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Variational Methods for Viscous Ergodic Mean Field Games

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Author: **Alessandro Cosenza**

Student ID: 945289

Advisor: Prof. Gianmaria Verzini

Co-advisors: Prof. Marco Cirant

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Abstract

In this thesis work we present an existence result for Viscous Ergodic Mean Field Games using Variational Methods.

Mean Field Games theory was developed in recent years to study differential games involving a large number of rational agents, each controlling their path according to some common strategy which is affected by the state of other agents. As the number n of agents goes to infinity, the evolution of the system is described by a Nash equilibrium of a game with a continuum of players. This results into a coupled system of PDEs, a Hamilton-Jacobi equation for the value function u and a Fokker-Planck equation for the agents' density m . In this thesis work we focus on Variational Mean Field Games, a specific class of Mean Field Games that admits a variational structure. This structure allows to find solutions of the system as critical points of an energy \mathcal{E} under a constraint \mathcal{K} , relying on techniques of convex analysis, properties of the Legendre-Fenchel transform and the Gagliardo-Nirenberg Inequality. In particular, we study Ergodic Mean Field Games, which are stationary systems that arise when considering the long time behavior of a Mean Field Game. Moreover, we consider the viscous case, in which a regularizing laplacian term is present.

In this thesis work, we extend an existence result for these systems to the case of stronger nonlinearities in the coupling term f in the Hamilton-Jacobi equation. In order to find a global minimum for \mathcal{E} , f needs to have polynomial growth under a certain "mass critical" exponent \bar{q} . We prove that, with smallness assumptions on the coefficients of f , it is possible to find local minima for \mathcal{E} considering nonlinearities up to the "Sobolev critical" exponent q^* included. We exploit some similarities between the variational structure of the problem and the one of Schrödinger equations, which were previously analyzed. Moreover, when the growth exponent of f matches exactly q^* , additional difficulties arise due to the critical scaling of the system. However, with additional smallness assumptions, we can use finer regularity results for the Hamilton-Jacobi and Fokker-Planck equations to deduce again the existence of a solution.

Keywords: Ergodic Mean Field Games, Hamilton-Jacobi equations, Fokker-Planck equa-

tions, Variational Methods, Sobolev critical exponent.

Abstract in lingua italiana

In questo lavoro di tesi presentiamo un risultato di esistenza per Giochi a Campo Medio Ergodici Viscosi tramite Metodi Variazionali.

La teoria dei Giochi a Campo Medio è stata sviluppata negli ultimi anni per studiare giochi differenziali con un grande numero di agenti, ognuno dei quali controlla la propria dinamica attraverso una strategia comune influenzata dalla posizione degli altri agenti. Quando il numero n di agenti tende all'infinito, l'evoluzione del sistema è descritta da un equilibrio di Nash per un gioco con un'infinità continua di giocatori. Questo si traduce in un sistema accoppiato di EDP, una equazione di Hamilton-Jacobi per la funzione valore u e una equazione di Fokker-Planck per la densità degli agenti m . In questo lavoro di tesi ci concentriamo sui Giochi a Campo Medio Variazionali, una classe specifica di Giochi a Campo Medio che ammette una struttura variazionale. Questa struttura permette di trovare soluzioni del sistema come punti critici di un'energia \mathcal{E} soggetta a un vincolo \mathcal{K} , attraverso tecniche di analisi convessa, proprietà della trasformata di Legendre-Fenchel e la disuguaglianza di Gagliardo-Nirenberg. In particolare, studiamo i Giochi a Campo Medio Ergodici, che sono sistemi stazionari che si presentano nello studio del comportamento per tempi lunghi di un Gioco a Campo Medio. Inoltre, consideriamo il caso viscoso, in cui è presente un termine regolarizzante dato dal Laplaciano.

In questo lavoro di tesi, estendiamo un risultato di esistenza per questi sistemi ad un caso di non linearità più forte per il termine di accoppiamento f nell'equazione di Hamilton-Jacobi. Per trovare un minimo globale per \mathcal{E} , f deve avere crescita polinomiale al di sotto di uno specifico "esponente critico di massa" \bar{q} . Noi dimostriamo che, con assunzioni di piccolezza per i coefficienti di f , è possibile trovare dei minimi locali per \mathcal{E} nei casi di non linearità fino allo "esponente di Sobolev critico" q^* incluso. Sfruttiamo alcune similarità tra la struttura variazionale del problema e quella delle equazioni di Schrödinger, che sono state precedentemente analizzate. Inoltre, nel caso in cui l'esponente di crescita di f coincida esattamente con q^* , si presentano ulteriori difficoltà dovute allo scaling critico del sistema. Tuttavia, con ulteriori assunzioni di piccolezza, è possibile utilizzare alcune stime di regolarità più raffinate per le equazioni di Hamilton-Jacobi e Fokker-Planck per dedurre ugualmente l'esistenza di una soluzione.

Parole chiave: Giochi a Campo Medio Ergodici, Equazioni di Hamilton-Jacobi, Equazioni di Fokker-Planck, Metodi Variazionali, Esponente di Sobolev critico.

Contents

Abstract	i
Abstract in lingua italiana	iii
Contents	v
Introduction	1
Notation	5
1 Mean Field Games	7
1.1 Heuristic Derivation of Mean Field Games	7
1.1.1 Stochastic Optimal Control	7
1.1.2 Heuristic Derivation	10
1.2 Standard existence theorems	13
1.2.1 Smoothing Couplings	13
1.2.2 Local Couplings	15
1.3 Application: n player games	16
1.4 Ergodic Limit and bounded domains	18
2 Fokker-Planck and Hamilton-Jacobi equations	21
2.1 Stationary Fokker-Planck equations	21
2.2 Viscous stationary Hamilton-Jacobi equations	27
3 Variational Methods For Viscous Ergodic Mean Field Games	35
3.1 Setting and Assumptions	36
3.1.1 Preliminary results on the Legendre-Fenchel transform	37
3.2 Existence of a Solution	38
3.2.1 Energy and Constraint	38
3.2.2 Regularization	42

3.2.3	Minimization of the functional	44
3.2.4	Convex Duality	48
3.2.5	Passage to the Limit	53
3.3	The Quadratic case	58
4	Conclusions	61
A	Appendix A	63
B	Appendix B	67
	Bibliography	69
	Ringraziamenti	75

Introduction

In this thesis work we present an original result of existence for Viscous Ergodic Mean Field Games using Variational Methods. The main idea is to expand known results to stronger non-linearities, up to the Sobolev critical case.

Mean Field Games (MFGs) theory models differential games involving infinitely many interacting players. Mean Field Game systems were introduced around 2005 by the seminal works of Lasry and Lions [36–38, 41] and independently by Caines, Huang and Malhamé [30–33]. The theory has developed rapidly in the years and various monographs were written covering both probabilistic methods [15, 16] and analytic methods [6, 29]. For an introduction to the subject, we refer to the notes of the lectures given by Pierre-Louis Lions in the years 2007-2012 at the Collège de France and the CIME lecture notes by Cardaliaguet and Porretta [13]. In this work, we focus on the PDE approach to Mean Field Games, namely interpreting the problem as a system of coupled PDEs, the Hamilton-Jacobi equation and the Fokker-Planck equation. This system arises by considering a differential game involving a large number of rational agents, each controlling their path according to some common strategy which is affected by the state of other agents. As the number n of agents goes to infinity, the evolution of the system is described by a Nash equilibrium of a game with a continuum of players. This, using techniques of optimal control, translates into a system where the value function u of a generic agent and the overall distribution of the agents m are coupled. The resulting system is

$$\begin{cases} -\partial_t u - \nu \Delta u + H(x, \nabla u) = f(x, m) & \text{in } (0, T) \times \mathbb{R}^N \\ \partial_t m - \nu \Delta m - \operatorname{div}(\nabla_p H(x, \nabla u)m) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ m(0) = m_0 \quad u(x, T) = G(x, m(T)) & \text{in } \mathbb{R}^N \end{cases} \quad (1)$$

In this thesis work we are interested in considering the long time behavior of the distribution of the agents, namely we consider Ergodic Mean Field Games. These are formally given by a limit of long-time horizon n -player games, which leads to a stationary version of (1) with an additional unknown Lagrange multiplier λ . Moreover, we suppose that the

agents play in a bounded state space Ω with reflection at the boundary. This translates into Neumann Boundary conditions for the PDEs. With this considerations we find the system

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(x, m(x)) & \text{on } \Omega \\ -\Delta m - \operatorname{div}(m \nabla H(\nabla u)) = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \frac{\partial m}{\partial n} + m \nabla H(\nabla u) \cdot n = 0 & \text{on } \partial\Omega \\ \int_{\Omega} m = 1, \quad \int_{\Omega} u = 0. \end{cases} \quad (2)$$

This is the system we focus on. For this system finding a solution means finding a triple $(u, \lambda, m) \in C^{2,\theta}(\bar{\Omega}) \times \mathbb{R} \times W^{1,p}(\Omega)$ for all $p > 1$ and all $\theta \in (0, 1)$ such that the first equation is solved in a classical sense and the second equation is solved in a weak sense. We assume that H is convex and satisfies some structure conditions, in particular $H(p) \leq C(|p|^\gamma + 1)$. We also assume that f is locally Lipschitz in x and

$$|f(x, m)| \leq C_f m^{q-1} + K_f. \quad (3)$$

Difficulties in the problem arise since f is unbounded and depends only on the local values of m . In particular for the model case $f(x, m) = -C_f a(x) m^{q-1}$, two values for q delineate different regimes, the "mass critical" exponent $\bar{q} = 1 + \frac{\gamma'}{N}$ (γ' here means the Hölder conjugate exponent of γ) and the "Sobolev critical" exponent $q^* = 1 + \frac{\gamma'}{N-\gamma'}$ if $N > \gamma'$ or $q^* = +\infty$ otherwise, which is the highest exponent that allows for the use of Sobolev embedding theorems. The main original result of this thesis work is the existence of a variational solution for this system in the case of $\bar{q} \leq q \leq q^*$, extending the work of Cesaroni and Cirant [18], in which they consider the case $q < \bar{q}$. We need to remark that solutions for $q < q^*$ were already found using fixed point methods in [23], however using variational methods we obtain more insight on the nature of such solutions. The case $q = q^*$, instead, was not treated before as far as we know.

Variational Mean Field Games are a particular class of MFGs that admits a variational structure which enables to find solutions to the system as critical point of a given energy under a specific constraint, relying on techniques of convex analysis and properties of the Legendre-Fenchel transform. For reference we mention [50]. Let us explain briefly the procedure for finding a solution of (2). This consists of several steps. To exploit the variational structure of the problem, we introduce the energy

$$\mathcal{E}(m, w) = \int_{\Omega} mL\left(-\frac{w}{m}\right) + F[m] dx \quad (4)$$

where L is the Legendre-Fenchel transform of H and $F = \int_0^m f(x, n) dn$. Both functions are extended in an opportune way to take into account the critical case $m = 0$. We also introduce the constraint

$$\begin{aligned} \mathcal{K} := \{ & (m, w) \in L^q(\Omega) \cap W^{1,r}(\Omega) \times L^1(\Omega) \text{ s.t.} \\ & \int_{\Omega} \nabla m \cdot \nabla \phi \, dx = \int_{\Omega} w \cdot \nabla \phi \, dx \, \forall \phi \in C^\infty(\bar{\Omega}), \\ & \int_{\Omega} m \, dx = 1, m \geq 0 \text{ a.e.} \} \end{aligned} \quad (5)$$

(here r is fixed and depends on q and γ). The goal is to find a minimizer (m, w) for \mathcal{E} on the constraint \mathcal{K} . Using the Gagliardo-Nirenberg inequality, it is possible to prove a priori bounds for functions in \mathcal{K} . Here, two cases are delineated. If $q < \bar{q}$, \mathcal{E} is bounded from below on \mathcal{K} and a global minimum is present. This is the case presented in [18]. If $\bar{q} \leq q \leq q^*$, \mathcal{E} is not bounded from below in general so no global minimum can be found. However, a local minimum may be present. The idea, coming from [45], is to restrict our minimization on a set $B_\alpha = \{\|m\|_q^q \leq \alpha\}$ for any $\alpha \geq 1$, on which \mathcal{E} is bounded from below. Once we have a minimum for any α , if we find a specific $\bar{\alpha}$ such that the minimum in the interior of $B_{\bar{\alpha}}$ is strictly less than the infimum on the boundary, then we can conclude that there exists a local minimum in the interior of $B_{\bar{\alpha}}$. This is possible assuming some restriction on C_f , the coefficient controlling the polynomial growth of f (see (3)).

Once we have a local or global minimum, we can employ techniques of convex analysis to construct a solution for (2). Due to the nature of f , which depends only on the local behavior of m , this procedure cannot be performed directly (unless one considers a dual problem in a relaxed form, which we wish to avoid). Our strategy is to approximate f with a sequence of smooth functions f_ε that have global behavior. For each of the resulting problems, we perform the above procedure and find a minimum $(m_\varepsilon, w_\varepsilon)$. Using this minimum we can now use convex analysis methods to construct an approximate smooth solution $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$. Then, the solution of the original problem can be found by a limit procedure. Here, the case $q = q^*$ is critical. Due to the critical scaling properties of the system an important a priori bound on the solutions, necessary to pass to limit, ceases to hold. To overcome this difficulty, we use finer regularity results on both the Hamilton-Jacobi equation and the Fokker-Planck equation, which are available imposing some additional constraint on C_f .

The work is structured as follows:

- In Chapter 1 we introduce the concept of Mean Field Games and give a heuristic derivation. Then, we present some classical existence results. Moreover, we show an application of Mean Field Games to differential games with a finite number n of agents. Lastly, we introduce Ergodic Mean Field Games with bounded state space, deriving the system that we analyze in Chapter 3.
- In Chapter 2 we present some results on the Hamilton-Jacobi and Fokker-Planck equations, which we exploit in Chapter 3.
- Chapter 3 is the core of the thesis. Here, the existence result is proved, following the procedure described above.

Notation

We summarize here some notation that we will use throughout the work.

- Ω —an open connected set.
- $\mathcal{P}(\Omega)$ —the space of probability measures over Ω .
- $L^p(\Omega)$ —the space of Lebesgue p -integrable functions over Ω .
- $W^{k,p}(\Omega)$ —the Sobolev space of functions with derivative up to order k in $L^p(\Omega)$.
- $C^{k,\alpha}(\bar{\Omega})$ —the space of functions with k -th derivative α -Hölder continuous.
- $\|\cdot\|_p$ —the $L^p(\Omega)$ norm.
- $\|\cdot\|_{k,p}$ —the $W^{k,p}(\Omega)$ norm.
- γ' —the Hölder conjugate exponent of γ , $\gamma' = \frac{\gamma}{\gamma-1}$.

We also denote by C a generic positive constant, while positive constants with a fixed value are denoted by a subscript ($C_f, C_H\dots$). This convention is due to the need to keep track of some constants in Chapter 3, where existence results are proved under specific constraints on the coefficients.

1 | Mean Field Games

Mean Field Games (MFGs) theory models differential games involving infinitely many interacting players. Mean Field Game systems were introduced around 2005 by the seminal works of Lasry and Lions [36–38, 41] and independently by Caines, Huang and Malhamé [30–33]. In this work, we focus on the PDE approach to MFGs, namely interpreting the problem as a system of coupled PDEs, the Hamilton-Jacobi equation and the Fokker-Planck equation. This chapter aims at introducing this approach. Firstly, we derive heuristically the system. Then, we present some classical existence result. In the sequel, we show that solutions of MFGs can be used to construct Nash equilibria for the n -players games. Lastly, we present the notion of Ergodic Mean Field Games, a particular case for the problem which translates into a stationary system. This system is then analyzed in Chapter 3, where an existence result is proved. Since the derivation of MFGs relies on several results of stochastic optimal control, we present the general ideas only without detailed proofs, providing references for the reader's convenience.

1.1. Heuristic Derivation of Mean Field Games

We start this chapter by giving a heuristic derivation of Mean Field Games. To do so, we need first to recall some basic notions of stochastic optimal control.

1.1.1. Stochastic Optimal Control

Let us introduce some results of Optimal Control Theory. We do not provide all the details and assumptions, as the subject is vast and we want to employ its methods only for a heuristic derivation. We refer to [47] for an introduction to the subject.

We are interested in a finite horizon control problem in which the state of the system is a stochastic process $(X_t)_t$ satisfying the SDE

$$X_s^{t,x,\alpha} = x + \int_t^s b(r, X_r^{t,x,\alpha}, \alpha_r) dr + \int_t^s \sigma(r, X_r^{t,x,\alpha}, \alpha_r) dB_r \quad (1.1)$$

where B_r is a standard M -dimensional Brownian motion adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$, $b : [0, T] \times \mathbb{R}^N \times A \rightarrow \mathbb{R}^N$ and $\sigma : [0, T] \times \mathbb{R}^N \times A \rightarrow \mathbb{R}^N$ are deterministic measurable functions and $(\alpha_r)_r$ is the control process, progressively measurable and taking values in some set A . We call \mathcal{A} the set of admissible control processes. In order for the equation to be well defined, assumptions have to be made both on b, σ and \mathcal{A} . A classical result in SDE theory asserts that taking b, σ uniformly lipschitz is sufficient for the existence of a solution. As for the set \mathcal{A} , we will not define it explicitly, supposing that the controls are such that all the quantities in play are well defined.

We aim at minimizing a certain cost functional J of our state

$$J(t, x, \alpha) = \mathbb{E} \left[\int_t^T L(s, X_s^{t,x,\alpha}, \alpha_s) ds + G(X_T^{t,x,\alpha}) \right] \quad (1.2)$$

where $L : [0, T] \times \mathbb{R}^N \times A \rightarrow \mathbb{R}$ is a running cost and $G : \mathbb{R}^N \rightarrow \mathbb{R}$ is a final cost. In order for the functional to be well defined, assumptions need to be made on L and g . The aim of the problem is to find $\alpha \in \mathcal{A}$ that minimizes J . Various approaches are possible, based on both probabilistic and analytic methods.

We focus here on the PDE approach. The idea is to introduce the value function $u : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ defined by

$$u(t, x) := \inf_{\alpha \in \mathcal{A}} J(t, x, \alpha). \quad (1.3)$$

This function represents the optimal value we can obtain starting at point x at time t . The value function u enjoys the following property, called *dynamic programming principle*. We have for all $t_1 < t_2$

$$u(t_1, x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_{t_1}^{t_2} L(s, X_s^{t_1,x,\alpha}, \alpha_s) dt + u(t_2, X_{t_2}^{t_1,x,\alpha}) \right]. \quad (1.4)$$

This heuristically means that, in order to play optimally at time t_1 , we just need to optimize up to the time t_2 , and then from t_2 onward. This is the most delicate step of this derivation, but remarkably under some assumptions on the probability space it can be proven. An important consequence of this property is that, in order to play optimally, we only need to know the current state and play accordingly. We hence expect a control which is a deterministic function of the state (a *Markovian control*). Moreover, the dynamic programming principle has an infinitesimal version. We can describe the local behavior by introducing the backward partial differential equation

$$\begin{cases} -\partial_t u(t, x) + H(x, \nabla u(t, x), D^2 u(t, x)) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ u(x, T) = G(x) & \text{in } \mathbb{R}^N \end{cases} \quad (1.5)$$

where

$$H(t, x, p, M) := \sup_{a \in A} \left[-L(t, x, a) - p \cdot b(t, x, a) - \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, x, a) M) \right]. \quad (1.6)$$

This equation is called *Hamilton-Jacobi(-Bellman)* equation. Ideally we would like to conclude that our value function u satisfies this equation. However, the value function is in general not very regular. The most appropriate notion of solution for the Hamilton-Jacobi equation is the notion of viscosity solution. In any case, when a regular solution is present we have the following theorem (which will be sufficient for us).

Theorem 1.1 (Verification Theorem). *Let $w \in C^{1,2}([0, T] \times \mathbb{R}^N) \cap C^0([0, T] \times \mathbb{R}^N)$ satisfy quadratic growth conditions as $|x| \rightarrow \infty$. Suppose that w is a (classical) solution of (1.5). Suppose moreover that there exists a measurable function $\alpha^*(t, x)$ with values in A such that*

$$H(t, x, \nabla u(t, x), D^2 u(t, x)) = \left[-L(t, x, \alpha^*(t, x)) - \nabla u(t, x) \cdot b(t, x, \alpha^*(t, x)) - \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, x, \alpha^*(t, x)) D^2 u(t, x)) \right], \quad (1.7)$$

the equation

$$dX_s = b(s, X_s, \alpha^*(s, X_s)) ds + \sigma(s, X_s, \alpha^*(s, X_s)) dB_s$$

given $X_t = x$ has an unique solution denoted by X_s^{t,x,α^*} and $\alpha_s^* = \alpha(t, X_s^{t,x,\alpha^*}) \in A$.

Then

$$w(t, x) = v(t, x) \quad \text{on } [0, T] \times \mathbb{R}^N \quad (1.8)$$

and α^* is the optimal Markovian control.

Proof. See [47, Th. 5.3.2]. We try to give here a heuristic argument of why Equation (1.5) appears. If we use (1.4) with t and $t+h$, we have

$$u(t, x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_t^{t+h} L(s, X_s^{t,x,\alpha}, \alpha_s) dt + u(t+h, X_{t+h}^{t,x,\alpha}) \right].$$

Now using Ito's formula

$$u(t, x) = \inf_{\alpha \in \mathcal{A}} \left[\int_t^{t+h} L(s, X_s^{t,x,\alpha}, \alpha_s) ds + u(t, x) + \int_t^{t+h} \partial_t u(s, X_s^{t,x,\alpha}) + \nabla u(s, X_s^{t,x,\alpha}) \cdot b(s, X_s^{t,x,\alpha}, \alpha_s) + \frac{1}{2} \text{Tr}(\sigma \sigma^T(s, X_s^{t,x,\alpha}, \alpha_s) D^2(u(s, X_s^{t,x,\alpha}))) ds \right].$$

We can now simplify $u(t, x)$ and let $h \rightarrow 0$ formally. We obtain

$$0 = \inf_{a \in A} \left[L(t, x, a) + \partial_t u(t, x) + \nabla u(t, x) \cdot b(t, x, a) + \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, x, a) D^2 u(t, x)) \right]$$

which recalling (1.6) becomes (1.5). \square

We can see that the control $\alpha^* = \alpha(s, X_s^{t,x,\alpha^*})$ is indeed a deterministic function of the state at a fixed time. The function $\alpha^*(t, x)$ is called *optimal feedback* and the process α^* , besides Markovian, is also called *closed loop* control. We remark lastly that, whenever α^* is uniquely defined by (1.7), it is possible to prove that H is differentiable at $(\nabla u(t, x), D^2 u(t, x))$ and

$$\begin{cases} H_p(t, x, \nabla u(t, x), D^2 u(t, x)) = -b(t, x, \alpha^*(t, x)) \\ H_M(t, x, \nabla u(t, x), D^2 u(t, x)) = -\frac{1}{2} \sigma \sigma^T(t, x, \alpha^*(t, x)). \end{cases} \quad (1.9)$$

1.1.2. Heuristic Derivation

We describe here a general derivation of the Mean Field Game system, when the state space is \mathbb{R}^N . This derivation can be found in [13] and is heuristic, but can be made rigorous using appropriately stochastic optimal control theorems as mentioned before. Let us introduce a system of infinitely many indistinguishable agents, each one with a dynamics described by

$$X_s = x + \int_t^s b(X_r, \alpha_r, m(r)) dr + \int_t^s \sigma(X_r, \alpha_r, m(r)) dB_r. \quad (1.10)$$

X is the state variable and takes values in \mathbb{R}^N , α is the control exercised by the agent, taking values in a set A , and B is a M dimensional Brownian Motion. $m(t)$ is a distribution in $\mathcal{P}(\mathbb{R}^N)$ and represents the distribution of all other agents. At this stage, we can suppose this distribution to be fixed. The coefficients $b : \mathbb{R}^N \times A \times \mathcal{P}(\mathbb{R}^N) \rightarrow \mathbb{R}^N$ and $\sigma : \mathbb{R}^N \times A \times \mathcal{P}(\mathbb{R}^N) \rightarrow \mathbb{R}^{N \times M}$ are assumed to be regular enough for X_t to be well defined. In this derivation we assumed both coefficients to be independent of time, but this is not required and in general time-dependent coefficient could be included. Each agent aims at minimizing the cost

$$J(t, x, \alpha) = \mathbb{E} \left[\int_t^T L(X_s, \alpha_s, m(s)) ds + G(X_T, m(T)) \right] \quad (1.11)$$

where $T > 0$ is the Horizon of the problem, $L : \mathbb{R}^N \times A \times \mathcal{P}(\mathbb{R}^N) \rightarrow \mathbb{R}$ is the running cost and $G : \mathbb{R}^N \times \mathcal{P}(\mathbb{R}^N)$ is the final cost. We assume both maps to be regular enough. Let us now define u as

$$u(t, x) = \inf_{\alpha} J(t, x, \alpha). \quad (1.12)$$

Following Theorem 1.1, we can suppose that u satisfies the Hamilton-Jacobi Equation

$$\begin{cases} -\partial_t u(t, x) + H(x, \nabla u(t, x), D^2 u(t, x), m(t)) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ u(x, T) = G(x, m(T)) & \text{in } \mathbb{R}^N \end{cases} \quad (1.13)$$

where $H : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{N \times N} \times \mathcal{P}(\mathbb{R}^N)$ is called Hamiltonian and is defined as

$$H(x, p, M, m) := \sup_{a \in A} \left[-L(x, a, m) - p \cdot b(x, a, m) - \frac{1}{2} \text{Tr}(\sigma \sigma^T(x, a, m) M) \right]. \quad (1.14)$$

We can now introduce $\alpha^*(t, x) \in A$ as a maximum point of H for $p = Du(t, x)$ and $M = D^2 u(t, x)$. This means

$$\begin{aligned} H(x, \nabla u(t, x), D^2 u(t, x), m(t)) &= -L(x, \alpha^*(t, x), m(t)) - \nabla u(t, x) \cdot b(x, \alpha^*(t, x), m(t)) \\ &\quad - \frac{1}{2} \text{Tr}(\sigma \sigma^T(x, \alpha^*(t, x), m(t)) D^2 u(t, x)). \end{aligned}$$

Again by Theorem 1.1, α^* is the optimal feedback for the problem. In addition, in this case $u(t, x)$ and $\alpha^*(t, x)$ also depend on the distribution $m(t)$. Each player hence controls at optimum his/her dynamics with a closed loop control, resulting in the SDE

$$dX_s^* = b(X_s^*, \alpha^*(s, X_s^*), m(s)) ds + \sigma(X_s^*, \alpha^*(s, X_s^*), m(s)) dB_s. \quad (1.15)$$

We assume the initial positions of all agents are not necessarily coincident but are independent one another. We also assume that no "common noise" is present, or in general that all Brownian Motions are independent. It is then possible to deduce that the actual distribution of all agents $\bar{m}(t)$ is given by the law $\mathcal{L}(X_s^*)$, where we suppose to know the initial distribution of all agents $\mathcal{L}(X_0^*)$. We now want to understand which equation is satisfied by $\bar{m}(t)$. We consider $\phi : [0, T) \times \mathbb{R}^N \rightarrow \mathbb{R}$ smooth and compactly supported.

We can apply Ito's formula to $\phi(t, X_t^*)$ to deduce that

$$\begin{aligned} 0 &= \mathbb{E}[\phi(T, X_T^*)] = \mathbb{E}[\phi(0, X_0^*)] + \int_0^T \mathbb{E} \left[\partial_t \phi(s, X_s^*) + b(X_s^*, \alpha^*(s, X_s^*), m(s)) \cdot \nabla \phi(s, X_s^*) \right. \\ &\quad \left. + \frac{1}{2} \text{Tr}(\sigma \sigma^T(X_s^*, \alpha^*(s, X_s^*), m(s)) D^2 \phi(s, X_s^*)) \right] ds \\ &= \int_{\mathbb{R}^N} \phi(0, x) m_0(dx) + \int_0^T \int_{\mathbb{R}^N} \left[\partial_t \phi(s, x) + b(x, \alpha^*(s, x), m(s)) \cdot \nabla \phi(s, x) \right. \\ &\quad \left. + \frac{1}{2} \text{Tr}(\sigma \sigma^T(x, \alpha^*(s, x), m(s)) D^2 \phi(s, x)) \right] \bar{m}(t, dx) ds. \end{aligned}$$

This formula is the definition of the solution in the distributional sense of the following equation

$$\begin{cases} \partial_t \bar{m} - \frac{1}{2} \sum_{i,j} D_{i,j}^2 (\bar{m}(\sigma \sigma^T)_{i,j}) + \text{div}(\bar{m}b) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ \bar{m}(0) = m_0 & \text{in } \mathbb{R}^N. \end{cases} \quad (1.16)$$

This is the so called (*Kolmogorov-Fokker-Planck*) equation. If all agents play optimally, at equilibrium we expect $m = \bar{m}$.

Remark 1.2. The previous idea can be interpreted also as a fixed point argument, where we have to find a fixed point of the map

$$(m_t)_t \rightarrow \alpha^{*,m}(x, t) \rightarrow X_t^{*,m} \rightarrow \mathcal{L}(X_t^{*,m}). \quad (1.17)$$

This map is the common ground between all approaches to MFGs. In our case, the first step is represented by the Hamilton-Jacobi equation and the last two steps are represented by the Fokker-Planck equation. For example using probabilistic methods, the first step could be represented by the Pontryagin maximum principle and the last two by a McKean-Vlasov equation (see for instance [15]).

Replacing $m = \bar{m}$ leads to the MFG system

$$\begin{cases} -\partial_t u + H(x, \nabla u, D^2 u, m) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ \partial_t m - \frac{1}{2} \sum_{i,j} D_{i,j}^2 (m(\sigma \sigma^T)_{i,j}) + \text{div}(mb) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ m(0) = m_0, u(x, T) = G(x, m(T)) & \text{in } \mathbb{R}^N. \end{cases} \quad (1.18)$$

We can simplify the system supposing $M = N$ and that σ is not controlled, for instance

$\sigma \equiv \sqrt{2\nu}I_N$. In this setting the system becomes

$$\begin{cases} -\partial_t u - \nu \Delta u + H(x, \nabla u, m) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ \partial_t m - \nu \Delta m + \operatorname{div}(H_p(x, \nabla u, m)m) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ m(0) = m_0, u(x, T) = G(x, m(T)) & \text{in } \mathbb{R}^N. \end{cases} \quad (1.19)$$

Here we have adopted a small abuse of notation and defined

$$H(x, p, m) = \sup_{a \in A} [-L(x, a, m) - p \cdot b(x, a, m)]. \quad (1.20)$$

Moreover we exploited the fact that (see (1.9))

$$H_p(x, \nabla u(t, x), m(t)) = -b(x, \alpha^*(t, x), m(t)). \quad (1.21)$$

1.2. Standard existence theorems

We now give some classic existence results for the system (1.19). For this analysis we make two preliminary assumptions. The first one is that the term H is decoupled in the terms ∇u and m . The second one is that we perform our analysis on the state space $\mathbb{T}^N = \mathbb{R}^N \setminus \mathbb{Z}^N$ (we also denote $Q_T = (0, T) \times \mathbb{T}^N$). This is in some sense the easiest state space to consider, as we do not have to deal with boundary conditions while still dealing with a bounded set. The system then becomes

$$\begin{cases} -\partial_t u - \nu \Delta u + H(x, \nabla u) = f(x, m) & \text{in } (0, T) \times \mathbb{T}^N \\ \partial_t m - \nu \Delta m + \operatorname{div}(H_p(x, \nabla u)m) = 0 & \text{in } (0, T) \times \mathbb{T}^N \\ m(0) = m_0, u(x, T) = g(x, m(T)) & \text{in } \mathbb{T}^N. \end{cases} \quad (1.22)$$

In particular we will distinguish between two cases for f , either a *smoothing coupling* or a *local coupling*. For each of the two cases we will introduce appropriate notions of solution.

1.2.1. Smoothing Couplings

We talk about *smoothing couplings* when the behavior of f and g depends on the global behavior of m . To remark this, let us indicate them as $f[m](x)$ and $g[m(T)](x)$.

Definition 1.3. We say that f and g are smoothing couplings if

$$f : C^0([0, T], \mathcal{P}(\mathbb{T}^N)) \rightarrow L^\infty(0, T; W^{1, \infty}(\mathbb{T}^N)) \subset C^0(\overline{Q}_T) \quad (1.23)$$

is continuous with range into a bounded set and

$$g : \mathcal{P}(\mathbb{T}^N) \rightarrow W^{1, \infty}(\mathbb{T}^N) \subset C^0(\mathbb{T}^N) \quad (1.24)$$

is continuous with range into a bounded set.

We can now define some appropriate notions of solution for this system.

Definition 1.4. We say that (u, m) is a *classical* solution to (1.22) if

- $m \in C^0([0, T], \mathcal{P}(\mathbb{T}^N))$, $m(0) = m_0$, $u \in C^0(\overline{Q}_T)$ and $u(x, T) = g[m(T)](x)$.
- $u, m \in C^{2,1}(Q_T)$ and the equations are satisfied pointwise in Q_T .

Definition 1.5. We say that (u, m) is a solution to (1.22) (without the adjective classical) if

- $m \in C^0([0, T], \mathcal{P}(\mathbb{T}^N))$, $m(0) = m_0$, $u \in C(\overline{Q}_T)$ and $u(x, T) = g[m(T)](x)$.
- m and ∇u are locally bounded and the equations are satisfied in the sense of distribution, against functions $\phi \in C_0^1((0, T) \times \mathbb{T}^N)$.

We can now give a first existence result.

Theorem 1.6. *Suppose that f and g are smoothing couplings. Suppose that $H \in C^1(Q_T \times \mathbb{R}^N)$ is convex in p and at least one of the following holds:*

$$\exists c_0 > 0 : |H_p(x, p)| \leq c_0(1 + |p|) \quad \forall (x, p) \in \mathbb{T}^N \times \mathbb{R}^N \quad (1.25)$$

$$\exists c_1 > 0 : H_x(x, p) \cdot p \geq -c_1(1 + |p|^2) \quad \forall (x, p) \in \mathbb{T}^N \times \mathbb{R}^N. \quad (1.26)$$

Then, for every $m_0 \in \mathcal{P}(\mathbb{T}^N)$ there exist $u \in L^\infty(0, T; W^{1, \infty}(\mathbb{T}^N)) \cap C^0(\overline{Q}_T)$ and $m \in C^0([0, T], \mathcal{P}(\mathbb{T}^N))$ such that (u, m) is a solution to (1.22). If in addition $H_p \in C^1(Q_T \times \mathbb{R}^N)$ and F is a bounded function in the space of Hölder continuous functions, the (u, m) is also a classical solution.

Proof. See [13, Th. 4]. We just remark that this proof relies on a priori estimates on the Hamilton-Jacobi equation and the Fokker-Planck equation and a fixed point argument using Schauder's theorem. \square

In general, for this type of systems uniqueness is not to be expected. However, under monotonicity assumptions on F, G we can obtain the following result.

Theorem 1.7. *Suppose f, g satisfy*

$$\int_0^T \int_{\mathbb{T}^N} (f[m_1](t, x) - f[m_2](t, x)) (m_1 - m_2)(t, dx) \geq 0 \quad (1.27)$$

$$\int_{\mathbb{T}^N} (g[m_1](x) - g[m_2](x)) (m_1 - m_2)(dx) \geq 0 \quad (1.28)$$

$\forall m_1, m_2 \in C^0([0, T], \mathcal{P}(\mathbb{T}^N))$ and $\forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^N)$ respectively. Suppose moreover that one of the following holds true: either

$$\begin{cases} \int_0^T \int_{\mathbb{T}^N} (f[m_1](t, x) - f[m_2](t, x)) (m_1 - m_2)(t, dx) = 0 \implies f[m_1](t, x) = f[m_2](t, x) \\ \int_{\mathbb{T}^N} (g[m_1](x) - g[m_2](x)) (m_1 - m_2)(dx) = 0 \implies g[m_1](x) = g[m_2](x) \end{cases} \quad (1.29)$$

$\forall m_1, m_2 \in C^0([0, T], \mathcal{P}(\mathbb{T}^N))$ and $\forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^N)$ respectively or

$$H(x, p) - H(x, q) - H_p(x, q)(p - q) = 0 \implies H_p(x, p) = H_p(x, q) \quad \forall p, q \in \mathbb{R}^N. \quad (1.30)$$

Then the system (1.22) has an unique solution as found in Theorem 1.6.

Proof. See [13, Th. 4]. □

1.2.2. Local Couplings

We talk about *Local Couplings* when f and g depend on the local values of the distribution m . We suppose that both f and g are continuous. Moreover, a standard assumption is that both f and g are bounded from below:

$$\exists K \in \mathbb{R} : f(t, x, m) \geq K, \quad g(x, m) \geq K. \quad (1.31)$$

Whenever f, g are bounded or $|H_p(t, x, p)| \leq \beta$ for some $\beta > 0$, existence results similar to Theorem 1.6 are available (see [13, Th. 11]). However, we will be interested in results when H has a superlinear growth in p . Since f is not regularizing, one does not expect in general the existence of a classical solution, but rather a weak one. In this direction, the proof of existence follows different arguments. Let us consider for instance the following

quadratic assumptions for H

$$\begin{aligned} H(t, x, p) &\geq C|p|^2 - C \\ H_p(t, x, p) \cdot p - H(t, x, p) &\geq \alpha|p|^2 - \gamma \\ |\nabla H(t, x, p)| &\leq C(1 + |p|) \end{aligned} \tag{1.32}$$

for some $C > 0$. We can introduce the following notion of solution.

Definition 1.8. Assume that (1.32) hold. A couple (u, m) is a weak solution to (1.22) if

- $f(t, x, m) \in L^1(Q_T)$, $G(x, m(T)) \in L^1(\mathbb{T}^N)$ and $m|\nabla u|^2 \in L^1(Q_T)$
- $u \in L^2(0, T; H^1(\mathbb{T}^N))$ is a weak solution of the Hamilton-Jacobi equation.
- $m \in C^0([0, T]; L^1(\mathbb{T}^N))$ is a distributional solution of the Fokker-Planck equation.

We have the following theorem.

Theorem 1.9. Assume that H is convex in p and (1.32) hold. Assume (1.31) hold and g is non-decreasing in m . Then, given $m_0 \in L^\infty(\mathbb{T}^N)$, there exists a weak solution to (1.22).

Proof. See [48, Th. 4.13]. □

We do not give additional details on this topic. We limit ourselves to observe that in the case of local couplings the procedure to find a solution usually involves introducing sequences $f_\varepsilon, g_\varepsilon$ of smoothing couplings that converge to f, g and analyze the resulting approximating problem. Using stability properties of the equations and a priori bounds, hopefully the solutions of the approximating problem converges to a solution of the original problem. So far, it is not known in general whether such solutions are smooth or not. In our results, additional assumptions on the growth of f will be crucial.

1.3. Application: n player games

We now introduce differential problems with a fixed number n of agents. As we have said, MFGs model differential games when the number of agents tends to infinity. We thus show that the strategy α^* obtain above via the MFG system is in some sense a good approximation for the optimal strategy of the finite differential game.

Let us start by considering a game with n agents. The agent i controls his/her dynamics by the equation

$$\begin{cases} dX_t^i = b(X_t^i, \alpha_t^i, m_{X_t^i}^{n,i})dt + \sqrt{2\nu}dB_t^i \\ X_0^i = x_0^i \in \mathbb{R}^N \end{cases} \tag{1.33}$$

and seeks to minimize the cost

$$J_i^n(\alpha^i, (\alpha^j)_{j \neq i}) = \mathbb{E} \left[\int_0^T L(X_t^i, \alpha_t^i, m_{X_t^i}^{n,i}) dt + G(x, m_{X_T^i}^{n,i}) \right]. \quad (1.34)$$

Here we suppose that the interaction between the agents is non-local and depends only on the reciprocal position, introducing the empirical measure

$$m_X^{n,i} := \sum_{j \neq i} \delta_{X^j}. \quad (1.35)$$

Notice that $m_X^{n,i}$ is a random measure, meaning that for every outcome ω , $m_X^{n,i}(\omega) \in \mathcal{P}(\mathbb{R}^N)$.

Remark 1.10. The choice of this type of interaction is convenient because of the so-called *Glivenko-Cantelli law of large numbers*. We do not state it precisely, but heuristically this affirms that if $X^1 \dots X^n$ are i.i.d. with law m , we have for $n \rightarrow \infty$ that

$$m_X^n := \frac{1}{n} \sum_{i=1}^n \delta_{X^i} \rightarrow m. \quad (1.36)$$

This also heuristically explains why in the derivation of the MFG system we suppose that the optimal control problem depends on the law m .

For this type of system we are usually interested in finding *Nash equilibria*.

Definition 1.11. A *Nash equilibrium* for the n differential game is a family $\bar{\alpha} = (\bar{\alpha}^1 \dots \bar{\alpha}^n)$ of admissible controls such that for any $i = 1 \dots n$ and any α_i admissible control we have

$$J_i^n(\bar{\alpha}^i, (\bar{\alpha}^j)_{j \neq i}) \leq J_i^n(\alpha^i, (\bar{\alpha}^j)_{j \neq i}) \quad (1.37)$$

Let us now show that solving the system (1.19) gives us a tool to construct an approximate Nash equilibrium. Indeed once we have a solution (u, m) , we can find the optimal feedback $\alpha^*(t, x)$ using the equation $b(x, \alpha^*(t, x), m(t)) = -H_p(x, \nabla u(t, x), m(t))$ (see (1.9)). Let us now suppose that each agent in the differential game adopts the strategy α^* , namely their dynamics is described by

$$\begin{cases} dX_t^i = b(X_t^i, \alpha^*(t, X_t^i), m_{X_t^i}^{n,i}) dt + \sqrt{2\nu} dB_t^i \\ X_0^i = x_0^i \in \mathbb{R}^N \end{cases} \quad (1.38)$$

and let $\bar{\alpha}_t^i = \alpha^*(t, X_t^i)$. We have the following result.

Theorem 1.12. *For any $n > 0$, the strategy $(\bar{\alpha}^1, \dots, \bar{\alpha}^n)$ is a ε_n -Nash equilibrium for the differential game, namely*

$$J_i^n(\bar{\alpha}^i, (\bar{\alpha}^j)_{j \neq i}) \leq J_i^n(\alpha^i, (\bar{\alpha}^j)_{j \neq i}) + \varepsilon_n \quad (1.39)$$

where ε_n is constant such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. See [13, Th. 6]. □

As we can see, we are able to construct approximate Nash equilibria for the differential game. Since differential games can be difficult to solve when the number of players is large, both numerically and analytically, it could be thus convenient to study the associated MFG instead.

1.4. Ergodic Limit and bounded domains

We introduce now a special class of Mean Field Games, for which the results of Chapter 3 are proved. This class has two main features. On the one hand we want to investigate the behavior of n -player games when players are allowed to play for long times, hence we are interested in establishing *ergodic* properties for Mean Field Games. On the other hand we want to consider situations in which the agents play in a bounded state space with a boundary. In our heuristic derivation and in the existence theorems we stated this feature was not present, as we considered \mathbb{R}^N or \mathbb{T}^N as state space, which both have no boundary. In our derivation, we consider the case of *reflection* at the boundary. Also in this case the derivation will be heuristic.

Let us begin as before by introducing an optimal control problem for a typical agent. This time the agent plays in a bounded state space $\Omega \subset \mathbb{R}^N$. We suppose that an agent satisfies the controlled SDE system

$$\begin{cases} dX_t = -\alpha_t dt + \sqrt{2}dB_t - n(X_t)dl_t \\ l(t) = \int_0^t \chi_{\partial\Omega}(X_s)dl_s \\ X_0 = 0 \end{cases} \quad (1.40)$$

where $n(x)$ represents the outward normal to $\partial\Omega$. As we can see, the dynamics is now described by a coupled system. This system is called a *Skorokhod problem* and a solution for it is a couple (X, l) . It is known that this problem has an unique solution (see for instance [42]). The role played by l_t is to reflect the process X_t with velocity $-n(X_t)$

whenever this "hits" the boundary $\partial\Omega$. By the second equation we also observe that l_t is non-null only on $\partial\Omega$. In this way, we obtain a dynamics which matches with the "standard" dynamics in Ω and which never leaves $\bar{\Omega}$. We will not enter into the delicate details of this procedure, we only keep in mind that this reflection at the boundary will lead to a Hamilton-Jacobi equation with Neumann boundary conditions.

Each agent aims at minimizing the cost, in the form of long time average, given by

$$J(x, \alpha_t) = \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T [L(\alpha_t) + f(X_t)] dt \right]. \quad (1.41)$$

We associate to this problem a *stationary* Hamilton-Jacobi equation. We are interested in finding $(u, \lambda) \in C^2(\bar{\Omega}) \times \mathbb{R}$ that solve

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(x) & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} u = 0 \end{cases} \quad (1.42)$$

where

$$H(p) = \sup_{a \in \mathbb{R}^N} [ap - L(a)]. \quad (1.43)$$

This is coherent with the definition (1.14). Moreover in this special case we notice that H coincides with the *Legendre transform* of L . This will enable the convex duality arguments used in Chapter 3. It is possible to prove an appropriate verification theorem also for this type of problem.

Theorem 1.13. *Suppose there exists $(u, \lambda) \in C^2(\bar{\Omega}) \times \mathbb{R}$ that solve (1.42). Then*

$$\lambda \leq J(x, \alpha_t) \quad (1.44)$$

for all $x \in \Omega$ and α_t admissible control. Moreover, if there exists $\alpha^(x)$ such that*

$$\alpha^*(x) \in \operatorname{argmax}_a [a \nabla u(x) - L(a)] \quad (1.45)$$

and $\alpha_t^ = \alpha^*(X_t)$ is an admissible control, then $\lambda = J(x, \alpha_t^*)$. In particular, α_t^* is an optimal Markovian control.*

Proof. See [19, Th. 1.1]. □

As in the time dependent case, also here we remark that whenever $\alpha^*(x)$ is uniquely

defined we can obtain, coherently with 1.9, that

$$\alpha^*(x) = \nabla H(\nabla(u(x))). \quad (1.46)$$

We can now consider as before an infinite number of agents, each with the dynamics we just described and each minimizing the cost

$$J(x, \alpha_t) = \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T [L(\alpha_t) + f(X_t, m(X_t))] dt \right] \quad (1.47)$$

where m is a fixed distribution of probability. We notice that f depends on the local values of m so we are in the case of local coupling. We would like now to investigate the behavior of the law of X_t , as we did in the time-dependent case. It is possible to prove (see [34]) that the law of X_t converges for long times to the measure $\bar{m}(x)$, which solves the *stationary* Fokker-Planck equation

$$\begin{cases} -\Delta \bar{m} - \operatorname{div}(\bar{m}b) = 0 & \text{on } \Omega \\ \frac{\partial \bar{m}}{\partial n} + \bar{m}b \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.48)$$

As in the time-dependent case, we can suppose that at equilibrium $m = \bar{m}$. This, leads to the following *stationary* Mean Field Game system

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(x, m(x)) & \text{on } \Omega \\ -\Delta m - \operatorname{div}(m \nabla H(\nabla u)) = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \frac{\partial m}{\partial n} + m \nabla H(\nabla u) \cdot n = 0 & \text{on } \partial\Omega \\ \int_{\Omega} m = 1, \quad \int_{\Omega} u = 0. \end{cases} \quad (1.49)$$

Finding solutions (u, λ, m) to this system is the aim of Chapter 3. We remark that this system, as the time dependent one, can be exploited to obtain ε -Nash equilibria for n -player ergodic games (see [38]).

Finally, a natural question that may arise is if the system we just derived is in some sense the limit as $T \rightarrow \infty$ of the system (1.22). This is an open question. Convergence results have been found when f non-decreasing (see [13, Th. 14]), while the problem remains open in the general case, even being false in specific cases of antimonotonicity. Since in our case f is not monotone we will not enter into further details.

2 | Fokker-Planck and Hamilton-Jacobi equations

In this chapter we collect some results on the two equations involved in the system, the Hamilton-Jacobi equation and the Fokker-Planck equation. These equations are well known and studied. We give some standard results on existence and uniqueness for these equations, and in particular we present some a priori bounds on the solutions. We present the results in the stationary case and with Neumann boundary conditions, so that they may be applied directly in Chapter 3.

For the Hamilton-Jacobi equation, the most fitting notion of solution to be considered is the notion of viscosity solution. However, in the case studied in this work the notion of classical solution is sufficient, so we present results in this setting. To obtain classical solutions, it is also necessary to use some results on strong solutions. Moreover, we show that $\|\nabla u\|_\infty \leq C$ where C depends only on the L^∞ norm of f .

The Fokker-Planck equation is usually interpreted in the distributional sense, with m that could be in general a measure of probability. For this work we limit ourselves to the case of absolutely continuous measures, with m being the L^1 density of a probability. In this case we can interpret the equation in a weak $W^{1,2}$ sense. By regularity, this solution is also in $W^{1,p}$ for all $p > 1$ and we show that the $W^{1,p}$ norm is controlled by a constant depending only on p and the L^∞ norm of the data.

2.1. Stationary Fokker-Planck equations

We deal here with the Fokker-Planck equation

$$\begin{cases} -\Delta m - \operatorname{div}(mb) = 0 & \text{on } \Omega \\ \frac{\partial m}{\partial n} + mb \cdot n = 0 & \text{on } \partial\Omega \\ \int_\Omega m = 1 \end{cases} \quad (2.1)$$

where $b : \mathbb{R} \rightarrow \mathbb{R}^N$ and Ω is a $C^{2,\alpha}$ bounded domain for all $\alpha \in (0, 1)$. Let us clarify the notion of solution we will use.

Definition 2.1. A weak solution for (2.1) is a function $m \in W^{1,2}(\Omega)$ such that

$$\int_{\Omega} \nabla m \cdot \nabla \phi \, dx = \int_{\Omega} bm \cdot \nabla \phi \, dx \quad \forall \phi \in W^{1,2}(\Omega) \quad (2.2)$$

and $\int_{\Omega} m = 1$.

Remark 2.2. For (2.2) to make sense, only integrability of b is required (for instance $b \in L^\infty(\Omega; \mathbb{R}^N)$). Hence, the writing of Neumann boundary conditions in (2.1) is only formal, as of course bm in general has no trace on $\partial\Omega$. Nonetheless, when b is regular enough boundary conditions of (2.1) can be interpreted in the standard trace sense. This formal writing of boundary conditions is also employed in Lemma 2.4.

We begin with the following proposition, an useful a priori L^p bound for linear equations.

Proposition 2.3. *Let $u \in L^p(\Omega)$, with $p > 1$. Suppose*

$$\left| \int_{\Omega} u \Delta \phi \, dx \right| \leq K \|\nabla \phi\|_{p'} \quad (2.3)$$

for all $\phi \in C^\infty(\overline{\Omega})$, $\frac{\partial \phi}{\partial n} = 0$ for some $K > 0$. Then $u \in W^{1,p}(\Omega)$ and there exists $C_E > 0$ independent of u such that

$$\|u\|_{1,p} \leq C_E(K + \|u\|_p) \quad (2.4)$$

Proof. See [2, Th. 8.1]. □

This proposition is used directly in the following lemma, where we construct a solutions to the auxiliary equation $-\Delta u = \operatorname{div} w$.

Lemma 2.4. *Let $w \in L^p(\Omega; \mathbb{R}^N)$. There exists an unique weak solution $m \in W^{1,p}(\Omega)$ to*

$$\begin{cases} -\Delta m - \operatorname{div} w = 0 & \text{on } \Omega \\ \frac{\partial m}{\partial n} + w \cdot n = 0 & \text{on } \partial\Omega \\ \int_{\Omega} m = 1. \end{cases} \quad (2.5)$$

Moreover

$$\|m\|_{1,p} \leq C_E(\|w\|_p + \|m\|_p) \quad (2.6)$$

Proof. Consider the functional

$$J(m) = \frac{1}{2} \int_{\Omega} |\nabla m|^2 - w \cdot \nabla m$$

and the affine space $X = \{m \in W^{1,2}(\Omega), \int_{\Omega} m = 1\}$. J is clearly continuous and strictly convex on X which is a closed and convex subset of a reflexive space, hence J has an unique minimization point on X by the direct method of Calculus of Variations (in particular, the strict convexity comes from the constraint $\int_{\Omega} m = 1$). Performing first variations on J , we deduce that m must satisfy

$$\int_{\Omega} \nabla m \cdot \nabla \phi \, dx = \int_{\Omega} w \cdot \nabla \phi \, dx \quad \forall \phi \in W^{1,2}(\Omega), \quad \int_{\Omega} \phi = 0, \quad (2.7)$$

which clearly can be extended to all $\phi \in W^{1,2}(\Omega)$ and it implies that m is a solution of (2.5). If we now consider the last equality for $\phi \in C^{\infty}$, $\frac{\partial \phi}{\partial n} = 0$ we obtain

$$\int_{\Omega} m(-\Delta \phi) \, dx = \int_{\Omega} \nabla m \cdot \nabla \phi \, dx = \int_{\Omega} w \cdot \nabla \phi \, dx \leq \|w\|_p \|\nabla \phi\|_{p'}$$

hence using Proposition 2.3 we deduce the desired inequality. \square

We are now ready to prove an existence result for the Fokker-Planck equation. This is a well known result (see [5, Th. II.4.4, II.4.5, II.4.7]).

Theorem 2.5. *Let $b \in L^s(\Omega; \mathbb{R}^N)$ for $s > N$. Then there exist an unique weak solution to (2.1) and*

$$\|m\|_{1,p} \leq C(\|b\|_s)$$

for all $p < N$.

Moreover, if $b \in L^{\infty}(\Omega; \mathbb{R}^N)$, then $m \in W^{1,p}(\Omega)$ for all $p > 1$ and there exists $C = C(p, \|b\|_{\infty}) > 0$ such that

$$\|m\|_{1,p} \leq C$$

Lastly, $m \in C^{0,\alpha}(\bar{\Omega})$ for all $\alpha \in (0, 1)$ and there exists $C = C(\|b\|_{\infty}) > 0$ such that

$$C^{-1} \leq m(x) \leq C$$

for all $x \in \Omega$.

Proof. Assume first b to be smooth. Let $K > 0$ and consider the equation

$$-\Delta v - b \cdot \nabla v + Kv = \psi \quad (2.8)$$

for any $\psi \in L^2(\Omega)$, with natural boundary conditions. If K is large enough, Lax-Milgram Lemma assures the existence of an unique weak solution $v \in W^{1,2}(\Omega)$, which is also in $W^{2,2}(\Omega)$ by our assumptions on the data. Define now $\mathcal{G}_K : L^2(\Omega) \rightarrow L^2(\Omega)$ such that $v = \mathcal{G}_K \psi$. By Sobolev Embeddings (Theorem A.1) \mathcal{G}_K is compact. We can now rewrite (2.1) as

$$(I - K\mathcal{G}_K^*) m = 0 \quad (2.9)$$

where \mathcal{G}_K^* is the adjoint operator of \mathcal{G}_K .

Using Fredholm's alternative we find

$$\dim \ker (I - K\mathcal{G}_K) = \dim \ker (I - K\mathcal{G}_K^*).$$

Therefore the number of linearly independent solutions of (2.1) is the same as the number of solutions of $(I - K\mathcal{G}_K) m = 0$, which is equivalent to

$$-\Delta v - b \cdot \nabla v = 0. \quad (2.10)$$

in a weak sense with natural boundary conditions. Since we are assuming b smooth, any v solving (2.10) has to be in $C^2(\overline{\Omega})$ by elliptic regularity. Moreover using Hopf's Lemma (Theorem B.1) and the hypothesis of homogeneous natural boundary conditions we can deduce that u must have at least one maximum in Ω . We can then use the strong maximum principle to conclude that v is constant. Hence solutions of (2.1) form a one dimensional vector space in $L^2(\Omega)$ and there exists an unique solution m (which therefore is in $W^{1,2}(\Omega)$) such that $\int_{\Omega} m \, dx = 1$. Moreover, $\|m\|_{1,p} \leq C(\|b\|_s)$ for all $p < N$. Indeed, if we use Lemma 2.4 we obtain

$$\|m\|_{1,p} \leq C \left(\|bm\|_p + \|m\|_p \right).$$

Using Hölder inequality, Sobolev embedding and interpolation we get for some $r < p^*$

$$\|m\|_{1,p} \leq \left(\|b\|_s \|m\|_r + \|m\|_p \right) \leq \|b\|_s \|m\|_{p^*}^{\theta_1} + \|m\|_{p^*}^{\theta_2}$$

for $\theta_1, \theta_2 < 1$, which gives

$$\|m\|_{1,p} \leq C(\|b\|_s).$$

Now if we have a general $b \in L^s(\Omega)$ we can consider a sequence of smooth functions $b_n \rightarrow b$ in $L^s(\Omega)$ and perform a standard approximation argument to deduce existence and uniqueness.

If moreover $b \in L^\infty(\Omega)$, we can use the same arguments as above to conclude that $\|m\|_{1,2} \leq C(\|b\|_\infty)$. Once we have a bound in $W^{1,2}(\Omega)$ we notice that by Sobolev embedding $bm \in L^{2^*}(\Omega)$, hence by Lemma 2.4 we conclude that $m \in W^{1,2^*}(\Omega)$. Iterating the argument, we find that $m \in W^{1,p}(\Omega)$ for all $p > 1$ and therefore $m \in C^{0,\alpha}(\bar{\Omega})$ for all $\alpha \in (0, 1)$.

Let us now prove the positivity of m . To do so, assume again that b is smooth, fix $z_0 \in C^0(\bar{\Omega})$ nonnegative and consider

$$\begin{cases} \partial_t z - \Delta z - b \cdot \nabla z = 0 & \text{on } \Omega \times (0, \infty) \\ \frac{\partial z}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty) \\ z(x, 0) = z_0(x). \end{cases} \quad (2.11)$$

Existence and uniqueness of a solution is standard (see for instance [27, Ch. 7.2.2]). Using Lemma B.3, we get that the solution z satisfies

$$0 \leq \min z_0 \leq z(x, t) \leq \max z_0$$

for all t . Moreover, using [4, Th. 4.1], we can deduce $z(\cdot, t) - \bar{\lambda}t - \bar{z}(\cdot) \rightarrow 0$ uniformly in x for $t \rightarrow \infty$, where $(\bar{\lambda}, \bar{z})$ is the solution of

$$\begin{cases} -\Delta \bar{z} - b \cdot \nabla \bar{z} = \bar{\lambda} & \text{on } \Omega \\ \frac{\partial \bar{z}}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

If z achieves both its maximum and its minimum only in $\partial\Omega$, then, using Hopf's Lemma, we can reach a contradiction with the hypothesis of homogeneous Neumann boundary conditions. This means that z has at least a maximum or a minimum in Ω , which implies using the Strong Maximum principle that z is constant and that necessarily $\lambda = 0$. Hence we have $z(\cdot, t) \rightarrow \bar{z}$ and $\min z_0 \leq \bar{z} \leq \max z_0$. We can now multiply (2.11) by m and equation (2.1) by z , integrate and subtract the two equations to obtain

$$\int_{\Omega} \partial_t z(x, t) m(x) dx = 0$$

for all $t > 0$. Thus we deduce that

$$\int_{\Omega} z_0(x) m(x) dx = \int_{\Omega} z(x, t) m(x) dx \rightarrow \int_{\Omega} \bar{z} m(x) dx = \bar{z}$$

which implies

$$0 \leq \min z_0 \leq \int_{\Omega} z_0(x)m(x) dx \leq \max z_0$$

for any $z_0 \in C^0(\bar{\Omega})$. From this, if $m < 0$ in some set A , that we can assume open by continuity of m , we can deduce an easy contradiction choosing z_0 with support in A . Thus, $m \geq 0$. By the same approximation argument as before, this holds also for $b \in L^\infty$. To prove that $m > 0$, finer results on equation (2.11) are needed. In particular, it is possible to prove (see [5, Th. II.4.6]) that there exists a weak solution z to (2.11) for any z_0 positive measure, which is continuous for $t > 0$ and such that

$$\int_{\Omega} m(x) dz_0(x) = \int_{\Omega} z(x, 1)m(x) dx \quad (2.12)$$

with $z(x, 1) > 0$. Suppose that $m(x_0) = 0$ for some x_0 . Then, choosing $z_0 = \delta_{x_0}$ yields

$$0 = \int_{\Omega} m(x)z(x, 1)dx \geq \min_{\bar{\Omega}} z(x, 1) > 0. \quad (2.13)$$

Now that we know $m > 0$ in $\bar{\Omega}$ we can prove that $m(x) \geq C(\|b\|_\infty)$. Suppose that it is not the case. Then, there exists a sequence b_n bounded in $L^\infty(\Omega)$ such that the respective solutions m_n satisfy

$$\min m_n(x) \rightarrow 0. \quad (2.14)$$

Up to a subsequence and using the uniform bound on the solutions we can conclude that

$$b_n \rightarrow b \text{ weakly}^* \text{ in } L^\infty(\Omega) \quad m_n \rightarrow m \text{ weakly in } W^{1,p}(\Omega) \quad (2.15)$$

for all $p < \infty$. Thus $m_n \rightarrow m$ in $C^{0,\alpha}$ for all $\alpha \in (0, 1)$ and we can conclude that m and b satisfy (2.1) with $\min m = 0$, which is a contradiction. \square

To conclude the section, we present a sharper estimate which will be useful in the Sobolev critical case that we will treat in Chapter 3. This result has the advantage of giving us a uniform bound in $L^N(\Omega)$. The application we keep in mind is the case of a sequence of b_ε which are all in $L^\infty(\Omega)$ but that are bounded uniformly only in $L^N(\Omega)$.

Proposition 2.6. *Let $b \in L^\infty(\Omega)$ such that $\|b\|_N < \frac{1}{C_E C_S}$, where C_E is defined in (2.4) and C_S is the p Sobolev Embedding constant (see Theorem A.1). Then for all $p < N$ the solution m of (2.1) satisfies*

$$\|m\|_{1,p} \leq C(\|b\|_N) \quad (2.16)$$

Proof. Arguing as before, by Lemma 2.4 we have that

$$\|m\|_{1,p} \leq C_E(\|bm\|_p + \|m\|_p).$$

by Hölder inequality, Sobolev Embedding and Interpolation

$$\|bm\|_p + \|m\|_p \leq \|b\|_N \|m\|_{p^*} + \|m\|_1 \|m\|_{p^*}^\theta \leq C_S \|b\|_N \|m\|_{1,p} + \|m\|_{1,p}^\theta$$

from which we can deduce the thesis provided our assumptions on $\|b\|_N$. \square

2.2. Viscous stationary Hamilton-Jacobi equations

We now consider the Hamilton-Jacobi equation

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(x) & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} u = 0 \end{cases} \quad (2.17)$$

We will assume throughout this section that $H : \mathbb{R}^N \rightarrow \mathbb{R}$ is strictly convex and $H \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C^1(\mathbb{R}^N)$.

Moreover we suppose it holds

$$\begin{aligned} C|p|^\gamma - C &\leq H(p) \leq C(|p|^\gamma + 1) \\ \nabla H(p) \cdot p - H(p) &\geq C|p|^\gamma - C \\ |\nabla H(p)| &\leq C|p|^{\gamma-1} \end{aligned} \quad (2.18)$$

for some $C > 0, \gamma > 1$. We also suppose that $\Omega \subset \mathbb{R}^N$ is convex and $C^{3,\alpha}$ for some $\alpha < 1$. These are standard assumptions that arise in the case of MFGs with local coupling (for instance in (1.32) we consider the quadratic case), and they will be assumed also in Chapter 3.

A solution of (2.17) is a pair $(u, \lambda) \in C^2(\bar{\Omega}) \times \mathbb{R}$ that satisfies the equation pointwise. To find a solution the idea is to rely on existence results for strong solutions for an auxiliary problem where λ is fixed and then perform a limit procedure. We use the following theorem.

Theorem 2.7. *Let $f \in W^{1,\infty}(\Omega)$ and $\lambda > 0$. Then, there exists an unique solution in*

$W^{3,p}(\Omega)$ for all $p < \infty$ to the problem

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda u = f(x) & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \quad (2.19)$$

Proof. See [39, Cor. II.1]. □

Remark 2.8. The previous theorem holds in general when Ω is just convex. In our case since we have a smooth domain, by Sobolev embeddings u is also a classical solution.

We also introduce the following crucial a priori bound. This bound makes use of the Bernstein method and we require u to be regular.

Theorem 2.9. *Let $f \in L^s(\Omega)$ for some $s > N$ and $u \in C^3(\bar{\Omega})$ solution of (2.17). Let $r > 1$ fixed. Suppose $\lambda \geq \lambda_0 \in \mathbb{R}$, $\|f\|_s \leq f_0$ for some $f_0 > 0$. Then*

$$\|\nabla u\|_r \leq C$$

with $C = C(r, H, f_0, \lambda_0) > 0$.

Proof. See [35, Th. A.1] and also [22]. □

This theorem holds in a more general setting where the regularity of u is not needed. If u is in $C^3(\bar{\Omega})$, clearly f is at least $C^1(\Omega)$. However, ultimately we are interested in finding uniform bounds in $L^s(\Omega)$ even if f already has additional regularity.

We can now use these two results to prove the existence of a solution to (2.17). This is a well known result (see [40]).

Theorem 2.10. *Let $f \in C^2(\bar{\Omega})$. There exists an unique constant $\lambda \in \mathbb{R}$ such that (2.17) has a unique solution in $C^2(\bar{\Omega})$ and*

$$\lambda = \sup \left\{ c \in \mathbb{R} \text{ s.t. } \exists u \in C^2(\bar{\Omega}), \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega : -\Delta u + H(\nabla u) + c \leq f \right\}. \quad (2.20)$$

Moreover there exists $K = K(\|f\|_\infty, \lambda)$ such that

$$\|\nabla u\|_\infty \leq K. \quad (2.21)$$

Proof. Let $\delta > 0$ and consider the problem

$$\begin{cases} \delta u_\delta - \Delta u_\delta + H(\nabla u_\delta) = f & \text{on } \Omega \\ \frac{\partial u_\delta}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 2.7 this system has an unique solution in $W^{3,p}(\Omega)$ for all $p < \infty$. Using Sobolev embeddings and elliptic regularity, by our assumptions on H and f , $u_\delta \in C^{3,\alpha}(\bar{\Omega})$ for some $\alpha < 1$. We now want to prove that $\|\delta u_\delta\|_\infty \leq \|f\|_\infty + |H(0)|$. If u_δ achieves its maximum and minimum in two interior points, x_M, x_m respectively, this is immediate, since by simple evaluation we have

$$\delta u_\delta(x_M) \leq f(x_M) - H(0) \tag{2.22}$$

$$\delta u_\delta(x_m) \geq f(x_m) - H(0). \tag{2.23}$$

If one of the extrema is on $\partial\Omega$ the inequality holds by Hopf's lemma (Theorem B.1). Indeed, let us suppose that (2.22) does not hold and there exists a maximum point $x_M \in \partial\Omega$. By convexity of H we have

$$H(\nabla u_\delta) - H(0) \geq \nabla H(0) \cdot \nabla u_\delta \tag{2.24}$$

Then in a neighborhood of x_M we have

$$-\Delta u_\delta + \nabla H(0) \cdot \nabla u_\delta \leq f - H(0) - \delta u_\delta < 0 \tag{2.25}$$

from which we can use Hopf's lemma to reach a contradiction with the homogeneous Neumann boundary conditions at x_M . If instead (2.23) does not hold, in a similar way

$$H(0) - H(\nabla u_\delta) \geq -\nabla H(\nabla u_\delta) \cdot \nabla u_\delta. \tag{2.26}$$

and we get in a neighborhood of x_m that

$$-\Delta u_\delta + \nabla H(\nabla u_\delta) \cdot \nabla u_\delta \geq f - H(0) - \delta u_\delta > 0. \tag{2.27}$$

Since u_δ is regular, $\nabla H(\nabla u_\delta)$ is bounded and, again by Hopf's lemma, we reach a contradiction.

We can now apply Theorem 2.9 to deduce that for all $r \in (0, \infty)$ there exists $C > 0$ such that $\|\nabla u_\delta\|_r \leq C$ with C depending only on constants, on $\|f\|_\infty$ and $\|\delta u_\delta\|_\infty$ which is itself controlled by $\|f\|_\infty + |H(0)|$. For r large enough using a bootstrap argument and

the estimates on H we can conclude that $-\Delta u_\delta \in L^q(\Omega)$ with $q > N$, hence using elliptic regularity and Sobolev embeddings

$$\|\nabla u_\delta\|_\infty \leq K \quad (2.28)$$

with K depending only on universal constants and $\|f\|_\infty$.

Define now $v_\delta(x) = u_\delta - u_\delta(0)$. We have

$$\|v_\delta\|_\infty = \|u_\delta\|_\infty \leq K$$

and also for every $x \in \Omega$

$$|u_\delta(x) - u_\delta(0)| \leq \|\nabla u_\delta\|_\infty |x| \leq C.$$

hence $\|v_\delta\|_\infty \leq C$. We notice that v_δ solves

$$\begin{cases} \delta v_\delta - \Delta v_\delta + H(\nabla v_\delta) = f - \delta u_\delta(0) & \text{on } \Omega \\ \frac{\partial v_\delta}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Using Elliptic regularity, since v_δ is uniformly bounded in $L^\infty(\Omega)$, we can deduce that v_δ is uniformly bounded in $C^{2,\alpha}$. We can now use Ascoli-Arzelà theorem to extract a subsequence which converges to v in C^2 and pass to the limit in the above equation. Noticing that $\delta u_\delta(0)$ converges up to subsequences to a constant that we call λ , we can conclude that v is a solution of (2.17). Moreover,

$$\|\nabla v\|_\infty \leq C. \quad (2.29)$$

To prove uniqueness, suppose by contradiction that there exists $(v_1, \lambda_1), (v_2, \lambda_2)$ solutions with $\lambda_1 > \lambda_2$ (we suppose for the moment that these solutions do not satisfy the constraint $\int_\Omega u = 0$). Up to the addition of constants, since both are bounded, we can assume $v_1 \leq v_2$. Then $\lambda_1 t + v_1$ is a solution to

$$\begin{cases} v_t - \Delta v + H(\nabla v) = f(x) & \text{on } \Omega, t > 0 \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, t > 0 \\ v(0, x) = v_1(x). \end{cases}$$

However, since $v_1 \leq v_2$, $v_2 + \lambda_2 t$ is a subsolution of the same problem. By convexity we

have

$$H(\nabla v_1) - H(\nabla v_2) \geq \nabla H(\nabla v_2) \cdot \nabla(v_1 - v_2), \quad (2.30)$$

and by (2.29) $\nabla H(\nabla v_2)$ is in $L^\infty(\Omega)$. Thus, setting $w = v_2(x) - v_1(x) + (\lambda_2 - \lambda_1)t$ and using Lemma B.3, we conclude that $v_1(x) + \lambda_1 t \leq v_2(x) + \lambda_2 t$ for $t > 0$. Letting $t \rightarrow \infty$, since the solutions are bounded, we conclude $\lambda_1 > \lambda_2$ which is a contradiction and proves λ is unique. Now suppose there exist two solutions u, v satisfying the equation with the same λ . Let $w = u - v$. We find that w satisfies

$$\begin{cases} -\Delta w + H(\nabla u) - H(\nabla v) = 0 & \text{on } \Omega \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \quad (2.31)$$

Again by convexity $H(\nabla u) - H(\nabla v) \geq \nabla H(\nabla v) \cdot \nabla w$. Hence we can use Hopf's lemma to deduce that w achieves a maximum in Ω . Then by the strong maximum principle (Theorem B.2) w is constant, thus it is 0. Lastly, let us prove (2.20). One inequality is trivial. For the other, if we suppose that there exists $c > \lambda$, we can consider the associate subsolution u_c and let $w = u_c - u$. Then, an argument perfectly analogue to the one for uniqueness allows to reach a contradiction. \square

We are now interested in finding extra regularity results. This results will be used in the Sobolev critical case that will arise in Chapter 3. In particular we are interested in finding bounds for ∇u when we have uniform boundness of f in $L^{\frac{N}{\gamma}}$. By simple Sobolev embedding arguments, we expect to find bounds for $\|\nabla u\|_{N(\gamma-1)}$. We start by proving the following lemma for linear equations.

Lemma 2.11. *Let $f \in L^p(\Omega)$ for $p > 1$ and let $(u, \lambda) \in W^{2,p}(\Omega) \times \mathbb{R}$ be a solution of*

$$\begin{cases} -\Delta u + \lambda = f & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} u = 0. \end{cases} \quad (2.32)$$

Then there exists $C > 0$ independent of λ such that

$$\|u\|_{2,p} \leq C \|f\|_p. \quad (2.33)$$

Proof. Using elliptic regularity (see for instance [51, Th. 3.17], a more general result than

the one found in Appendix A), we have that

$$\|u\|_{2,p} \leq C(\|f - \lambda\|_p + \|u\|_{1,2}). \quad (2.34)$$

Moreover by Lax-Milgram we have $\|u\|_{1,2} \leq C \|f - \lambda\|_2$. Hence, we can deduce that

$$\|u\|_{2,p} \leq C \|f\|_p + C' |\lambda|. \quad (2.35)$$

To conclude, notice that the role of λ is to enforce compatibility conditions on f . If we test (2.32) with a constant function, we get that necessarily

$$\lambda = \frac{1}{|\Omega|} \int_{\Omega} f \, dx \quad (2.36)$$

from which we can deduce the thesis. \square

Using this estimate the idea is to construct a "barrier" for ∇u which we employ in a topological fixed point argument. This means finding a value M such that no solution exists with $\|\nabla u\|_{N(\gamma-1)} = M$.

Proposition 2.12. *Let (u, λ) be the unique solution of (2.17) under the hypothesis of Theorem 2.10. Then there exists $\delta_0, M > 0$ such that if $\|f\|_{\frac{N}{\gamma}} < \delta_0$ then*

$$\|\nabla u\|_{N(\gamma-1)} < M \vee \|\nabla u\|_{N(\gamma-1)} > M. \quad (2.37)$$

Moreover, δ_0 can be chosen small so that M is arbitrarily small.

Proof. Let $p = \frac{N}{\gamma}$. Then $p^* = p\gamma$. Hence by Sobolev embedding, Lemma 2.11 and assumptions on H

$$y := \|\nabla u\|_{p\gamma} \leq \|u\|_{1,p\gamma} \leq C \|u\|_{2,p} \leq C(\|\nabla u\|_{p\gamma}^{\gamma} + \|f\|_p) \leq Cy^{\gamma} + C'\delta_0 \quad (2.38)$$

We now choose δ_0 such that there exists two positive solutions y_1, y_2 to $y = Cy^{\gamma} + C'\delta_0$. In this way we can choose M such that $y_1 < M < y_2$. Moreover, y_1 can be made arbitrarily small if δ_0 is small, hence also $M > 0$ can be chosen arbitrarily small. \square

Now that we have the barrier, we can use Leray-Schauder fixed point theorem to deduce the following theorem. We remark that our goal is to find a uniform bound for ∇u controlled by $\|f\|_{\frac{N}{\gamma}}$, but that in general the hypothesis of Theorem 2.10 hold and we can

suppose u and f in $C^2(\overline{\Omega})$.

Theorem 2.13. *Let (u, λ) be a solution of (2.17). Then there exists $\delta_0, M > 0$ such that if $\|f\|_{\frac{N}{\gamma}} < \delta_0$ then*

$$\|\nabla u\|_{N(\gamma-1)} < M. \quad (2.39)$$

Moreover, δ_0 can be chosen small so that M is arbitrarily small.

Proof. We use Leray-Schauder fixed point theorem (see for instance [1, Th. 5.1]). Let $U = \{w \in C^2(\overline{\Omega}) : \|\nabla w\|_{N(\gamma-1)} < M\}$ and consider the operator $T : \overline{U} \rightarrow C^2(\overline{\Omega})$ where $u = T(w)$ is defined by the solution of the system

$$\begin{cases} -\Delta u + H(\nabla w) + \lambda = f(x) & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} u = 0. \end{cases} \quad (2.40)$$

We claim that T is continuous and compact. Leray-Schauder theorem asserts that either T has a fixed point or there exists $s \in (0, 1)$ and $u \in \partial\Omega$ such that $u = sT(u)$. However, the latter is not possible by exactly the same reasoning of Proposition 2.12. Hence T has a fixed point in U , which by uniqueness is the solution of (2.17), from which we can deduce the desired estimate. Let us now see that T is compact. Clearly $T_1 : C^2(\overline{\Omega}) \rightarrow C^1(\overline{\Omega})$ defined by $u = H(\nabla u)$ is continuous by our assumptions on H . Moreover let $T_2 : C^{0,\alpha}(\overline{\Omega}) \rightarrow C^{2,\alpha}(\overline{\Omega})$ be the operator defined by $u = T_2 w$ the solution of

$$\begin{cases} -\Delta u + w + \lambda = f(x) & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} u = 0. \end{cases} \quad (2.41)$$

□

for any $\alpha \in (0, 1)$. This is well defined and continuous by elliptic regularity and because we supposed $f \in C^2$. Finally, by Ascoli-Arzelà the immersion $T_3 : C^{2,\alpha}(\overline{\Omega}) \rightarrow C^2(\overline{\Omega})$ is continuous and compact. Hence we conclude that $T = T_3 \circ T_2 \circ T_1$ is continuous and compact.

3 | Variational Methods For Viscous Ergodic Mean Field Games

In this chapter we present an original existence result for Viscous Ergodic Mean Field Games with local couplings using variational methods. We deal with the system derived in Section 1.4

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(x, m(x)) & \text{on } \Omega \\ -\Delta m - \operatorname{div}(m \nabla H(\nabla u)) = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \frac{\partial m}{\partial n} + m \nabla H(\nabla u) \cdot n = 0 & \text{on } \partial\Omega \\ \int_{\Omega} m = 1, \quad \int_{\Omega} u = 0. \end{cases} \quad (3.1)$$

We consider the case of f with local coupling, unbounded and with polynomial growth exponent q . The strategy for the resolution of the system is to introduce an energy function \mathcal{E} , which we minimize on a specific constraint \mathcal{K} . This minimum point will be related by convex duality arguments to the solution of (3.1). A crucial role will be played by the behavior of f . Since f is a local coupling, it is necessary to perform an approximation procedure using a sequence f_{ε} of smoothing couplings. Moreover, the variational structure of the energy \mathcal{E} depends on the growth of f . In particular two limit cases appear. If the growth is under the *mass critical exponent* \bar{q} , the energy \mathcal{E} admits a global minimum. If $q > \bar{q}$ the energy in general could be unbounded, however if q is smaller than the *Sobolev critical exponent* q^* a local minimum may be present, provided some smallness assumption on the coefficient C_f . In both cases, the estimates for the minimization of \mathcal{E} are based on the *Gagliardo-Nirenberg inequality* (see Appendix A).

Existence results for $q < \bar{q}$ were proved in [18] for the case of $\Omega = \mathbb{T}^N$. This chapter presents those arguments and extends them to the cases $\bar{q} \leq q \leq q^*$ and to the case of Neumann Boundary conditions. We need to remark that solutions for $q < q^*$ were already found using fixed point methods in [23], however using variational methods we

obtain more insight on the nature of such solutions. The case $q = q^*$ instead, was not treated before as far as we know. The motivation for the search of local minima also resides in [45], where similar arguments are carried out for the case of the Schrödinger equation. The Schrödinger equation with a normalization constraint and system (3.1) share a similar variational structure, and we will see in Section 3.3 that in the quadratic case they are connected by the so-called *Hopf-Cole transformation*. This motivates the idea to adopt these methods also for the non quadratic case, where a transformation is available only in specific cases (see [21]).

3.1. Setting and Assumptions

Let us clarify what it means to find a solution to this system.

Definition 3.1. A solution to the system (3.1) is a triple $(u, \lambda, m) \in C^{2,\theta}(\overline{\Omega}) \times \mathbb{R} \times W^{1,p}(\Omega)$ for all $\theta \in (0, 1)$ and all $p > 1$ such that (u, λ) is a classical solution to the Hamilton-Jacobi equation and m is a weak solution to the Fokker-Planck equation.

Remark 3.2. In general solutions for the Fokker-Planck equation are in $W^{1,2}(\Omega)$ and solutions for the Hamilton-Jacobi equation are in $C^2(\overline{\Omega})$. However, due to the coupling of the problem, once we have solutions in those spaces we can extract additional regularity with elliptic methods (See Appendix A).

Let us now collect the assumptions taken on the data. We consider a domain $\Omega \subset \mathbb{R}^N$, bounded and convex, with a boundary of class $C^{3,\alpha}$ for all $\alpha \in (0, 1)$. Boundedness and smoothness of the domain are standard assumptions necessary to enable the use of embedding and regularity theory (See Appendix A), while convexity is crucial for the preliminary estimates for Hamilton Jacobi equations (Theorem 2.9). To make calculations easier, we also take $|\Omega| = 1$, but this hypothesis is not restrictive and all the following arguments hold provided some rescaling of the coefficients.

Since we are dealing with local couplings, we take the following assumptions. We consider $H : \mathbb{R}^N \rightarrow \mathbb{R}$, strictly convex and $H \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C^1(\mathbb{R}^N)$.

Moreover we suppose it holds

$$\begin{aligned} C_H |p|^\gamma - C_H^{-1} &\leq H(p) \leq C_H(|p|^\gamma + 1) \\ \nabla H(p) \cdot p - H(p) &\geq C_H |p|^\gamma - K_H \\ |\nabla H(p)| &\leq C_H |p|^{\gamma-1} \end{aligned} \tag{3.2}$$

for some $C_H > 0, K_H > 0, \gamma > 1$. We can see that this assumptions are coherent with standard assumptions for the case of local coupling (1.32), however we consider a general exponent of growth γ . A classic example of a function satisfying these assumptions is

$H(p) = \frac{1}{\gamma} |p|^\gamma$. For the right-hand side we suppose $f : \mathbb{R}^N \times [0, +\infty) \rightarrow \mathbb{R}$ is locally Lipschitz and it holds

$$-C_f m^{q-1} - K_f \leq f(x, m) \leq C_f m^{q-1} + K_f \quad (3.3)$$

For some $C_f > 0, K_f > 0, q > 1$. The assumptions of f are coherent with the one found in Section 1.2, but we remark that f is in general unbounded from below, unlike (1.31). An example to keep in mind is $f(x) = -C_f a(x) m^{q-1}$ with $0 \leq a(x) \leq 1$. The most general interval we consider for q is

$$1 < q \leq 1 + \frac{\gamma'}{N - \gamma'} \quad \text{for } \gamma' < N \quad (3.4)$$

or any $q > 1$ if $\gamma' \geq N$. As we said, the variational structure of the problem depends strongly on the exponent q . We will have an existence result for any f in the case

$$1 < q < \bar{q} = 1 + \frac{\gamma'}{N} \quad (3.5)$$

where $q = \bar{q}$ is the *mass critical* exponent. If $q \geq \bar{q}$, existence will depend on smallness assumptions on the coefficient C_f . Moreover, we remark that special care will be needed for the *Sobolev critical* exponent

$$q = q^* = 1 + \frac{\gamma'}{N - \gamma'}$$

(of course, only for $\gamma' < N$), for which additional restrictions on C_f will be needed.

3.1.1. Preliminary results on the Legendre-Fenchel transform

Before stating the existence theorem, we collect some results which will be necessary for the construction of the energy associated to the system. We introduce the Legendre-Fenchel Transform, which plays a key role in the construction of the energy and enables the convex duality arguments.

Definition 3.3. The *Legendre-Fenchel* Transform of a convex function f is the function

$$f^*(q) := \sup_{p \in \mathbb{R}^N} [p \cdot q - f(p)]$$

We refer to [8] or [49] for the properties of this transform. We just remark that given $f_1 \leq f_2$ by definition $f_1^* \geq f_2^*$. Moreover, using standard properties of convex functions,

whenever f is lower-semicontinuous and proper we have $f^{**} = f$.

Let us now denote by $L_H = H^*$. We have the following results for L_H under our assumptions on H .

Proposition 3.4. *There exists $C_L > 0$ such that for all $p, b \in \mathbb{R}^N$*

1. $L_H \in C^2(\mathbb{R}^N \setminus \{0\})$ and it is strictly convex.
2. $0 \leq C_L |q|^{\gamma'} \leq L_H(q) \leq C_L^{-1}(|q|^{\gamma'} + 1)$
3. $\nabla L_H(q) \cdot q - L_H(q) \geq C_L |q|^{\gamma'} - C_L^{-1}$
4. $C_L |q|^{\gamma'-1} - C_L^{-1} \leq |\nabla L_H(q)| \leq C_L^{-1}(|q|^{\gamma'-1} + 1)$

Proof. See [20]. □

3.2. Existence of a Solution

We are now ready to state our main result.

Theorem 3.5. *Suppose (3.2) and (3.3) hold. Suppose that either*

- $1 < q < \bar{q}$,
- $\bar{q} \leq q^*$ and $C_f < C_q$,
- $q = q^*$ and $C_f < C_{crit}$.

Then there exists a solution (u, λ, m) to the system (3.1), and $m > 0$.

The constants C_q and C_{crit} are explicitly calculated throughout the chapter, here we just remark that they depend on all the fixed constants of the problems $(q, C_H, C_L, K_f, \gamma)$ and also on some regularity and embedding constants $(C_S, C_E, C_p, \delta_0)$. The proof of this theorem consists of several steps, which are explained in detail in the following sections.

3.2.1. Energy and Constraint

We begin by introducing a constrained optimization problem. Let

$$\frac{1}{r} := \frac{1}{\gamma'} + \frac{1}{\gamma q}. \quad (3.6)$$

Observe that by our assumptions on γ and q

$$1 < r \leq \frac{N}{N - \gamma' + 1} \quad (3.7)$$

and consider the constraint

$$\begin{aligned} \mathcal{K} := \{ & (m, w) \in L^q(\Omega) \cap W^{1,r}(\Omega) \times L^1(\Omega) \text{ s.t.} \\ & \int_{\Omega} \nabla m \cdot \nabla \phi \, dx = \int_{\Omega} w \cdot \nabla \phi \, dx \quad \forall \phi \in C^\infty(\bar{\Omega}), \\ & \int_{\Omega} m \, dx = 1, m \geq 0 \text{ a.e.} \}. \end{aligned} \quad (3.8)$$

The functions involved are m and an auxiliary function w . The constraint enforces m to be a probability density and connects the two functions to satisfy $-\Delta m = \operatorname{div} w$ in a weak sense with Neumann boundary conditions. We expect the function w to be in some sense related with the drift term $m \nabla H(\nabla u)$ and to encapsulate in \mathcal{K} the information of the Fokker-Planck equation. We can now associate to the system the functional

$$\mathcal{E}(m, w) := \begin{cases} \int_{\Omega} mL\left(-\frac{w}{m}\right) + F(x, m) \, dx & \text{if } (m, w) \in \mathcal{K} \\ +\infty & \text{otherwise} \end{cases} \quad (3.9)$$

where

$$mL\left(-\frac{w}{m}\right) := \begin{cases} mL_H\left(-\frac{w}{m}\right) & \text{if } m > 0 \\ 0 & \text{if } m = 0, w = 0, \\ +\infty & \text{otherwise} \end{cases} \quad (3.10)$$

and

$$F(x, m) := \begin{cases} \int_0^m f(x, n) \, dn & \text{if } m \geq 0 \\ +\infty & \text{if } m < 0. \end{cases} \quad (3.11)$$

We are interested in minimizing this energy. Minima of this function will be related to solutions of (3.1) by convex duality arguments. We remark that the definition of \mathcal{E} is adjusted so that only couples $(m, w) \in \mathcal{K}$ are considered and also the terms (3.10) and (3.11) are adjusted so that only positive values for m are considered. We also remark that by the properties of the Legendre transform, L_H coincides with the function L of Section 1.4.

Let us now use the properties of the Legendre-Fenchel transform and our assumptions on

f and H to deduce some a priori bounds.

Proposition 3.6. *The function*

$$(m, w) \longrightarrow mL\left(-\frac{w}{m}\right)$$

is convex, and strictly convex if restricted to $m > 0$. We have,

$$mH(p) = \begin{cases} \sup_{w \in \mathbb{R}^N} \left[-p \cdot w - mL\left(-\frac{w}{m}\right) \right] & \text{if } m \neq 0 \\ 0 & \text{if } m = 0. \end{cases} \quad (3.12)$$

Moreover

$$C_L \frac{|w|^{\gamma'}}{m^{\gamma'-1}} \leq mL\left(-\frac{w}{m}\right) \leq C_L^{-1} \frac{|w|^{\gamma'}}{m^{\gamma'-1}} + C_L^{-1} m. \quad (3.13)$$

Lastly

$$-\frac{C_f}{q} m^q - K_f m \leq F(x, m) \leq \frac{C_f}{q} m^q + K_f m. \quad (3.14)$$

Proof. Convexity and Formula 3.12 come directly from the definitions. In particular convexity follows since $mL\left(-\frac{w}{m}\right)$ is the sup of a family of affine functions. For strict convexity, we know by Proposition 3.4 that L_H is strictly convex. If restricted to $m > 0$, we can also conclude that $mL\left(-\frac{w}{m}\right)$ is strictly convex. Estimate (3.13) comes directly from Proposition 3.4 and finally estimate (3.14) comes directly from (3.3). \square

To conclude this section, let us investigate the regularity satisfied by couples $(m, w) \in \mathcal{K}$. We can use the Gagliardo-Nirenberg inequality (see Appendix A) to deduce the following crucial a priori bound.

Proposition 3.7. *Let $(m, w) \in \mathcal{K}$ and E defined by*

$$E := \int_{\Omega} \frac{|w|^{\gamma'}}{m^{\gamma'-1}} dx.$$

Then there exists $C_p > 0$ such that:

$$\|w\|_{1,r} \leq C_p(E + 1) \quad (3.15)$$

$$\|m\|_q \leq C_p(E + 1). \quad (3.16)$$

Moreover, if (3.5) holds, we have that there also exists $\delta > 0$ such that

$$\|m\|_q^{q(1+\delta)} \leq C_p(E+1). \quad (3.17)$$

Proof. We can suppose without loss of generality that E is finite. By (3.8), let us take $\phi \in C^\infty(\bar{\Omega})$ with $\frac{\partial \phi}{\partial n} = 0$. We have:

$$\begin{aligned} \int_{\Omega} m(-\Delta u) dx &= \int_{\Omega} \nabla m \cdot \nabla \phi dx = \int_{\Omega} w \cdot \nabla \phi dx \\ &\leq \int_{\Omega} \left(\frac{|w|^{\gamma'}}{m^{\gamma'-1}} \right)^{\frac{1}{\gamma'}} m^{\frac{1}{\gamma}} |\nabla \phi| dx \leq \left(\int_{\Omega} \frac{|w|^{\gamma'}}{m^{\gamma'-1}} dx \right)^{\frac{1}{\gamma'}} \|m\|_{\frac{q}{\gamma}}^{\frac{1}{\gamma}} \|\nabla \phi\|_{r'} \\ &\leq E^{\frac{1}{\gamma'}} \|m\|_{\frac{q}{\gamma}}^{\frac{1}{\gamma}} \|\nabla \phi\|_{r'} \end{aligned}$$

where r is defined in (3.6). Using Proposition 2.3, we get that

$$\|m\|_{1,r} \leq C \left(E^{\frac{1}{\gamma'}} \|m\|_{\frac{q}{\gamma}}^{\frac{1}{\gamma}} + \|m\|_r \right)$$

Moreover, by using Interpolation (Theorem A.4), we get

$$\|m\|_r \leq \|m\|_{\frac{q}{\gamma}}^{\frac{1}{\gamma}} \|m\|_1^{\frac{\gamma-1}{\gamma}} = \|m\|_{\frac{q}{\gamma}}^{\frac{1}{\gamma}}$$

and so

$$\|m\|_{1,r} \leq C \left(E^{\frac{1}{\gamma'}} + 1 \right) \|m\|_{\frac{q}{\gamma}}^{\frac{1}{\gamma}}. \quad (3.18)$$

Now let r^* be such that $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{N}$ if $r < N$ and $r^* = +\infty$ if $r \geq N$. Since we have by direct computation that $q \leq r^*$, using Sobolev embeddings (Theorem A.1) we find that there exists $C > 0$ such that

$$\|m\|_q \leq C \|m\|_{1,r}$$

which combined with (3.18) yields

$$\|m\|_q \leq C \left(E^{\frac{1}{\gamma'}} + 1 \right) \|m\|_{\frac{q}{\gamma}}^{\frac{1}{\gamma}}$$

which gives (3.16) after some basic estimates. Using (3.16) in (3.18) again then yields (3.15).

To obtain the last result let us use Gagliardo-Nirenberg inequality (Theorem A.5). That yields:

$$\|m\|_q \leq C \|\nabla m\|_r^\alpha \|m\|_1^{1-\alpha} + C \|m\|_1 = C \|\nabla m\|_r^\alpha + C$$

where α is given by

$$\alpha = \frac{1 - \frac{1}{q}}{1 - \frac{1}{r} + \frac{1}{N}}.$$

We need to check that $0 \leq \alpha < 1$. The first inequality is trivially satisfied, while the second, after some calculations, yields

$$q < 1 + \frac{\gamma'}{N - \gamma'} \quad \text{for } \gamma' < N$$

which is satisfied by the stronger hypothesis (3.5). Hence using (3.18) we get

$$\|m\|_q \leq C \|\nabla m\|_r^\alpha + C \leq C \left(1 + E^{\frac{\alpha}{\gamma'}}\right) \|m\|_q^{\frac{\alpha}{\gamma'}} + C$$

so

$$\|m\|_q^{\frac{\gamma'}{\alpha}} \leq C(1 + E) \|m\|_q^{\frac{\gamma'}{\alpha}} + C^{\frac{\gamma'}{\alpha}}$$

which gives

$$\|m\|_q^{\frac{\gamma'}{\alpha} - \frac{\gamma'}{\alpha}} \leq C(E + 1) + \frac{C}{\|m\|_q^{\frac{\gamma'}{\alpha}}}.$$

Since we have by Hölder inequality that $\|m\|_q \geq C \|m\|_1 = C$, this gives (3.17) with

$$\delta = \frac{1}{q - 1} \left(\frac{\gamma' + N}{N} - q \right)$$

which is positive thanks to (3.5). □

3.2.2. Regularization

In this section we introduce a regularization for our problem. Since we are dealing with local couplings, we cannot directly perform the duality procedure for the system (3.1). The strategy is to introduce a regularized system with smooth global couplings, associate an energy \mathcal{E}_ε to that system and minimize that energy. Once we have a minimum point for the regularized energy, we perform the convex duality argument to deduce the existence of a regular solution. The solution to the initial problem is then found by an appropriate limit procedure for the regularized solutions.

Let us now consider for $\varepsilon > 0$ the following regularized system

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f_\varepsilon[m](x) & \text{on } \Omega \\ -\Delta m - \operatorname{div}(m \nabla H(\nabla u)) = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \frac{\partial m}{\partial n} + m \nabla H(\nabla u) \cdot n = 0 & \text{on } \partial\Omega \\ \int_\Omega m = 1, \quad \int_\Omega u = 0, \end{cases} \quad (3.19)$$

where

$$\begin{aligned} f_\varepsilon[m](x) &:= f(\cdot, m * \chi_\varepsilon(\cdot)) * \chi_\varepsilon(x) \\ &= \int_{\mathbb{R}^N} \chi_\varepsilon(x-y) f\left(y, \int_{\mathbb{R}^N} m(z) \chi_\varepsilon(y-z) dz\right) dy \end{aligned} \quad (3.20)$$

and χ_ε is a sequence of standard symmetric mollifiers approximating the unit (f and m are extended as 0 outside of Ω). We can notice that given

$$F_\varepsilon[m] := \int_\Omega F(x, m * \chi_\varepsilon(x)) dx \quad (3.21)$$

we have that it holds

$$F_\varepsilon[m'] - F_\varepsilon[m] = \int_0^1 \int_\Omega f_\varepsilon[(1-t)m + tm'](x)(m' - m)(x) dx \quad (3.22)$$

for $m, m' \in L^1(\Omega)$ and $\int_\Omega m = \int_\Omega m' = 1$. This means that the problem admits a potential, and we talk about *potential* Mean Field Games (see [11]). Equation (3.22) describes the notion of derivative in the space of measures, which is often used in MFGs (see [12]). We do not enter into details here, as in our case m is an L^1 density and it can be proven that this notion of derivative coincides with the notion of Fréchet derivative. (see e.g. [3]).

Let us observe that using the properties of mollifiers and (3.14) we have

$$-\frac{C_f}{q} \|m\|_q^q - K_f \leq F_\varepsilon[m] \leq \frac{C_f}{q} \|m\|_q^q + K_f \quad (3.23)$$

and also

$$-\frac{C_f}{q} \sup_\Omega \chi_\varepsilon^q - K_f \leq F(x, m * \chi_\varepsilon(x)) \leq \frac{C_f}{q} \sup_\Omega \chi_\varepsilon^q + K_f \quad (3.24)$$

We can now introduce the following energy for the approximate system:

$$\mathcal{E}_\varepsilon(m, w) := \begin{cases} \int_{\Omega} mL\left(-\frac{w}{m}\right) dx + F_\varepsilon[m] & \text{if } (m, w) \in \mathcal{K} \\ +\infty & \text{otherwise.} \end{cases} \quad (3.25)$$

3.2.3. Minimization of the functional

Our goal now is to find a minimizer for the energy of the regularized system. Here, the exponent \bar{q} comes into play. If $q < \bar{q}$, the energy can be proven to admit a global minimum. This is the case addressed in [18]. If $q \geq \bar{q}$, this is false in general and we may look for local minima. Following the idea introduced in [45] let us introduce for $\alpha \geq 1$

$$B_\alpha := \{(m, w) \in \mathcal{K} : \|m\|_q^q \leq \alpha\} \quad (3.26)$$

and

$$U_\alpha := \{(m, w) \in \mathcal{K} : \|m\|_q^q = \alpha\}. \quad (3.27)$$

Let us also define, for $\varepsilon > 0$ fixed,

$$c_\alpha = \inf_{(m, w) \in \mathcal{K} \cap B_\alpha} \mathcal{E}_\varepsilon(m, w) \quad (3.28)$$

and

$$\hat{c}_\alpha = \inf_{(m, w) \in \mathcal{K} \cap U_\alpha} \mathcal{E}_\varepsilon(m, w) \quad (3.29)$$

We start by proving the existence of a minimum in the sets B_α , and also adapt the arguments to prove the existence of a global minimum under the stricter assumptions.

Lemma 3.8. *For all $\alpha \geq 1$, c_α is achieved. Moreover, if (3.5) holds, then \mathcal{E}_ε has a global minimum on \mathcal{K} .*

Proof. Let us begin by bounding \mathcal{E}_ε by below. Using (3.13), (3.16) and (3.23) we have:

$$\mathcal{E}_\varepsilon(m, w) \geq \frac{C_L}{C_p} \|m\|_q - C_L - \frac{C_f}{q} \|m\|_q^q - K_f \geq K' \quad (3.30)$$

since we know using Hölder inequality that $\|m\|_q \geq C$ and we know $\|m\|_q^q \leq \alpha$ in B_α . Consider now a minimizing sequence (m_n, w_n) for c_α . Eventually, $\mathcal{E}_\varepsilon(m_n, w_n) \leq c_\alpha + 1$.

Hence, again by (3.13) and (3.23) we have:

$$\int_{\Omega} \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}} \leq C_L^{-1} (c_{\alpha} + 1 - F_{\varepsilon}[m_n]) \leq C_L^{-1} (c_{\alpha} + 1 + K_f + \frac{C_f}{q} \alpha) \quad (3.31)$$

which implies that $\left(\int_{\Omega} \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}} \right)$ is bounded. Using (3.15), we have that $\|m_n\|_{1,r} \leq C$. We can use Theorem A.1 to conclude that up to subsequences

$$m_n \rightarrow m \text{ on } L^1(\Omega), \quad m_n \rightharpoonup m \text{ on } W^{1,r}(\Omega).$$

Then, using Hölder inequality

$$\int_{\Omega} |w_n|^{\frac{\gamma'q}{\gamma'+q-1}} dx \leq \left(\int_{\Omega} \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}} \right)^{\frac{q}{\gamma'+q-1}} \|m_n\|_q^{\frac{\gamma'-1}{q(\gamma'+q-1)}},$$

hence w_n is equibounded in $L^{\frac{\gamma'q}{\gamma'+q-1}}(\Omega)$ and so $w_n \rightharpoonup w$ in $L^{\frac{\gamma'q}{\gamma'+q-1}}(\Omega)$. By $L^1(\Omega)$ convergence of m_n we can conclude in a standard way that $m \geq 0$ and that $\int_{\Omega} m = 1$. Moreover, the convergences are strong enough to pass to the limit in the weak constraint in \mathcal{K} , hence we conclude that $(m, w) \in \mathcal{K}$.

To conclude, we know that $\int_{\Omega} m L\left(-\frac{w}{m}\right) dx$ is lower-semicontinuous with respect to the weak convergence of $W^{1,r}(\Omega) \times L^{\frac{\gamma'q}{\gamma'+q-1}}(\Omega)$. Indeed, we can exploit its convexity and adapt classical results linking convexity and semicontinuity (see for instance [26, Th. 2.2.1]). Moreover, using (3.24) and Dominated Convergence we can deduce that F_{ε} is strong continuous with respect to $L^1(\Omega)$ convergence, up to a subsequence. Hence:

$$\mathcal{E}_{\varepsilon}(m, w) \leq \liminf_n \int_{\Omega} m_n L\left(-\frac{w_n}{m_n}\right) dx + \lim_n F_{\varepsilon}[m_n] \leq \liminf_n \mathcal{E}_{\varepsilon}(m_n, w_n) = e$$

Now suppose that (3.5) holds. Then (3.17) holds. The proof is perfectly analogue, but we do not need to restrict ourselves on the set B_{α} . Indeed, instead of (3.30), we can directly infer using (3.17) that

$$\mathcal{E}_{\varepsilon}(m, w) \geq \frac{C_L}{C_p} \|m\|_q^{q(1+\delta)} - C_L - \frac{C_f}{q} \|m\|_q^q - K_f \geq K'. \quad (3.32)$$

Moreover, in (3.31) we can again use (3.17) to conclude that

$$\int_{\Omega} \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}} \leq C_L^{-1} \left(e + 1 + K_f + \frac{C_f}{q} \left(\int_{\Omega} \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}} \right)^{\frac{1}{1+\delta}} \right)$$

which again implies that $\int_{\Omega} \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}}$ is bounded. \square

We can see in the above lemma the role played by \bar{q} . When we have $q > \bar{q}$, looking at (3.30) we may suppose that $\mathcal{E}_{\varepsilon}$ is not bounded from below and that no global minima exists. The same quantity however is bounded from below in any B_{α} , hence the idea to look for a local minimum. The above theorem assures the existence of a minima on any B_{α} , which could be realized on $U_{\alpha} \subset B_{\alpha}$. To find a local minimum in the interior we thus need to find some $\bar{\alpha} > 1$ for which $c_{\bar{\alpha}} < \hat{c}_{\bar{\alpha}}$.

Let us first notice that if q does not satisfy (3.5), \mathcal{E} (and also $\mathcal{E}_{\varepsilon}$) could indeed be unbounded.

Proposition 3.9. *Let $f(x, m(x)) = -C_f m(x)^{q-1} - K_f$ and $q > 1 + \frac{\gamma'}{N}$. Then, there exists (m_n, w_n) such that $\mathcal{E}(m_n, w_n) \rightarrow -\infty$.*

Proof. Choose $m_0 \in C_0^{\infty}(B_1(0))$ non-negative, such that $\int_{B_1(0)} m_0 dx = 1$ and $\int_{B_1(0)} \frac{|\nabla m_0|^{\gamma'}}{m_0^{\gamma'-1}} dx$ is finite. Now choose $x_0 \in \Omega$ and define

$$m_{\lambda}(x) = \lambda^N m_0(\lambda(x - x_0)) \quad w_{\lambda}(x) = \nabla m_{\lambda}(x) = \lambda^{N+1} \nabla m_0(\lambda(x - x_0)).$$

We notice that for $\lambda > \frac{1}{\text{dist}(x_0, \partial\Omega)}$ we have that $\text{supp}(m_{\lambda}) \subset B_{\frac{1}{\lambda}}(x_0)$ and $(m_{\lambda}, w_{\lambda}) \in \mathcal{K}$. Moreover $\|m_{\lambda}\|_q^q = \lambda^{N(q-1)} \|m_0\|_q^q$. Now we have, using our assumption and (3.13),

$$\begin{aligned} \mathcal{E}(m_{\lambda}, w_{\lambda}) &= \int_{\Omega} m_{\lambda} L\left(-\frac{w_{\lambda}}{m_{\lambda}}\right) + F(x, m_{\lambda}) dx \leq \\ C_L^{-1} \int_{\Omega} \frac{|w_{\lambda}|^{\gamma'}}{m_{\lambda}^{\gamma'-1}} dx - \frac{C_f}{q} \|m_{\lambda}\|_q^q - C &= \lambda^{\gamma'} C_L^{-1} \int_{B_1(0)} \frac{|\nabla m_0|^{\gamma'}}{m_0^{\gamma'-1}} dt - \lambda^{N(q-1)} \frac{C_f}{q} \|m_0\|_q^q - C \end{aligned}$$

and we can see that under our assumptions on q for $\lambda \rightarrow +\infty$ this quantity tends to $-\infty$. \square

Remark 3.10. By the proof above, we can see that \mathcal{E} may be unbounded also in the case $q = 1 + \frac{\gamma'}{N}$, provided C_f is large enough.

Let us now turn to the existence of a local minimum. To do so, we need to assume some restriction on the coefficient C_f , so that an α such that $c_{\alpha} < \hat{c}_{\alpha}$ may be found.

Theorem 3.11. *Let*

$$\bar{\alpha} = \left(\frac{C_L}{C_f C_p} \right)^{q'} \tag{3.33}$$

and suppose

$$C_f < \min \left\{ \frac{C_L}{C_p}, (K'q')^{\frac{1}{1-q}} \left(\frac{C_L}{C_p} \right)^q \right\} \quad (3.34)$$

where

$$K' := C_L + C_L^{-1} + 2K_f + \frac{C_L}{qC_p}. \quad (3.35)$$

Then, there exists $(m_\varepsilon, w_\varepsilon) \in \mathcal{K} \cap B_{\bar{\alpha}}$ such that $(m_\varepsilon, w_\varepsilon)$ is a local minimum for \mathcal{E}_ε . Moreover it holds

$$\mathcal{E}_\varepsilon(m_\varepsilon, w_\varepsilon) = c_{\bar{\alpha}-\delta} = c_{\bar{\alpha}} \quad (3.36)$$

for all δ less than some $\bar{\delta}$ small enough.

Proof. Let us start the proof by the simple consideration that if we find $\alpha_2 > \alpha_1$ such that $\hat{c}_{\alpha_2} > \hat{c}_{\alpha_1}$, then we have:

$$c_{\alpha_2} = \min_{\alpha} \{ \hat{c}_{\alpha} : 1 \leq \alpha \leq \alpha_2 \} \leq \hat{c}_{\alpha_1} < \hat{c}_{\alpha_2}$$

and so we can conclude the existence of an interior minimum in B_{α_2} . We choose $\alpha_1 = 1$ and $\alpha_2 = \bar{\alpha}$. By our assumptions on C_f , we are sure that $\alpha_2 > \alpha_1$.

We now consider estimates for \hat{c}_{α_1} . We begin by noticing that by Hölder inequality we have

$$1 = \|m\|_1 \leq \|m\|_q |\Omega|^{\frac{1}{q'}} = 1$$

which implies $m = k$ a.e. and hence $m = 1$ a.e. . We can now notice that $(m, w) \equiv (1, 0) \in \mathcal{K} \cap U_{\alpha_1}$. Thus, using (3.13) and (3.23) we find

$$\hat{c}_{\alpha_1} \leq \mathcal{E}_\varepsilon(1, 0) \leq C_L^{-1} + \frac{C_f}{q} + K_f < C_L^{-1} + \frac{C_L}{qC_p} + K_f.$$

thanks to our assumptions on C_f . Next, we can reinterpret (3.30) for $\alpha = \bar{\alpha}$ as

$$\hat{c}_{\bar{\alpha}} \geq \frac{C_L}{C_p} \bar{\alpha}^{\frac{1}{q}} - C_L - \frac{C_f}{q} \bar{\alpha} - K_f.$$

Now concluding means checking that under our assumptions on C_f

$$\phi(\alpha) := \frac{C_L}{C_p} \alpha^{\frac{1}{q}} - \frac{C_f}{q} \alpha \geq C_L + C_L^{-1} + 2K_f + \frac{C_L}{qC_p} = K'$$

for $\alpha = \bar{\alpha}$ (notice that $\bar{\alpha}$ maximizes ϕ). Moreover since ϕ is continuous, we get that some δ small enough exists so that also $\hat{c}_{\bar{\alpha}-\delta} > \hat{c}_{\alpha_1}$. \square

These assumption on C_f are chosen to have the largest possible intervals such that $B_{\bar{\alpha}}$ admits an interior minimizer. In this construction we can see that $\bar{\alpha}$ depends on the value of C_f . In the sequel, we will be interested in having a value of $\bar{\alpha}$ independent on C_f . This is possible under different assumptions on C_f .

Theorem 3.12. *Let*

$$\hat{\alpha} = \left(\frac{C_p}{C_L} K'' + 1 \right)^q \quad (3.37)$$

where

$$K'' = C_L + C_L^{-1} + K_f$$

and suppose

$$C_f < \frac{qC_L}{C_p(\hat{\alpha} - 1)} \quad (3.38)$$

then, the results of Theorem 3.11 hold.

Proof. The proof is analogous to the one of Theorem 3.11. □

Let us conclude this section by giving an uniform estimate for the minima we found. This estimates are crucial in the limit procedure.

Corollary 3.13. *Let $(m_\varepsilon, w_\varepsilon)$ be a minimizer of \mathcal{E}_ε . Then, there exists $C > 0$ independent of ε such that*

$$\|m_\varepsilon\|_q \leq C, \quad \|m_\varepsilon\|_{1,r} \leq C \quad (3.39)$$

and

$$\|w_\varepsilon\|_{\frac{\gamma'q}{\gamma'+q-1}} \leq C \quad (3.40)$$

Proof. Suppose $q < \bar{q}$. We can use the fact that

$$\mathcal{E}_\varepsilon(m_\varepsilon, w_\varepsilon) \leq \mathcal{E}_\varepsilon(1, 0) \leq C_L^{-1} + \frac{C_f}{q} + K_f \quad (3.41)$$

and follow the same reasoning of Lemma 3.8 to conclude that inequalities (3.39) and 3.40 hold with a constant independent on ε . If $q \geq \bar{q}$, the same results hold with the proof being even easier since $m_\varepsilon \in B_{\bar{\alpha}}$ where $\bar{\alpha}$ is independent of ε . □

3.2.4. Convex Duality

We now want to employ some convex duality arguments. These consists into exploiting an exchange theorem for convex functionals. This idea was introduced in [49] and was

already applied to MFGs in several settings (we follow [18], see also [11]). However, our functional \mathcal{E}_ε is not convex in general. Thus, the idea is to linearize the functional around the minimizer we found. Given $(m_\varepsilon, w_\varepsilon)$, a global minimizer of \mathcal{E}_ε on \mathcal{K} or a local minimizer on $\mathcal{K} \cap B_{\bar{\alpha}}$, let us introduce the following linearized functional:

$$J_\varepsilon(m, w) = \int_{\Omega} mL\left(-\frac{w}{m}\right) + f_\varepsilon[m_\varepsilon](x)m \, dx. \quad (3.42)$$

We can notice that this functional is convex. We now prove that this functional admits the same minimizer as \mathcal{E}_ε .

Proposition 3.14. *Let $(m_\varepsilon, w_\varepsilon)$ be a global minimizer of \mathcal{E}_ε on \mathcal{K} or a local minimizer on $\mathcal{K} \cap B_{\bar{\alpha}}$ as constructed above. Then*

$$\min_{(m,w) \in \mathcal{K}} J_\varepsilon(m, w) = J_\varepsilon(m_\varepsilon, w_\varepsilon) \quad (3.43)$$

Proof. Let $(m, w) \in \mathcal{K}$ and consider for $0 < \lambda < 1$

$$m_\lambda = \lambda m + (1 - \lambda)m_\varepsilon$$

If $(m_\varepsilon, w_\varepsilon)$ is a local minimum, since by (3.36) $m_\varepsilon \in B_{\bar{\alpha}-\delta}$, we can conclude that for λ small enough, $m_\lambda \in B_{\bar{\alpha}}$. If $(m_\varepsilon, w_\varepsilon)$ is a global minimum this argument holds for all λ . Hence, by minimality and convexity

$$\begin{aligned} F_\varepsilon[m_\varepsilon] - F_\varepsilon[m_\lambda] &\leq \int_{\Omega} m_\lambda L\left(-\frac{w_\lambda}{m_\lambda}\right) \, dx - \int_{\Omega} m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) \, dx \\ &\leq \lambda \int_{\Omega} mL\left(-\frac{w}{m}\right) \, dx + (1 - \lambda) \int_{\Omega} m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) \, dx - \int_{\Omega} m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) \, dx \\ &= \lambda \left(\int_{\Omega} mL\left(-\frac{w}{m}\right) \, dx - \int_{\Omega} m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) \, dx \right). \end{aligned}$$

Now, by (3.22), we have

$$\begin{aligned} F_\varepsilon[m_\varepsilon] - F_\varepsilon[m_\lambda] &= \int_0^1 \int_{\Omega} f_\varepsilon[(1-t)m_\lambda + tm_\varepsilon](x)(m_\varepsilon - m_\lambda)(x) \, dx \\ &= -\lambda \int_0^1 \int_{\Omega} f_\varepsilon[m_\varepsilon + \lambda(1-t)(m - m_\varepsilon)](x)(m - m_\varepsilon)(x) \, dx \end{aligned}$$

Combining the two expressions, we can use Lipschitz estimates for $f_\varepsilon[\cdot](x)$ near m_ε and

send λ to 0 to conclude that

$$\int_{\Omega} mL\left(-\frac{w}{m}\right) dx - \int_{\Omega} m_{\varepsilon}L\left(-\frac{w_{\varepsilon}}{m_{\varepsilon}}\right) dx \geq - \int_{\Omega} f_{\varepsilon}[m_{\varepsilon}](x)(m_{\varepsilon} - m)(x) dx$$

which is equivalent to the minimality of J_{ε} . \square

Now that we have a convex functional, we can construct a solution for (3.19).

Theorem 3.15. *Let $(m_{\varepsilon}, w_{\varepsilon})$ be a minimizer of J_{ε} . Then $m_{\varepsilon} \in W^{1,p}(\Omega)$ for all $p > 1$ and there exists $\lambda_{\varepsilon} \in \mathbb{R}$ and $u_{\varepsilon} \in C^2(\bar{\Omega})$ such that $(u_{\varepsilon}, \lambda_{\varepsilon}, m_{\varepsilon})$ is a solution to (3.19). Moreover:*

$$w_{\varepsilon} = -\nabla H(\nabla(u_{\varepsilon})) \quad (3.44)$$

and finally there exists $C > 0$ independent of ε such that

$$\|m_{\varepsilon}\|_q \leq C, \quad \|m_{\varepsilon}\|_{1,r} \leq C \quad (3.45)$$

and

$$|\lambda_{\varepsilon}| \leq C. \quad (3.46)$$

Proof. We start by introducing

$$\mathcal{A}(m, w, u, c) = \int_{\Omega} [mL\left(-\frac{w}{m}\right) + f_{\varepsilon}[m_{\varepsilon}](x)m + m\Delta u + \nabla u \cdot w - cm] dx + c. \quad (3.47)$$

By Corollary 3.13, we have estimates (3.45) and $(m_{\varepsilon}, w_{\varepsilon}) \in W^{1,r}(\Omega) \times L^{\frac{\gamma'q}{\gamma'+q-1}}(\Omega)$. Hence, we can say that

$$J_{\varepsilon}(m_{\varepsilon}, w_{\varepsilon}) = \inf_{(m,w) \in W^{1,r}(\Omega) \times L^{\frac{\gamma'q}{\gamma'+q-1}}(\Omega)} \sup_{(u,c) \in C^2(\bar{\Omega}) \times \mathbb{R}, \frac{\partial u}{\partial n} = 0} \mathcal{A}(m, w, u, c)$$

where $\frac{\partial u}{\partial n} = 0$ is enforced on $\partial\Omega$. Indeed, $\sup_{(u,c) \in C^2(\bar{\Omega}) \times \mathbb{R}, \frac{\partial u}{\partial n} = 0} \mathcal{A}(m, w, u, c) < +\infty$ only if (m, w) satisfy the integral constraints in \mathcal{K} , as we can easily construct divergent sequences (u_n, c_n) otherwise. Moreover, $m \geq 0$ is enforced since $mL\left(-\frac{w}{m}\right) = +\infty$ otherwise. Thus we can conclude that the infimum on \mathcal{K} is actually achieved. We can now perform the duality argument. Note that $\mathcal{A}(\cdot, \cdot, u, c)$ is convex and weakly lower semicontinuous and $\mathcal{A}(m, w, \cdot, \cdot)$ is linear (and thus concave). Hence we can use the Min-Max exchange The-

orem [8, Th. 2.3.7] and deduce that

$$\begin{aligned} & \min_{(m,w) \in W^{1,r}(\Omega) \times L^{\frac{\gamma'q}{\gamma'+q-1}}(\Omega)} \sup_{(u,c) \in C^2(\bar{\Omega}) \times \mathbb{R}, \frac{\partial u}{\partial n} = 0} \mathcal{A}(m, w, u, c) \\ &= \sup_{(u,c) \in C^2(\bar{\Omega}) \times \mathbb{R}, \frac{\partial u}{\partial n} = 0} \min_{(m,w) \in W^{1,r}(\Omega) \times L^{\frac{\gamma'q}{\gamma'+q-1}}(\Omega)} \mathcal{A}(m, w, u, c). \end{aligned}$$

Moreover, we can use Rockafellar Integral Exchange Theorem [49, Th. 14.60] and (3.12) to get that

$$\begin{aligned} & \min_{(m,w) \in W^{1,r}(\Omega) \times L^{\frac{\gamma'q}{\gamma'+q-1}}(\Omega)} \mathcal{A}(m, w, u, c) \\ &= \int_{\Omega} \min_{(m,w) \in \mathbb{R} \times \mathbb{R}^N} \left[mL \left(-\frac{w}{m} \right) + f_{\varepsilon}[m_{\varepsilon}](x)m + m\Delta u + \nabla u \cdot w - cm \right] dx + c \\ &= \int_{\Omega} \min_{m \in \mathbb{R}} m[-H(\nabla u) + f_{\varepsilon}[m_{\varepsilon}](x) + \Delta u - c] dx + c. \end{aligned}$$

Now observe that the minimization can be done on $m \geq 0$ using the same reasoning as before and that

$$\min_{m \geq 0} m[-H(\nabla u) + f_{\varepsilon}[m_{\varepsilon}](x) + \Delta u - c] = \begin{cases} 0 & \text{if } \Delta u - H(\nabla u) + f_{\varepsilon}[m_{\varepsilon}] - c \geq 0 \\ -\infty & \text{if } \Delta u - H(\nabla u) + f_{\varepsilon}[m_{\varepsilon}] - c < 0. \end{cases}$$

Putting all together we get that

$$\begin{aligned} J_{\varepsilon}(m_{\varepsilon}, w_{\varepsilon}) &= \sup_{(u,c) \in C^2(\bar{\Omega}) \times \mathbb{R}, \frac{\partial u}{\partial n} = 0} \int_{\Omega} \min_{m \in \mathbb{R}} m[-H(\nabla u) + f_{\varepsilon}[m_{\varepsilon}](x) + \Delta u - c] dx + c \\ &= \sup \left\{ c \in \mathbb{R} \mid \exists u \in C^2(\bar{\Omega}), \frac{\partial u}{\partial n} = 0 \text{ s.t. } -\Delta u + H(\nabla u) + c \leq f_{\varepsilon}[m_{\varepsilon}](x) \right\}. \end{aligned}$$

Since $f_{\varepsilon}[m_{\varepsilon}](x)$ is smooth, we can use Theorem 2.10 to conclude that there exists $(u_{\varepsilon}, \lambda_{\varepsilon}) \in C^2(\bar{\Omega}) \times \mathbb{R}$ such that

$$\begin{cases} -\Delta u_{\varepsilon} + H(\nabla u_{\varepsilon}) + \lambda_{\varepsilon} = f_{\varepsilon}[m_{\varepsilon}](x) & \text{on } \Omega \\ \frac{\partial u_{\varepsilon}}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} u_{\varepsilon} dx = 0 \end{cases}$$

and

$$\lambda_{\varepsilon} = J_{\varepsilon}(m_{\varepsilon}, w_{\varepsilon}) = \int_{\Omega} m_{\varepsilon} \left[L \left(-\frac{w_{\varepsilon}}{m_{\varepsilon}} \right) + f_{\varepsilon}[m_{\varepsilon}(x)] \right] dx. \quad (3.48)$$

Combining this last two equations and the constraint (3.8) tested with u_{ε} (to be precise

the testing space we considered was $C^\infty(\bar{\Omega})$, but we can still use u_ε as a test function by a standard density argument) we can deduce

$$\begin{aligned} 0 &= \int_{\Omega} m_\varepsilon \left[L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) + f_\varepsilon[m_\varepsilon(x)] - \lambda_\varepsilon \right] dx \\ &= \int_{\Omega} m_\varepsilon \left[L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) + \Delta u_\varepsilon + H(\nabla u_\varepsilon) \right] dx \\ &= \int_{\{m_\varepsilon > 0\}} m_\varepsilon \left[L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) + \nabla u_\varepsilon \cdot \frac{w_\varepsilon}{m_\varepsilon} + H(\nabla u_\varepsilon) \right] dx. \end{aligned} \quad (3.49)$$

We notice now that the term in square brackets is non-negative since H is the Legendre transform of L . Hence we have that a.e. on $\{m_\varepsilon > 0\}$,

$$m_\varepsilon H(\nabla u_\varepsilon) = -m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) - \nabla u_\varepsilon \cdot w_\varepsilon$$

which means that $-\frac{w_\varepsilon}{m_\varepsilon}$ achieves the maximum in (3.12). This implies in a standard way (see also [49, Ex. 11.9]) that

$$-\frac{w_\varepsilon}{m_\varepsilon} = \nabla H(\nabla u_\varepsilon) \quad (3.50)$$

a.e. on $\{m_\varepsilon > 0\}$. On $\{m_\varepsilon = 0\}$, the definition of L and (3.49) force $w_\varepsilon = 0$. Substituting in (3.8) we get that m_ε solves (2.1) with $b = \nabla H(\nabla u_\varepsilon)$. Using Theorem 2.9, we have that $\nabla u_\varepsilon \in L^\infty(\Omega)$ and thus $\nabla H(\nabla u_\varepsilon) \in L^\infty(\Omega)$. We can then apply Theorem 2.5 to conclude that $m_\varepsilon \in W^{1,p}(\Omega)$ for every $p > 1$ and $m_\varepsilon > 0$, hence $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ is a solution to (3.19). Lastly, to prove the bound for λ_ε , by (3.13), (3.3) and properties of mollifiers we have

$$\begin{aligned} \lambda_\varepsilon &= J_\varepsilon(m_\varepsilon, w_\varepsilon) \geq \int_{\Omega} m_\varepsilon f(\cdot, m_\varepsilon * \chi_\varepsilon(\cdot)) * \chi_\varepsilon dx \\ &= \int_{\Omega} m_\varepsilon * \chi_\varepsilon f(x, m_\varepsilon * \chi_\varepsilon(x)) dx \geq -C \|m_\varepsilon * \chi_\varepsilon\|_q^q - C \geq -C \|m_\varepsilon\|_q^q - C \geq -C' \end{aligned}$$

where the last inequality comes from (3.39). To conclude we get by minimality and again by (3.3) and properties of mollifiers

$$\begin{aligned} \lambda_\varepsilon &= J_\varepsilon(m_\varepsilon, w_\varepsilon) \leq J_\varepsilon(1, 0) = \int_{\Omega} f_\varepsilon[m_\varepsilon](x) dx \\ &= \int_{\Omega} f(\cdot, m_\varepsilon * \chi_\varepsilon(\cdot)) * \chi_\varepsilon(x) dx \leq \int_{\Omega} f(x, m_\varepsilon * \chi_\varepsilon(x)) dx \\ &\leq C \int_{\Omega} (m_\varepsilon * \chi_\varepsilon)^{q-1} dx - C \leq C \int_{\Omega} m_\varepsilon^{q-1} dx - C \\ &\leq C \|m_\varepsilon\|_q^\theta - C \leq C \end{aligned}$$

where in the last inequalities we used Interpolation and (3.39) again. \square

3.2.5. Passage to the Limit

We now want to let $\varepsilon \rightarrow 0$. To do so we need some a priori estimate. We need to distinguish two cases: if $q < q^*$, we can use the following blow-up argument to deduce an a priori L^∞ bound on m_ε . If $q = q^*$, the argument fails and we need to require some extra smallness on C_f to obtain an estimate.

Proposition 3.16. *Let $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ be a solution to (3.19) and suppose $q < q^*$. Then there exist $C > 0$ independent of ε such that*

$$\|m_\varepsilon\|_\infty \leq C \quad (3.51)$$

Proof. Suppose by contradiction that

$$M_\varepsilon = \max_{\Omega} m_\varepsilon = m_\varepsilon(x_\varepsilon) \rightarrow +\infty.$$

Define

$$\mu_\varepsilon := M_\varepsilon^{-\beta} \quad \beta := (q-1)\frac{\gamma-1}{\gamma}.$$

We have that $\mu_\varepsilon \rightarrow 0$. Define the following rescaling

$$\begin{cases} v_\varepsilon(x) &= \mu_\varepsilon^{\frac{2-\gamma}{\gamma-1}} u_\varepsilon(\mu_\varepsilon x + x_\varepsilon) \\ n_\varepsilon(x) &= M_\varepsilon^{-1} m_\varepsilon(\mu_\varepsilon x + x_\varepsilon). \end{cases}$$

We notice that $n_\varepsilon(0) = 1$ and that $0 \leq n_\varepsilon(x) \leq 1$. Define also

$$H_\varepsilon(q) = \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} H(\mu_\varepsilon^{\frac{1}{1-\gamma}} q) \quad \nabla H_\varepsilon(q) = \mu_\varepsilon \nabla H(\mu_\varepsilon^{\frac{1}{1-\gamma}} q) \quad (3.52)$$

By (3.2) we get

$$C_H |p|^\gamma - C_H^{-1} \leq H_\varepsilon(p) \leq C_H (|p|^\gamma + 1) \quad |\nabla H_\varepsilon(p)| \leq C_H |p|^{\gamma-1} \quad (3.53)$$

Then, define

$$\tilde{f}_\varepsilon(x) := \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} f_\varepsilon[m_\varepsilon](x_\varepsilon + \mu_\varepsilon x). \quad (3.54)$$

Since $m_\varepsilon(x) \leq m_\varepsilon$, we can use (3.3) to get that

$$\begin{aligned} \left\| \tilde{f}_\varepsilon[m_\varepsilon] \right\|_\infty &\leq \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} \|f(\cdot, m * \chi_\varepsilon(\cdot)) * \chi_\varepsilon\|_\infty \leq \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} \|f(\cdot, m_\varepsilon * \chi_\varepsilon)\|_\infty \\ &\leq \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} C \|m_\varepsilon * \chi_\varepsilon\|_\infty^{q-1} + C \leq \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} (C \|m_\varepsilon\|_\infty^{q-1} + C) \\ &\leq C + CM_\varepsilon^{q-1-\beta\frac{\gamma}{\gamma-1}} \leq C \end{aligned}$$

by our definitions of μ_ε and β . Lastly, define

$$\tilde{\lambda}_\varepsilon = \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} \lambda_\varepsilon. \quad (3.55)$$

Clearly by (3.46), $|\tilde{\lambda}_\varepsilon| \leq C$. Now after some computations we have

$$\left\{ \begin{array}{l} \Delta v_\varepsilon(x) = \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} \Delta u_\varepsilon(x\mu_\varepsilon + x_\varepsilon) \\ H_\varepsilon(\nabla v_\varepsilon(x)) = \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} H(\nabla u_\varepsilon(x\mu_\varepsilon + x_\varepsilon)) \\ \Delta n_\varepsilon(x) = \mu_\varepsilon^{\frac{1}{\beta}+2} \Delta m_\varepsilon(x\mu_\varepsilon + x_\varepsilon) \\ \nabla H_\varepsilon(\nabla v_\varepsilon(x)) = \mu_\varepsilon \nabla H(\nabla u_\varepsilon(x\mu_\varepsilon + x_\varepsilon)) \\ \operatorname{div}(n_\varepsilon \nabla H_\varepsilon(\nabla v_\varepsilon(x))) = \mu_\varepsilon^{\frac{1}{\beta}+2} \operatorname{div}(m_\varepsilon(x\mu_\varepsilon + x_\varepsilon) \nabla H(\nabla u_\varepsilon(x\mu_\varepsilon + x_\varepsilon))) \\ \frac{\partial v_\varepsilon(x)}{\partial n} = \mu_\varepsilon^{\frac{1}{\gamma-1}} \frac{\partial u_\varepsilon(\mu_\varepsilon x + x_\varepsilon)}{\partial n}(\mu_\varepsilon x + x_\varepsilon) \\ \frac{\partial n_\varepsilon}{\partial n} = \mu_\varepsilon^{\frac{1}{\beta}+1} \frac{\partial m_\varepsilon(\mu_\varepsilon x + x_\varepsilon)}{\partial n}. \end{array} \right.$$

from which we deduce that $(v_\varepsilon, \tilde{\lambda}_\varepsilon, n_\varepsilon)$ is a solution of

$$\left\{ \begin{array}{ll} -\Delta v_\varepsilon + H_\varepsilon(\nabla v_\varepsilon) + \tilde{\lambda}_\varepsilon = \tilde{f}_\varepsilon(x) & \text{on } \Omega_\varepsilon \\ -\Delta n_\varepsilon - \operatorname{div}(n_\varepsilon \nabla H_\varepsilon(\nabla v_\varepsilon)) = 0 & \text{on } \Omega_\varepsilon \\ \frac{\partial v_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon \\ \frac{\partial n_\varepsilon}{\partial n} + n_\varepsilon \nabla H_\varepsilon(\nabla v_\varepsilon) \cdot n = 0 & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} n_\varepsilon = M_\varepsilon^{-1-\beta}, \quad \int_{\Omega} v_\varepsilon = 0. & \end{array} \right. \quad (3.56)$$

where $\Omega_\varepsilon = \{x : \mu_\varepsilon x + x_\varepsilon \in \Omega\}$. We now have to distinguish two cases. Suppose first that

$$\lim_{\varepsilon \rightarrow 0} \frac{d(x_\varepsilon, \partial\Omega)}{\mu_\varepsilon} = +\infty.$$

From that we have that $\Omega_\varepsilon \uparrow \mathbb{R}^N$, hence for ε small enough we have that $\Omega_\varepsilon \supset B_R(0)$ for an $R > 0$ independent on ε . We know that $\tilde{\lambda}_\varepsilon$ and $\tilde{f}_\varepsilon(x)$ are uniformly bounded, thus using Theorem 2.10 we can conclude that there exists C independent of ε such that

$\|\nabla v_\varepsilon\|_\infty \leq C$. Now, using (3.53) we can deduce that $\|\nabla H_\varepsilon(\nabla v_\varepsilon)\|_\infty \leq C$. Hence, by Theorem 2.5, for ε small enough n_ε is equibounded in $W^{1,p}(B_R(0))$ for all $p > 1$, and by Sobolev Embedding also in $C^\theta(\overline{B_R(0)})$. We know that $n_\varepsilon(0) = 1$, therefore using equiboundedness in $C^\theta(\overline{B_R(0)})$ we can deduce that there exists $\delta > 0$ and $r < R$ such that $\int_{B_r(0)} n_\varepsilon^q(x) dx > \delta > 0$. Thus we have that

$$0 < \delta \leq \int_{B_r(0)} n_\varepsilon^q(x) dx \leq \|n_\varepsilon\|_q^q = M_\varepsilon^{-q} \mu_\varepsilon^{-N} \|m_\varepsilon\|_q^q = M_\varepsilon^{-q+\beta N} \|m_\varepsilon\|_q^q. \quad (3.57)$$

Since $q < q^*$ then $-q + \beta N < 0$, and using (3.39) we have that

$$0 < \delta \leq M_\varepsilon^{-q+\beta N} \|m_\varepsilon\|_q^q \leq C M_\varepsilon^{-q+\beta N} \rightarrow 0$$

which is a contradiction.

Suppose now that

$$\lim_{\varepsilon \rightarrow 0} \frac{d(x_\varepsilon, \partial\Omega)}{\mu_\varepsilon} \leq C.$$

Up to subsequences we can suppose that $x_\varepsilon \rightarrow p \in \partial\Omega$ as $\varepsilon \rightarrow 0$. Moreover, up to an affine transformation we can suppose $p = 0 \in \partial\Omega$ and $n(0) = -e_N$. Define $x' = (x_1, \dots, x_{N-1})$. By the smoothness of Ω there exists $U \in \mathbb{R}^N$, $\Gamma \in \mathbb{R}^{N-1}$ and $\phi(x') \in C^{2,\alpha}(\Gamma)$ such that

$$\begin{aligned} \phi(0) &= 0, & \nabla\phi(0) &= 0, \\ \partial\Omega \cap U &= \{(x', x_N) : x_N = \phi(x')\}, \\ \Omega \cap U &= \{(x', x_N) : x_N > \phi(x')\}. \end{aligned}$$

Let us now define a diffeomorphism $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ that "straightens" the boundary. We set

$$(\Psi(x))_i := \begin{cases} x_i - x_N \frac{\partial\phi}{\partial x_i}(x') & \text{for } 1 \leq i \leq N-1 \\ x_N - \phi(x') & \text{for } i = N. \end{cases} \quad (3.58)$$

We can see that Ψ is invertible in a neighborhood of 0. We now extend with an even reflection v_ε and m_ε . We set

$$w_\varepsilon(x) = v_\varepsilon \circ \Psi(x)^{-1}(x', |x_N|) \quad (3.59)$$

$$\rho_\varepsilon(x) = n_\varepsilon \circ \Psi(x)^{-1}(x', |x_N|). \quad (3.60)$$

Due to the homogeneous Neumann boundary conditions, we can derive (see for instance [7, Lem. 3.3] or [43, Sec. 3]) that $w_\varepsilon, \rho_\varepsilon$ satisfy an equation similar to (3.56) in a fixed neighborhood of the boundary point p independent of ε . From this, we can repeat the

above argument and reach a contradiction. \square

We can see in this proof the criticality of the case $q = q^*$. Looking at (3.57), in the case $q = q^*$ we lose all information and it is not possible to reach a contradiction. This is coherent to the critical rescaling properties of the Sobolev critical exponent. To tackle this problem, the idea is to obtain additional regularity using finer estimates for both the Fokker-Planck and the Hamilton-Jacobi equation. Once we find uniform bounds for m_ε in some L^p with $p > q^*$, the above arguments can be used to conclude again that we have an uniform L^∞ bound. This procedure requires additional assumptions on C_f . Moreover we need to have a value for $\bar{\alpha}$ which is independent from C_f , as we have constructed in Theorem 3.12.

Proposition 3.17. *Let $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ be a solution to (3.19) and suppose $q = q^*$. Then*

$$\|\nabla u_\varepsilon\|_{N(\gamma-1)} \leq \frac{1}{(C_E C_S C_H)^{\frac{1}{\gamma-1}}} \quad (3.61)$$

provided that

$$C_f < \left(\frac{C_p}{C_L} K' + 1 \right)^{-(q-1)} \delta_0 \quad (3.62)$$

where δ_0 is defined in Theorem 2.13, C_E is defined in Proposition 2.3 and C_S is a properly chosen Sobolev Embedding constant.

Proof. We use Theorem 2.13, choosing δ_0 small enough so that $M < (C_E C_S C_H)^{\frac{-1}{\gamma-1}}$. Let us compute the norm of f_ε in $L^{\frac{N}{\gamma}}(\Omega)$. Using 3.3, convolution properties and the definition of q^* we have

$$\|f_\varepsilon\|_{\frac{N}{\gamma}} = \|f(x, m * \chi_\varepsilon(x))\|_{\frac{N}{\gamma}} \leq C_f \|m_\varepsilon^{q-1}\|_{\frac{N}{\gamma}} + K_f = C_f (\|m_\varepsilon\|_q^q)^{\frac{1}{q}} + K_f.$$

Now using Theorem 3.12, we know that

$$\|m\|_q^q \leq \bar{\alpha} = \left(\frac{C_p}{C_L} K' + 1 \right)^q.$$

Hence if

$$C_f \leq \bar{\alpha}^{-\frac{1}{q}} \delta_0$$

we have that $\|f\|_\varepsilon \leq \delta_0 + K_f$. Thus, we can apply Theorem 2.13 with $f = f_\varepsilon - K_f$ to conclude. \square

Now everything is ready to prove the main result. Thanks to the uniform bounds, we

use a bootstrap procedure to obtain the regularity necessary to pass to the limit into the equations.

Proof of Theorem 3.5. The constants C_q and C_{crit} are given by the estimates (3.34) and (3.38), (3.62) respectively. Let $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ be a solution as constructed in Theorem 3.15 and suppose $q < q^*$. By (3.46) we have that, up to subsequences, $\lambda_\varepsilon \rightarrow \lambda$. Moreover, by Proposition 3.16 we have that $\|m_\varepsilon\|_\infty \leq C$. Hence by properties of mollifiers and (3.3) we get

$$\begin{aligned} \|f_\varepsilon[m_\varepsilon]\|_\infty &= \|f(\cdot, m * \chi_\varepsilon(\cdot)) * \chi_\varepsilon\|_\infty \leq \|f(\cdot, m_\varepsilon * \chi_\varepsilon)\|_\infty \\ &\leq C \|m_\varepsilon * \chi_\varepsilon\|_\infty^{q-1} + C \leq C \|m_\varepsilon\|_\infty^{q-1} + C \leq C \end{aligned}$$

Thus, we can conclude by Theorem 2.10, that $\|\nabla u_\varepsilon\|_\infty \leq K$ for some $K > 0$ independent of ε . Using the estimates on H and elliptic regularity in the Hamilton-Jacobi equation we can conclude that $\|u_\varepsilon\|_{1,p} \leq C$ for all $p > 1$. Moreover, by Sobolev embeddings, u_ε is equibounded in $C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, 1)$. Since $\nabla H(\nabla u_\varepsilon) \leq C$, we now use Theorem 2.5 to conclude that $\|m_\varepsilon\|_{1,p} \leq C$ for all $p > 1$, and therefore by Sobolev embedding m_ε is equibounded in $C^\theta(\overline{\Omega})$ for all $\theta \in (0, 1)$. Hence, we get that, up to subsequences, $m_\varepsilon \rightarrow m$ in $W^{1,p}$ for all $p > 1$ and $m_\varepsilon \rightarrow m$ uniformly. We can then go back to the Hamilton-Jacobi equation and, with a similar reasoning, conclude that $f_\varepsilon[m_\varepsilon](x)$ is equibounded in $C^\theta(\Omega)$ for all $\theta \in (0, 1)$. Hence u_ε is equibounded in $C^{2,\theta}(\overline{\Omega})$ for all $\theta \in (0, 1)$. Finally, by Ascoli-Arzelà we can conclude that up to subsequences $u_\varepsilon \rightarrow u$ in $C^2(\overline{\Omega})$. Now the convergences are strong enough to pass to the limit in the equations, so we can conclude that (u, λ, m) is a solution of (3.1), with the positivity of m coming from Theorem 2.5 and pointwise convergence.

Suppose now that $q = q^*$. Using Proposition 3.17 we have that $\|\nabla u_\varepsilon\|_{N(\gamma-1)} \leq \frac{1}{(C_E C_S C_H)^{\frac{1}{\gamma-1}}}$. Hence $\|\nabla H(\nabla u_\varepsilon)\|_N \leq \frac{1}{(C_E C_S)}$ and we can use Proposition 2.6 to conclude that m_ε are uniformly bounded in $W^{1,p}(\Omega)$ for some $p < N$, such that by Sobolev Embeddings they are uniformly bounded in $L^q(\Omega)$ for some $q > q^*$ (we suppose that C_S is chosen accordingly). Once we have this bound, we can adapt the reasoning of Proposition 3.16 to conclude that m_ε is bounded in L^∞ also in this case, and start the bootstrap procedure. \square

3.3. The Quadratic case

Let us now talk briefly about the case $\gamma = 2$. Let us suppose that $H(p) = \frac{1}{2}|p|^\gamma$ and $f(x) = -C_f a(x) |m(x)|^{q-1}$ with $a(x) \geq a_0 > 0$ and $\|a\|_\infty = 1$. The system becomes

$$\begin{cases} -\Delta u + \frac{1}{2} |\nabla u|^2 + \lambda = -C_f a(x) m^{q-1} & \text{in } \Omega \\ -\Delta m - \operatorname{div}(m \nabla u) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \frac{\partial m}{\partial n} + m \nabla u \cdot n = 0 & \text{on } \partial\Omega \\ \int_\Omega m = 1, \quad \int_\Omega u = 0. \end{cases} \quad (3.63)$$

In this special case, it is possible to use the so called *Hopf-Cole transformation* to decouple the system. Let us introduce the function $\phi = ce^{-\frac{u}{2}}$. A computation shows that ϕ satisfies

$$\begin{cases} -2\Delta\phi - C_f a(x) |m|^{q-1} \phi = \lambda\phi & \text{on } \Omega \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \Omega. \end{cases} \quad (3.64)$$

If now we set $c^2 = (\int_\Omega e^{-u})^{-1}$ we can observe that ϕ^2 satisfies the Fokker-Planck equation. Since we expect uniqueness in the Fokker-Planck equation, we can conclude that $m = \phi^2$ and thus the problem reduces to finding ϕ and λ such that

$$\begin{cases} -2\Delta\phi - C_f a(x) |\phi|^{2(q-1)} \phi = \lambda\phi & \text{on } \Omega \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \Omega \\ \int_\Omega \phi^2 = 1 \end{cases} \quad (3.65)$$

This system appears in the study of Schrödinger equations and was intensively studied. In [45], variational techniques are used to deduce the existence of a solution for (3.65) (in the case of multi-populations and Dirichlet boundary conditions). We explain briefly the main ideas. Let us simplify the notation setting $\omega = -\frac{\lambda}{2}$, $\mu_f = \frac{C_f}{2}$ and $p = 2q - 1$. We need to find ϕ and ω such that

$$\begin{cases} -\Delta\phi + \omega\phi = \mu_f a(x) |\phi|^{p-1} \phi & \text{on } \Omega \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \Omega \\ \int_\Omega \phi^2 = 1 \end{cases} \quad (3.66)$$

In this case no regularization procedure is needed and the variational structure of the problem comes directly from the Lagrange multipliers rule in Banach spaces (see for instance [52, Sec. 4.14]). This implies that any solution of (3.66) is a critical point of the energy

$$\mathcal{E}(u) := \frac{1}{2} \int_{\Omega} |u|^2 + |\nabla u|^2 \, dx - \frac{\mu_f}{p+1} \int_{\Omega} a(x) |u|^{p+1} \, dx \quad (3.67)$$

on the constraint

$$\mathcal{K} = \{u \in W^{1,2}(\Omega) : \int_{\Omega} u^2 = 1\} \quad (3.68)$$

We can now state an existence theorem. Let us set $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N < 3$, coherently with Theorem A.1.

Theorem 3.18. *Suppose*

$$1 < p < 1 + \frac{4}{N} \quad (3.69)$$

Then \mathcal{E} admits a global minimizer on \mathcal{K} . Suppose instead that

$$1 + \frac{4}{N} \leq p < 2^* - 1. \quad (3.70)$$

Define $r := \frac{N}{4}(p-1) > 1$ and consider $\lambda_2(\Omega) > 0$ the second eigenvalue of $-\Delta$ on Ω with Neumann boundary conditions. Then, if

$$\mu_f < \frac{p+1}{2} \frac{(r-1)^{r-1}}{r^r} (\lambda_2(\Omega) + 1)^{r-1}, \quad (3.71)$$

\mathcal{E} admits a local minimum in \mathcal{K} .

Proof. The key is again the Gagliardo-Nirenberg inequality, which in this case coincides with Interpolation and Sobolev Embeddings. We find that

$$\|u\|_{p+1} \leq \|u\|_2^{1-\theta} \|u\|_{1,2}^{\theta} \quad (3.72)$$

for the usual $\theta \in (0, 1)$. Using the properties of u and a we get

$$\int_{\Omega} a(x) u^{p+1} \, dx \leq \|u\|_{p+1}^{p+1} \leq \|u\|_{1,2}^{\frac{N}{2}(p-1)} \quad (3.73)$$

Applying (3.72) to \mathcal{E} we find

$$\mathcal{E}(u) = \frac{1}{2} \int_{\Omega} |u|^2 + |\nabla u|^2 dx - \frac{\mu_f}{p+1} \int_{\Omega} a(x) |u|^{p+1} dx \geq \frac{1}{2} \|u\|_{1,2}^2 - \frac{\mu_f}{p+1} \|u\|_{1,2}^{\frac{N}{2}(p-1)}. \quad (3.74)$$

Here, two cases are delineated. If (3.69) holds, then \mathcal{E} is bounded from below and we can use the direct method of Calculus of Variations to conclude the existence of a minimum. Indeed \mathcal{K} is closed and convex, the first term of (3.67) is weakly lower semicontinuous since it is convex and the second term is weakly continuous because of the compact Sobolev embedding.

Now suppose (3.70) holds. We introduce for $\alpha \geq 0$ the set

$$B_{\alpha} = \{u \in W^{1,2}(\Omega) : \|u\|_{1,2}^2 \leq \alpha\}. \quad (3.75)$$

The energy \mathcal{E} is bounded on $B_{\alpha} \cap \mathcal{K}$, thus by the previous reasoning there exists a minimum for every α . Arguing as in Theorem 3.11, to prove the existence of a local minimum it is sufficient to find $\alpha_1 < \alpha_2$ such that

$$\hat{c}_{\alpha_1} := \inf_{\partial B_{\alpha_1} \cap \mathcal{K}} \mathcal{E} < \inf_{\partial B_{\alpha_2} \cap \mathcal{K}} \mathcal{E} =: \hat{c}_{\alpha_2} \quad (3.76)$$

. We let $\alpha_1 = \lambda_2(\Omega) + 1$ and consider a normalized eigenfunction ϕ_2 of $\lambda_2(\Omega)$. We have

$$\hat{c}_{\alpha_1} \leq \mathcal{E}(\phi_2) = \frac{\lambda_2(\Omega) + 1}{2} - \int_{\Omega} a(x) |\phi|^{p+1} dx \leq \frac{\lambda_2(\Omega) + 1}{2}. \quad (3.77)$$

Now all is left to do is find $\alpha_2 = \bar{\alpha}$ such that $\hat{c}_{\bar{\alpha}} > \frac{\lambda_2(\Omega)+1}{2}$. Using (3.74), we need to find $\bar{\alpha} > \lambda_2(\Omega) + 1$ such that

$$\frac{\bar{\alpha}}{2} - \frac{\mu_f}{p+1} \bar{\alpha}^r > \frac{\lambda_2(\Omega) + 1}{2}. \quad (3.78)$$

By a direct computation, if (3.71) holds, then $\bar{\alpha} = \frac{r}{r-1}(\lambda_2(\Omega) + 1)$ satisfies the inequality. \square

Remark 3.19. In this proof we can see the variational structure this problem shares with the MFG system, as (3.74) closely resembles the estimate (3.32).

Remark 3.20. We can notice that also in this setting the "Sobolev critical" case $p = 2^* - 1$ presents difficulties. In particular, we lose the semicontinuity of \mathcal{E} due to the lack of compactness. Nonetheless, also this case can be treated (see [45]), using ideas developed by Brezis and Nirenberg and the so called Brezis-Lieb Lemma (see [9, 10]).

4 | Conclusions

In this thesis work we presented the concept of Mean Field Games, presented some results on Hamilton-Jacobi and Fokker-Planck equations and proved an original existence result for a Viscous Ergodic Mean Field Game system using Variational Methods. We proved that the system

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(x, m(x)) & \text{on } \Omega \\ -\Delta m - \operatorname{div}(m \nabla H(\nabla u)) = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \frac{\partial m}{\partial n} + m \nabla H(\nabla u) \cdot n = 0 & \text{on } \partial\Omega \\ \int_{\Omega} m = 1, \quad \int_{\Omega} u = 0. \end{cases} \quad (4.1)$$

where

$$-C_f m^{q-1} - K_f \leq f(x, m) \leq C_f m^{q-1} + K_f, \quad (4.2)$$

has a solution for $1 < q \leq q^*$. If $q < \bar{q}$, it has a solution for every f , if $q \geq \bar{q}$ the solution exists under smallness hypothesis on C_f . If $q = q^*$ additional smallness hypothesis on C_f have to be made in order to account for the critical scaling properties of the Sobolev critical exponent. In addition, we performed the analysis in the case of Neumann Boundary conditions. This result is an extension of [18], where results were found for $q < \bar{q}$ and in the case $\Omega = \mathbb{T}^N$. We remark that solutions for $q < q^*$ were already found using fixed point methods in [23], however using variational methods we obtain more insight on the nature of such solutions. The case $q = q^*$, instead, is original and is based on finer regularity results for the Hamilton-Jacobi and Fokker-Planck equation, which are possible under smallness hypothesis on C_f .

To conclude, let us talk briefly about possible further developments. Generalizations of the problem can follow different paths. A first generalization is to consider evolutive systems. In this case it is still possible to preserve the variational structure of the problem and look for solutions, but we have to deal with possible finite time blow up [24]. Continuing the analogy with Schrödinger equations, it could be possible to prove existence results under smallness assumptions on the data.

A further step would be to include common noise for the system. When deriving the

MFG system, a key assumption is that brownian motions of the agents are independent. This allows to use a generalized version of the law of large number when performing the limit to derive a system of PDEs. When this hypothesis is not present we cannot perform this procedure, and in general SPDEs have to be considered.

A parallel path to be explored would be to consider systems of MFGs, modelling games where different populations are present [25]. In this case, the coupling term between the populations is a key factor and influences the existence results (as one can see for the Schrödinger systems in [45]). Finally, we need to remark that MFGs find various applications in Finance [14] and can be solved using Neural Networks [17]. Thus, it could be interesting, once we have developed the existence theory for these new models, to test their performance in these fields of study.

A | Appendix A

We collect here some useful inequalities and embeddings.

Let us recall initially the various embeddings that occur in Sobolev spaces.

Theorem A.1 (Sobolev embeddings). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with C^1 boundary.*

- If $1 \leq p < N$, let

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

Then, for $1 \leq q \leq p^*$

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega).$$

Moreover, for $1 \leq q < p^*$

$$W^{1,p}(\Omega) \xhookrightarrow{c} L^q(\Omega).$$

- If $p = N$,

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega).$$

for all $q \in [1, +\infty)$.

- If $p > N$,

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}).$$

for some $\alpha(N, p) \in (0, 1)$.

Another compactness result is given by the well known Ascoli-Arzelà theorem.

Theorem A.2 (Ascoli-Arzelà). *Let f_n be an uniformly bounded sequence of continuous functions defined on $\Omega \subset \mathbb{R}^N$ such that $\forall \varepsilon > 0 \exists \delta > 0$ such that*

$$|x - y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon$$

for all $n > 0, x, y \in \Omega$. Then there exists a subsequence f_{n_k} that converges uniformly.

This theorem enables the following corollary.

Corollary A.3. *Let $k \in \mathbb{N}$ and $\alpha > \beta$. Then*

$$C^{k,\alpha}(\overline{\Omega}) \xrightarrow{c} C^{k,\beta}(\overline{\Omega}).$$

Proof. Without loss of generality we suppose $k = 0$. Suppose $(u_n)_n$ is bounded in $C^{0,\alpha}(\overline{\Omega})$. Then clearly the hypothesis of Ascoli-Arzelà hold and there exists a subsequence (which we denote again by u_n) such that $u_n \rightarrow u$ uniformly. Let $w_n = u - u_n$. We have

$$\frac{|w_n(x) - w_n(y)|}{|x - y|^\beta} = \left(\frac{|w_n(x) - w_n(y)|}{|x - y|^\alpha} \right)^{\frac{\beta}{\alpha}} |w_n(x) - w_n(y)|^{1 - \frac{\beta}{\alpha}} \leq C \|w_n\|_\infty \quad (\text{A.1})$$

from which we can conclude that $w_n \rightarrow 0$ in $C^{0,\beta}(\overline{\Omega})$. \square

Let us now collect some useful inequalities. The Interpolation inequality is a useful consequence of the Hölder inequality.

Theorem A.4 (Interpolation inequality). *Let $1 \leq p \leq r \leq q \leq +\infty$ Suppose $u \in L^p(\Omega) \cap L^q(\Omega)$.*

Then $u \in L^r$ for all $r \in [p, q]$ and

$$\|u\|_r \leq \|u\|_p^\alpha \|u\|_q^{1-\alpha}$$

where:

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$$

A key theorem which generalises both Theorem A.1 and Theorem A.4 is the following.

Theorem A.5 (Gagliardo-Nirenberg inequality). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with C^1 boundary. Suppose j, m are non-negative integers, $1 \leq p, r \leq +\infty$, $1 \leq q < +\infty$. Suppose $u \in L^q(\Omega)$ and $D^m u \in L^r(\Omega)$. Then there exist $C_1, C_2 > 0$ independent of u such that the following inequality holds:*

$$\|D^j u\|_q \leq C_1 \|D^m u\|_r^\alpha \|u\|_p^{1-\alpha} + C_2 \|u\|_p$$

where

$$\frac{1}{q} = \frac{j}{N} + \alpha \left(\frac{1}{r} - \frac{m}{N} \right) + (1-\alpha) \frac{1}{p}$$

for α such that $\frac{j}{m} \leq \alpha \leq 1$, unless $1 < r < +\infty$ and $m - j - \frac{N}{r}$ is a non-negative integer, in which case the inequality holds only if also $\alpha \neq 1$.

Proof. See [44]. □

We will use the theorem in the case $j = 0, m = 1, \alpha \neq 1$ which yields

$$\|u\|_q \leq C_1 \|\nabla u\|_r^\alpha \|u\|_p^{1-\alpha} + C_2 \|u\|_p$$

where:

$$\frac{1}{q} = \alpha \left(\frac{1}{r} - \frac{1}{N} \right) + (1 - \alpha) \frac{1}{p}$$

for $0 \leq \alpha < 1$.

Finally, let us recall both the classical and the L^p approach to Elliptic regularity.

Theorem A.6 (Elliptic regularity). *Let $\Omega \in C^{2,\theta}$. Suppose u is a solution of $-\Delta u = f$ on Ω , $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Then*

- *If $f \in C^{0,\alpha}(\overline{\Omega})$ for some α and u is a classical solution then $u \in C^{2,\alpha}(\overline{\Omega})$ and*

$$\|u\|_{C^{2,\alpha}} \leq C(\|f\|_{C^{0,\alpha}} + \|u\|_\infty)$$

- *If $f \in L^p(\Omega)$ for some $p \in (1, \infty)$ and u is a strong solution then $u \in W^{2,p}(\Omega)$ and*

$$\|u\|_{2,p} \leq C(\|f\|_p + \|u\|_p)$$

Proof. See [28, Th. 6.30, Th. 9.11] . □

B | Appendix B

Let us collect here some results related with maximum principles for second order operators. We define

$$Lu := A(x) \cdot D^2u + b(x) \cdot \nabla u + c(x)u$$

with $A(x)$ symmetric, uniformly elliptic and positive definite, $b \in L^\infty(\Omega; \mathbb{R}^N)$ and $c \in L^\infty(\Omega)$. We will suppose $u \in C^2(\bar{\Omega})$ and that Ω is regular enough. We also set $Q_T = \Omega \times (0, T)$ and $S_T = \partial\Omega \times (0, T)$

Theorem B.1 (Hopf's lemma). *Suppose $Lu \geq 0$ in Ω and $c = 0$. Let $x_0 \in \partial\Omega$ such that*

- *u is continuous at x_0 ;*
- *$u(x_0) > u(x)$ for all $x \in \Omega$;*
- *$\partial\Omega$ satisfies the interior ball condition.*

Then the normal derivative of u at x_0 , if it exists, satisfies

$$\frac{\partial u}{\partial n}(x_0) > 0.$$

If $c \leq 0$, the same results hold provided that $u(x_0) > 0$, and if $u(x_0) = 0$, then same results hold irrespective of the sign of c .

Proof. See [28, Th. 3.4] □

Theorem B.2 (Strong Maximum Principle). *Suppose $Lu \geq 0$ (≤ 0) in Ω and $c = 0$. Then if u achieves a maximum (minimum) in Ω , then u is constant. If $c \leq 0$, then if u achieves a non-negative maximum (non-positive minimum) in Ω , then u is constant.*

Proof. See [28, Th. 3.5]. □

Let us also recall the following useful lemma for parabolic equations, which is a consequence of maximum principles.

Lemma B.3. *Let $u \in C^2(Q_T) \cap C^1(\overline{Q}_T)$ be such that*

$$\begin{cases} u_t - Lu \geq 0 & \text{on } Q_T \\ \alpha \frac{\partial u}{\partial n} + \beta u \geq 0 & \text{on } S_T \\ u(0, x) \geq 0 & \text{on } \Omega \end{cases}$$

where $\alpha, \beta \geq 0$. Then $u \geq 0$ in \overline{Q}_T .

Proof. See [46, Lemma 2.1].

□

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