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Stable Minimal Hypersurfaces

Tesi di Laurea Magistrale in
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Abstract

This work includes an introduction to the problem of minimal hypersurfaces and the mathematical tools needed to approach it, in the last chapter a flatness condition for minimal stable hypersurfaces is proved and from the results in [2] an extension of [17, Theorem 2] in the case of stability is presented. After an introduction to vector bundles, sections, and the Laplacian operator on sections some well-known equalities and inequalities are proved. Three different mathematical descriptions of hypersurfaces are then introduced, Caccioppoli sets, currents, and classical submanifolds, and the area functional is defined for each description. Then the first variation is computed in the setting of Caccioppoli sets. Then the second variation formula is fully computed in the case of submanifolds embedded in general manifolds, as the stability will be studied in this setting. The Bernstein Theorem, that is flatness of minimizing hypersurfaces in $\mathbb{R}^{n+1}$ for $n \leq 6$, is proved using blow up and blow down methods. Stable minimal submanifolds are then considered, the stability is related to the positivity of an operator of the form $-\Delta - V$, the eigenvalue problem is hence studied and the Morse Index Theorem which states the behavior of eigenvalues under contraction of the domain is presented. In order to have the counter-example for the Bernstein Theorem also when only smooth competitors are considered, it is important the smoothing of minimal cones. The smoothing of cones with a singularity at the origin is found in [14] the result is presented and an outline of the proof using minimizing currents is given. At last, it is proved that if the second form of a minimal hypersurface in $\mathbb{R}^{n+1}$ with $n < 6$ is bounded by the first eigenvalue of the Laplacian $|A|^2(x) \leq \lambda_1(-\Delta)$ then the hypersurface is stable and flat. In conclusion from the results in [2] is derived a theorem extending [17, Theorem 2] in the case of stability, with which it is proved flatness of stable minimal hypersurfaces for $n = 2$ and as in [2] for $n = 3$.

Keywords: stable minimal hypersurfaces, rigidity, currents, bounded variation functions, Morse index theorem, Schrödinger operator, Bernstein theorem, sets of finite perimeter

Abstract in lingua italiana

Questo lavoro comprende una introduzione al problema delle ipersuperfici minime e agli strumenti matematici necessari a descriverle, nell'ultimo capitolo una condizione affinché una ipersuperficie minima stabile sia piana è dimostrato, inoltre dai risultati in [2] un'estensione di [17, Theorem 2] per la stabilità è presentato. Dopo una introduzione ai vector bundles, sezioni, e l'operatore Laplaciano sulle sezioni alcune delle più note uguaglianze e disuguaglianze sono dimostrate. Tre differenti descrizioni matematiche di ipersuperficie sono introdotte, insiemi di Caccioppoli, le correnti, e sotto varietà classiche, inoltre viene introdotto il funzionale di area per ogni descrizione. La variazione prima è calcolata nel caso di insiemi di Caccioppoli. La variazione seconda è poi calcolata nel caso di sotto varietà immerse in varietà generiche, in quanto la stabilità verrà studiata su varietà. Il teorema di Bernstein, cioè che ipersuperfici minimizzanti in $\mathbb{R}^{n+1}$ sono piane per $n \leq 6$, è dimostrato usando i metodi di blow up e blow down. Si passa quindi al caso stabile, la condizione di stabilità è legata alla positività di un'operatore della forma $-\Delta - V$, è quindi studiato il problema agli autovalori e il teorema dell'indice di Morse, il quale discute il comportamento degli autovalori quando il dominio è contratto, viene presentato. Per avere il controesempio del teorema di Bernstein quando ipersuperfici lisce sono considerate, è importante lo smoothing di coni minimi. Questo risultato per coni con singolarità nell'origine si trova in [14], viene quindi ripresentato e la dimostrazione discussa. In fine viene dimostrato che se la seconda forma di una ipersuperficie minima in $\mathbb{R}^{n+1}$ con $n < 6$ è controllata dal primo autovalore del Laplaciano $|A|^2(x) \leq \lambda_1(-\Delta)$ allora la ipersuperficie è stabile e piana. Concludendo dai risultati in [2] è derivato un teorema che estende [17, Theorem 2] nel caso della stabilità, con tale teorema viene dimostrato che ipersuperfici minime stabili sono piane per $n = 2$, e come in [2] anche per $n = 3$.

Keywords: ipersuperfici minime stabili, rigidità, correnti, funzioni a variazione finita, teorema dell'indice di Morse, operatore di Schrödinger, teorema di Bernstein, sets di perimetro finito

Contents

1 | Introduction

The main goal of this thesis is to describe rigidity results for stable minimal hypersurfaces, in the Euclidean space. This problem has attracted a lot of interest in the Geometric Analysis community. In this chapter, three different mathematical abstractions of the concept of hypersurface will be introduced. The different properties of these approaches will be briefly mentioned together with the pros and cons of each. Then the area functional is introduced and the concepts of minimal, minimizing, and stable minimal hypersurfaces are presented.

1.1. Mathematical Description of Hypersurfaces

It is sought a description of n-dimensional orientable hypersurfaces in $\mathbb{R}^{n+k}$ which enables calculations and definition of functionals like the measure of area. A desired property would also be that the space arising from such a description had a nice topology so the tools of calculus of variations, like the direct method, may be used. The first thing that comes to mind is to use the concept of manifolds and submanifolds. Hence an n-dimensional orientable hypersurface would be identified with a smooth manifold (M, g) embedded in $\mathbb{R}^{n+k}$ where g is the metric induced by such embedding. With this description the concept of an orientable hypersurface is easily extended on any smooth ambient space, in fact given an $(n+k)$-manifold (N, h) then a smooth hypersurface in it can be described as (M, g) an n-dimensional submanifold of N where g is the metric induced by the embedding. This description is restricted to smooth hypersurfaces, hence excluding hypersurfaces with edges and singularities. In order to extend the description to singular hypersurfaces it must be introduced the concept of integer multiplicity currents, a subset of the dual space of smooth differential n-forms in $\mathbb{R}^{n+k}$. It will be only mentioned that they are in some sense integer multiplicity varifolds equipped with an orientation, hence to each integer multiplicity current, there corresponds a varifold. This description covers all orientable hypersurfaces given by rectifiable sets, that is subsets of $\mathbb{R}^{n+k}$ which are the union of C^1 hypersurfaces. With currents, the theory for non-flat ambient space is less natural and is approached by considering an $(n+k)$-dimensional submanifold N in $\mathbb{R}^{n+k'}$ and restricting the currents on it. The topology as dual space of a separable Banach space is exceptional for compactness, and in fact, with this description, an existence result for the area functional minimizer is achieved. In the case in which hypersurfaces of codimension one in flat ambient space are considered the tool of sets of finite perimeter can be used to describe boundless hypersurfaces, also in this case the topology of bounded variation function comes with good compactness properties, and in fact sets of finite perimeter coincide with codimension one currents that are the boundary of a set.

1.2. Minimal, Minimizing, and Stable Hypersurfaces

In all the descriptions of hypersurfaces presented above is possible to introduce the area functional. In the case of a manifold M, it will be given by the volume form as

$$\mathcal{A}(M) = \int_M \omega_g.$$

For current the dual norm of *continuous* differential form does the trick, notice that this gives rise to a different topology than the dual one, hence it is defined the mass of a current T with respect to the compact set W as

$$\mathbf{M}_W(T) = \sup_{\substack{|\omega|\leq 1 \\ \mathrm{supp}(\omega)\in W}} T(\omega)$$

which extend the concept of area. In the case of Caccioppoli sets the measure of area is given by the perimeter of the set.
Variations of the hypersurface can be introduced through diffeomorphisms. In particular, a family $f_t : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ of smooth compactly supported functions defined in a neighborhood of the hypersurface and with f_0 the identity, can be interpreted as a transformation of the ambient space which induce also a deformation of the hypersurface. Denoting formally with $S_t = f_t(S)$ the hypersurface S variated through the family $f_t = I + tX + o(t^2)$ with X a smooth vector field, compact and null on ∂S, it will be proved that

$$\frac{d}{dt}\mathcal{A}(S_t)|_{t=0} = \int_S \mathrm{div}(X)$$

where this integral in the case of currents represents the duality of the varifold's measure related to the current with the continuous function $\mathrm{div}(X)$, the same holds in the case of a set of finite perimeter with the measure associated with it. It is easily shown that the integral obtained can be used to extend the concept of mean curvature $\boldsymbol{H}$ specifically

$$\int_S \operatorname{div}(X) = -\int_S \langle X, \boldsymbol{H} \rangle \,,$$

from which it may be asserted that a null value for the first variation of the area is achieved in hypersurfaces with zero mean curvature. In an unhappy labeling choice, the stationary hypersurfaces for the area functional are called *minimal*, hence minimal hypersurfaces may be non-stable, let alone achieve the minimum value for the functional. When indeed the minimal value for the functional is achieved the hypersurface will be called *minimizing*. The main result of this topic is Bernstein's Theorem, which states that an n-dimensional hypersurface S without boundary and minimizing for any compact set in $\mathbb{R}^{n+1}$ is a plane for $n \leq 6$. Then the minimizing Simons cone

$$C_S = \{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

gives the counterexample for higher dimensions.
A natural generalization of the classical Bernstein problem is the *stable* Bernstein problem, that is if it's true that given $M \hookrightarrow \mathbb{R}^{n+1}$ a complete, orientable, isometrically immersed, stable minimal hypersurface of co-dimension one, then M is a hyperplane. In the case $n = 2$ the (positive) answer was given in three different papers, which appeared between 1979 and 1981 (see do Carmo and Peng [5], Fischer-Colbrie and Schoen [8] and Pogorelov [16], while under the condition of controlled volume growth Schoen, Simon and Yau [17]). Up until recently, without additional hypothesis, the remaining cases $3 \leq n \leq 6$ were still open, even if the study of minimal (in particular stable or in general with finite index) hypersurfaces immersed into a general Riemannian manifold (not only flat) is a very active field and has attracted a lot of interest. Then, in 2021, Chodosh and Li [3] (see also [4]) proved the flatness of M for $n = 3$ and, soon after, Catino, Mastrolia and Roncoroni [2] presented a completely different proof for the same result.
The concept of stable hypersurfaces is introduced through the second variation of the area functional, a minimal hypersurface S is stable if

$$\frac{d^2}{dt^2}\mathcal{A}(S_t)|_{t=0} \geq 0.$$

For this part, codimension one and the submanifold approach will be considered, with ambient space given by a general manifold. In such settings, the second variation of a minimal hypersurface takes the form

$$\frac{d^2}{dt^2}\mathcal{A}(S_t)|_{t=0} = \int_S |\nabla f|^2 - (|A|^2 + Ric(\nu,\nu))f^2$$

with A second fundamental form and Ric the Ricci tensor of the ambient manifold which is calculated along the hypersurface's normal ν. Note how the second variation corresponds to the energy of the operator

$$L_S = -\Delta f - (|A|^2 + Ric(\nu,\nu))f$$

and hence the stability condition corresponds to asking non-negative eigenvalues of such operator. The spectrum of L_S is countable and diverging to infinity, furthermore from the Morse Index Theorem, in the case of stable minimal hypersurfaces for any compact set in $\mathbb{R}^{n+1}$, the inequality for stability must be strict. It is then proved a result extensively used when studying problems of this kind, that is the existence of a strictly positive function $u > 0$ satisfying $L_S = 0$. The existence of such function u is used in [2] to show that the Bernstein theorem is extended to the stable case for $n = 3$, that is stable minimal hypersurfaces with no boundary in $\mathbb{R}^4$ are hyperplanes, the same method is here extended to $n = 2$. It is also proved a flatness result for $n < 6$ under the condition $|A|^2(x) \leq \lambda_1(-\Delta)$ which, if shown to be true a priori for stable hypersurfaces, would close the stable Bernstein problem apart from $n = 6$.

2 | Preliminaries

In the following will be introduced the concept of vector bundles and sections on them, these are of relevance as the variation of a manifold may be described by a compact section in the normal bundle. Following, it is constructed the Laplacian operator on sections of vector bundles, necessary to have a clear view of the operator L_S presented in the introduction. Then some useful formulas for tensor fields on submanifolds are proved. A quick introduction to differential forms is made as they are at the basis of the definition of currents.

2.1. Vector Bundles

Given a smooth manifold M it is possible to associate a real vector space to each point of M, in particular given a $p \in M$ there is $V_p \cong \mathbb{R}^k$ a vector space associated with p. As the name states the *Vector Bundle* V is the bundle of all such vector spaces, that is $V = \bigcup_{p \in M} V_p$, and each V_p is called the fiber of V in p. The dimension of the vector spaces determines the rank, in the case above V is a k-rank vector bundle. Notice that $V \cong M \times \mathbb{R}^k$, it is hence possible to construct an Atlas of coordinates system on V giving it the structure of a manifold[1]. Now that an idea of what vector bundles are let's give a rigorous definition.

Definition 2.1 (Vector Bundle). *A k-rank Vector Bundle is characterized by a manifold M called the base space of the bundle, a manifold V called the total space of the bundle, and a smooth surjective map $\pi : V \to M$ called the projection of the bundle, in relation with one another in the following way:*

- $\forall p \in M \quad \pi(p)^{-1} =: V_p$ *is a k-dimensional real vector space*

- $\forall p \in M \quad \exists U \ni p \quad \exists\ \Phi : \pi(U)^{-1} \to U \times \mathbb{R}^k$ *such that* $\pi_U : U \times \mathbb{R}^k \to U$ *the*

[1] Making all these steps rigorous holds the *Vector Bundle Chart Lemma* which is in general introduced after the rigorous definition of a Vector Bundle. Here it has been used before in order to give a more intuitive and geometric notion for Vector Bundle.

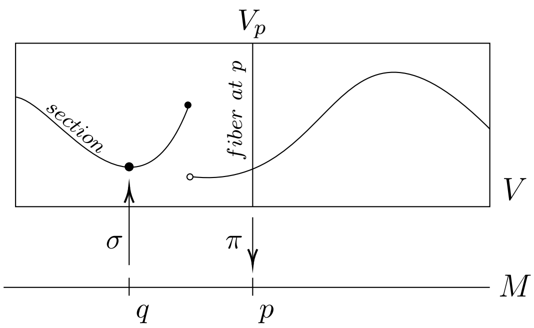
projection onto the first factor satisfies $\pi_U \circ \Phi = \pi$. That is it is required $\forall p$ the existence of a neighborhood U and a diffeomorphism Φ such that the diagram below commute.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

Furthermore $\forall p \in M \quad \Phi(p)$ is a linear isomorphism from V_p to $\{p\} \times \mathbb{R}^k$.

It is hence possible to construct the vector bundle arising from the tangent space T_pM of each point, from now on called TM. In the case of S being a submanifold of M for each point $p \in S \subset M$ the vector space T_pM can be decomposed into tangential T_pS and normal N_pS vector spaces of S. Specifically, T_pS is given by $\text{span}(di_p)$ with i the immersion map, the normal space is simply the orthogonal space of the tangent one. The definition of the normal bundle of a submanifold easily follows as $NS = \bigcup_{p \in S} N_pS$.[2]
The concept of vector fields may now be extended to vector bundles.

Definition 2.2 (Section of a Vector Bundle)**.** *Let V be a vector bundle on M, a section of V is a map (not necessarily continuous) $\sigma : M \to V$ such that $\sigma \circ \pi = Id_M$. That is the only requirement is that $\sigma(p) \in V_p$, i.e. the image lies in the corresponding fiber. A section is called smooth in the case σ is such, and the space of smooth sections over V is identified as $\mathfrak{X}(V)$.*



A section of a vector bundle V corresponds to a vector field over M associated with V. Posing $V = TM$ a section is just the usual concept of a tangential vector field. Less trivial examples in the embedded case $S \hookrightarrow M$ are normal vector fields, that is sections of $V = NS$, and also $V = TM\big|_S$ whose sections are vector fields on S with values in the

[2] An abuse of notation by not differentiating between S and its embedding has been used in order to convey more clearly the concept.

tangent space of M.
A vector bundle in which each fiber is equipped with an inner product $\langle \cdot, \cdot \rangle_V$ and a compatible connection ∇, takes the name *Riemannian vector bundle*. By compatible connection it is meant that the connection satisfies for any $p \in M$:

$$\forall x \in T_pM \ \forall \psi, \phi \in \mathfrak{X}(V) \quad x(\langle \psi_p, \phi_p \rangle) = \langle \nabla_x \psi, \phi_p \rangle + \langle \psi_p, \nabla_x \phi \rangle$$

The exact notation should be $(\nabla_x \phi)_p$ but given that from $x \in T_pM$ the point of calculation is clear it is dropped in the notation of the connection. To understand what kind of object ∇ is consider fixing $\psi \in \mathfrak{X}(V)$ and a point p so:

$$(\nabla_{(\cdot)} \psi)_p \in \mathcal{L}(T_pM, V_p) \tag{2.1}$$

hence considering the section as input

$$(\nabla(\cdot))_p : \mathfrak{X}(V) \to \mathcal{L}(T_pM, V_p)$$

at last introducing the vector bundle $\mathcal{L}(TM, V) = \bigcup_{p \in M} \mathcal{L}(TM, V)_p = \bigcup_{p \in M} \mathcal{L}(TM_p, V_p)$ is obtained

$$\nabla(\cdot) : \mathfrak{X}(V) \to \mathfrak{X}(\mathcal{L}(TM, V))$$

Being a derivation, a further property that a connection must satisfy is the product rule, that is given $f \in C^\infty(M)$ and $\psi \in \mathfrak{X}(V)$:

$$\nabla(f\psi) = df(\cdot)\psi + f\nabla\psi$$

this is the most general way to treat the connection.

Definition 2.3 (Riemannian Vector Bundle)**.** *A vector bundle in which each fiber is equipped with an inner product $\langle \cdot, \cdot \rangle$ and there exists a connection*

$$\nabla : \mathfrak{X}(V) \to \mathfrak{X}(\mathcal{L}(TM, V))$$

satisfying

$$\forall p \in M \ \forall x \in T_pM \ \forall \psi, \phi \in \mathfrak{X}(V) \quad x(\langle \psi_p, \phi_p \rangle) = \langle \nabla_x \psi, \phi_p \rangle + \langle \psi_p, \nabla_x \phi \rangle$$

and

$$\forall f \in C^\infty(M) \ \forall \psi \in \mathfrak{X}(V) \quad \nabla(f\psi) = df(\cdot)\psi + f\nabla\psi$$

is called Riemannian vector bundle.

Given that $\nabla\psi \in \mathfrak{X}(\mathcal{L}(TM, V))$ for $\psi \in \mathfrak{X}(V)$, are now introduced the natural scalar product and connection on $\mathcal{L}(\otimes^m TM, V)$. Let $s, r \in \mathcal{L}(\otimes^m TM, V)_p$, choosing a coordinates system $\{e_i\}$ in T_pM the scalar product is given by:

$$\langle s, r\rangle = g^{i_1 j_1} \cdots g^{i_m j_m} \langle s(e_{i_1}, ..., e_{i_m}), r(e_{j_1}, ..., e_{j_m})\rangle \tag{2.2}$$

where the scalar product on the right is evidently in V.
The connection will have the form

$$\nabla : \mathfrak{X}(\mathcal{L}(\otimes^m TM, V)) \to \mathfrak{X}(\mathcal{L}(TM, \mathcal{L}(\otimes^m TM, V))) \equiv \mathfrak{X}(\mathcal{L}(\otimes^{m+1} TM, V))$$

and letting $\mathcal{H} \in \mathfrak{X}(\mathcal{L}(\otimes^m TM, V))$, $x \in T_pM$ and $Y_i \in \mathfrak{X}(TM)$ the connection in p is defined as[3]:

$$\nabla_x(\mathcal{H})(Y_1, ..., Y_m) = \nabla_x(\mathcal{H}(Y_1, ..., Y_m)) - \sum_i \mathcal{H}(Y_1, ..., \nabla_x Y_i, ..., Y_m) \tag{2.3}$$

It is important to notice that the dependence on Y_i is actually only on $(Y_i)_p$, just passing in coordinates makes this clear. The two definitions are in line with the ones for tensors, which are indeed retrieved by posing $V = \mathbb{R}$. An interesting way to see the definition of connection is to notice that it is in fact the product rule for the connection in V, in fact by rearranging the terms:

$$\nabla_x(\mathcal{H}(Y_1, ..., Y_m)) = \nabla_x(\mathcal{H})(Y_1, ..., Y_m) + \sum_i \mathcal{H}(Y_1, ..., \nabla_x Y_i, ..., Y_m) \tag{2.4}$$

and this is precisely how the derivative of a linear function is expected to behave, in fact by passing to coordinates it is evident that defining the connection in this way means precisely asking for the product rule to hold.

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[3]The symbol ∇_x is used unchanged to indicate connections in different spaces because the argument specifies unambiguously the space on which the connection operates.

2.2. Hessian

It has been presented the Riemannian structure for the vector bundle $\mathcal{L}(TM, V)$ in which $\nabla\psi$ with $\psi \in V$ lies. Applying the connection in $\mathcal{L}(TM, V)$ on $\nabla\psi$ holds:

$$\nabla(\nabla\psi) : \mathfrak{X}(TM) \to \mathfrak{X}(\mathcal{L}(TM, V)) \tag{2.5}$$

$$\nabla_x(\nabla\psi)(Y) = \nabla_x(\nabla_Y\psi) - \nabla_{\nabla_x Y}\psi \tag{2.6}$$

again the dependence on Y is limited to $Y_p = y$. It can hence be defined in a straightforward way as a bilinear form $\nabla_{x,y}(\psi) \coloneqq \nabla_x(\nabla\psi)(y)$ which is the Hessian of ψ evaluated along x and y.

Definition 2.4 (Hessian of Sections). *Let $\psi \in \mathfrak{X}(V)$ then the Hessian is the bilinear form obtained by iterative application of the connection first in V then in $\mathcal{L}(TM, V)$ holding*

$$\nabla^2\psi = \nabla_{\cdot,\cdot}\psi \in \mathcal{L}(TM \otimes TM, V) \tag{2.7}$$

$$\nabla_{X,Y}\psi = \nabla_X(\nabla_Y\psi) - \nabla_{\nabla_X Y}\psi \tag{2.8}$$

where $X, Y \in \mathfrak{X}(TM)$.

The definition of Laplacian of sections is now trivial

Definition 2.5 (Laplacian of Sections). *Let $\psi \in \mathfrak{X}(V)$ then the Laplacian $\Delta\psi \in \mathfrak{X}(V)$ is given by the trace of the Hessian*

$$\Delta : \mathfrak{X}(V) \to \mathfrak{X}(V) \tag{2.9}$$

$$(\Delta\psi)_p = \mathrm{Tr}(\nabla^2\psi)_p = g^{ij}\nabla_{e_i,e_j}\psi \tag{2.10}$$

where it has been introduce a basis $\{e_i\}$ for T_pM.

Also, for the Laplacian on sections of vector bundles holds a result analogous to the Green formula

Proposition 2.1. *Let $\psi, \varphi \in V(M)$ if either M has no boundary or ψ and φ vanish on it then*

$$\int_M \langle \Delta\psi, \varphi \rangle = \int_M \langle \Delta\varphi, \psi \rangle = -\int_M \langle \nabla\varphi, \nabla\psi \rangle \tag{2.11}$$

implying that the Laplacian is a symmetric and negative operator in the sections.

Proof. To simplify the calculations notice that taking a parallel orthonormal frame $\{E_i\}$

satisfying $\nabla_{E_i}E_j = 0$ in p for any i, j then (2.8) loses the second term on the right-hand side hence

$$\langle \Delta\psi, \varphi\rangle = \sum_i \langle \nabla_{E_i}\nabla_{E_i}\psi, \varphi\rangle = \sum_i \nabla_{E_i}\langle \nabla_{E_i}\psi, \varphi\rangle - \langle \nabla_{E_i}\psi, \nabla_{E_i}\varphi\rangle \tag{2.12}$$

by (2.2) with the orthonormal frame the second term corresponds with $-\langle \nabla\psi, \nabla\varphi\rangle$ while the first is the codifferential of the one form $\omega = \langle \nabla_{(\cdot)}\psi, \varphi\rangle$ hence for Stokes Theorem

$$\int_M \langle \Delta\psi, \varphi\rangle = -\int_M \langle \nabla\psi, \nabla\varphi\rangle + \int_{\partial M} \omega^* \tag{2.13}$$

but by hypothesis, ω^* is null at the boundary. □

This discussion on vector bundles encloses also the cases of Tensor Bundles, Vector Fields, and Co-Vector Fields, it is enough to pose $V = \mathbb{R}$, and up to raising the indices all these cases are covered. From the definition in (2.3) it is obtained a product rule for the tensor product. In fact, given T and F two tensor fields and representing with v, w the respective inputs and $\nabla v, \nabla w$ the sum of the connection applied once for every respective input, then

$$\begin{aligned} \nabla_x(T \otimes F)(v, w) &= x(T(v)F(w)) - T(\nabla_x v)F(w) - T(v)F(\nabla_x w) = \\ &= \nabla_x T \otimes F(v, w) + T \otimes \nabla_x F(v, w) \end{aligned} \tag{2.14}$$

Furthermore, if the Levi Civita connection is used in TM then the metric tensor has a null covariant derivative, implying that the connection commute with the trace

$$\nabla_x g(v, w) = x(g(v, w)) - g(\nabla_x v, w) - g(v, \nabla_x w) = 0 \tag{2.15}$$

Hence it is retrieved the unique covariant derivative for tensors presented in [13].

Hessian Commutation

Being known that, for smooth scalar functions in flat space, the Hessian is symmetric, it is spontaneous the question of how the Hessian commutes in the curved case and on vectors, co-vectors, and tensors. The results of these commutations are known as Ricci Identities.

Starting with $u \in C^\infty(M)$ and using the torsion-free property of the Levi-Civita connection:

$$\begin{aligned} &\nabla_{x,y}(u) - \nabla_{y,x}(u) = \nabla_x(\nabla_y u) - \nabla_{\nabla_x Y} u - \nabla_y(\nabla_x u) + \nabla_{\nabla_y X} u = \\ &x(y(u)) - y(x(u)) - \nabla_x Y(u) + \nabla_y X(u) = [x, y](u) - [x, y](u) = 0 \end{aligned}$$

as expected the Hessian of a scalar function commute. Going forward calculations in coordinates will be used for clarity, the notation used is:

$$\nabla_{j,i}(\cdot) = (\cdot)_{,ij} \tag{2.16}$$

notice that the order is inverted, as after the comma the derivation coordinates are in the order of execution. Furthermore the notation $(\cdot)_{,i}$ identifies the connection as element of $\mathcal{L}(TM, \cdot)$. With this notation, the commutation of the Hessian for scalar functions becomes:

$$u_{,ij} - u_{,ji} = 0 \tag{2.17}$$

For vector fields $V \in \mathfrak{X}(TM)$ the commutation gives the definition of curvature tensor:

$$\nabla_{x,y}(V) - \nabla_{x,y}(V) = \nabla_x(\nabla_y V) - \nabla_y(\nabla_x V) - \nabla_{[x,y]}V = R(x,y)V \tag{2.18}$$

which in coordinates notation becomes

$$V_{,ij}{}^{k} - V_{,ji}{}^{k} = -R_{ijm}{}^{k}V^{m} \tag{2.19}$$

the minus sign in the coordinates formula is due to the fact that, between the two notations, the order of derivatives is inverted. Hence to keep the same order of i and j in the curvature tensor the two indices have been swapped and being R minor antisymmetric a negative is added.
The case of covectors is the more contrived because the connection in the dual space is again defined by Formula(2.3) and must hence be used twice together also with its reordering Formula(2.4), so for $\alpha \in T^*M$ and $z \in T_pM$:

$$\begin{aligned}
&\nabla_{x,y}(\alpha)(z) = x(\nabla_y(\alpha)(z)) - \nabla_{\nabla_x Y}(\alpha)(z) - \nabla_y(\alpha)(\nabla_x z) = \\
&x(y(\alpha(z)) - \alpha(\nabla_y Z)) - \nabla_x Y(\alpha(z)) + \alpha(\nabla_{\nabla_x Y} Z) - \nabla_y(\alpha)(\nabla_x z) = \\
&xy(\alpha(z)) - \nabla_x(\alpha)(\nabla_y Z) - \alpha(\nabla_x \nabla_y Z) - \nabla_x Y(\alpha(z)) + \alpha(\nabla_{\nabla_x Y} Z) - \nabla_y(\alpha)(\nabla_x z)
\end{aligned}$$

now inverting the role of x and y and subtracting from the above, most term cancels out and it is obtained:

$$\begin{aligned}
&\nabla_{x,y}(\alpha)(z) - \nabla_{y,x}(\alpha)(z) = -\alpha(R(x,y)Z) \\
&\alpha_{k,ij} - \alpha_{k,ji} = \alpha_m R_{ijk}{}^{m}
\end{aligned}$$

The formula found so far will be used to find the commutation formula for general tensors. Formula (2.3) applied to the connection in the tensor bundle V becomes for any $T \in \mathfrak{X}(V)$:

$$\nabla_{x,y}T = \nabla_x(\nabla_y T) - \nabla_{\nabla_x y}T$$

Furthermore from the product rule with the tensor product it is a matter of calculations to show that:

$$\nabla_{x,y}(T \otimes F) - \nabla_{y,x}(T \otimes F) = (\nabla_{x,y} - \nabla_{y,x})(T) \otimes F + T \otimes (\nabla_{x,y} - \nabla_{y,x})(F)$$

Hence representing a $\binom{k}{l}$-tensor as $V_1 \otimes \cdots \otimes V_k \otimes \omega^1 \otimes \cdots \otimes \omega^l$ it holds:

$$\nabla_{x,y}T - \nabla_{y,x}T = R(x,y)V_1 \otimes \cdots \otimes \omega^l + ... + V_1 \otimes \cdots \otimes (-R(x,y) * \omega^1) \otimes \cdots \omega^l + ...$$

where $R(x,y) * \omega$ is the 1-form defined by $(R(x,y) * \omega)(z) = \omega(R(x,y)Z)$. Written in coordinates

$$T^{i_1 \dots i_k}_{j_1 \dots j_k, pq} - T^{i_1 \dots i_k}_{j_1 \dots j_k, qp} = -R_{pqm}{}^{i_1} T^{m \dots i_k}_{j_1 \dots j_k} - ... + R_{pqj_1}{}^{m} T^{i_1 \dots i_k}_{m \dots j_k} + ... \tag{2.20}$$

This concludes the discussion on the Hessian commutativity.

2.3. Second Fundamental Form

Let $S \hookrightarrow M$ be a submanifold of M. It has already been discussed how T_pM can be split into T_pS and N_pS, hence for $X, Y \in \mathfrak{X}(TM)$ and $p \in S$

$$\nabla^M_X(Y)_p = (\nabla^M_X(Y)_p)^\top + (\nabla^M_X(Y)_p)^\perp \tag{2.21}$$

which is the decomposition into T_pS component and N_pS component. When the two sections X, Y are taken such that they have only the T_pS component for any $p \in M$ then (2.21) does not depend on the values of X, Y outside of S and it can be shown that $(\nabla^M_X(Y)|_S)^\top$ is the unique Levi-Civita connection consistent with the induced metric. The second term in (2.21) is the so-called second fundamental form.

Definition 2.6 (Second Fundamental Form)**.** *Let $S \hookrightarrow M$ be a submanifold of M, then the mapping $A : \mathfrak{X}(TS) \times \mathfrak{X}(TS) \to \mathfrak{X}(NS)$ defined as:*

$$A(X,Y) = (\nabla^M_X Y|_S)^\perp \tag{2.22}$$

is a bilinear form and called second fundamental form.
The definition is well-posed as the following properties show

- *$A(X,Y) = A(Y,X)$ in fact $A(X,Y) - A(Y,X) = [X,Y]^\perp = 0$ because $[X,Y] \in TS$.*
- *Thanks to the linearity of $\nabla^M_{(\cdot)}$ the map A is linear in the first argument. Symmetry*

implies linearity also on the second argument, proving bilinearity of A.

- *Dependence only on X_p and Y_p, proved in an analogous fashion of bilinearity, the dependence on just X_p is induced by the ∇^M, symmetry provides the conclusion also for the second argument.*

After this discussion (2.21) for $X, Y \in \mathfrak{X}(TS)$ can be rewritten as

$$\nabla^M_X Y = \nabla_X Y + A(X,Y) \tag{2.23}$$

From the second fundamental form, two other objects are derived. First a bilinear form with values in the real numbers $A^N(X,Y) : \mathfrak{X}(TS) \times \mathfrak{X}(TS) \to C^\infty(S)$ with $N \in NS$ defined as

$$A^N(X,Y) = \langle A(X,Y), N \rangle \tag{2.24}$$

and the linear operator associated to this bilinear form $\mathcal{W}^N : \mathfrak{X}(TM) \to \mathfrak{X}(TM)$ that is

$$\langle \mathcal{W}^N(X), Y \rangle = A^N(X,Y) = \langle A(X,Y), N \rangle \tag{2.25}$$

which is called Wirtinger operator. This operator is easily characterized, in fact

$$0 = X(\langle Y, N \rangle) = \langle \nabla^M_X Y, N \rangle + \langle Y, \nabla^M_X N \rangle \tag{2.26}$$

holding

$$\mathcal{W}^N(X) = -(\nabla_X N)^\top \tag{2.27}$$

notice that $\nabla_X N$ is orthogonal to N hence in codimension one the projection can be omitted, in fact, $1 = \langle N, N \rangle$ hence $0 = X(\langle N, N \rangle) = 2\langle \nabla_X N, N \rangle$. Also in the case of co-dimension one, being M orientable, the distinction between A and A^N vanishes as NM is identifiable with $M \times \mathbb{R}$.

2.4. Main equations and inequalities

In the following are recalled some useful formulas in Riemannian Geometry which are going to be used frequently. In particular are presented and proved: the Bochner formula, the Gauss equation, the Codazzi equation, the improved Kato inequality, the Simons' equality and the Simon's inequality. With M will be identified a Riemannian manifold.

Theorem 2.1 (Bochner Formula). *Let $u \in C^\infty(M)$ then*

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\nabla^2 u|^2 + \langle \nabla(\Delta u), \nabla u\rangle + Rc(\nabla u, \nabla u) \tag{2.28}$$

Proof. let us proceed in coordinates, furthermore, it is clearer if the metric is not written (being in no way relevant for calculation):

$$\begin{aligned}\frac{1}{2}(u_{,i}u_{,i})_{,kk} &= (u_{,ik}u_{,i})_{,k} = u_{,ikk}u_{,i} + u_{,ik}u_{,ik} \\ &= u_{,kik}u_{,i} + |\nabla^2 u|^2 = u_{,kki}u_{,i} + R_{ikk}{}^{m}u_{,m}u_{,i} + |\nabla^2 u|^2\end{aligned} \tag{2.29}$$

in the first and second equalities trace commutativity with the connection and product rule for tensor product has been used; the third equality is the definition of metric in tensor space and the commutation of scalar function's Hessian; the fourth is the Hessian commutation formula of 1-form. The symmetries of curvature tensor give that $R_{ikk}{}^{m} = R_{im}$ (disregarding again the metric tensor), achieving the thesis. □

The following results are regarding embedded submanifolds. So $S \hookrightarrow M$ will be equipped with the induced metric, furthermore, the objects in M will be denoted with a tilde. When a connection in NS is needed it will be used the one given by $(\widetilde{\nabla}|_S)^\perp$.

Theorem 2.2 (Gauss Equation). *The relation between the ambient and the submanifold curvatures is given by Gauss Equation, so for all $W, X, Y, Z \in \mathfrak{X}(TS)$:*

$$Rm(W,X,Y,Z) = \widetilde{Rm}(W,X,Y,Z) + \langle A(W,Z), A(X,Y)\rangle - \langle A(W,Y), A(X,Z)\rangle \tag{2.30}$$

Proof. Using (2.23)

$$\begin{aligned}&\widetilde{Rm}(W,X,Y,Z) = \langle \widetilde{\nabla}_W\widetilde{\nabla}_X Y - \widetilde{\nabla}_X\widetilde{\nabla}_W Y - \widetilde{\nabla}_{[W,X]}Y, Z\rangle = \\ &\langle \widetilde{\nabla}_W(\nabla_X Y + A(X,Y)) - \widetilde{\nabla}_X(\nabla_W Y + A(W,Y)) - \widetilde{\nabla}_{[W,X]}Y, Z\rangle\end{aligned}$$

Notice that

$$\langle \widetilde{\nabla}_W A(X,Y), Z\rangle = \langle -\mathcal{W}^{A(X,Y)}(W), Z\rangle = -\langle A(X,Y), A(W,Z)\rangle \tag{2.31}$$

hence

$$\langle \widetilde{\nabla}_W(\nabla_X Y) - \widetilde{\nabla}_X(\nabla_W Y) - \widetilde{\nabla}_{[W,X]}Y, Z\rangle - \langle A(X,Y), A(W,Z)\rangle + \langle A(W,Y), A(X,Z)\rangle$$

which is the thesis because $\langle \widetilde{\nabla}_X(\cdot), Z\rangle = \langle \nabla_X(\cdot), Z\rangle$ with $Z \in \mathfrak{X}(TS)$. $\square$

Theorem 2.3 (Codazzi Equation). *For all $W, X, Y \in \mathfrak{X}(TS)$ it holds*

$$(\widetilde{R}(W,X)Y)^{\perp} = (\nabla_W A)(X,Y) - (\nabla_X A)(W,Y) \tag{2.32}$$

Proof. Proved by considering the product with any $N \in NM$:

$$\widetilde{Rm}(W,X,Y,N) = \langle \widetilde{\nabla}_W \widetilde{\nabla}_X Y - \widetilde{\nabla}_X \widetilde{\nabla}_W Y - \widetilde{\nabla}_{[W,X]} Y, N\rangle =$$
$$\langle \widetilde{\nabla}_W (\nabla_X Y + A(X,Y)) - \widetilde{\nabla}_X (\nabla_W Y + A(W,Y)) - \widetilde{\nabla}_{[W,X]} Y, N\rangle$$

Notice that $\langle \widetilde{\nabla}_W A(X,Y), N\rangle = \langle \nabla_W A(X,Y), N\rangle$ connection in NM and $\langle \widetilde{\nabla}_W (\nabla_X Y), N\rangle = \langle A(W, \nabla_X Y), N\rangle$

$$\langle A(W,\nabla_X Y) + \nabla_W A(X,Y) - A(X,\nabla_W Y) - \nabla_X A(W,Y) - A([W,X],Y), N\rangle$$

from Formula(2.4)

$$\langle (\nabla_W A)(X,Y) - (\nabla_X A)(W,Y) + A(\nabla_W X, Y) - A(\nabla_X W, Y) - A([W,X],Y), N\rangle$$

and $A(\nabla_W X, Y) - A(\nabla_X W, Y) - A([W,X],Y) = 0$ gives the thesis. $\square$

Notice that in a flat ambient space, the Codazzi equation states the full symmetry of the tensor $h_{ij,k}$ where $h_{ij} = A(e_i, e_j)$. Two symmetric tensors with such full symmetries are called Codazzi tensors.

The next results are set in a flat ambient space and for co-dimension one.

Theorem 2.4 (Improved Kato inequality). *Given b a symmetric, Codazzi, trace-free tensor in $S \hookrightarrow \mathbb{R}^{n+1}$ submanifold of co-dimension one then:*

$$|\nabla |b||^2 \leq \frac{n}{n+2} |\nabla b|^2 \tag{2.33}$$

furthermore the constant is optimal.

Proof. From chain rule

$$\nabla |b| = \frac{1}{2|b|} \nabla(|b|^2) \tag{2.34}$$

hence it will be proved

$$\frac{1}{4}|\nabla(|b|^2)|^2 \leq \frac{n}{n+2}|b|^2|\nabla b|^2 \tag{2.35}$$

passing now to coordinates that diagonalize b, considering the left hand side:

$$|\nabla(|b|^2)|^2 = (b_{ij}b_{ij})_{,k}(b_{pq}b_{pq})_{,k} = 2b_{ij,k}b_{ij}2b_{pq,k}b_{pq} = 4\sum_k\left(\sum_i b_{ii,k}b_{ii}\right)^2 \tag{2.36}$$

considering the right-hand side, thanks to Codazzi and symmetry the indices may commute:

$$|\nabla b|^2 = b_{ij,k}b_{ij,k} = \sum_k {b_{kk,k}}^2 + \sum_{k,i\neq k}\left({b_{ik,k}}^2 + {b_{ki,k}}^2 + {b_{kk,i}}^2\right) + \sum_{i\neq k\neq j} {b_{ij,k}}^2 \tag{2.37}$$

basically, it has been separated the sum in three parts: the one in which all the indices coincide; the second where two indices coincide; and at last when no indices coincide. Hence Inequality(2.35) is satisfied if:

$$\sum_k\left(\sum_i b_{ii,k}b_{ii}\right)^2 \leq \frac{n}{n+2}{b_{ii}}^2\sum_k\left({b_{kk,k}}^2 + 3\sum_{i\neq k}{b_{ii,k}}^2\right) \tag{2.38}$$

naming $b_{ii} = x_i$ and $b_{ii,k} = y_i$, notice free trace implies $\sum y_i = \sum x_i = 0$, the inequality would be achieved if, for every k:

$$\frac{n+2}{n} \leq \frac{\sum x_i^2\left(y_k^2 + 3\sum_{i\neq k} y_i^2\right)}{\left(\sum y_i x_i\right)^2} \tag{2.39}$$

$$\sum y_i = 0 \implies y_k^2 = \left(\sum_{i\neq k} y_i\right)^2 \leq (n-1)\sum_{i\neq k} y_i^2 \tag{2.40}$$

where the last inequality is Jensen applied to x^2.
Now separating $3\sum_{i\neq k} y_i^2 = (3-t)\sum_{i\neq k} y_i^2 + t\sum_{i\neq k} y_i^2$ choosing $t = \frac{n+2}{n}$ holds $3-t = 2\frac{n-1}{n}$ hence from the Jensen inequality

$$\frac{n+2}{n}\frac{\sum x_i^2\sum y_i^2}{\left(\sum y_i x_i\right)^2} \leq \frac{\sum x_i\left(y_k^2 + 3\sum_{i\neq k} y_i^2\right)}{\left(\sum y_i x_i\right)^2} \tag{2.41}$$

and Cauchy Schwarz conclude the proof $\left(\sum y_i x_i\right)^2 \leq \sum x_i^2 \sum y_i^2$. □

Theorem 2.5 (Simons' Equality). *If S is a minimal hypersurface then the second form* *is a trace-free Codazzi tensor and satisfies:*

$$\frac{1}{2}\Delta|A|^2 + |A|^4 = |\nabla A|^2 \tag{2.42}$$

Proof.

$$\frac{1}{2}\Delta|A|^2 = \frac{1}{2}(h_{ij}h_{ij})_{,kk} = h_{ij,kk}h_{ij} + h_{ij,k}h_{ij,k} \tag{2.43}$$

and if $h_{ij,kk} = -(h_{pq}h_{pq})h_{ij}$ the thesis holds:

$$\frac{1}{2}\Delta|A|^2 = -(h_{pq}h_{pq})h_{ij}h_{ij} + h_{ij,k}h_{ij,k} = -|A|^4 + |\nabla A| \tag{2.44}$$

using Codazzi (full symmetry of $h_{ij,k}$) and the commutativity of the Hessian for tensors:

$$h_{ij,kk} = h_{ik,jk} = h_{ik,kj} + R_{jki}{}^{m}h_{mk} + R_{jkk}{}^{m}h_{im} \tag{2.45}$$

Using Gauss Equation $R_{jkim} = h_{jm}h_{ki} - h_{ji}h_{km}$ and using coordinates that diagonalize h:

$$h_{ij,kk} = h_{kk,ij} + (h_{jk}h_{ki} - h_{ji}h_{kk})h_{kk} + (h_{jm}h_{kk} - h_{jk}h_{km})h_{mi} \tag{2.46}$$

using $h_{ii} = 0$

$$h_{ij,kk} = h_{jk}h_{ki}h_{kk} - h_{ji}h_{kk}{}^2 - h_{jk}h_{km}h_{mi} \tag{2.47}$$

but $h_{jk}h_{km}h_{mi} = h_{jk}h_{kk}h_{ki}$ hence it remains only

$$h_{ij,kk} = -h_{ji}h_{kk}{}^2 \tag{2.48}$$

which is what was left to prove written in diagonal coordinates. □

Theorem 2.6 (Simons' Inequality)**.** *The improved Kato inequality together with Simons' Equality holds the so-called Simons' Inequality*

$$|A|\Delta|A| + |A|^4 \geq \frac{2}{n}|\nabla|A||^2 \tag{2.49}$$

Proof. Applying Kato inequality to the right-hand side of Simons' Equality

$$|\nabla|A||^2 \geq \frac{n+2}{n}|\nabla|A||^2 \tag{2.50}$$

to alleviate the notation consider $f = |A|$, then

$$\frac{1}{2}\Delta(f^2) = \frac{1}{2}(f^2)_{,jj} = \frac{1}{2}(2ff_{,j})_{,j} = \sum_j (f_{,j})^2 + ff_{,jj} \tag{2.51}$$

which inserted into Simons's equality together with the bound given by Kato holds the result. □

2.5. Differential Forms

Another fundamental vector bundle on a manifold M is generated by the space of k-covectors in T_pM denoted $\bigwedge^k(T_p^*M)$, in the case of real manifolds they can be identified with fully antisymmetric covariant tensors. In this case, the operation corresponding to the tensor product is the wedge product $\wedge$ which is obtained by antisymmetrization of the tensor product, so for ω a k-covector and η an l-covector $\omega \wedge \eta \in \bigwedge^{(k+l)}(T_p^*M)$. Hence as for tensors, a basis can be introduced starting from a dual basis of the tangent space $\{\epsilon^i\}$ and $\omega \in \bigwedge^k(T_p^*M)$ can be represented as

$$\omega = \sum_{\alpha \in I} a_\alpha \epsilon^\alpha \tag{2.52}$$

where $I = \{(i_1, i_2, ..., i_k) \in \mathbb{N}^k : 1 \leq i_1 < i_2 < ... < i_k \leq n\}$, $\epsilon^\alpha = \epsilon^{i_1} \wedge ... \wedge \epsilon^{i_k}$ and $a_\alpha \in \mathbb{R}$ $\forall \alpha \in I$. Einstein convention can be used also with multi-indices holding $\omega = a_\alpha \epsilon^\alpha$. A scalar product in $\bigwedge^k(T_p^*M)$ is induced by requiring the basis $\{\epsilon^\alpha\}$ to be orthonormal whenever $\{\epsilon^i\}$ are orthonormal. The same discussion can be done for k-vectors which corresponds to antisymmetric contravariant tensors, this space is denoted $\bigwedge^k(T_pM)$. It is easy to see that k-vectors and k-covectors are dual to each other. Having a vector space defined on each $p \in M$ a vector bundle can be generated through the union, the bundle of k-covectors is $\bigwedge^k(T^*M)$ and smooth sections on it are called differential k-forms and denoted with $\Omega^k(M) = \mathfrak{X}(\bigwedge^k(T^*M))$. Two operations on $\Omega^k(M)$ are needed for later discussions and are introduced here. The exterior derivative introduces a differentiation in $\Omega^k(M)$.

Definition 2.7 (Exterior Derivative). *Let $\{x^i\}$ be a coordinate system in M then $\{\frac{\partial}{\partial x^i}\}$ defines a basis in TM and $\{dx^i\}$ a basis in T^*M, the exterior derivative of $\omega \in \Omega^k(M)$ is then defined as*

$$d\omega = \sum_{\alpha \in I} \frac{\partial a_\alpha}{\partial x^i} dx^i \wedge dx^\alpha \tag{2.53}$$

where it is used Einstein convention over i. Notice that for every i and j fixed

$$\frac{\partial^2 a_\alpha}{\partial x^i x^j} dx^i \wedge dx^j = -\frac{\partial^2 a_\alpha}{\partial x^j x^i} dx^j \wedge dx^i \tag{2.54}$$

implying

$$dd\omega = 0 \tag{2.55}$$

The second operation is the pullback which is simply induced by the pullback of one form.

Definition 2.8 (Differential Forms Pullback). *Let M and N be two manifolds and $f : M \to N$ a smooth mapping between them. Consider the coordinates $\{y^i\}$ in N and $a_\alpha dy^\alpha = \omega \in \Omega^k(N)$ then the pullback $f^\# \omega \in \Omega^k(M)$ is given by*

$$f^\# \omega = \sum_{\alpha \in I} a_\alpha \circ f \; dy^{i_1}(df(\cdot)) \wedge ... \wedge dy^{i_k}(df(\cdot)) \tag{2.56}$$

notice that in the case in which $N = M$ and ω is multiple of the volume form

$$dy^{i_1}(df(\cdot)) \wedge ... \wedge dy^{i_k}(df(\cdot)) = \det(f) dy^{i_1} \wedge ... \wedge dy^{i_k}$$

For the definition of currents it is necessary a set of sections smaller than $\Omega^k(M)$, that is the space of smooth k-forms with compact support in $U \subset M$ open, this new space of sections in $\bigwedge^k(T^*M)$ will be denoted $\mathscr{D}^n(U)$. In coordinates a section $\omega \in \mathscr{D}^n(U)$ is characterized as

$$\omega = \sum_{\alpha \in I} a_\alpha \epsilon^\alpha \quad \text{where} \quad a_\alpha \in C_c^\infty(U) \quad \forall \alpha \in I \tag{2.57}$$

This space of sections is equipped with a topology that extends the one of smooth functions with compact support.

Definition 2.9 (Topology of $\mathscr{D}^n(U)$). *Let $\omega^{(k)} \in \mathscr{D}^n(U)$ be a sequence in $\mathscr{D}^n(U)$ then*

$$\omega^{(k)} \to \omega \iff \begin{array}{l} \exists K \Subset U : \operatorname{spt}(a_\alpha^{(k)}) \subset K \quad \forall \alpha, k \\ D^\beta a_\alpha^{(k)} \to D^\beta a_\alpha \text{ uniformly} \quad \forall \alpha, \beta \end{array} \tag{2.58}$$

where β is a multi-index of any order.

This concludes all the notions of differential forms necessary to introduce the currents.

3 | Hypersurfaces

This chapter aims to introduce the different descriptions of hypersurfaces, in particular, to present the concept of currents and how they can describe hypersurfaces and the family of Caccioppoli sets whose boundaries define a hypersurface. The case of submanifold will not be discussed, it is evident that the embedding induces a metric from which a volume form is defined giving the area functional, and all the tools presented in the preliminaries can be used to analyze them.

3.1. Introduction to Currents

The main reference of this introduction is "Lectures on Geometric measure Theory" by Simon Leon [18]. Let us consider the setting of n-dimensional hypersurfaces in $\mathbb{R}^{n+k}$. The goal is to generalize the concept of smooth manifolds to describe a much greater set of embedded hypersurfaces with singularities. At the basis of this theory lies the concept of countably n-rectifiable sets.

Definition 3.1 (Countably n-rectifiable set). *A set $M \subset \mathbb{R}^{n+k}$ is countably n-rectifiable if there exists a countable family of Lipschitz functions $F_j : \mathbb{R}^n \to \mathbb{R}^{n+k}$, $j \in \mathbb{N}$, such that*

$$\mathcal{H}^n\left(M \setminus \bigcup_{j\in\mathbb{N}} F_j(\mathbb{R}^n)\right) = 0, \tag{3.1}$$

where $\mathcal{H}$ denotes the Hausdorff measure.
Thanks to the approximation of Lipschitz functions through C^1 functions an analogous definition of countably n-rectifiable set is

$$\mathcal{H}^n\left(M \setminus \bigcup_{j\in\mathbb{N}} N_j\right) = 0, \tag{3.2}$$

where N_j is a countable family of C^1 n-dimensional submanifolds. of $\mathbb{R}^{n+k}$.

This new tool enables the description of immersed hypersurfaces with singularities while maintaining the properties of C^1 submanifolds $\mathcal{H}^n$-almost everywhere. In fact, an important characterization of countably n-rectifiable sets is through approximate tangent space which is now defined

Definition 3.2 (Tangent space). *Let $M \subset \mathbb{R}^{n+k}$ be an $\mathcal{H}^n$-measurable set, with $\mathcal{H}^n(M \cap K) < \infty\ \forall K \Subset \mathbb{R}^{n+k}$, considering the homothety $\eta_{x,\lambda}(y) = (x-y)/\lambda$, the n-dimensional subspace $P \subset \mathbb{R}^{n+k}$ such that*

$$\lim_{\lambda \to 0} \int_{\eta_{x,\lambda}(M)} f(y)\, d\mathcal{H}^n(y) = \int_P f(y)\, d\mathcal{H}^n(y) \quad \forall f \in C_c^0(\mathbb{R}^{n+k}) \tag{3.3}$$

is called the approximate tangent space for M ar $x \in \mathbb{R}^{n+k}$.

Notice that P is unique if it exists, and will be denoted as $T_x M$. The characterization mentioned above follows

Theorem 3.1. *Let $M \subset \mathbb{R}^{n+k}$ be a $\mathcal{H}^n$-measurable set with $\mathcal{H}^n(M \cap K) < \infty\ \forall K \Subset \mathbb{R}^{n+k}$. Then M is countably n-rectifiable iff the approximate tangent space $T_x M$ exists for $\mathcal{H}^n$-a.e. $x \in M$.*

Also, a differential structure can be extended to n-rectifiable sets through the approximating C^1 manifolds N_j.

Currents

The currents are used to introduce a topology on the rectifiable sets.

Definition 3.3 (Currents). *Consider $U \subset \mathbb{R}^{n+k}$ open and $\mathscr{D}^n(U)$ the space of $C_c^\infty(U)$ sections of the vector bundle $\bigwedge^n(T^*\mathbb{R}^{n+k})$ as introduced in the previous chapter. Then elements of the dual of $\mathscr{D}^n(U)$ are called currents and will be denoted with $\mathscr{D}_n(U)$.*

Let us introduce the concept of support of a current, for $T \in \mathscr{D}_n(U)$ it is identified by $\mathrm{spt}(T)$ and defined as the relatively closed subset of U given by the complement of the union of open sets W for which $T(\omega) = 0\ \forall \omega$ with support in W

$$\mathrm{spt}(T) = U \sim \bigcup W. \tag{3.4}$$

The space of currents is incredibly huge, being an extension of classical distributions, and for it to be useful some restrictions must be introduced. Notice that $\mathscr{D}_0(U)$ identifies the classical distributions, furthermore the structure theorem of distributions is valid also for currents and the concept of order can be thence extended. In fact from [6, 4.1.1] for each

$T \in \mathscr{D}_n(U)$ and each $K \Subset U$ it must exist $C \in \mathbb{R}$ and $m \in \mathbb{N}$ positive such that

$$T(\omega) \leq C \sum_{\beta \leq m} \sup_{x \in K} \langle D^\beta \omega(x), D^\beta \omega(x) \rangle^{1/2} \quad \forall \omega \in \mathscr{D}^n(U) \tag{3.5}$$

where with $D^\beta \omega$ is identified the k-form given by $D^\beta a_\alpha dx^\alpha$. The order of the current is hence the smaller value for m which is independent of K, if non-existent the order is said to be infinite. Currents of order zero with $n \geq 1$ can be interpreted as a generalization of n-dimensional oriented submanifolds $M \subset U$ with finite $\mathcal{H}^n$ measure. Indeed each n-dimensional oriented submanifold M with orientation $\xi = \tau_1 \wedge ... \wedge \tau_n$ where $\{\tau_i\}$ is an orthonormal basis of $T_x M$ has an associated current $[\![M]\!]$ defined as follows

$$[\![M]\!](\omega) = \int_M \langle \omega, \xi \rangle \; d\mathcal{H}^n, \quad \omega \in \mathscr{D}^n(U). \tag{3.6}$$

Leveraging this relation with classical submanifold Stokes' Theorem is used to extend the definition of boundary. For any n-dimensional smooth oriented and compact submanifold M with smooth boundary and any ω differential $(n-1)$-form on it, then

$$\int_M d\omega = \int_{\partial M} \omega \tag{3.7}$$

hence in analogy, for any $T \in \mathscr{D}_n(U)$ it is defined its boundary $\partial T \in \mathscr{D}_{n-1}(U)$ as

$$\partial T(\omega) = T(d\omega) \quad \forall \omega \in \mathscr{D}^{n-1}(U) \tag{3.8}$$

with $\partial T = 0$ when $n = 0$. Notice that $\partial^2 T = T(dd\omega) = 0$.
In the same fashion the measure of a submanifold is extended to the currents, in fact for M submanifold its measure coincides with

$$\int_M d\mathcal{H}^n = \sup_{\substack{|\omega| \leq 1 \\ \omega \in \mathscr{D}^n(U)}} [\![M]\!](\omega) \tag{3.9}$$

where $|\omega| = \sup_x \langle \omega, \omega \rangle^{1/2}$. So it is defined the mass $\mathbf{M}_W(T)$ of the current T with respect to the compact set $W \Subset U$ as

$$\mathbf{M}_W(T) = \sup_{\substack{|\omega| \leq 1 \\ \omega \in \mathscr{D}^n(U) \\ \mathrm{supp}(\omega) \Subset W}} T(\omega) \tag{3.10}$$

it is of relevance to notice how the requirement $\mathbf{M}_W(T) < \infty$, $\forall W \Subset U$ relates with the requirement of a distribution to be of zeroth order, in fact, $\mathbf{M}_W(T)$ is the smaller value of C so that (3.5) holds with $m = 0$. Finite mass currents are dual to the $C_c^0(U)$ differential n-forms, as they cannot depend on the derivatives otherwise they would not have bounded mass. Hence the general Riesz Theorem [18, Theorem 4.1] holds

Lemma 3.1. *Let $T \in \mathscr{D}_n(\mathbb{R}^{n+k})$ such that $\mathbf{M}_W(T) < \infty$ $\forall W \Subset \mathbb{R}^{n+k}$ then there exist a Radon measure $\mu_T : \mathcal{B}(\mathbb{R}^{n+k}) \to \mathbb{R}$ and a μ_T-measurable function $\vec{T} : \mathbb{R}^{n+k} \to \bigwedge^n(T^*\mathbb{R}^{n+k})$ such that*

$$T(\omega) = \int_{R^{n+k}} \langle \omega(x), \vec{T}(x) \rangle \; d\mu_T(x) \tag{3.11}$$

Furthermore, the space of $C_c^0(U)$ differential n-forms is a separable Banach space when endowed with the norm $|\omega| = \sup_x \langle \omega, \omega \rangle^{1/2}$, that is the same norm with which the mass is previously defined. Hence the space of currents with bounded mass is the dual space of a separable Banach space, on which Banach-Alaoglu holds giving

Theorem 3.2 (Compactness of finite mass currents)**.** *Let $T^{(k)}$ be a sequence in $\mathscr{D}_n(U)$ such that $\mathbf{M}_W < C$, $\forall W \Subset U$ for some $C \in \mathbb{R}$ then for Banach-Alaoglu there exist $T \in \mathscr{D}_n(U)$ such that*

$$T^{(k)} \rightharpoonup T \tag{3.12}$$

where the weak topology on the space of currents is considered

$$T^{(k)} \rightharpoonup T \iff T^{(k)}(\omega) \to T(\omega) \quad \forall \omega \in \mathscr{D}^n(U) \tag{3.13}$$

The lower semi-continuity of the mass with respect to the weak topology is trivial in fact

$$T(\omega) = \lim_k T^{(k)}(\omega) \leq \liminf_k \sup_{\substack{|\xi|\leq 1 \\ \xi \in \mathscr{D}^n(U) \\ \mathrm{supp}(\xi) \Subset W}} T(\xi) = \liminf_k \mathbf{M}_W(T^{(k)}) \quad \forall \omega \in \mathscr{D}^n(U) \tag{3.14}$$

and by taking the supremum over ω the lower semi-continuity is achieved. Another tool that will be used in the following is the push-forward of a current, the definition is trivially the application of the pull-back to the input, but in order to maintain the compactness of the support some complications are necessary. Given a smooth map $f : U \subset \mathbb{R}^{n+k} \to V \subset \mathbb{R}^{n+k'}$ such that $f|_{\mathrm{spt}(T)}$ is *proper* that is $f^{-1}(K) \cap \mathrm{spt}(T) \Subset U$ when $K \Subset V$ (as otherwise, the pull-back wouldn't have compact support) then

$$f_\# T(\omega) = T(\zeta f^\# \omega) \quad \forall \omega \in \mathscr{D}^n(V) \tag{3.15}$$

where ζ any $C_c^\infty(U)$ such that $\zeta = 1$ in $\mathrm{spt}(T) \cap \mathrm{spt}(f^\# \omega)$. The introduction of ζ is necessary to have $\zeta f^\# \omega \in \mathscr{D}^n(U)$ when $k' < k$. So in the case in which $k' = k$ the function ζ can be removed.
Let us now restrict further the set of currents considered

Definition 3.4 (Integer multiplicity rectifiable n-currents). *Let $T \in \mathscr{D}_n(U)$, if there exist an n-rectifiable subset $M \subset U$, a positive integer-valued $\mathcal{H}^n$-integrable function $\theta(x)$, and an $\mathcal{H}^n$-measurable function $\xi : M \to \bigwedge_n(\mathbb{R}^{n+k})$ such that for $\mathcal{H}^n$ almost everywhere $\xi(x) = \tau_1 \wedge ... \wedge \tau_n$ with $\{\tau_i\}$ orthonormal basis for $T_x M$, and*

$$T(\omega) = \int_M \langle \omega(x), \xi(x) \rangle \, \theta(x) \, d\mathcal{H}^n(x) \tag{3.16}$$

then T is called integer multiplicity rectifiable n-current, or simply integer multiplicity current. The notation $T = \tau(M, \theta, \xi)$ may be used, from which is evident that these types of currents have an associated integer n-varifold $V = v(M, \theta)$ on which they introduce an orientation.

These currents achieve the goal of introducing a topology into the n-rectifiable sets, while also taking into account a possible folding of the manifold through the multiplicity function. Most importantly the subset of integer multiplicity currents is closed with respect to the weak topology as the following compactness theorem states.

Theorem 3.3 (Federer-Fleming. Compactness of integer multiplicity currents).
Let $T^{(k)} \in \mathscr{D}_n(\mathbb{R}^{n+k})$ be a sequence of integer multiplicity currents such that

$$\mathbf{M}_W(T^{(k)}) + \mathbf{M}_W(\partial T^{(k)}) \leq C \quad \forall W \Subset \mathbb{R}^{n+k} \tag{3.17}$$

with $C \in \mathbb{R}$. Then there exists an integer multiplicity current $T \in \mathscr{D}_n(\mathbb{R}^{n+k})$ such that

$$T^{(k)} \rightharpoonup T \tag{3.18}$$

up to subsequences.

This result together with mass lower semicontinuity holds by application of the direct method the existence of a minimal integer multiplicity current for any integer multiplicity boundary.

Theorem 3.4 (Existence of Minimizing Current). *Given $S \in \mathscr{D}_{n-1}(\mathbb{R}^{n+k})$ integer multiplicity with compact support and $\partial S = 0$. Then there is a compact minimizing integer* *multiplicity current* $T \in \mathscr{D}_n(\mathbb{R}^{n+k})$ *with* $\partial T = S$ *and* $\mathbf{M}(T) \leq \mathbf{M}(R)$ *for any* R *integer multiplicity such that* $\partial R = S$.

Proof. The proof is the application of the direct method as mentioned above. The competitor's space is not empty given that there is at least the cone generated by S, and the minimizer will stay in any ball containing S given that the projection on such balls reduces the mass. $\square$

These minimal currents will be the last topic of this introduction.

Definition 3.5 (Area Minimizing Currents). *An integer multiplicity current* T *in* U *is minimizing with respect to a set* A *if*

$$\mathbf{M}_W(T) \leq \mathbf{M}_W(S) \quad \forall S \textit{ int. mult. s.t. } \partial S = \partial T \textit{ and } \operatorname{spt}(S - T) \subset A \cap W \tag{3.19}$$

is satisfied for any compact $W \Subset U$.

Note that even if not explicitly stated in the name, minimizing currents regards only integer multiplicity currents.
Amazingly enough, there is a compactness result for minimizing currents

Theorem 3.5 (Compactness of Minimizing currents). *Let* $T^{(k)}$ *be a sequence of minimizing currents such that*

$$\mathbf{M}_W(T^{(k)}) + \mathbf{M}_W(\partial T^{(k)}) \leq C \quad \forall W \Subset \mathbb{R}^{n+k} \tag{3.20}$$

with $C \in \mathbb{R}$. *Then up to subsequence, the limit integral current* T *is minimizing and*

$$\mu_T^{(k)} \to \mu_T \tag{3.21}$$

in the Radon measure sense.

Notice that in the case in which the currents considered are of the form $T_j = \partial [\![E_j]\!]$ it is enough a bound on the mass, without the one on the boundary's mass. At last, let us introduce the concept of singular set of a current T. The singular set $\operatorname{sing}(T)$ is given by those points $x \in \operatorname{supp}(T)$ for which it does not exist a neighborhood W so that $W \cap \operatorname{supp}(T)$ is a C^2 connected n-manifold.
The next topic is a way of representing currents of the type $T = \partial [\![E]\!]$ in a different way using Bounded Variation Functions.

3.2. BV functions and Cacciopoli sets

Functions of bounded variation are of great importance when free boundary problems are studied, for the description of n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ just a small set of BV functions will be considered.

Definition 3.6 (Functions of Bounded Variation)**.** *A function $f \in L^1(\Omega)$ is of bounded variation $f \in BV(\Omega)$ if the distributional derivative is a vector-valued Radon measure, that is $Df \in \mathcal{M}(\Omega)$:*

$$BV(\Omega) = \{f \in L^1(\Omega) : Df \in \mathcal{M}(\Omega)\} \tag{3.22}$$

The space of bounded variation functions is endowed with the norm

$$||f||_{BV} = |f|_{L^1(\Omega)} + |Df|(\Omega) \tag{3.23}$$

where $|Df|(\Omega)$ is the total variation of the measure Df given by

$$|Df|(\Omega) = \sup_{\substack{\phi \in C_c^1(\Omega) \\ |\phi| \leq 1}} \int_\Omega \langle \phi, Df \rangle \, dx = \sup_{\substack{\phi \in C_c^1(\Omega) \\ |\phi| \leq 1}} \int_\Omega \operatorname{div}(\phi) f dx \tag{3.24}$$

where the second equality comes from the definition of weak derivative. With the norm (3.23) the space of BV functions is a Banach space, but the most relevant thing for the calculus of variations is that it happens to be the dual of a separable Banach space.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^m$ open, the space $BV(\Omega)$ is isomorphic to the dual of the quotient X/Y with $X = [C_c^0(\Omega)]^{m+1}$ and Y the closure in X of*

$$E = \{(\phi, \tilde{\phi}) \in X : \phi = \operatorname{div}(\tilde{\phi})\} \tag{3.25}$$

The proof is given by showing that the map $T : BV(\Omega) \to X^*$ given by

$$Tu = (u\mathcal{L}^m, Du) \tag{3.26}$$

is an isomorphism with $(X/Y)^*$. It is hence clear that E comes from $\ker(Tu)$, and with further considerations, the proof is achieved. To arrive at a nice compactness result the last ingredient is a Sobolev-type embedding theorem for $BV(\Omega)$, in particular, the statement needed is that

$$BV(\Omega) \subset L^p(\Omega) \quad 1 \leq p < 1^* = \frac{n}{n-1} \quad \text{with compact embeddings} \tag{3.27}$$

under enough regularity of Ω. It can be now stated the compactness result

Theorem 3.7 (Compactness in $BV(\Omega)$). *Let Ω Lipschitz and $\{f_n\} \in BV(\Omega)$ bounded $||f_n||_{BV} < M$ then there exist a function $f \in BV(\Omega)$ such that*

$$f_n \xrightarrow{L^1} f \quad \textit{and} \quad Df \rightharpoonup Df_n \tag{3.28}$$

Proof. The result with weak convergence in $L^1(\Omega)$ is given by Banach-Alaoglu theorem being $BV(\Omega)$ a dual space, from the compactness embedding in $L^1(\Omega)$ the weak convergence is actually strong. □

In the space of bounded variation functions three different convergence can be considered. All have strong convergence in L^1 and they differentiate regarding the convergence of the derivative as follows

$$Df_n \rightharpoonup Df \quad \text{weak convergence} \tag{3.29}$$

$$|Df_n|(\Omega) \to |Df|(\Omega) \quad \text{intermediate convergence} \tag{3.30}$$

$$Df_n \to Df \quad \text{strong convergence} \tag{3.31}$$

the strong convergence would block approximation by smooth functions due to the excessively strong requirement, as an example $\delta_{1/n}$ doesn't converge strongly to δ_0. On the other hand, the weak convergence wouldn't preserve the total variation, in fact, it holds a lower semicontinuity result for the total variation

$$\int u \mathrm{div}\phi = \lim_n \int u_n \mathrm{div}\phi \leq \liminf_n \sup_\phi \int u_n \mathrm{div}\phi \tag{3.32}$$

which by taking the supremum over ϕ holds lower semicontinuity. Notice that this lower semicontinuity result holds also when only strong L^1 convergence is assumed. With the intermediate convergence, BV functions can be approximated by smooth functions while maintaining the total variation, this will be of use in the following. Now that BV functions have been properly introduced the Caccioppoli sets or sets of finite perimeter are easily defined

Definition 3.7 (Caccioppoli Sets). *Let E be a Borel set in $\mathbb{R}^m$, then the perimeter with* *respect to an open set* $\Omega \subset \mathbb{R}^m$ *is defined as*

$$P(E,\Omega) = |D\psi_E|(\Omega) \tag{3.33}$$

where ψ_E *is the characteristic function of the set* E*. Then if* $P(E,\Omega) < \infty$ *for any open and bounded* Ω*, or analogously* $\psi_E \in BV(\Omega)$*, the set* E *is a Caccioppoli set (or set of finite perimeter).*

The perimeter differs from the Hausdorff measure of the boundary $\mathcal{H}^{m-1}(\partial E \cap \Omega) = |\partial E \cap \Omega|$ but among the sets with the same characteristic function it can always be chosen one for which equality holds. In fact, this asymmetry arises from the fact that BV functions as a subset of L^1 are defined almost everywhere, this is not so for sets. A counterexample is obtained just by removing all points with rational components to the set, this holds an infinite $\mathcal{H}^{m-1}$ measure for the boundary. To solve this problem is generally introduced the concept of reduced boundary

Definition 3.8 (Reduced Boundary). *Let* $E \subset \Omega$ *be a set of finite perimeter then the reduced boundary* $\partial^* E$ *is given by the* $x \in \Omega$ *satisfying:*

$$|D\psi_E|(B_r(x)) > 0 \quad \forall r > 0 \tag{3.34}$$

$$\nu_E(x) = \lim_{r\to 0} \frac{D\psi_E(B_r(x))}{|D\psi_E|(B_r(x))} \quad \textit{exists finite} \tag{3.35}$$

$$|\nu_E|(x) = 1 \tag{3.36}$$

The vector field $\nu_E : \partial^* E \to S^{m-1}$ *is denoted as generalized inner unit normal vector of* E*.*

There is a structure theorem for BV functions associated with Caccioppoli sets from which the relation to currents of the type $T = \partial[\![E]\!]$ is evident.

Theorem 3.8 (De Giorgi's Structure Theorem). *If* E *is a Caccioppoli set in* $\mathbb{R}^{n+1}$ *then* $\partial^* E$ *is countably* n*-rectifiable,*

$$D\psi_E = \nu_E \mathcal{H}^n \llcorner \partial^* E \tag{3.37}$$

and a generalized Green theorem holds

$$\int_E \operatorname{div}(\phi)\, dx = \int_{\partial^* E} \phi \nu_E \, d\mathcal{H}^n \quad \forall \phi \in C^1_c(\mathbb{R}^{n+1}) \tag{3.38}$$

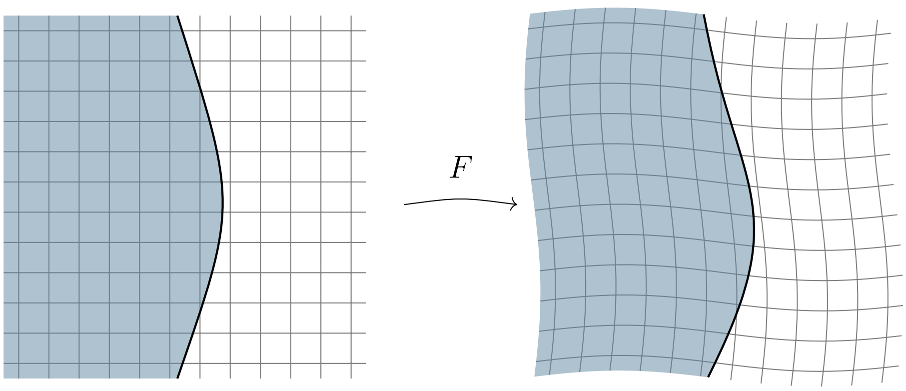
To conclude the relation with the current $T = \partial[\![E]\!]$ is given by $T = \tau(\partial^* E, 1, \omega_{\mathcal{E}} \llcorner \nu_E)$ where $\omega_{\mathcal{E}} \llcorner \nu_E$ is the restriction of the euclidean volume form with the normal ν_E.

4 | Variations

In the next chapter will be proved the Bernstein Theorem using the set of finite perimeters, for this reason, the first variation is here introduced using Bounded Variation functions. On the other hand, the second variation is presented and calculated in the case of submanifolds of codimension one embedded in a general manifold, as the stability will be discussed in such a setting.

4.1. First variation

For these calculations, the main references are [12, 15]. In order to introduce a variation of a set E of finite perimeter the whole ambient space is warped through a diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$. More precisely a family F_t of diffeomorphisms with $F_0 = I$ is considered to then differentiate with respect to the family parameter t.



It must then be studied the behavior of the total variation of a BV function under diffeomorphism, this is done in [12, Lemma 10.1]

Lemma 4.1. *Let $f \in BV_{loc}(\Omega)$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorfism. Considering $A \Subset \Omega$ and calling $f_* = f \circ F^{-1}$, $A_* = F(A)$ then:*

$$\int_{A_*} |Df_*| = \int_A |HDf| \tag{4.1}$$

with $H = |det(DF)|[DF]^{-1}$.

Proof. Consider the approximating sequence $f_j \in C^1(\Omega)$ converging to f in intermediate convergence, then for any $g \in C_0^1(A, \mathbb{R}^n)$ and representing $F^{-1} = \Phi$:

$$\int_{A_*} \langle g_*, Df_{j*} \rangle \, dx = \int_{A_*} \langle g \circ \Phi, D\Phi Df_j \circ \Phi \rangle \, dx = \tag{4.2}$$

$$\int_A \langle g, (D\Phi \circ F) Df_j \rangle \, |\det(DF)| dy = \int_A \langle g, HDf_j \rangle \, dy \tag{4.3}$$

passing to the limit the equation still holds thanks to L^1 convergence and the expressions being linear bounded functionals of f_j hence

$$\int_{A_*} \langle g_*, Df_{j*} \rangle = \int_A \langle g, HDf_j \rangle \tag{4.4}$$

For $|g| \leq 1$ taking the supremum on the left-hand side

$$\int_{A_*} |Df_*| \geq \int_A \langle g, HDf \rangle \tag{4.5}$$

taking the supremum on g holds the inequality between total variations, the opposite inequality is obtained analogously considering first $|g_*| \leq 1$ hence

$$\int_{A_*} |Df_*| = \int_A |HDf| = \int_A |H\nu||Df| \tag{4.6}$$

where $\nu = \frac{Df}{|Df|}$ has unitary norm, and in the case in which f is the characteristic function of a set of finite perimeter E it corresponds with the normal to the boundary ν_E. □

Let us now consider a family of parametric diffeomorphisms F_t such that $F_0 = I$ and such that they differ from the identity only in a compact subset of A. These will represent a variation contained in A and hence $A_* = A$ simplifying the next calculations. Hence:

$$\left.\frac{d}{dt}\right|_{t=0} \int_A |Df_*| = \left.\frac{d}{dt}\right|_{t=0} \int_A |H\nu||Df| = \int_A \frac{d\,|H\nu|}{dt} |Df| \tag{4.7}$$

It is left to show the derivative of the norm:

$$\left.\frac{d\sqrt{\langle H\nu, H\nu \rangle}}{dt}\right|_{t=0} = \left.\frac{\langle H\nu, \dot{H}\nu \rangle}{|H\nu|}\right|_{t=0} = \langle \nu, \dot{H}\nu \rangle \tag{4.8}$$

for the last inequality it has been used $F_0 = I \implies H|_{t=0} = I$ and $|\nu| = 1$.

Applying this result to $f = \phi_E$

$$\delta P(E, A) = \int_{A \cap \partial^* E} \langle \nu, \dot{H} \nu \rangle \, d\mathcal{H}^n \tag{4.9}$$

where ν is the normal to the hypersurface and H depends on the diffeomorphism used to variate.
To find a more informative expression for the variation the family of diffeomorphism discussed above can be expanded using Taylor:

$$F_t(x) = x + tT(x) + O(t^2) \tag{4.10}$$

where $T \in C_c^\infty(A, \mathbb{R}^{n+1})$ and $T(x) = \partial_t F_t(x)\big|_{t=0}$. Conversely given $T \in C_c^\infty(A, \mathbb{R}^{n+1})$ it is possible to construct a family of diffeomorphisms. Hence considering without loss of generality the variation $F_t(x) = x + tT(x)$:

$$DF_t(x) = I + t\nabla T(x) \tag{4.11}$$

and as proved in [15, 17.2] for tensor of this form holds:

$$\det(DF_t(x)) = 1 + t\mathrm{div}(T(x)) + \frac{t^2}{2}(\mathrm{div}(T(x))^2 - \mathrm{Tr}(\nabla T(x)^2)) + O(t^3) \tag{4.12}$$

$$[DF_t(x)]^{-1} = I - t\nabla T(x) + t^2 \nabla T(x)^2 + O(t^3) \tag{4.13}$$

The product of which is H. We hence can find the expression of the variation:

$$\dot{H}\big|_{t=0} = \mathrm{div}(T(x))I - \nabla T(x) \tag{4.14}$$

$$\langle \nu, \dot{H}\nu \rangle = \mathrm{div}(T(x))\mathbb{I}_{\partial E} - \langle \nu, \nabla_\nu T(x) \rangle \tag{4.15}$$

where $\mathbb{I}_{\partial^* E}$ is the characteristic function of the reduced boundary. The right-hand side of (4.15) corresponds to the divergence over $\partial^* E$. As discussed previously on n-rectifiable sets it can be introduced $x \in \partial^* E$ $\mathcal{H}^n$ almost everywhere a differential structure, then using local orthogonal coordinates in x for the ambient space $\{x_i, \nu\}$ where $\{x_i\}$ are coordinates for $\partial^* E = N$ it is evident that:

$$\mathrm{div}(T(x)) - \langle \nu, \nabla_\nu T(x) \rangle = \sum_{v \in \{x_i, \nu\}} \langle v, \nabla_v T(x) \rangle - \langle \nu, \nabla_\nu T(x) \rangle = \mathrm{div}^N T(x) \tag{4.16}$$

Arriving at the variation:

$$\delta P(E, A) = \int_{A \cap \partial^* E} \mathrm{div}^N T(x) \, d\mathcal{H}^n \tag{4.17}$$

The last step to arrive at the statement "Minimal hypersurfaces have zero mean curvature" is the divergence theorem for the case of non-tangential vector fields. It is enough to introduce it in a smooth manifold then it is generalized on n-rectifiable sets $\mathcal{H}^n$ almost everywhere.

Theorem 4.1 (Divergence Theorem for non tangent vector bundle)**.** *Let $M \hookrightarrow \mathbb{R}^{n+1}$ n-dimensional manifold with smooth boundary and $T \in \mathfrak{X}(T\mathbb{R}^{n+1})$ a vector field on M, then:*

$$\int_M div^M T = -\int_M \langle T, \boldsymbol{H}^M \rangle \, d\mathcal{H}^n + \int_\Gamma \langle T, \nu_\Gamma^M \rangle \, d\mathcal{H}^{n-1} \tag{4.18}$$

where $\Gamma = \partial M$, ν_Γ^M is the normal to Γ with respect to M and $\boldsymbol{H} = H\nu$ is the mean curvature vector obtained multiplying the mean curvature to the normal, the mean curvature is given by the trace of the second fundamental form $H = \mathrm{tr}(A)$. Notice that in the case in which T has values in $T(M)$ it is obtained the Stokes Theorem.

Proof. By separating the field into tangent and normal parts it is possible to apply the Stokes Theorem on the tangential part to limit the proof to only normal fields, in particular, consider $T = \phi\nu$. Choosing the *principal* orthonormal basis $\{\tau_i\}$ completed with ν in the ambient space, with associated curvatures $\{k_i\}$, then the Weingarten operator satisfies $\nabla_{\tau_i}\nu = k_i\tau_i$ hence:

$$\nabla_{\tau_i}(\phi\nu) = \tau_i(\phi)\nu + \phi\nabla_{\tau_i}(\nu) = \tau_i(\phi)\nu - \phi k_i \tau_i \tag{4.19}$$

$$\mathrm{div}^M(\phi\nu) = \sum_i \langle \tau_i, \tau_i(\phi)\nu - \phi k_i \tau_i \rangle = -\phi \sum_i k_i \tag{4.20}$$

□

Having this generalization of the divergence theorem:

$$\delta P(E, A) = -\int_{A \cap \partial E} \langle T(x), \boldsymbol{H} \rangle \, d\mathcal{H}^{n-1} \tag{4.21}$$

It is simple to obtain the formula for the second variation in terms of H in fact:

$$\delta^2 P(E, A) = \frac{d}{dt}\bigg|_{t=0} \frac{d}{dt} \int_A |Df_*| = \int_A \frac{d}{dt} \frac{\langle H\nu, \dot{H}\nu \rangle}{|H\nu|}\bigg|_{t=0} |Df| \tag{4.22}$$

Calculating the derivative:

$$\frac{d}{dt}\frac{\langle H\nu,\dot{H}\nu\rangle}{|H\nu|}\bigg|_{t=0} = \frac{|\dot{H}\nu|^2+\langle H\nu,\ddot{H}\nu\rangle}{|H\nu|}\bigg|_{t=0} - \frac{\langle H\nu,\dot{H}\nu\rangle^2}{|H\nu|^3}\bigg|_{t=0} =$$
$$|\dot{H}\nu|^2+\langle \nu,\ddot{H}\nu\rangle - \langle \nu,\dot{H}\nu\rangle^2$$

arriving at

$$\delta^2 P(E,A) = \int_A |\dot{H}\nu|^2+\langle \nu,\ddot{H}\nu\rangle - \langle \nu,\dot{H}\nu\rangle^2 \tag{4.23}$$

no further calculations will be made for the second variation in this case as it will be now discussed in the submanifold case.

4.2. Monotonicity Formula and Bound of Minimizer Area

Here are presented two important results regarding minimal hypersurfaces which will be used in the following. First is the monotonicity formula for minimal hypersurfaces

Theorem 4.2 (Monotonicity Formula)**.** *Let E be a Caccioppoli set with minimal boundary in Ω then for $x_0 \in \partial E \cap \Omega$ and a.e. $r \in (0, \operatorname{dist}(x_0, \partial A))$*

$$\frac{d}{dr}\frac{P(E,B_r(x_0))}{r^n} = \frac{d}{dr}\int_{B_r\cap\partial E}\frac{(\langle \nu, x-x_0\rangle)^2}{|x-x_0|^{n+2}}d\mathcal{H}^n(x) \tag{4.24}$$

The proof can be found in [15, Theorem 28.9] or in [18, 17] in the more general case of n-varifolds. Basically, the proof is given by choosing in the first variation formula (4.21) the variation $T(x) = \gamma((x-x_0)/r)(x-x_0)$, then after some calculations the formula is obtained by letting γ tend to 1.
The area of an area minimizer of codimension one will vary continuously with respect to the given boundary, this is a consequence of the more general [12, Lemma 5.6] regarding bounded variation functions. First, let us introduce the infimum value among BV functions of the total variation of the derivative when the boundary is fixed by $f \in BV(\Omega)$ that is

$$\nu(f,\Omega) = \inf\left\{\int_\Omega |Dg| : g \in BV(\Omega), \operatorname{spt}(g-f) \subset \Omega\right\} \tag{4.25}$$

for the previous discussions on compactness in BV the minimum is achieved, the functions that achieve the minimum are called of least gradient and are used in [1] to show the minimality of the Simons cone. With this concept the Lemma states

Lemma 4.2. *Given $f, g \in BV(B_R)$ and $\rho < R$ then*

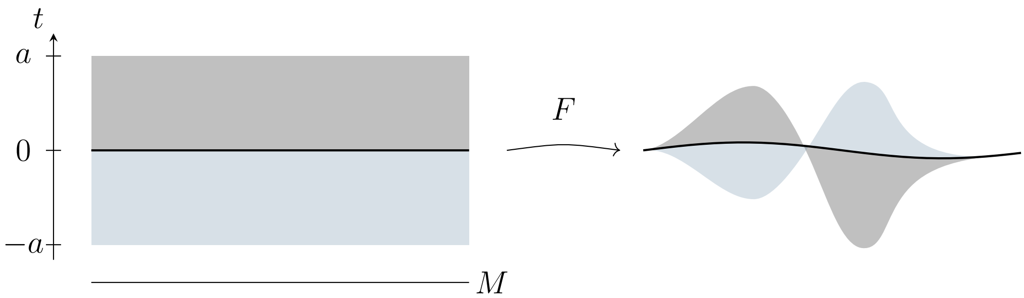
$$|\nu(f, B_\rho) - \nu(g, B_\rho)| \leq \int_{\partial B_\rho} |f^- - g^-| \, d\mathcal{H}^n \tag{4.26}$$

where the minus at superscript indicates the trace from inside the ball.

Now it is clear that applying this theorem to characteristic functions of minimizing Caccioppoli sets holds a continuity result of the minimizer's area with respect to the given boundary. The bounding term is a way of measuring the distance between the two boundaries, notice how it is similar to the flat metric in the currents representation of hypersurfaces.

4.3. Second Variation

Considering now a hypersurface represented by a submanifold S embedded in a manifold M through $i : S \to M$. The variation will be given by a map $F : S \times [-a, a] \to M$ such that $F(\cdot, t) = F_t$ is an embedding of S in M and satisfying $F_0 = i$ and $F_t|_{\partial S} = i|_{\partial S}$. Basically $\mathrm{Im}(F_t)$ represents a variation of the hypersurface.



It is possible without loss of generality to consider families of embedding such that the vector field given by the pushforward of $\frac{\partial}{\partial t}$ through F_0 lies in NS, this field will be denoted by $V = dF_0(\frac{\partial}{\partial t}) \in \mathfrak{X}(NS)$ and is the analogous of T before. The first variation will not be computed again in this setting as it coincides with the one found previously hence

$$\delta_V \mathcal{A}(S) = -\int_S \langle V, \boldsymbol{H} \rangle \, d\omega_{vol} \tag{4.27}$$

When variations are considered the integral will be always on S but the induced metric will change as the embedding varies. So introducing the dependence on t and deriving

$$\delta_V^2 \mathcal{A}(S) = -\int_S \frac{\partial}{\partial t}(\langle V, \boldsymbol{H} \rangle) d\omega_{vol}^t - \int_S \langle V, \boldsymbol{H} \rangle \, \frac{\partial}{\partial t}(d\omega_{vol}^t)\bigg|_{t=0} \tag{4.28}$$

Being interested in stable minimal hypersurfaces the calculation will be done for $\boldsymbol{H} = 0$ in $t = 0$, so the second part of the right-hand side vanishes. For clarity in the following calculations, the connection ∇ is the one in the ambient manifold M. By definition of V and compatibility with the metric

$$\delta_V^2 \mathcal{A}(S) = -\int_S \langle \nabla_V V, \boldsymbol{H} \rangle + \langle V, \nabla_V \boldsymbol{H} \rangle \, d\omega_{vol}^t \Big|_{t=0} \tag{4.29}$$

again $H = 0$ and only the second part is non-zero. Let us then study $\langle \nabla_V \boldsymbol{H}, V \rangle$, to do so the basis $\{e_i\}$ of $i(S)$ is extended to all the variations through the pushforward $e_i^t = dF_t(e_i)$ which generates a basis $\{e_i^t\}$ in the immersion $F_t(S)$. This choice of coordinates is convenient as it implies $[V, e_i] = 0$ from orthogonality. The mean curvature is given by the trace of the second fundamental form hence $\boldsymbol{H} = g^{ij} A_{ij}$ where both depend on t as noticed before

$$\langle \nabla_V \boldsymbol{H}, V \rangle = \langle \nabla_V (g^{ij} A_{ij}), V \rangle = \langle \dot{g}^{ij} A_{ij}, V \rangle + \langle g^{ij} \nabla_V A_{ij}, V \rangle \tag{4.30}$$

Let us first study the first term, notice that from $g^{ik} g_{kl} = \delta_l^i$

$$\dot{g}^{ik} g_{kl} = -g^{im} \dot{g}_{ml} \tag{4.31}$$

inverting g_{kl} by multiplying for g^{lj}

$$\dot{g}^{ij} = -g^{im} g^{jl} \dot{g}_{ml} \tag{4.32}$$

it is now easy to calculate the derivative of $g_{ml} = \langle e_m, e_l \rangle$, using $\nabla_V e_i = \nabla_{e_i} V$ from vanishing Lie brackets and the definition of Wirtinger operator

$$\dot{g}_{ml} = \langle \nabla_V e_m, e_l \rangle + \langle e_m, \nabla_V e_l \rangle = \langle \nabla_{e_m} V, e_l \rangle + \langle e_m, \nabla_{e_l} V \rangle = -2 \langle A_{ml}, V \rangle \tag{4.33}$$

hence the first term in (4.30) is given by $2g^{im} g^{jl} \langle A_{ml}, V \rangle \langle A_{ij}, V \rangle$ and introducing the restriction of codimension one $V = uN$ with $u \in C_c^\infty(S)$ hence

$$\langle \dot{g}^{ij} A_{ij}, V \rangle = 2u^2 |A|^2 \tag{4.34}$$

It is left to study the second term in (4.30), at first notice that from vanishing mean curvature in $t=0$

$$\langle g^{ij}\nabla_V A_{ij}, V\rangle = g^{ij}\nabla_V \langle A_{ij}, V\rangle = g^{ij}\langle \nabla_V\nabla_{e_i}e_j, V\rangle = g^{ij}\langle \nabla_{e_i}\nabla_V e_j, V\rangle - g^{ij}\langle {R_{iVj}}^k\tilde{e}_k, V\rangle \tag{4.35}$$

where Hessian commutation formula has been used and $\{\tilde{e}_k\} = \{e_i, V\}$. From connection compatibility and Ricci tensor definition

$$\langle g^{ij}\nabla_V A_{ij}, V\rangle = g^{ij}e_i(\langle \nabla_V e_j, V\rangle) - g^{ij}\langle \nabla_V e_j, \nabla_{e_i}V\rangle + Ric(V,V) \tag{4.36}$$

considering now $V = uN$, being $\{e_i\}$ orthonormal $\nabla_{e_i}e_j = 0$ for any i and j, it is obtained

$$g^{ij}e_i(\langle \nabla_V e_j, V\rangle) = g^{ij}e_i(\langle \nabla_{e_j}(uN), uN\rangle) = g^{ij}e_i(\langle e_j(u)N + u\nabla_{e_j}N, uN\rangle) \tag{4.37}$$

$$g^{ij}e_i(e_j(u)u) = g^{ij}e_i(e_j(u))u + g^{ij}e_i(u)e_j(u) = u\Delta^S u + |\nabla^S u|^2 \tag{4.38}$$

and

$$-g^{ij}\langle \nabla_{e_j}(uN), \nabla_{e_i}(uN)\rangle = -g^{ij}e_i(u)e_j(u) - u^2 g^{ij}\langle \nabla_{e_i}N, \nabla_{e_j}N\rangle = -|\nabla^S u|^2 - u^2|A|^2 \tag{4.39}$$

where $g^{ij}\langle \nabla_{e_i}N, \nabla_{e_j}N\rangle$ is the norm squared of the Wirtinger operator, which coincides with that of A as

$$|\mathcal{W}|^2 = g^{ij}\mathcal{W}_i^k\mathcal{W}_j^l g_{kl} = g^{ij}g^{km}A_{im}g^{ln}A_{jn}g_{kl} = g^{ij}g^{nm}A_{im}A_{jn} = |A|^2 \tag{4.40}$$

concluding (4.30) is the sum of (4.34), (4.38), (4.39) and $Ric(V,V)$ holding the second variation formula for minimal hypersurfaces of codimension one

$$\delta_V^2\mathcal{A}(S) = \int_S -u\Delta^S u - u^2|A|^2 - u^2 Ric(N,N)d\omega_{vol}^t \tag{4.41}$$

5 | The Bernstein Theorem

The most important result in the topic of minimal hypersurfaces is the Bernstein Theorem, which will be here exposed and proved using the description of hypersurfaces as the boundary of sets of finite perimeter, hence achieving the result among also non-regular competitors. The proof will follow the blow-up and blow-down procedure by Fleming [9].

Theorem 5.1 (Bernstein Theorem)**.** *Minimizing hypersurfaces in the whole $\mathbb{R}^{n+1}$ are hyperplane for $n < 7$. A counter-example in dimension 8 is given by the minimizing Simons Cone*

$$C_S = \left\{x \in \mathbb{R}^n \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\right\} \tag{5.1}$$

which gives counterexamples also in higher dimensions as $S \times \mathbb{R}^{n-8}$.

Proof. Denoting with M a global minimal hypersurface among Cacciopoli sets in $\mathbb{R}^{n+1}$. That is for any $K \Subset \mathbb{R}^{n+1}$ and all Cacciopoli sets L s.t. $L \equiv M$ in $\mathbb{R}^{n+1} \backslash K$ it holds

$$|D\psi_M|(K) \leq |D\psi_L|(K)$$

To improve notation it is used the structure theorem to relate total variation and Hausdorff measure $|D\psi_E|(K) = \int_K |D\psi_E| = \mathcal{H}^n(\partial E \cap K) = |\partial E \cap K|$, as noted previously this equivalence is true for $\partial^* E$ but again for clarity the asterisk will be omitted. The idea of the proof is to blow up M and show that the result is a cone, the same by blowing down, from the non-existence of non-flat stable minimal cones for $n < 7$ the theorem is proved. Consider

$$M_j = \left\{x \in \mathbb{R}^{n+1} \mid \lambda_j x \in M\right\} = \frac{M}{\lambda_j} \tag{5.2}$$

which is a blow-down sequence if λ_j tends to infinity and a blow-up sequence if it tends to zero, the calculations will not depend on the behavior of λ_j until the very end of the proof so the two cases are analyzed at the same time.

Minimality of M_j by inversion if it wasn't minimal it would exist a set L so that

$|\partial M_j \cap B_\rho| \geq |\partial L \cap B_\rho|$ which would imply:

$$\lambda_j^{-n}|\partial M \cap B_{\lambda_j \rho}| = |\partial M_j \cap B_\rho| \geq |\partial L \cap B_\rho| = |\partial(\lambda_j L) \cap B_{\lambda_j \rho}|\lambda_j^{-n} \tag{5.3}$$

which contradicts the minimality of M.

Convergence of $\boldsymbol{M_j}$ fixing ρ the sequence is bounded in BV in fact the bound in L^1 is given by the measure of B_ρ and for the bound on total variation minimality implies:

$$|\partial M_j \cap B_\rho| \leq \rho^n \sigma_{n+1} \tag{5.4}$$

where σ_{n+1} identifies the measure of the $n+1$-dimensional sphere's shell.

Hence from the compactness result there exist a converging subsequence. This gives the convergence for ρ fixed. To extend consider $\rho_k \to +\infty$ for $k \in \mathbb{N}$ strictly increasing. Starting from $k = 0$ there is a converging subsequence M_{0j} being ρ fixed, passing to $k = 1$ it is considered the subsequence M_{0j} and, from this subsequence, it is again extracted a converging subsequence in the new ball M_{1j} iterating this process is obtained at each step a subsequence of the previous subsequence, each element can be identified by M_{kj}. Taking the diagonal sequence M_{jj} holds a sequence converging for every compact set in $\mathbb{R}^n$. This proves the convergence, and the limit is identified as C. From now on M_j will identify the diagonal subsequence.

Minimality of $\boldsymbol{C}$ limits of minimizing hypersurfaces are minimizing. This result is simply the compactness of minimizing currents, in here it is proved for sets of finite perimeters. To show this it is introduced a measure of how much the total variation of $f \in BV(\Omega)$ differs from the minimal one:

$$\Psi(f, \Omega) = \int_\Omega |Df| - \nu(f, \Omega)$$

where ν as introduced before is

$$\nu(f, \Omega) = \inf \left\{ \int_\Omega |Dg| \ : \ g \in \mathrm{BV}(\Omega), \ spt(f-g) \subset \Omega \right\}$$

So $\Psi(f, \Omega) = 0$ means that f is of least gradient, and in the case of characteristic functions of Caccioppoli sets means that the set minimizes perimeter. To prove this compactness result then it is sufficient lower semi-continuity of $\Psi(M_j, B_r) = \Psi(\psi_{M_j}, B_r)$ with respect to the first argument converging in L^1. From lower semi-continuity of the total variation it is enough to show a continuity result for $\nu(M_j, B_r)$. This result is given by (4.26) applied to the sequence and its limit, that is

$$|\nu(\psi_C, B_r) - \nu(\psi_{M_j}, B_r)| \leq \int_{\partial B_r} |\psi_C^- - \psi_{M_j}^-| \, d\mathcal{H}^{n-1} \tag{5.5}$$

by showing that L^1 convergence implies the right-hand side limit to be zero. In fact taking $r = \rho + \epsilon$, then for a.e. ϵ near 0 there exist a subsequence M_{k_j} s.t.:

$$\lim_j \int_{B_{\rho+\epsilon}} |\psi_C - \psi_{M_{k_j}}| \, dx = 0 \implies \lim_j \int_{\partial B_{\rho+\epsilon}} |\psi_C - \psi_{M_{k_j}}| \, d\mathcal{H}^{n-1} = 0 \tag{5.6}$$

and taking a sequence $\epsilon_l \to 0$ for which the implication always holds, the limit with respect to l of the right-hand side is the integral of the traces, concluding the proof. To show (5.6) assume the negation, then there would exist a set I of positive measure in the right neighborhood of 0 such that for $\epsilon \in I$ the inferior limit of the shell integral would be greater than 0. Then integrating over I and for Fatou's Lemma:

$$0 < \int_I \liminf_j \int_{\partial B_{\rho+\epsilon}} |\psi_C - \psi_{M_j}| \, d\mathcal{H}^{n-1} \, d\epsilon \leq \liminf_j \int_I \int_{\partial B_{\rho+\epsilon}} |\psi_C - \psi_{M_j}| \, d\mathcal{H}^{n-1} \, d\epsilon \tag{5.7}$$

which contradicts the L^1 convergence. The continuity up to subsequence of $\nu(M_j, B_r)$ is proved, hence the minimality of C.
From the continuity of $\nu(M_j, B_r)$ a further useful result is obtained, in fact, minimality of C and M_j means $\int_{B_r} |Df| = \nu(f, B_r)$ with f the appropriate characteristic function, and continuity of ν implies

$$\lim_j \int_{\partial B_r} |D\phi_{M_j}| = \int_{\partial B_r} |D\phi_C| \tag{5.8}$$

***C* is a cone** The monotonicity formula for a set E with stationary perimeter considering $x_0 = 0$ states

$$\frac{d}{dr}\frac{P(E, B_r)}{r^n} = \frac{d}{dr}\int_{B_r \cap \partial E} \frac{(\langle \nu, x\rangle)^2}{|x|^{n+2}} d\mathcal{H}^n(x) \tag{5.9}$$

From this equation, two useful pieces of information for later use are extracted. First the fact that

$$\frac{P(E, B_r)}{r^n}$$

is an increasing function of r. Second integrating the equation from r_1 to r_2 holds:

$$\frac{P(E, B_{r_2})}{r_2^n} - \frac{P(E, B_{r_1})}{r_1^n} = \int_{(B_{r_2} - B_{r_1}) \cap \partial E} \frac{\langle \nu, x\rangle^2}{|x|^{n+2}} d\mathcal{H}^n(x) \tag{5.10}$$

To prove that C is a cone it would be sufficient to show that $\frac{P(C,B_r)}{r^{n-1}}$ is actually independent of r as this would imply that the right-hand side of the integrated monotonicity formula is null for every r_1 and r_2 implying $\langle \nu, x\rangle = 0$ for any $x \in \partial C$ which is a characterization of Cones. To prove independence on r is used (5.8) to get this chain of *equalities*:

$$\frac{|\partial C \cap B_r|}{r^n} = \lim_j \frac{|\partial M_j \cap B_r|}{r^n} = \lim_j \frac{|\partial M \cap B_{\lambda_j r}|}{(\lambda_j r)^n} = \lim_{\rho\to+\infty} \frac{|\partial M \cap B_\rho|}{\rho^n} \tag{5.11}$$

Now the two cases of blow-up and blow-down must be separated as the last limit in this chain is increasing for the blow-down and decreasing for the blow-up. From Inequality (5.4) the limit has an upper bound given by σ_{n+1}, so in the blow-down case the limit is increasing and bounded hence converges and the independence on r is proved. For the blow up the bound from below is simply given by zero, so a decreasing sequence bounded from below converges, and again independence from r is achieved. This concludes the proof that C is indeed a Cone in both the case of blow up and blow down.

Conclusions For $n < 7$ there are no singular stable minimal cones, hence both the blow-up and the blow-down are hyperplanes and the two equality chains above are equal to ω_n, this implies thanks to monotonicity:

$$\omega_n = \lim_{\rho\to 0} \frac{|\partial M \cap B_\rho|}{\rho^n} \leq \frac{|\partial M \cap B_r|}{r^n} \leq \lim_{\rho\to+\infty} \frac{|\partial M \cap B_\rho|}{\rho^n} = \omega_n \tag{5.12}$$

Hence also for M itself, the monotonicity formula does not depend on r and it is hence a cone, in particular a hyperplane for $n < 7$. $\square$

Regarding the statement that for $n < 7$ there are no stable singular minimal cones, the most general result is a result on the singular set of minimal hypersurfaces. The singular set of a hypersurface S with associated measure μ_S is given by those points $x \in \mathrm{supp}(\mu_S)$ for which it does not exist a neighborhood W so that $W \cap \mathrm{supp}(\mu_S)$ is a C^2 connected n-manifold, this set is denoted $\mathrm{sing}(S)$. The regularity result for area minimizers states that if S is local area minimizing in U then

$$\mathcal{H}^{n-7+\alpha}(\mathrm{sing}(S) \cap U) = 0 \quad \forall \alpha > 0 \tag{5.13}$$

which means that for $n < 7$ the singular set is empty, for $n = 7$ it is at most discrete. The proof can be found in [15, Theorem 28.1] and follows from two results: the non-existence of minimizing cones with a singular point in the origin for $n < 7$ known as Simons Theorem [19], proved by choosing the variation $u(x) = \phi(x)|A|(x)$ with $\mathrm{spt}(\phi \cap \{0\}) = \emptyset$ in the second variation formula; and from Federer's dimension reduction theorem which states

that for minimizing cones with non-discrete singular set the blow up in singular points holds $F \times \mathbb{R}$ with F minimizing cone of one dimension lower.
To properly conclude the Bernstein problem the minimality of the Simons Cone should be proven, the result was first proved in [1] using BV functions of least gradient. In fact level sets of such functions are Caccioppoli sets that minimize perimeter, this is a consequence of the fact that BV functions of least gradient are a lattice that is if $f \in BV(\Omega)$ then both $\max(f-t,0)$ and $\min(f,t)$ are $BV(\Omega)$ of least gradient, taking

$$\phi_{\varepsilon,t} = \frac{1}{\varepsilon}\min(\varepsilon, \max(f-t,0))$$

for $\varepsilon \to 0$ it converges in L^1 to the t-level set of f and from lower semi continuity of $\Psi(\phi_{\varepsilon,t},\Omega)$ it is of least gradient. In the paper, they find a function of least gradient for which $\{f=0\}$ coincides with the Simons Cone concluding the proof.

6 | Stability

This chapter presents the stability condition for minimal hypersurfaces. Throughout the chapter with submanifold or S will be considered an orientable, connected, and complete submanifold, so only smooth regular hypersurfaces are under consideration.

From the second variation formula up to the application of Green Theorem, given a minimal oriented complete submanifold S embedded in a manifold M of codimension one, varying the hypersurface by $T = u\nu$ with u compact support holds:

$$\delta^2 \mathcal{A}(S) = \int_S |\nabla u|^2 - (|A|^2 + Ric(\nu, \nu))u^2 \, dV_g \tag{6.1}$$

To have a more compact notation let us introduce

$$V = |A|^2 + Ric(\nu, \nu) \tag{6.2}$$

and

$$E(u) = \int_S |\nabla u|^2 - Vu^2 \, dV_g \tag{6.3}$$

Consider now Ω open bounded subset of S, and $u, w = 0$ on $\partial\Omega$, then using Green's Theorem for sections (2.11) in which the vector bundle is simply $\mathbb{R}$, it is obtained:

$$\delta_w E(u) = \frac{d}{d\epsilon}\bigg|_{t=0} \int_\Omega |\nabla(u + \epsilon w)|^2 - V(u + \epsilon w)^2 \, dV_g = \int_\Omega 2 < \nabla u, \nabla w > -2Vuw \, dV_g =$$
$$= \int_\Omega 2(-\Delta u - Vu)w \, dV_g = \langle 2L_\Sigma u, w\rangle_{L^2} = \mathcal{I}_\Omega(u, w)$$

Hence the second variation of the area corresponds to double the energy of the operator:

$$L_S = -\Delta - V \tag{6.4}$$

which is strongly elliptic and has uniqueness in the Cauchy problem [19, Prop. 1.2.3.]. The properties of such operators are now studied.

6.1. Schrödinger Operator $-\Delta - V$

The stability requirement for the hypersurface is intertwined with the spectrum of the operator L_S, it is evident that $\mathcal{I}_\Omega(u,w) = \mathcal{I}_\Omega(w,u)$ hence the operator L_S is symmetric implying real spectrum, also the spectrum is discrete and all its elements have an associated eigenfunction, this will be proved in the following discussion. That said the following theorem shows the relation between $\sigma(L_S)$ and stability.

Theorem 6.1 (Variational Characterization of Stability). *Given a minimal embedded submanifold S. Then it is stable with respect to the area functional iff $\sigma(L_S) \subset [0, +\infty)$.*

Proof. Consider the minimization of $E(u)$ under the constraint $\int_\Omega u^2 dV_g = 1$

$$\lambda_1 = \inf_{||u||_2=1} \int_\Omega |\nabla u|^2 - V u^2 \tag{6.5}$$

In order to have nice topological properties and apply the direct method let us ambient this problem in the Hilbert space $H_0^1(S)$. To show that the infimum is achieved consider a minimizing sequence $u^{(k)} \in H_0^1(S)$ with $||u^{(k)}||_2 = 1$, evidently the sequence is bounded being $E(u) \propto ||u||_{H^1}$ and hence from Banach-Alaoglu there is a subsequence weakly converging. It is left to prove lower semi-continuity, notice that $H_0^1(S) \Subset L^2(S)$ hence the subsequence converges strongly in $L^2(S)$ implying continuity of L^2 norm, this fact together with lower semi-continuity of the norm with respect to weak convergence means that E is the sum of a lower semi-continuous function and a continuous one, implying it is lower semi-continuous. Hence $u^{(k)} \rightharpoonup u$ with $||u||_2 = 1$ and the minimum is achieved by u. Proceeding with the Lagrange multiplier method the problem of finding the minimizer can be expressed as the minimization of the functional

$$\Lambda(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 - V v^2 - \gamma v^2 \tag{6.6}$$

where γ is the Lagrange multiplier. The minimizer u is a critical point of the Lagrange functional, hence the variation of $\Lambda(v)$, calculated in the same way it has been done for E, will vanish in u along any direction $w \in H_0^1(S)$

$$\delta_w \Lambda(u) = \int_\Omega \langle \nabla u, \nabla w \rangle - V u w - \gamma u w = \int_\Omega (-\Delta u - V u - \gamma u) w = 0 \tag{6.7}$$

From the rightmost integral of (6.7) is obtained that the minimizer u of (6.5) is an eigenfunction of the operator L_S. To obtain the eigenvalue associated to u consider $w = u$

then from the first integral in (6.7)

$$\int_\Omega |\nabla u|^2 - Vu^2 = \gamma ||u||_2 \tag{6.8}$$

and being u the minimizer of (6.5) evidently $\gamma = \lambda_1$ and no eigenvalue is lower otherwise its eigenfunction would be the minimizer, so λ_1 is the first eigenvalue of L_S. To conclude from the definition of λ_1 for any $w \in C_c^\infty(S)$:

$$\int_\Omega |\nabla w|^2 - Vw^2 = ||w||_2^2 \int_\Omega \left|\nabla \frac{w}{||w||_2}\right|^2 - V\left(\frac{w}{||w||_2}\right)^2 \geq ||w||_2^2 \lambda_1 \tag{6.9}$$

which being λ_1 the smallest eigenvalue means that stability is satisfied if the operator L_S has no negative eigenvalues for any subset Ω proving the *if* part.
For the *only if* part is trivial from density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$. □

Eigenvalue Problem

The goal is now to study the eigenvalues of L_S. Let $\Omega \subset S$ consider the eigenvalue problem:

$$\begin{cases} -\Delta u - Vu = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6.10}$$

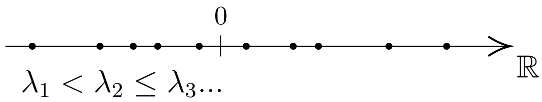
it is convenient to make a slight modification to the problem. Let $s = \max_\Omega(V)$ notice that:

$$-\Delta u - Vu = \lambda u \iff -\Delta u + (s - V)u = (\lambda + s)u \tag{6.11}$$

and $s - V \geq 0$. Denoting $L_S^+ = -\Delta + (s - V)$ evidently $\sigma(L_S) = \sigma(L_S^+) - s$ so it is enough studying the new operator. The operator L_S^+ was constructed to be positive and coercive as can be easily checked. Hence Lax-Milgram Theorem holds so given $f \in L^2(S)$ the weak solution of $L_S^+ w = f$ exists unique and $||w||_H \leq C||f||_2$, this means well-posedness and boundedness of the inverse Green Operator $G_S = (L_S^+)^{-1}$

$$G_S : L^2(S) \to H_0^1(S) \Subset L^2(S) \tag{6.12}$$

so G_S is a compact, symmetric and positive operator from $L^2(S)$ in itself. From the spectral theorem, its eigenvalues are countable, positive, and converging to 0. This implies that the eigenvalues of L_S^+ are countable, positive, and diverging to infinity. As already mentioned the eigenvalues of L_S correspond to the one of this new operator but shifted to the left by $s = \max_\Omega(V)$. Hence $\sigma(L_S)$ is given by a sequence $\{\lambda_k\}$ of real numbers with smaller value λ_1 and diverging to infinity.



A condition on V that implies stability is easily found.

Proposition 6.1. *Given $\Omega \subset S$ then S (minimal) is stable in Ω if $V(x) \leq \lambda_1(-\Delta)$ $\forall x \in \Omega$ where $\lambda_1(-\Delta)$ is the first eigenvalue of the Dirichlet problem in Ω for the operator $-\Delta$.*

Proof. As it has been already shown stability coincides with having $\lambda_1(L_S) \geq 0$. Consider $s = \max_{\Omega}(V) \leq \lambda_1(-\Delta)$ then this chain of inequality is satisfied

$$\inf_{||u||_2=1} \int_{\Omega} |\nabla u|^2 - Vu^2 \geq \inf_{||u||_2=1} \int_{\Omega} |\nabla u|^2 - su^2 = \inf_{||u||_2=1} \int_{\Omega} |\nabla u|^2 - s \tag{6.13}$$

where the left most part is $\lambda_1(L_S)$ and the rightmost is $\lambda_1(-\Delta) - s \geq 0$ concluding. $\square$

Notice that the inverse implication of this proposition is generally not true but studying this specific case could give some insights into the rigidity of the Bernstein Theorem, in fact, it will be later proved that with the hypothesis $V(x) \leq \lambda_1(-\Delta)$ for $n < 6$ and flat ambient space the stable hypersurface is flat.
A general concept in the subject of operators on manifolds is the index of an operator.

Definition 6.1 (Index). *Given a manifold M on which is defined an operator L, the index $\mathrm{Ind}_L(\Omega)$ of Ω with respect to L is the number of negative eigenvalues of L in Ω. In the following, $\mathrm{Ind}(\Omega)$ will identify the index of the stability operator.*

With the definition of index, it is evident that stability in Ω coincides with $\mathrm{Ind}(\Omega) = 0$.

Morse Index Theorem

Given a connected manifold S and a subset $\Omega \subset S$ with regular boundary consider a family of diffeomorphisms $g_t : \Omega \to \Omega$ such that $\Omega_t = g_t(\Omega) \subset \Omega_s = g_s(\Omega)$ for any $t > s$, hence g_t is a contraction. The family is called of ϵ-type if there exist a $\bar{t}$ such that for every $t > \bar{t}$ then $\mathcal{A}(\Omega_t) \leq \epsilon$. The theorem presented in this section, proved by Smale [20, Lemma 2, Lemma 7], exposes the behavior of the eigenvalues of operators under such deformations.

Theorem 6.2 (Morse Index Theorem). *Let $L : H^{2k}(\Omega) \to L^2(\Omega)$ be a self-adjoint, strongly elliptic operator of order $2k$. Then the spectral theorem discussed before holds* *(strongly elliptic gives coercivity, the order $2k$ is needed for symmetry). And the eigenvalues λ_k^t of the Dirichlet problem in $\Omega_t = g_t(\Omega)$ where g_t as above, are non-decreasing functions of t and are strictly increasing if L has uniqueness in the Cauchy problem. Furthermore, there exists ε for which if g_t is of ε-type then all the eigenvalues are positive for all $t > \bar{t}$.*

To give a glimpse of the proof, the fact that the eigenvalues are non-decreasing functions of t is trivial, being the Rayleigh quotient of a subset the infimum over a smaller family of functions. The result of strict increasing is obtained by showing that if $\lambda_n^t = \lambda_n^s$ then the eigenfunctions coincide up to trivial expansion [20, Lemma 6], which is absurd if there is uniqueness in the Cauchy problem. Lastly, the proof of the stability of small sets for the specific case of L_S is presented here, for the general case the calculations are similar but it is also used a Gardings inequality [20, Lemma 7]. Considering again $s = \max_\Omega V(x)$ then

$$\int_{\Omega_t} |\nabla f|^2 \geq \int_{\Omega_t} sf$$

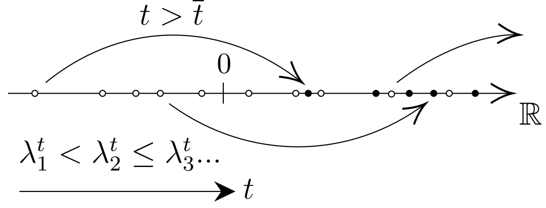
is a stronger result than stability, this inequality is exactly the Poincaré inequality with coefficient $1/s$ so it holds if the Poincaré constant for Ω_t is such that $C_p < 1/s$. This is indeed true as C_p can get as small as wanted by decreasing the area of the domain, in fact from Sobolev embeddings

$$||f||_{L^r} \leq C_s ||f||_{H^1} \quad \forall r > \frac{2n}{n-2}$$

and choosing $r > 2$ from holder inequality

$$||f||_{L^2} \leq |\Omega_t|^{1/2-1/r} ||f||_{L^r} \leq |\Omega_t|^{1/2-1/r} C_s ||f||_{H^1}$$

so $|\Omega_t|^{1/2-1/r} C_s = C_p$ and for a sufficiently small $\varepsilon > |\Omega_t|$ stability is achieved.
In a more direct way, this theorem states that on smaller and smaller domains the eigenvalues increase, and if $\mathrm{Ind}_L(\Omega) > 0$ then any negative eigenvalue as it increases will annihilate and then become positive.



There is an analogy between Jacobi fields and domain in which there is a null eigenvalue, in fact, Jacobi fields on a geodesic represent the displacement of it that up to the second order does not change the length. Among Jacobi fields, there may be also displacements that pass from one geodesic to another. By considering the case of hypersurfaces, the Jacobi fields are solutions to $L_S = 0$ that is the second variation of the area is 0 along them, among these fields there are also the ones that displace the hypersurface to another minimal stable hypersurface. If zero is not an eigenvalue the solution of $L_S = 0$ will be unique from Fredolm's alternative, in such cases having a homogeneous boundary condition the unique solution vanishes. For domains Ω in which there is indeed a null eigenvalue the boundary $\partial\Omega$ is called conjugate boundary, in analogy to the conjugate point, and elements of the eigenspace are Jacobi fields with homogeneous boundary condition.

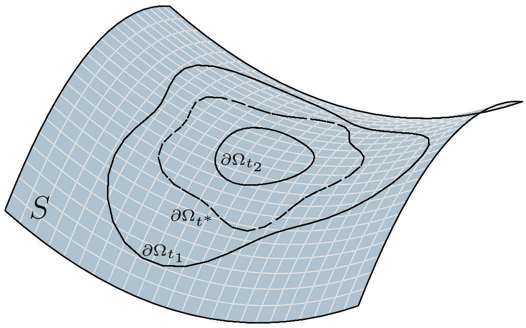


Figure 6.1: The various curves on the surface S represent the boundaries of a contraction Ω_t for different values of t. If the dashed curve, for $t = t^*$, is a conjugate boundary, then zero is an eigenvalue of L_S and there are variations (eigenfunctions of 0) supported on Ω_{t^*} that does not change the area functional up to the second order. Also domains contained in Ω_{t^*} like Ω_{t_2} have strictly lower Morse index than domains, like Ω_{t_1}, containing Ω_{t^*}, as the conjugate boundary identifies the passage of an eigenvalue from negative to positive.

Thanks to the Morse Index Theorem the index of the whole hypersurface S defined as $\mathrm{Ind}(S) = \lim_{R\to\infty} \mathrm{Ind}(B_R)$ is well-posed and independent either on the point and on the actual exhaustion of S chosen. Notice also that the limit could converge or diverge to infinity, holding a further separation between minimal non-stable hypersurfaces, that is Finite Index minimal hypersurfaces and Non Finite index minimal hypersurfaces. In the case of finite index minimal hypersurfaces it can be proved [7, Proposition 1] that by removing a compact subset K from S then $S\backslash K$ is stable, this is done by constructing Jacobi fields on successive rings, once this process holds a number of Jacobi fields equal to $\mathrm{Ind}(S)$ then K is any compact containing the union of such rings given that the support of all the eigenfunctions of the negative eigenvalues would be inside K.

Stable Hypersurfaces

Consider now S stable minimal hypersurface, that is:

$$\lambda_1(L_S) \geq 0 \text{ for any } \Omega \tag{6.14}$$

The first thing to notice is that non-compact stable minimal hypersurfaces are strongly stable, in fact from Morse Index Theorem for any exhaustion $\Omega_i \subset \Omega_{i+1}$:

$$0 \leq \lambda_1(\Omega_{i+1}) < \lambda_1(\Omega_i) \tag{6.15}$$

but the first inequality is actually strict for any i otherwise it would negate the result on the strict decreasing value of eigenvalues under expansion. Notice that such a result is achievable only in the case of non-compact hypersurfaces, and is actually true for any stability operator L.
The following theorem on positive operators will be later used, it is presented in [8] for L_S but it has a more general validity, as proved here.

Theorem 6.3. *Let S be a complete, non-compact manifold on which is defined a second order differential operator L self-adjoint, strongly elliptic, and positive on any $\Omega \subset S$. Then there exists a function $u > 0$ on S such that $Lu = 0$.*

Proof. As already shown the operator is strictly positive hence from Fredholm's alternative there is uniqueness in the problem:

$$\begin{cases} Lv_i = 0 & \text{in } \Omega_i \\ v_i = 1 & \text{in } \partial\Omega_i \end{cases} \tag{6.16}$$

Furthermore $v_i \geq 0$, in fact, if otherwise the domain in which $v_i < 0$ would be a conjugate boundary which is absurd being the operator positive. From maximum principle $v_i \geq 0$ implies $v_i > 0$. It is now fixed $p \in \Omega_0$ and defined $u_i = \frac{v_i}{v_i(p)}$ satisfying:

$$\begin{cases} Lu_i = 0 \quad u_i > 0 & \text{in } \Omega_i \\ u_i(p) = 1 \end{cases} \tag{6.17}$$

The hypothesis of [11, Theorem 8.20] are satisfied holding the Harnack's inequality:

$$\sup_{B_\sigma(p)} u_i \leq C \inf_{B_\sigma(p)} u_i \tag{6.18}$$

for σ such that $B_{4\sigma}(p) \subset \Omega_i$ and C dependent on σ. This implies:

$$\sup_{B_\sigma(p)} u_i \leq C u_i(p) = C \tag{6.19}$$

$$C^{-1} \leq \inf_{B_\sigma(p)} u_i \tag{6.20}$$

The first inequality gives a uniform bound on $||u_i||_\infty$, for Schauder interior a priori estimates [11, Theorem 6.2] the bound extends to the second derivative norm, these two bounds together bound the first derivative norm. Hence Ascoli-Arzelà compactness theorem holds for both the function and its first derivative, and a converging subsequence may be extracted for any σ. With the diagonalization argument, there is a u to which u_i converges uniformly in every compact subset of Ω. Hence $Lu = 0$ and thanks to the lower bound (6.20) $u > 0$ concluding. □

Notice that the behavior of the Harnack constant is exponential in σ, hence it is not true in general that there is ε such that $u \geq \varepsilon > 0$ on S.

7 | Smoothing of Cones

The discussion on stable minimal hypersurfaces was and will be limited to the smooth case, on the other hand, the counterexample of Bernstein Theorem was presented as a singular cone, it is hence missing a counter-example in the smooth case. For this reason, it is now made an overview of [14, Theorem 2.1] which is the smoothing of any minimizing cone with singularity in the origin. An interesting property used in the proof and presented in [14, Lemma 1.18] is that given two minimizing currents $T = \partial[\![E]\!]\llcorner W$ and $S = \partial[\![F]\!]\llcorner W$ if $\operatorname{spt}(\partial S) \subset \bar{E}$ then $\operatorname{spt}(S) \subset \bar{E}$ which in simple words it says that if the boundary of a minimizing current is on one side of another one then the whole current will stay on that side.

Theorem 7.1. *Let C be a minimizing hypercone in $\mathbb{R}^{n+1}$ with $\Gamma = C \cap \partial B_1$ a smooth compact submanifold of S^n and $\operatorname{sing}(C) = \{0\}$, suppose that C separates $\mathbb{R}^{n+1}$ in exactly two open sets E_+ and E_- hence $C = \partial[\![E_+]\!]$. Then denoting E either E_+ or E_-, there exist an oriented connected embedded minimizing hypersurface $S \subset E$ such that $S = \partial[\![F]\!]$, $\bar{F} \subset E$ open and $\operatorname{sing}(S) = \emptyset$, $\operatorname{dist}(S, 0) = 1$.*

Corollary 7.1. *Choosing as C the Simons Cone*

$$C = \left\{x \in \mathbb{R}^n \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\right\} \tag{7.1}$$

it satisfies the hypothesis of the theorem holding the existence of a non-flat smooth minimizing (hence stable) hypersurface S. Hence it is not necessarily true that stable minimal smooth hypersurfaces in $\mathbb{R}^{n+1}$ are flat when $n \geq 7$, answering to the stable Bernstein problem.

Proof. Only the outline of the proof will be presented.

Step 1. The first step is to generate a sequence of minimizing currents in B_1 converging to the cone and with support only on one side. To do so consider a family of C^2 functions

$$\varphi_j : \Gamma \to E \cap \partial B_1 \text{ such that } |\varphi_j - i_\Gamma|_{C^2} \leq 1/j \tag{7.2}$$

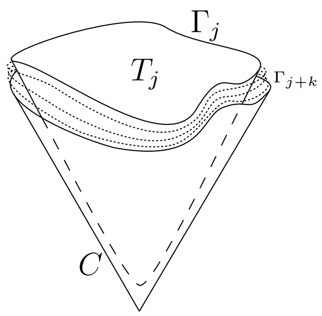


Figure 7.1: T_j lying on one side of C at a distance

It will be denoted $\Gamma_j = \varphi_j(\Gamma)$. Then for sufficiently large j there is a minimizing current T_j with $\partial T_j = \Gamma_j$ and there exists E_j open $\bar{E}_j \subset E$ such that

$$T_j = \partial [\![E_j]\!] \llcorner B_1 \text{ hence } \operatorname{supp}(T_j) \cap \operatorname{supp}(C) = \emptyset \tag{7.3}$$

$$T_j \rightharpoonup C \tag{7.4}$$

So the T_j lies on one side of the cone and are at a distance from it.

Step 2. Through homothety of the above sequence a minimizing current S is obtained, this is exactly the smooth hypersurface sought and its regularity is left to be proven. Indeed it is possible to choose λ_j converging to zero such that

$$\operatorname{dist}\left(\operatorname{spt}\left(\frac{1}{\lambda_j} T_j\right), 0\right) = 1 \tag{7.5}$$

the mass of the rescaled currents is bounded for every B_ρ, in fact from the push-forward and the fact that $\operatorname{spt}(f(cx)) = 1/c \operatorname{spt}(f(x))$

$$\mathbf{M}_{B_\rho}\left(\frac{1}{\lambda_j} T_j\right) = \sup_{\substack{|\omega| \leq 1 \\ \operatorname{spt}(\omega) \in B_\rho}} \frac{1}{\lambda_j} T_j\left(\omega(x)\right) = \sup_{\substack{|\omega| \leq 1 \\ \operatorname{spt}(\omega) \in B_{\rho\lambda_j}}} T_j\left(\frac{1}{\lambda_j^n}\omega(x)\right) = \frac{1}{\lambda_j^n} \mathbf{M}_{B_{\rho\lambda_j}}(T_j) \tag{7.6}$$

and being T_j minimizing its mass is certainly bounded by the mass of $[\![B_{\rho\lambda}]\!] = \rho^n \lambda^n \omega_n$ hence bounding the above chain by $\rho^n \omega_n$. From compactness, the rescaled currents converge to $S = \partial [\![F]\!]$ with $\bar{F} \subset E$.

Step 3. To conclude it is sought to show that $x \cdot \nu_S(x) > 0$ for all $x \in \operatorname{spt}(S)$, from which the support is locally given by the graph of a minimal hypersurface function concluding the proof. First using again homothety it is shown that C is the tangent cone for S at infinity. Hence there exists a great enough R_0 and a C^2 function v on $C \backslash B_{R_0}$ such that

$$\operatorname{spt}(S) \backslash B_{2R_0} \subset \operatorname{graph}_C(v) \tag{7.7}$$

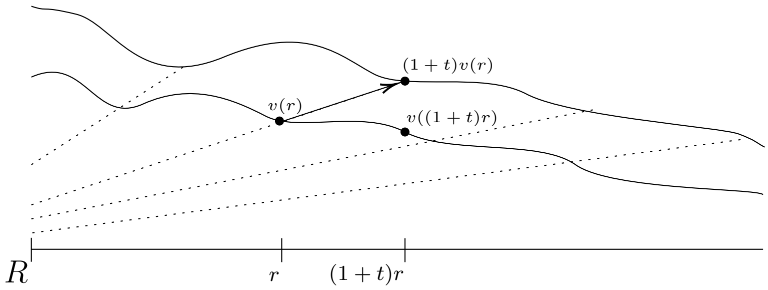


Figure 7.2: Rescale of $v(r\sigma)$ by $(1+t)$ for σ fixed

Being S a minimizing hypersurface v must be a Jacobi field and the asymptotic of the Jacobi fields on C are known to be such that

$$\frac{\partial}{\partial r}\left(\frac{v(r\sigma)}{r}\right) < 0 \tag{7.8}$$

for sufficiently large r, where σ represents the vectors pointing to Γ. By fixing σ inequality (7.8) implies

$$\frac{\partial v(r\sigma)}{\partial r} < \frac{v(r\sigma)}{r} \tag{7.9}$$

and from Grönwall's Inequality

$$v(r_2\sigma) < v(r_1\sigma)\frac{r_2}{r_1} \tag{7.10}$$

choosing $r_2 = (1+t)r_1$ and $r_1 = r$

$$v((1+t)r\sigma) < v(r\sigma)(1+t) \tag{7.11}$$

Notice that this inequality implies that $\operatorname{spt}((1+t)S)\backslash B_R \subset F$ as shown in Figure(7.2), so in particular

$$\operatorname{dist}(\partial B_r \cap \operatorname{spt}(S), \operatorname{spt}((1+t)S)) > 0 \tag{7.12}$$

Also for σ fixed $x = r\sigma + v(r)\nu_C$

$$x \cdot \nu_v = c(r,v) \cdot \left(-\frac{\partial v}{\partial r}, 1\right) = cv - c\frac{\partial v}{\partial r} r > 0 \tag{7.13}$$

from Inequality(7.9) where ν_v is the normal to the graph of v in the plane σ, ν_C and $1/c = \sqrt{1 + \frac{\partial v}{\partial r}^2}$. Hence $x \cdot \nu_S(x) > 0$ for every σ so

$$\inf_{\operatorname{spt}(S)\cap\partial B_r} x \cdot \nu_S(x) > 0 \tag{7.14}$$

It is left to extend this property to any r. To do so notice that

$$\operatorname{dist}(B_r \cap \operatorname{spt}(S), \operatorname{spt}((1+t)S)) = \operatorname{dist}(\partial B_r \cap \operatorname{spt}(S), \operatorname{spt}((1+t)S)) \tag{7.15}$$

otherwise, it would be possible to translate S along the direction connecting the nearer inner points by a factor slightly higher than the distance, in this way the boundary is on one side but the neighborhood of the nearer point is on the other side. This contradicts the fact that for two minimal hypersurfaces if the boundary of one lies on one side of the other so does the whole hypersurface. From (7.14)

$$f(t) = \operatorname{dist}(\partial B_r \cap \operatorname{spt}(S), \operatorname{spt}((1+t)S)) \geq c(r)t$$

for small t. Because $f(t) = x \cdot \nu_S \ t + o(t)$ and from (7.14) there is a $c(r)$ bounding $x \cdot \nu_S > c(r)$. For the same reason, the limit of the distance divided by t in the limits gives the scalar product hence from (7.15)

$$\frac{\operatorname{dist}(B_r \cap \operatorname{spt}(S), \operatorname{spt}((1+t)S))}{t} \geq c(r) \implies \inf_{B_r} x \cdot \nu_S(x) > 0 \tag{7.16}$$

To conclude (7.16) implies that each ray $\ell = \{\lambda x : x \in C, \lambda > 0\}$ intersect $\operatorname{spt}(S)$ only in one point y. As a consequence for any $y \in \operatorname{spt}(S)$ there is a ball $B_\rho(y)$ for which $\operatorname{reg}(S) \cap B_\rho(y)$ is the graph of a function u defined on $\ell^\perp$. But being $\mathcal{H}^n(\operatorname{reg}(S)) = 0$ from regularity of weak solution of the area functional $\operatorname{sing}(S) = \emptyset$. $\square$

8 | Rigidity Result

Throughout the chapter is considered a smooth, complete, connected, and orientable hypersurface $(M^n, g) \hookrightarrow \mathbb{R}^{n+1}$ where g identifies the induced metric on M and $n \geq 2$. Two results regarding the flatness of stable minimal hypersurfaces are given, the first is a conditional result up to $n < 6$, while the second is unconditional but only up to $n < 4$.

8.1. Condition for flatness $n < 6$

The condition required is $|A|^2(x) \leq \lambda_1(-\Delta)$ which is not as strong as it seems, in fact, $\lambda_1(-\Delta)$ on a manifold with negative Ricci curvature will not even tend to zero as the domain increases.

Theorem 8.1. *Let M be a minimal submanifold as above. Suppose that for any ball $B(p,r)$ $p \in M$ it holds the bound $|A|^2(x) \leq \lambda_1(-\Delta)$ $\forall x \in B(p,r)$, where $\lambda_1(-\Delta)$ is the first eigenvalue of $-\Delta$ in $B(p,r)$. Then the hypersurface is stable and, for $n < 6$, it is flat.*

The first step in the proof is to analyze the behavior of Laplace-Beltrami eigenvalues. A bound from above for $\lambda_1(-\Delta)$ can be found in [10, Theorem 5.2].

Theorem 8.2. *Consider $\Omega = B(p, r_0)$ the geodetic ball of center p and radius r_0. If $Ric \geq -(n-1)\beta^2$ in Ω, $\beta > 0$ then*

$$\lambda_1(-\Delta) \leq \frac{(n-1)^2\beta^2}{4} + \inf_{0<t<1} \left\{ \frac{\pi^2}{(1-t)^2 r_0^2} + \frac{(n-1)(n-3)\beta^2}{4\sinh^2(t\beta r_0)} \right\} \tag{8.1}$$

for $n \geq 2$.

In order to use this bound a lower bound on Ric must be evaluated.

Proposition 8.1. *Being S a minimal hypersurface then*

$$Ric = -A^2 \geq -\frac{n-1}{n}|A|^2 \tag{8.2}$$

where the inequality is in the sense of quadratic form that is it holds when is given as input to the tensor a unitary vector. This is a local result, the result in a set Ω is obtained by considering $M = \max_\Omega |A|^2$ then

$$Ric = -A^2 \geq -\frac{n-1}{n}M \tag{8.3}$$

Proof. From Gauss Equation in $\mathbb{R}^{n+1}$ and codimension one

$$R_{ijkl} = A_{il}A_{jk} - A_{ik}A_{jl} \tag{8.4}$$

The Ricci tensor is obtained by contracting the first and last indices i and l

$$Ric_{jk} = \sum_i A_{ii}A_{jk} - A_{ik}A_{ji} = \sum_i -A_{ik}A_{ji} = -A^2 \tag{8.5}$$

where it has been used $\sum_i A_{ii} = 0$ from minimality. To show the bound consider a sequence $\sum_i x_i = 0$ so that $x_1 = -\sum_i^{n-1} x_i$, notice that Jensen inequality on $f(x) = x^2$ implies

$$\frac{1}{n-1}x_1^2 = \frac{1}{n-1}\left(\sum_i^{n-1} x_i\right)^2 \leq \sum_i^{n-1} x_i^2 \tag{8.6}$$

adding x_1^2 to both sides

$$\frac{n}{n-1}x_1^2 \leq \sum_i^n x_i^2 \tag{8.7}$$

To conclude consider coordinates that diagonalize the tensor A then

$$A^2 = \sum_i A_{ik}A_{ji} = A_{jj}^2 \leq \frac{n-1}{n}\sum_i A_{ii}^2 = \frac{n-1}{n}|A|^2 \tag{8.8}$$

□

So Proposition (8.1) in Theorem (8.2) implies $-(n-1)\beta^2 = -\frac{n-1}{n}M$ hence $\beta^2 = \frac{M}{n}$ hence

$$\max_\Omega |A|^2 = M \leq \frac{(n-1)^2 M}{4n} + \inf_{0<t<1}\left\{\frac{\pi^2}{(1-t)^2 r_0^2} + \frac{(n-1)(n-3)M/n}{4\sinh^2(r_0 t\sqrt{M/n})}\right\} \tag{8.9}$$

for any Ω. The proof will be concluded by inversion, suppose that $|A|^2$ is not always null, that is $M > 0$ for some set Ω. Then considering p where $|A|^2 > 0$ and $B(p, r_0)$, identifying $M(r_0) = \max_{B(p,r_0)} |A|^2$. Then the quantity inside the brackets once t is fixed can be sent as low as wanted by increasing r_0 because

$$\frac{f(x)}{4\sinh^2(x\sqrt{f(x)})} \to 0 \quad \text{as} \quad x \to +\infty \tag{8.10}$$

for any $f(x) > 0$ and non decreasing. So for any ε there is a choice of r_0 such that

$$M(r_0) \leq \frac{(n-1)^2 M(r_0)}{4n} + \varepsilon \tag{8.11}$$

which is absurd if $\frac{(n-1)^2}{4n} < 1$, that is for $n < 6$. This concludes the proof.

8.2. Flatness result $n = 2, 3$

In this section, the results in [2] are repurposed as an extension to stable hypersurfaces of Theorem 2 in [17] in the case of flat ambient space. This extension will hold as a corollary the flatness of stable minimal hypersurfaces in $\mathbb{R}^4$ and $\mathbb{R}^3$, the same result in the case of $\mathbb{R}^5$ is blocked by the requirement of completeness for the conformal metric. The method is not applicable in $\mathbb{R}^7$ as the volume growth requirement is too strong. In this section with u it is identified the strictly positive ($u > 0$) solution of $L_M u = -\Delta u + u|A|^2 = 0$ introduced and proved to exist previously.

Theorem 8.3. *Let $f = [2\beta - k(q-n)]\log(u)$ with $k > 0$, $q + \delta \in [4, 4 + \sqrt{8/n}]$ for $\delta > 0$ small enough and $\beta > 0$ satisfying $|\beta - 1| < \sqrt{2 - \frac{q-2}{4(q-4+2/n)}}$. Introducing the conformal metric $\tilde{g} = u^{2k} g$. Suppose that the conformal metric is complete and*

$$\lim_{R\to\infty} \frac{1}{R^{q+\delta}} \int_{B^{\tilde{g}}_{2R}} e^{-f}\, dV_{\tilde{g}} = 0 \tag{8.12}$$

then M is totally geodesic (being restricted to flat ambient space it is flat).

Notice that the restriction on q implies that this method can be potentially applied to $n \leq 5$ because at most $\int_{B^{\tilde{g}}_{2R}} e^{-f}\, dV_{\tilde{g}} = O(R^n)$. This theorem corresponds to the union of [2, Lemma 2.5], which in turn is [17, Theorem 1] weighted, and the Final Estimate of [2].

Lemma 8.1. *For δ small enough, β and q as above, there exists $C > 0$ such that*

$$\int_M |A|^{q+\delta} u^{-2\beta-k\delta} \psi^{q+\delta}\, dV_g \leq C \int_M u^{-2\beta-k\delta} |\nabla\psi|^{q+\delta}\, dV_g \qquad \forall \psi \in C_0^\infty(M).$$

Proof. From [17, (2.8)] in flat case

$$\int_M |A|^p \varphi^2 \leq C \int_M |A|^{p-2} |\nabla \varphi|^2 \qquad \forall \varphi \in C_0^\infty(M), \tag{8.13}$$

for every $p \in [4, 4+\sqrt{8/n}]$ and for some $C = C(n,p) > 0$. Choosing $\varphi = u^\alpha \psi$, with ψ smooth with compact support and since applying the product rule and from Cauchy-Schwarz and Young's inequalities,

$$|\nabla(u^\alpha \psi)|^2 \leq \psi^2 |\nabla u^\alpha|^2 + u^{2\alpha} |\nabla \psi|^2 + 2|\psi \nabla u^\alpha||u^\alpha \nabla \psi| \leq 2\psi^2 |\nabla (u^\alpha)|^2 + 2u^{2\alpha}|\nabla \psi|^2 \,, \tag{8.14}$$

then (8.13) becomes

$$\int_M |A|^p u^{2\alpha} \psi^2 \leq 2C \left[\int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 + \int_M |A|^{p-2} u^{2\alpha} |\nabla \psi|^2 \right] \quad \forall \psi \in C_0^\infty(M). \tag{8.15}$$

The goal is now to bound the first integral on the right-hand side of (8.15) with the second one. Consider the compactly supported field $|A|^{p-2}\psi^2 u^\alpha \nabla u^\alpha$, applying the divergence theorem to it holds

$$\begin{aligned} \int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 = -\int_M |A|^{p-2} \psi^2 u^\alpha \Delta u^\alpha - \int_M u^\alpha \psi^2 \langle \nabla u^\alpha, \nabla |A|^{p-2} \rangle \\ -2 \int_M |A|^{p-2} u^\alpha \psi \langle \nabla u^\alpha, \nabla \psi \rangle \,, \end{aligned}$$

then it is used the fact that

$$\Delta u^\alpha = \alpha u^{\alpha-1} \Delta u + \alpha(\alpha-1) u^{\alpha-2} |\nabla u|^2 \quad \text{and} \quad |\nabla u^\alpha|^2 = \alpha^2 u^{2\alpha-2} |\nabla u|^2 \,,$$

together with Cauchy-Schwarz and Young's inequalities to get

$$\begin{aligned} \int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 \leq -\alpha \int_M |A|^{p-2} u^{2\alpha-1} \psi^2 \Delta u - \frac{\alpha-1}{\alpha} \int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 \\ - \int_M u^\alpha \psi^2 \langle \nabla u^\alpha, \nabla |A|^{p-2} \rangle + \varepsilon \int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 + \frac{1}{\varepsilon} \int_M |A|^{p-2} u^{2\alpha} |\nabla \psi|^2 \,, \end{aligned}$$

for all $\varepsilon > 0$. From the definition of u

$$\begin{aligned} \int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 \leq \alpha \int_M |A|^p u^{2\alpha} \psi^2 - \frac{\alpha-1}{\alpha} \int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 \\ - \int_M u^\alpha \psi^2 \langle \nabla u^\alpha, \nabla |A|^{p-2} \rangle + \varepsilon \int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 + \frac{1}{\varepsilon} \int_M |A|^{p-2} u^{2\alpha} |\nabla \psi|^2 \,, \end{aligned}$$

i.e.

$$\left(1-\varepsilon+\frac{\alpha-1}{\alpha}\right)\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \leq \alpha\int_M |A|^p u^{2\alpha}\psi^2$$
$$-\int_M u^\alpha\psi^2\langle\nabla u^\alpha, \nabla|A|^{p-2}\rangle + \frac{1}{\varepsilon}\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\,,$$

Now, since

$$\nabla|A|^{p-2} = (p-2)|A|^{p-3}\nabla|A| = (p-2)|A|^{\frac{p-2}{2}}|A|^{\frac{p-4}{2}}\nabla|A|\,,$$

then, from Cauchy-Schwarz and Young's inequalities

$$\left(1-\varepsilon+\frac{\alpha-1}{\alpha}-\frac{p-2}{2t_1}\right)\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \leq \alpha\int_M |A|^p u^{2\alpha}\psi^2 \tag{8.16}$$
$$+\frac{(p-2)t_1}{2}\int_M |A|^{p-4}\psi^2u^{2\alpha}|\nabla|A||^2 + \frac{1}{\varepsilon}\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\,,$$

for every $t_1 > 0$. Now, multiplying by $|A|^{p-4}f^2$ the Simons inequality (2.49), integrating by parts and using Young's inequality holds

$$\int_M |A|^p f^2 \geq \left(\frac{2}{n}+p-3-t_2\right)\int_M |A|^{p-4}|\nabla|A||^2 f^2 - \frac{1}{t_2}\int_M |A|^{p-2}|\nabla f|^2$$

for every $t_2 > 0$. Choosing $f = u^\alpha\psi$ it is obtained

$$\int_M |A|^p u^{2\alpha}\psi^2 \geq \left(\frac{2}{n}+p-3-t_2\right)\int_M |A|^{p-4}|\nabla|A||^2u^{2\alpha}\psi^2$$
$$-\left(\frac{1}{t_2}+\varepsilon\right)\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 - \frac{1}{t_2}\left(1+\frac{1}{t_2\varepsilon}\right)\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2 \tag{8.17}$$

for every $\varepsilon > 0$, since

$$|\nabla(u^\alpha\psi)|^2 \leq (1+t_2\varepsilon)\psi^2|\nabla(u^\alpha)|^2 + \left(1+\frac{1}{t_2\varepsilon}\right)u^{2\alpha}|\nabla\psi|^2\,.$$

Now let $\delta > 0$. Using (8.17) in (8.16) with

$$\alpha = -\beta - \frac{k\delta}{2} < 0$$

$\beta, k > 0$ holds

$$\left(1+\frac{1+\beta+\frac{k\delta}{2}}{\beta+\frac{k\delta}{2}}-\left(1+\beta+\frac{k\delta}{2}\right)\varepsilon-\frac{p-2}{2t_1}-\frac{\beta+\frac{k\delta}{2}}{t_2}\right)\int_M |A|^{p-2}\psi^2|\nabla u^{-\beta-\frac{k\delta}{2}}|^2$$
$$\leq \left[\frac{1}{\varepsilon}+\frac{\beta+\frac{k\delta}{2}}{t_2}\left(1+\frac{1}{t_2\varepsilon}\right)\right]\int_M |A|^{p-2}u^{-2\beta-k\delta}|\nabla\psi|^2$$
$$+\left[\frac{(p-2)t_1}{2}-\frac{2\beta+k\delta}{n}-\left(\beta+\frac{k\delta}{2}\right)p+3\beta+3\frac{k\delta}{2}+\left(\beta+\frac{k\delta}{2}\right)t_2\right]\int_M |A|^{p-4}\psi^2u^{-2\beta-k\delta}|\nabla|A||^2\,,$$

for all $\varepsilon, t_1, t_2 > 0$. Let

$$p = q+\delta, \quad t_1 = \frac{2}{p-2}\left(\beta+\frac{k\delta}{2}\right)\left(\frac{2}{n}+p-4\right), \quad t_2 = 1$$

then

$$\frac{(p-2)t_1}{2}-\frac{2\beta+k\delta}{n}-\left(\beta+\frac{k\delta}{2}\right)p+3\beta+3\frac{k\delta}{2}+\left(\beta+\frac{k\delta}{2}\right)t_2 = 0$$

and the hypothesis on β implies

$$1+\frac{1+\beta+\frac{k\delta}{2}}{\beta+\frac{k\delta}{2}}-\left(1+\beta+\frac{k\delta}{2}\right)\varepsilon-\frac{p-2}{2t_1}-\frac{\beta+\frac{k\delta}{2}}{t_2} > 0.$$

Thus

$$\int_M |A|^{q+\delta-2}\psi^2|\nabla u^{-\beta-\frac{k\delta}{2}}|^2 \leq C\int_M |A|^{q+\delta-2}u^{-2\beta-k\delta}|\nabla\psi|^2\,,$$

for some $C > 0$ achieving the goal of bounding the first integral with the second one in (8.15), substituting

$$\int_M |A|^{q+\delta}u^{-2\beta-k\delta}\psi^2 \leq C\int_M |A|^{q+\delta-2}u^{-2\beta-k\delta}|\nabla\psi|^2 \tag{8.18}$$

and from Young's inequality

$$a^\lambda b^{1-\lambda} \leq a + b(1-\lambda)\lambda^{\lambda/(1-\lambda)} \qquad \forall\lambda\in[0,1] \tag{8.19}$$

taking

$$\begin{aligned} a &= |A|^{q+\delta}\varepsilon'^{1/\lambda}\psi^2 \\ b &= |\nabla\psi|^{q+\delta}\frac{1}{\varepsilon'^{1/(1-\lambda)}}\psi^{-(q+\delta-2)} \\ \lambda &= \frac{q+\delta-2}{q+\delta} \end{aligned} \tag{8.20}$$

then (8.19) reads

$$|A|^{q+\delta-2}|\nabla\psi|^2 \leq \varepsilon'^{1/\lambda}|A|^{q+\delta}\psi^2 + \frac{C}{\varepsilon'^{1/(1-\lambda)}}|\nabla\psi|^{q+\delta}\psi^{-(q+\delta-2)} \tag{8.21}$$

using (8.21) into (8.18) holds

$$\int_M |A|^{q+\delta}u^{-2\beta-k\delta}\psi^2 \leq \varepsilon'^{1/\lambda}\int_M |A|^{q+\delta}u^{-2\beta-k\delta}\psi^2 + \frac{C}{\varepsilon'^{1/(1-\lambda)}}\int_M u^{-2\beta-k\delta}|\nabla\psi|^{q+\delta}\psi^{-(q+\delta-2)} \tag{8.22}$$

for all $\varepsilon' > 0$ and $\psi \in C_0^\infty(M)$. Therefore

$$\begin{aligned}\int_M |A|^{q+\delta}u^{-2\beta-k\delta}\psi^2 &\leq C\int_M u^{-2\beta-k\delta}|\nabla\psi|^{q+\delta}\psi^{-(q+\delta-2)}\\ &= C\int_M u^{-2\beta-k\delta}|\nabla\psi^{\frac{2}{q+\delta}}|^{q+\delta}.\end{aligned}$$

The conclusion now follows immediately by replacing ψ with $\psi^{\frac{2}{q+\delta}}$. □

For the final estimate let $x_0 \in M$ and let $\tilde{r}$ the distance function from x_0 with respect to the metric $\tilde{g} = u^{2k}g$. Choosing $\psi := \eta(\tilde{r})$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $[0, R]$, $\eta \equiv 0$ on $[2R, +\infty)$ and $|\eta'| \leq C/R$ on $[R, 2R]$, for some $C > 0$ and $R > 0$. From Lemma 8.1, for some δ small enough

$$\begin{aligned}\int_M |A|^{q+\delta}u^{-2\beta-k\delta}\eta^{q+\delta}\,dV_g &\leq C\int_M u^{-2\beta-k\delta}|\nabla\psi|_g^{q+\delta}\,dV_g\\ &= C\int_M u^{-2\beta-k\delta+k(q+\delta)}|\tilde{\nabla}\psi|_{\tilde{g}}^{q+\delta}\,dV_g\\ &\leq \frac{C}{R^{q+\delta}}\int_{B^{\tilde{g}}_{2R}(x_0)} u^{-2\beta+kq-nk}\,dV_{\tilde{g}}\end{aligned}$$

where it has been used the fact that $|\tilde{\nabla}\tilde{r}|_{\tilde{g}} \equiv 1$. This proves the theorem.
In the following Theorem 8.3 is used together with a generalization of the classical Bishop-Gromov Volume estimate to prove flatness.

Volume Comparison

The weighted Bishop-Gromov volume comparison that will be used is based on an estimate of the N-Bakry-Emery-Ricci curvature tensor instead of the Ricci tensor, this new tensor is defined as

$$Ric_g^{N,f} = Ric_g + \nabla f - \frac{1}{N} df \otimes df \tag{8.23}$$

then the volume comparison states

Theorem 8.4. *Given (M, g) complete n-dimensional Riemannian manifold. If the N-Bakry-Emery-Ricci curvature tensor of $f \in C^\infty(M)$ satisfies $Ric_g^{N,f} \geq (n-1)\eta$ for some real η then the volume growth is controlled by the volume of the geodesic ball in the dummy space of dimension $n + N$ and constant curvature equal to η.*

So in the case of $\eta = 0$ considering balls B_R it holds

$$\int_{B_R} e^{-f}\, dV_g \leq C R^{n+N} \tag{8.24}$$

This comparison is an extension of the classical Bishop-Gromov comparison which is obtained for $f = 0$.

For proving flatness the volume comparison is needed in the conformal metric $\tilde{g}$, hence this metric must be complete. The following result of completeness can be found in [2].

Proposition 8.2. *Consider the conformal metric $\tilde{g} = u^{2k} g$ with $\frac{n-1}{n} \leq k < 1$, then if*

$$P_{k,n}(t) = \frac{k(t-1)^2}{1-k} - 2t + (n-1) < 0 \tag{8.25}$$

for some $t > 1$ the conformal metric $\tilde{g}$ is complete.

The requirement (8.25) is satisfied for $n = 2, 3$ but not for $n = 4$.

At last, the bound on $Ric_f^{N,\tilde{g}}$ must be given for the application of the volume estimate. Given that the coefficient of f differs in general from the one in [2] will be here computed in the case of a general $f = l\log(u)$. From the properties of the connection it holds:

$$df = l\frac{du}{u} \tag{8.26}$$

$$\nabla_g^2 f = l\nabla\left(\frac{du}{u}\right) = -l\frac{du \otimes du}{u^2} + \frac{l}{u}\nabla_g^2 u \tag{8.27}$$

$$\Delta_g f = l\left(\frac{\Delta_g u}{u} - \frac{|\nabla_g u|^2}{u^2}\right) \tag{8.28}$$

from the formulas of conformal metric with $\tilde{g} = u^{2k} g$

$$\begin{aligned} Ric_{\tilde{g}} =& Ric_g - (n-2)k\left(-\frac{du\otimes du}{u^2} + \frac{\nabla^2_g u}{u} - k\frac{du\otimes du}{u^2}\right) \\ & - \left[k\left(\frac{\Delta_g u}{u} - \frac{|\nabla_g u|^2_g}{u^2}\right) + (n-2)k^2\frac{|\nabla_g u|^2_g}{u^2}\right] g \end{aligned} \tag{8.29}$$

and

$$\nabla^2_{\tilde{g}} f = \nabla^2_g f - 2lk\frac{du\otimes du}{u^2} + \frac{lk}{u^2}|\nabla u|^2 g \tag{8.30}$$

hence using $|A|^2 = -\Delta_g u/u$

$$\begin{aligned} Ric_{\tilde{g}} + \nabla^2_{\tilde{g}} f =& Ric_g + \left[(n-2)k^2 - 2kl\right]\frac{du\otimes du}{u^2} + k|A|^2 g \\ & + \left[k + kl - (n-2)k^2\right]\frac{|\nabla_g u|^2_g}{u^2} g + (l-(n-2)k)\left[\frac{\nabla^2_g u}{u} - \frac{du\otimes du}{u^2}\right] \end{aligned} \tag{8.31}$$

from Cauchy Schwartz

$$|\nabla_g f|^2_g g \geq df \otimes df \tag{8.32}$$

and from $Ric = -A^2 \geq -\frac{n-1}{n}|A|^2 g$ if $k + kl - (n-2)k^2 > 0$ it is obtained

$$Ric_{\tilde{g}} + \nabla^2_{\tilde{g}} f - \frac{k}{l^2}\left[n - l - \frac{l}{k} - 1\right]\frac{du\otimes du}{u^2} \geq \left(k - \frac{n-1}{n}\right)|A|^2 g + [l - (n-2)k]\frac{\nabla^2_g u}{u} \tag{8.33}$$

where in the case of Theorem (8.3) $l = 2\beta - k(q-n)$.

The flatness is now just a matter of applying these results. In particular, it must be found a choice of variables that satisfies the hypothesis of the theorem, the restriction for completeness, and a lower bound with $\eta = 0$ on the N-Bakry-Emery tensor with $n + N < q + \delta$.

Case n=3

Notice that if $l = k(n-2) = k$ the Hessian in (8.33) vanishes. Furthermore the best choice of k for $P_{k,n}(t) < 0$ is $k = \frac{n-1}{n} = \frac{2}{3}$, the inequality is satisfied at $t = \frac{3}{2}$. Hence the bound (8.33) reads $Ric_g^{2,f} \geq 0$ and the volume comparison gives

$$\int_{B_R^{\tilde{g}}} e^{-f}\, dV_{\tilde{g}} \leq CR^5 \tag{8.34}$$

Being $n + N = 5$ it is taken $q = 5$ implying that $\beta = 1$ which satisfies the hypothesis and

$$\lim_{R\to\infty} \frac{1}{R^{5+\delta}} \int_{B_{2R}^{\tilde{g}}} e^{-f}\, dV_{\tilde{g}} = 0 \tag{8.35}$$

hence from Theorem (8.3) the hypersurface is flat.

Case n=2

For this case the classical Gromov Estimate will be used, hence $f = 0$ that is $l = 0$, and a bound on Ricci is needed. From (8.29) with $n = 2$:

$$Ric_{\tilde{g}} = Ric_g - k\left(\frac{\Delta_g u}{u} - \frac{|\nabla_g u|_g^2}{u^2}\right) g \tag{8.36}$$

again $\frac{\Delta_g u}{u} = -|A|^2$ and $Ric_g = -A^2 \geq -\frac{n-1}{n}|A|^2 g$ hence:

$$Ric_{\tilde{g}} \geq \left(k - \frac{n-1}{n}\right)|A|^2 + k\frac{|\nabla_g u|_g^2}{u^2} g \geq 0 \tag{8.37}$$

Hence the classical Gromov volume comparison theorem holds

$$\int_{B_{2R}^{\tilde{g}}} dV_{\tilde{g}} \leq CR^2$$

Taking again $k = \frac{1}{2} = \frac{n-1}{n}$ with $t = 2$ $P_{k,n}(t) < 0$. Then the choice of $q = 4$ holds $\beta = 1/2$ that is admissible and

$$\lim_{R\to\infty} \frac{1}{R^{4+\delta}} \int_{B_{2R}^{\tilde{g}}} e^{-f}\, dV_{\tilde{g}} = 0 \tag{8.38}$$

concluding.

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