HOLOMORPHIC FUNCTIONAL CALCULUS FOR SECTORIAL OPERATORS AND ANALYTIC SEMIGROUPS: PERSPECTIVES TOWARD STATISTICAL LEARNING

A Dissertation Submitted
in Partial Fulfilment of the Requirements
for the Degree of

MASTER OF SCIENCE

in **Statistical Learning**

by

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POLITECNICO
DI MILANO

June 2026

Acknowledgements

I would like to warmly thank my supervisors, **Pr. Fabrizio Colombo** and **Pr. Antonino De Martino**, for their guidance, trust, and support throughout this thesis. Their advice and their expertise in Operator Theory were essential in shaping this work, especially given the ambitious and exploratory nature of the topic.

I am grateful to Politecnico di Milano for the stimulating academic environment in which this thesis was developed, and for giving me the opportunity to deepen my knowledge at the intersection of mathematics, analysis, and learning.

I also wish to thank Centrale Nantes, whose double-degree programme allowed me to study at Politecnico di Milano and to benefit from this rich academic and international experience.

Finally, I would like to thank my family and friends for their encouragement and support during my studies and throughout the writing of this thesis.

Abstract

This thesis studies the operator-theoretic foundations of the holomorphic functional calculus for sectorial operators and its connection with analytic semigroups. The starting point is the problem of giving a rigorous meaning to expressions of the form $f(A)$, where A is a linear operator on a Banach space and f is a complex-valued function. While this is elementary for scalars and finite-dimensional matrices, it becomes substantially more delicate for unbounded operators on infinite-dimensional spaces, such as differential operators arising in evolution equations.

The thesis first recalls the functional calculus for bounded operators, from polynomials and power series to the Dunford–Riesz holomorphic functional calculus. This highlights the central role of the resolvent operator and the link between complex analysis, spectral theory, and functions of operators.

The main part of the work is devoted to sectorial operators and the associated H^∞-functional calculus. We introduce the resolvent estimates defining sectoriality, suitable classes of holomorphic functions on sectors, regularization procedures, boundedness properties, and spectral mapping results. These tools provide a rigorous framework for treating functions of unbounded operators beyond the classical bounded setting.

We then study analytic C_0-semigroups and their connection with sectorial operators. Operators arising from abstract evolution equations can be related to time-evolution families $(T(t))_{t\geq 0}$, formally written as $T(t) = e^{-tA}$. This correspondence connects the spectrum, the resolvent, semigroup theory, and the functional calculus.

Finally, the thesis discusses perspectives toward statistical learning and data-driven operator approximation. It explains how semigroups and resolvent representations may provide a natural operator-theoretic framework for future learning approaches, while emphasizing the analytical difficulties and open problems involved.

Keywords: functional calculus, sectorial operators, holomorphic functional calculus, analytic semigroups, resolvent operator, evolution equations, operator theory, statistical learning.

Riassunto

Questa tesi studia i fondamenti operatoriali del calcolo funzionale olomorfo per operatori settoriali e il suo legame con i semigruppi analitici. Il punto di partenza è il problema di attribuire un significato rigoroso a espressioni della forma $f(A)$, dove A è un operatore lineare su uno spazio di Banach e f è una funzione a valori complessi. Mentre ciò è elementare per scalari e matrici finite-dimensionali, diventa sostanzialmente più delicato per operatori non limitati su spazi infinito-dimensionali, come gli operatori differenziali che compaiono nelle equazioni di evoluzione.

La tesi richiama innanzitutto il calcolo funzionale per operatori limitati, dai polinomi e dalle serie di potenze fino al calcolo funzionale olomorfo di Dunford–Riesz. Questo mette in evidenza il ruolo centrale dell'operatore risolvente e il legame tra analisi complessa, teoria spettrale e funzioni di operatori.

La parte principale del lavoro è dedicata agli operatori settoriali e al calcolo funzionale H^∞ associato. Vengono introdotte le stime del risolvente che definiscono la settorialità, opportune classi di funzioni olomorfe su settori, procedure di regolarizzazione, proprietà di limitatezza e risultati di mappatura spettrale. Questi strumenti forniscono un quadro rigoroso per trattare funzioni di operatori non limitati oltre il contesto classico degli operatori limitati.

Successivamente, si studiano i semigruppi analitici C_0 e la loro connessione con gli operatori settoriali. Gli operatori che emergono dalle equazioni di evoluzione astratte possono essere collegati a famiglie di evoluzione temporale $(T(t))_{t\geq 0}$, formalmente scritte come $T(t) = e^{-tA}$. Questa corrispondenza mette in relazione lo spettro, il risolvente, la teoria dei semigruppi e il calcolo funzionale.

Infine, la tesi discute alcune prospettive verso l'apprendimento statistico e l'approssimazione di operatori guidata dai dati. Essa spiega come i semigruppi e le rappresentazioni mediante il risolvente possano fornire un quadro operatoriale naturale per futuri approcci di apprendimento, sottolineando al contempo le difficoltà analitiche e i problemi aperti coinvolti.

Parole chiave: calcolo funzionale, operatori settoriali, calcolo funzionale olomorfo, semigruppi analitici, operatore risolvente, equazioni di evoluzione, teoria degli operatori, apprendimento statistico.

Contents

List of Figures

Chapter 1

Introduction

1.1 Motivation

A central problem in modern analysis is to give a rigorous meaning to expressions of the form $f(A)$, where A is a linear operator acting on a Banach space and f is a complex-valued function. While such expressions are immediate in the scalar setting, their extension to operators is far from trivial and requires a careful analytical framework.

This question arises naturally in many contexts. For instance, the exponential of an operator e^{tA} plays a fundamental role in the study of evolution equations, where it describes the time propagation of a system. More generally, functions of operators appear in spectral theory, in the analysis of partial differential equations, and in quantum mechanics, where observables are modeled by operators and their functional transformations encode physical quantities.

In many of these applications, the relevant operators are not bounded. Differential operators such as the Laplacian, which arise naturally in partial differential equations, typically act as unbounded operators on infinite-dimensional function spaces. This makes the construction of $f(A)$ substantially more delicate than in the finite-dimensional or bounded-operator setting. A rigorous framework must therefore take into account the spectral geometry of the operator, the behavior of its resolvent, and the analytic properties of the functions to which the calculus is applied.

The aim of this thesis is to study the operator-theoretic foundations of the holomorphic functional calculus for sectorial operators and its connection with analytic semigroups. The functional calculus will serve as a unifying tool connecting spectral analysis, resolvent estimates, the theory of analytic semigroups, and the representation of evolution equations.

This point of view also provides a natural foundation for future interactions with statistical learning and data-driven operator approximation. Indeed, in many applications one observes the time evolution of a system rather than the underlying generator itself. Understanding the analytical relationship between generators, resolvents, semigroups, and functions of operators is therefore a necessary preliminary step before addressing possible learning or reconstruction procedures.

Accordingly, the purpose of this thesis is not to develop a complete data-driven reconstruction theory, but rather to present the functional analytic framework in which such questions can be formulated rigorously. The final part of the work discusses how the operator-theoretic material developed here may open perspectives toward statistical learning and data-driven approaches to evolution equations.

1.2 From Scalars to Operators: An Intuitive Perspective

Before introducing the general theory, it is useful to recall how functions of operators arise in simple settings and to briefly introduce the main objects involved.

Definition 1.2.1 (Bounded linear operator)**.** Let X and Y be complex Banach spaces. A linear map

$$A : X \to Y$$

is called a bounded linear operator if there exists a constant $C \geq 0$ such that

$$\|Ax\|_Y \leq C\|x\|_X, \qquad \forall x \in X.$$

Equivalently, A is continuous on X (or equivalently continuous at 0). The space of all bounded linear operators from X to Y is denoted by

$$\mathcal{L}(X, Y).$$

Definition 1.2.2 (Operator norm)**.** Let X and Y be Banach spaces, and let

$$A \in \mathcal{L}(X, Y).$$

The operator norm of A is defined by

$$\|A\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X \leq 1} \|Ax\|_Y.$$

Equivalently,

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}.$$

Definition 1.2.3. When $X = Y$, we simply write

$$\mathcal{L}(X) := \mathcal{L}(X, X).$$

Elements of $\mathcal{L}(X)$ are called bounded linear operators on X.

Typical examples of bounded operators include matrices acting on $\mathbb{C}^n$, but also operators acting on infinite-dimensional function spaces. A fundamental example arises in the study of partial differential equations. Consider the heat equation

$$\partial_t u(t, x) - \Delta u(t, x) = 0, \quad (t, x) \in (0, +\infty) \times \Omega,$$

with suitable boundary conditions on Ω. This equation can be written abstractly as

$$\frac{d}{dt} u(t) + Au(t) = 0,$$

where the operator $A = -\Delta$ acts on a suitable function space such as $L^2(\Omega)$. This illustrates how differential operators naturally fit into the operator-theoretic framework.

If $\lambda \in \mathbb{C}$ is a scalar, then $f(\lambda)$ is simply the evaluation of the function f at λ.

If A is a diagonalizable matrix, say $A = PDP^{-1}$ with $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, one can define

$$f(A) = P \operatorname{diag}(f(\lambda_1), \dots, f(\lambda_n)) P^{-1}.$$

In this case, applying a function to an operator reduces to applying the function to its eigenvalues.

This observation naturally leads to the notion of the spectrum.

Definition 1.2.4. Let $A \in \mathcal{L}(X)$. The *resolvent set* of A is defined by

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A) \text{ is invertible in } \mathcal{L}(X)\}.$$

The *spectrum* of A is the complement of the resolvent set:

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

The spectrum generalizes the notion of eigenvalues and plays a central role in operator theory. In infinite-dimensional spaces, it may contain points that are not eigenvalues, reflecting more subtle analytical phenomena.

The previous diagonalization formula suggests that $f(A)$ should depend only on the values of f on $\sigma(A)$. However, in infinite-dimensional spaces, operators are in general not diagonalizable, and even when they are, their spectral decomposition may be highly nontrivial.

Therefore, one seeks a definition of $f(A)$ that does not rely on diagonalization but instead uses analytic tools. The key idea is to replace the scalar kernel appearing in complex analysis by an operator-valued analogue, leading to the notion of the holomorphic functional calculus.

1.3 Functional Calculus for Bounded Operators

We now recall the construction of the functional calculus for bounded linear operators, which provides the foundation for the general theory. We progressively extend the class of admissible functions, starting from polynomials and culminating in holomorphic functions.

1.3.1 Polynomial Functional Calculus

Definition 1.3.1. Let X be a complex Banach space and $A \in \mathcal{L}(X)$. For any polynomial

$$p(\lambda) = a_n \lambda^n + \cdots + a_1 \lambda + a_0,$$

the *polynomial functional calculus* is defined by

$$p(A) = a_n A^n + \cdots + a_1 A + a_0 I.$$

The polynomial functional calculus satisfies the expected algebraic properties.

Proposition 1.3.2. *Let $A \in \mathcal{L}(X)$ and let p, q be polynomials. For all $\alpha, \beta \in \mathbb{C}$, the following properties hold:*

- *(Linearity)* $(\alpha p + \beta q)(A) = \alpha p(A) + \beta q(A)$,
- *(Multiplicativity)* $(pq)(A) = p(A)q(A)$,
- *(Identity)* $1(A) = I$.

Proof. All properties follow directly from the linearity of the operator A and the associativity of operator composition. □

This construction provides a natural extension of scalar polynomials to operators. However, it remains limited to algebraic functions.

1.3.2 Extension via Power Series

Definition 1.3.3. Let $A \in \mathcal{L}(X)$ and let f be a function admitting a power series expansion

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

with infinite radius of convergence. The *functional calculus via power series* is defined by

$$f(A) = \sum_{n=0}^{\infty} c_n A^n,$$

provided that the series converges in the operator norm.

A fundamental example is given by the exponential function:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

While this approach enlarges the class of admissible functions, it still depends on convergence properties and does not apply to general holomorphic functions.

1.3.3 Resolvent and Spectral Properties

We now introduce the resolvent operator, which plays a central role in the holomorphic functional calculus.

Definition 1.3.4. Let $A \in \mathcal{L}(X)$. The *resolvent operator* is defined by

$$R(\lambda, A) = (\lambda I - A)^{-1},$$

for all λ such that $(\lambda I - A)$ is invertible.

A Reminder on Neumann Series

We recall a classical result on Neumann series, which plays a key role in the study of resolvent operators.

Proposition 1.3.5 (Neumann series)**.** *Let X be a Banach space and let $T \in \mathcal{L}(X)$ such that $\|T\| < 1$. Then the series*

$$\sum_{n=0}^{\infty} T^n$$

converges in the operator norm, and we have

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Proof. Since $\|T\| < 1$, the series $\sum \|T^n\| \leq \sum \|T\|^n$ converges, so the series $\sum T^n$ converges in $\mathcal{L}(X)$.

Let $S = \sum_{n=0}^{\infty} T^n$. Then

$$(I - T)S = \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n = I.$$

Similarly, $S(I - T) = I$, which proves that $S = (I - T)^{-1}$. □

We directly apply this result to the resolvent :

Proposition 1.3.6. *Let $A \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$ such that $|\lambda| > \|A\|$. Then*

$$(\lambda I - A)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n.$$

Proof. We write

$$(\lambda I - A) = \lambda \left(I - \frac{1}{\lambda} A \right).$$

Since $\left\| \frac{1}{\lambda} A \right\| < 1$, the Neumann series applies and gives

$$\left(I - \frac{1}{\lambda} A \right)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n} A^n.$$

Multiplying by λ^{-1} yields the result. □

Resolvent Properties

A key identity satisfied by the resolvent is the following.

Proposition 1.3.7 (Resolvent identity)**.** *Let $A \in \mathcal{L}(X)$ and let λ, μ belong to the resolvent set of A. Then*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

Proof. Starting from the identity

$$(\lambda I - A) - (\mu I - A) = (\lambda - \mu)I,$$

we multiply on the left by $R(\lambda, A)$ and on the right by $R(\mu, A)$ to obtain

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

□

We will also be using the following property :

Proposition 1.3.8. *Let A be a closed operator on a Banach space X, and let $\lambda \in \rho(A)$. Then*

$$AR(\lambda, A) = \lambda R(\lambda, A) - I.$$

Proof. Since $\lambda \in \rho(A)$, the resolvent operator

$$R(\lambda, A) = (\lambda I - A)^{-1}$$

is well defined and bounded on X. Moreover, its range is contained in $D(A)$, so the composition $AR(\lambda, A)$ is well defined.

Starting from the identity

$$(\lambda I - A)R(\lambda, A) = I,$$

we expand the left-hand side:

$$\lambda R(\lambda, A) - AR(\lambda, A) = I.$$

Rearranging the terms yields

$$AR(\lambda, A) = \lambda R(\lambda, A) - I,$$

which is the desired formula. □

Spectral Properties

We end this subsection with a few more results. We first give the *spectral mapping theorem* for bounded operators (see for instance [19]):

Theorem 1.3.9 (Spectral Mapping Theorem for polynomials)**.** *Let X be a Banach space and let*

$$A \in \mathcal{L}(X).$$

If p is a complex polynomial, then

$$\sigma(p(A)) = p(\sigma(A)).$$

In particular, for every $n \in \mathbb{N}$,

$$\sigma(A^n) = \{\lambda^n : \lambda \in \sigma(A)\}.$$

We now give the definition of the *spectral radius*:

Definition 1.3.10 (Spectral radius)**.** Let X be a Banach space and let

$$A \in \mathcal{L}(X).$$

The *spectral radius* of A is defined by

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ denotes the spectrum of A.

Then, an important result that will be used later is the following (see Theorem 10.13 in [19]):

Theorem 1.3.11 (Spectral Radius Formula)**.** *Let X be a Banach space and let $A \in \mathcal{L}(X)$. Then*

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n},$$

where

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

is the spectral radius of A.

Proof. Since the spectrum satisfies

$$\sigma(A^n) = \{\lambda^n : \lambda \in \sigma(A)\},$$

we have

$$r(A)^n = r(A^n) \leq \|A^n\|.$$

Hence

$$r(A) \leq \|A^n\|^{1/n}$$

for every n, and therefore

$$r(A) \leq \liminf_{n\to\infty} \|A^n\|^{1/n}.$$

Conversely, let $|\lambda| > \limsup_{n\to\infty} \|A^n\|^{1/n}$. Then there exists $\rho < |\lambda|$ and $N \in \mathbb{N}$ such that

$$\|A^n\| \leq \rho^n, \qquad n \geq N.$$

Thus the Neumann series (see Proposition 1.3.6)

$$\sum_{n=0}^{\infty} \lambda^{-n-1} A^n$$

converges in $\mathcal{L}(X)$, and

$$(\lambda I - A)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n.$$

Hence $\lambda \in \rho(A)$, so $|\lambda| > r(A)$. This proves

$$\limsup_{n\to\infty} \|A^n\|^{1/n} \leq r(A).$$

Combining the two inequalities gives the result. □

1.3.4 A Reminder from Complex Analysis

We recall a fundamental result from complex analysis which serves as the basis for the holomorphic functional calculus.

Theorem 1.3.12 (Cauchy's theorem)**.** *Let $U \subset \mathbb{C}$ be an open set and let f be a holomorphic function on U. If γ is a closed contour contained in U, then*

$$\int_\gamma f(\lambda)\, d\lambda = 0.$$

Theorem 1.3.13 (Cauchy's integral formula)**.** *Let $U \subset \mathbb{C}$ be an open set, f holomorphic on U, and let γ be a positively oriented simple closed contour in U. If z_0 lies inside γ, then*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\lambda)}{\lambda - z_0}\, d\lambda.$$

These results suggest that a function can be reconstructed from its values on a contour via the Cauchy kernel $(\lambda - z_0)^{-1}$. The key idea of the holomorphic functional calculus is to replace this scalar kernel by the resolvent operator $(\lambda I - A)^{-1}$.

1.3.5 The Dunford–Riesz Functional Calculus

We now introduce the holomorphic functional calculus for bounded operators, which generalizes the previous constructions.

Definition 1.3.14. Let $A \in \mathcal{L}(X)$ and let f be holomorphic on an open set U such that $\sigma(A) \subset U$. Let ∂U be a positively oriented contour enclosing $\sigma(A)$. The *Dunford–Riesz functional calculus* is defined by

$$f(A) = \frac{1}{2\pi i} \int_{\partial U} f(\lambda)(\lambda I - A)^{-1} \, d\lambda.$$

This definition is directly inspired by the Cauchy integral formula, where the scalar kernel $(\lambda - z)^{-1}$ is replaced by the resolvent operator.

Theorem 1.3.15 (Algebraic properties)**.** *Let f, g be holomorphic functions defined on a neighborhood of $\sigma(A)$, and let $\alpha, \beta \in \mathbb{C}$. Then*

$$(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A),$$
$$(fg)(A) = f(A)g(A).$$

Proof. Linearity follows from the linearity of the integral. For multiplicativity, one uses the resolvent identity together with Fubini's theorem and Cauchy's integral formula to reduce the double integral to a single contour integral involving fg. □

Thus, the holomorphic functional calculus extends the polynomial and power series constructions to a large class of holomorphic functions, providing a powerful and flexible tool for the analysis of bounded operators.

1.4 Limitations of the Bounded Framework

Although the Dunford–Riesz functional calculus provides a powerful tool, it relies crucially on the boundedness of the operator A.

Indeed, for bounded operators, the spectrum $\sigma(A)$ is compact, and the resolvent exists outside a sufficiently large disk. This ensures that the contour integral defining $f(A)$ is well-defined and that the resolvent behaves nicely at infinity.

However, in many important applications, the relevant operators are *unbounded.* This is the case, for instance, for differential operators such as the Laplacian, which naturally arise in the study of partial differential equations and evolution problems.

In this setting, several difficulties appear:

- The spectrum is no longer compact and may extend to infinity,
- The resolvent may only be defined on a restricted region of the complex plane,
- The integral defining $f(A)$ may fail to converge.

These limitations show that the classical holomorphic functional calculus cannot be directly applied to unbounded operators. This motivates the introduction of a more general framework, capable of handling operators with unbounded spectrum while preserving the key analytical and algebraic properties of the calculus.

This leads naturally to the theory of sectorial operators and the associated holomorphic functional calculus, which will be developed in the next chapter.

1.5 Link with Analytic Semigroups

One of the main motivations for introducing the holomorphic functional calculus is its deep connection with the theory of evolution equations. Consider the abstract Cauchy problem

$$\frac{d}{dt}u(t) + Au(t) = 0, \quad t > 0, \qquad u(0) = u_0,$$

where A is a (possibly unbounded) operator on a Banach space X.

In many situations, the solution of this problem can be written formally as

$$u(t) = e^{-tA}u_0,$$

where the exponential of the operator is defined via a suitable functional calculus. This expression provides a natural link between operator theory and time evolution.

A family of operators $(T(t))_{t\geq 0}$ defined by $T(t) = e^{-tA}$ is called a *semigroup*. When the map $t \mapsto T(t)$ extends analytically to a sector of the complex plane, the semigroup is said to be *analytic*.

A fundamental result, which will be developed in detail in Chapter 3, is that sectorial operators are precisely those generating analytic semigroups. This establishes a deep correspondence between spectral properties of operators and their associated time-evolution behavior.

The holomorphic functional calculus provides the appropriate framework to define and study objects such as e^{-tA} and more general functions of A. In particular, it allows one to rigorously connect spectral analysis, semigroup theory, and the regularity properties of solutions to evolution equations.

1.6 Structure of the Thesis

The remainder of this thesis is organized as follows.

In Chapter 2, we introduce the class of sectorial operators and develop the holomorphic functional calculus in this setting. We define the appropriate classes of holomorphic functions on sectors and study the main properties of the calculus, including regularization procedures, boundedness properties, algebraic features, and spectral mapping results.

In Chapter 3, we study analytic semigroups and their connection with sectorial operators. We present the main results on analytic C_0-semigroups, the identification of their generators, sectoriality conditions, and spectral properties of the associated evolution operators.

The thesis concludes by discussing how the operator-theoretic framework developed in the preceding chapters may provide a foundation for future applications to statistical learning and data-driven operator approximation. In particular, the connection between semigroups, resolvents, and the holomorphic functional calculus suggests possible directions for learning problems involving evolution equations, while also highlighting the analytical difficulties that remain open.

Chapter 2

Sectorial Operators and the Holomorphic Functional Calculus

In this chapter we build the analytical foundations of the thesis. Our aim is to introduce the class of sectorial operators and to construct the holomorphic functional calculus which allows us to define $f(A)$ for a large class of holomorphic functions f. This calculus, provides the abstract framework in which semigroup theory, regularity of evolution equations, and possibly the future work on learning operators can be rigorously formulated.

2.1 Sectorial Operators

We saw in the introduction (see definition 1.3.14) that in the classical setting of bounded operators on a Banach space X, the holomorphic functional calculus is constructed through the Riesz–Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_{\partial U} f(\lambda)(\lambda I - A)^{-1}\, d\lambda,$$

where the spectrum $\sigma(A)$ is compact and contained in the open set U. This calculus provides a powerful tool to define $f(A)$ for any analytic function f defined in a neighbourhood of $\sigma(A)$. However, this framework relies crucially on the boundedness of A: the resolvent $(\lambda I - A)^{-1}$ must exist as a bounded operator on the whole space X for λ outside the spectrum.

In many analytical applications — in particular, when studying evolution equations of the form

$$\frac{du}{dt} + Au = f(t), \qquad u(0) = u_0,$$

the relevant operator A is *unbounded*, typically a differential operator such as the Laplacian or an elliptic generator. In this case, we do not have any guaranties that the construction of Riesz-Dunford is still correct, since the spectrum is no longer bounded and $(\lambda I - A)^{-1}$ may only exist for large values of λ.

To overcome this limitation, one seeks a more general class of operators for which a holomorphic functional calculus can still be defined in a consistent way. This leads to the notion of *sectorial operators*. These are closed, densely defined operators whose resolvent satisfies appropriate bounds outside a sector of the complex plane. They play a central role in the theory of analytic semigroups and in the modern approach to evolution equations.

The holomorphic functional calculus for sectorial operators, developed by McIntosh and Haase respectively in [14] and [4], extends the Riesz–Dunford construction to this unbounded setting. It preserves most of the algebraic properties of the bounded calculus, while providing access to fractional powers, semigroups of the form e^{-tA}, and many regularity estimates. It thus forms the analytical backbone of our subsequent study.

2.1.1 Spectral Geometry and Resolvent Bounds

Definition 2.1.1 (Sectorial operator)**.** Let X be a complex Banach space and $\omega \in [0, \pi)$. A densely defined closed operator $A : D(A) \subset X \to X$ is called *sectorial of angle* ω, written $A \in \mathrm{Sect}(\omega)$, if its spectrum satisfies $\sigma(A) \subset \overline{\Sigma_\omega}$ and for every $\phi > \omega$ there exists $M_\phi > 0$ such that

$$\|(\lambda I - A)^{-1}\| \leq \frac{M_\phi}{|\lambda|}, \qquad \lambda \in \mathbb{C} \setminus \overline{\Sigma_\phi}, \tag{2.1}$$

where $\Sigma_\phi = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \phi\}$.

Example 2.1.2 (The Laplace operator)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary $\partial\Omega$, and consider the Laplace operator

$$A = -\Delta, \qquad D(A) = H^2(\Omega) \cap H^1_0(\Omega),$$

acting on the Hilbert space $X = L^2(\Omega)$, where

$$H^2(\Omega) = \{u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega) \text{ for all multi-indices } |\alpha| \leq 2\},$$

and

$$H^1_0(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}.$$

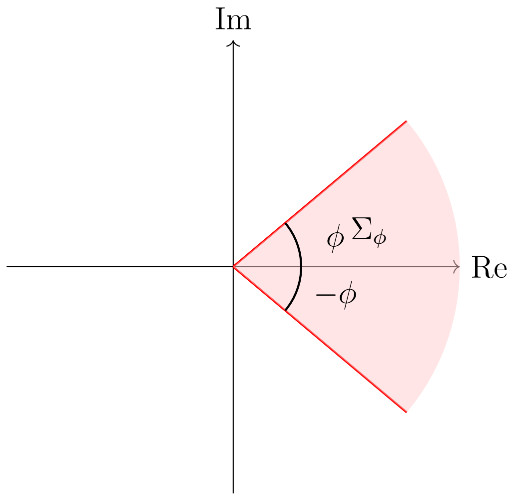

Figure 2.1: Sector $\Sigma_\phi = \{z \in \mathbb{C} : |\arg z| < \phi\}$

A is self-adjoint and positive. Since A is defined with homogeneous Dirichlet boundary conditions, integration by parts yields, for $u, v \in D(A)$,

$$\langle Au, v\rangle_{L^2(\Omega)} = \int_\Omega \nabla u \cdot \nabla v \, dx = \langle u, Av\rangle_{L^2(\Omega)}.$$

Hence A is symmetric, and since $D(A)$ is dense in $L^2(\Omega)$, A is self-adjoint. Moreover,

$$\langle Au, u\rangle_{L^2(\Omega)} = \int_\Omega |\nabla u|^2 \, dx \geq 0,$$

so A is nonnegative.

Spectrum and resolvent. Since $A = -\Delta$ with Dirichlet boundary conditions is self-adjoint and nonnegative on $L^2(\Omega)$, the spectral theorem implies that

$$\sigma(A) \subset [0, +\infty).$$

In particular, for every $\lambda \in \mathbb{C} \setminus [0, +\infty)$, the resolvent operator $(\lambda I - A)^{-1}$ exists.

For a self-adjoint operator, the norm of the resolvent is controlled by the distance from λ to the spectrum:

$$\|(\lambda I - A)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(A))}.$$

Let $\lambda = re^{i\theta}$ with $r > 0$ and $\theta \neq 0$. Since the spectrum lies on the positive real axis, the distance from λ to $[0, +\infty)$ is the length of the perpendicular from λ to this axis, namely

$$\operatorname{dist}(\lambda, [0, +\infty)) = r|\sin\theta|.$$

Consequently,
$$\|(\lambda I - A)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(A))} \leq \frac{1}{r|\sin\theta|}.$$

Sectorial estimate. Let $\phi \in (0, \pi)$ and choose $\omega = 0$. If $\lambda \notin \Sigma_\phi$, then by definition

$$|\arg(\lambda)| \geq \phi.$$

Writing again $\lambda = re^{i\theta}$, this implies

$$|\sin\theta| \geq \sin\phi.$$

Using the previous resolvent estimate, we obtain

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{r|\sin\theta|} \leq \frac{1}{r\sin\phi} = \frac{1}{|\lambda|\sin\phi}.$$

Hence
$$\|(\lambda I - A)^{-1}\| \leq \frac{M_\phi}{|\lambda|}, \qquad \lambda \in \mathbb{C} \setminus \Sigma_\phi,$$

with $M_\phi = \frac{1}{\sin\phi}$. Therefore the Laplacian satisfies the sectorial resolvent estimate and is sectorial with angle 0.

The definition of sectorial operators captures two key features: the localization of the spectrum within a sector and the decay of the resolvent along rays outside this sector. We now collect several basic analytical properties of such operators.

Proposition 2.1.3 (Basic resolvent estimates)**.** *If $A \in \operatorname{Sect}(\omega)$, then for every $\phi > \omega$ there exists $M_\phi > 0$ such that*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M_\phi}{|\lambda|}, \qquad \lambda \in \mathbb{C} \setminus \overline{\Sigma_\phi}. \tag{2.2}$$

Moreover, there exists $C_\phi > 0$ such that

$$\|A(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq C_\phi, \qquad \lambda \in \mathbb{C} \setminus \overline{\Sigma_\phi}.$$

Proof. Fix $\phi > \omega$. Since $A \in \operatorname{Sect}(\omega)$, by definition there exists $M_\phi > 0$ such that

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M_\phi}{|\lambda|} \qquad \text{for all } \lambda \in \mathbb{C} \setminus \overline{\Sigma_\phi}.$$

This is exactly the first claim.

For the second estimate, let $\lambda \in \mathbb{C} \setminus \overline{\Sigma_\phi}$. Then $\lambda \in \rho(A)$, so $(\lambda I - A)^{-1} \in \mathcal{L}(X)$ is well-defined. We start from the elementary identity, valid on $D(A)$:

$$(\lambda I - A)(\lambda I - A)^{-1} = I.$$

Expanding the left-hand side gives

$$\lambda(\lambda I - A)^{-1} - A(\lambda I - A)^{-1} = I,$$

hence

$$A(\lambda I - A)^{-1} = \lambda(\lambda I - A)^{-1} - I.$$

Now take operator norms in $\mathcal{L}(X)$ and apply the triangle inequality:

$$\|A(\lambda I - A)^{-1}\| \le \|\lambda(\lambda I - A)^{-1}\| + \|I\|.$$

Since λ is a scalar, $\|\lambda(\lambda I - A)^{-1}\| = |\lambda| \, \|(\lambda I - A)^{-1}\|$, and $\|I\| = 1$, so

$$\|A(\lambda I - A)^{-1}\| \le |\lambda| \, \|(\lambda I - A)^{-1}\| + 1.$$

Using the sectorial resolvent bound 2.2, we obtain

$$|\lambda| \, \|(\lambda I - A)^{-1}\| \le |\lambda| \, \frac{M_\phi}{|\lambda|} = M_\phi,$$

and therefore

$$\|A(\lambda I - A)^{-1}\| \le M_\phi + 1.$$

Thus the second estimate holds with $C_\phi := 1 + M_\phi$ (which is > 1). □

Remark 2.1.4. Let $A \in \mathrm{Sect}(\omega)$. Then, one could prove that for any $\lambda_0 > 0$, the shifted operator $A + \lambda_0 I$ is sectorial of the same angle ω.

Theorem 2.1.5 (Characterization via analytic semigroups)**.** *Let A be a densely defined operator on X. Then the following are equivalent:*

1. *A is sectorial of angle $\omega < \pi/2$;*
2. *$-A$ generates a bounded analytic semigroup $(e^{-tA})_{t>0}$ on X;*
3. *For some $\phi > \omega$, there exists M_ϕ such that $\|(\lambda I - A)^{-1}\| \le M_\phi/|\lambda|$ for all $\lambda \notin \overline{\Sigma_\phi}$.*

Remark 2.1.6. This equivalence highlights the dual analytical nature of sectorial operators: they are simultaneously characterized by their spectral geometry and by their time-evolution properties. In particular, the semigroup $(e^{-tA})_{t>0}$ associated with A

will later serve as a bridge between the holomorphic functional calculus and the theory of evolution equations. It will be partially proved in the next chapter.

2.2 Holomorphic Functions of Polynomial Limits

In order to define the holomorphic functional calculus for unbounded operators, we must specify a suitable class of functions for which the Cauchy integral

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - A)^{-1}\, d\lambda$$

is well defined. Since A is sectorial, the integrand is typically singular near 0 and decays only along rays of the sector at infinity. Thus, the function f must vanish (or at least decay) sufficiently fast at both the origin and at infinity. Following Haase in [4], we formalize this behavior through the notion of *polynomial limits.*

Polynomial limits at 0 and ∞

Definition 2.2.1 (Polynomial limits)**.** Let $\phi \in (0, \pi]$ and f be a holomorphic function on the sector Σ_ϕ.

1. We say that f has a *polynomial limit* $c \in \mathbb{C}$ at 0 if there exist $\alpha > 0$ and $C > 0$ such that
$$|f(z) - c| \leq C|z|^\alpha, \qquad z \to 0,\ z \in \Sigma_\phi.$$
2. We say that f has a *polynomial limit* $d \in \mathbb{C} \cup \{\infty\}$ at ∞ if $f(z^{-1})$ has a polynomial limit d at 0.
3. If f has polynomial limit 0 at both 0 and ∞, we say that f is *regularly decaying* at 0 and ∞.

Remark 2.2.2. Intuitively, a polynomial limit means that f approaches its limit (at 0 or ∞) at least at a polynomial rate. For example, $f(z) = (1 + z)^{-1}$ has polynomial limit 1 at 0 and 0 at infinity, whereas $f(z) = e^z$ has polynomial limit ∞ at infinity but is not polynomially bounded on Σ_ϕ.

2.2.1 The Dunford–Riesz class

We denote by

$$H^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\}, \qquad \|f\|_{\infty,\Sigma_\phi} = \sup_{z\in\Sigma_\phi} |f(z)|.$$

Within this algebra, we define the subclass of functions that decay polynomially both at the origin and at infinity.

Definition 2.2.3 (Dunford–Riesz class)**.** The *Dunford–Riesz class* on Σ_ϕ (see 2.1.1) is defined by

$$H_0^\infty(\Sigma_\phi) = \left\{ f \in H^\infty(\Sigma_\phi) : \exists\, C, s > 0, \quad |f(z)| \leq C \min(|z|^s, |z|^{-s}), \ \forall z \in \Sigma_\phi \right\}.$$

Functions in $H_0^\infty(\Sigma_\phi)$ are precisely those that are regularly decaying at both ends. They guarantee integrability of the Cauchy integral in the definition of $f(A)$ and form a Banach algebra ideal within $H^\infty(\Sigma_\phi)$.

Lemma 2.2.4 (Equivalent descriptions)**.** *Let $\phi \in (0, \pi]$ and $f : \Sigma_\phi \to \mathbb{C}$ be holomorphic. The following are equivalent:*

1. *$f \in H_0^\infty(\Sigma_\phi)$;*
2. *There exist $C, s > 0$ such that $|f(z)| \leq C(1 + |z|)^{-s}$ for $|z|$ large and $|f(z)| \leq C|z|^s$ for $|z|$ small;*
3. *f and $f(1/z)$ are both bounded and vanish polynomially at 0.*

Proof. We show (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2). Assume $f \in H_0^\infty(\Sigma_\phi)$. Then there exist $C, s > 0$ such that

$$|f(z)| \leq C \min(|z|^s, |z|^{-s}) \qquad (z \in \Sigma_\phi).$$

If $|z| \leq 1$, then $\min(|z|^s, |z|^{-s}) = |z|^s$, hence $|f(z)| \leq C|z|^s$. If $|z| \geq 1$, then $\min(|z|^s, |z|^{-s}) = |z|^{-s}$, hence $|f(z)| \leq C|z|^{-s}$. In particular, for $|z|$ large we have $|f(z)| \leq C(1 + |z|)^{-s}$: let $g(x) = \frac{(1+x)^s}{x^s}$ for $x \geq 1$. Derivating g yields that $g'(x) \leq 0$ for $x \geq 1$, showing that g is decreasing and bounded by $g(1) = 2^s$. We only need to set new $C_1 := 2^s C$ to get the result for $|z|$ large.

(2) $\Rightarrow$ (1). Assume (2). Let $s_0 > 0$ and $C_0 > 0$ be such that

$$|f(z)| \leq C_0 |z|^{s_0} \quad \text{for } |z| \leq 1, \qquad |f(z)| \leq C_0 (1 + |z|)^{-s_0} \quad \text{for } |z| \geq 1.$$

Since $(1 + |z|)^{-s_0} \leq |z|^{-s_0}$ for $|z| \geq 1$, we obtain

$$|f(z)| \leq C_0 |z|^{s_0} \ \ (|z| \leq 1), \qquad |f(z)| \leq C_0 |z|^{-s_0} \ \ (|z| \geq 1),$$

hence $|f(z)| \leq C_0 \min(|z|^{s_0}, |z|^{-s_0})$ for all $z \in \Sigma_\phi$, i.e. $f \in H_0^\infty(\Sigma_\phi)$.

(1) $\Rightarrow$ (3). If $f \in H_0^\infty(\Sigma_\phi)$, then f is bounded by definition. Moreover, for $|z| \to 0$ we have $|f(z)| \lesssim |z|^s$, so f vanishes polynomially at 0. For $g(z) := f(1/z)$ (holomorphic on Σ_ϕ), the bound $|f(w)| \lesssim |w|^{-s}$ as $|w| \to \infty$ becomes

$$|g(z)| = |f(1/z)| \lesssim |z|^s \qquad (|z| \to 0),$$

so g also vanishes polynomially at 0, and in particular g is bounded near 0. On the other hand, as $|z| \to \infty$, $1/z \to 0$ so $g(z) = f(1/z)$ stays bounded. Thus f and $f(1/z)$ are bounded and both vanish polynomially at 0.

(3) $\Rightarrow$ (1). Assume f and $g(z) := f(1/z)$ are bounded and both satisfy $|f(z)| \lesssim |z|^s$ and $|g(z)| \lesssim |z|^s$ as $|z| \to 0$. Then for $|z| \leq 1$ we get $|f(z)| \lesssim |z|^s$. For $|z| \geq 1$, write $w = 1/z$ so $|w| \leq 1$ and

$$|f(z)| = |g(w)| \lesssim |w|^s = |z|^{-s}.$$

Combining the two regions yields $|f(z)| \lesssim \min(|z|^s, |z|^{-s})$ on Σ_ϕ, hence $f \in H_0^\infty(\Sigma_\phi)$. □

Remark 2.2.5. Typical examples include $f(z) = e^{-tz}$ and $f(z) = z(1+z)^{-2}$, both of which decay polynomially at infinity and vanish at the origin.

2.2.2 The extended Dunford–Riesz class

Some functions of practical interest, such as $(1+z)^{-1}$ or constants, do not belong to $H_0^\infty(\Sigma_\phi)$ since they do not vanish at both 0 and ∞. To accommodate them, we enlarge our function space.

Definition 2.2.6 (Extended Dunford–Riesz class)**.** The *extended Dunford–Riesz class* on Σ_ϕ is defined as

$$\mathcal{E}(\Sigma_\phi) := H_0^\infty(\Sigma_\phi) + \operatorname{span}\{1, (1+z)^{-1}\}$$

since the main components missing in $H_0^\infty(\Sigma_\phi)$ are 1 and $\frac{1}{1+z}$.

Lemma 2.2.7 (Characterization of the extended Dunford–Riesz class)**.** *Let $\phi \in (0, \pi]$ and let $f : \Sigma_\phi \to \mathbb{C}$ be holomorphic. The following assertions are equivalent:*

1. *$f \in \mathcal{E}(\Sigma_\phi)$;*

2. *f is bounded on Σ_ϕ and has finite polynomial limits at both 0 and ∞.*

Proof. $(i) \Rightarrow (ii)$ follows directly from the definition of $\mathcal{E}(\Sigma_\phi)$: boundedness is part of the assumption, and the finite polynomial limits at 0 and ∞ are required by construction.

Conversely, assume (ii). If f is bounded and possesses finite polynomial limits c_0 at 0 and c_∞ at ∞, then $f - c_0(1+z)^{-1} - c_\infty z(1+z)^{-1}$ decays polynomially at both ends. Hence this difference belongs to $H_0^\infty(\Sigma_\phi)$, which implies that $f \in \mathcal{E}(\Sigma_\phi)$ by definition of the extended class. □

Remark 2.2.8. This lemma shows that $\mathcal{E}(\Sigma_\phi)$ can be viewed as the smallest subalgebra of $H^\infty(\Sigma_\phi)$ containing $H_0^\infty(\Sigma_\phi)$ and all constant functions. In particular, the functions $f(z) \equiv 1$ and $f(z) = (1+z)^{-1}$, which will play a role in the regularization procedure, both belong to $\mathcal{E}(\Sigma_\phi)$ by definition.

The classes $H_0^\infty(\Sigma_\phi)$ and $\mathcal{E}(\Sigma_\phi)$ provide the natural domain for the holomorphic functional calculus of sectorial operators.

- For $f \in H_0^\infty(\Sigma_\phi)$, the Cauchy integral defining $f(A)$ converges absolutely thanks to the polynomial decay at 0 and ∞.

- For $f \in \mathcal{E}(\Sigma_\phi)$, one can still define $f(A)$ through a small computation we will depict in the next section.

Hence, these spaces constitute the analytic setting in which the holomorphic functional calculus for sectorial operators can be consistently extended beyond bounded operators.

2.3 The H^∞-Functional Calculus

This section depicts the definition and the main properties of the so-called H^∞-Functional Calculus for sectorial operators. For further details on the construction of the calculus, one can refer to Haase.

2.3.1 Definition for Holomorphic functions

For bounded operators $T \in \mathcal{L}(X)$ and f holomorphic on a neighbourhood of $\sigma(T)$, the Dunford–Riesz formula

$$f(T) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - T)^{-1}\, d\lambda$$

follows from Cauchy's theorem and the polynomial approximation of f (see 1.3). For sectorial (unbounded) $A \in \mathrm{Sect}(\omega)$ (see Definition 2.1.1), the spectrum is unbounded but contained in a sector $\overline{\Sigma_\phi}$, $\phi > \omega$, and the resolvent satisfies

$$\|(\lambda I - A)^{-1}\| \leq \frac{M_\phi}{|\lambda|} \qquad (\lambda \in \mathbb{C} \setminus \overline{\Sigma_\phi}).$$

Proposition 2.3.1. *Let $A \in \mathrm{Sect}(\omega)$ and let $\phi > \omega$. For every $f \in H_0^\infty(\Sigma_\phi)$ and every $\phi' \in (\omega, \phi)$, define*

$$f(A) := \frac{1}{2\pi i} \int_{\Gamma_{\phi'}} f(\lambda)(\lambda I - A)^{-1}\, d\lambda,$$

where $\Gamma_{\phi'}$ is a contour consisting of the two rays $\lambda = re^{\pm i\phi'}$ $(r \in [\varepsilon, R])$, closed by circular arcs $\{|\lambda| = \varepsilon\}$ and $\{|\lambda| = R\}$, and where the limits $\varepsilon \to 0$ and $R \to \infty$ are taken.

Then:

1. *the integral defining $f(A)$ converges absolutely (hence defines a bounded operator);*
2. *the value of $f(A)$ is independent of the choice of $\phi' \in (\omega, \phi)$ and of the particular admissible contour inside $\Sigma_{\phi'}$.*

Proof. prove the first point. Since $f \in H_0^\infty(\Sigma_\phi)$, by definition there exist constants $C, s > 0$ such that

$$|f(\lambda)| \leq C \min(|\lambda|^s, |\lambda|^{-s}), \qquad \lambda \in \Sigma_\phi.$$

Fix $\phi' \in (\omega, \phi)$. For $\lambda = re^{\pm i\phi'}$, the sectorial resolvent estimate yields

$$\|(\lambda I - A)^{-1}\| \leq \frac{M_{\phi'}}{|\lambda|} = \frac{M_{\phi'}}{r}.$$

Hence, along each ray,

$$\|f(\lambda)(\lambda I - A)^{-1}\| \leq C\, \frac{M_{\phi'}}{r} \min(r^s, r^{-s}) = CM_{\phi'} \min(r^{s-1}, r^{-s-1}).$$

Since $s > 0$, the function $r \mapsto \min(r^{s-1}, r^{-s-1})$ is integrable on $(0, \infty)$: it behaves like r^{s-1} near 0 and like r^{-s-1} at infinity. This proves absolute convergence of the integral on the rays.

We prove now the second point. On the inner arc $|\lambda| = \varepsilon$, we have $|f(\lambda)| \leq C|\lambda|^s =$

$C\varepsilon^s$ and $\|(\lambda I - A)^{-1}\| \leq M_{\phi'}/|\lambda| = M_{\phi'}/\varepsilon$. Therefore

$$\int_{|\lambda|=\varepsilon} \|f(\lambda)(\lambda I - A)^{-1}\| \, |d\lambda| \leq (2\pi\varepsilon) \, C\varepsilon^s \, \frac{M_{\phi'}}{\varepsilon} \lesssim \varepsilon^s \xrightarrow[\varepsilon \to 0]{} 0.$$

Similarly, on the outer arc $|\lambda| = R$, we have $|f(\lambda)| \leq C|\lambda|^{-s} = CR^{-s}$ and $\|(\lambda I - A)^{-1}\| \leq M_{\phi'}/R$, hence

$$\int_{|\lambda|=R} \|f(\lambda)(\lambda I - A)^{-1}\| \, |d\lambda| \leq (2\pi R) \, CR^{-s} \, \frac{M_{\phi'}}{R} \lesssim R^{-s} \xrightarrow[R \to \infty]{} 0.$$

Thus the arc contributions vanish, and the improper integral defining $f(A)$ is well-defined.

Let finally Γ and Γ' be two admissible contours contained in $\Sigma_{\phi'}$ and oriented counterclockwise. Let D be the domain enclosed between Γ and Γ'. For $z \in D$, we have $z \in \rho(A)$, so the map

$$z \longmapsto f(z)(zI - A)^{-1}$$

is holomorphic on D with values in $\mathcal{L}(X)$ (product of holomorphic operator-valued functions). By Cauchy's theorem for Banach-valued holomorphic functions,

$$\int_{\partial D} f(z)(zI - A)^{-1} \, dz = 0.$$

Since $\partial D = \Gamma - \Gamma'$ (with the induced orientations), it follows that

$$\int_{\Gamma} f(z)(zI - A)^{-1} \, dz = \int_{\Gamma'} f(z)(zI - A)^{-1} \, dz.$$

Dividing by $2\pi i$ yields that $f(A)$ is independent of the contour, and in particular independent of the choice of $\phi' \in (\omega, \phi)$. $\square$

This yields the following definition :

Definition 2.3.2. Let $A \in \mathrm{Sect}(\omega)$ and $f \in H_0^\infty(\Sigma_\phi)$ for some $\phi > \omega$. The H^∞-functional calculus is defined by

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda I - A)^{-1} \, d\lambda, \tag{2.3}$$

where Γ is a contour surrounding the spectrum of A inside Σ_ϕ.

Moreover, one can enlarge the definition to the constant functions, by just setting $c(A) = cI$ for some $c \in \mathbb{C}$.

Theorem 2.3.3 (Algebra homomorphism)**.** *Let $A \in \mathrm{Sect}(\omega)$ and $\phi > \omega$. The map*

$$\Phi : H_0^\infty(\Sigma_\phi) \to \mathcal{L}(X), \qquad \Phi(f) = f(A),$$

defined by the contour integral (2.3), is a homomorphism of algebra:

(i) Linearity: $(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A)$.

(ii) Multiplicativity: $(fg)(A) = f(A)g(A)$.

(iii) Unit: $1(A) = I$.

Proof. (i) and (iii) follow from linearity of the integral and the setting for the constant functions.

For (ii), choose two contours Γ_λ and Γ_μ lying on the boundary rays of the sector, with Γ_μ strictly inside Γ_λ. Then, by definition of the functional calculus,

$$f(A)g(A) = \Big(\frac{1}{2\pi i}\int_{\Gamma_\lambda} f(\lambda)R(\lambda, A)\, d\lambda\Big)\Big(\frac{1}{2\pi i}\int_{\Gamma_\mu} g(\mu)R(\mu, A)\, d\mu\Big),$$

By Fubini's theorem, the integrability conditions allow us to interchange the order of integration, and we can apply the resolvent identity (see 1.3.7).

$$R(\lambda, A)R(\mu, A) = \frac{R(\lambda, A) - R(\mu, A)}{\mu - \lambda}.$$

Hence,

$$\begin{aligned}
f(A)g(A) &= \frac{1}{(2\pi i)^2}\int_{\Gamma_\lambda}\int_{\Gamma_\mu} f(\lambda)g(\mu)R(\lambda, A)R(\mu, A)\, d\mu\, d\lambda \\
&= \frac{1}{(2\pi i)^2}\int_{\Gamma_\lambda}\int_{\Gamma_\mu} \frac{f(\lambda)g(\mu)}{\mu - \lambda}\big(R(\lambda, A) - R(\mu, A)\big)\, d\mu\, d\lambda \\
&= \frac{1}{(2\pi i)^2}\int_{\Gamma_\lambda}\int_{\Gamma_\mu} \frac{f(\lambda)g(\mu)}{\mu - \lambda}R(\lambda, A)\, d\mu\, d\lambda - \frac{1}{(2\pi i)^2}\int_{\Gamma_\lambda}\int_{\Gamma_\mu} \frac{f(\lambda)g(\mu)}{\mu - \lambda}R(\mu, A)\, d\mu\, d\lambda.
\end{aligned}$$

Now, for fixed λ (outside Γ_μ), the inner integral in the first term is

$$\int_{\Gamma_\mu} \frac{g(\mu)}{\mu - \lambda}\, d\mu = 2\pi i\, g(\lambda)$$

by Cauchy's integral formula. Similarly, in the second term, for fixed μ (inside Γ_λ), the inner integral equals

$$\int_{\Gamma_\lambda} \frac{f(\lambda)}{\mu - \lambda}\, d\lambda = 0,$$

by Cauchy Theorem.

Combining both contributions, we obtain

$$f(A)g(A) = \frac{1}{2\pi i}\int_{\Gamma_\lambda} f(\lambda)g(\lambda)R(\lambda, A)\, d\lambda = (fg)(A).$$

This proves the multiplicative property of the functional calculus for sectorial operators. $\square$

2.3.2 Full definition for the extended Riesz-Dunford class

Following Haase, to extend the formula to the full class, we introduce the auxiliary class

$$H^{(0)}(\Sigma_\phi) := \Big\{\, J \in H^\infty(\Sigma_\phi) \;:\; J \text{ is regularly decaying at } \infty \text{ and holomorphic at } 0 \,\Big\}.$$

By construction $H^{(0)}(\Sigma_\phi) \subset \mathcal{E}(\Sigma_\phi)$ (see Definition 2.2.6), and every $J \in H^{(0)}(\Sigma_\phi)$ can be represented near 0 by a holomorphic Taylor expansion, while its decay at ∞ guarantees integrability on the sector rays.

Lemma 2.3.4 (Cauchy formula with a small circle)**.** *Let $A \in \mathrm{Sect}(\omega)$ and fix $\omega < \phi' < \phi$. Let $J \in H^{(0)}(\Sigma_\phi)$ and choose $\delta > 0$ so small that J is holomorphic on $\overline{B_\delta(0)}$ and $B_\delta(0) \subset \Sigma_\phi$. Let Γ be the positively oriented boundary of $\Sigma_{\phi'} \cup B_\delta(0)$, i.e. the two rays $\arg z = \pm\phi'$ (traversed outwards) closed by the arc $|z| = \delta$. Then*

$$J(A) \;=\; \frac{1}{2\pi i}\int_\Gamma J(z)\,(\,zI - A\,)^{-1}\, dz.$$

Proof. Let first be $J \in H^{(0)}(\Sigma_\phi) \cap H_0^\infty(\Sigma_\phi)$. Then, one can shrink the small ball of ray δ to 0, so that the integral over this arc is going to 0, and we get the result.

Now, let $J \in H^{(0)}(\Sigma_\phi)$, then by definition, one can write $J = g + c(1+z)^{-1}$, with $g \in H^{(0)}(\Sigma_\phi) \cap H_0^\infty(\Sigma_\phi)$. Indeed, since $J \in H^{(0)}(\Sigma_\phi)$, by definition J is holomorphic on $\Sigma_\phi \setminus \{0\}$ and has at most a simple pole at 0. Hence there exist $c \in \mathbb{C}$ and a function g holomorphic on Σ_ϕ such that

$$J(z) = g(z) + \frac{c}{1+z}.$$

Moreover, the function g vanishes polynomially at both 0 and ∞, so that $g \in H_0^\infty(\Sigma_\phi)$. Thus, we only have to prove the result for $(1+z)^{-1}$. Set $R > 1$, and the new contour Γ' as the negatively oriented boundary of $\Sigma_{\phi'} \cup B_R(0)$. Then, by construction of the contour (avoiding the pole -1), and by the Cauchy Theorem, one gets :

$$\int_{\Gamma'} (1+z)^{-1} R(z,A)\, dz = 0$$

Thus, we can write :

$$\int_{\Gamma} (1+z)^{-1} R(z,A)\, dz = \int_{\Gamma} (1+z)^{-1} R(z,A)\, dz + \int_{\Gamma'} (1+z)^{-1} R(z,A)\, dz$$

By construction of the contours, the rays $\arg z = \pm\phi'$ are traversed in opposite directions in Γ and Γ'. Therefore the integrals over these rays cancel each other.

Consequently, the only remaining part of the contour is a closed curve surrounding the point $z = -1$. We denote this remaining contour by Γ''. It is negatively oriented.

Hence

$$\int_{\Gamma} (1+z)^{-1} R(z,A)\, dz = \int_{\Gamma''} (1+z)^{-1} R(z,A)\, dz.$$

Applying then the Cauchy integral formula, the remaining integral gives:

$$\frac{1}{2\pi i} \int_{\Gamma''} (1+z)^{-1} R(z,A)\, dz = -R(-1,A)$$

since $\Gamma^{''}$ is negatively oriented. This is exactly the formula we were seeking since $-R(-1,A) = (1+A)^{-1}$.

By linearity, the statement follows for $J = g + c(1+z)^{-1}$, concluding the proof. $\square$

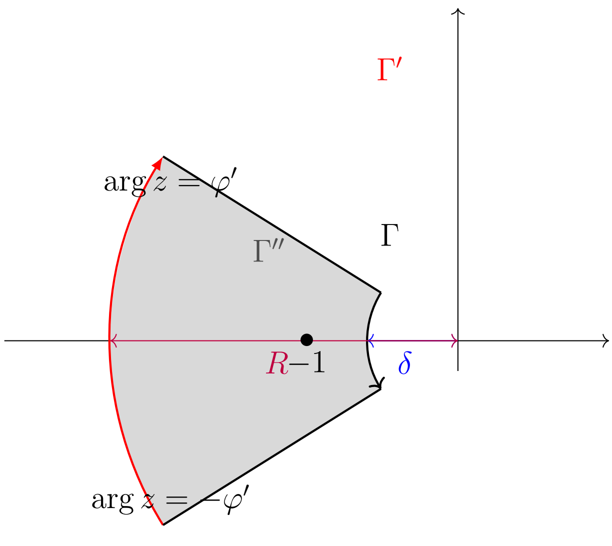

Figure 2.2: The contours $\Gamma^{'}$ and Γ are compensating

We are now in a position to extend the Dunford–Riesz functional calculus to a larger class of holomorphic functions.

Theorem 2.3.5 (Primary functional calculus on the extended class)**.** *Let $A \in \mathrm{Sect}(\omega)$ on X and fix $\phi \in (\omega, \pi)$. Define $\Phi : \mathcal{E}(\Sigma_\phi) \to \mathcal{L}(X)$ by*

$$\Phi(J) = J(A) := \frac{1}{2\pi i} \int_\Gamma J(z)\, R(z, A)\, dz, \qquad R(z, A) = (zI - A)^{-1},$$

where Γ is any admissible contour as in Lemma 2.3.4. Then Φ is a homomorphism of algebra. Moreover:

(a) $(z(1+z)^{-1})(A) = A(1+A)^{-1}$.

(b) If B is a closed operator commuting with all resolvents $R(\lambda, A)$ for $\lambda \in \rho(A)$, then B commutes with $J(A)$ for every $J \in \mathcal{E}(\Sigma_\phi)$. In particular, each $J(A)$ commutes with A.

(c) If $x \in N(A) = \{x \in D(A) | Ax = 0\}$ and $J \in \mathcal{E}(\Sigma_\phi)$, then $J(A)x = J(0)\, x$.

Proof. Let $J \in \mathcal{E}(\Sigma_\phi)$. By definition of $\mathcal{E}(\Sigma_\phi)$, there exist $f \in H_0^\infty(\Sigma_\phi)$ and $a, b \in \mathbb{C}$ such that

$$J(z) = f(z) + a + b(1+z)^{-1}.$$

Using the linearity of the contour integral and Lemma 2.3.4, we define

$$J(A) = f(A) + aI + b(1+A)^{-1}.$$

This coincides with the Dunford–Riesz representation

$$J(A) = \frac{1}{2\pi i} \int_\Gamma J(z)\, R(z, A)\, dz$$

for every admissible contour Γ.

We now prove that Φ is multiplicative. Let

$$J_1(z) = f_1(z) + a_1 + b_1(1+z)^{-1}, \qquad J_2(z) = f_2(z) + a_2 + b_2(1+z)^{-1},$$

with $f_1, f_2 \in H_0^\infty(\Sigma_\phi)$ and $a_i, b_i \in \mathbb{C}$. Expanding the product $J_1 J_2$ shows that it suffices to establish multiplicativity for the elementary products involving $f \in H_0^\infty(\Sigma_\phi)$ and $(1+z)^{-1}$.

For two functions $f_1, f_2 \in H_0^\infty(\Sigma_\phi)$, multiplicativity

$$(f_1 f_2)(A) = f_1(A) f_2(A)$$

is the standard property of the H_0^∞–functional calculus (see Theorem 2.3.3). Indeed, using the Dunford representation for both operators gives

$$f_1(A)f_2(A) = \frac{1}{(2\pi i)^2} \int_{\Gamma_\lambda} \int_{\Gamma_\mu} f_1(\lambda)f_2(\mu)\, R(\lambda, A)R(\mu, A)\, d\mu\, d\lambda.$$

Since the integrals converge absolutely, Fubini's theorem allows interchanging the order of integration. The resolvent identity

$$R(\lambda, A)R(\mu, A) = \frac{R(\lambda, A) - R(\mu, A)}{\mu - \lambda}$$

reduces the double integral to a contour integral in the inner variable, to which Cauchy's formula applies. This yields precisely the Dunford integral defining $(f_1f_2)(A)$.

Next consider a product involving $(1 + z)^{-1}$. Let $f \in H_0^\infty(\Sigma_\phi)$. Since $(1 + z)^{-1}$ is holomorphic on Σ_ϕ and behaves like $|z|^{-1}$ at infinity, the function $f(1 + z)^{-1}$ again belongs to $H_0^\infty(\Sigma_\phi)$. Using the contour representations for $f(A)$ and $(1 + A)^{-1}$ gives

$$f(A)(1 + A)^{-1} = \frac{1}{(2\pi i)^2} \int_{\Gamma_\lambda} \int_{\Gamma_\mu} f(\lambda)\frac{1}{1 + \mu} R(\lambda, A)R(\mu, A)\, d\mu\, d\lambda.$$

Applying Fubini's theorem and the resolvent identity yields an inner integral of the form

$$\frac{1}{2\pi i} \int_{\Gamma_\mu} \frac{1}{(1 + \mu)(\mu - \lambda)}\, d\mu.$$

The integrand has a simple pole at $\mu = \lambda$ inside the contour, so Cauchy's formula gives

$$\frac{1}{2\pi i} \int_{\Gamma_\mu} \frac{1}{(1 + \mu)(\mu - \lambda)}\, d\mu = \frac{1}{1 + \lambda}.$$

Substituting this into the previous expression yields

$$f(A)(1 + A)^{-1} = \frac{1}{2\pi i} \int_{\Gamma_\lambda} \frac{f(\lambda)}{1 + \lambda} R(\lambda, A)\, d\lambda = (f(1 + z)^{-1})(A).$$

The same argument shows that

$$((1 + z)^{-1}f)(A) = (1 + A)^{-1}f(A).$$

Finally, applying the same double-contour argument to the function $(1 + z)^{-2}$ shows that

$$((1 + z)^{-2})(A) = (1 + A)^{-2}.$$

Collecting the previous identities proves that

$$(J_1 J_2)(A) = J_1(A) J_2(A),$$

so Φ is an algebra homomorphism.

Property (a) follows from the identity

$$z(1+z)^{-1} = 1 - (1+z)^{-1}.$$

Applying the functional calculus gives

$$(z(1+z)^{-1})(A) = I - (1+A)^{-1}.$$

Since

$$A(1+A)^{-1} = (1+A-1)(1+A)^{-1} = I - (1+A)^{-1},$$

we conclude that

$$(z(1+z)^{-1})(A) = A(1+A)^{-1}.$$

For (b), assume that B is closed and commutes with every resolvent $R(\lambda, A)$. Let $f \in H_0^\infty(\Sigma_\phi)$. For $x \in D(B)$ one has

$$Bf(A)x = \frac{1}{2\pi i}\int_\Gamma f(z)\, BR(z,A)x\, dz = \frac{1}{2\pi i}\int_\Gamma f(z)\, R(z,A)Bx\, dz = f(A)Bx.$$

Thus $Bf(A)x = f(A)Bx$ for all $x \in D(B)$. Since $f(A)$ is bounded and B is closed, this implies $Bf(A) = f(A)B$. Constants commute trivially with B, and $(1+A)^{-1} = R(-1, A)$ also commutes with B by assumption. Consequently B commutes with $J(A)$ for every $J \in \mathcal{E}(\Sigma_\phi)$.

Taking $B = A$ yields that A commutes with $J(A)$ on $D(A)$.

For (c), let $x \in N(A)$. Then $(zI - A)x = zx$, hence

$$R(z, A)x = \frac{1}{z}x.$$

Using the contour representation of $J(A)$ gives

$$J(A)x = \frac{1}{2\pi i}\int_\Gamma \frac{J(z)}{z}\, dz\, x.$$

Since J is holomorphic at 0 and Γ surrounds the origin, Cauchy's formula yields

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{J(z)}{z}\,dz = J(0).$$

Therefore $J(A)x = J(0)x$. □

Remark 2.3.6. In [4], Haase defined what we just depicted as a primary functional calculus. Indeed, he introduced a wider set of concepts, including the one of abstract functional calculus; it allows him to give a more abstract sense to the functional calculus, so that he can give very general lemmas and properties. For our purposes, the definition of the primary function calculus will be enough to understand the underlying properties and the way to compute $f(A)$.

2.4 The Regularization Procedure

The class $\mathcal{E}(\Sigma_\phi)$ provides a natural setting for defining $f(A)$ through the Cauchy integral, but it is too restrictive for many applications. In particular, functions with polynomial growth do not belong to $\mathcal{E}(\Sigma_\phi)$ since they fail to have finite limit at ∞. To handle such functions, one introduces the notion of *regularization*, following the path built by [4].

2.4.1 Definition and properties

Definition 2.4.1 (Regularizable functions)**.** Let $A \in \mathrm{Sect}(\omega)$ and $\phi > \omega$. A holomorphic function f on Σ_ϕ is said to be *regularizable* (with respect to A) if there exists $e \in \mathcal{E}(\Sigma_\phi)$ such that:

1. $e(A)$ is injective (i.e. $\ker e(A) = \{0\}$);
2. $ef \in \mathcal{E}(\Sigma_\phi)$.

In this case, e is called a *regularizer* for f, and we define

$$f(A) := e(A)^{-1}\,(ef)(A). \tag{2.4}$$

Intuitively, the regularizer e serves as a "decaying envelope" that renders the product ef integrable along the contour defining the functional calculus. The operator $f(A)$ is then obtained by inverting $e(A)$ after applying the calculus to $ef \in \mathcal{E}(\Sigma_\phi)$.

Proposition 2.4.2 (Well-posedness of the regularized calculus)**.** *Let f be regularizable with respect to A. Then:*

1. *The definition (2.4) is independent of the choice of e;*
2. *The mapping $f \mapsto f(A)$ satisfies*

$$(fg)(A) = f(A)g(A), \qquad (f+g)(A) = f(A) + g(A),$$

whenever both sides are well-defined.

Proof. Suppose e_1, e_2 are two regularizers for f. Then, by definition, both $e_1 f$ and $e_2 f$ belong to $\mathcal{E}(\Sigma_\phi)$, and $e_1(A)$, $e_2(A)$ are injective. Moreover, $(e_1e_2)(A)$ is injective and $(e_1e_2f) \in \mathcal{E}(\Sigma_\phi)$. By definition and properties of the $H^\infty-calculus$ (see Theorem 2.3.5), we have

$$e_1(A)e_2(A) = (e_1e_2)(A) = (e_2e_1)(A) = e_2(A)e_1(A)$$

Thus, inverting gives $(e_1)(A)^{-1}(e_2)(A)^{-1} = (e_2)(A)^{-1}(e_1)(A)^{-1}$. Now :

$$\begin{aligned} f(A) = (e_1)(A)^{-1}(e_1f)(A) &= (e_1)(A)^{-1}(e_2)(A)^{-1}(e_2)(A)(e_1f)(A) \\ &= (e_2)(A)^{-1}(e_1)(A)^{-1}(e_2e_1f)(A) \\ &= (e_2)(A)^{-1}(e_1)(A)^{-1}(e_1e_2f)(A) \\ &= (e_2)(A)^{-1}(e_1)(A)^{-1}e_1(A)(e_2f)(A) \\ &= (e_2)(A)^{-1}(e_2f)(A) \end{aligned}$$

The algebraic properties follow from the fact that multiplication is preserved for all functions in $\mathcal{E}(\Sigma_\phi)$, and regularizers can be chosen consistently for sums and products. □

Remark 2.4.3. The construction (2.4) allows one to define $f(A)$ for a wide range of functions, even those that grow polynomially or logarithmically at infinity, provided one can "compensate" the growth by a decaying factor e. This greatly extends the class of admissible symbols beyond bounded functions, so it has a really great theoritical power.

Some examples are :

1. For $f(z) = z^{-\alpha}$ $(\alpha > 0)$, choose $e(z) = \dfrac{z}{(1+z)^2}$. Then $ef(z) = \dfrac{z^{1-\alpha}}{(1+z)^2} \in \mathcal{E}(\Sigma_\phi)$ by Lemma 2.2.7 and $e(A) = A(1+A)^{-2}$ is injective.

2. For $f(z) = log(1+z)$, take $e(z) = (1+z)^{-2}$. Then $ef(z) = \dfrac{log(1+z)}{(1+z)^2} \in \mathcal{E}(\Sigma_\phi)$, hence $f(A) = log(1+A)$ is well-defined.

2.4.2 Extension to polynomially growing functions

As previously mentioned, the regularization procedure can be used to extend the H^∞-functional calculus to functions of controlled polynomial growth. For $\alpha \geq 0$, define

$$H^\infty_\alpha(\Sigma_\phi) = \{f \in \mathcal{O}(\Sigma_\phi) : \exists C > 0,\ |f(z)| \leq C(1+|z|)^\alpha,\ \forall z \in \Sigma_\phi\}.$$

The parameter α measures the growth rate of f at infinity.

Proposition 2.4.4 (Regularization of polynomially growing functions)**.** *Let $A \in$ Sect(ω) and $\phi > \omega$. Let $\alpha \geq 0$ and $f \in H^\infty_\alpha(\Sigma_\phi)$, then f is* regularizable *with any regularizer of the form*

$$e(z) = (1+z)^{-m}, \qquad m > \alpha.$$

In particular, $e \in \mathcal{E}(\Sigma_\phi)$ and $ef \in \mathcal{E}(\Sigma_\phi)$. Hence, the regularized value

$$f(A) := e(A)^{-1}(ef)(A)$$

is well defined.

Proof. Fix $m > \alpha$ and define $e(z) = (1+z)^{-m}$. Since e is holomorphic on Σ_ϕ and satisfies

$$|e(z)| \leq C(1+|z|)^{-m},$$

it follows that e has finite polynomial limits both at 0 and ∞. Hence $e \in \mathcal{E}(\Sigma_\phi)$.

Next, using the growth of f, we have

$$|e(z)f(z)| \leq C(1+|z|)^{\alpha-m}.$$

Because $\alpha - m < 0$, the right-hand side decays at infinity and has polynomial limit at 0. Thus $ef \in \mathcal{E}(\Sigma_\phi)$.

Now, $(1+A)^{-m}$ is injective: indeed, $(1+A)$ is invertible for every sectorial operator A ($-1 \in \rho(A)$ since we choose an angle for the sector that is different from π), its powers are injective as well. Therefore $e(A)$ is invertible in $\mathcal{L}(X)$, so e is an admissible regularizer for f. Finally, since $ef \in \mathcal{E}(\Sigma_\phi)$, the Cauchy integral

$$(ef)(A) = \frac{1}{2\pi i} \int_\Gamma e(z)f(z)R(z,A)\,dz$$

is well defined, and so is $e(A)$. Hence $f(A) = e(A)^{-1}(ef)(A)$ is well defined. $\square$

Remark 2.4.5. The use of $e \in \mathcal{E}(\Sigma_\phi)$ rather than $H_0^\infty(\Sigma_\phi)$ is essential here: it allows regularizers such as $(1+z)^{-1}$ or $(1+z)^{-m}$, which are not vanishing at 0 and therefore lie outside $H_0^\infty(\Sigma_\phi)$. This extended regularization is the cornerstone of the natural functional calculus in the sense of [4].

One can easily see that $H^\infty(\Sigma_\phi)$ is a particular case here, setting $\alpha = 0$. The extension procedure allows us now to consider a wider class of functions for the functional calculus. Indeed, setting $\mathcal{M}(\Sigma_\phi)$ as the set of the meromorphic functions on Σ_ϕ, i.e. the set of functions that are holomorphic almost everywhere (except in a finite number of poles), one can now define :

$$\mathcal{M}(\Sigma_\phi)_A = \{f \in \mathcal{M}(\Sigma_\phi) \text{ s.t } \exists\ e \in \mathcal{E}(\Sigma_\phi) \mid ef \in \mathcal{E}(\Sigma_\phi)\}$$

The previous proof then immediatly rises $H^\infty(\Sigma_\phi) \subset M(\Sigma_\phi)_A$. We also state the following theorem to end this subsection :

Theorem 2.4.6 (Fundamental properties of the functional calculus)**.** *Let A be a closed operator on a Banach space X, and let Φ be the functional calculus for A as we defined previously. For every $f \in \mathcal{M}(\Sigma_\omega)_A$, the following hold:*

1. *If $T \in \mathcal{L}(X)$ commutes with A, then T also commutes with $f(A)$. In particular, if $f(A)$ is bounded, then $f(A)$ commutes with A.*

2. *The calculus preserves constants and the identity:*

$$1(A) = I, \qquad z(A) = A.$$

3. *For $g \in \mathcal{M}(\Sigma_\omega)_A$, one has*

$$(f+g)(A) \subset f(A) + g(A), \qquad (fg)(A) \subset f(A)g(A),$$

whenever the right-hand side is well-defined and bounded.

4. *If $g \in H(A) = \{f \in \mathcal{M}(\Sigma_\omega)_A \mid f(A) \in \mathcal{L}(X)\}$ and $g(A)$ is invertible, then*

$$f(A) = g(A)^{-1}f(A)g(A).$$

5. *For $\lambda \in \mathbb{C}$, one has*

$$\lambda \notin \sigma(f(A)) \quad \Longleftrightarrow \quad (\lambda - f)^{-1} \in H(A),$$

and in this case

$$(\lambda - f(A))^{-1} = (\lambda - f)^{-1}(A).$$

Proof. Recall that for $f \in \mathcal{M}(\Sigma_\phi)_A$ there exists a *regularizer* $e \in \mathcal{E}(\Sigma_\phi)$ such that $ef \in \mathcal{E}(\Sigma_\phi)$ and such that $e(A) \in \mathcal{L}(X)$ is injective. The extended calculus is then defined by

$$f(A) := e(A)^{-1}(ef)(A),$$

with domain $D(f(A)) = \mathrm{Ran}(e(A))$, and this definition is independent of the choice of the regularizer.

1. Assume $T \in \mathcal{L}(X)$ commutes with A. Then T commutes with every resolvent $R(\lambda, A)$, hence by Theorem 2.3.5(b) it commutes with $J(A)$ for every $J \in \mathcal{E}(\Sigma_\phi)$. In particular, for a regularizer e as above, T commutes with $e(A)$ and with $(ef)(A)$. Therefore, on $\mathrm{Ran}(e(A))$,

$$Tf(A) = Te(A)^{-1}(ef)(A) = e(A)^{-1}T(ef)(A) = e(A)^{-1}(ef)(A)T = f(A)T.$$

If moreover $f(A) \in \mathcal{L}(X)$ is bounded, then $f(A)$ commutes with A by applying the previous statement with $T = f(A)$.

2. For the primary calculus on $\mathcal{E}(\Sigma_\phi)$ one has $1(A) = I$ and $z(A) = A$. For the extended calculus, choose $e \equiv 1$ as regularizer for constants. For $f(z) = z$, take any $e \in \mathcal{E}(\Sigma_\phi)$ such that $ez \in \mathcal{E}(\Sigma_\phi)$ and use $z(A) = e(A)^{-1}(ez)(A) = A$.

3. Let $f, g \in \mathcal{M}(\Sigma_\phi)_A$. Choose regularizers $e_f, e_g \in \mathcal{E}(\Sigma_\phi)$ for f and g respectively. Then $e := e_f e_g$ is a regularizer for f, g, $f+g$ and fg, and $ef, eg, e(f+g), e(fg) \in \mathcal{E}(\Sigma_\phi)$. On $\mathrm{Ran}(e(A))$ we have

$$(f+g)(A) = e(A)^{-1}e(f+g)(A) = e(A)^{-1}\big((ef)(A) + (eg)(A)\big),$$

and similarly

$$(fg)(A) = e(A)^{-1}e(fg)(A) = e(A)^{-1}(ef \cdot g)(A).$$

Using that Φ is a homomorphism on $\mathcal{E}(\Sigma_\phi)$, one gets

$$e(fg)(A) = (ef \cdot eg)(A) = (ef)(A)(eg)(A).$$

These identities yield

$$(f+g)(A) \subset f(A)+g(A), \qquad (fg)(A) \subset f(A)g(A),$$

where the inclusions reflect the fact that all operators are initially defined on the common core $\mathrm{Ran}(e(A))$, and that the right-hand sides require boundedness to be unambiguously defined on the whole space.

4. Let $g \in H(A)$ so that $g(A) \in \mathcal{L}(X)$, and assume $g(A)$ is invertible. Pick a regularizer e for f and note that $g \in \mathcal{M}(\Sigma_\phi)_A$ with the trivial regularizer 1. Using (3) on $\mathcal{E}(\Sigma_\phi)$ and the fact that Φ is a homomorphism there, one obtains on $\mathrm{Ran}(e(A))$:

$$g(A)f(A) = g(A)e(A)^{-1}(ef)(A) = e(A)^{-1}g(A)(ef)(A) = e(A)^{-1}(g{\cdot}ef)(A) = (gf)(A).$$

Multiplying on the left by $g(A)^{-1}$ gives

$$f(A) = g(A)^{-1}(gf)(A).$$

Applying the same argument to fg and using associativity yields the announced conjugation identity

$$f(A) = g(A)^{-1}f(A)g(A).$$

5. Let $\lambda \in \mathbb{C}$. If $(\lambda - f)^{-1} \in H(A)$, then $(\lambda - f)^{-1}(A) \in \mathcal{L}(X)$ and, by point *3.*, we have

$$(\lambda I - f(A))(\lambda - f)^{-1}(A) = I = (\lambda - f)^{-1}(A)(\lambda I - f(A))$$

on the natural domain, hence $\lambda \notin \sigma(f(A))$ and

$$(\lambda I - f(A))^{-1} = (\lambda - f)^{-1}(A).$$

Conversely, if $\lambda \notin \sigma(f(A))$, then $(\lambda I - f(A))^{-1} \in \mathcal{L}(X)$. Defining $h := (\lambda - f)^{-1}$, one checks that $h \in \mathcal{M}(\Sigma_\phi)_A$ and that $h(A) = (\lambda I - f(A))^{-1}$, hence $h \in H(A)$. This proves the equivalence and the resolvent identity. □

2.5 Boundedness of the Functional Calculus

For functions in $H^\infty(\Sigma_\phi)$, even if f is bounded, it may happen that $f(A)$ is not bounded. Thus, in this section, we try to give an overview of the topic of boundedness of the Functional Calculus as previously defined. Of course, one could go further (as Haase do, by depicting a complete description of how to build an operator that has no bounded calculus), but for the purpose of applications, we only need a few characterization to go on. In all the following, we denote for A sectorial operator, $D(A) \subset \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. We also recall that by definition, A is densely defined, so that the closure of $D(A)$ equals $\mathcal{H}$.

Definition 2.5.1. Following Haase in [4], the natural $\mathcal{F}$–calculus for A is said to be *bounded* if

$$f(A) \in \mathcal{L}(X), \quad \|f(A)\|_{\mathcal{L}(X)} \leq C\|f\|_{\infty,\Sigma_\phi}, \qquad f \in \mathcal{F},$$

where $\mathcal{F}$ is a subalgebra of $H^\infty(\Sigma_\phi)$ and $C > 0$.

Remark 2.5.2 (Closed graph argument). Assume that $\mathcal{F}$ is a closed subalgebra of $H^\infty(\Sigma_\phi)$ and that A is injective. Suppose that $f(A)$ is well-defined and bounded for every $f \in \mathcal{F}$.

Consider the map

$$\Phi : \mathcal{F} \longrightarrow \mathcal{L}(X), \qquad \Phi(f) = f(A).$$

We claim that Φ has a closed graph. Let (f_n) be a sequence in $\mathcal{F}$ such that

$$f_n \to f \quad \text{in } H^\infty(\Sigma_\phi), \qquad f_n(A) \to T \quad \text{in } \mathcal{L}(X).$$

Since $\mathcal{F}$ is closed in $H^\infty(\Sigma_\phi)$, we have $f \in \mathcal{F}$. Using the definition of the functional calculus and the convergence of f_n to f, one checks that for every x in a core of the calculus (for instance $\mathrm{Ran}(e(A))$) for a suitable regularizer e) one has

$$f_n(A)x \longrightarrow f(A)x.$$

Hence $Tx = f(A)x$ on this dense set. Since both operators are bounded, it follows that $T = f(A)$. Therefore the graph of Φ is closed.

Since $\mathcal{F}$ and $\mathcal{L}(X)$ are Banach spaces, the Closed Graph Theorem implies that Φ is bounded. Consequently there exists a constant $C > 0$ such that

$$\|f(A)\|_{\mathcal{L}(X)} \leq C\|f\|_{\infty,\Sigma_\phi}, \qquad f \in \mathcal{F}.$$

Before going on with the characterization, let us give some definition and results that will be useful in its proof.

Definition 2.5.3. Let $(Q, \preceq)$ be a **directed set**, and let X be a topological space. A **net** in X is a map ϕ:

$$\phi : Q \to X$$

The net is usually denoted by $(\phi_\alpha)_{\alpha\in A}$ or simply (ϕ_α).

One can think a net as a generalized notion of a sequence indexed by real numbers. Now we state the following :

Theorem 2.5.4. *Let A be a sectorial operator of angle ω acting on a Hilbert space $\mathcal{H}$. We denote by*

$$\mathcal{R}(A) = \{Ax \mid x \in D(A)\}$$

the range of A.

Let $0 \leq \omega < \mu < \pi$, and let (ψ_α) be a net in $H_0^\infty(\Sigma_\mu)$ such that $\|\psi_\alpha\|_\infty \to 0$ with α. Then:

- *(i) If there exists $c > 0$ and $s > 0$ such that*

$$|\psi_\alpha(t)| \leq c|t|^s(1 + |t|^{2s})^{-1} \qquad (t \in \Sigma_\mu),$$

  then $\|\psi_\alpha(A)\| \to 0$ with α.

- *(ii) If there exist c, M and $s > 0$ such that $|\psi_\alpha(t)| \leq c|t|^s$ for $|t| < 1$ and $\|\psi_\alpha(A)\| \leq M$ for all α, then if $u \in \mathcal{H}$, $\psi_\alpha(A)u \to 0$.*

- *(iii) If there exists M such that $\|\psi_\alpha(A)\| \leq M$ and if $u \in \overline{\mathcal{R}(A)}$, then $\psi_\alpha(A)u \to 0$.*

Proof. • For point (i), since A is a sectorial operator of angle ω, fix ν such that $\omega < \nu < \mu$. Then $\partial\Sigma_\nu \subset \Sigma_\mu$, and for every $\psi_\alpha \in H_0^\infty(\Sigma_\mu)$ we have the functional calculus representation (see 2.3.1)

$$\psi_\alpha(A) = \frac{1}{2\pi i} \int_{\Gamma_{\delta,\Lambda}} \psi_\alpha(z)(z - A)^{-1}\, dz,$$

where $\Gamma_{\delta,\Lambda}$ consists of the two rays $\{re^{\pm i\nu} : \delta \leq r \leq \Lambda\}$ together with the circular arcs of radii δ and Λ, all lying inside Σ_μ.

By sectoriality of A, there exists $C > 0$ such that

$$\|(z - A)^{-1}\| \leq \frac{C}{|z|} \qquad (z \in \partial\Sigma_\nu).$$

The hypothesis on ψ_α yields, for $z \in \Gamma_{\delta,\Lambda}$,

$$|\psi_\alpha(z)| \leq c\,\frac{|z|^s}{1 + |z|^{2s}}.$$

Therefore,

$$\|\psi_\alpha(A)\| \le \frac{1}{2\pi}\int_{\Gamma_{\delta,\Lambda}} |\psi_\alpha(z)|\,\|(z-A)^{-1}\|\,|dz| \le M_\omega \int_{\Gamma_{\delta,\Lambda}} \frac{|z|^{s-1}}{1+|z|^{2s}}\,|dz|.$$

Parametrizing the rays by $z = re^{\pm i\nu}$ leads us to the function

$$f(r) = M_\omega \frac{r^{s-1}}{1+r^{2s}}, \qquad r>0.$$

Since $s>0$, we have $f(r)\sim M_\omega r^{s-1}$ as $r\to 0$ and $f(r)\sim M_\omega r^{-(s+1)}$ as $r\to\infty$. By the standard integrability criterion for power functions, f is integrable on $(0,\infty)$. Hence, for every $\varepsilon>0$, we may choose $0<\delta<1<\Lambda$ such that

$$\int_0^\delta f(r)\,dr < \frac{\varepsilon}{3}, \qquad \int_\Lambda^\infty f(r)\,dr < \frac{\varepsilon}{3},$$

uniformly in α.

On the bounded part of the contour, which has finite length, we have

$$\int_{\Gamma_{[\delta,\Lambda]}} |\psi_\alpha(z)|\,\|(z-A)^{-1}\|\,|dz| \le C_{\delta,\Lambda}\|\psi_\alpha\|_\infty,$$

where $C_{\delta,\Lambda}>0$ is independent of α. Since $\|\psi_\alpha\|_\infty\to 0$, there exists α_0 such that for $\alpha\ge\alpha_0$,

$$C_{\delta,\Lambda}\|\psi_\alpha\|_\infty < \frac{\varepsilon}{3}.$$

Combining the three contributions, we obtain for $\alpha\ge\alpha_0$

$$\|\psi_\alpha(A)\| < \varepsilon.$$

Since $\varepsilon>0$ was arbitrary, this proves that

$$\|\psi_\alpha(A)\| \longrightarrow 0 \qquad (\alpha\to\infty).$$

- For point (ii), define

$$\widetilde{\psi}_\alpha(z) := (1+z)^{-1}\psi_\alpha(z), \qquad z\in\Sigma_\mu.$$

Since $-1\notin\overline{\Sigma_\mu}$ (because $\mu<\pi$), the function $z\mapsto(1+z)^{-1}$ is bounded and holomorphic on Σ_μ. Hence each $\widetilde{\psi}_\alpha$ belongs to $H_0^\infty(\Sigma_\mu)$, and we have

$$\|\widetilde{\psi}_\alpha\|_\infty \le C\|\psi_\alpha\|_\infty \longrightarrow 0 \quad (\alpha\to\infty),$$

for some constant $C > 0$ independent of α.

Moreover, by the local bound $|\psi_\alpha(z)| \le c|z|^s$ for $|z| < 1$, and the fact that multiplication by $(1+z)^{-1}$ improves the decay at infinity, one checks that there exist constants $c' > 0$ and $s' > 0$, independent of α, such that

$$|\widetilde{\psi}_\alpha(z)| \le c'|z|^{s'}(1+|z|^{2s'})^{-1}, \qquad z \in \Sigma_\mu.$$

Therefore the family $(\widetilde{\psi}_\alpha)$ satisfies the hypothesis of part (i). By (i) we obtain

$$\|\widetilde{\psi}_\alpha(A)\| \longrightarrow 0 \quad (\alpha \to \infty).$$

On the other hand, by the calculus (see Theorem 2.3.3),

$$\widetilde{\psi}_\alpha(A) = (1+A)^{-1}\psi_\alpha(A) = \psi_\alpha(A)(1+A)^{-1},$$

Hence for every $v \in \mathcal{H}$,

$$\psi_\alpha(A)(1+A)^{-1}v = \widetilde{\psi}_\alpha(A)v \longrightarrow 0.$$

In particular, if $u \in D(A)$, then $(1+A)u \in \mathcal{H}$ and

$$\psi_\alpha(A)u = \psi_\alpha(A)(1+A)^{-1}(1+A)u = \widetilde{\psi}_\alpha(A)(1+A)u \longrightarrow 0.$$

Thus

$$\psi_\alpha(A)u \to 0 \quad \text{for all } u \in D(A).$$

Now $D(A)$ is dense in $\mathcal{H}$ (since A is sectorial), and the operators $\psi_\alpha(A)$ are uniformly bounded: $\|\psi_\alpha(A)\| \le M$ for all α. Let $u \in H$ and $\varepsilon > 0$. Choose $u_0 \in D(A)$ such that $\|u - u_0\| < \varepsilon/(3M)$. Then for all α, we have

$$\|\psi_\alpha(A)u\| \le \|\psi_\alpha(A)(u-u_0)\| + \|\psi_\alpha(A)u_0\| \le M\|u-u_0\| + \|\psi_\alpha(A)u_0\|.$$

By the choice of u_0, the first term is less than $\varepsilon/3$. The second term tends to 0 as $\alpha \to \infty$, since $u_0 \in D(A)$. Hence there exists α_0 such that for all $\alpha \ge \alpha_0$,

$$\|\psi_\alpha(A)u_0\| < \frac{2\varepsilon}{3},$$

and consequently

$$\|\psi_\alpha(A)u\| < \varepsilon.$$

This shows that $\psi_\alpha(A)u \to 0$ for every $u \in H$.

- Point (iii) follows by using similar arguments. $\square$

This theorem is the first result that will lead us to the final characterization. Now we state the second theorem :

Theorem 2.5.5. *Let A be a sectorial operator of angle ω on a Hilbert space $\mathcal{H}$, then for $0 \leq \omega < \mu < \pi$, let (f_α) be a net in $H^\infty(\Sigma_\mu)$ and let $f \in H^\infty(\Sigma_\mu)$ such that $\exists M > 0, \|f_\alpha(A)\| \leq M$. Furthermore, assume :*

$$\forall\ 0 < \delta < \Lambda,\ sup\{|f_\alpha(z) - f(z)|,\ z \in \Sigma_\mu \text{ and } \delta \leq |z| \leq \Lambda\} \to 0$$

Then $f(A) \in \mathcal{L}(\mathcal{H})$ and $f_\alpha(A)u \to f(A)u$ for all $u \in \mathcal{H}$, and $\|f(A)\| \leq M$.

Proof. Define

$$\varphi(z) := z(1+z)^{-2}, \qquad z \in \Sigma_\mu.$$

Since $-1 \notin \overline{\Sigma_\mu}$ and $\varphi(z) \to 0$ both as $z \to 0$ and $z \to \infty$ within Σ_μ, we have $\varphi \in H_0^\infty(\Sigma_\mu)$. For each α, set

$$g_\alpha := f_\alpha \varphi, \qquad g := f\varphi.$$

Then, one easily checks that $g_\alpha, g \in H_0^\infty(\Sigma_\mu)$, and by the assumed local uniform convergence of f_α to f on every set $\{z : \delta \leq |z| \leq \Lambda\}$, we also have $g_\alpha \to g$ locally uniformly in Σ_μ. Moreover, by the multiplicativity of the holomorphic functional calculus on $H_0^\infty(\Sigma_\mu)$ (see Theorem 2.3.3), $g_\alpha(A) = (f_\alpha\varphi)(A) = f_\alpha(A)\varphi(A)$, and therefore $\|g_\alpha(A)\| = \|f_\alpha(A)\varphi(A)\| \leq M\,\|\varphi(A)\|$ for all α. Hence the net (g_α) satisfies the assumptions of Theorem 2.5.4 with uniform bound on $\|g_\alpha(A)\|$ and $g_\alpha \to g$ locally uniformly.

By part (iii) of Theorem 2.5.4 applied to (g_α) and g, we obtain

$$g_\alpha(A)u \longrightarrow g(A)u \quad \text{for all } u \in H,$$

and $\|g(A)\| \leq M\|\varphi(A)\|$. In terms of f_α and f, this reads

$$f_\alpha(A)\varphi(A)u \longrightarrow (f\varphi)(A)u \quad \text{for all } u \in H.$$

By multiplicativity of the calculus,

$$(f\varphi)(A) = f(A)\varphi(A),$$

so we have

$$f_\alpha(A)\varphi(A)u \longrightarrow f(A)\varphi(A)u \quad \text{for all } u \in H \tag{2.5}$$

Since $\varphi \not\equiv 0$, $\mathrm{Ran}(\varphi(A))$ is dense in $\mathcal{H}$.

Define an operator T on $\operatorname{Ran}(\varphi(A))$ by

$$T\big(\varphi(A)u\big) := f(A)\varphi(A)u, \qquad u \in H.$$

By equation (2.5), we have

$$f_\alpha(A)\varphi(A)u \;\longrightarrow\; T\big(\varphi(A)u\big) \quad \text{for all } u \in H, \tag{2.6}$$

and moreover

$$\|f_\alpha(A)\varphi(A)u\| \leq \|f_\alpha(A)\|\,\|\varphi(A)u\| \leq M\,\|\varphi(A)u\|.$$

Taking the limit in (2.6), we obtain $\|T(\varphi(A)u)\| \leq M\,\|\varphi(A)u\|$ Hence T is bounded on the dense subspace $\operatorname{Ran}(\varphi(A))$ with $\|T\| \leq M$, and therefore T extends uniquely to a bounded operator on all of $\mathcal{H}$, still denoted $f(A)$, with $\|f(A)\| \leq M$. We already know from (2.5) that

$$f_\alpha(A)y \;\longrightarrow\; f(A)y \quad \text{for all } y \in \operatorname{Ran}(\varphi(A)),$$

and $\{f_\alpha(A)\}$ is uniformly bounded in operator norm: $\|f_\alpha(A)\| \leq M$ for all α. Since $\operatorname{Ran}(\varphi(A))$ is dense in $\mathcal{H}$, let $u \in \mathcal{H}$ and $\varepsilon > 0$. Choose $y \in \operatorname{Ran}(\varphi(A))$ such that $\|u - y\| < \varepsilon/(3M)$. Then for every α we have

$$\begin{aligned}\|f_\alpha(A)u - f(A)u\| &\leq \|f_\alpha(A)(u-y)\| + \|f_\alpha(A)y - f(A)y\| + \|f(A)(y-u)\| \\ &\leq M\|u-y\| + \|f_\alpha(A)y - f(A)y\| + M\|y-u\|.\end{aligned}$$

Hence

$$\|f_\alpha(A)u - f(A)u\| \leq \frac{2\varepsilon}{3} + \|f_\alpha(A)y - f(A)y\|.$$

Since $y \in \operatorname{Ran}(\varphi(A))$, we know that

$$f_\alpha(A)y \longrightarrow f(A)y.$$

Therefore there exists α_0 such that for all $\alpha \geq \alpha_0$,

$$\|f_\alpha(A)y - f(A)y\| < \frac{\varepsilon}{3}.$$

Consequently,

$$\|f_\alpha(A)u - f(A)u\| < \varepsilon.$$

Thus

$$f_\alpha(A)u \longrightarrow f(A)u \qquad \text{for all } u \in \mathcal{H},$$

and by construction $\|f(A)\| \leq M$. $\square$

The boundedness of the H^∞–functional calculus can now be characterized in various equivalent ways, particularly in the setting of Hilbert spaces; McIntosh [14] depicted this in the following. Define

$$H_{0+}(\Sigma_\mu) = \{\psi \in H_0^\infty(\Sigma_\mu), \psi(t) > 0 \; \forall t > 0\}$$

Theorem 2.5.6. *Let $0 \leq \omega < \mu < \pi$ and let A be a sectorial operator on $\mathcal{H}$, so that it has dense range. Assume that there exist functions $\psi, \tilde{\psi} \in H_{0+}(\Sigma_\mu)$ such that T and T^* satisfy quadratic estimates with respect to ψ and $\tilde{\psi}$, for some real numbers q_1 and q_2:*

$$\textbf{I.} \int_0^\infty \|\psi(tT)u\|^2 \frac{dt}{t} \leq q_1^2 \|u\|^2, \qquad \forall u \in \mathcal{H},$$
$$\textbf{II.} \int_0^\infty \|\tilde{\psi}(tT^*)v\|^2 \frac{dt}{t} \leq q_2^2 \|v\|^2, \qquad \forall v \in \mathcal{H}.$$

Then, for every $f \in H^\infty(\Sigma_\mu)$, the operator $f(T)$ is bounded on $\mathcal{H}$, and there exists a constant $C > 0$ (independent of f) such that

$$\|f(T)\| \leq C \|f\|_\infty, \qquad \forall f \in H^\infty(\Sigma_\mu)$$

Proof. We will use, as a general notation for this proof, for a given function φ,

$$\varphi_t(z) := \varphi(tz).$$

Let $\theta \in H_0^\infty(\Sigma_\mu)$, such that $\int_0^\infty \varphi(t) \frac{dt}{t} = 1$, where $\varphi = \psi\tilde{\psi}\theta$. We give an example of how to find such a function θ in the Remark 2.5.7. Now, define, for $z \in \Sigma_\mu$, $R > \epsilon > 0$:

$$f_{\epsilon,R}(z) = \int_\epsilon^R (f\varphi_t)(z) \frac{dt}{t}$$

Then we get the following by ***I*** & ***II***: for each $u, v \in \mathcal{H}$, we have

$$\begin{aligned}
\Big|\langle f_{\varepsilon,R}(A)u,\, v\rangle\Big| &= \Big| \int_\varepsilon^R \Big\langle (f\varphi_t)(A)u,\, v\Big\rangle \,\frac{dt}{t}\Big| \\
&= \Big| \int_\varepsilon^R \Big\langle (f\theta_t)(A)\,\psi_t(A)u,\, \widetilde{\psi}_t(A^*)v\Big\rangle \,\frac{dt}{t}\Big| \qquad \text{(commutation from 2.3.3)} \\
&\le \int_\varepsilon^R \Big|\Big\langle (f\theta_t)(A)\,\psi_t(A)u,\, \widetilde{\psi}_t(A^*)v\Big\rangle\Big| \,\frac{dt}{t} \\
&\le \int_\varepsilon^R \|(f\theta_t)(A)\|\,\|\psi_t(A)u\|\,\|\widetilde{\psi}_t(A^*)v\|\,\frac{dt}{t} \qquad \text{(Cauchy–Schwarz in } \mathcal{H}) \\
&\le \Big(\sup_{t\in(\varepsilon,R)} \|(f\theta_t)(A)\|\Big) \int_\varepsilon^R \|\psi_t(A)u\|\,\|\widetilde{\psi}_t(A^*)v\|\,\frac{dt}{t} \\
&\le \Big(\sup_{t\in(\varepsilon,R)} \|(f\theta_t)(A)\|\Big)\Big(\int_\varepsilon^R \|\psi_t(A)u\|^2\,\frac{dt}{t}\Big)^{1/2}\Big(\int_\varepsilon^R \|\widetilde{\psi}_t(A^*)v\|^2\,\frac{dt}{t}\Big)^{1/2} \\
&\le q_1q_2\,\|u\|\,\|v\|\,\sup_{t\in(\varepsilon,R)} \|(f\theta_t)(A)\| \qquad \text{(quadratic estimates).}
\end{aligned}$$

Now, one can see that $\theta_t \in H_0^\infty(\Sigma_\mu)$ for a fixed t so that it regularizes f, and we can write by (2.3) :

$$(f\theta_t)(A) = \frac{1}{2\pi i}\int_\Gamma f(t)\theta(tz)(zI-A)^{-1}\,dz$$

and using the estimates of the resolvent and the fact that $\theta \in H_0^\infty(\Sigma_\mu)$, one gets :

$$\begin{aligned}
\|(f\theta_t)(A)\| &\le \frac{1}{2\pi}\|f\|_\infty \int_\Gamma M_\mu C|z|^{-1}\frac{t^s|z|^s}{1+t^{2s}|z|^{2s}}|dz| \\
&\le \kappa\|f\|_\infty
\end{aligned}$$

where κ does not depend on f. This leads to :

$$| < f_{\epsilon,R}(A)u, v > | \le \kappa q_1 q_2 \|u\|\|v\|\|f\|_\infty$$

for every $R > \epsilon > 0$.

Furthermore, one can see that:

$$\begin{aligned}
|f_{\epsilon,R}(z) - f(z)| &= \Big|\int_\epsilon^R f(z)\varphi(tz)\frac{dt}{t} - f(z)\Big| \\
&\le ||f||_\infty\Big|\int_\epsilon^R \varphi(tz)\frac{dt}{t} - 1\Big| \to 0
\end{aligned}$$

when $\epsilon \to 0$, $R \to \infty$. Indeed, with the change of variables $w = tz$,

$$I_{\epsilon,R}(z) = \int_{[\epsilon z,\, Rz]} \frac{\varphi(w)}{w}\, dw,$$

write $z = |z|e^{i\alpha}$ with $\alpha = \arg(z)$. Consider the closed contour

$$\Gamma = [\epsilon, R] \,\cup\, \gamma_R \,\cup\, [Rz, \epsilon z] \,\cup\, \gamma_\epsilon,$$

where γ_R and γ_ϵ are the circular arcs of radii R and ϵ joining R to Rz and ϵz to ϵ, respectively; one can see it on figure 2.3. The function

$$g(w) := \frac{\varphi(w)}{w}$$

is holomorphic near Γ, hence by Cauchy's theorem,

$$\int_\Gamma g(w)\, dw = 0.$$

Therefore,

$$I_{\epsilon,R}(z) = \int_\epsilon^R \varphi(t)\, \frac{dt}{t} \;+\; \int_{\gamma_R} \frac{\varphi(w)}{w}\, dw \;+\; \int_{\gamma_\epsilon} \frac{\varphi(w)}{w}\, dw.$$

Parametrizing γ_R by $w(\phi) = Re^{i\phi}$,

$$\left| \int_{\gamma_R} \frac{\varphi(w)}{w}\, dw \right| \leq \alpha \sup_{|w|=R} |\varphi(w)| \xrightarrow[R\to\infty]{} 0,$$

and similarly

$$\left| \int_{\gamma_\epsilon} \frac{\varphi(w)}{w}\, dw \right| \leq \alpha \sup_{|w|=\epsilon} |\varphi(w)| \xrightarrow[\epsilon\to 0]{} 0,$$

since by definition $\varphi(w) \to 0$ as $|w| \to 0$. Then, taking the limits $\epsilon \to 0$ and $R \to \infty$, we obtain

$$\lim_{\epsilon\to 0,\, R\to\infty} I_{\epsilon,R}(z) = \int_0^\infty \varphi(t)\, \frac{dt}{t} = 1.$$

Hence, for every $z \neq 0$,

$$\lim_{\epsilon\to 0,\, R\to\infty} \int_\epsilon^R \varphi(tz)\, \frac{dt}{t} = 1.$$

This shows that the net $f_{\epsilon,R}$ fits the requirements of the Theorem 2.5.5, leading to the result and the end of the proof. □

Remark 2.5.7. Let ψ and $\tilde{\psi} \in H_{0+}(\Sigma_\mu)$, we show how to find the function θ from the proof in the case where $\psi(z) \sim z^\alpha(1+z^\beta)^{-1}$ and $\tilde{\psi}(z) \sim z^\gamma(1+z^\epsilon)^{-1}$ for some $\alpha, \beta, \gamma, \epsilon > 0$.

We search for θ such that $\int_0^\infty \frac{t^{\alpha+\gamma-1}}{(1+t^\beta)(1+t^\epsilon)}\theta(t)dt = 1$. This looks like we want to make

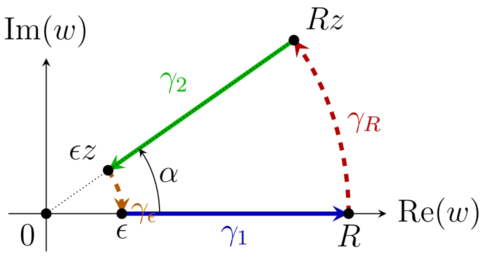

Figure 2.3: The contour used in the proof of theorem 2.5.6

the integrand being a density of probability, with support in $[0, \infty[$.

Set for instance $\theta(t) = \frac{1}{\Gamma(\alpha+\gamma)}(1+t^{\beta})(1+t^{\epsilon})e^{-t}$, it belongs to $H_0^{\infty}(\Sigma_\mu)$, and the integrand is exactly the density of a random variable $X \sim Gamma(\alpha+\gamma, 1)$, thus it sums to 1 and we get the result.

Remark 2.5.8. Here, through the proofs provided, one can easily see the high importance of the assumption that states that A is densely-defined. In fact, McIntosh works with this assumption because they are necessary in the case where we want to have the characterization for the boundedness. However, Haase works with a larger setting, without this assumption (and indeed, we do not use in any proof such an assumption). It means that all what we depicted through this chapter, except this section, can be also said if the operator is not densely defined and is more general.

Example 2.5.9 (Negative Laplacian on L^2)**.** The most prominent example of a sectorial operator that satisfies the required quadratic estimates and admits a bounded H^{∞} functional calculus is the negative laplacian $A = -\Delta$ on L^2. We already know that it is sectorial and densely defined. We also have $A^* = A$, and therefore $\tilde{\psi} = \psi$ and $q_1 = q_2 = q$. The Quadratic Estimate (QE) is guaranteed if the constant q^2 (denoted K_ψ^2) is finite, where q^2 is defined by:

$$q^2 = K_\psi^2 = \int_0^{\infty} \frac{|\psi(s)|^2}{s} ds$$

Choice of the function ψ : We select a suitable holomorphic function ψ that vanishes at $s = 0$ and decays rapidly enough at infinity:

$$\psi(s) = \tilde{\psi}(s) = \frac{s}{(1+s)^{3/2}}$$

Calculation of the constant q^2 : Substituting $\psi(s)$ into the integral yields:

$$K_\psi^2 = \int_0^{\infty} \frac{\left(\frac{s}{(1+s)^{3/2}}\right)^2}{s} ds = \int_0^{\infty} \frac{s}{(1+s)^3} ds$$

Using the substitution $u = 1 + s$, we have $s = u - 1$ and $ds = du$. The bounds are $u = 1$ to $u = \infty$.

$$K_\psi^2 = \int_1^\infty \frac{u-1}{u^3} du = \int_1^\infty \left(u^{-2} - u^{-3}\right) du$$

Evaluating the integral:

$$K_\psi^2 = \left[-u^{-1} + \frac{1}{2}u^{-2}\right]_1^\infty = \lim_{u\to\infty} \left(-\frac{1}{u} + \frac{1}{2u^2}\right) - \left(-\frac{1}{1} + \frac{1}{2(1)^2}\right)$$

$$K_\psi^2 = 0 - \left(-1 + \frac{1}{2}\right) = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$$

The Quadratic Estimates are satisfied with the bound $q^2 = 1/2$. Consequently, the required constants for the theorem are:

$$q_1 = q_2 = q = \frac{1}{\sqrt{2}}$$

Remark 2.5.10. Cowling, Doust, McIntosh and Yagi provided another complete characterization of bounded H^∞–calculi in the Hilbert setting in [2]. It connects analytic estimates (square function inequalities), spectral properties (bounded imaginary powers), and geometric operator theory (via similarity to self-adjoint models). In particular, sectorial operators with bounded imaginary powers are precisely those that admit a bounded holomorphic functional calculus.

2.6 The Spectral Mapping Theorem

In this section we establish a spectral mapping theorem for the natural (holomorphic) functional calculus of sectorial operators. All results are stated in terms of the *extended spectrum*

$$\widetilde{\sigma}(A) := \sigma(A) \cup \{0, \infty\} \tag{2.7}$$

which captures both the local spectral behaviour and the asymptotics of A near 0 and ∞.

Throughout, $A \in \mathrm{Sect}(\omega)$ still denotes a sectorial operator, and $H(A)$ denotes the algebra of all functions f that are in $\mathcal{M}(\Sigma_\phi)_A$ with $\omega < \phi < \pi$ and for which the operator $f(A)$ is well-defined and bounded in the sense of the H^∞- functional calculus.

2.6.1 The Spectral Inclusion Theorem

We first establish a spectral inclusion result for $f \in \mathcal{M}(\Sigma_\omega)_A$.

Lemma 2.6.1. *Let $A \in \mathrm{Sect}(\omega)$ and $\lambda \in \mathbb{C}$ be such that $\lambda I - A$ is injective. Let $e \in H(A)$ and $c \neq 0$ such that*

$$f(z) := \frac{e(z) - c}{\lambda - z} \in H(A).$$

Then

$$e(A)(\lambda I - A)^{-1} = (\lambda I - A)^{-1} e(A) \tag{2.8}$$

Proof. The inclusion $e(A)(\lambda I - A)^{-1} \subset (\lambda I - A)^{-1} e(A)$ follows directly from point 3 of the fundamental properties (see Theorem 2.4.6) of the calculus. Conversely, if $e(A)x \in \mathrm{Ran}((\lambda I - A)^{-1})$, there exists $z \in D(A)$ such that $e(A)x = (\lambda I - A)z$. Using $e = (\lambda - z)f + c$, one obtains

$$(\lambda I - A)z = cx + (\lambda I - A)f(A)x,$$

hence $cx = (\lambda I - A)(z - f(A)x)$. As $c \neq 0$, we conclude $x \in \mathrm{Ran}((\lambda I - A)^{-1})$, proving equality. $\square$

Lemma 2.6.2. *Let $A \in \mathrm{Sect}(\omega)$ and $f \in \mathcal{M}(\Sigma_\omega)_A$. Then for any $\lambda \neq 0$, there exists $e \in H(A)$ that regularizes f and satisfies $e(\lambda) \neq 0$.*

Proof. Let g be any regularizer for f, i.e. $g, fg \in H(A)$ and $g(A)$ injective. If g has a zero of order n at λ, define

$$e(z) := \frac{g(z)}{(\lambda - z)^n}.$$

Then $e(\lambda) \neq 0$, $e(A)$ remains injective, and $ef \in H(A)$. Thus e is a regularizer for f with $e(\lambda) \neq 0$. $\square$

Proposition 2.6.3. *Let $A \in \mathrm{Sect}(\omega)$ and $f \in \mathcal{M}(\Sigma_\omega)_A$. Assume that $f(\lambda) = 0$ and that $f(A)$ is invertible. Then $\lambda \in \rho(A)$.*

Proof. Since $f \in \mathcal{M}(\Sigma_\omega)_A$, choose a regulariser e for f as given by Lemma 2.6.2, so that $e \in \mathcal{E}$, $e(\lambda) \neq 0$, and $e(A)$ is injective with dense range. Define $g := ef$. Then $g \in \mathcal{E}$ and

$$g(\lambda) = e(\lambda) f(\lambda) = 0.$$

Since $g(\lambda) = 0$, we may factor

$$g(z) = (z - \lambda) h(z)$$

for some $h \in \mathcal{E}$. By the multiplicative property of the functional calculus (see Theorem 2.3.3), we obtain

$$g(A) = (A - \lambda I)\, h(A).$$

On the other hand,

$$g(A) = e(A)f(A).$$

Since $f(A)$ is invertible and $e(A)$ is injective with dense range, it follows that $g(A)$ is injective and has dense range.

From

$$g(A) = (A - \lambda I)h(A)$$

we deduce that $A - \lambda I$ is injective.

To prove surjectivity, observe that

$$f(A) = e(A)^{-1}g(A) = e(A)^{-1}(A - \lambda I)h(A).$$

Since $f(A)$ is surjective and $h(A)$ is bounded, it follows that the range of $A - \lambda I$ is dense. Using Lemma 2.6.1, we have

$$e(A)(A - \lambda I)^{-1} = (A - \lambda I)^{-1}e(A),$$

which implies that the range of $A - \lambda I$ is closed. Hence $\mathrm{Ran}(A - \lambda I) = \mathcal{H}$.

Thus $A - \lambda I$ is bijective, and therefore $\lambda \in \rho(A)$. □

Now we can deduce the first result concerning the spectrum :

Theorem 2.6.4 (Spectral Inclusion Theorem)**.** *Let $A \in \mathrm{Sect}(\omega)$ and $f \in \mathcal{M}(\Sigma_\omega)_A$. Then*

$$f(\sigma(A) \setminus \{0\}) \subset \widetilde{\sigma}(f(A))$$

where $\widetilde{\sigma}(f(A))$ is defined by formula (2.7).

Proof. Let $\lambda \in \sigma(A) \setminus \{0\}$ and set $\mu = f(\lambda)$. Suppose $\mu \neq \infty$, if $\mu \notin \widetilde{\sigma}(f(A))$, then $f(A) - \mu I$ would be invertible, which contradicts Proposition 2.6.3 applied to $f - \mu$ (indeed, λ should be in the resolvent of A then, which contradicts our assumption of $\lambda \in \sigma(A) \setminus \{0\}$.

Suppose now $\mu = \infty \notin \widetilde{\sigma}(f(A))$, then $\exists \lambda_0 \in \mathbb{C}$ such that $f(A) - \lambda_0$ is invertible. Define then $g = 1/(f - \lambda_0)$, we get $g(\lambda) = \frac{1}{\mu - \lambda_0} = 0$ and $g(A)$ is invertible by definition. Applying again Proposition 2.6.3 to g, we get that λ belongs to the resolvent of A, which is not possible. Thus $\mu \in \widetilde{\sigma}(f(A))$. □

The above result leaves aside the points 0 and ∞, which require a refined argument based on the notion of polynomial limits.

Theorem 2.6.5. *Let $A \in \mathrm{Sect}(\omega)$ and $\lambda_0 \in \{0, \infty\}$. If $f \in \mathcal{E}(\Sigma_\omega)$ admits a polynomial limit μ at λ_0 and $\lambda_0 \in \widetilde{\sigma}(A)$, then $\mu \in \widetilde{\sigma}(f(A))$.*

Proof. We omit the proof here, but one can refer to the Theorem 2.7.5 of [4] for details. □

2.6.2 The Full Spectral Mapping Theorem

We now extend the inclusion to an equality, controlling the poles of f and its asymptotic limits.

Lemma 2.6.6. *Let $A \in \mathrm{Sect}(\omega)$ and let $f \in \mathcal{M}(\Sigma_\omega)_A$ have only finitely many poles, all contained in $\rho(A)$, and finite polynomial limits at each point of $\widetilde{\sigma}(A) \cap \{0, \infty\}$. Then $f(A)$ is well-defined and bounded.*

Proof. Let λ_0 be a pole of f. For $n_0 \in \mathbb{N}$ sufficiently large, the factor $(\lambda_0 - z)^{n_0}$ cancels the pole of f at λ_0. To control the behaviour at infinity, we simultaneously divide by $(1+z)^{n_0}$ and define

$$r_{\lambda_0}(z) := \frac{(\lambda_0 - z)^{n_0}}{(1+z)^{n_0}}, \qquad \tilde{f}(z) := r_{\lambda_0}(z)\, f(z).$$

Then $\tilde{f}$ has one pole less than f, and still has finite polynomial limits at the points of $\widetilde{\sigma}(A) \cap \{0, \infty\}$, since r_{λ_0} is a rational function with at most polynomial growth/decay at 0 and ∞.

Because $\lambda_0 \in \rho(A)$ and $-1 \in \rho(A)$ for a sectorial operator, the operators $(\lambda_0 I - A)$ and $(1 + A)$ are boundedly invertible, hence so is

$$r_{\lambda_0}(A) = (\lambda_0 I - A)^{n_0}(1 + A)^{-n_0}.$$

Repeat this to get the finite product

$$r(z) := \prod_{\lambda_0 \text{ pole of } f} r_{\lambda_0}(z),$$

that is a rational function bounded on Σ_ω, such that $r(A)$ is bounded and invertible, and such that rf is holomorphic on Σ_ω with no poles. Furthermore, rf has finite polynomial limits at the points of $\widetilde{\sigma}(A) \cap \{0, \infty\}$, such that $rf \in \mathcal{E}(\Sigma_\omega)$.

We define as usual then:

$$f(A) := r(A)^{-1}(rf)(A).$$

This is well-defined and bounded as a product of two bounded operators, and the result shows up. □

Proposition 2.6.7 (Refined spectral inclusion)**.** *Let* $A \in \mathrm{Sect}(\omega)$ *and let* f *satisfy the assumptions of Lemma 2.6.6. Then*

$$\widetilde{\sigma}(f(A)) \subset f\big(\widetilde{\sigma}(A)\big).$$

Proof. We first deal with finite points $\mu \in \mathbb{C}$. Let $\mu \notin f(\widetilde{\sigma}(A))$ and consider $g(z) := (\mu - f(z))^{-1}$. Then g is meromorphic on Σ_ω, with poles exactly at those $\lambda \in \Sigma_\omega$ such that $f(\lambda) = \mu$. By the assumption $\mu \notin f(\widetilde{\sigma}(A))$, no such λ lies in $\widetilde{\sigma}(A)$, hence all poles of g lie in $\rho(A)$.

At the endpoints $\lambda_0 \in \widetilde{\sigma}(A) \cap \{0, \infty\}$, the function f has a finite polynomial limit, say $f(\lambda_0) = L \in \mathbb{C}$. Since $\mu \neq L$ (otherwise $\mu \in f(\widetilde{\sigma}(A))$), the denominator $\mu - f(z)$ does not vanish near λ_0, and we see that g has again a finite polynomial limit at λ_0. Thus g satisfies the hypotheses of Lemma 2.6.6, so $g(A)$ is well-defined and bounded.

By the functional calculus, we have $g(A) = (\mu I - f(A))^{-1}$. Hence $\mu I - f(A)$ is invertible with bounded inverse, so $\mu \notin \widetilde{\sigma}(f(A))$. This proves the inclusion for all finite μ.

Now let $\mu = \infty$. The condition $\infty \notin f(\widetilde{\sigma}(A))$ means that f is finite on $\widetilde{\sigma}(A)$, in particular every pole of f lie in $\rho(A)$. With the finite polynomial limits at the points of $\widetilde{\sigma}(A) \cap \{0, \infty\}$, Lemma 2.6.6 applies to f, and $f(A)$ is bounded. Thus, $\infty \notin \widetilde{\sigma}(f(A))$.

Combining the two cases, we get:

$$\widetilde{\sigma}(f(A)) \subset f\big(\widetilde{\sigma}(A)\big).$$

□

Theorem 2.6.8 (Spectral Mapping Theorem)**.** *Let* $A \in \mathrm{Sect}(\omega)$ *and let* $f \in \mathcal{M}(\Sigma_\phi)_A$*, with finitely many poles contained in* $\rho(A)$ *and finite polynomial limits at* $\widetilde{\sigma}(A) \cap \{0, \infty\}$*. Then*

$$\widetilde{\sigma}(f(A)) = f(\widetilde{\sigma}(A)).$$

Proof. The inclusion "$\subset$" follows from Proposition 2.6.7, and the reverse inclusion from Theorem 2.6.5. □

Remark 2.6.9. This theorem generalizes the classical spectral mapping formula

$$\sigma(f(A)) = f(\sigma(A))$$

to the setting of sectorial operators via the extended spectrum $\widetilde{\sigma}(A)$. It ensures that the holomorphic functional calculus preserves both the spectral geometry of A and its asymptotic behaviour at 0 and ∞.

Example 2.6.10 (Heat semigroup generated by the Laplacian)**.** Let $A = -\Delta$ on $L^2(\Omega)$ with Dirichlet boundary conditions. Then $\widetilde{\sigma}(A) = [0, \infty]$, and for $f(z) = e^{-tz}$ one has

$$\widetilde{\sigma}(f(A)) = e^{-t[0,\infty)} = [0, 1].$$

2.7 Comments

In this thesis, we restrict ourselves to cases where the H^∞-functional calculus is bounded, since this property provides a robust analytical framework in which stability, regularity, and compatibility with semigroup theory can be properly formulated. This boundedness assumption will play an important role in connecting the abstract functional calculus with the theory of analytic semigroups developed in the next chapter.

The theory developed in this chapter provides the analytical backbone of the thesis. By introducing sectorial operators and constructing the H^∞-functional calculus, we have built a rigorous framework that extends spectral methods far beyond the bounded or self-adjoint setting. This extension is not merely technical: it captures the type of phenomena arising in parabolic and elliptic evolution equations, where the governing operators are often unbounded, possibly non-selfadjoint, and act on infinite-dimensional Banach spaces.

From a conceptual point of view, the class of sectorial operators forms a natural domain for analytic semigroup theory. The resolvent estimates inherent in sectoriality are precisely the estimates that allow one to control the behaviour of the associated semigroup and to connect spectral information with time evolution. This provides the crucial link between the algebraic and analytic structure of the H^∞-functional calculus and the time-evolution viewpoint characteristic of partial differential equations. In particular, the representation formula

$$e^{-tA} = \frac{1}{2\pi i} \int_\Gamma e^{-t\lambda} (\lambda I - A)^{-1} \, d\lambda$$

shows explicitly how the exponential semigroup appears as a particular instance of the functional calculus applied to the holomorphic function $f(z) = e^{-tz}$.

The regularization procedure discussed previously is particularly important from both a theoretical and a practical standpoint. Many functions of interest, such as resolvents, exponentials, or polynomially growing functions, do not necessarily belong to the Dunford–Riesz class $H_0^\infty(\Sigma_\phi)$ introduced in Definition 2.2.3. Regularization provides a systematic way to extend the calculus to such functions by embedding them into a larger algebra $\mathcal{E}(\Sigma_\phi)$ (see Definition 2.2.6), where the Cauchy-type integral remains meaningful after a suitable modification. This makes it possible to treat operators of physical relevance, such as diffusion generators, Stokes operators, or fractional Laplacians, within a unified analytical framework.

The boundedness of the H^∞-functional calculus plays a central role in determining whether the abstract theory leads to effective operator estimates. In the Hilbert space setting, this boundedness is closely related to square function estimates, as described in Theorem 2.5.6. Such characterizations provide powerful tools for verifying boundedness through energy estimates, Fourier multiplier methods, or geometric properties of the underlying domain. When the calculus is bounded, one obtains a stable framework for studying regularity properties of evolution equations and for controlling functions of unbounded operators.

In the context of this thesis, the main role of the H^∞-functional calculus is therefore to provide a rigorous operator-theoretic foundation for the study of unbounded operators and their associated semigroups. It allows one to pass from spectral and resolvent information to functions of operators in a coherent way, and it prepares the ground for the semigroup-theoretic results of the next chapter. Possible connections with statistical learning and data-driven operator approximation will be discussed only at the end of the thesis, as perspectives motivated by the analytical framework developed here.

To sum up, this chapter establishes the mathematical infrastructure on which the rest of the thesis is based. It identifies the class of operators and functions for which the H^∞-functional calculus is well behaved, explains the role of regularization and boundedness, and clarifies the connection between resolvent estimates, functions of operators, and semigroup theory. This provides a solid and coherent foundation for the subsequent study of analytic semigroups and for the final discussion of possible perspectives toward data-driven operator theory.

Chapter 3

Analytic Semigroups Theory

This chapter introduces the functional analytic framework used to study linear evolution equations of the form

$$\begin{cases} u'(t) = Au(t), & t > 0, \\ u(0) = x \in X, \end{cases}$$

where $A : D(A) \subset X \to X$ is a (possibly unbounded) linear operator on a complex Banach space X. The main objective is to construct and analyse the analytic semigroup generated by A, which provides the natural solution operator for such problems. The main part of the material provided here comes from Lunardi in [13].

In the theory of analytic semigroups, one usually adopts a slightly different notion of sectoriality than in the purely functional-analytic framework of Haase. The difference lies not in the nature of the operator, but in the geometric convention used to describe the sector of resolvent boundedness.

Definition 3.0.1 (Sectorial operator in the sense of Lunardi in [13]). Let $A : D(A) \subset X \to X$ be a closed linear operator on a Banach space X. We say that A is *sectorial* if there exist real numbers $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$ and $M > 0$ such that:

1. The resolvent set $\rho(A)$ contains the sector

$$S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, \, |\arg(\lambda - \omega)| < \theta\};$$

2. For all $\lambda \in S_{\theta,\omega}$ one has the resolvent estimate

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|}.$$

The main difference with Haase's Definition 2.1.1 is geometric: Lunardi's sector $S_{\theta,\omega}$ is a region of the complex plane where the *resolvent* of A is well defined and uniformly bounded, while in Haase's convention the *spectrum* of A is required to lie inside a sector Σ_ϕ around the positive real axis. Equivalently, if we define the shifted operator

$$B := -(A - \omega I),$$

then the two notions coincide:

$$A \text{ is sectorial in Lunardi's sense of angle } \theta \quad \Longleftrightarrow \quad B \in \mathrm{Sect}(\theta) \text{ in Haase's sense.}$$

Indeed, the resolvent identity $R(\lambda, A) = R(\lambda - \omega, B)$ immediately transfers Lunardi's estimate to Haase's form

$$\|R(\mu, B)\| \leq \frac{M}{|\mu|}, \qquad \mu \in S_{\theta,0}.$$

Hence both definitions describe the same class of operators, up to a horizontal translation of the spectrum by ω.

This change of viewpoint is motivated by semigroup theory. If A is sectorial in Lunardi's sense with parameters (θ, ω, M), then A generates an analytic C_0-semigroup $(e^{tA})_{t\geq 0}$ satisfying the bound

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq M e^{\omega t}, \qquad t > 0.$$

Here ω plays the role of a *growth rate*: the resolvent is analytic in the region $\Re(\lambda) > \omega$, which guarantees exponential control of the semigroup. From a dynamical viewpoint, this normalization is natural because the time evolution problem

$$\frac{du}{dt} = Au, \qquad u(0) = u_0,$$

is well posed precisely when the resolvent exists in a right half-plane beyond the spectral bound ω.

In contrast, Haase's ([4]) convention (where $\omega = 0$ and the sector opens around the positive real axis) is better suited to the H^∞-functional calculus, which focuses on spectral localization rather than temporal growth. The two settings are thus equivalent modulo the translation $A \mapsto -(A - \omega I)$, but Lunardi's version aligns more directly with the analysis of parabolic and evolutionary equations, where the semigroup estimate above is central.

3.1 Analytic C_0-semigroups: definition and basic properties

In the section, we apply the functional calculus as seen previously to define the analytic semi-group generated by A, and we derive its main properties.

3.1.1 Analytic semi-group and properties

Definition 3.1.1. $\{T(t)\}_{t\geq 0}$ is an analytic semigroup, if it satisfies:

$$T(0) = I, \quad T(t+s) = T(t)T(s), \quad t \mapsto T(t) \text{ is analytic in } (0, \infty).$$

Definition 3.1.2 (Analytic semigroup generated by a sectorial operator)**.** Let A be a sectorial operator. For $t > 0$, define

$$e^{tA} := \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A)\, d\lambda, \tag{3.1}$$

where $r > 0$, $\eta \in (\pi/2, \theta)$, and $\gamma_{r,\eta}$ is the contour consisting of the rays $\{\lambda : |\arg \lambda| = \eta,\ |\lambda| \geq r\}$ and the arc $\{\lambda : |\arg \lambda| \leq \eta,\ |\lambda| = r\}$, oriented counterclockwise. We also set $e^{0A}x := x$ for all $x \in X$. The family $\{e^{tA}\}_{t\geq 0}$ is called the *analytic semigroup generated by A*.

As we have seen in the previous chapter (see Proposition 2.3.1), the integral in (3.1) is independent of the parameters r, η, and $t \mapsto e^{tA}$ is analytic on $(0, \infty)$ in operator norm.

Remark 3.1.3. If A is sectorial, then $\{e^{tA}\}_{t\geq 0}$ is analytic, and it is strongly continuous (C_0) if and only if $D(A)$ is dense in X.

Now we can depict some of the most important properties of e^{tA}.

Proposition 3.1.4 (Fundamental properties)**.** *Let A be a sectorial operator on a Banach space X, and let $\{e^{tA}\}_{t\geq 0}$ be the analytic semigroup generated by A. Then the following statements hold.*

1. *For every $t > 0$, $x \in X$, and $k \in \mathbb{N}$, one has $e^{tA}x \in D(A^k)$, and for all $x \in D(A^k)$,*

$$A^k e^{tA} x = e^{tA} A^k x.$$

2. The semigroup property holds:

$$e^{tA}e^{sA} = e^{(t+s)A}, \qquad t, s \geq 0.$$

3. There exist constants $M_k > 0$ such that

$$\|e^{tA}\|_{L(X)} \leq M_0 e^{\omega t}, \qquad \|t^k(A - \omega I)^k e^{tA}\|_{L(X)} \leq M_k e^{\omega t}, \quad t > 0,$$

where ω is the constant appearing in the sectorial estimate

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}$$

for $\lambda \in S_{\theta,\omega}$. In particular, for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists $C_{k,\varepsilon} > 0$ such that

$$\|t^k A^k e^{tA}\|_{L(X)} \leq C_{k,\varepsilon} e^{(\omega+\varepsilon)t}, \quad t > 0.$$

4. The mapping $t \mapsto e^{tA}$ belongs to

$$C^\infty((0, +\infty); L(X)) = \{f : (0, +\infty) \to L(X) \mid f \text{ is infinitely differentiable on } (0, +\infty)\}$$

and satisfies

$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA}, \qquad t > 0.$$

Moreover, it admits an analytic extension to the sector

$$S = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta - \pi/2\}.$$

Proof. We use the Dunford integral representation

$$e^{tA} = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A)\, d\lambda, \qquad t > 0,$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$, $r > 0$, and $\eta \in (\pi/2, \theta)$.

1. We first treat the case $k = 1$. Let $x \in X$. Using the resolvent identity (see Proposition 1.3.8),

$$AR(\lambda, A) = \lambda R(\lambda, A) - I,$$

we obtain formally

$$Ae^{tA}x = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} AR(\lambda, A)x\, d\lambda = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} \big(\lambda R(\lambda, A) - I\big)x\, d\lambda.$$

Now the map

$$\lambda \longmapsto e^{t\lambda}\big(\lambda R(\lambda, A) - I\big)x$$

is integrable along the contour, since for $\lambda \in \omega + \gamma_{r,\eta}$ one has

$$\|\lambda R(\lambda, A) - I\| \leq |\lambda| \, \|R(\lambda, A)\| + 1$$

and the sectorial estimate gives $\|R(\lambda, A)\| \lesssim |\lambda - \omega|^{-1}$, while $e^{t\Re\lambda}$ decays exponentially on the two rays of the contour because $\eta > \pi/2$. Hence the above integral converges in X, which shows that $e^{tA}x \in D(A)$ and

$$Ae^{tA}x = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \lambda e^{t\lambda} R(\lambda, A)x \, d\lambda - \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} x \, d\lambda.$$

The second integral vanishes by Cauchy's theorem, since $\lambda \mapsto e^{t\lambda}$ is entire. Therefore

$$Ae^{tA}x = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \lambda e^{t\lambda} R(\lambda, A)x \, d\lambda.$$

Repeating the same argument inductively, one gets for every $k \in \mathbb{N}$

$$A^k e^{tA}x = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \lambda^k e^{t\lambda} R(\lambda, A)x \, d\lambda,$$

and in particular $e^{tA}x \in D(A^k)$. Now let $x \in D(A^k)$. Since $(\lambda I - A)^{-1}$ commutes with A on $D(A)$, one proves by iteration that $R(\lambda, A)$ commutes with A^k on $D(A^k)$, namely

$$A^k R(\lambda, A)x = R(\lambda, A)A^k x, \qquad x \in D(A^k).$$

Therefore,

$$A^k e^{tA}x = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} A^k R(\lambda, A)x \, d\lambda = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A)A^k x \, d\lambda = e^{tA}A^k x.$$

2. Set

$$B := A - \omega I.$$

Since A is sectorial of angle θ and vertex ω, the operator B is sectorial of the same angle θ, but with vertex 0. Indeed,

$$\lambda I - B = \lambda I - (A - \omega I) = (\lambda + \omega)I - A,$$

hence

$$R(\lambda, B) = R(\lambda + \omega, A).$$

Therefore, if $\lambda \in S_{\theta,0}$, then $\lambda + \omega \in S_{\theta,\omega}$, and the sectorial estimate for A yields

$$\|R(\lambda, B)\| = \|R(\lambda + \omega, A)\| \leq \frac{M}{|\lambda + \omega - \omega|} = \frac{M}{|\lambda|}.$$

So B is sectorial with vertex 0. We now show that

$$e^{tB} = e^{-\omega t} e^{tA}.$$

By Dunford's formula for B,

$$e^{tB} = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} R(\lambda, B)\, d\lambda.$$

Using $R(\lambda, B) = R(\lambda + \omega, A)$ and the change of variable $\mu = \lambda + \omega$, we obtain

$$e^{tB} = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t(\mu-\omega)} R(\mu, A)\, d\mu = e^{-\omega t} \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\mu} R(\mu, A)\, d\mu = e^{-\omega t} e^{tA}.$$

It remains to prove the semigroup property for B:

$$e^{tB} e^{sB} = e^{(t+s)B}.$$

Using Dunford's representation twice,

$$e^{tB} e^{sB} = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}} \int_{\gamma_{r',\eta'}} e^{t\lambda + s\mu} R(\lambda, B) R(\mu, B)\, d\mu\, d\lambda.$$

By the sectorial estimates and the exponential decay on the rays, the double integral is absolutely convergent, so Fubini's theorem applies. Now use the resolvent identity

$$R(\lambda, B) R(\mu, B) = \frac{R(\lambda, B) - R(\mu, B)}{\mu - \lambda}.$$

Substituting this into the double integral gives

$$e^{tB} e^{sB} = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}} \int_{\gamma_{r',\eta'}} e^{t\lambda + s\mu} \frac{R(\lambda, B) - R(\mu, B)}{\mu - \lambda}\, d\mu\, d\lambda.$$

We split this into two terms:

$$e^{tB} e^{sB} = I_1 - I_2,$$

where

$$I_1 = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}} \int_{\gamma_{r',\eta'}} \frac{e^{t\lambda + s\mu}}{\mu - \lambda} R(\lambda, B)\, d\mu\, d\lambda,$$

and

$$I_2 = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}} \int_{\gamma_{r',\eta'}} \frac{e^{t\lambda+s\mu}}{\mu-\lambda} R(\mu, B)\, d\mu\, d\lambda.$$

For I_1, fix λ. As a function of μ, the integrand

$$\mu \longmapsto \frac{e^{s\mu}}{\mu-\lambda}$$

is meromorphic with a simple pole at $\mu = \lambda$. Choosing the contours so that $\gamma_{r',\eta'}$ surrounds $\gamma_{r,\eta}$, Cauchy's integral formula gives

$$\frac{1}{2\pi i} \int_{\gamma_{r',\eta'}} \frac{e^{s\mu}}{\mu-\lambda}\, d\mu = e^{s\lambda}.$$

Hence

$$I_1 = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} e^{s\lambda} R(\lambda, B)\, d\lambda = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{(t+s)\lambda} R(\lambda, B)\, d\lambda = e^{(t+s)B}.$$

For I_2, fix μ. Then the map

$$\lambda \longmapsto \frac{e^{t\lambda}}{\mu-\lambda}$$

is holomorphic in the region enclosed by $\gamma_{r,\eta}$ (because the pole $\lambda = \mu$ lies outside that contour, by the choice of nested contours). Therefore Cauchy's theorem gives

$$\frac{1}{2\pi i} \int_{\gamma_{r,\eta}} \frac{e^{t\lambda}}{\mu-\lambda}\, d\lambda = 0,$$

and thus $I_2 = 0$. Combining the two identities, we obtain

$$e^{tB} e^{sB} = e^{(t+s)B}.$$

Finally, using $e^{tA} = e^{\omega t} e^{tB}$, we get

$$e^{tA} e^{sA} = e^{\omega t} e^{tB}\, e^{\omega s} e^{sB} = e^{\omega(t+s)} e^{(t+s)B} = e^{(t+s)A}.$$

3. We first prove the estimate for $\|e^{tA}\|_{L(X)}$. From the Dunford formula,

$$e^{tA} = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A)\, d\lambda,$$

hence

$$\|e^{tA}\|_{L(X)} \leq \frac{1}{2\pi} \int_{\omega+\gamma_{r,\eta}} e^{t\Re\lambda}\, \|R(\lambda, A)\|\, |d\lambda|.$$

Using the sectorial estimate,

$$\|R(\lambda, A)\| \le \frac{M}{|\lambda - \omega|},$$

we get

$$\|e^{tA}\|_{L(X)} \le \frac{M}{2\pi} \int_{\omega+\gamma_{r,\eta}} \frac{e^{t\Re\lambda}}{|\lambda - \omega|} \, |d\lambda|.$$

Write the contour as the union of the circular arc

$$arc = \omega + \{re^{i\varphi} : |\varphi| \le \eta\}$$

and the two rays

$$rays = \omega + \{\rho e^{\pm i\eta} : \rho \ge r\}.$$

On the circular arc, $|\lambda - \omega| = r$ and $\Re\lambda \le \omega + r$, so

$$\int_{\text{arc}} \frac{e^{t\Re\lambda}}{|\lambda - \omega|} \, |d\lambda| \le \int_{-\eta}^{\eta} \frac{e^{(\omega+r)t}}{r} \, r \, d\varphi = 2\eta \, e^{(\omega+r)t}.$$

Since r is fixed, this is bounded by $Ce^{\omega t}$ for t in bounded intervals, and for large t it is absorbed into the contribution of the rays by choosing r fixed once and for all.

On each ray, let $\lambda = \omega + \rho e^{\pm i\eta}$, $\rho \ge r$. Then

$$\Re\lambda = \omega + \rho\cos\eta, \qquad |\lambda - \omega| = \rho, \qquad |d\lambda| = d\rho.$$

Because $\eta > \pi/2$, one has $\cos\eta < 0$, hence

$$\int_{\text{ray}} \frac{e^{t\Re\lambda}}{|\lambda - \omega|} \, |d\lambda| = e^{\omega t} \int_{r}^{\infty} \frac{e^{t\rho\cos\eta}}{\rho} \, d\rho.$$

Setting $u = t\rho$, this becomes

$$e^{\omega t} \int_{tr}^{\infty} \frac{e^{u\cos\eta}}{u} \, du \le C_\eta e^{\omega t},$$

since $\cos\eta < 0$. Summing the three pieces of the contour, we obtain

$$\|e^{tA}\|_{L(X)} \le M_0 e^{\omega t}$$

for some constant $M_0 > 0$ independent of $t > 0$. We now estimate $t^k(A - \omega I)^k e^{tA}$. Since $A - \omega I = B$, one has

$$e^{tA} = e^{\omega t} e^{tB}, \qquad (A - \omega I)^k e^{tA} = e^{\omega t} B^k e^{tB}.$$

Thus it is enough to estimate $t^k B^k e^{tB}$. By Dunford's formula for B,

$$B^k e^{tB} = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} \lambda^k e^{t\lambda} R(\lambda, B)\, d\lambda.$$

Therefore

$$\|t^k B^k e^{tB}\|_{L(X)} \leq \frac{t^k}{2\pi} \int_{\gamma_{r,\eta}} |\lambda|^k e^{t\Re\lambda} \|R(\lambda, B)\|\, |d\lambda|.$$

Since B is sectorial with vertex 0,

$$\|R(\lambda, B)\| \leq \frac{M}{|\lambda|},$$

hence

$$\|t^k B^k e^{tB}\|_{L(X)} \leq \frac{Mt^k}{2\pi} \int_{\gamma_{r,\eta}} |\lambda|^{k-1} e^{t\Re\lambda}\, |d\lambda|.$$

Again we split the contour into the arc and the rays. On the arc, $|\lambda| = r$, so

$$t^k \int_{\text{arc}} |\lambda|^{k-1} e^{t\Re\lambda}\, |d\lambda| \leq C\, t^k r^k e^{tr}.$$

Since r is fixed, this contribution is bounded uniformly in $t > 0$ after adjusting the contour in the standard way (equivalently, one may choose $r = 1/t$ in the usual proof). On the rays, with $\lambda = \rho e^{\pm i\eta}$, $\rho \geq r$, we get

$$t^k \int_r^\infty \rho^{k-1} e^{t\rho\cos\eta}\, d\rho.$$

With the change of variable $u = t\rho$,

$$t^k \int_r^\infty \rho^{k-1} e^{t\rho\cos\eta}\, d\rho = \int_{tr}^\infty u^{k-1} e^{u\cos\eta}\, du \leq C_{k,\eta},$$

again because $\cos\eta < 0$. Hence

$$\|t^k B^k e^{tB}\|_{L(X)} \leq M_k$$

for some constant $M_k > 0$, and therefore

$$\|t^k (A - \omega I)^k e^{tA}\|_{L(X)} \leq M_k e^{\omega t}.$$

Finally, we prove the estimate involving A^k. Since

$$A = (A - \omega I) + \omega I = B + \omega I,$$

we expand by the binomial formula:

$$A^k = \sum_{j=0}^{k} \binom{k}{j} \omega^{k-j} B^j.$$

Therefore

$$t^k A^k e^{tA} = e^{\omega t} \sum_{j=0}^{k} \binom{k}{j} \omega^{k-j} t^k B^j e^{tB} = e^{\omega t} \sum_{j=0}^{k} \binom{k}{j} \omega^{k-j} t^{k-j} \big(t^j B^j e^{tB}\big).$$

Now, for every $\varepsilon > 0$,

$$t^{k-j} \le C_{k,j,\varepsilon} e^{\varepsilon t}, \qquad t > 0,$$

since every polynomial growth is dominated by any exponential. Using the previous bound

$$\|t^j B^j e^{tB}\|_{L(X)} \le M_j,$$

we deduce

$$\|t^k A^k e^{tA}\|_{L(X)} \le e^{\omega t} \sum_{j=0}^{k} \binom{k}{j} |\omega|^{k-j}\, C_{k,j,\varepsilon} M_j\, e^{\varepsilon t} \le C_{k,\varepsilon} e^{(\omega+\varepsilon)t}.$$

4. Differentiating under the integral sign gives

$$\frac{d^k}{dt^k} e^{tA} = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \lambda^k e^{t\lambda} R(\lambda, A)\, d\lambda = A^k e^{tA}, \quad t > 0,$$

showing that $t \mapsto e^{tA}$ is C^∞. Finally, for any $\varepsilon > 0$ choose $\eta = \theta - \varepsilon$; then the same contour integral defines a holomorphic map

$$z \mapsto e^{zA} = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{z\lambda} R(\lambda, A)\, d\lambda$$

for all $z \in S_\varepsilon := \{z \neq 0 : |\arg z| < \theta - \pi/2 - \varepsilon\}$. The union of these sectors as $\varepsilon \downarrow 0$ gives the full analytic extension claimed. □

Remark 3.1.5. The estimates in (iii) and the differentiability in (iv) are crucial later when studying the *regularity of solutions* to evolution equations of the form $u'(t) = Au(t)$ or $u'(t) = Au(t) + f(t)$. They ensure that the semigroup e^{tA} smooths initial data: for instance, e^{tA} maps X into $D(A^k)$ for every $t > 0$, which provides instant regularization.

A key tool in the analysis of unbounded operators and evolution equations is the *Yosida approximation.* It provides a sequence of bounded operators that approximate an unbounded generator A in a strong and resolvent-consistent way, while preserving the essential analytic properties of the semigroup it generates.

Proposition 3.1.6 (Yosida approximation and convergence of analytic semigroups)**.** *Let $A : D(A) \subset X \to X$ be a sectorial operator in the sense of Lunardi, with parameters (ω, θ, M). For each $n > \omega$, define the bounded operator*

$$A_n := n\, A\, R(n, A) = n^2 R(n, A) - nI \in \mathcal{L}(X).$$

Then $\rho(A) \subset \rho(A_n)$ and, for every $\lambda \in \rho(A)$ and $n \in \mathbb{N}$,

$$R(\lambda, A_n) = \frac{n^2}{(\lambda+n)^2}\, R\Big(\frac{\lambda n}{\lambda+n},\, A\Big) + \frac{1}{\lambda+n}\, I. \tag{3.2}$$

In particular, $R(\lambda, A_n) \to R(\lambda, A)$ in $\mathcal{L}(X)$ as $n \to \infty$ for every $\lambda \in \rho(A)$. Moreover, for each $t > 0$,

$$\|e^{tA_n} - e^{tA}\|_{\mathcal{L}(X)} \longrightarrow 0 \quad \textit{as } n \to \infty.$$

Proof. Without loss of generality, one can suppose that $\omega < 0$. Since $\eta > \frac{\pi}{2}$, it automatically gives, on the contour : $\Re(\lambda) < 0$.
Since $A_n = nAR(n, A)$, the operator A_n is bounded on X. We claim that for every $\lambda \in \rho(A)$,

$$R(\lambda, A_n) = \frac{n^2}{(\lambda+n)^2} R\Big(\frac{\lambda n}{\lambda+n}, A\Big) + \frac{1}{\lambda+n} I.$$

To verify this, define $\mu := \mu(\lambda, n) = \dfrac{\lambda n}{\lambda+n}$. Then $\mu \in \rho(A)$ whenever $\lambda \in \rho(A)$. Let

$$S(\lambda) := \frac{n^2}{(\lambda+n)^2} R(\mu, A) + \frac{1}{\lambda+n} I.$$

We show that $(\lambda I - A_n)S(\lambda) = I = S(\lambda)(\lambda I - A_n)$.

Since $A_n = n^2 R(n, A) - nI$, one has

$$\lambda I - A_n = (\lambda+n)I - n^2 R(n, A).$$

Hence

$$\begin{aligned}(\lambda I - A_n)S(\lambda) &= \big((\lambda+n)I - n^2 R(n, A)\big)\Big(\frac{n^2}{(\lambda+n)^2} R(\mu, A) + \frac{1}{\lambda+n} I\Big)\\ &= \frac{n^2}{\lambda+n} R(\mu, A) + I - \frac{n^4}{(\lambda+n)^2} R(n, A)R(\mu, A) - \frac{n^2}{\lambda+n} R(n, A).\end{aligned}$$

Now we use the *resolvent identity*

$$R(n, A) - R(\mu, A) = (\mu - n)\, R(n, A)R(\mu, A).$$

Since $\mu - n = -\dfrac{n^2}{\lambda + n}$, we have

$$R(n,A)R(\mu,A) = -\frac{\lambda+n}{n^2}\big(R(n,A) - R(\mu,A)\big).$$

Substituting this into the previous expression yields

$$(\lambda I - A_n)S(\lambda) = \frac{n^2}{\lambda+n}R(\mu,A) + I + \frac{n^2}{\lambda+n}\big(R(n,A) - R(\mu,A)\big) - \frac{n^2}{\lambda+n}R(n,A) = I.$$

Hence $S(\lambda) = R(\lambda, A_n)$ and formula (3.2) holds.

Fix $\lambda \in \rho(A)$. From (3.2),

$$R(\lambda, A_n) - R(\lambda, A) = \frac{n^2}{(\lambda+n)^2}R\Big(\frac{\lambda n}{\lambda+n}, A\Big) + \frac{1}{\lambda+n}I - R(\lambda, A).$$

As $n \to \infty$,

$$\frac{n^2}{(\lambda+n)^2} \to 1, \qquad \frac{1}{\lambda+n} \to 0, \qquad \frac{\lambda n}{\lambda+n} \to \lambda,$$

and since $\zeta \mapsto R(\zeta, A)$ is holomorphic on $\rho(A)$, it follows that $R(\lambda, A_n) \to R(\lambda, A)$ in $\mathcal{L}(X)$.

By sectoriality of A, there exist $\theta_A \in (\pi/2, \pi)$ and $c > 0$ such that

$$\|R(\zeta, A)\| \le \frac{c}{|\zeta|}, \quad |\arg \zeta| < \theta_A.$$

For such λ and n large enough, $\mu(\lambda, n)$ lies in the same sector and satisfies $|\mu(\lambda, n)| \simeq |\lambda|$. Then by (3.2),

$$\|R(\lambda, A_n)\| \le \frac{n^2}{|\lambda+n|^2}\|R(\mu, A)\| + \frac{1}{|\lambda+n|} \le C\,|\lambda|^{-1},$$

for some C independent of n. This uniform bound ensures dominated convergence for the Dunford integral below.

For $t > 0$, the analytic semigroups generated by A and A_n are given by the Dunford representation

$$e^{tA} = \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}R(\lambda, A)\,d\lambda, \qquad e^{tA_n} = \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}R(\lambda, A_n)\,d\lambda,$$

where $\gamma_{r,\eta}$ is a contour along the sector $\{|\arg \lambda| = \eta\}$, $\pi/2 < \eta < \theta_A$. Subtracting the two integrals gives

$$e^{tA_n} - e^{tA} = \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}\big(R(\lambda, A_n) - R(\lambda, A)\big)\,d\lambda.$$

By what we have done previously, the integrand converges pointwise to zero, and :

$$\|e^{t\lambda}(R(\lambda, A_n) - R(\lambda, A))\| \leq 2\,|e^{t\lambda}|\,\frac{C}{|\lambda|},$$

which is integrable along $\gamma_{r,\eta}$ thanks to $\Re(\lambda) < 0$ on the rays of the contour. Hence, by the dominated convergence theorem,

$$\|e^{tA_n} - e^{tA}\|_{\mathcal{L}(X)} \longrightarrow 0, \qquad n \to \infty,$$

for all $t > 0$. $\square$

Remark 3.1.7. The Yosida approximation allows one to treat unbounded generators by approximating them with bounded operators, for which many results (existence, differentiability, estimates) are easier to obtain.

Proposition 3.1.8 (Behaviour near $t = 0$)**.** *Let A be a sectorial operator on X and $\{e^{tA}\}_{t\geq 0}$ its analytic semigroup. Then:*

1. *If $x \in D(A)$, then $\lim\limits_{t\to 0^+} e^{tA}x = x$. Conversely, if there exists $y = \lim_{t\to 0^+} e^{tA}x$, then $x \in D(A)$ and $y = x$.*

2. *For every $x \in X$ and $t > 0$, the integral $\displaystyle\int_0^t e^{sA}x\,ds$ belongs to $D(A)$, and*

$$A\int_0^t e^{sA}x\,ds = e^{tA}x - x.$$

If moreover the map $s \mapsto Ae^{sA}x$ belongs to $L^1(0,t;X)$, then

$$e^{tA}x - x = \int_0^t Ae^{sA}x\,ds.$$

3. *If $x \in D(A)$ and $Ax \in D(A)$, then*

$$\lim_{t\to 0^+} \frac{e^{tA}x - x}{t} = Ax.$$

Conversely, if there exists $z = \lim_{t\to 0^+}(e^{tA}x - x)/t$, then $x \in D(A)$ and $z = Ax$.

4. *If $x \in D(A)$ and $Ax \in D(A)$, then $\lim\limits_{t\to 0^+} Ae^{tA}x = Ax$. Conversely, if there exists $v = \lim_{t\to 0^+} Ae^{tA}x$, then $x \in D(A)$ and $v = Ax$.*

Proof. **(1)** Let $\xi > \omega$ and $0 < r < \xi - \omega$. For $x \in D(A)$ define $y = \xi x - Ax \in X$, i.e,

$x = R(\xi, A)y$. Using the Dunford representation of e^{tA} we have

$$e^{tA}x = e^{tA}R(\xi, A)y = \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}R(\lambda, A)R(\xi, A)y\, d\lambda.$$

Applying the resolvent identity (see proposition 1.3.8)

$$R(\lambda, A)R(\xi, A) = \frac{R(\lambda, A) - R(\xi, A)}{\xi - \lambda},$$

we obtain

$$e^{tA}x = \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} \frac{e^{t\lambda}}{\xi - \lambda}R(\lambda, A)y\, d\lambda - \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} \frac{e^{t\lambda}}{\xi - \lambda}R(\xi, A)y\, d\lambda.$$

The second integral vanishes because $R(\xi, A)$ is constant in λ and the contour $\omega + \gamma_{r,\eta}$ encloses no poles of the integrand. Hence

$$e^{tA}x = \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} \frac{e^{t\lambda}}{\xi - \lambda}R(\lambda, A)y\, d\lambda.$$

For $t \to 0^+$, the factor $e^{t\lambda} \to 1$ uniformly on the contour, and since $\left\|\frac{R(\lambda,A)y}{\xi-\lambda}\right\| \leq \frac{C}{|\lambda|^2}$ for some $C > 0$, by dominated convergence,

$$\lim_{t\to 0^+} e^{tA}x = \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} \frac{1}{\xi - \lambda}R(\lambda, A)y\, d\lambda = R(\xi, A)(\xi x - Ax) = x.$$

Hence $\lim_{t\to 0^+} e^{tA}x = x$ for every $x \in D(A)$. Since $D(A)$ is dense in its graph norm, the same limit holds for all such x.

Conversely, assume that $y = \lim_{t\to 0^+} e^{tA}x$ exists. Then $e^{tA}x \in D(A)$ for all $t > 0$, and applying $R(\xi, A)$ gives

$$R(\xi, A)y = \lim_{t\to 0^+} R(\xi, A)e^{tA}x = \lim_{t\to 0^+} e^{tA}R(\xi, A)x = R(\xi, A)x,$$

where we used that $R(\xi, A)$ and e^{tA} commute on $D(A)$. Applying $(\xi I - A)$ yields $y = x$, hence $x \in D(A)$.

(2) Let $\xi \in \rho(A)$ and $x \in X$. For $\varepsilon \in (0, t)$ we write

$$\int_{\varepsilon}^{t} e^{sA}x\, ds = \int_{\varepsilon}^{t} (\xi - A)R(\xi, A)e^{sA}x\, ds.$$

Integrating by parts gives

$$\int_{\varepsilon}^{t} e^{sA}x\,ds = \xi \int_{\varepsilon}^{t} R(\xi, A)e^{sA}x\,ds - \int_{\varepsilon}^{t} \frac{d}{ds}\big(R(\xi, A)e^{sA}\big)x\,ds \tag{3.3}$$

$$= \xi \int_{\varepsilon}^{t} R(\xi, A)e^{sA}x\,ds - e^{tA}R(\xi, A)x + e^{\varepsilon A}R(\xi, A)x \tag{3.4}$$

Since $R(\xi, A)x \in D(A)$ and e^{tA} is continuous on $[0, t]$, letting $\varepsilon \to 0$ yields

$$\int_0^t e^{sA}x\,ds = \xi R(\xi, A) \int_0^t e^{sA}x\,ds - R(\xi, A)(e^{tA}x - x).$$

Applying $(\xi I - A)$ to both sides gives

$$A \int_0^t e^{sA}x\,ds = e^{tA}x - x.$$

In particular, the integral belongs to $D(A)$.

(3) If $x \in D(A)$ and $Ax \in D(A)$, then using (2),

$$\frac{e^{tA}x - x}{t} = \frac{1}{t}A \int_0^t e^{sA}x\,ds = \frac{1}{t} \int_0^t e^{sA}Ax\,ds.$$

Since $s \mapsto e^{sA}Ax$ is continuous near $s = 0$, we have $\lim_{t\to 0^+}(e^{tA}x - x)/t = Ax$. Conversely, if $z = \lim_{t\to 0^+}(e^{tA}x - x)/t$ exists, then $x = \lim_{t\to 0^+} e^{tA}x$, so $x, z \in D(A)$ and a direct computation gives $z = Ax$.

(4) If $x, Ax \in D(A)$, statement (1) gives $Ae^{tA}x \to Ax$. Conversely, if $v = \lim_{t\to 0^+} Ae^{tA}x$ exists, then $s \mapsto Ae^{sA}x$ is continuous at $s = 0$, and

$$v = \lim_{t\to 0^+} \frac{1}{t} \int_0^t Ae^{sA}x\,ds = \lim_{t\to 0^+} \frac{e^{tA}x - x}{t} = Ax,$$

so $x \in D(A)$ by (3) and $v = Ax$. □

3.1.2 Identification of the generator

We now want to prove some results about the generator. In particular, we are interested in knowing if two sectorial operators can give birth to the same semigroup, and, furthermore, if given an analytic semigroup, we are able to find a sectorial operator that generates it.

Lemma 3.1.9. *If A is sectorial with parameters (ω, θ, M) and $\Re\lambda > \omega$, then*

$$R(\lambda, A) \;=\; \int_0^{+\infty} e^{-\lambda t}\, e^{tA}\, dt$$

Proof. Fix $\eta \in (\pi/2, \theta)$ and $r \in (0, \Re\lambda - \omega)$. Let $\gamma_{r,\eta}$ be the standard sectorial contour consisting of the two rays $\{\omega + \rho e^{\pm i\eta} : \rho \geq r\}$ and the circular arc $\{\omega + re^{i\varphi} : \varphi \in [-\eta, \eta]\}$, positively oriented. By the Dunford representation of the analytic semigroup,

$$e^{tA} \;=\; \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{tz}\, R(z, A)\, dz \qquad (t > 0).$$

Justification of Fubini's theorem. For $z \in \omega + \gamma_{r,\eta}$, the sectorial estimate gives

$$\|R(z, A)\| \leq \frac{M}{|z - \omega|}.$$

Moreover, since $r < \Re\lambda - \omega$ and $\eta \in (\pi/2, \theta)$, there exists

$$\delta := \min\{\Re\lambda - \omega - r,\ \Re\lambda - \omega\} > 0$$

such that

$$\Re(\lambda - z) \geq \delta \qquad \text{for every } z \in \omega + \gamma_{r,\eta}.$$

Indeed, on the circular arc $z = \omega + re^{i\varphi}$ one has

$$\Re(\lambda - z) = \Re\lambda - \omega - r\cos\varphi \geq \Re\lambda - \omega - r > 0,$$

and on the rays $z = \omega + \rho e^{\pm i\eta}$, $\rho \geq r$, one has

$$\Re(\lambda - z) = \Re\lambda - \omega - \rho\cos\eta \geq \Re\lambda - \omega > 0,$$

because $\cos\eta < 0$. Hence, for all $t > 0$ and all $z \in \omega + \gamma_{r,\eta}$,

$$\left\|e^{-\lambda t} e^{tz} R(z, A)\right\| = e^{-t\Re(\lambda - z)} \|R(z, A)\| \leq e^{-\delta t} \frac{M}{|z - \omega|}.$$

Therefore

$$\int_{\omega+\gamma_{r,\eta}} \int_0^\infty \left\|e^{-\lambda t} e^{tz} R(z, A)\right\| dt\, |dz| \leq M \int_{\omega+\gamma_{r,\eta}} \frac{1}{|z - \omega|} \left(\int_0^\infty e^{-\delta t}\, dt\right) |dz|.$$

Since

$$\int_0^\infty e^{-\delta t}\, dt = \frac{1}{\delta},$$

it remains to check that

$$\int_{\omega+\gamma_{r,\eta}} \frac{|dz|}{|z-\omega|} < \infty.$$

This is true because on the circular arc $|z-\omega| = r$, so its contribution is finite, and on each ray $z = \omega + \rho e^{\pm i\eta}$, $\rho \geq r$, one has $|z-\omega| = \rho$, hence

$$\int_r^\infty \frac{d\rho}{\rho}$$

would diverge. Thus this bound is not sufficient by itself on the rays, and one must keep the exponential factor before integrating in t. More precisely,

$$\int_0^\infty e^{-t\Re(\lambda-z)}\,dt = \frac{1}{\Re(\lambda-z)},$$

so that

$$\int_{\omega+\gamma_{r,\eta}} \int_0^\infty \left\| e^{-\lambda t} e^{tz} R(z,A) \right\| dt\,|dz| \leq M \int_{\omega+\gamma_{r,\eta}} \frac{|dz|}{|z-\omega|\,\Re(\lambda-z)}.$$

Now on the circular arc this is finite since the contour is compact, while on the rays $z = \omega + \rho e^{\pm i\eta}$, $\rho \geq r$, we have

$$|z-\omega| = \rho, \qquad \Re(\lambda-z) = \Re\lambda - \omega - \rho\cos\eta \geq c(1+\rho)$$

for some $c > 0$, because $\cos\eta < 0$. Hence along each ray

$$\frac{1}{|z-\omega|\,\Re(\lambda-z)} \leq \frac{C}{\rho(1+\rho)},$$

which is integrable on $[r,\infty)$. Therefore

$$\int_{\omega+\gamma_{r,\eta}} \int_0^\infty \left\| e^{-\lambda t} e^{tz} R(z,A) \right\| dt\,|dz| < \infty,$$

and Fubini's theorem applies.

Compute the integral. Using Fubini and the Dunford formula,

$$\int_0^\infty e^{-\lambda t} e^{tA}\,dt \;=\; \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \left(\int_0^\infty e^{-(\lambda-z)t}\,dt \right) R(z,A)\,dz \;=\; \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \frac{1}{\lambda-z} R(z,A)\,dz,$$

since $\Re(\lambda-z) > 0$ on the contour.

Cauchy's integral and identification. The map $z \mapsto R(z,A)$ is holomorphic on the resolvent set, and

$$z \longmapsto \frac{1}{\lambda-z} R(z,A)$$

is holomorphic on $\mathbb{C} \setminus \{\lambda\}$ with a simple pole at $z = \lambda$ whose residue is $R(\lambda, A)$. The contour $\omega + \gamma_{r,\eta}$ is homologous (in the resolvent set) to a small circle around $z = \lambda$ (because $\Re\lambda > \omega$), hence by Cauchy's integral formula,

$$\frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} \frac{1}{\lambda - z} R(z, A)\, dz \;=\; R(\lambda, A).$$

Combining the steps gives the desired identity

$$R(\lambda, A) \;=\; \int_0^\infty e^{-\lambda t} e^{tA}\, dt.$$

□

This lemma allows us to state and prove the following important result :

Corollary 3.1.10. *If A and B are sectorial and $e^{tA} = e^{tB}$ for all $t > 0$, then $A = B$.*

Proof. Let $\omega_A, \omega_B \in \mathbb{R}$ be the growth bounds associated with A and B in the sectorial estimate. Fix λ with $\Re\lambda > \max\{\omega_A, \omega_B\}$. By Lemma 3.1.9, the Laplace transform of the semigroup gives

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} e^{tA}\, dt = \int_0^\infty e^{-\lambda t} e^{tB}\, dt = R(\lambda, B).$$

Hence $R(\lambda, A) = R(\lambda, B)$ for all λ in the open half-plane $\{\Re\lambda > \max\{\omega_A, \omega_B\}\}$.

Fix one such λ. Since $R(\lambda, A) = R(\lambda, B)$, we have $\operatorname{Ran} R(\lambda, A) = \operatorname{Ran} R(\lambda, B)$, so $D(A) = D(B)$ because $D(A) = \operatorname{Ran} R(\lambda, A)$ and likewise for B. Moreover, on this common domain,

$$A = \lambda I - R(\lambda, A)^{-1} = \lambda I - R(\lambda, B)^{-1} = B.$$

Thus $A = B$. □

This result insures that two different operators cannot generate the same analytic semigroup. Furthermore, the previous results have characterized the local behaviour of analytic semigroups and clarified how the generator A governs the evolution near $t = 0$. We now turn to a structural property of operators which ensures the very existence of such semigroups: *sectoriality*. In particular, we show that an operator whose resolvent is uniformly bounded on a right half-plane automatically satisfies the sectorial resolvent estimates required in the definition. This result links the abstract semigroup theory to the analytic geometry of the resolvent set.

3.1.3 A sufficient condition for sectoriality

Lemma 3.1.11 (Resolvent Neumann series and local openness)**.** *Let $A : D(A) \subset X \to X$ be a closed linear operator and let $\lambda_0 \in \rho(A)$. Then*

$$\left\{\lambda \in \mathbb{C} : \ |\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}\right\} \subset \rho(A),$$

and for every such λ one has the Neumann series expansion

$$R(\lambda, A) = R(\lambda_0, A)\,[I + (\lambda - \lambda_0)R(\lambda_0, A)]^{-1} = \sum_{n=0}^{\infty}(-1)^n(\lambda - \lambda_0)^n R(\lambda_0, A)^{n+1}. \quad (3.5)$$

In particular, $\rho(A)$ is open and $\lambda \mapsto R(\lambda, A)$ is analytic on $\rho(A)$.

Proof. Fix $\lambda_0 \in \rho(A)$. For $y \in X$, the equation $(\lambda I - A)x = y$ is equivalent to

$$(\lambda - \lambda_0)x + (\lambda_0 I - A)x = y.$$

Set $z := (\lambda_0 I - A)x \in X$. Since $x = R(\lambda_0, A)z$, the above reads

$$\left[I + (\lambda - \lambda_0)R(\lambda_0, A)\right] z = y.$$

If $\|(\lambda - \lambda_0)R(\lambda_0, A)\| < 1$, then $I + (\lambda - \lambda_0)R(\lambda_0, A)$ is invertible with bounded inverse given by the Neumann series (see proposition 1.3.6), hence

$$z = \sum_{n=0}^{\infty}(-1)^n(\lambda - \lambda_0)^n R(\lambda_0, A)^n y, \qquad x = R(\lambda_0, A)z,$$

which yields (3.5). Thus the ball of radius $\|R(\lambda_0, A)\|^{-1}$ centered at λ_0 is contained in $\rho(A)$, and (3.5) shows analyticity. □

Proposition 3.1.12. *Let $A : D(A) \subset X \to X$ be a linear operator such that the half-plane*

$$H_\omega := \{\lambda \in \mathbb{C} : \ \Re\lambda \geq \omega\}$$

is contained in $\rho(A)$ and

$$\|\lambda R(\lambda, A)\| \leq M \qquad \text{for all } \ \Re\lambda \geq \omega, \quad (3.6)$$

for some $\omega \in \mathbb{R}$ and $M > 0$. Then A is sectorial.

Proof. Since, by assumption, $H_\omega \subset \rho(A)$, it remains to prove that the resolvent set contains a sector strictly larger than the right half-plane.

Fix $r > 0$ and set

$$\lambda_0 := \omega + ir.$$

Since $\Re\lambda_0 = \omega$, we have $\lambda_0 \in H_\omega \subset \rho(A)$. From (3.6) we get

$$\|\lambda_0 R(\lambda_0, A)\| \leq M.$$

Therefore

$$\|R(\lambda_0, A)\| = \frac{\|\lambda_0 R(\lambda_0, A)\|}{|\lambda_0|} \leq \frac{M}{|\lambda_0|}.$$

Now

$$|\lambda_0| = |\omega + ir| = \sqrt{\omega^2 + r^2},$$

hence

$$\|R(\lambda_0, A)\| \leq \frac{M}{\sqrt{\omega^2 + r^2}}. \tag{3.7}$$

By Lemma 3.1.11, if $\mu \in \mathbb{C}$ satisfies

$$|\mu - \lambda_0| < \|R(\lambda_0, A)\|^{-1},$$

then $\mu \in \rho(A)$. Since (3.7) implies

$$\|R(\lambda_0, A)\|^{-1} \geq \frac{\sqrt{\omega^2 + r^2}}{M},$$

it follows that

$$B\left(\omega + ir, \frac{\sqrt{\omega^2 + r^2}}{M}\right) \subset \rho(A).$$

Applying the same argument to $\omega - ir$, we also obtain

$$B\left(\omega - ir, \frac{\sqrt{\omega^2 + r^2}}{M}\right) \subset \rho(A) \qquad (r > 0).$$

Now let

$$S := \{\lambda \neq \omega : \ |\arg(\lambda - \omega)| < \pi - \arctan M\}.$$

We claim that

$$S \subset \bigcup_{r>0} B\left(\omega \pm ir, \frac{\sqrt{\omega^2 + r^2}}{M}\right),$$

which will imply that $S \subset \rho(A)$. Let $\lambda \in S$, and write

$$\lambda - \omega = x + iy, \qquad x, y \in \mathbb{R}.$$

We first consider the case $y > 0$. Then $\lambda - \omega$ lies in the upper half-plane. Since

$$|\arg(\lambda - \omega)| < \pi - \arctan M,$$

and since in the present case $\arg(\lambda - \omega) \in (0, \pi)$, we have in fact

$$\arg(\lambda - \omega) < \pi - \arctan M.$$

Moreover, because $\pi - \arctan M > \frac{\pi}{2}$, the condition above forces $\arg(\lambda - \omega)$ to lie in the interval

$$\left(\frac{\pi}{2}, \pi - \arctan M\right),$$

hence $x < 0$ and $y > 0$. Set

$$r := y > 0.$$

We now show that there exists $\theta \in (0, 1)$ such that

$$\lambda = \omega + ir - \frac{\theta r}{M}.$$

Since $\lambda - \omega = x + iy = x + ir$, it is enough to write

$$x = -\frac{\theta r}{M},$$

that is,

$$\theta := -\frac{Mx}{r} = -\frac{Mx}{y}.$$

Because $x < 0$ and $y > 0$, we immediately get

$$\theta > 0.$$

It remains to prove that $\theta < 1$. For this, let

$$\beta := \arg(\lambda - \omega).$$

Since $\lambda - \omega = x + iy$ with $x < 0$ and $y > 0$, we are in the second quadrant, so

$$\beta \in \left(\frac{\pi}{2}, \pi\right).$$

By the definition of S,

$$\beta < \pi - \arctan M.$$

Subtracting β from π, we obtain

$$\pi - \beta > \arctan M.$$

Now both sides belong to $\left(0, \frac{\pi}{2}\right)$, and the function tan is strictly increasing on this interval. Therefore

$$\tan(\pi - \beta) > \tan(\arctan M) = M.$$

Since β is the argument of $x + iy$ in the second quadrant, we have

$$\tan(\pi - \beta) = \frac{y}{-x}.$$

Hence

$$\frac{y}{-x} > M.$$

Because $-x > 0$, we may multiply both sides by $-x$ and divide by $y > 0$, obtaining

$$1 > \frac{-Mx}{y}.$$

But

$$\frac{-Mx}{y} = \theta,$$

so indeed

$$\theta < 1.$$

Thus

$$0 < \theta < 1.$$

Therefore

$$\lambda = \omega + x + iy = \omega - \frac{\theta r}{M} + ir = \omega + ir - \frac{\theta r}{M}.$$

It follows that

$$\lambda - (\omega + ir) = -\frac{\theta r}{M},$$

and hence

$$|\lambda - (\omega + ir)| = \frac{\theta r}{M}.$$

Since $\theta \in (0, 1)$, we have

$$\frac{\theta r}{M} < \frac{r}{M}.$$

Also, since $\omega^2 \geq 0$, we have

$$r^2 \leq \omega^2 + r^2,$$

and taking square roots gives

$$r \leq \sqrt{\omega^2 + r^2}.$$

Dividing by $M > 0$, we obtain

$$\frac{r}{M} \leq \frac{\sqrt{\omega^2 + r^2}}{M}.$$

Combining the last two inequalities yields

$$|\lambda - (\omega + ir)| = \frac{\theta r}{M} < \frac{r}{M} \leq \frac{\sqrt{\omega^2 + r^2}}{M}.$$

Therefore

$$\lambda \in B\left(\omega + ir, \frac{\sqrt{\omega^2 + r^2}}{M}\right) \subset \rho(A).$$

We now consider the case $y < 0$. Set

$$r := -y > 0.$$

Then $\lambda - \omega = x - ir$. Arguing exactly as above, one shows that $x < 0$ and that there exists $\theta \in (0, 1)$ such that

$$\lambda = \omega - ir - \frac{\theta r}{M}.$$

Consequently,

$$|\lambda - (\omega - ir)| = \frac{\theta r}{M} < \frac{r}{M} \leq \frac{\sqrt{\omega^2 + r^2}}{M},$$

so

$$\lambda \in B\left(\omega - ir, \frac{\sqrt{\omega^2 + r^2}}{M}\right) \subset \rho(A).$$

We have thus proved that every $\lambda \in S$ belongs to $\rho(A)$. Hence

$$S \subset \rho(A).$$

Define then

$$V := \Big\{\lambda : \ \Re\lambda < \omega, \ |\arg(\lambda - \omega)| \leq \pi - \arctan(2M)\Big\}.$$

Let $\lambda \in V$. As above, restricting ourselves to the case $r > 0$, one can pick $r > 0$ and $\theta \in (0, 1/2]$ (to adjust the factor $1/2M$) such that

$$\lambda = \omega + ir - \frac{\theta r}{M}.$$

Apply the Neumann series formula (3.5) with $\lambda_0 = \omega + ir$:

$$R(\lambda, A) = \sum_{n=0}^{\infty} (-1)^n \big(\lambda - (\omega + ir)\big)^n R(\omega + ir, A)^{n+1}.$$

By (3.7), we have $\|R(\omega + ir, A)\| \leq M/\sqrt{\omega^2 + r^2} \leq M/r$, and since $\theta \leq \frac{1}{2}$, we get

$$\|R(\lambda, A)\| \ \leq \ \sum_{n=0}^{\infty} \frac{|\lambda - \lambda_0|^n \, M^{n+1}}{r^{n+1}} = \frac{M}{r} \sum_{n=0}^{\infty} \theta^n \ \leq \ \frac{2M}{r}$$

Furthermore, for $\lambda = \omega + ir - \frac{\theta r}{M}$ one can easily depict the geometric relation

$$r \;\geq\; \Big(1 + \frac{1}{4M^2}\Big)^{-1/2} |\lambda - \omega|,$$

so we get:

$$\|R(\lambda, A)\| \;\leq\; \frac{2M}{r} \;\leq\; \frac{2M}{\big(1+\frac{1}{4M^2}\big)^{-1/2}} \frac{1}{|\lambda - \omega|} \;=:\; \frac{C}{|\lambda - \omega|}.$$

Previous steps show that there exists $\theta \in (\pi/2, \pi)$ such that $S_{\theta,\omega} \subset \rho(A)$ and the sectorial bound $\|R(\lambda, A)\| \leq C/|\lambda - \omega|$ holds on $S_{\theta,\omega}$. Thus A is sectorial. For a better understanding of the proof, one can refer to the figure 3.1. $\square$

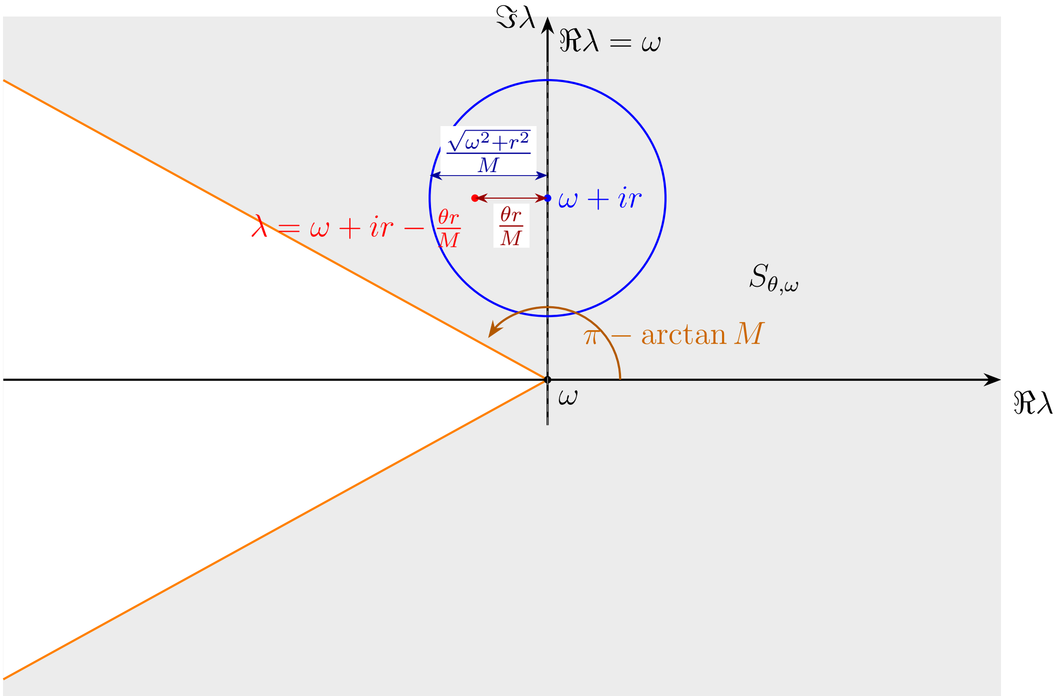

Figure 3.1: The way the proof of Proposition 3.1.12 is thought; for a given λ with real part less than ω, one can include it in the resolvent through a ball.

3.2 Spectral Properties of e^{TA}

We end this chapter with a spectral result for analytic semigroups. The objective is to relate the spectrum of the infinitesimal generator A to the spectrum of the time-T evolution operator e^{TA}. This will be useful later when interpreting finite-dimensional approximations of the semigroup operator learned from data.

We first isolate an elementary identity which will be used in the proof of the main result.

Lemma 3.2.1. *Let $A : D(A) \subset X \to X$ be the generator of the analytic semigroup $(e^{tA})_{t\geq 0}$ defined in Definition 3.1.2. Let $T > 0$ and $\lambda \in \mathbb{C}$. Define*

$$B_{\lambda,T}x := \int_0^T e^{\lambda(T-s)} e^{sA} x \, ds, \qquad x \in X.$$

Then $B_{\lambda,T} \in \mathcal{L}(X)$, $B_{\lambda,T}x \in D(A)$ for every $x \in X$, and

$$(\lambda I - A)B_{\lambda,T} = e^{T\lambda} I - e^{TA}. \tag{3.8}$$

Proof. We first prove that $B_{\lambda,T}$ is well defined and bounded. Let $x \in X$. By Definition 3.1.2, the family $(e^{tA})_{t\geq 0}$ is the analytic semigroup generated by A. In particular, the map

$$s \longmapsto e^{sA}x$$

is strongly continuous on $[0, T]$. Since the scalar map

$$s \longmapsto e^{\lambda(T-s)}$$

is continuous, the function

$$s \longmapsto e^{\lambda(T-s)} e^{sA} x$$

is continuous from $[0, T]$ into X. Hence the Bochner integral defining $B_{\lambda,T}x$ is well defined.

Moreover, by the estimate in Proposition 3.1.4, item (3), there exists a constant $M_T > 0$ such that

$$\sup_{0\leq s\leq T} \|e^{sA}\|_{\mathcal{L}(X)} \leq M_T.$$

Therefore,

$$\begin{aligned} \|B_{\lambda,T}x\| &\leq \int_0^T \left| e^{\lambda(T-s)} \right| \|e^{sA}x\| \, ds \\ &\leq M_T \left(\int_0^T e^{\mathrm{Re}(\lambda)(T-s)} \, ds \right) \|x\|. \end{aligned}$$

Since the scalar integral is finite, we obtain

$$\|B_{\lambda,T}x\| \leq C_{\lambda,T}\|x\|$$

for some constant $C_{\lambda,T} > 0$. Thus

$$B_{\lambda,T} \in \mathcal{L}(X).$$

We now prove that $B_{\lambda,T}x \in D(A)$ and identify $AB_{\lambda,T}x$. By the definition of the generator, it is enough to prove that the limit

$$\lim_{h\to 0^+} \frac{e^{hA}B_{\lambda,T}x - B_{\lambda,T}x}{h}$$

exists in X.

Using the semigroup property from Proposition 3.1.4, item (2), we have

$$e^{hA}B_{\lambda,T}x = \int_0^T e^{\lambda(T-s)}e^{hA}e^{sA}x\,ds = \int_0^T e^{\lambda(T-s)}e^{(s+h)A}x\,ds.$$

With the change of variables $u = s + h$, this becomes

$$e^{hA}B_{\lambda,T}x = \int_h^{T+h} e^{\lambda(T-u+h)}e^{uA}x\,du = e^{\lambda h}\int_h^{T+h} e^{\lambda(T-u)}e^{uA}x\,du.$$

Hence

$$\begin{aligned} e^{hA}B_{\lambda,T}x - B_{\lambda,T}x &= e^{\lambda h}\int_h^{T+h} e^{\lambda(T-u)}e^{uA}x\,du - \int_0^T e^{\lambda(T-u)}e^{uA}x\,du \\ &= (e^{\lambda h}-1)\int_h^T e^{\lambda(T-u)}e^{uA}x\,du \\ &\quad + e^{\lambda h}\int_T^{T+h} e^{\lambda(T-u)}e^{uA}x\,du \\ &\quad - \int_0^h e^{\lambda(T-u)}e^{uA}x\,du. \end{aligned}$$

Dividing by h, we get

$$\begin{aligned} \frac{e^{hA}B_{\lambda,T}x - B_{\lambda,T}x}{h} &= \frac{e^{\lambda h}-1}{h}\int_h^T e^{\lambda(T-u)}e^{uA}x\,du \\ &\quad + e^{\lambda h}\frac{1}{h}\int_T^{T+h} e^{\lambda(T-u)}e^{uA}x\,du \\ &\quad - \frac{1}{h}\int_0^h e^{\lambda(T-u)}e^{uA}x\,du. \end{aligned}$$

We now pass to the limit as $h \to 0^+$. First,

$$\frac{e^{\lambda h}-1}{h} \longrightarrow \lambda.$$

Moreover, by the strong continuity of the semigroup from Definition 3.1.2, we have

$$\int_h^T e^{\lambda(T-u)}e^{uA}x\,du \longrightarrow \int_0^T e^{\lambda(T-u)}e^{uA}x\,du = B_{\lambda,T}x.$$

Again by strong continuity,

$$\frac{1}{h}\int_{T}^{T+h} e^{\lambda(T-u)}e^{uA}x\,du \longrightarrow e^{TA}x,$$

and

$$\frac{1}{h}\int_{0}^{h} e^{\lambda(T-u)}e^{uA}x\,du \longrightarrow e^{T\lambda}x.$$

Therefore,

$$\lim_{h\to 0^+}\frac{e^{hA}B_{\lambda,T}x - B_{\lambda,T}x}{h} = \lambda B_{\lambda,T}x + e^{TA}x - e^{T\lambda}x.$$

By the definition of the generator, this proves that

$$B_{\lambda,T}x \in D(A)$$

and that

$$AB_{\lambda,T}x = \lambda B_{\lambda,T}x + e^{TA}x - e^{T\lambda}x.$$

Rearranging the terms gives

$$(\lambda I - A)B_{\lambda,T}x = e^{T\lambda}x - e^{TA}x.$$

Since this holds for every $x \in X$, we obtain

$$(\lambda I - A)B_{\lambda,T} = e^{T\lambda}I - e^{TA}.$$

□

We now introduce the contour which will be used in the resolvent representation of $(I - e^{TA})^{-1}$. Let $T > 0$. The poles of the scalar function

$$z \longmapsto \frac{e^{Tz}}{1 - e^{Tz}}$$

are exactly the points

$$\frac{2k\pi i}{T}, \qquad k \in \mathbb{Z}.$$

We choose a contour $\gamma^{\#}$ surrounding $\sigma(A)$, adapted to the analytic semigroup generated by A, and avoiding all the points

$$\frac{2k\pi i}{T}, \qquad k \in \mathbb{Z}.$$

More precisely, $\gamma^{\#}$ is obtained from a sectorial contour surrounding the spectrum of A, with small circles removed around the finitely many points $\frac{2k\pi i}{T}$ which lie inside the contour. This construction ensures that the function

$$z \longmapsto \frac{e^{Tz}}{1-e^{Tz}}$$

is holomorphic on the region enclosed by $\gamma^{\#}$, except for the removed singularities, and that the resolvent $R(z,A)$ is well-defined on the contour.

Proposition 3.2.2. *Let $T>0$. Assume that*

$$\frac{2k\pi i}{T} \in \rho(A), \qquad \forall k \in \mathbb{Z}.$$

Then

$$1 \in \rho(e^{TA}),$$

and the inverse of $I - e^{TA}$ is given by

$$(I-e^{TA})^{-1} = \frac{1}{2\pi i}\int_{\gamma^{\#}} \frac{e^{Tz}}{1-e^{Tz}} R(z,A)\,dz + I. \tag{3.9}$$

Conversely, if

$$1 \in \rho(e^{TA}),$$

then

$$\frac{2k\pi i}{T} \in \rho(A), \qquad \forall k \in \mathbb{Z}.$$

Proof. We first assume that

$$\frac{2k\pi i}{T} \in \rho(A), \qquad \forall k \in \mathbb{Z}.$$

The scalar function

$$\varphi(z) := \frac{e^{Tz}}{1-e^{Tz}}$$

is meromorphic on $\mathbb{C}$, and its poles are precisely the points

$$\frac{2k\pi i}{T}, \qquad k \in \mathbb{Z}.$$

By assumption, none of these poles belongs to $\sigma(A)$. Therefore, by the choice of $\gamma^{\#}$, the function φ is holomorphic on a neighbourhood of $\sigma(A)$ enclosed by the contour. Hence the holomorphic functional calculus developed in Chapter 2 allows us to define

$$\varphi(A) = \frac{1}{2\pi i}\int_{\gamma^{\#}} \frac{e^{Tz}}{1-e^{Tz}} R(z,A)\,dz.$$

The convergence of the integral follows from the resolvent estimates available for the generator of an analytic semigroup and from the exponential decay of e^{Tz} along the unbounded parts of the contour.

At the scalar level, we have

$$(1 - e^{Tz})\left(1 + \frac{e^{Tz}}{1 - e^{Tz}}\right) = 1.$$

Equivalently,

$$(1 - e^{Tz})(1 + \varphi(z)) = 1.$$

Applying the algebra property of the holomorphic functional calculus, we obtain

$$(I - e^{TA})(I + \varphi(A)) = I.$$

Since both e^{TA} and $\varphi(A)$ are functions of the same operator A, they commute, and therefore

$$(I + \varphi(A))(I - e^{TA}) = I.$$

Thus $I - e^{TA}$ is invertible and

$$(I - e^{TA})^{-1} = I + \varphi(A).$$

Using the integral expression of $\varphi(A)$, this gives

$$(I - e^{TA})^{-1} = \frac{1}{2\pi i} \int_{\gamma^{\#}} \frac{e^{Tz}}{1 - e^{Tz}} R(z, A)\, dz + I.$$

We now prove the converse. Assume that

$$1 \in \rho(e^{TA}).$$

Let

$$\lambda_k := \frac{2k\pi i}{T}, \qquad k \in \mathbb{Z}.$$

Then

$$e^{T\lambda_k} = 1.$$

Applying Lemma 3.2.1 with $\lambda = \lambda_k$, we obtain

$$(\lambda_k I - A)B_{\lambda_k,T} = e^{T\lambda_k} I - e^{TA} = I - e^{TA}. \tag{3.10}$$

We first prove that $\lambda_k I - A$ is injective. Let $x \in D(A)$ be such that

$$(\lambda_k I - A)x = 0.$$

Then

$$Ax = \lambda_k x.$$

Using the differentiability result for $t \mapsto e^{tA}x$ when $x \in D(A)$, (see Proposition 3.1.4), the function $u(t) = e^{tA}x$ satisfies

$$u'(t) = Au(t) = \lambda_k u(t), \qquad u(0) = x.$$

Hence

$$e^{tA}x = e^{t\lambda_k}x.$$

Taking $t = T$, we obtain

$$e^{TA}x = e^{T\lambda_k}x = x.$$

Thus

$$(I - e^{TA})x = 0.$$

Since $1 \in \rho(e^{TA})$, the operator $I - e^{TA}$ is injective. Hence $x = 0$, and $\lambda_k I - A$ is injective.

We now prove surjectivity. Let $y \in X$. Since $1 \in \rho(e^{TA})$, the operator $I - e^{TA}$ is surjective. Hence there exists $x \in X$ such that

$$(I - e^{TA})x = y.$$

Using (3.10), we get

$$y = (I - e^{TA})x = (\lambda_k I - A)B_{\lambda_k,T}x.$$

By Lemma 3.2.1,

$$B_{\lambda_k,T}x \in D(A).$$

Therefore $y \in \mathrm{Ran}(\lambda_k I - A)$. Since $y \in X$ was arbitrary,

$$\mathrm{Ran}(\lambda_k I - A) = X.$$

Thus $\lambda_k I - A : D(A) \to X$ is bijective.

Finally, since A is closed, $\lambda_k I - A$ is a closed operator. A closed bijective operator from $D(A)$, endowed with the graph norm, onto X has a bounded inverse by the closed graph theorem. Therefore

$$\lambda_k \in \rho(A).$$

Since $k \in \mathbb{Z}$ was arbitrary, we conclude that

$$\frac{2k\pi i}{T} \in \rho(A), \qquad \forall k \in \mathbb{Z}.$$

$\square$

As a consequence, we obtain the spectral mapping theorem for analytic semigroups.

Corollary 3.2.3 (Spectral Mapping Theorem)**.** *For every $T > 0$,*

$$\sigma(e^{TA}) \setminus \{0\} = e^{T\sigma(A)}.$$

Proof. We prove both inclusions.

Let first

$$\mu \in \sigma(e^{TA}) \setminus \{0\}.$$

Since $\mu \neq 0$, there exists $\lambda \in \mathbb{C}$ such that

$$\mu = e^{T\lambda}.$$

Assume by contradiction that

$$\lambda + \frac{2k\pi i}{T} \in \rho(A), \qquad \forall k \in \mathbb{Z}.$$

Equivalently,

$$\frac{2k\pi i}{T} \in \rho(A - \lambda I), \qquad \forall k \in \mathbb{Z}.$$

Applying Proposition 3.2.2 to $A - \lambda I$, we obtain

$$1 \in \rho\left(e^{T(A-\lambda I)}\right).$$

Since λI commutes with A, we have

$$e^{T(A-\lambda I)} = e^{-T\lambda} e^{TA}.$$

Thus

$$1 \in \rho\left(e^{-T\lambda} e^{TA}\right),$$

which is equivalent to

$$e^{T\lambda} \in \rho(e^{TA}).$$

But $e^{T\lambda} = \mu$, contradicting the fact that

$$\mu \in \sigma(e^{TA}).$$

Hence there exists $k \in \mathbb{Z}$ such that

$$\lambda + \frac{2k\pi i}{T} \in \sigma(A).$$

Therefore

$$\mu = e^{T\lambda} = e^{T\left(\lambda + \frac{2k\pi i}{T}\right)} \in e^{T\sigma(A)}.$$

This proves

$$\sigma(e^{TA}) \setminus \{0\} \subset e^{T\sigma(A)}.$$

Conversely, let

$$\mu \in e^{T\sigma(A)}.$$

Then there exists $\lambda \in \sigma(A)$ such that

$$\mu = e^{T\lambda}.$$

Assume by contradiction that

$$\mu \in \rho(e^{TA}).$$

Then

$$e^{T\lambda} \in \rho(e^{TA}),$$

or equivalently,

$$1 \in \rho\left(e^{-T\lambda}e^{TA}\right).$$

Since

$$e^{-T\lambda}e^{TA} = e^{T(A-\lambda I)},$$

we obtain

$$1 \in \rho\left(e^{T(A-\lambda I)}\right).$$

Applying Proposition 3.2.2 to $A - \lambda I$, we get

$$\frac{2k\pi i}{T} \in \rho(A - \lambda I), \qquad \forall k \in \mathbb{Z}.$$

In particular, for $k = 0$,

$$0 \in \rho(A - \lambda I).$$

This is equivalent to

$$\lambda \in \rho(A),$$

which contradicts $\lambda \in \sigma(A)$. Hence

$$\mu \in \sigma(e^{TA}).$$

Therefore

$$e^{T\sigma(A)} \subset \sigma(e^{TA}) \setminus \{0\}.$$

Combining the two inclusions gives

$$\sigma(e^{TA}) \setminus \{0\} = e^{T\sigma(A)}.$$

The point 0 is excluded because the exponential map never takes the value 0, although 0 may belong to $\sigma(e^{TA})$ in infinite-dimensional spaces. $\square$

This result shows that, apart from the possible spectral value 0, the spectrum of the semigroup operator e^{TA} is obtained from the spectrum of its generator through the exponential map. In the learning framework developed in the next chapter, this explains why approximating the one-step evolution operator e^{TA} also provides indirect spectral information on the underlying generator A.

Remark 3.2.4. A similar result can be proved for the point spectrum.

3.3 Comments

This chapter has recalled the semigroup framework needed to connect the operator-theoretic material of Chapter 2 with the analysis of evolution equations. The central idea is that many evolution problems can be rewritten in the abstract form

$$\frac{d}{dt}u(t) = Au(t), \qquad u(0) = u_0,$$

or, depending on the sign convention,

$$\frac{d}{dt}u(t) + Au(t) = 0, \qquad u(0) = u_0.$$

When the operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$, the solution of the homogeneous problem is represented by

$$u(t) = T(t)u_0.$$

In the analytic case, this representation is enriched by stronger regularity and spectral properties. Thus, the semigroup provides a rigorous bridge between an unbounded differential operator and the bounded evolution operators describing the system at positive times.

The chapter also emphasized the connection between semigroups and sectorial operators. This connection is important because sectoriality is the analytical framework in which the holomorphic functional calculus was developed in Chapter 2. In particular, when $-A$ generates a bounded analytic semigroup, the time evolution can be written as

$$T(t) = e^{-tA},$$

and this exponential is not merely formal: it is justified by semigroup theory and is consistent with the holomorphic functional calculus for sectorial operators. This establishes the link between the objects introduced in Chapter 2 and the time-evolution viewpoint developed in the present chapter.

A second key ingredient recalled in this chapter is the Laplace representation of the resolvent. Under suitable assumptions, the resolvent of the generator can be expressed in terms of the semigroup by

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt.$$

This identity provides a way to pass from the time-domain object $T(t)$ to the spectral object $R(\lambda, A)$. It is therefore one of the main analytical mechanisms linking semigroup theory, resolvent estimates, and the functional calculus. In particular, it shows that the resolvent is not an isolated spectral object, but can also be understood through the time evolution generated by the operator.

The spectral mapping property for e^{TA} was also recalled. It shows that, apart from the possible spectral value 0, the spectrum of the time-T evolution operator is obtained by exponentiating the spectrum of the generator:

$$\sigma(e^{TA}) \setminus \{0\} = e^{T\sigma(A)}.$$

This result clarifies how spectral information is transformed when passing from the generator to the associated evolution operator. It also explains why the semigroup viewpoint is especially useful in the study of evolution equations: instead of treating the generator only as an abstract unbounded operator, one can study its bounded time-evolution operators and recover part of the spectral information through the semigroup.

In summary, the role of this chapter is to provide the analytical mechanism that connects three levels of description:

$$\text{generator } A, \qquad \text{semigroup } T(t), \qquad \text{resolvent } R(\lambda, A).$$

Together with the holomorphic functional calculus of Chapter 2, this chain of relations provides a coherent framework for treating functions of unbounded operators arising in evolution equations. It also prepares the ground for the final discussion of possible perspectives toward statistical learning and data-driven operator theory, where semigroup observations and resolvent representations may serve as natural operator-theoretic tools.

Chapter 4

Conclusion and Perspectives

The aim of this thesis has been to present the operator-theoretic framework needed for the holomorphic functional calculus of sectorial operators and its connection with analytic semigroups. The central question was how to give a rigorous meaning to expressions of the form $f(A)$, where A is possibly unbounded and acts on an infinite-dimensional Banach space. The preceding chapters have shown that this question naturally leads from bounded functional calculus to sectorial operators, resolvent estimates, analytic semigroups, and the Riesz-Dunford representation of operator functions.

The purpose of this final chapter is not to develop a complete data-driven reconstruction theory. Rather, it is to summarize the mathematical framework established in the thesis and to explain how it may serve as a starting point for a future research project at the interface between operator theory and statistical learning. The main idea is that, in many applications, one observes the time evolution of a system rather than the unbounded generator itself. Since semigroups, resolvents, and functional calculi are analytically linked, it is natural to ask whether trajectory data could provide indirect access to these operator-theoretic objects.

The discussion below should therefore be understood as a perspective. It describes a possible route from measurements of solutions of an evolution equation to a finite-dimensional approximation of the associated semigroup, and then to resolvent and functional-calculus approximations. Each step would require additional analytical and statistical estimates to become a complete theory. These estimates are not proved here; they are left as part of a possible future project.

4.1 Summary of the Mathematical Framework

The starting point of the thesis is the classical problem of defining functions of operators. For bounded operators on Banach spaces, the Dunford–Riesz functional calculus gives a natural extension of the scalar Cauchy integral formula. If $A \in \mathcal{L}(X)$ and f is holomorphic in a neighbourhood of the spectrum $\sigma(A)$, one defines

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - A)^{-1} \, d\lambda,$$

where Γ is a contour surrounding the spectrum of A. This construction shows that the resolvent operator

$$R(\lambda, A) = (\lambda I - A)^{-1}$$

is the fundamental object through which holomorphic functions of operators are constructed.

However, the bounded framework is not sufficient for many operators arising in partial differential equations and evolution problems. Differential operators such as the Laplacian are naturally unbounded on infinite-dimensional function spaces. Their spectra may be unbounded, and the classical Dunford–Riesz contour integral cannot be applied without additional structure. This motivates the introduction of sectorial operators.

Chapter 2 developed the holomorphic functional calculus for sectorial operators. A sectorial operator is characterized by a spectral localization condition together with resolvent bounds outside a sector of the complex plane. These estimates ensure that Cauchy-type integrals remain meaningful for suitable classes of holomorphic functions. The sectorial framework makes it possible to define functions of unbounded operators, including resolvents, fractional powers, and exponentials. It also provides the natural setting for the H^∞-functional calculus, where one seeks boundedness estimates of the form

$$\|f(A)\| \leq C\|f\|_\infty$$

for appropriate bounded holomorphic functions on sectors.

Chapter 3 then connected this resolvent-based approach with analytic semigroup theory. If $-A$ generates an analytic C_0-semigroup $(T(t))_{t\geq 0}$, one formally writes

$$T(t) = e^{-tA}.$$

In this setting, the exponential is not simply a notation: it is justified by the semigroup theory and is compatible with the holomorphic functional calculus. This connection is essential because it links three complementary descriptions of the same evolution problem:

$$\text{generator } A, \qquad \text{semigroup } T(t), \qquad \text{resolvent } R(\lambda, A).$$

A key identity relating these objects is the Laplace representation of the resolvent. Under the usual growth assumptions on the semigroup, and for $\operatorname{Re}\lambda$ sufficiently large, one has

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt. \tag{4.1}$$

This formula was recalled in Chapter 3; see Lemma 3.1.9. It is particularly important for the perspective developed below, since it suggests a way to pass from time-evolution information to resolvent information.

Thus, the mathematical framework of the thesis can be summarized as follows. The holomorphic functional calculus constructs $f(A)$ from the resolvent. The resolvent can be represented through the semigroup by a Laplace transform. The semigroup describes the time evolution of the system. Therefore, if trajectory data provide information about the semigroup, one may hope to use this information as an entry point toward resolvent and functional-calculus approximation. This is the basic motivation for the data-driven perspective discussed in the next sections.

4.2 A Data-Driven Perspective from Evolution Equations

We now explain, at a formal level, how trajectory data could be related to the operator-theoretic objects studied in this thesis. Consider an abstract evolution equation

$$\frac{d}{dt}u(t) + Au(t) = 0, \qquad u(0) = u_0, \tag{4.2}$$

where A is a sectorial operator on a Banach space X. If $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$, the solution is given by

$$u(t) = T(t)u_0 = e^{-tA}u_0. \tag{4.3}$$

In applications, however, the operator A is often unknown or too complicated to handle directly. What may be available instead are measurements of the solution $u(t)$ at selected spatial points and selected times.

Suppose for instance that the state $u(t)$ is a function on a spatial domain Ω, and that we observe it at points

$$x_1, \dots, x_M \in \Omega.$$

For a fixed time t, the observed state can be represented by the vector

$$\mathbf{u}(t) = \big(u(t, x_1), \ldots, u(t, x_M)\big)^\top \in \mathbb{R}^M.$$

This finite-dimensional vector is a sampled version of the infinite-dimensional state $u(t)$. Abstractly, this corresponds to a sampling operator

$$P_M : X \to X_M \simeq \mathbb{R}^M,$$

so that

$$\mathbf{u}(t) = P_M u(t).$$

Conversely, one may introduce a reconstruction operator

$$J_M : X_M \to X,$$

which maps a discrete vector back to a function or approximate state in the original space. The pair (P_M, J_M) is used to compare finite-dimensional approximations with the continuous dynamics.

At the discrete level, one may consider a finite-dimensional semigroup

$$S_M(t) : X_M \to X_M,$$

obtained from a spatial discretization of the evolution equation. This operator is not assumed to be exactly equal to $P_M T(t) J_M$. Rather, it is the finite-dimensional dynamical object that should approximate $T(t)$ after lifting back to X. The relevant deterministic consistency statement is of the form

$$J_M S_M(t) P_M x \longrightarrow T(t)x, \qquad x \in X, \tag{4.4}$$

possibly uniformly for t in compact intervals.

A classical way to justify such a convergence is provided by Trotter–Kato type approximation results. In the projection–injection framework, one compares the finite-dimensional generators with the original generator through their resolvents. The following statement is not used here as a new result, but only as a guiding principle for a future project; see, for example, [5].

Theorem 4.2.1 (Trotter–Kato approximation theorem)**.** *Let X be a Banach space, and for each $M \in \mathbb{N}$, let X_M be a finite-dimensional Banach space. Let*

$$P_M : X \to X_M, \qquad J_M : X_M \to X$$

be bounded linear operators such that

$$P_M J_M = I_{X_M}, \qquad J_M P_M x \to x \quad \text{for every } x \in X.$$

Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X, and let A_M be the generator of a strongly continuous semigroup $(S_M(t))_{t\geq 0}$ on X_M.

Assume that the semigroups are uniformly exponentially bounded, in the sense that there exist constants $C \geq 1$ and $\omega \in \mathbb{R}$, independent of M, such that

$$\|T(t)\|_{\mathcal{L}(X)} \leq Ce^{\omega t}, \qquad \|S_M(t)\|_{\mathcal{L}(X_M)} \leq Ce^{\omega t}, \qquad t \geq 0.$$

Then the following assertions are equivalent.

1. *There exists $\lambda_0 \in \rho(A)$ with $\operatorname{Re}\lambda_0 > \omega$, and $\lambda_0 \in \rho(A_M)$ for all sufficiently large M, such that*

$$J_M(\lambda_0 I_{X_M} - A_M)^{-1} P_M x \longrightarrow (\lambda_0 I_X - A)^{-1} x \quad \text{for every } x \in X.$$

2. *For every $x \in X$ and every $T_0 > 0$,*

$$\sup_{0\leq t\leq T_0} \|J_M S_M(t) P_M x - T(t)x\|_X \longrightarrow 0 \qquad \text{as } M \to \infty.$$

In other words, convergence of the lifted resolvents at one point of the resolvent set is equivalent to strong convergence of the lifted semigroups, uniformly on compact time intervals.

The important point for the present discussion is that the passage from a discretized semigroup $S_M(t)$ to the original semigroup $T(t)$ is not automatic. It requires analytical assumptions, typically expressed in terms of the resolvents of the corresponding generators. This is one of the reasons why the data-driven part is kept here as a perspective rather than as a complete reconstruction theorem.

4.3 Recovering a Discrete Semigroup from Measurements

We now describe the finite-dimensional algebraic mechanism that motivates a possible learning procedure. Fix a time step $\Delta t > 0$. At the discretized level, the object of interest is the one-step operator

$$S_M(\Delta t) \in \mathcal{L}(X_M).$$

If we had complete knowledge of this matrix, then we would know how the discrete state evolves over one time step:

$$\mathbf{u}(t + \Delta t) = S_M(\Delta t)\mathbf{u}(t).$$

The data-driven question is whether this matrix can be inferred from measured trajectory pairs.

Assume that one observes several initial states and their evolution after one time step. At the discrete level, this gives pairs

$$u_0^{(k)} \in X_M, \qquad u_1^{(k)} = S_M(\Delta t)u_0^{(k)}, \qquad k = 1, \dots, N.$$

Here $u_0^{(k)}$ and $u_1^{(k)}$ may be interpreted as sampled measurements of solutions at two consecutive times:

$$\begin{aligned} u_0^{(k)} &= \left(u^{(k)}(0, x_1), \dots, u^{(k)}(0, x_M)\right)^\top, \\ u_1^{(k)} &= \left(u^{(k)}(\Delta t, x_1), \dots, u^{(k)}(\Delta t, x_M)\right)^\top. \end{aligned}$$

Equivalently, they are the columns of the data matrices

$$U = [u_0^{(1)}, \dots, u_0^{(N)}], \qquad V = [u_1^{(1)}, \dots, u_1^{(N)}]$$

in $\mathbb{R}^{M \times N}$. Since each pair satisfies

$$u_1^{(k)} = S_M(\Delta t)u_0^{(k)},$$

one has the matrix relation

$$V = S_M(\Delta t)U. \tag{4.5}$$

This identity already contains the basic learning idea: the one-step semigroup is the linear operator that maps the cloud of initial observations to the cloud of evolved observations.

A covariance formulation makes this relation more symmetric and closer to standard operator-learning and reduced-order modeling techniques. Define the empirical covariance and cross-covariance matrices by

$$C_U^{(N)} = \frac{1}{N}UU^\top, \qquad C_{UV}^{(N)} = \frac{1}{N}VU^\top. \tag{4.6}$$

Using (4.5), we obtain the exact algebraic relation

$$C_{UV}^{(N)} = S_M(\Delta t) C_U^{(N)}. \tag{4.7}$$

Thus, the discrete semigroup is encoded in the covariance relation between the observed states and their one-step evolution.

If $C_U^{(N)}$ is invertible, then (4.7) formally gives

$$S_M(\Delta t) = C_{UV}^{(N)} \big(C_U^{(N)}\big)^{-1}. \tag{4.8}$$

If $C_U^{(N)}$ is not invertible, or if some directions are poorly observed, one may instead use a Moore–Penrose pseudo-inverse (see [1]):

$$\Phi_{N,M} = C_{UV}^{(N)} \big(C_U^{(N)}\big)^{+}. \tag{4.9}$$

The operator $\Phi_{N,M}$ should then be interpreted as a finite-dimensional estimate of the one-step semigroup on the subspace effectively explored by the data. In practice, small singular values or small covariance eigenvalues would have to be handled carefully, for instance by spectral truncation. This is a standard issue in inverse problems and data-driven reduced modeling; see, for example, [3, 15, 9].

This covariance-based viewpoint is closely related to methods such as Dynamic Mode Decomposition, where one fits a linear time-advance operator from snapshot pairs [20, 10]. It is also conceptually related to Koopman-based approaches, where the time evolution is represented by a linear operator acting on observables [6]. The difference in the present perspective is that the learned finite-dimensional evolution is not the final object of interest. Rather, it is only a first step toward approximating more refined operator-theoretic quantities, namely resolvents and, ultimately, functions of the generator.

This also distinguishes the present point of view from neural operator approaches such as DeepONet and Fourier Neural Operators [12, 11, 8], or from physics-informed neural networks [18]. These methods aim to learn input-output maps or solution operators with strong predictive performance. Here, the perspective is more structural: the goal would be to preserve the analytic links between semigroups, resolvents, and functional calculus.

At this level, no statistical convergence theorem is claimed. A complete theory would need to specify the sampling model, the observation noise, the excitation of the initial data, and the regularization of the covariance inverse. The point of (4.7) is only to exhibit the algebraic mechanism by which trajectory data can contain information about the discretized semigroup.

4.4 From a Learned Semigroup to Resolvent Approximation

Once a finite-dimensional approximation of the one-step semigroup is available, the next question is whether it can be used to approximate the resolvent of the generator. The analytical motivation comes from the Laplace formula (4.1). Formally, if the semigroup is known, then one may recover the resolvent by integrating the time evolution against the kernel $e^{-\lambda t}$.

In practice, this integral would have to be discretized. For a time step $\Delta t > 0$, a first approximation of the Laplace integral is the Riemann sum

$$R(\lambda, A)x \approx \Delta t \sum_{k=0}^{K-1} e^{-\lambda k \Delta t} T(k\Delta t)x. \tag{4.10}$$

The truncation index K is chosen so that $K\Delta t$ is large enough to capture the main contribution of the integral. This approximation involves two classical sources of error: the time-discretization error due to the Riemann approximation, and the truncation error due to replacing the infinite interval $[0, \infty)$ by $[0, K\Delta t]$.

At the discretized level, if $S_M(t)$ is a genuine finite-dimensional semigroup, then

$$S_M(k\Delta t) = S_M(\Delta t)^k.$$

This suggests the finite-dimensional discrete Laplace proxy

$$R_M^{\Delta t, K}(\lambda) = \Delta t \sum_{k=0}^{K-1} e^{-\lambda k \Delta t} S_M(\Delta t)^k. \tag{4.11}$$

If $S_M(\Delta t)$ is replaced by the learned operator $\Phi_{N,M}$, one obtains the data-driven proxy

$$R_{N,M}^{\Delta t, K}(\lambda) = \Delta t \sum_{k=0}^{K-1} e^{-\lambda k \Delta t} \Phi_{N,M}^k. \tag{4.12}$$

This formula is the natural discrete analogue of the Laplace representation of the resolvent.

If the discrete operator is stable enough and the infinite series converges, one may also formally write

$$R_{N,M}^{\Delta t}(\lambda) = \Delta t \big(I - e^{-\lambda \Delta t} \Phi_{N,M}\big)^{-1}. \tag{4.13}$$

This is simply the closed form of the geometric series associated with (4.12). Such a formula is attractive because it transforms the reconstruction of the resolvent into the inversion of a finite-dimensional matrix. However, its validity requires spectral stability conditions ensuring that the inverse exists and that the series is well behaved.

This point of view is related to recent works in which resolvents or Laplace-domain representations are used for data-driven analysis of dynamical systems. Susuki and Mezić introduce the Koopman resolvent as a Laplace-domain object associated with nonlinear autonomous dynamics [21]. Meng, Zhou, Ornik and Liu propose resolvent-type methods for learning generators from data [17, 16]. Kostic, Lounici, Halconruy, Devergne, Novelli and Pontil use Laplace-transform techniques for learning continuous Markov semigroups [7]. The perspective of the present thesis is different but complementary: the resolvent is not only a tool for generator identification, but also the object that enters the holomorphic functional calculus.

To turn (4.12) into a rigorous approximation of $(\lambda I - A)^{-1}$, one would need to combine several limits. First, for fixed M, the learned operator $\Phi_{N,M}$ should approximate $S_M(\Delta t)$ on the observed subspace. Second, the lifted finite-dimensional semigroup should approximate the original semigroup through a Trotter–Kato type argument, as indicated in Theorem 4.2.1. Third, the Riemann sum should converge to the Laplace integral as $\Delta t \to 0$ and $K\Delta t \to \infty$. These steps are not established here, but they indicate the analytical structure of the future problem.

4.5 From Resolvents to the Functional Calculus

The final step in the proposed perspective is to use approximate resolvents inside the holomorphic functional calculus. For a sectorial operator A, the functional calculus is based on contour integral formulas of the form

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z)(zI - A)^{-1}\, dz, \tag{4.14}$$

where Γ is an admissible contour and f belongs to a suitable class of holomorphic functions. Formula (4.14) shows that, if the resolvent can be approximated along the contour, then one may try to approximate $f(A)$.

A practical version of this idea would require discretizing the contour integral. For quadrature points $z_1, \ldots, z_J$ on Γ and weights $w_1, \ldots, w_J$, one obtains a formal approximation

$$f(A) \approx \frac{1}{2\pi i} \sum_{j=1}^{J} w_j f(z_j)(z_j I - A)^{-1}. \tag{4.15}$$

If each resolvent $(z_j I - A)^{-1}$ is replaced by a learned Laplace proxy, then one is led to the finite-dimensional approximation

$$f(A) \approx \frac{1}{2\pi i} \sum_{j=1}^{J} w_j f(z_j) J_M R_{N,M}^{\Delta t,K}(z_j) P_M. \tag{4.16}$$

This expression should not be interpreted as a theorem in the present thesis. It is a formal representation of what a data-driven functional-calculus approximation might look like if all the intermediate steps were controlled.

Another way to view the same idea is through rational approximation. Suppose that f is approximated on a suitable sector or contour by a rational function of the form

$$r_J(z) = \sum_{j=1}^{J} \frac{a_j}{z_j - z}. \tag{4.17}$$

Then the functional calculus gives

$$r_J(A) = \sum_{j=1}^{J} a_j (z_j I - A)^{-1}. \tag{4.18}$$

In such a representation, approximating $f(A)$ reduces to approximating finitely many resolvents. This is precisely where a learned Laplace-resolvent proxy could be inserted. The rational approximation viewpoint is natural in numerical functional calculus because it reduces a holomorphic functional-calculus problem to a finite number of resolvent evaluations.

The challenge is that each approximation layer introduces an error. The learned semigroup introduces a data error. The discrete Laplace transform introduces time quadrature and truncation errors. The passage from finite-dimensional dynamics to the infinite-dimensional semigroup introduces a projection and discretization error. Finally, the contour quadrature or rational approximation introduces an additional functional-calculus error. A complete future theory would have to propagate these errors through the chain

$$\text{data} \longrightarrow \text{semigroup} \longrightarrow \text{resolvent} \longrightarrow \text{functional calculus.}$$

This chain is the main conceptual contribution of the perspective proposed here. It shows how the analytical results of the thesis could become relevant for statistical learning: not by directly learning the unbounded operator A, but by learning finite-dimensional time-evolution operators and then using the semigroup–resolvent–functional calculus structure to recover more refined operator functions.

4.6 Analytical Difficulties and Future Work

The discussion above deliberately avoids claiming a complete reconstruction result. Several substantial difficulties would need to be addressed before the proposed pipeline could become a rigorous method.

First, the finite-dimensional semigroups must be connected to the original semigroup. This is not a purely numerical issue. In infinite dimensions, convergence of sampled or projected operators does not automatically imply convergence of the corresponding evolution families. A Trotter–Kato type theorem gives a natural criterion in terms of resolvent convergence, but verifying this criterion depends on the discretization and on the properties of the generator.

Second, the data must excite the relevant directions of the finite-dimensional space. If the measured trajectories only explore a low-dimensional subspace, then the covariance relation (4.7) can identify the semigroup only on that subspace. This is not merely a numerical limitation; it is an identifiability issue. Regularization or spectral truncation may stabilize the estimator, but it cannot recover directions that are absent from the data.

Third, the use of powers $\Phi_{N,M}^k$ in (4.12) may amplify errors. Even if $\Phi_{N,M}$ is close to $S_M(\Delta t)$ for one time step, long-time iterations can be sensitive to spectral perturbations. This is particularly important because the Laplace approximation involves multiple time steps. Stability estimates would therefore be required to control the learned discrete semigroup over the time window used in the Laplace sum.

Fourth, the approximation of the resolvent must be uniform enough to be inserted into the functional calculus. A contour integral such as (4.14) requires control of the resolvent on an entire contour, not only at a single value of λ. Similarly, rational approximations require estimates at all poles z_j. Thus, a complete analysis would need resolvent error bounds that are uniform over the relevant complex parameters.

Finally, the entire construction is sensitive to the fact that A is unbounded. The main reason for using sectorial operators and the holomorphic functional calculus is precisely that unbounded operators require careful analytical control. A future learning theory should therefore preserve the sectorial structure, resolvent bounds, and semigroup properties, rather than treating the operator as a purely finite matrix object.

These difficulties explain why the present thesis does not claim to solve the learning problem. Instead, it identifies the operator-theoretic framework in which such a problem could be formulated. The results of Chapters 2 and 3 provide the necessary analytical language: sectoriality, resolvent estimates, analytic semigroups, Laplace representations, and contour formulas for $f(A)$.

4.7 Final Remarks

The thesis has focused on the mathematical foundations of the holomorphic functional calculus for sectorial operators and its relation to analytic semigroups. Starting from the classical bounded functional calculus, we moved to the sectorial framework, where functions of unbounded operators can be defined through resolvent estimates and contour integrals. We then studied analytic semigroups and their connection with sectoriality, emphasizing the role of the semigroup as the time-domain counterpart of the generator and its resolvent.

The final perspective discussed in this chapter suggests that these ideas may be useful for future work in statistical learning and data-driven operator approximation. If one observes trajectories of an evolution equation, it may be possible to learn a finite-dimensional approximation of the semigroup. Through the Laplace representation, such a learned semigroup could then provide approximations of the resolvent. Through the holomorphic functional calculus, these resolvent approximations could finally be used to approximate functions of the underlying operator.

This program remains open. Its rigorous development would require combining operator theory, numerical analysis, inverse problems, and statistical learning. The role of the present thesis is to provide the operator-theoretic foundation for such a future direction, while making clear that the data-driven reconstruction itself is not fully developed here.

Bibliography

[1] Adi Ben-Israel and Thomas NE Greville. *Generalized inverses: theory and applications*. Springer, 2003.

[2] Michael Cowling et al. "Banach space operators with a bounded H functional calculus". In: *journal of the australian mathematical society* 60.1 (1996), pp. 51–89.

[3] Heinz W. Engl, Martin Hanke, and Andreas Neubauer. *Regularization of Inverse Problems*. Springer, 1996.

[4] Markus Haase. "The functional calculus for sectorial operators". In: *The functional calculus for sectorial operators*. Springer, 2006, pp. 19–60.

[5] Kazufumi Ito and Franz Kappel. "The Trotter-Kato theorem and approximation of PDEs". In: *Mathematics of computation* 67.221 (1998), pp. 21–44.

[6] Bernard O. Koopman. "Hamiltonian systems and transformation in Hilbert space". In: *Proceedings of the National Academy of Sciences* 17.5 (1931), pp. 315–318.

[7] Vladimir R. Kostic et al. "Laplace Transform Based Low-Complexity Learning of Continuous Markov Semigroups". In: *Proceedings of the 42nd International Conference on Machine Learning*. Vol. 267. Proceedings of Machine Learning Research. PMLR, 2025, pp. 31560–31589.

[8] Nikola Kovachki et al. "Neural operator: Learning maps between function spaces". In: *arXiv preprint arXiv:2108.08481* (2021).

[9] Boris Kramer, Benjamin Peherstorfer, and Karen E. Willcox. "Learning Nonlinear Reduced Models from Data with Operator Inference". In: *Annual Review of Fluid Mechanics* 56 (2024), pp. 521–548. DOI: 10.1146/annurev-fluid-121021-025220.

[10] J. Nathan Kutz et al. *Dynamic Mode Decomposition: Data-Driven Modeling of Complex Systems*. SIAM, 2016.

[11] Zongyi Li et al. "Fourier neural operator for parametric partial differential equations". In: *arXiv preprint arXiv:2010.08895* (2020).

[12] Lu Lu et al. "Learning nonlinear operators via DeepONet based on the universal approximation theorem of operators". In: *Nature Machine Intelligence* 3.3 (2021), pp. 218–229.

[13] Alessandra Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Springer Science & Business Media, 2012.

[14] Alan McIntosh. "Operators which have an H^∞ functional calculus". In: *Miniconference on operator theory and partial differential equations*. Vol. 14. Australian National University, Mathematical Sciences Institute, 1986, pp. 210–232.

[15] Shane A. McQuarrie, Panos Khodabakhshi, and Karen E. Willcox. "Nonintrusive Reduced-Order Models for Parametric Partial Differential Equations via Data-Driven Operator Inference". In: *SIAM Journal on Scientific Computing* 45.4 (2023), A1917–A1946. DOI: 10.1137/21M1452810.

[16] Yiming Meng et al. "Koopman-Based Learning of Infinitesimal Generators without Operator Logarithm". In: *arXiv preprint arXiv:2403.15688* (2024).

[17] Yiming Meng et al. "Resolvent-Type Data-Driven Learning of Generators for Unknown Continuous-Time Dynamical Systems". In: *arXiv preprint arXiv:2411.00923* (2024).

[18] Maziar Raissi, Paris Perdikaris, and George Karniadakis. "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations". In: *Journal of Computational Physics* 378 (2019), pp. 686–707.

[19] Walter Rudin. *Functional Analysis*. 2nd ed. McGraw–Hill, 1991.

[20] Peter J. Schmid. "Dynamic mode decomposition of numerical and experimental data". In: *Journal of Fluid Mechanics* 656 (2010), pp. 5–28.

[21] Yoshihiko Susuki and Igor Mezić. "Koopman Resolvent: A Laplace-Domain Analysis of Nonlinear Autonomous Dynamical Systems". In: *SIAM Journal on Applied Dynamical Systems* 20.4 (2021), pp. 2013–2036. DOI: 10.1137/20M1335935.