac{1}{2}J_{\eta\eta}(b_1(\eta)'b_1(\eta) + b_2(\eta)'b_2(\eta))\} \end{aligned} \tag{4.9}$$

where the indexes denote partial derivatives and $J(W,\eta)$ is the value function, indicating the maximum utility given wealth W and pollution level η. Consider that the function J refers to a maximum consumption, while the utility process V is for any permitted consumption process.

4.1.2. Individual portfolio optimization

The optimal consumption and portfolio weights can be determined by solving the problem described in equation (9) in the first order condition.

Proposition 4.1. *The optimal consumption follows from*

$$f_C = J_W \tag{4.10}$$

While the investor can manage his portfolio by using the weights given by the equation

$$w = \Omega^{-1}(R - r1)\theta + \Omega^{-1}k\kappa + \Omega^{-1}\Omega^{\frac{1}{2}}b_1 H \tag{4.11}$$

with $\theta = -\frac{J_W(W,\eta)}{J_{WW}(W,\eta)W}$, $\kappa = -\frac{J_\eta(W,\eta)}{J_{WW}(W,\eta)W}$ *and* $H = -\frac{J_{W\eta}(W,\eta)}{J_{WW}(W,\eta)W}$

Equation (10) can be divided in 3 different parts:

- myopic demand
- externality
- hedging

The myopic demand is characterized by the first term of the equation is the one described in Merton [19, 21].
The second term describes how pollution produced by the specific asset induce a negative externality on the consumption. Weights associated with the individual assets are related to the technological parameter k capturing the CO_2 output per dollar invested.
The last term refers to the hedging part of the optimal portfolio. The investor wants to hedge his portfolio against unexpected changes in η.

As a result the investor wants to invest in a four-fund separation portfolio, as the literature suggests, composed by

- a risk-free asset
- a myopic fund
- an externality fund
- a hedge fund

It's important to note that the myopic fund corresponds to the usual tangency portfolio emerging from mean-variance analysis.

Proposition 4.2. *The representative investor is indifferent between investing in the $n+1$ assets described above and investing in four funds:*

1. *a risk-free asset*
2. *a first portfolio of risky assets with weights,*

$$w_{myopic} = \frac{\Omega^{-1}(R - r1)}{1'\Omega^{-1}(R - r1)} \tag{4.12}$$

3. *a second portfolio of risky assets with weights*

$$w_{ext} = \frac{\Omega^{-1}k\kappa}{1'\Omega^{-1}k\kappa} \tag{4.13}$$

4. *a third portfolio of risky assets with weights*

$$w_{hedge} = \frac{\Omega^{-1}\Omega^{\frac{1}{2}}b_1 H}{1'\Omega^{-1}\Omega^{\frac{1}{2}}b_1 H} \tag{4.14}$$

In this model, the investor overweights the green companies and punishes the brown ones by shorting their assets.

4.1.3. Equilibrium asset pricing implications

The equilibrium risk premiums of risky assets 1 through $(n-2)$ can be written as linear combinations of the market risk premium and the risk premiums of assets $(n-1)$ and n, respectively. Of course, the ordering of risky assets is arbitrary, and one can create assets as linear combinations of the original assets, so it suffices that there are two assets or portfolios of assets that satisfy the invertibility condition.

Proposition 4.3. *Suppose the $(n-1)$th asset's returns have an instantaneous covariance of $\sigma_{(n-1)\eta}$ with changes in η, while the nth asset's returns have an instantaneous covariance* *of $\sigma_{n\eta}$ with changes in η. We also define the 3 × 3 matrix*

$$\Xi = \begin{bmatrix} \sigma_M^2 & k_M & \sigma_{M\eta} \\ \sigma_{M(n-1)} & k_{n-1} & \sigma_{(n-1)\eta} \\ \sigma_{Mn} & k_n & \sigma_{n\eta} \end{bmatrix} \tag{4.15}$$

where M stands for the market portfolio and $k_M = w'_M k$ (and the rest of the notation is self-explanatory). Provided that Ξ^{-1} exists, the equilibrium, conditionally expected return of assets $i = 1, 2, ..., (n-2)$ can be expressed as

$$R_i - r = \begin{bmatrix} \sigma_{iM} & k_i & \sigma_{i\eta} \end{bmatrix} \Xi^{-1} \begin{bmatrix} R_M - r \\ R_{n-1} - r \\ R_n - r \end{bmatrix} \tag{4.16}$$

The analysis on the expected returns aims to provide investors with a framework to evaluate the trade-off between financial returns and environmental impact,

$$R_i - r = \begin{bmatrix} \sigma_{iM} & k_i & \sigma_{i\eta} \end{bmatrix} \begin{bmatrix} \frac{1}{\theta} \\ -\frac{\kappa}{\theta} \\ -\frac{H}{\theta} \end{bmatrix} \tag{4.17}$$

This result shows that an investor is prepared to accept lower risk premium to hedge the increase of pollution. This fact is justified by $-\frac{H}{\theta} < 0$ and $-\frac{\kappa}{\theta} > 0$. The first depends on the effect of wealth on indirect utility that is increasing in pollution, so $J_{W\eta} > 0$. The second one derive from $J_W > 0$ and $J_\eta < 0$. A lower risk premium characterizes stocks that are positively correlated to the amount of carbon in the atmosphere.

When the carbon in the atmosphere (η) is not correlated with traded assets, it is impossible to use them to hedge the uncertainty associated with η. In this scenario, the nth asset has a beta of zero with the market ($\sigma_{nM} = 0$), and investing in the nth asset induces carbon emissions ($k_n > 0$). The risk premium of assets 1 to $n-1$ can be calculated using the given formula

$$R_i - r = \frac{1}{\theta}\sigma_{iM} - \frac{\kappa}{\theta}k_i. \tag{4.18}$$

Focusing on the risk premium of the market and of the asset n, we obtain a system of equations:

$$\begin{cases} R_i - r = \frac{1}{\theta}\sigma_{iM} - \frac{\kappa}{\theta}k_i \\ R_M - r = \frac{1}{\theta}\sigma_M^2 - \frac{\kappa}{\theta}k_M \\ R_n - r = \frac{\kappa}{\theta}k_n \end{cases} \tag{4.19}$$

It's important to note that taking the last equation and setting $k_n = 1$ is obtained the Social Cost of Carbon [28]. Instead of introducing a carbon tax, this formulation gives an increased cost of capital.

Solving the system of equations, the result is

$$R_i - r = \left(\frac{k_i - \beta_i k_M}{k_n}\right)(R_n - r) + \beta_i(R_M - r) \tag{4.20}$$

Where $\beta_i = \frac{\sigma iM}{\sigma_M^2}$.

If a market as a k_i that is grater than $\beta_i k_m$, which means that an asset has more emissions than what is suggested by its market exposure, it's α_i relative to the CAPM is positive. That's another evidence that the brown stocks have higher returns in respect to the green ones [3, 4].

4.1.4. A green-brown dichotomy

In this section, we consider the presence of two risky assets, a green asset and a brown asset with a larger carbon footprint ($k_g > 0$ and $k_g < k_b$). It is assumed that the returns of the two assets are independent of each other, and each asset has an instantaneous variance of the return of σ_i^2, where $i \in g, b$. In addition, the representative agent also invests in a short-term risk-free asset with an instantaneous return of r, where $0 < r < R_i$, and has a zero carbon impact ($k_f = 0$). The total amount of investments is divided among these three financial instruments ($w_g + w_b + w_f = 1$). The relationship between stochastic carbon and return fluctuations is defined as

$$\mathbf{b}_1(\eta) = \sigma_\eta \cdot \eta \cdot \begin{bmatrix} \rho_{g\eta} \\ \rho_{b\eta} \end{bmatrix} \tag{4.21}$$

where $\rho_{g\eta}^2 + \rho_{b\eta}^2 = 1$, and is assumed that $\mathbf{b}_2 = 0$.

Assuming a Stochastic Differential Utility [7, 8], the HJB equation can be written in scalar notation as

$$\begin{aligned} \widetilde{\rho} J \ln([1-\gamma]J) = \underset{C,\omega_g,\omega_b}{max} \{ & \widetilde{\rho}(1-\gamma) J \ln(C) \\ & + [rW + \omega_g(R_g - r)W + \omega_b(R_b - r)W - A_0\psi\eta + A_0 - C]J_W \\ & + [(\omega_g k_g + \omega_b k_b)W - \delta\eta]J_\eta + \frac{1}{2}J_{WW}(\omega_g^2\sigma_g^2 + \omega_b^2\sigma_b^2)W^2 \\ & + J_{W\eta}(\omega_b\sigma_b\sigma_\eta\rho_{b\eta} + \omega_g\sigma_g\sigma_\eta\rho_{g\eta})\eta W + \frac{1}{2}J_{\eta\eta}\sigma_\eta^2\eta^2 \} \end{aligned} \tag{4.22}$$

Proposition 4.4 (Optimal rules in two asset example). *Optimal consumption is given by:*

$$C = \tilde{\rho} X \tag{4.23}$$

where $\tilde{\rho}$ is the rate of time preference and the state $X = W + \frac{A_0}{r} + \xi\eta$. The portfolio weight allocated to the green asset is

$$\omega_g = \frac{R_g - r + \xi k_g}{\gamma \sigma_g^2} \frac{X}{W} - \frac{\xi\eta}{W} \frac{\sigma_\eta}{\sigma_g} \rho_{g\eta} \tag{4.24}$$

and the portfolio weight allocated to the brown asset amounts to

$$\omega_b = \frac{R_b - r + \xi k_b}{\gamma \sigma_b^2} \frac{X}{W} - \frac{\xi\eta}{W} \frac{\sigma_\eta}{\sigma_b} \rho_{b\eta} \tag{4.25}$$

The valuation of the environmental externality is given by

$$\xi = \frac{1}{2a}(-b + \sqrt{b^2 - 4ac}) < 0 \tag{4.26}$$

where $a = \sigma_\eta \left[\frac{k_g \rho_{g\eta}}{\sigma_g} + \frac{k_b \rho_{b\eta}}{\sigma_b}\right]$, $b = r + \delta + \sigma_\eta \left[\rho_{b\eta} S_b + \rho_{g\eta} S_g\right]$ and $c = A_0 \psi$ with $S_i = \frac{R_i - r}{\sigma_i}$

The determination of consumption is contingent upon two key factors: the individual's financial wealth (W) and the current value of non-financial income, adjusted for negative environmental externalities and discounted at the risk-free rate. In addition, the state variable X, which drives the consumption decision, is impacted by the carbon concentration level (η) multiplied by a negative factor ($\xi < 0$). Thus, heightened levels of pollution serve to decrease the optimal consumption level for the investor.

The influence of carbon concentration is not confined to the consumption decision alone,

as it also affects the myopic demands for risky assets that are determined by the state variable X. Consequently, the rising incidence of climate change will prompt individuals to adopt a more conservative investment strategy, with a reduced appetite for risky assets across the board.
Conversely, there exists a countervailing effect that operates via the hedging term. If the correlation coefficient $\rho_{i\eta}$ exceeds a certain threshold value ($\rho_{i\eta} > 0$), investors will opt to hold a larger proportion of risky assets, including those that are classified as brown assets. It's important also to note that the hedging demand does not depend on the coefficient of rick aversion γ. It can be notice that from the above expression, for a company with a tiny or even neutral carbon footprint ($k_g \simeq 0$) but with a negative correlation with η, the hedging part tends to make the investor holding less stocks of this firm [12].

A deeper analysis of ξ can be done to see the dependences of this term on its different factors.
Assuming $\sigma_\eta \simeq 0$ the absolute value of ξ is decreasing in δ, in fact, when this value is high, which means a high absorption capacity of the ecosystem, the weight attached to the carbon concentration is lower.

$$\lim_{\sigma_\eta \to 0} \xi = -\frac{A_0 \psi}{r + \delta}. \tag{4.27}$$

With the hypothesis that carbon concentration does not affect non-financial income, the investor does not care about it, in fact implies $\xi = 0$ and so that the value function does not depend on η.

The general asset pricing can be rewritten as

$$R_i - r = \sigma_{im} \gamma \frac{W}{X} - k_i \xi + \sigma_{i\eta} \gamma \frac{\xi}{X} \tag{4.28}$$

The first term is characterized by the relative risk aversion γ, the second one is the negative externality of carbon emissions captured by ξ, the third one is the hedging motive. This last term, dominated by $\sigma_{i\eta}$, has the influences of both the risk aversion γ and the negative externality ξ.

From the last formulation, there can be given to ξ the interpretation of Social Cost of Carbon [28]

$$SCC = -\frac{J_\eta}{f_C} = -\frac{J_\eta}{J_W} = -\xi > 0 \tag{4.29}$$

4.2. Model integration with ESG factor and abatement strategy

4.2.1. Model setup

As in the starting model, the investor can choose from n risky assets and a riskless one. We first want to present the dynamics of the price of asset i

$$\frac{dS_i}{S_i} = (R_i - \phi_i \eta_t)dt + \sigma_i' dZ_t \tag{4.30}$$

We add a negative dependence on η by a parameter ϕ_i, associates to the asset i, which means the intensity of pollution on its return.

We consider the Stochastic Utility framework [7, 8] with f(C, V) as normalized aggregator function.

$$V_t = \mathbf{E}\left[\int_t^\infty f(C_s, V_s)ds\right] \tag{4.31}$$

where $f(C_s, V_s) = U(C) - \tilde{\rho}J$, and $U(C)$ is the utility function

With the dynamics of the prices described before, the investor's wealth now becomes

$$dW_t = (rW_t + w_t'(R - \Phi\eta_t - r1)W_t + Y_t - C_t)dt + w_t'\Omega^{\frac{1}{2}}W_t dZ_t \tag{4.32}$$

The investor's wealth has in its dynamics the additional influence of the CO_2 concentration.

There is a vast literature that tries to predict the dynamics of the pollution. We want to joint the formulation of pollution of Fischer [9] with the one of Hambel [15] introducing the growth rate g_t and the abatement strategy α_t. In other words, g_t is the growth rate if society does not take additional actions to reduce carbon dioxide emissions, α_t is the emissions' reduction which derives from policies introduced by the society.

$$d\eta_t = (\mu_t + (g_t - \gamma)\eta_t - \alpha_t\eta_t)dt + b_1(\eta)'dZ_t + b_2(\eta)'dB_t \tag{4.33}$$

As before, we consider the vector of independent Brownian motions B_t, mutually independent of the Brownian motions in Z_t. The vectors $b_1(\eta)$ and $b_2(\eta)$ determine stochastic fluctuations in the stock of CO_2.
We introduced g_m in the expression of pollution capturing the business as usual drift and α_t as control variable to model the abatement strategy used to decrease the concentration of CO_2. This value can drastically change the evolution of the pollution. An $\alpha = 0$ strategy means that the business continues without additional activities to reduce the CO_2. The value of η is strongly affected by the abatement strategy.

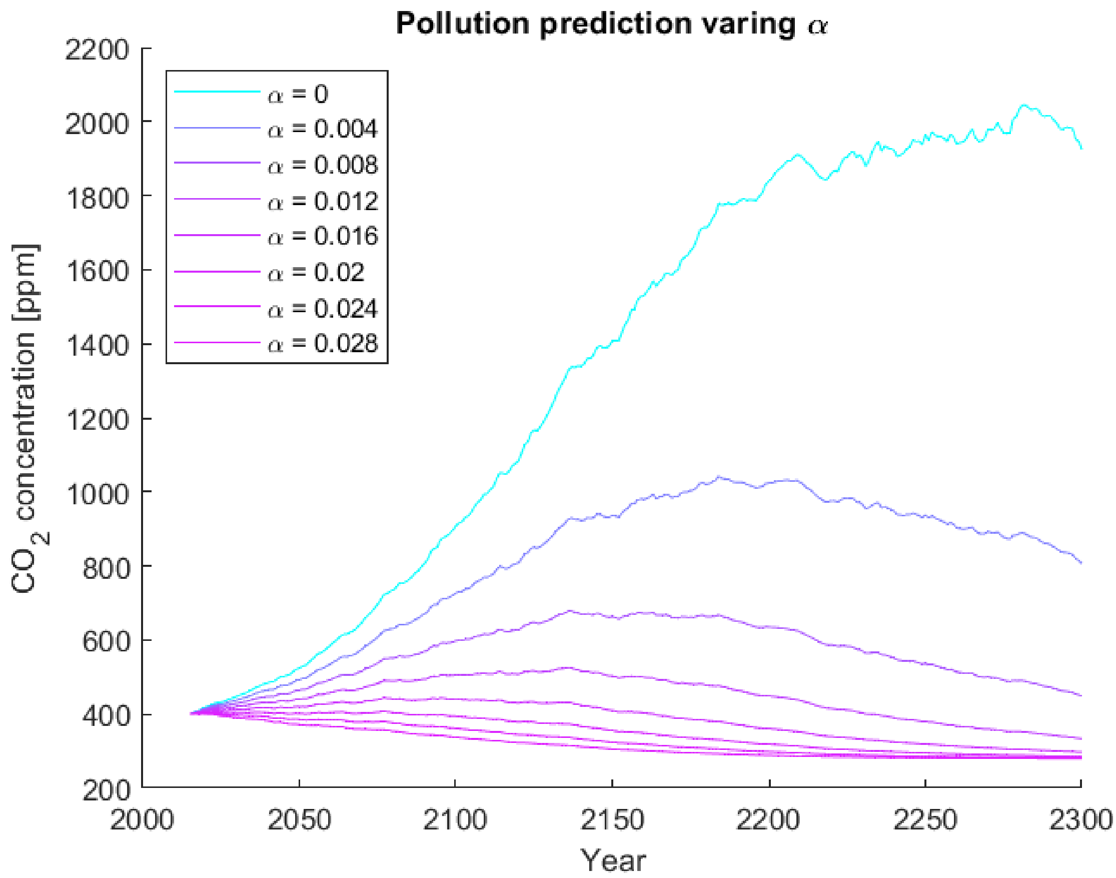

Figure 4.1: Comparison on the evolution of the pollution considering different $\alpha > 0$

To find the actual α, we make the hypothesis that in the last 7 years[1] the alpha can be assumed as constant, so we take the α that minimizes the quadratic error with the values in last years. In addition, the volatility of the pollution is simplified in the value of literature $\sigma_\eta = 0.78\%$.

[1]We choose to start the comparison from 2016 because Hambel, who's dynamics of pollution is the one at which we inspire, considers the hypothesis that 2015 is the t_0 of the model

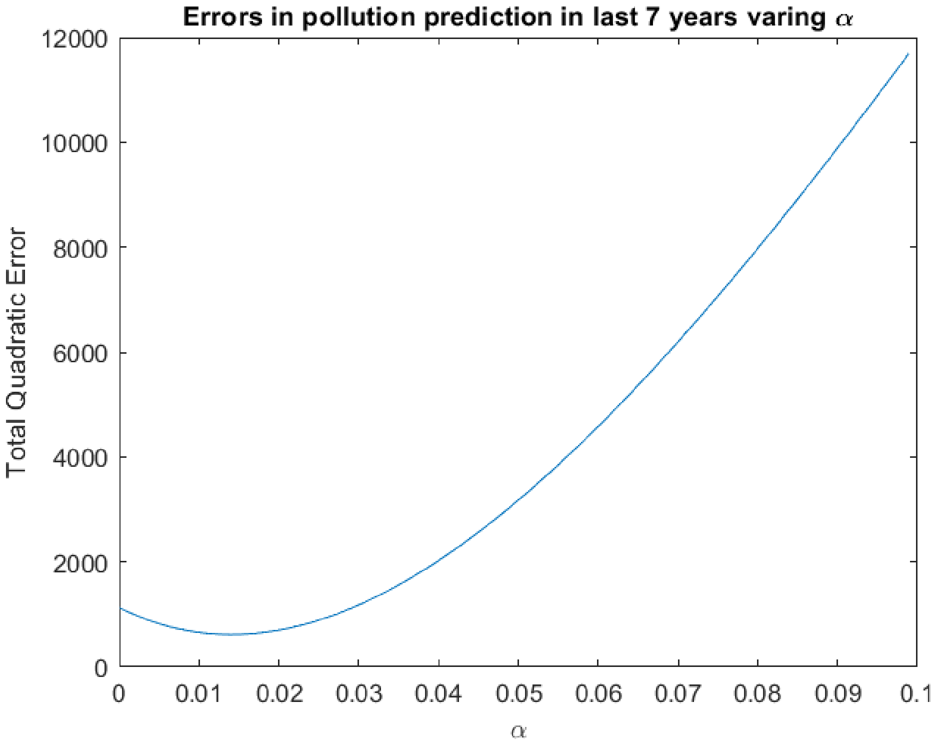

Figure 4.2: Sum of quadratic errors of different α predictions

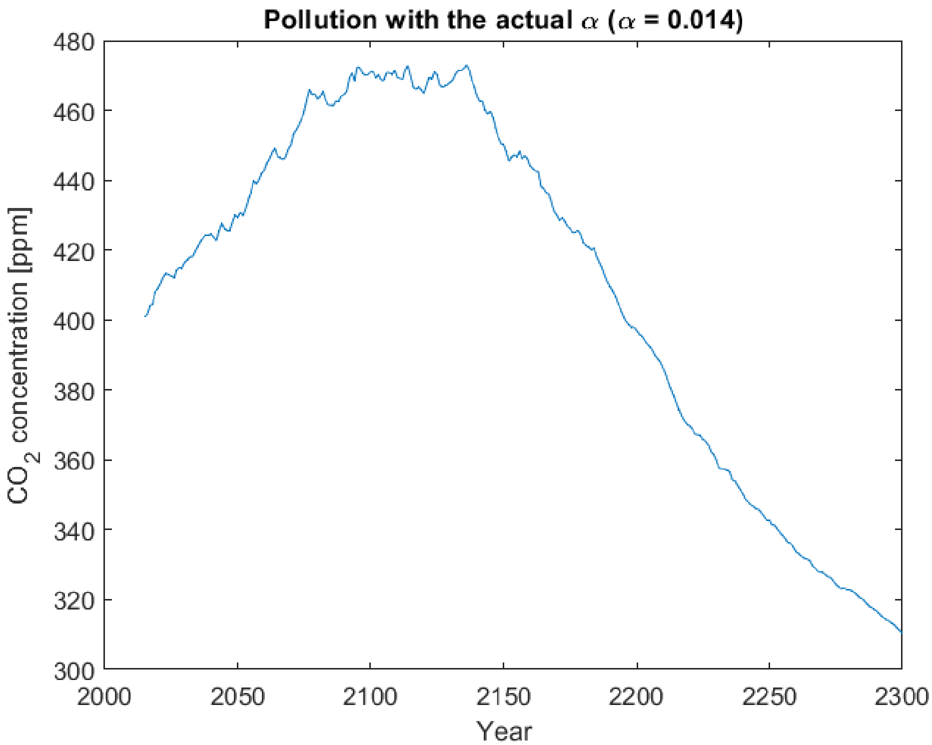

Figure 4.3: Evolution of pollution up to 2300

From the non-financial aspect, we maintain the exponential damage function [9, 13].

$$Y_t = A_0(1 - D(\eta_t)) \tag{4.34}$$

$$D(\eta_t) = 1 - exp(-\psi\eta_t) \tag{4.35}$$

The different formulation of stochastic prices with the dependences both in volatility and drift on the pollution, combined to the new dynamics of concentration of CO_2 in the atmosphere, results in the HJB equation [7, 8]

$$\begin{aligned} 0 = \max_{C,w,\alpha}\{f(C,J) &+ [rW + w'(R - \Phi\eta_t - r1)W - A_0\psi\eta + (A_0 - C)]J_W \\ &+ [w'kW + (g-\gamma)\eta - \alpha]J_\eta + \frac{1}{2}J_{WW}w'\Omega wW^2 + J_{W\eta}w'\Omega^{\frac{1}{2}}b_1(\eta)W \\ &+ \frac{1}{2}J_{\eta\eta}(b_1(\eta)'b_1(\eta) + b_2(\eta)'b_2(\eta))\} \end{aligned} \tag{4.36}$$

4.2.2. Individual portfolio optimization

The optimal consumption and portfolio weights now comes from the new HJB equation. So the new proposition becomes

Proposition 4.5. *The optimal consumption follows from*

$$f_C = J_W \tag{4.37}$$

The optimal abatement strategy is given by

$$b_1(\eta)_\alpha = \frac{J_W}{J_{W\eta}w'\Omega^{\frac{1}{2}}W + J_{\eta\eta}b_1(\eta)} \tag{4.38}$$

While the investor can manage his portfolio by using the weights given by the equation

$$w = \Omega^{-1}(R - \Phi\eta - r1)\theta + \Omega^{-1}k\kappa + \Omega^{-1}\Omega^{\frac{1}{2}}b_1 H \tag{4.39}$$

with $\theta = -\frac{J_W(W,\eta)}{J_{WW}(W,\eta)W}$, $\kappa = -\frac{J_\eta(W,\eta)}{J_{WW}(W,\eta)W}$ *and* $H = -\frac{J_{W\eta}(W,\eta)}{J_{WW}(W,\eta)W}$

The result consists again in a four-fund separation portfolio, as the literature suggests, composed by

- a risk-free asset
- a myopic fund
- an externality fund
- a hedge fund

Proposition 4.6. *The representative investor is indifferent between investing in the $n+1$ assets described above and investing in four funds:*

1. *a risk-free asset*
2. *a first portfolio of risky assets with weights,*

$$w_{myopic} = \frac{\Omega^{-1}(R - r1 - \Phi\eta)}{1'\Omega^{-1}(R - r1 - \Phi\eta)} \tag{4.40}$$

3. *a second portfolio of risky assets with weights*

$$w_{ext} = \frac{\Omega^{-1}k\kappa}{1'\Omega^{-1}k\kappa} \tag{4.41}$$

4. *a third portfolio of risky assets with weights*

$$w_{hedge} = \frac{\Omega^{-1}\Omega^{\frac{1}{2}}b_1 H}{1'\Omega^{-1}\Omega^{\frac{1}{2}}b_1 H} \tag{4.42}$$

The myopic term does not refer to the usual tangency portfolio from the mean-variance analysis, now it takes in addition also the sustainability of the firm trough its ESG factor. The investor's aim in this fund is focused on short term returns, so, as literature suggests (see Section 2.3), he gives more importance to the brown assets instead of the green ones. This fund is apparently the only fund that changes his formulation, but it's not. In fact, the hedging term also changes because of the dependence on pollution of the parameter $b_1(\eta)$. Now the society can control the evolution of b_1 with the abatement strategy, driving the hedging funds of investors.

4.2.3. Equilibrium asset pricing implications

The equilibrium model resulting from the new formulation differs from Fischer's outcome in that it accounts not only for the returns of individual assets and risk-free assets, but also for the sustainability component of each asset. This modification alters the equilibrium model, which then integrates with the formulations of Pedersen and Pastor by introducing the environmental factor into the equilibrium returns [24, 25].

Proposition 4.7. *Suppose the $(n-1)$th asset's returns have an instantaneous covariance of $\sigma_{(n-1)\eta}$ with changes in η, while the nth asset's returns have an instantaneous covariance of $\sigma_{n\eta}$ with changes in η. We also define the 3 × 3 matrix*

$$\Xi = \begin{bmatrix} \sigma_M^2 & k_M & \sigma_{M\eta} \\ \sigma_{M(n-1)} & k_{n-1} & \sigma_{(n-1)\eta} \\ \sigma_{Mn} & k_n & \sigma_{n\eta} \end{bmatrix} \tag{4.43}$$

where M stands for the market portfolio and $k_M = w'_M k$ (and the rest of the notation is self-explanatory). Provided that Ξ^{-1} exists, the equilibrium, conditionally expected return of assets $i = 1, 2, ..., (n-2)$ can be expressed as

$$R_i - r = \begin{bmatrix} \sigma_{iM} & k_i & \sigma_{i\eta} \end{bmatrix} \Xi^{-1} \begin{bmatrix} R_M - r \\ R_{n-1} - r \\ R_n - r \end{bmatrix} + \begin{bmatrix} \sigma_{iM} & k_i & \sigma_{i\eta} \end{bmatrix} \Xi^{-1} \begin{bmatrix} \phi_M \\ \phi_{n-1} \\ \phi_n \end{bmatrix} \eta \tag{4.44}$$

The simplified case where η is uncorrelated with the treaded assets, and also $\sigma_{nM} = 0$, produces the following system of equations

$$\begin{cases} R_i - r = \frac{1}{\theta}\sigma_{iM} - \frac{\kappa}{\theta}k_i + \phi_i\eta \\ R_M - r = \frac{1}{\theta}\sigma_M^2 - \frac{\kappa}{\theta}k_M + \phi_M\eta \\ R_n - r = \frac{\kappa}{\theta}k_n + \phi_n\eta \end{cases} \tag{4.45}$$

The final result of the system is

$$R_i - r = \left(\frac{k_i - \beta_i k_M}{k_n}\right)(R_n - \phi_n\eta - r) + \beta_i(R_M - \phi_M\eta - r) + \phi_i\eta \tag{4.46}$$

where $\beta_i = \frac{\sigma iM}{\sigma_M^2}$.

4.2.4. A green-brown dichotomy

In this section, we want to study the evolution of the dichotomy between green stocks and brown stocks ($k_g < k_b$).

Assuming

$$\mathbf{b}_1(\eta) = \sigma_\eta \cdot \eta \cdot \begin{bmatrix} \rho_{g\eta} \\ \rho_{b\eta} \end{bmatrix} \tag{4.47}$$

where $\rho_{g\eta}^2 + \rho_{b\eta}^2 = 1$, and also $\mathbf{b}_2 = 0$.

We consider that the abatement strategy is given and not mutable, that's because the single investor cannot modify or influence it, but he needs to consider it as a pre-determined parameter.

With Stochastic Differential Utility, the HJB equation can be written in scalar notation as [7, 8]

$$\begin{aligned}\widetilde{\rho} J \ln([1-\gamma]J) = \max_{C,\omega_g,\omega_b} \{\widetilde{\rho}(1-\gamma) J \ln(C) \\ + [rW + \omega_g(R_g - r - \phi_g\eta)W + \omega_b(R_b - r - \phi_b\eta)W - A_0\psi\eta + A_0 - C]J_W \\ + [(\omega_g k_g + \omega_b k_b)W + (g-\delta)\eta - \alpha\eta]J_\eta + \frac{1}{2}J_{WW}(\omega_g^2\sigma_g^2 + \omega_b^2\sigma_b^2)W^2 \\ + J_{W\eta}(\omega_b\sigma_b\sigma_\eta\rho_{b\eta} + \omega_g\sigma_g\sigma_\eta\rho_{g\eta})\eta W + \frac{1}{2}J_{\eta\eta}\sigma_\eta^2\eta^2\}\end{aligned} \quad (4.48)$$

Proposition 4.8 (Optimal rules in two asset example). *Optimal consumption is given by:*

$$C = \tilde{\rho} X \quad (4.49)$$

where $\tilde{\rho}$ is the rate of time preference and the state $X = W + \frac{A_0}{r} + \xi\eta$. The portfolio weight allocated to the green asset is

$$\omega_g = \frac{R_g - r - \phi_b\eta + \xi k_g}{\gamma\sigma_g^2}\frac{X}{W} - \frac{\xi\eta}{W}\frac{\sigma_\eta}{\sigma_g}\rho_{g\eta} \quad (4.50)$$

and the portfolio weight allocated to the brown asset amounts to

$$\omega_b = \frac{R_b - r - \phi_b\eta + \xi k_b}{\gamma\sigma_b^2}\frac{X}{W} - \frac{\xi\eta}{W}\frac{\sigma_\eta}{\sigma_b}\rho_{b\eta} \tag{4.51}$$

The valuation of the environmental externality is given by

$$\xi = \frac{1}{2a}(-b + \sqrt{b^2 - 4ac}) < 0 \tag{4.52}$$

Where $a = \sigma_\eta \left[\frac{k_g \rho_{g\eta}}{\sigma_g} + \frac{k_b \rho_{b\eta}}{\sigma_b}\right]$, $b = r + \delta + \sigma_\eta \left[\rho_{b\eta} S_b + \rho_{g\eta} S_g\right]$ *and* $c = A_0\psi$ *with* $S_i = \frac{R_i - \phi_i \eta - r}{\sigma_i}$

The impact of carbon concentration extends beyond the sole determination of consumption decisions. It also influences the myopic demands for risky assets, which are determined by the state variable X. Therefore, the increasing prevalence of climate change is expected to induce individuals to adopt a more cautious investment approach, resulting in a decreased appetite for risky assets across all sectors.
A contrasting influence exists through the hedging term. When the correlation coefficient $\rho_{i\eta}$ surpasses a certain threshold value ($\rho_{i\eta} > \frac{k_i X}{\gamma\sigma_i\sigma_\eta\eta} > 0$), investors will choose to allocate a greater proportion of risky assets, including those classified as brown assets. The decisions concerning the abatement strategy for the concentration of carbon in the atmosphere have a direct impact on investors' decisions. A strategy that effectively combats pollution prompts the investor to increase their myopic portfolio and adopt a more risk-taking attitude. At the same time, however, it progressively increases the amount of risky assets in the hedging portfolio, thus maintaining equilibrium.

An interesting case is when the firm considered does not produce emissions $k_i = 0$. This hypothesis implies that $a = 0$ but does not mean that the climate change does not affect investor's decisions. In fact, even if the investor chooses between 2 different firms without emissions, he has to take into account the ESG score which is related to b.

$$\xi = -\frac{c}{b} = -\frac{A_0\psi}{r + \delta + \sigma_\eta\left(\rho_{b\eta}S_b + \rho_{g\eta}S_g\right)} \tag{4.53}$$

Now we can rewrite the general asset pricing as

$$R_i - r = \sigma_{im}\gamma\frac{W}{X} - k_i\xi + \sigma_{i\eta}\gamma\frac{\xi}{X} \tag{4.54}$$

So ξ assumes the interpretation of Social Cost of Carbon [28]

$$SCC = -\frac{J_\eta}{f_C} = -\frac{J_\eta}{J_W} = -\xi > 0 \tag{4.55}$$

The condition of optimal consumption founded in 4.2.2 is used.

We study the evolution of Social Cost of Carbon on the simplified case considering $\sigma_\eta = 0$, which is not far from the real value considered in literature: $\sigma_\eta = 0.0078$ [15].

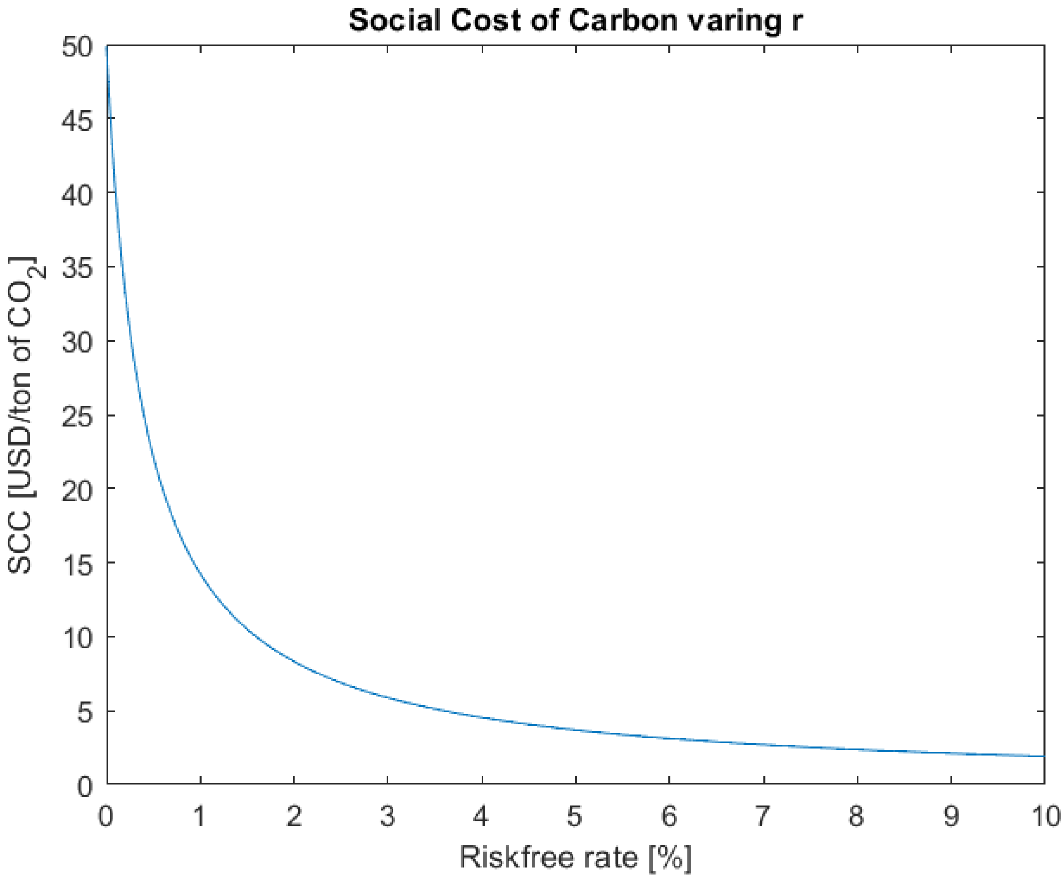

Figure 4.4: In the X axis the risk-free rate is presented, in the Y axis the corresponding Social Cost of Carbon expressed in USD/ton of CO_2 in the USA

The SCC can vary widely depending on the assumptions and inputs used in the calculation. Also, the impact of the damage caused by climate change is crucial in the formulation. We consider a medium damage condition, suggested by literature, with a damage elasticity ψ equal to 5.3%, but changing it the SCC increases linearly.

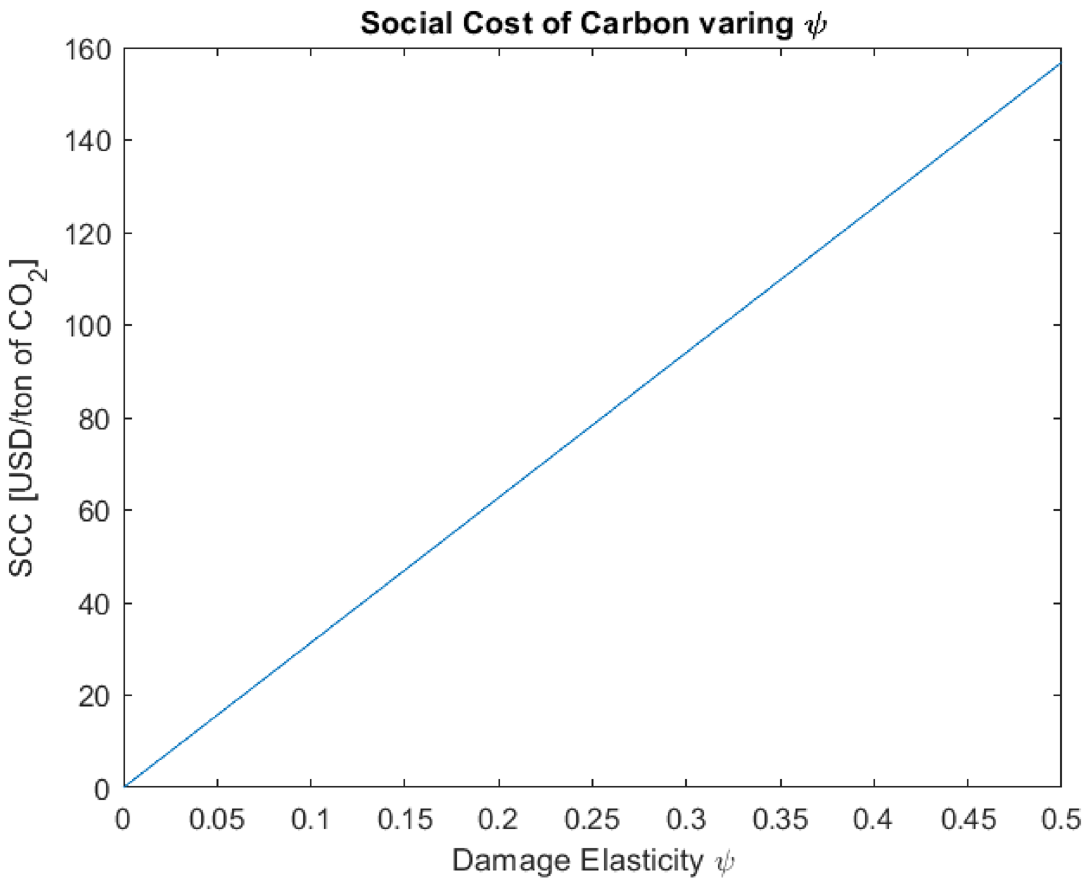

Figure 4.5: In the figure is represented the linear dependence of Social Cost of Carbon on the damage elasticity ψ. Risk-free rate is setted at 0.8%

Comparing the results obtained with the literature presented in Section 3.2, we can see that the value of SCC is coherent with literature, even if it's a simplification.

5 | Conclusion

Initially, in our work, we have meticulously reviewed and presented the findings of the relevant literature pertaining to the area of Sustainable Finance.
Initially, the ESG factor and the ESG frontier were introduced, which we have seen to allow for the maximization of the Sharpe Ratio. Subsequently, the utility of this parameter was demonstrated as it enables the construction of models that, taking into consideration this factor, predict higher returns in the short term for brown investments, whereas a sustainable investment approach allows the investor to protect against climate risk.
Our analysis then focused on modelling and predicting the dynamics of carbon dioxide presence in the atmosphere and the economic damages it may cause to an investor. Additionally, the Social Cost of Carbon metric was presented, which is often equivalent to the carbon price.

We started from a model presented by Fischer that considers a representative investor who aims to optimize his investments.
The modifications that we have presented consist of, firstly, modifying the expression of stock prices, we introduced the ESG factor into the drift of the stochastic process of such prices. Secondly, we introduced a control variable defined as the abatement strategy. This allows us to control the trend of pollution and hypothetically manipulate it to see how it can modify the investment strategy.

By changing the price drift through the introduction of the ESG factor, our model assumes new characteristics compared to the initial one. Firstly, one of the funds in the portfolio changes its structure. Specifically, the myopic fund is modified by considering the ESG factors of the stocks. Looking at the literature, a brown company, compared to a green one, will have a higher return in the short term. The investor's optimal portfolio, taking such information into account, will tend to invest more in brown stocks for their own economic benefit. However, it should be noted that the externality fund and the hedging fund are not influenced by the ESG factor, and the investor must still consider climate risk and try to mitigate it. In the green-brown dichotomy, it can be observed how the contribution of the ESG factor favours green assets more in the hedging part but disadvantages them in the myopic part; conversely, it favours brown assets more in the myopic part but disadvantages them in the hedging part.

The modification of the pollution dynamics by introducing the control variable allows us to influence the optimal investment dynamics by varying the abatement strategy. In this way, it is more evident how the decision to reduce emissions leads to clear changes in the weights of risky assets. If measures are not taken to reduce the concentration of CO_2, this will lead to greater pollution and, consequently, a lower amount of risky assets in the myopic fund and an increased interest in the hedging fund.
This last modification is interesting from an institutional point of view. In this way, the effects of government actions on the market are clear, both in terms of combating climate change and in terms of choosing not to act to reduce it.

Moreover, the calculated Social Cost of Carbon is consistent with literature. Given the almost negligible value of σ_η (0.78%), the simplified version is capable of calculating the value with accuracy.

Bibliography

[1] M. Barnett, W. Brock, and L. P. Hansen. Pricing uncertainty induced by climate change. *The Review of Financial Studies*, 33(3):1024–1066, 2020.

[2] E. Barucci. *Ingegneria finanziaria: un'introduzione quantitativa*. Egea, 2009.

[3] P. Bolton and M. Kacperczyk. Global pricing of carbon-transition risk. Technical report, National Bureau of Economic Research, 2021.

[4] P. Bolton and M. Kacperczyk. Do investors care about carbon risk? *Journal of financial economics*, 142(2):517–549, 2021.

[5] Y. Cai and T. S. Lontzek. The social cost of carbon with economic and climate risks. *Journal of Political Economy*, 127(6):2684–2734, 2019.

[6] R. E. Center. Mitigation of climate change. 1454:147, 2014.

[7] D. Duffie and L. G. Epstein. Asset pricing with stochastic differential utility. *The Review of Financial Studies*, 5(3):411–436, 1992.

[8] D. Duffie and L. G. Epstein. Stochastic differential utility. *Econometrica: Journal of the Econometric Society*, pages 353–394, 1992.

[9] T. Fischer and F. Lundtofte. Green portfolios. *SSRN Electronic Journal*, 2022.

[10] C. Geczy and J. Guerard. Esg and expected returns on equities: The case of environmental ratings. *Wharton Pension Research Council Working Paper*, (2021-15), 2021.

[11] C. Geczy, R. F. Stambaugh, and D. Levin. Investing in socially responsible mutual funds. *Available at SSRN 416380*, 2005.

[12] S. Giglio, B. Kelly, and J. Stroebel. Climate finance. *Annual Review of Financial Economics*, 13:15–36, 2021.

[13] M. Golosov, J. Hassler, P. Krusell, and A. Tsyvinski. Optimal taxes on fossil fuel in general equilibrium. *Econometrica*, 82(1):41–88, 2014.

[14] C. Hambel, H. Kraft, and R. van der Ploeg. Asset diversification versus climate action. 2020.

[15] C. Hambel, H. Kraft, and E. Schwartz. Optimal carbon abatement in a stochastic equilibrium model with climate change. *European Economic Review*, 132:103642, 2021.

[16] C. Hambel, H. Kraft, and E. Schwartz. The social cost of carbon in a non-cooperative world. *Journal of International Economics*, 131:103490, 2021.

[17] H. Hong, N. Wang, and J. Yang. Welfare consequences of sustainable finance. Technical report, National Bureau of Economic Research, 2021.

[18] H. D. Matthews, N. P. Gillett, P. A. Stott, and K. Zickfeld. The proportionality of global warming to cumulative carbon emissions. *Nature*, 459(7248):829–832, 2009.

[19] R. C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, pages 247–257, 1969.

[20] R. C. Merton. An intertemporal capital asset pricing model. *Econometrica: Journal of the Econometric Society*, pages 867–887, 1973.

[21] R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. In *Stochastic optimization models in finance*, pages 621–661. Elsevier, 1975.

[22] W. D. Nordhaus. Rolling the 'dice': an optimal transition path for controlling greenhouse gases. *Resource and Energy Economics*, 15(1):27–50, 1993.

[23] N. A. of Sciences. *Valuing climate damages: updating estimation of the social cost of carbon dioxide*. National Academies Press, 2017.

[24] L. Pástor, R. F. Stambaugh, and L. A. Taylor. Sustainable investing in equilibrium. *Journal of Financial Economics*, 142(2):550–571, 2021.

[25] L. H. Pedersen, S. Fitzgibbons, and L. Pomorski. Responsible investing: The esg-efficient frontier. *Journal of Financial Economics*, 142(2):572–597, 2021.

[26] N. Stern. Stern review: The economics of climate change. 2006.

[27] I. Van den Bijgaart, R. Gerlagh, and M. Liski. A simple formula for the social cost of carbon. *Journal of Environmental Economics and Management*, 77:75–94, 2016.

[28] T. S. Van den Bremer and F. Van der Ploeg. The risk-adjusted carbon price. *American Economic Review*, 111(9):2782–2810, 2021.

[29] M. L. Weitzman. Ghg targets as insurance against catastrophic climate damages. *Journal of Public Economic Theory*, 14(2):221–244, 2012.