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Deformation Monitoring Using the Inverse Finite Element Method (iFEM): A System- atic Approach to Sensor Network Design

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1 | Abstract

This thesis presents a comprehensive methodology for structural health monitoring (SHM) of composite radome panels through the inverse Finite Element Method (iFEM), addressing the critical need for real-time deformation reconstruction in aerospace applications. The study focuses on a laminated Kevlar/epoxy composite plate subjected to bending-dominated loading scenarios, serving as a simplified geometric counterpart to full-scale radome structures. The research systematically develops both direct and inverse finite element frameworks. High-fidelity direct FEM models simulate the mechanical response under Three-Point Bending (3PB) and Cantilever Plate (CP) configurations, evaluated through both Displacement Control and Load Control conditions. These models provide synthetic strain data that replicate Fiber Bragg Grating (FBG) sensor measurements, enabling rigorous validation of the inverse algorithm. A quadrilateral inverse-shell element (iQS4) formulated on First-Order Shear Deformation Theory serves as the foundation for displacement reconstruction from discrete strain measurements. Extensive numerical sensitivity analyses optimize the sensor network design, revealing that while shear strain contributions are negligible in bending scenarios, strategic placement of sensors capturing both longitudinal and transverse strains significantly enhances reconstruction accuracy. The study demonstrates that double-sided instrumentation, though more complex, provides superior noise resilience compared to single-surface configurations. Two strain pre-extrapolation strategies are evaluated: Physics-Based methods employing biharmonic spline interpolation and Data-Driven approaches utilizing polynomial surface fitting. Furthermore, noise analysis reveals that polynomial approaches exhibit inherent smoothing effects that effectively filter measurement perturbations, whereas Physics-Based methods show higher sensitivity to localized errors. The optimized sensor network achieves displacement reconstruction errors below 1.5% for 3PB configurations and below 5% for CP configurations under ideal conditions. Under realistic noise levels ($15 \mu\epsilon$), errors remain bounded within acceptable thresholds, validating the methodology's robustness for operational implementation. This work establishes a validated numerical framework that bridges theoretical formulation with practical SHM requirements, providing essential groundwork for experimental validation and eventual deployment on full-scale aerospace structures.

2 | Introduction

In recent years, Structural Health Monitoring (SHM) has gained increasing relevance in the design and maintenance of advanced engineering structures. The ability to continuously monitor the structural integrity of components allows for early detection of damage, optimization of maintenance schedules, and significant improvements in safety and lifecycle cost reduction. SHM systems are particularly valuable in aerospace, civil, and marine applications, where structures are often subjected to complex and variable loading conditions that can lead to fatigue, delamination, or impact damage over time.

Among the various approaches developed to interpret strain data collected from embedded or surface-mounted sensors, the inverse Finite Element Method (iFEM) has emerged as a powerful tool. Unlike conventional finite element analysis, which predicts displacements and strains from known loads, iFEM reconstructs the full-field displacement and strain fields from discrete strain measurements. This inverse approach enables real-time reconstruction of structural states without requiring prior knowledge of external loads or boundary conditions, making it highly suitable for integration into sensor-based SHM frameworks. When combined with high-resolution sensors such as Fiber Bragg Gratings (FBGs), iFEM provides a robust, model-driven pathway for damage detection and condition assessment in composite and anisotropic structures.

Recently, the iFEM approach has gained a lot of attention in shape sensing because it aligns well with the general implementation strategy of SHM. Many of the methods presented in the literature require knowledge of the loading conditions and the mechanical properties of materials, which are information that are often difficult to establish without laboratory tests or empirical models. For these reasons, such approaches are generally unsuitable for use in the SHM framework because a valid algorithm should be general, robust, stable, and precise in a wide range of loads, materials, inertial characteristics, inherent errors in deformation measurements and it also has to be fast enough for real-time applications. Sensor systems are used to collect structural data that are then processed within iFEM to draw meaningful conclusions about the state of health and integrity of the structure. The initial framework for iFEM was firstly introduced by Tessler and Spangler [1–3] for the reconstruction of three-dimensional displacement in real-time full field using instrumented strain data. This alternative method has attracted significant interest from the scientific community and in the following years its mathematical formulation has been tested to evaluate its capability in accurately reconstructing full-field displacement and strain using both numerically generated and in-situ acquired strain data, as will be showed in

the following review.

A meaningful reconstruction of the full field is crucial for the identification of the damage entity. The damage's feature assessment depends on the level adaptation of the general baseline to the characteristic of the structure under exam. For this purpose, many IE were proposed to implement algorithms that are able to exploit the iFEM methodology. Tessler and Spangler [3] proposed the first three-node IE iMIN3 for the full-field displacement reconstruction of plate and shell structures undergoing general loading conditions. Kefal et al. [4] proposed a two-dimensional four-node quadrilateral inverse shell element iQS4, which was successfully validated in real-case scenario [5–8] and an eight-node curved inverse shell element iCS8 [9, 10] with quadratic isoparametric shape functions. These elements have been compared in a study [11], which demonstrated that the iQS4 outperforms the iMIN3 in terms of accuracy for plane problems, whereas for curved structures, the highest accuracy is achieved by the iCS8.

Cerracchio et al. reformulated element iMIN3 [3] to include the kinematic assumptions of the Refined Zigzag Theory (RZT), defining a new three-node IE iRZT3, able to deal with composite panel and sandwich structures [12, 13], along with Kefal [14] which formulated a four-node element iRZT4 accounting the RZT as well, later validated both numerically and experimentally [15, 16].

The iRZT3 element was later refined by Kefal et al. extending its range of applications to problems that involve sparse in situ strain data, developing a new IE i3-RZT [17–19]. de Mooij et al. [20] validated the inverse formulation for conventional solid elements by developing a 20-node hexahedral implementation (C3D20R), demonstrating that standard FEM shape functions could be directly employed in iFEM analyses.

To address the discretization challenges that have arisen in the transition from flat to curved elements, different isogeometric IEs iKLS [21, 22], iKP4 [23] and IgaiMin [24] were formulated for iFEM analysis of shell structures under reduced sensor arrangement. In parallel with the formulation of dedicated finite elements, other studies have also explored the possibility of combining iFEM with isogeometric analysis [25–27]

Niu et al. formulated a triangular IE iMITC3+ coupling Mixed Interpolation of Tensorial Components (MITC) technique with iFEM [28] and a five-node inverse plate element to allow shape sensing in single sensorized surfaces [29]. Zhu et al. [30] formulated an inverse beam element with a 2nd order shape function for axial deformation. Shang et al. [31] proposed the iQAC method to account for excessive distortion in isoparametric context. While the vast majority of plate IE are based on Mindlin theory, Wu et al. [32] adapted the formulation to Kirchhoff-Love theory to better describe the behavior of thin shell structures, reducing the functional to be minimized to fewer terms

The need to properly model cracks in structure led researchers to formulate different crack adaptive IEs. Li et al. [33] proposed an eight-node triangular element accounting for stress

singularities at crack's tip, Khalid et al. a six-nodes triangular element iTP6 able to overcome singularity at crack edges in a computationally efficient way . A study by Xiao et al. [34] integrates iFEM with the Mixed Interpolation of MITC method to develop a novel four-node quadrilateral inverse curved shell element, named iMICS4. Huang et al. [35] developed two-node inverse Hermite beam element iHB2 with Hermite polynomial shape function, to implement a piecewise iFEM able to discretize the structure into a combination of rigid and elastic parts. Another study for complex structure with variable elastic properties (stiffened Aircraft structures) was done by Chen et al. [36], which improved the shape sensing capability by coupling the iFEM with Eringen's differential nonlocal theory. Shang et al. [37] proposed an innovative three-dimensional sixteen-nodes framework (16-iSBFE) that unifies displacement field reconstruction for varying thickness plates, overcoming classical iFEM limitations through integration of scaled boundary element theory and degenerated shell elements. Recent advances in iFEM for shell structures include polygonal elements iSPS with MITC and strain smoothing, enabling flexible mesh refinement and high accuracy with sparse sensor data [38]. Recently, Li et al. proposed a two-dimensional eight-node quadrilateral element iQP8 [39] and a two-dimensional four-node quadrilateral element iQP4 [40] suitable for plane stress loading condition and validated it against different in-plane loading conditions with reduced sensor arrangement. Khalid et al. [41] defined an IE iKS4 which aims to overcome discretization and shear-locking issues in shell structures and compared it numerically against the already existing element iQS4.

Although it deviates from the scope of this work, it is worth noting that the described methodology has been validated with numerical and experimental validation also for a wide span of slender beam-like structure, ranging numerous different configurations, like classical constrained beam, varying cross-section, complex cross-sections, composite beams, beams subjected to non-linear displacement or dynamic loads, airfoil beams and many others [26, 42–79]. In order to properly assess the beam shape sensing it was formulated a two-node IE iBeam3 for beam-like structures [80, 81], which shape-sensing capability has been compared with the pre-existing iQS4 element [50]. Recently, a newer IE iEBT2 was formulated such that the coefficient matrix was made independent of strain sensors location and so the existence of a solution is always guaranteed , overcoming ill-posed issues of the coefficient matrix [82]. As for what was done for plate and shell structures, isogeometric analysis based on nonuniform rational B-spline (NURBS) exploitation was carried out for beam structures [57, 58, 83], and another IE for beams named iPCB was formulated exploiting nonuniform rational B-spline (NURBS) as shape functions to interpolate displacement fields across nodes [84]. Savino e Tondolo [85] expanded the existing IEs by introducing three different elements named iCB3, iCB4 and iCB5 (respectively three-node, four-node and five-node elements) by considering the exact solution of the curved beam theory that corresponds to the membrane-bending coupling and the explicit statement of the rigid-body motions.

The primary limitation emerged from the theoretical formulation for the shell structures iFEM approach is its inherent requirement for triaxial strain measurements at each element of the mesh,

a condition that is often impractical due to cost and spatial constraints. To mitigate this issue, weighting factors were originally integrated into the initial iFEM formulation [1, 2]. To avoid setting the strain to zero in the elements missing experimental measurements input, several studies have been conducted with the objective of identifying the most robust and efficient interpolation and reconstruction techniques of in-situ acquired data, thereby enabling the use of nonzero weighting factors. Among the different strain interpolation techniques proposed, Smoothing Element Analysis (SEA) [18, 19, 86–88] has proven to be one of the most effective options after being compared to polynomial fittings [89]. Recent papers have also studied physics-based strain interpolation for handling interpolation in cases of sharp strain gradients induced by holes and rivets [90]. Poloni et al. [91] proposed a thickness-normalized strain pre-extrapolation for iFEM, overcoming key limitations of SEA and polynomial fitting. Unlike these methods—which require piecewise treatment of thickness variations—their approach enables global strain interpolation by removing thickness-induced discontinuities, significantly simplifying sensor networks (SN) for variable-thickness structures. Kobayashi et al. [92] demonstrated that integrating distributed fiber-optic sensors with cubic spline strain interpolation techniques enables an improvement in global strain field reconstruction. Zhu et al. [68] provided a refinement of iFEM formulation with high-order Lagrange shape functions to reconstruct full-field displacements/strains from sparse sensor data, prioritizing interpolation accuracy over sensor placement optimization. To overcome the uncertainties inherent in interpolation, it has been proposed a Gaussian Process as interpolation technique, which is then used as a weighting metric in the iFEM functional [93].

The theoretical consideration made so far highlighted how the accuracy and efficiency of inverse Finite Element Method (iFEM) for structural health monitoring rely on optimal sensor placement, as strategic positioning of strain sensors ensures robust strain field reconstruction while trying to minimize measurement uncertainties and computational costs. Considering this multiple studies have addressed this aspect. Zhao et al. introduced an eigenvalue-based optimization framework for strain sensor placement in iFEM applications, leveraging Particle Swarm Optimization (PSO) [94, 95], building on these studies, the author advanced beyond single-objective approaches by introducing a multi-objective optimization framework (MOPSO), which unlike earlier eigenvalue-focused methods explicitly accounts for practical installation errors [55]. Further refining this direction, Li et al. proposed an improved adaptive MOPSO (IAMOPSO) that enhances exploration-exploitation balance through dynamic inertia weights, guided particle selection, and Pareto-front maintenance, achieving superior accuracy and robustness in complex structural reconstructions [96]. Building on these advancements, Niu et al. [97] developed a unified shape sensing method (iSBFEM) that extends applicability to thick and moderately thick plates, leverages single-surface strain measurements, and incorporates transverse shear effects via stress equilibrium and SEA. Niu et al. [29] proposed a coupled iFEM-Scaled Boundary Analysis approach for plate structures, eliminating the need for symmetric sensor deployment while maintaining reconstruction accuracy. Their method leverages a five-node inverse plate element and a bi-objective optimization framework to minimize sensor count and enhance robustness against measurement noise. Zhao et al. then developed the improved adaptive large-scale co-

operative coevolution (IALSCC) algorithm, which introduces adaptive region partitioning and dynamic gbest selection to handle large-scale sensor optimization for iFEM [98]. Roy et al. [99] addressed the problem with a systematic approach to sensor pattern selection by testing different structured FBG configurations for iFEM-based plate deformation reconstruction. Regarding the handling of inherent sensor errors, a study by [100] presented a novel two-step calibration approach combining error distribution to nodal degrees of freedom (DOF) and NURBS-based data augmentation to enhance deformation reconstruction [100]. This method was later enhanced by introducing a Bayesian-regularized DOF error correction model and NURBS-based approximation for noisy data expansion, enabling reconstruction errors reduction not only on extreme deformation points but across the entire structure [101]. Ghasemzadeh et al. [102] exploited the Genetic Algorithm inside the iFEM framework for automated sensor placement optimization in plate/shell structures. Lu et al. [103] proposed a two-stage calibration method (CFTSC) combining single-objective PSO for coarse sensor position correction and self-constructed fuzzy networks for fine error compensation, significantly improving iFEM displacement accuracy under installation offsets. Del Priore and Lampani [104] introduced an innovative approach combining iFEM with multi-objective sensor placement optimization and strain pre-extrapolation via SEA for plate structures.

The remarkable effectiveness of iFEM in reconstructing deformations compared to other shape sensing techniques has led to an extensive research into its application within SHM, aiming to refine the original algorithm to tailor it to specific structural requirements. Different studies introduced an incremental iFEM framework to reconstruct large displacements under quasi-static or dynamic loads, iteratively updating geometry and strain measurements [105, 106]. Zhu et al. [107] coupled the iFEM algorithm with a nodal strain vector formulation and an element strain function, which allowed for direct strain field reconstruction without relying on displacement-derived solutions, significantly improving accuracy in bending-torsion scenarios. Wu e Tan [108] combined iFEM with energy-based regularization to adapt the method to new available uniaxial Distributed Fiber Optical Sensing. The very nature of iFEM algorithm. The very nature of the iFEM algorithm makes it particularly suitable for implementation as sensory information within a digital-twin model. this possibility has been explored in a naval structure by Wei et al. [109]. Ji et al. [110] proposed a single-side iFEM algorithm coupled with a flexible sensing film, avoiding the need for symmetric sensor placement. Esposito [111] proposed a novel two-step shape sensing approach that couples the inverse Finite Element Method (iFEM) with Modal Method (MM), where the latter is adopted as a Virtual Sensor Expansion (VSE) method for the pre-extrapolation to a large number of strain information from a small amount of measured strains. Dai and Xie [112] proposed iXFEM, merging iFEM with XFEM's enrichment strategies to monitor cracked structures—overcoming classical iFEM's inability to handle discontinuities. Their approach enables direct fracture parameter estimation from sparse strain data. Belur et al. [113] propose an extension of iFEM, termed iFEM-MoRe, which integrates Welch's Fourier transformation transforming full-field displacement data into the frequency domain, enabling accurate extraction of modal parameters and high-resolution mode shape reconstruction under

random excitation. Recently, Poloni et al. [62] introduced a reference-free deflection monitoring method for multi-span bridges, combining distributed fiber optic sensing (BOTDA) with the Inverse Finite Element Method (iFEM). In the context of population-based SHM (PBSHM), Delo et al. have recently advanced the integration of iFEM with PBSHM through two key studies: the first [114] enabled the integration of fiber-optic strain data with classical modal analysis and the harmonization of heterogeneous data to be shared between distinct structures' sensorized domains, the second [115] extended this framework by leveraging Transfer Component Analysis to enhance knowledge transfer across populations with varying sensor setups, validating the approach on experimental aircraft models. Together, these works pave the way for more accurate PBSHM. Biscotti et al. [116] proposed a closed-loop control strategy for morphing wings, exploiting the iFEM to reconstruct global displacements from strain data. Their approach bridges the gap between shape sensing and active actuation control, enabling real-time adaptation to aerodynamic loads. Wu et al. [117] introduced an Extended iFEM (EiFEM) framework that integrates solid linear tetrahedron elements and energy-based regularization, enabling accurate shape reconstruction of geometrically complex structures. Their method resolves ill-posedness caused by sparse fiber-optic strain data through the imposing of mechanical consistency via deformation energy minimization. Bardiani et al. [118] presented a SHM framework for large-scale containerships SHM that combines iFEM with a multi-objective optimized FBG SN, capable of minimizing displacement reconstruction errors and decoupling weight matrices for top/bottom shell surfaces. Zhang et al. [119] proposed an enhanced iFEM leveraging Singular Value Decomposition (SVD)-based strain field pre-regression to overcome sensors number and placement regularity constraints, their data-physics hybrid approach achieved superior noise immunity compared to conventional polynomial fitting or SEA methods. The displacement-gradient-based variable-size iFEM (Dong et al. [120]) introduces an adaptive element discretization strategy to optimize computational efficiency for large-scale structures, achieving real-time shape sensing with minimal loss of accuracy. This approach addresses a critical limitation of classical iFEM, where uniform meshing often leads to unnecessary computational overhead in mildly deformed regions. Guo et al. [121] proposed the iLRPIM, a mesh-free inverse method that overcomes the limitations of Jacobian transformations in iFEM, achieving higher accuracy for irregular plate structures through radial point interpolation. Li et al. [122] proposed a combination of iFEM and Radial Basis Functions as framework for distributed dynamic load identification, bypassing transfer matrices to avoid ill-posedness. The rise of artificial intelligence has opened new possibilities for enhancing the iFEM, offering powerful tools for real-time structural health monitoring and damage identification. The integration of AI with iFEM for SHM is exemplified by Li et al. [123], who employed a CNN trained on strain mode differences derived from iFEM to achieve accurate damage localization and quantification. Another study by Zhu et al [124]. employed a machine learning model which was trained with FBG spectral data as input data to predict decoupled temperature and strain values. Huang et al. [125] proposed a strain surrogate model based on a parallel Multi-Layer Perceptron (MLP) which was able to establish a nonlinear mapping between limited experimental data and full-field strains, enabling accu-

rate deformation monitoring under thermal loads for large-scale aerospace structures. Bardiani et al. [79] introduced a probabilistic strain interpolation and extrapolation method based on Gaussian Process (GP) regression to address the limitations of deterministic 1D iFEM, which typically requires two strain measurements per element. By leveraging a squared exponential kernel to model the strain field, the framework provides non-parametric approximations and naturally yields uncertainty estimates that are converted into functional weights for the iFEM least-square minimization. This stochastic weighting strategy prioritizes direct sensor measurements while proportionally discounting extrapolated data characterized by higher uncertainty, ensuring a more robust displacement field estimation.

Multiple different techniques have been developed and integrated by researchers to adapt iFEM to focus on a paramount aspect of SHM which is damage quantification and identification, by recognizing that inconsistencies between the physical structure and the iFEM model result in erroneous displacement and strain field reconstructions [126, 127]. One of the original algorithm extensions was made through the field of modal analysis, which was exploited by individuating its relative parameters inconsistencies [123, 128, 129]. iFEM was also coupled with others already existing damage identification methods, such as pseudo-excitation approach [130]. Specifically to crack damage context, various studies were conducted to assess both static and dynamic state of cracks in plate structure. Colombo et al. provided an anomaly identification method on pre-cracked plate structures under different loading conditions, verifying the capability of the iFEM algorithm to detect and localize the presence of such defects [131, 132]. Poloni et al. [133] applied the concept of anomaly index also to debonding quantification in adhesive-bonded joints, successfully monitoring the presence and the advancing of the delamination entity. Li et al. [134] successfully applied iFEM to detect dent damage in stiffened cylindrical offshore structures, introducing a novel damage parameter derived from von Mises strain variations. Kefal et al. studied the capability of the iFEM to predict crack propagation by using as input for the algorithm point-wise strain data collected via DIC [135] and proposed a novel algorithm that couples iFEM and PD for damage propagation prediction [136]. Roy et al. [137] addressed the problem of damage detection in a thin cracked plate under the action of in-plane loading by reconstructing the full-field strain using discrete uniaxial strain data. Oboe et al. [138, 139] were also able to deal with the estimation of the actual crack size by systematically introducing the damage into the iFEM model and assessing the crack's size by means of a maximum likelihood principle. Ganjdoust et al. [16] developed an iFEM-RZT-based approach for delamination detection in composites, introducing layer-wise damage indices that combine in-plane and through-thickness strain measures, overcoming the limitations of classical plate theories. Qiu et al. [140] were able to perform damage localization exploiting anomaly index based on classical iFEM framework, and moreover coupled classical iFEM with Bayesian Learning, enabling to integrate the damage identification also with damage quantification information. Ghasemzadeh et al. [141] demonstrated the effectiveness of iFEM in detecting pitting corrosion in marine structures, leveraging a novel strain-based damage index to localize defects even with complex morphologies. Zhao et al. [142] integrated iFEM with Support Vector Re-

gression (SVR) approach to automate crack detection and depth quantification in steel beams, leveraging a novel anomaly index to decouple substrate strain from crack-induced strain. Chen et al. [143] combined iFEM with Variational Mode Decomposition (VMD)-Hilbert signal processing to localize low-velocity impacts on carbon fiber reinforced polymer (CFRP) plates using FBG sensors. A normalized damage indication factor based on von Mises equivalent strain variations was utilized by Abdollahzadeh et al. [144] to effectively localize and identify crack initiation zones in automotive suspension components.

Since the first mathematical formulation was presented, iFEM has emerged as a pivotal tool in the context of Structural Health Monitoring (SHM), offering enhanced capabilities for real-time assessment and damage detection in structural systems. For this reason, the validated SHM framework has been implemented in the shape sensing of numerous real systems where real-time deformation monitoring, damage detection, and load assessment are essential to prevent catastrophic failures under dynamic or extreme operational conditions. The application of iFEM in SHM of marine structures is crucial for real-time monitoring and damage assessment under harsh operational conditions, leading to several studies validating its effectiveness in accurately reconstructing structural deformations and detecting anomalies, helping to well-design a monitoring system suitable for their characteristics through numerical validation. Validation of this method are present in literature on a variety of structures, such as submarine pressure hull [145], offshore measurements towers [9], wind turbines [77, 134, 146–148], cylindrical structure susceptible to corrosion [141], chemical tankers [6], container ships [7, 118, 149, 150], pipes and sub-sea pipelines [30, 61, 151], floating reflector antenna [8], bulk carriers [152], hull and relative stiffeners [36, 107, 153, 154], submarine cables [155].

The integration of iFEM into SHM for aerospace and aeronautical structures plays a key role in enabling accurate deformation monitoring and damage detection while adhering to the strict constraints of weight and minimal sensor intrusiveness required in such applications. Given the demanding environmental conditions these structures face, significant research efforts have been dedicated to assessing the reliability of iFEM in reconstructing full-field deformations and identifying structural anomalies on a span of different systems, helicopters [156], wings and airfoil structures [35, 37, 49, 64, 72, 94, 110, 116, 157–161], sensing satellites [120, 124, 125, 162].

Following the comprehensive review of iFEM fundamentals, spanning elements formulations, validation protocols, damage detection capabilities, interpolation techniques, sensor placement optimization, and applications in marine and aerospace structures, The focus now shifts to a critical specialization: iFEM implementations for shape sensing in FBG-instrumented composite plates. Over the past decades, the extensive adoption of composite materials has led to significant advantages in terms of weight reduction and enhanced performance, particularly within the aeronautical and aerospace sectors. Nevertheless, these materials are vulnerable to various forms of damage and degradation, notably delamination, debonding, and crack propagation through both the fiber and the matrix. Consequently, aeronautical structures require frequent and costly inspections through nondestructive testing (NDT) techniques to ensure their structural integrity,

which in turn limits their operational availability. In response to these challenges, Structural Health Monitoring (SHM) systems have been increasingly implemented with the aim of enhancing the safety and availability of structures. By enabling real-time diagnosis and prognosis of degradation processes, SHM systems offer the potential to extend service life, optimize maintenance scheduling, and ultimately reduce operational costs. However, this emerging application domain presents unique challenges, stemming from the non-isotropic behavior of composite materials and the complex mechanisms of strain transfer in layered structures, which significantly complicate the accurate detection and interpretation of damage phenomena.

Colombo et al. [127] applied the inverse Finite Element Method (iFEM) to a Carbon Fiber Reinforced Polymer (CFRP) stiffened panel subjected to ballistic impact and fatigue loading, aiming to reconstruct the strain field and detect damage. A distributed fiber optic network was used both for strain input and testing, enabling a nearly continuous strain acquisition. To address non-trivial boundary conditions, the authors introduced a boundary effect superimposition technique based on the weighted contribution of multiple simplified models. Damage identification was performed through a load-independent Anomaly Index and Mahalanobis distance.

Kefal et al. [163] experimentally implemented the inverse Finite Element Method (iFEM), enhanced with Refined Zigzag Theory (RZT), to perform real-time shape and strain sensing on a thick wing-shaped sandwich structure composed of woven composite face sheets, subjected to bending load applied by means of a loaded tip coincident with a corner of the wing structure. The structure was instrumented with both embedded FBG sensors and surface-mounted strain gauges, and the strain data were processed through iFEM to reconstruct the full-field displacements and strains. Following this study, also Abdollahzadeh et al. [15] applied the method for shape sensing of thick sandwich structures using iFEM-RZT. Both numerical and experimental validations were conducted, utilizing embedded FBG sensors and surface-mounted strain gauges, with polynomial smoothing of sparse strain data before iFEM processing. The experimental setup in this case was made by a wing-shaped sandwich panel but under torsional loading, realized through a similar loaded tip

Kefal and Tessler [164] proposed a novel delamination damage identification strategy for laminated composite and sandwich shell structures by integrating the inverse Finite Element Method (iFEM) with Refined Zigzag Theory (RZT). Their method reconstructs principal section strains and curvatures based only on strain gauge measurements, without requiring any prior knowledge of loading conditions. The numerical setup consisted in plates and shallow shells with embedded delaminations, and for this structures they were addressed damage size, in-plane location, and through-thickness position when subjected to torsional loads. Following this study, also Ganjdoust et al. [16] performed a numerical campaign on detecting delamination through-the-thickness damage in composite structures by employing iFEM enhanced with RZT. The numerical setup was made by plate, stiffened plate and shell with different damage locations and under tensional, bending and torsional loads.

Another study from [5] applied iFEM to reconstruct large, geometrically non-linear deformations

of thin laminated composite plates and shells. The iQS4 quadrilateral inverse shell element was used, integrating a polynomial strain pre-extrapolation to enhance sparse strain measurements. Experimental validation was performed on a clamped flat composite plate under increasing compressive line load.

Oboe et al. [89] applied the inverse Finite Element Method (iFEM) to reconstruct displacements and strains in a CFRP plate subjected to compression buckling, addressing the limitations imposed by incomplete SN. Strain pre-extrapolation was carried out using both Smoothing Element Analysis (SEA) and polynomial fitting before feeding the data into the iFEM. Validation was performed through laser measurements and FEM simulations. The authors addressed how proper pre-extrapolation is crucial to achieve accurate iFEM displacement reconstruction under sensor-sparse conditions. Experimental and numerical validations confirm that iFEM can effectively reconstruct full-field deformations under buckling of layered composite structures.

Poloni et al. [91] proposed a novel variable-thickness strain pre-extrapolation method to improve the accuracy and cost-efficiency of iFEM-based Structural Health Monitoring (SHM) for composite structures with non-uniform thickness. The authors introduced a normalization technique scaling membrane strains and curvatures based on laminate thickness, allowing a global extrapolation of strain fields across regions of varying thickness. A numerical case study on a clamped composite plate subjected to traction, tip loads, uniform pressure, and shear loads validated the methodology, highlighting substantial reductions in required sensors without compromising shape-sensing accuracy.

Roy and Gherlone [165] developed a novel damage detection strategy for composite structures by combining iFEM with SEA to reconstruct displacements and identify damage without requiring prior healthy-state information. Their method relies on establishing a real-time baseline from the smoothed iFEM results and comparing it to the current state to identify damage, such as delaminations and skin-spar debonds. The strategy was numerically validated on a CFRP plate with delamination and on a composite wing box under static tip loading

Poloni et al. [133] proposed an innovative application of iFEM for quantifying debonding in adhesively bonded CFRP joints using sparse strain measurements. Two strategies were compared: a damage detection approach based on anomaly indexes and a model-updating strategy where multiple iFEM models with different assumed debond lengths are compared through a maximum likelihood estimation. Experiments were conducted on cracked lap shear specimens subjected to fatigue loading, equipped with distributed fiber optic sensors (OBR) and FBG sensors.

Kobayashi et al. [92] have studied the shape sensing of CFRP and aluminum honeycomb sandwich panels using iFEM combined with distributed fiber-optic sensors. The study included both numerical simulations and experimental validations. Numerical analyses were performed to evaluate the influence of sensor types (point, quasi-distributed, and distributed) and densities on reconstruction accuracy, while experimental tests involved a CFRP sandwich panel with embedded all-grating fibers subjected to four static bending load cases, including localized

and distributed loads. The results demonstrated that distributed sensing provided the highest accuracy in displacement reconstruction.

Li et al.[166] have studied the full-field deformation reconstruction of large-scale cryogenic CFRP composite tanks using iFEM combined with limited strain data from distributed optical fiber sensors. The aim of their work was to develop a robust methodology for online deformation monitoring of aerospace composite tanks, which are subjected to complex thermo-mechanical loads and exhibit critical strain gradients in the junction area between the head and cylindrical shell. The study employed numerical simulations to validate the proposed strain reconstruction method, focusing on deriving inner-surface strains from outer-surface measurements using analytical formulations. The composite tank was subjected to uniform internal pressure and cryogenic thermal loads, simulating realistic operating conditions. The results demonstrated that the iFEM, when combined with optimized sensor placement and strain interpolation techniques, could accurately reconstruct full-field deformations, even with sparse strain data. Additionally, they highlighted the importance of sensor placement strategy, showing that denser sensor spacing in the junction area significantly improves deformation prediction.

Esposito [111] has studied a novel shape sensing approach that couples Modal Virtual Sensor Expansion (VSE) with iFEM to reconstruct the displacement field of composite stiffened structures using a reduced number of strain sensors. The study aimed to overcome the limitations of standalone iFEM (requiring dense sensor arrays) and the Modal Method (MM) (limited accuracy) by combining their strengths. Numerical simulations and experimental tests were conducted on TWILL T-300 carbon-fiber fabric prepreg composite stiffened panels and a wing-shaped structure, respectively, under static concentrated loads applied at different locations to induce bending and torsional deformations.

Li and Sharif-Khodaei [167] have studied shape sensing in composite shell structures using distributed fiber optic sensing combined with iFEM, combined with a strain interpolation technique leveraging single image super-resolution to preserve high spatial frequency features in non-smooth strain fields. Experimental validation was conducted on three CFRP specimens with varying dimensions, materials, and layouts. The specimens were subjected to tensile, three-point-bending (3PB), and compression loads to evaluate the method's accuracy under different loading conditions. Distributed fiber optic sensors, employing optical frequency domain reflectometry, were used to capture strain data, while iFEM was applied to reconstruct displacement fields.

Abdollahzadeh et al. [144] extended the application of the iQS4 element to complex automotive geometries by evaluating a 17-layer carbon-fabric laminated control arm, demonstrating that iFEM framework maintains high fidelity in reconstructing structural responses—achieving a 1% deviation in displacement while effectively managing the orthotropic behavior and complex stacking sequences inherent to advanced composite architectures.

The reviewed literature demonstrates the widespread adoption of iFEM for shape sensing and

SHM of composite structures, particularly those instrumented with FBG sensors. Previous studies have successfully explored iFEM applications across a variety of loading scenarios, such as tension, bending, torsion, and buckling, and diverse structural configurations, including flat plates, stiffened panels, and curved shells. Furthermore, key theoretical advancements have been established in recent years, encompassing strain pre-extrapolation techniques, integration with the Refined Zigzag Theory (RZT) for thick composites, and robust damage detection strategies.

In light of the aforementioned literature, this thesis aims to bridge the gap between theoretical formulations and the practical implementation of shape sensing techniques for advanced composite structures. Specifically, the present work introduces the mathematical framework of the iFEM and systematically applies it to the deformation reconstruction of a composite plate designed for radome applications. This specific application represents a critical, yet heavily underexplored, research area. While most existing literature has predominantly focused on metallic structures or Carbon Fiber-Reinforced Polymers (CFRP), this study differentiates itself by analyzing sandwich panels composed of aramid fibers embedded in an epoxy matrix.

Radome structures encounter unique operational and environmental challenges, including strict electromagnetic transparency requirements, degradation due to ultraviolet exposure and moisture absorption, as well as severe impact risks, such as bird strikes. Consequently, the proposed methodology closely aligns with the predictive maintenance requirements for these specific components, where real-time deformation monitoring is paramount to preventing both electromagnetic signal distortion and catastrophic structural failure.

To ensure high fidelity and reliability in the future experimental validation, a second key feature of this study is the exclusive use of standardized testing machines and experimental procedures governed by international standards. This rigorous approach guarantees precise control over the testing conditions and the repeatability of the results.

Ultimately, this investigation serves as a crucial foundational step toward the development and physical deployment of a robust, real-time SHM setup. By meticulously evaluating the sensor network design and validating the shape reconstruction accuracy under various operational conditions, this study lays the essential groundwork for future in-situ continuous monitoring systems.

The remainder of this thesis is structured as follows:

- **Chapter 3** details the theoretical background and the mathematical formulation of the iFEM. It covers the derivation of the quadrilateral inverse-shell element, the definition of the least-squares functional based on in-situ strain sensor data, and discusses advanced solutions to overcome the limitations of the original iFEM formulation, including the weighted formulation and strain pre-extrapolation techniques for uninstrumented elements.
- **Chapter 4** is dedicated to the core case study of this research, detailing the application of the shape sensing framework to the composite radome panel. The chapter systemati-

cally outlines the direct Finite Element Model (FEM) design, which is developed to obtain the high-fidelity reference structural response, and the inverse FEM model design, which establishes the discretization used for the shape reconstruction. Furthermore, it comprehensively describes the iFEM algorithmic implementation and the operational workflow of the developed code, elucidating how the simulated strain data are processed to accurately recover the full-field displacement of the structure.

- **Chapter 5** is dedicated to the presentation and critical discussion of the numerical results. It evaluates the performance of the proposed shape sensing framework, highlighting the accuracy and reliability of the iFEM algorithm across different sensor configurations and load cases.

3 | Inverse Finite Element Formulation

3.1. Quadrilateral inverse-shell element

This section is dedicated to present the mathematical framework developed for the application of iFEM, in particular it is shown the formulation proposed for the inverse element iQS4 [4], the element that will be used to discretize the structures under exam. It is noted that this is intended to be the generic original formulation, its specific variations and adaptations to the case study will be highlighted and discussed in the following sections.

The four-node quadrilateral inverse-shell element iQS4 is developed on the basis of a weighted-least-squares iFEM formulation [1, 2], and has six displacement DOF per node, as shown in Fig. 1b.

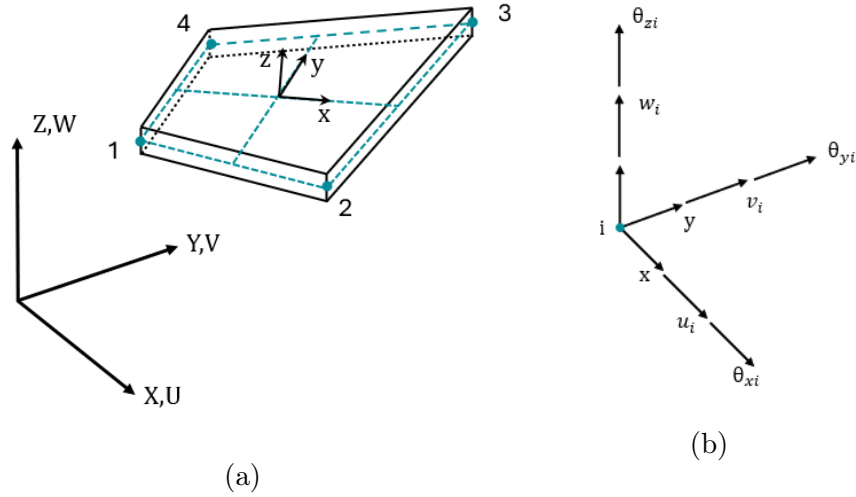


Figure 1: (a) Four-node quadrilateral inverse-shell element, iQS4, represented inside global (X, Y, Z) and local (x, y, z) frames of reference; (b) Nodal DOF corresponding to local coordinates.

The first step is to define a set of convenient coordinate frame of reference to guarantee the geometric uniqueness of the assembled shell structure. A local coordinate system (x, y, z) is adopted as the reference frame for the element, with its origin at the center of the quadrilateral mid-plane $(0, 0, 0)$. The shell is considered to have a constant thickness $2h$, with the thickness direction defined by the coordinate $z \in [-h, +h]$ (see Fig. 1a). The positions of the element nodes are expressed in the global coordinate system (X, Y, Z) , and appropriate transformation

matrices are constructed to map local quantities into the global system. These transformations follow standard finite element procedures and are necessary for assembling element-level matrices into the overall system of equations (refer to [168]).

The (x, y) reference plane of the iQS4 element can be uniquely defined in terms of bilinear isoparametric shape functions $N_q(s, t)$ and the element local nodal coordinates (x_q, y_q) ($q = 1, 2, 3, 4$), where s and t are dimensionless isoparametric coordinates (refer to Fig. 2). The mapping functions can be expressed as follows:

$$x(s, t) \equiv x = \sum_{q=1}^4 N_q x_q \quad (1a)$$

$$y(s, t) \equiv y = \sum_{q=1}^4 N_q y_q \quad (1b)$$

where the N_q shape functions are listed in appendix A. The nodal DOF, expressed as positive x translations u_q , positive y translations v_q , and positive clockwise rotations θ_{zq} , describe the u and v membrane displacements by

$$u(x, y) \equiv u = \sum_{q=1}^4 N_q u_q + \sum_{q=1}^4 L_q \theta_{zq} \quad (2a)$$

$$v(x, y) \equiv v = \sum_{q=1}^4 N_q v_q + \sum_{q=1}^4 M_q \theta_{zq} \quad (2b)$$

where L_q and M_q are the shape functions that establish the relation between the nodal rotation DOF and the membrane displacements of the element. These kind of shape functions have been already provided by Cook [169] and are listed in appendix A.

The transverse displacement w and two bending rotations θ_x and θ_y are defined by the DOF of positive z translation w_q and positive counter clockwise rotations around the in-plane x and y axes, θ_{xq} and θ_{yq} . These transverse displacement field and rotation fields are interpolated through nodal values as

$$w(x, y) \equiv w = \sum_{q=1}^4 N_q w_q - \sum_{q=1}^4 L_q \theta_{xq} - \sum_{q=1}^4 M_q \theta_{yq} \quad (3a)$$

$$\theta_x(x, y) \equiv \theta_x = \sum_{q=1}^4 N_q \theta_{xq} \quad (3b)$$

$$\theta_y(x, y) \equiv \theta_y = \sum_{q=1}^4 N_q \theta_{yq} \quad (3c)$$

By combining Eq. 2 and Eq. 3, the three components of the displacement vector of any point inside the element can be described as:

$$u_x(x, y, z) \equiv u_x = u + z\theta_y \quad (4a)$$

$$u_y(x, y, z) \equiv u_y = v - z\theta_x \quad (4b)$$

$$u_z(x, y, z) \equiv u_z = w \quad (4c)$$

Once the displacement field is derived, the corresponding strain field is given by the strain-displacement relations of linear elasticity theory, as

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u}{\partial x} + z \frac{\partial \theta_y}{\partial x} \quad (5a)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial v}{\partial y} + z \frac{\partial \theta_x}{\partial y} \quad (5b)$$

$$\gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + z \left(\frac{\partial \theta_y}{\partial y} + \frac{\partial \theta_x}{\partial x} \right) \quad (5c)$$

$$\gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} = \frac{\partial w}{\partial x} + \theta_y \quad (5d)$$

$$\gamma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = \frac{\partial w}{\partial y} + \theta_x \quad (5e)$$

The mathematical framework of the iFEM for shell structures is grounded on the FSDT. A fundamental assumption within this kinematic model is the treatment of the transverse normal stress. In accordance with the plane stress hypothesis ($\sigma_{zz} = 0$), which is widely accepted for thin and moderately thick structures, the transverse normal strain ε_{zz} is omitted from the formulation.

This simplification is physically justified as the thickness dimension of the radome panel is significantly smaller than its characteristic global dimensions. Consequently, the contribution of ε_{zz} to the total variance of the least-squares functional is negligible compared to the membrane, bending, and transverse shear strain components. By neglecting this term, the computational efficiency of the algorithm is preserved, facilitating the displacement reconstruction which is a primary objective of this study.

Plugging Eqs. 3 and 4 into Eq. 5 results in the strain-displacement relations expressed in terms of the i -th element nodal displacement vector, $\mathbf{u}^i$, as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \equiv \mathbf{e}(\mathbf{u}^i) + z\mathbf{k}(\mathbf{u}^i) = \mathbf{B}^m\mathbf{u}^i + z\mathbf{B}^b\mathbf{u}^i \quad (6a)$$

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \equiv \mathbf{g}(\mathbf{u}^i) = \mathbf{B}^s\mathbf{u}^i \quad (6b)$$

where

$$\mathbf{u}^i = [\mathbf{u}_1^i \quad \mathbf{u}_2^i \quad \mathbf{u}_3^i \quad \mathbf{u}_4^i]^T \quad (7a)$$

$$\mathbf{u}_q^i = [u_q \quad v_q \quad w_q \quad \theta_{xq} \quad \theta_{yq} \quad \theta_{zq}]^T \quad (q = 1, 2, 3, 4) \quad (7b)$$

The membrane strains $\mathbf{e}(\mathbf{u}^i)$, related to the stretching of the mid-surface, are governed by the matrix $\mathbf{B}^m$, which contains derivatives of the relative shape functions. The bending curvatures and transverse shear strains are represented by $\mathbf{k}(\mathbf{u}^i)$ and $\mathbf{g}(\mathbf{u}^i)$, respectively. Accordingly, the matrices $\mathbf{B}^b$ and $\mathbf{B}^s$ include the corresponding derivatives associated with bending and shear behavior. The explicit expressions for $\mathbf{B}^m$, $\mathbf{B}^b$, and $\mathbf{B}^s$ are provided in appendix appendix A.

3.2. Input data from in-situ strain sensors

Discrete in-situ strain measurements obtained from embedded or on-board sensors play a key role in the iFEM formulation. Traditional strain rosettes or advanced fiber-optic SN, such as FBG sensors, allow the acquisition of a large volume of strain data directly from the structure, in particular this case will be analyzed by means of a FBG sensors pattern, which will be discussed later.

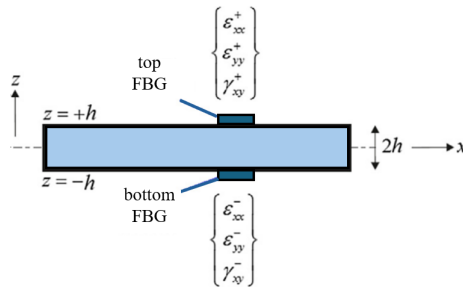


Figure 2: Top and bottom surface strains measured by FBG sensors within iQS4 element at $\mathbf{x}_j = (x_j, y_j, \pm h)$ locations.

The iFEM formulation allows for the definition of n_s discrete measurement points $\mathbf{x}_j = (x_j, y_j, \pm h)$, ($j = 1, \dots, n$) within each i -th element, without imposing constraints on the type of sensor employed. For instance, a possible measurement sensor could be strain rosettes, capable of recording mul-

multiple components of the strain tensor, such as ε_{xx} , ε_{yy} , and γ_{xy} , where x and y denote the local coordinate directions. .

In practical scenarios, where only surface measurements are feasible, the membrane and bending strain components of the mid-plane can still be estimated as described in Eq. 8, provided that strain data are acquired on both external surfaces of the plate Fig. 2, i.e., at $z = \pm h$

$$\mathbf{e}_{i,j}^\varepsilon = \frac{1}{2} \left\{ \begin{array}{c} \varepsilon_{xx}^+ + \varepsilon_{xx}^- \\ \varepsilon_{yy}^+ + \varepsilon_{yy}^- \\ \gamma_{xy}^+ + \gamma_{xy}^- \end{array} \right\}_{i,j}, \quad (j = 1, \dots, n_s) \quad (8a)$$

$$\mathbf{k}_{i,j}^\varepsilon = \frac{1}{2h} \left\{ \begin{array}{c} \varepsilon_{xx}^+ - \varepsilon_{xx}^- \\ \varepsilon_{yy}^+ - \varepsilon_{yy}^- \\ \gamma_{xy}^+ - \gamma_{xy}^- \end{array} \right\}_{i,j}, \quad (j = 1, \dots, n_s) \quad (8b)$$

The measured surface strains are denoted as $(\varepsilon_{xx}^+, \varepsilon_{yy}^+, \gamma_{xy}^+)_{i,j}$ and $(\varepsilon_{xx}^-, \varepsilon_{yy}^-, \gamma_{xy}^-)_{i,j}$, where the superscripts $+$ and $-$ refer to values at the top (positive z values) and bottom (negative z values) surfaces, respectively.

Although these measurements can be used to compute the in-situ membrane strains $\mathbf{e}^\varepsilon$ and bending curvatures $\mathbf{k}^\varepsilon$, they are not directly suitable for evaluating the in-situ transverse shear strains $\mathbf{g}^\varepsilon$. However, in the context of thin shell deformation, the contribution of $\mathbf{g}^\varepsilon$ is significantly smaller compared to the bending curvatures $\mathbf{k}^\varepsilon$, so the influence of $\mathbf{g}^\varepsilon$ can be safely neglected in the iFEM formulation.

3.3. Least-squares functional of inverse Finite Element Method

The iFEM reconstructs the deformed shape by minimizing a weighted least-squares functional with respect to the nodal DOF. For an individual inverse element, this functional $\Phi_i(\mathbf{u}^i)$ is expressed by

$$\Phi_i(\mathbf{u}^i) = \|\mathbf{e}(\mathbf{u}^i) - \mathbf{e}_i^\varepsilon\|^2 + \|\mathbf{k}(\mathbf{u}^i) - \mathbf{k}_i^\varepsilon\|^2 + \|\mathbf{g}(\mathbf{u}^i) - \mathbf{g}_i^\varepsilon\|^2 \quad (9)$$

The squared norms expressed in Eq. 9 can be written in the form of the normalized Euclidean norms

$$\|\mathbf{e}(\mathbf{u}^i) - \mathbf{e}_i^\varepsilon\|^2 = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} (\mathbf{e}(\mathbf{u}^i)_j - \mathbf{e}_{i,j}^\varepsilon)^2 dx dy \quad (10a)$$

$$\|\mathbf{k}(\mathbf{u}^i) - \mathbf{k}_i^\varepsilon\|^2 = \frac{(2h)^2}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} (\mathbf{k}(\mathbf{u}^i)_j - \mathbf{k}_{i,j}^\varepsilon)^2 dx dy \quad (10b)$$

$$\|\mathbf{g}(\mathbf{u}^i) - \mathbf{g}_i^\varepsilon\|^2 = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} (\mathbf{g}(\mathbf{u}^i)_j - \mathbf{g}_{i,j}^\varepsilon)^2 dx dy \quad (10c)$$

where A^i represents the mid-plane area of the i -th element and n_s the number of sensors within the element. It becomes evident how this mathematical formulation is able to describe analytically both multi-sensorized elements and sensorless elements structures.

A more compact form of the functional formulation is now built by grouping the integral expressions and summations into single matrix terms. This is achieved by applying the principle of the norm of the difference between vectors to Eq. 10, which result in

$$\|\mathbf{e}(\mathbf{u}^i) - \mathbf{e}_i^\varepsilon\|^2 = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} (\mathbf{e}(\mathbf{u}^i)_j - \mathbf{e}_{i,j}^\varepsilon)^\top (\mathbf{e}(\mathbf{u}^i)_j - \mathbf{e}_{i,j}^\varepsilon) dx dy \quad (11a)$$

$$\|\mathbf{k}(\mathbf{u}^i) - \mathbf{k}_i^\varepsilon\|^2 = \frac{(2h)^2}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} (\mathbf{k}(\mathbf{u}^i)_j - \mathbf{k}_{i,j}^\varepsilon)^\top (\mathbf{k}(\mathbf{u}^i)_j - \mathbf{k}_{i,j}^\varepsilon) dx dy \quad (11b)$$

$$\|\mathbf{g}(\mathbf{u}^i) - \mathbf{g}_i^\varepsilon\|^2 = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} (\mathbf{g}(\mathbf{u}^i)_j - \mathbf{g}_{i,j}^\varepsilon)^\top (\mathbf{g}(\mathbf{u}^i)_j - \mathbf{g}_{i,j}^\varepsilon) dx dy \quad (11c)$$

After substituting Eq. 6 into Eq. 11, performing the necessary transpositions of the matrices and vectors involved and factoring the vectors $\mathbf{u}^{iT}$ and $\mathbf{u}^i$ respectively before and after the expression considering the sum of all three components, the functional takes the following form:

$$\Phi_i(\mathbf{u}^i) = \mathbf{u}^{iT} \mathbf{k}^i \mathbf{u}^i - 2\mathbf{u}^{iT} \mathbf{f}^i + \xi^i \quad (12)$$

in which:

$$\mathbf{k}^i = \iint_{A_i} \left(\mathbf{B}^{m\top} \mathbf{B}^m + (2h)^2 \mathbf{B}^{b\top} \mathbf{B}^b + \mathbf{B}^{s\top} \mathbf{B}^s \right) dx dy \quad (13a)$$

$$\mathbf{f}^i = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} \left(\mathbf{B}^{m\top} \mathbf{e}_{i,j}^\varepsilon + (2h)^2 \mathbf{B}^{b\top} \mathbf{k}_{i,j}^\varepsilon + \mathbf{B}^{s\top} \mathbf{g}_{i,j}^\varepsilon \right) dx dy \quad (13b)$$

$$\xi^i = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} \left(\mathbf{e}_{i,j}^{\varepsilon\top} \mathbf{e}_{i,j}^\varepsilon + (2h)^2 \mathbf{k}_{i,j}^{\varepsilon\top} \mathbf{k}_{i,j}^\varepsilon + \mathbf{g}_{i,j}^{\varepsilon\top} \mathbf{g}_{i,j}^\varepsilon \right) dx dy \quad (13c)$$

The $\mathbf{k}^i$ is not calculated upon a summation over all the sensors since the $\mathbf{B}$ matrices are constant across all the i elements and an average will result again in the same matrices. It is worth observing, as this will prove useful in the subsequent steps, that the term ξ^i in the functional Φ_i of Eq. 13 does not depend on the element nodal displacement vector $\mathbf{u}^i$. Moreover, while the vector $\mathbf{f}^i$ is a function of both the number and the values of the in-situ strain measurements, the matrix $\mathbf{k}^i$ remains unaffected by the measured strain values.

All the equations so far have been formulated in the local frame of reference of the element, in order to make them suitable for numerical implementation it is convenient to consider them as an assemble of all the discretized structure elements and referring them to the global frame of reference. The global coordinates nodal displacement vector $\mathbf{U}$ of the whole structure is therefore defined as the assembly of all the $\mathbf{u}^i$, and it is related to the element nodal displacement by

means of an extraction matrix $\mathbf{E}^i$, before being able to be assembled, the $\mathbf{u}^i$ vectors need to be transposed into the global coordinate system by through a transformation matrix $\mathbf{T}^i$ (both these matrices can be found in appendix A), giving rise to the following relations :

$$\mathbf{u}^i = \mathbf{E}_i \mathbf{U} \quad (14a)$$

$$\mathbf{U}^i = \mathbf{T}^i \mathbf{u}^i \quad (14b)$$

The global functional is then defined as

$$\Phi = \sum_{i=1}^{n_{el}} \Phi_i(\mathbf{u}^i) \quad (15)$$

By following standard finite element procedures for assembling the contribution of each inverse element in a global system, $\Phi(\mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^{n_{el}}) \rightarrow \Phi(\mathbf{U})$, the global functional of Eq. 15 is reduced to

$$\Phi(\mathbf{U}) = \mathbf{U}^\top \mathbf{K} \mathbf{U} - 2\mathbf{U}^\top \mathbf{F} + \Xi \quad (16)$$

with

$$\mathbf{K} = \sum_{i=1}^{n_{el}} \mathbf{E}_i^\top \mathbf{T}^{i\top} \mathbf{k}^i \mathbf{T}^i \mathbf{E}_i \quad \mathbf{F} = \sum_{i=1}^{n_{el}} \mathbf{E}_i^\top \mathbf{T}^{i\top} \mathbf{f}^i \quad \Xi = \sum_{i=1}^{n_{el}} \xi^i \quad (17)$$

Setting null the partial derivative of the global weighted functional Eq. 16 with respect to the global displacements vector of the structure $\mathbf{U}$, the global system of equations can be obtained:

$$\frac{\partial \Phi(\mathbf{U})}{\partial \mathbf{U}} = \mathbf{K} \mathbf{U} - \mathbf{F} = 0 \quad \Rightarrow \quad \mathbf{K} \mathbf{U} = \mathbf{F} \quad (18)$$

where $\mathbf{K}$ is a symmetric matrix which is only a function of shape functions derivatives, transformation and extraction matrices , thus only of the geometry, while $\mathbf{F}$ is a vector function also of the measured strain values. However, the matrix $\mathbf{K}$ is singular, since it includes the rigid body motion of the structure. This singularity is overcome by including in the formulation the displacement boundary conditions of the specific case considered, reducing Eq. 18 to:

$$\mathbf{K}_{\mathbf{FF}} \mathbf{U}_{\mathbf{F}} = \mathbf{F}_{\mathbf{F}} \quad (19)$$

where $\mathbf{K}_{\mathbf{FF}}$ is a positive definite matrix which is now always non-singular, guaranteeing the existence of a solution. As for the unconstrained matrix, also $\mathbf{K}_{\mathbf{FF}}$ is a function of only the mesh discretization following the sensor layout. At this point the original objective of the formulation can be explicitated, and the displacement field is obtained by

$$\mathbf{U}_{\mathbf{F}} = \mathbf{K}_{\mathbf{FF}}^{-1} \mathbf{F}_{\mathbf{F}} \quad (20)$$

It can be observed that the nature of the two factors required to compute the displacement field highlights the advantage of using this method.

- $\mathbf{K}_{\mathbf{FF}} = f(\mathbf{B}^m, \mathbf{B}^b, \mathbf{B}^s, \mathbf{T}^i, \mathbf{E}^i, \mathbf{k}^i, \mathbf{n}_{el}, \mathbf{n}_s) :$ symmetric, positive definite left-hand-side matrix with a guaranteed non-singularity imposed by boundary conditions constraints, only depends on the sensor layout and the resulting mesh discretization. This matrix can be fully assembled once the structure's geometry and the elements' kinematic relations are known. For this reason, it is unaffected by material properties and their possible degradations (mechanical, chemical, etc.), load variations, or any events acting on the system that do not concretely affect the structure geometry itself. Consequently, it requires only a one-time, offline computation (including its inversion)
- $\mathbf{F}_{\mathbf{F}} = f(\mathbf{B}^m, \mathbf{B}^b, \mathbf{B}^s, \mathbf{T}^i, \mathbf{E}^i, \mathbf{k}^i, \mathbf{n}_{el}, \mathbf{n}_s, \mathbf{e}^\varepsilon, \mathbf{k}^\varepsilon, \mathbf{g}^\varepsilon) :$ right-hand-side vector function of the same parameters of the left-hand-side matrix and of the measured strain values. Given this, the vector will need to be updated during the service with an appropriate frequency.

Considering this, the single version of the matrix $\mathbf{K}_{\mathbf{FF}}$, the rapid update of the vector $\mathbf{F}_{\mathbf{F}}$ upon changes in input strain data, and the computation of the strain field via a simple matrix-vector product highlight the adaptability of this algorithm for computationally efficient real-time applications.

3.4. Original iFEM formulation's limitation and relative solutions

A practical limitation of the method described thus far lies in the assumption that each of the elements is equipped with sensors, effectively leading to a discretization with mono- or multi-sensorized elements. In real applications, especially for medium and large scale structures, it is often unfeasible to instrument every element of the inverse mesh. As a result, several elements may lack any measurement data.

3.4.1. Weighted formulation

To overcome this issue, a modification to the original formulation was introduced. According to Eq. 10, in the absence of a strain component $\mathbf{e}_{i,j}^\varepsilon$, $\mathbf{k}_{i,j}^\varepsilon$ or $\mathbf{g}_{i,j}^\varepsilon$, a zero value would be implicitly assigned, and the corresponding functional would reduce accordingly to

$$\|\mathbf{e}(\mathbf{u}^i)\|^2 = \iint_{A_i} (\mathbf{e}(\mathbf{u}^i))^2 dx dy \quad (21a)$$

$$\|\mathbf{k}(\mathbf{u}^i)\|^2 = (2h)^2 \iint_{A_i} (\mathbf{k}(\mathbf{u}^i))^2 dx dy \quad (21b)$$

$$\|\mathbf{g}(\mathbf{u}^i)\|^2 = \iint_{A_i} (\mathbf{g}(\mathbf{u}^i))^2 dx dy \quad (21c)$$

The proposed approach introduces weighting factors w ranging from 0 to 1, associated with each element in the mesh. Elements equipped with sensors are assigned a weight $w = 1$, while unsensorized ones are given a significantly smaller weight, in the order of 10^{-4} . As a result, the original formulation of Eq. 9 is reformulated as follows:

$$\Phi_i(\mathbf{u}^i) = \|\mathbf{e}(\mathbf{u}^i) - \mathbf{e}_i^\varepsilon\|_{\mathbf{W}_m^i}^2 + \|\mathbf{k}(\mathbf{u}^i) - \mathbf{k}_i^\varepsilon\|_{\mathbf{W}_b^i}^2 + \|\mathbf{g}(\mathbf{u}^i) - \mathbf{g}_i^\varepsilon\|_{\mathbf{W}_g^i}^2 \quad (22)$$

Where $\|\cdot\|_{\mathbf{W}}^2$ denotes the weighted Euclidean norm with weight matrix $\mathbf{W}$. So, we have:

$$\|\mathbf{x}\|_{\mathbf{W}}^2 = \mathbf{x}^\top \mathbf{W} \mathbf{x} \quad (23)$$

All the weight matrices are diagonal:

$$\mathbf{W}_m^i = \begin{bmatrix} \delta_i^{\varepsilon_{xx}} & 0 & 0 \\ 0 & \delta_i^{\varepsilon_{yy}} & 0 \\ 0 & 0 & \delta_i^{\varepsilon_{xy}} \end{bmatrix}, \quad \mathbf{W}_b^i = \begin{bmatrix} \delta_i^{k_{xx}} & 0 & 0 \\ 0 & \delta_i^{k_{yy}} & 0 \\ 0 & 0 & \delta_i^{k_{xy}} \end{bmatrix}, \quad \mathbf{W}_g^i = \mathbf{W}_g = \begin{bmatrix} \delta^{\gamma_{xz}} & 0 \\ 0 & \delta^{\gamma_{yz}} \end{bmatrix} \quad (24)$$

The weights δ_i are set equal to one if inside the element there is an experimental measurement of the corresponding strain component, otherwise they are set to a smaller value (i.e., $10^{-4} \div 10^{-2}$). Since the out-of-plane shear strains cannot be directly measured, the weights related to these strain components are always set equal to a small value, i.e., 10^{-4} , regardless the element to improve the displacement reconstruction.

Thus, also in the formulation of the aforementioned matrices there will be included the weighting factors, as demonstration here it is shown the change in the formulation of the functional $\Phi_i(\mathbf{u}^i)$ terms, the $\mathbf{k}^i$ matrix, the vector $\mathbf{f}^i$ and the scalar ξ^i :

$$\|\mathbf{e}(\mathbf{u}^i) - \mathbf{e}_i^\varepsilon\|_{\mathbf{W}_m^i}^2 = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} (\mathbf{e}(\mathbf{u}^i)_j - \mathbf{e}_{i,j}^\varepsilon)^\top \mathbf{W}_m^i (\mathbf{e}(\mathbf{u}^i)_j - \mathbf{e}_{i,j}^\varepsilon) dx dy \quad (25a)$$

$$\|\mathbf{k}(\mathbf{u}^i) - \mathbf{k}_i^\varepsilon\|_{\mathbf{W}_b^i}^2 = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} (\mathbf{k}(\mathbf{u}^i)_j - \mathbf{k}_{i,j}^\varepsilon)^\top \mathbf{W}_b^i (\mathbf{k}(\mathbf{u}^i)_j - \mathbf{k}_{i,j}^\varepsilon) dx dy \quad (25b)$$

$$\|\mathbf{g}(\mathbf{u}^i) - \mathbf{g}_i^\varepsilon\|_{\mathbf{W}_s^i}^2 = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} (\mathbf{g}(\mathbf{u}^i)_j - \mathbf{g}_{i,j}^\varepsilon)^\top \mathbf{W}_s^i (\mathbf{g}(\mathbf{u}^i)_j - \mathbf{g}_{i,j}^\varepsilon) dx dy \quad (25c)$$

$$\mathbf{k}^i = \iint_{A_i} \left(\mathbf{B}^{m\top} \mathbf{W}_m^i \mathbf{B}^m + (2h)^2 \mathbf{B}^{b\top} \mathbf{W}_b^i \mathbf{B}^b + \mathbf{B}^{s\top} \mathbf{W}_s^i \mathbf{B}^s \right) dx dy \quad (25d)$$

$$\mathbf{f}^i = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} \left(\mathbf{B}^{m\top} \mathbf{W}_m^i \mathbf{e}_{i,j}^\varepsilon + (2h)^2 \mathbf{B}^{b\top} \mathbf{W}_b^i \mathbf{k}_{i,j}^\varepsilon + \mathbf{B}^{s\top} \mathbf{W}_s^i \mathbf{g}_{i,j}^\varepsilon \right) dx dy \quad (25e)$$

$$\xi^i = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} \left(\mathbf{e}_{i,j}^{\varepsilon\top} \mathbf{W}_m^i \mathbf{e}_{i,j}^\varepsilon + (2h)^2 \mathbf{k}_{i,j}^{\varepsilon\top} \mathbf{W}_b^i \mathbf{k}_{i,j}^\varepsilon + \mathbf{g}_{i,j}^{\varepsilon\top} \mathbf{W}_s^i \mathbf{g}_{i,j}^\varepsilon \right) dx dy \quad (25f)$$

3.4.2. Strain Pre-Extrapolation for Uninstrumented Elements

Besides the weighting strategy, another approach developed to enhance displacement reconstruction is pre-extrapolation, which augments the iFEM input data by interpolating or extrapolating the discrete strains measured at sensor locations onto a larger domain (e.g., at the centroids of uninstrumented elements) [88, 89]. The use of pre-extrapolation allows for improved iFEM performance, particularly when employing a coarser inverse mesh [89]. However, it must be acknowledged that pre-extrapolated strains may lack direct physical fidelity and can deviate from the true strain values, especially in regions far from sensor locations [90]. To account for this, the weighting strategy is employed concurrently: elements receiving pre-extrapolated data are assigned a weight that is non-zero but still less than unity (e.g., 10^{-2}), signifying less confidence than in a directly-measured element (weight 1.0) but more than in an element with no data (weight 10^{-4}) [156].

Among the many categories, it is possible to distinguish two in particular that are used in this study:

- **Physics-Based Methods:** These methods integrate the measured strain data with knowledge of the structure's governing equations or known geometric discontinuities [90]. This approach aims for a more accurate reconstruction, particularly in regions with high gradients or complex, physically-governed behavior.
- **Data-Driven Methods:** These methods rely solely on the data provided by measured strain field, typically yielding a smooth and continuous strain distribution across the domain [89]. They learn the field's behavior directly from the *in-situ* data without incorporating the structure's governing physical equations (e.g., Polynomial fitting).

Given the nature of the experimental campaign (3PB and torsion tests) conducted on a thin plate, both approaches are theoretically viable and will be explored.

3.4.2.1. Physics-Based Approach: Biharmonic Spline Interpolation

The strategy for pre-extrapolating strain input is based on the fundamentals of Kirchhoff's classical plate theory (CPT) for thin, isotropic plates [170]. CPT governs the behavior of a thin

plate subjected to a transverse distributed load, $q(x, y)$. The equilibrium equation is given by:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q(x, y) = 0 \quad (26)$$

where M_x , M_y , and M_{xy} are the bending and twisting moments per unit width. For an isotropic, linear elastic plate, these moments are related to the plate's curvatures via the constitutive relations:

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (27a)$$

$$M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (27b)$$

$$M_{xy} = -D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \quad (27c)$$

Here, $w(x, y)$ is the transverse displacement, ν is Poisson's ratio, and D is the flexural rigidity, defined as:

$$D = \frac{E(2h)^3}{12(1 - \nu^2)} \quad (28)$$

where E is Young's modulus and $2h$ is the total plate thickness.

Substituting the constitutive relations (Eq. 27) into the equilibrium equation yields the governing partial differential equation (PDE) for the plate's deflection:

$$\nabla^4 w \equiv \left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = \frac{q(x, y)}{D} \quad (29)$$

where ∇^4 is the biharmonic operator, defined as $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$.

The critical insight for this methodology is that in the 3PB and torsion experiments, loads are concentrated at discrete locations (supports and loading points) rather than being distributed. Consequently, in the regions of the plate *between* these concentrated loads, the distributed load $q(x, y)$ is zero. In these extensive unloaded domains, the governing PDE simplifies to the homogeneous biharmonic equation:

$$\nabla^4 w = 0 \quad (30)$$

This establishes that the transverse displacement field $w(x, y)$ is biharmonic.

Analytical Justification: The Biharmonic Strain Field The iFEM formulation requires strain inputs $\mathbf{e}^\varepsilon$ and $\mathbf{k}^\varepsilon$, which are derived from surface strain measurements $\varepsilon(x, y, \pm h)$. The next step is to demonstrate that these surface strain fields are also biharmonic, justifying the use of a biharmonic interpolator.

According to CPT, the in-plane strains $\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}$ at any point (x, y, z) are related to the

transverse displacement w and the mid-plane strains (u_0, v_0) as:

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy} = \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w}{\partial y^2}, \quad \gamma_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) - 2z \frac{\partial^2 w}{\partial x \partial y}$$

The surface strains measured at $z = \pm h$ are thus a linear superposition of mid-plane (membrane) strains and bending-induced strains (proportional to curvature). For example, for the xx -component:

$$\varepsilon_{xx}(x, y, \pm h) = \underbrace{\frac{\partial u_0}{\partial x}}_{\text{membrane}} \pm \underbrace{h \frac{\partial^2 w}{\partial x^2}}_{\text{bending}}$$

If we assume bending-dominated scenario, the membrane strain are null and consequently trivially biharmonic (e.g., $\nabla^4(0) = 0$).

The crucial part is the bending component. Since the biharmonic operator ∇^4 and partial derivative operators (e.g., $\frac{\partial^2}{\partial x^2}$) are linear and commute, we can apply ∇^4 to the curvature terms. For the xx -curvature $k_{xx} = -\frac{\partial^2 w}{\partial x^2}$:

$$\nabla^4(k_{xx}) = \nabla^4 \left(-\frac{\partial^2 w}{\partial x^2} \right) = -\frac{\partial^2}{\partial x^2} (\nabla^4 w)$$

Since w satisfies $\nabla^4 w = 0$ in the unloaded regions (Eq. ??), it follows that:

$$\nabla^4(k_{xx}) = 0$$

This proves that the curvature field k_{xx} is also biharmonic, the same logic applies to k_{yy} and k_{xy} .

Because the surface strain $\varepsilon_{xx}(x, y, \pm h)$ is a linear combination of the membrane strain (biharmonic) and the curvature (biharmonic), the resulting surface strain field is itself biharmonic:

$$\nabla^4 [\varepsilon_{xx}(x, y, \pm h)] = 0$$

This provides a strong physical and analytical justification for using biharmonic spline interpolation to reconstruct the complete surface strain fields from sparse sensor data.

Mathematical Implementation: Biharmonic Spline Interpolation Biharmonic spline interpolation (known as *thin plate spline* in 2D) reconstructs a scalar field, $\varepsilon(x, y)$, from a sparse set of N known values ε_i at locations (x_i, y_i) . The interpolant is formally defined as the function $\varepsilon(x, y)$ that minimizes the "bending energy" of the surface, defined by the functional $J(\varepsilon)$ [171], while passing exactly through the data points:

$$J(\varepsilon) = \iint_{\Omega} \left[\left(\frac{\partial^2 \varepsilon}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 \varepsilon}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 \varepsilon}{\partial y^2} \right)^2 \right] dx dy$$

This is a classic variational problem, the solution of which is found using the Green's function for the 2D biharmonic operator ∇^4 . The fundamental solution (Green's function) $G(r)$ satisfies $\nabla^4 G(r) = \delta(r)$, where $\delta(r)$ is the Dirac delta function and r is the Euclidean distance $r = \sqrt{x^2 + y^2}$. For the 2D case, this function is $G(r) = r^2 \log(r)$ (Fig. 3).

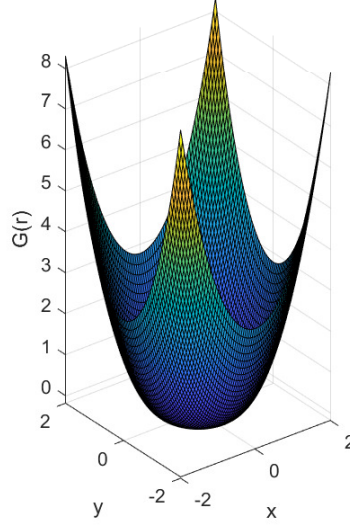


Figure 3: Graph of $G(r)$.

The interpolated field $\varepsilon(x, y)$ is then constructed as a linear combination of these Green's functions, centered at each data point (x_i, y_i) , plus a first-degree polynomial $P(x, y)$:

$$\varepsilon(x, y) = \sum_{i=1}^N \alpha_i \cdot G\left(\sqrt{(x - x_i)^2 + (y - y_i)^2}\right) + P(x, y) \quad (31)$$

The polynomial $P(x, y) = c_0 + c_1x + c_2y$ is necessary to ensure that the solution is invariant to translation and rotation and to guarantee a unique solution by spanning the null space of the problem, moreover a plane's function still respect Eq. 30.

The unknown coefficients, the N weights α_i and the 3 polynomial coefficients c_k , are found by solving a linear system of $N + 3$ equations. This system is formed by enforcing two sets of conditions:

1. **Interpolation Condition (N equations):** The surface must pass through all data points:

$$\varepsilon(x_j, y_j) = \varepsilon_j \quad \text{for } j = 1, \dots, N$$

2. **Orthogonality/Constraint Condition (3 equations):** To ensure uniqueness, the

weights α_i must be orthogonal to the polynomial basis:

$$\sum_{i=1}^N \alpha_i = 0, \quad \sum_{i=1}^N \alpha_i x_i = 0, \quad \sum_{i=1}^N \alpha_i y_i = 0$$

These three constraint equations are necessary to ensure a unique solution. The solution is composed of the spline part, $\sum \alpha_i G(r_i)$, and the polynomial part, $P(x, y)$. The polynomial $P(x, y)$ (a plane) represents the "null space" of the problem, as it has zero bending energy $J(\varepsilon)$ by definition.

An ambiguity would exist if the spline part could also contain a hidden linear trend. These conditions force the spline coefficients α_j to be orthogonal to the polynomial basis functions (1, x , and y) evaluated at sensor locations. This constraint ensures that the spline part contains no linear component, forcing $P(x, y)$ to capture the entire global linear trend while the spline models only the non-linear deviations. This guarantees that the $(N + 3) \times (N + 3)$ system has a unique solution for the coefficients $(\alpha_j, \mathbf{c}_k)$.

This forms a symmetric linear system:

$$\begin{bmatrix} \mathbf{K} & \mathbf{P} \\ \mathbf{P}^\top & \mathbf{0} \end{bmatrix} \begin{bmatrix} \alpha \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \varepsilon_{\text{data}} \\ \mathbf{0} \end{bmatrix}$$

where:

- $\mathbf{K}$ is an $N \times N$ matrix where $K_{ij} = G(r_{ij}) = r_{ij}^2 \log(r_{ij})$, with r_{ij} being the distance between point i and point j .
- $\mathbf{P}$ is an $N \times 3$ matrix of the polynomial terms, $\mathbf{P}_i = [1, x_i, y_i]$.
- $\alpha = [\alpha_1, \dots, \alpha_N]^\top$, $\mathbf{c} = [c_0, c_1, c_2]^\top$, and $\varepsilon_{\text{data}} = [\varepsilon_1, \dots, \varepsilon_N]^\top$.

Computational Logic of the 'v4' Method The Matlab 'v4' function is a computational implementation of this exact biharmonic spline interpolation. Its internal logic proceeds as follows:

1. **Build the System:** Given the N sensor coordinates (x_i, y_i) and their measured strain values ε_i , the code replicates the centroid value on the four Gaussian points of the same element then the **v4** is called and it constructs the $(4N + 3) \times (4N + 3)$ linear system shown above.
2. **Solve for Coefficients:** It solves this system for the unknown coefficient vectors α and $\mathbf{c}$.
3. **Evaluate at Query Points:** Once the coefficients are known, the function can evaluate the interpolated strain $\varepsilon(x_q, y_q)$ at any number of query points (x_q, y_q) (e.g., the gaussian

points) using the summation:

$$\varepsilon_q = \sum_{i=1}^{4N} \alpha_i \cdot G(r_{qi}) + (c_0 + c_1 x_q + c_2 y_q)$$

$$\text{where } r_{qi} = \sqrt{(x_q - x_i)^2 + (y_q - y_i)^2}.$$

This procedure is applied independently to each of the six required surface strain components (e.g., $\varepsilon_{xx}^+, \varepsilon_{yy}^+, \gamma_{xy}^+, \varepsilon_{xx}^-, \varepsilon_{yy}^-, \gamma_{xy}^-$). The result is a complete, smooth, and physically-consistent C^2 strain field over the entire domain, which can then be used to compute $\mathbf{e}^\varepsilon$ and $\mathbf{k}^\varepsilon$ for every element in the inverse mesh.

3.4.2.2. Data-Driven Approach: Polynomial Surface Fitting

As an alternative to physics-based interpolation, a purely Data-Driven approach can be employed. This method makes no assumptions about the governing physical equations. Instead, it treats the strain extrapolation as a mathematical regression problem: it seeks to find a function $\varepsilon(x, y)$ that best fits the sparse, measured sensor data in a least-squares sense.

The most common function family for this purpose is the 2D polynomial, as it is computationally efficient and easily implemented. The core assumption is that the true strain field, within the instrumented region, can be reasonably approximated by a low-order polynomial surface.

Mathematical Implementation: 2D Polynomial Models A general 2D polynomial model, denoted in Matlab syntax as ‘polyPQ’, defines the strain ε as a function of coordinates x and y :

$$\varepsilon(x, y) = \sum_{i=0}^P \sum_{j=0}^Q c_{ij} x^i y^j \quad (32)$$

where P and Q are the polynomial degrees in x and y , respectively, and c_{ij} are the coefficients to be determined. For example:

- ‘poly10’ (Linear in x): $\varepsilon(x, y) = c_{00} + c_{10}x$
- ‘poly11’ (Bilinear): $\varepsilon(x, y) = c_{00} + c_{10}x + c_{01}y + c_{11}xy$
- ‘poly22’ (Biquadratic): $\varepsilon(x, y) = c_{00} + c_{10}x + c_{01}y + c_{11}xy + c_{20}x^2 + c_{02}y^2 + c_{21}x^2y + c_{12}xy^2 + c_{22}x^2y^2$

Given N sensor measurements ε_k at locations (x_k, y_k) , and consequently $4N$ input data considering the gaussian points, the goal is to find the coefficient vector $\mathbf{c} = [c_{00}, c_{10}, c_{01}, \dots]^\top$ that minimizes the Sum of Squared Errors (SSE):

$$\min_{\mathbf{c}} \text{SSE} = \min_{\mathbf{c}} \sum_{k=1}^{4N} (\varepsilon_k - \varepsilon(x_k, y_k; \mathbf{c}))^2 \quad (33)$$

This is a standard linear least-squares problem. It can be expressed in matrix form as:

$$\min_{\mathbf{c}} \|\mathbf{V}\mathbf{c} - \boldsymbol{\varepsilon}_{\text{data}}\|_2^2 \quad (34)$$

where:

- $\boldsymbol{\varepsilon}_{\text{data}} = [\varepsilon_1, \dots, \varepsilon_{4N}]^T$ is the vector of measured strain values.
- $\mathbf{V}$ is the $4N \times M$ design matrix, where M is the total number of coefficients c_{ij} , calculated as $(P+1) \times (Q+1)$. Each row $\mathbf{v}_k$ corresponds to a sensor k and contains the polynomial basis functions evaluated at (x_k, y_k) . For a ‘poly11’ model, $M = 4$ and:

$$\mathbf{v}_k = [1, x_k, y_k, x_k y_k]$$

The least-squares solution $\mathbf{c}^*$ is formally given by the normal equations:

$$\mathbf{c}^* = (\mathbf{V}^T \mathbf{V})^{-1} \mathbf{V}^T \boldsymbol{\varepsilon}_{\text{data}} \quad (35)$$

For this system to be solvable, the number of sensors N must be at least as large as the number of coefficients M . For stability, it is strongly recommended that $N > M$.

Computational Logic and Model Selection (Matlab ‘fit’) In Matlab, this entire process is efficiently managed by the `fit` function (part of the Curve Fitting Toolbox).

1. **Function Call:** The user provides the sensor coordinates (as an $[x, y]$ matrix) and the corresponding strain values $\boldsymbol{\varepsilon}_{\text{data}}$.
2. **Model Definition:** The user specifies the desired polynomial model as a string (e.g., ‘poly22’).
3. **Solving:** The `fit` function automatically builds the design matrix $\mathbf{V}$ and solves the least-squares problem for $\mathbf{c}^*$. (Internally, it uses robust methods like QR decomposition, which are more numerically stable than directly computing the normal equations).
4. **Output:** The function returns a `cfit` object (a "fit object"). This object internally stores the coefficients $\mathbf{c}^*$ and the model structure.
5. **Evaluation:** This `cfit` object is now a callable function. It can be evaluated at any set of query points (x_q, y_q) to return the predicted strain ε_q .

4 | Case Study: Deformation Reconstruction of a Composite Radome Panel

This case study focuses on the development and experimental validation of a methodology for reconstruction of full-field displacements on composite structures, with a specific application to the structural health monitoring of aircraft radomes. The approach leverages iFEM, aiming to support predictive maintenance strategies in aerospace systems.

The primary objective of this study is the development and implementation of a iFEM algorithm capable of estimating the full-field displacement of a composite panel under mechanical loading.

The case study is structured into two main work packages (WPs):

- WP1 – Direct FEM Modeling

A direct FEM model is built using **Abaqus/CAE** to simulate the experimental setup. This model serves as a reference for defining expected displacements and for determining virtual sensor locations and enabling virtual testings.

- WP2 – iFEM Algorithm Implementation

The inverse algorithm is implemented through numerical software approach. It reconstructs the displacement field from strain measurements, exploiting virtually acquired points.

4.1. Direct FEM Modeling

The primary objective of this research work is the development of a robust SHM framework based on iFEM for shape sensing of complex aeronautical components, specifically a composite radome. However, the direct application of the shape-sensing algorithm to a doubly-curved, geometrically complex structure introduces significant challenges in decoupling the sources of reconstruction error. Geometric non-linearities, membrane-bending coupling, and complex boundary conditions inherent to the real radome structure could mask the intrinsic performance of the SN and the reconstruction algorithm.

Therefore, a stepwise validation strategy is adopted. With the intention of future addressing

of the full-scale radome, the methodology is preliminary validated on a simplified geometric counterpart: a flat rectangular plate extracted from the radome structure itself. This preliminary phase serves as a fundamental bridge between the theoretical formulation and the final application.

This chapter details the methodology for the Direct Finite Element Modeling (Direct FEM), which serves as the foundational platform for the subsequent iFEM implementation and validation. The primary objective of the direct simulation is twofold: first, to establish a high-fidelity numerical truth model of the test specimens under prescribed loading conditions, and second, to provide synthetic strain data for the optimal design and preliminary numerical assessment of the SN (as discussed in Chapter 5).

The direct FEM models replicate the two main mechanical tests of the experimental campaign: the Three-Point Bending (3PB) test and the Cantilever Plate (CP) test. Each test is analyzed under two distinct approaches to fully characterize the structural response: one involving a LC and another involving a DC.

The following sections provide a guide for the definition of the material properties, geometry, assembly, boundary condition, loading, and meshing procedures for both the 3PB and CP FEM simulations, ensuring their full reproducibility.

4.1.1. Part module

A part is created object through the following settings:

- **Part name:** Part-1 (default naming is acceptable for the Matlab parser)
- **Modelling space and base feature:** 3D - deformable
- **Base feature:**
 - ▷ Shape: Shell
 - ▷ Type: Planar
- **Sketch design:** as base sketch a rectangle is made in xy-plane (dimension 100x400 mm, as shown in Fig. 4). By doing so, the specimen will already be defined in a suitable coordinate system.
- **Partition:** Exploiting the command *Partition Face: Sketch* the surface need to be partitioned according to the position of the cylinder of 3PB machine or the clamp for the CP, in order to correctly assign the boundary condition later, as shown in Fig. 5a and Fig. 5b.
- **Datum CSYS:** Global (X, Y, Z). No local CSYS defined in this module.

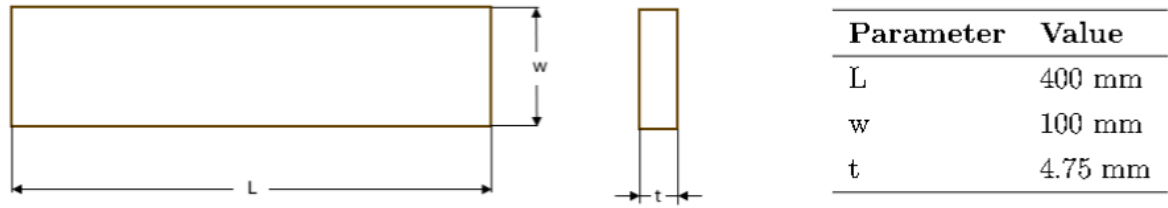
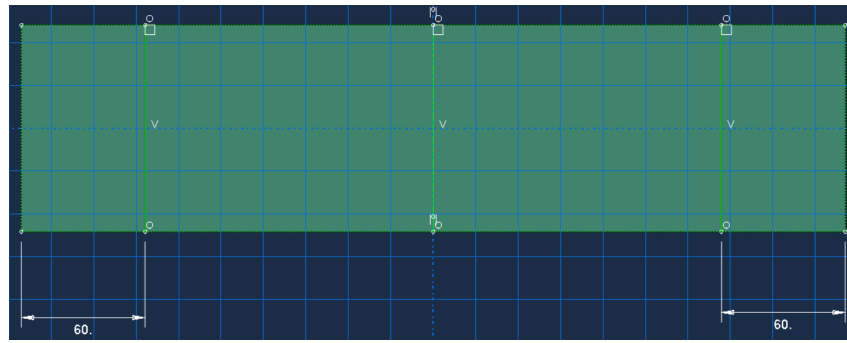
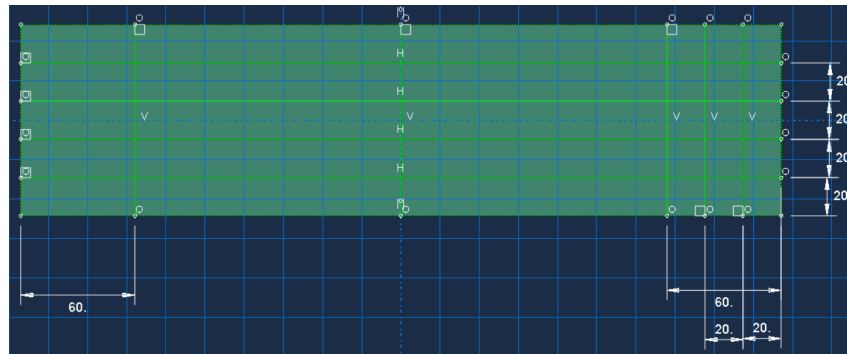


Figure 4: Geometric dimensions of the test specimen.



(a)



(b)

Figure 5: Sketch designs used for partitioning the specimen surface in Abaqus/CAE, ensuring precise assignment of boundary conditions for (a) the 3PB test and (b) the CP test.

4.1.2. Property module

Material In this section it defined the specimen's material through its elastic properties.

The material selected for the experimental campaign is a composite laminate made of Kevlar[®] 49 fibers embedded in an epoxy matrix, specifically designed for the manufacturing of radomes. This choice reflects a balance between mechanical robustness, light weight, and compatibility with electromagnetic requirements, making the material especially suitable for aerospace applications where structural and stealth functionalities must coexist.

The matrix is made of HexPly[®] F155, its mechanical properties shown in table 1 are provided by the manufacturer Hexcel[®] (see [172]).

Property	Value
Specific gravity	1.335
Tg dry	121 °C
Equilibrium moisture absorption	9.4 %
Linear coefficient of thermal expansion	$5.27 \times 10^{-5} \text{ cm}/^{\circ}\text{C}$
Tensile strength	80 MPa
Tensile modulus	3.24 GPa
Tensile strain	5.2 %
Fracture toughness	$1.65 \text{ MPa}\sqrt{\text{m}}$

Table 1: Neat HexPly[®] F155 neat resin properties.

HexPly[®] F155 is a modified epoxy resin system developed for advanced composite manufacturing through autoclave curing. Designed as a controlled-flow prepreg, it enables precise handling during layup and ensures consistent resin distribution throughout the laminate. The resin is formulated to undergo co-curing at 250°F (121°C), a temperature suitable for integrating the prepreg with other structural components during the cure cycle. This process allows for simultaneous consolidation and bonding, particularly effective when used with honeycomb cores or metallic substrates. The resin's controlled flow characteristics are essential for minimizing void content and ensuring high-quality laminate consolidation during the pressure and temperature conditions of autoclave processing.

The reinforcement used in this composite is a woven aramid fabric arranged in a four-harness crowfoot satin (Hexcel designation K285 in [172]) configuration which is more pliable than the plain weave and is easier to conform to curved surfaces. In this weave pattern there is a three-by-one interfacing where a filling yarn floats over three warp yarns and under one [173].

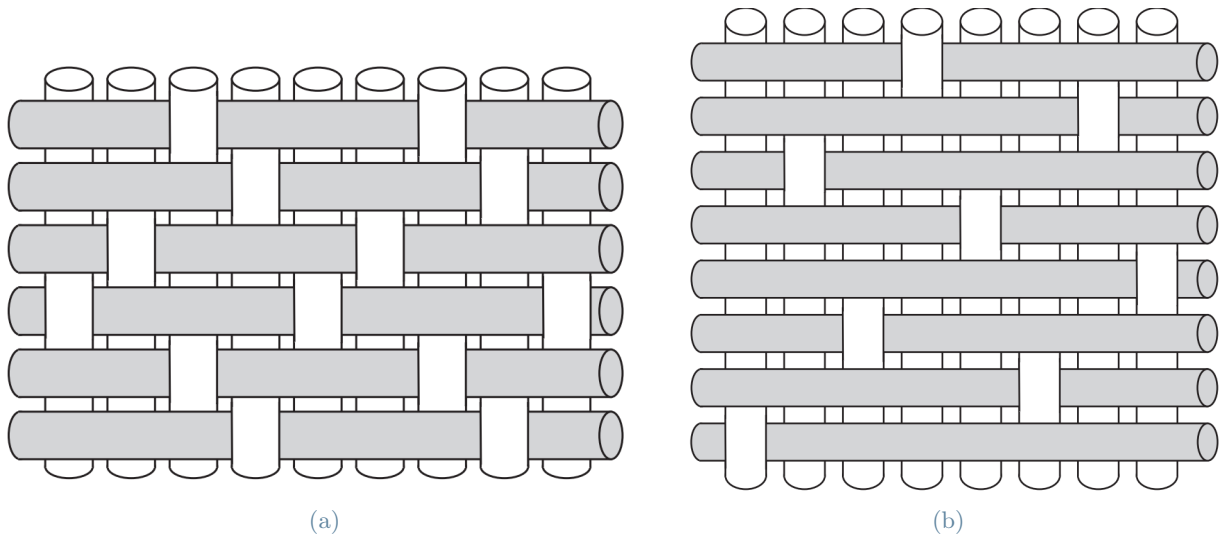


Figure 6: Satin weave pattern of aramid fibers in (a) Four Harness Satin (b) Eight Harness Satin.

Aramid fibers, chemically known as poly-para-phenylene terephthalamide (PPTA), are charac-

terized by aromatic rings linked by amide bonds, which provide exceptional thermal stability, low density, and high strength-to-weight ratio. Kevlar 49, in particular, is engineered for structural applications due to its high modulus and resistance to impact and fatigue. The crow foot satin weave adopted in the K285 variant offers a balanced combination of in-plane stiffness, damage tolerance, and drapeability, making it suitable for use in curved or complex geometries.

From a physical standpoint, the K285 prepreg has a resin content ranging between 49% and 55% (dry) and a flow range between 14% and 26% when cured at 250°F (121°C) under 50 psi (345 kPa).

The main mechanical properties of the K285 system are summarized in the table below:

Property	T [°C]	Value
Tensile strength	24	571 MPa
Tensile modulus	24	30.3 GPa
Tensile strength	71	547 MPa
Tensile modulus	71	26.9 GPa
Compression strength	24	261 MPa
Compression modulus	24	23.4 GPa
Compression strength	71	195 MPa
Compression modulus	71	23.4 GPa

Table 2: Mechanical Properties of Kevlar[®] 49 Fabric K285 (Crow Foot Satin).

The laminate used for the specimens is composed of multiple woven layers of Kevlar 49 fabric arranged in a symmetric layup configuration [0]₁₉. Previous investigations and analyses conducted by the research group on the same material from the same manufacturer provided the numerical characterization for the material under consideration, summarized in table 3. Where the subscript 1 and 2 refer to principal in-plane directions of the laminate, while subscript 3 refers to the transversal direction. Tension modulus E_1 and E_2 , along with shear modulus G_{13} and G_{23} , are equal. This is due to the orthotropic nature of the laminate, resulting from the stacking of plies in which the fibers are arranged in a crow foot satin weave, implying the presence of an orthogonal fiber network.

Parameter	Value
E_1	27872.44 MPa
E_2	27872.44 MPa
ν_{12}	0.0615
G_{12}	3400 MPa
G_{13}	2652.42 MPa
G_{23}	2652.42 MPa

Table 3: Orthotropic Material Properties of the Final Laminate.

Thermal Stability Kevlar[®] 49 demonstrates exceptional thermal stability, maintaining its me-

chanical performance across a broad temperature range. This property makes it particularly well-suited for high-temperature environments, such as those encountered in aerospace and industrial settings. Kevlar fibers begin to thermally degrade only between 427°C and 482°C, significantly higher than the decomposition temperatures of conventional synthetic fibers like nylon or polyester. Even under prolonged exposure to elevated temperatures—up to 300°C—Kevlar® 49 retains much of its tensile strength and stiffness, with minimal degradation. This resilience makes it an ideal material for components operating in extreme conditions, such as turbine engines or high-speed aircraft.

Dimensional Stability and Shrinkage Kevlar® 49 also excels in maintaining dimensional stability under thermal stress. Experimental data indicate that shrinkage is less than 0.1% when exposed to dry air at 177°C, confirming the material's ability to retain its geometry through thermal cycling. In aqueous environments at 100°C, dimensional changes remain negligible, ensuring structural consistency even when exposed to moisture and heat. Additionally, Kevlar's low coefficient of thermal expansion contributes to its dimensional predictability, making it highly suitable for precision applications where thermal deformation must be minimized.

Chemical and UV Resistance Kevlar® 49 displays broad chemical resistance, with stability against most acids, alkalis, and solvents. Although strong mineral acids like sulfuric acid can degrade the material over time, Kevlar remains chemically robust under typical exposure conditions. Its resistance to ultraviolet (UV) radiation further supports its use in outdoor and harsh-environment applications, such as aerospace and marine sectors. While extended UV exposure may lead to surface degradation, this effect can be mitigated through the application of protective coatings or UV-stabilizing additives.

Following this sequence it's possible to assign to the material the properties already mentioned in table 3: Create material (leave it named "Material-1" as default), define the properties inside Mechanical→Elasticity→Elastic, as shown in table 3. Measure units are expressed in N, mm and MPa respectively for forces, length and stresses.

Composite shell section Now the section is created and assigned. For operational reasons related to the code, the section must be defined using the "Create Section" command (and not "Create Composite Layup").

- **Section name:** Section-1 (as default)
- **Category:** Shell
- **Type:** Composite
- **Thickness Integration rule:** Simpson, 3 through-thickness points / ply (as default)
- **Layers definition:** defined as [0]₁₉, as provided by the producer and as shown in Fig. 7
- **Assign Section:** Assign the newly created section to the entire part (the shell-offset must be defined from "Middle surface")

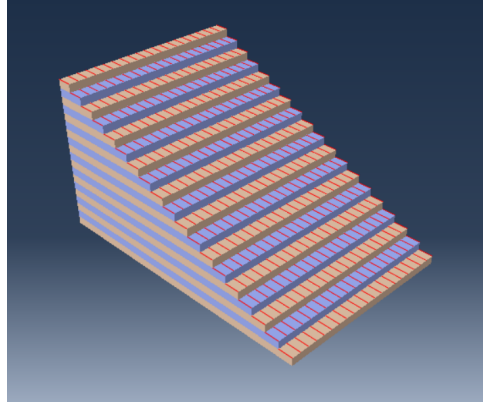


Figure 7: Ply Stack Plot of the specimen.

4.1.3. Assembly module

For the Assembly module, the component of interest must be introduced as a single instance, ensuring it is imported in dependent mode to facilitate subsequent meshing operations. The two assemblies used for the 3PB and CP configurations are illustrated in Fig. 8a and Fig. 8b, respectively.

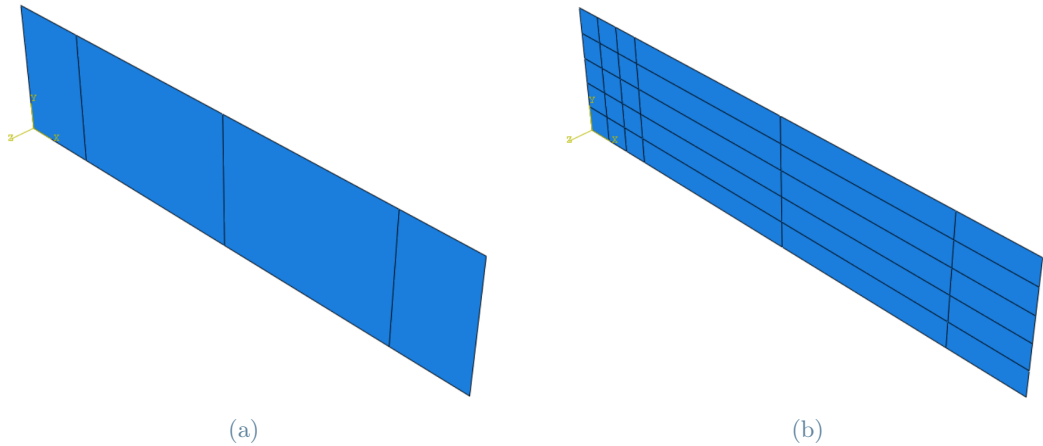


Figure 8: Assembly for (a) the 3PB test and (b) the CP test.

4.1.4. Step module

The simulation time step is defined within the **Step module** as a Static, General step and is conventionally named Step-1 (see Fig. 9). This static analysis approach is mandatory because the iFEM formulation presented in this work is independent of the system's dynamics. The mechanical tests (3PB and CP) are performed under quasi-static loading conditions, making a static solver appropriate.

Crucially, the option for Non-Linear Geometry is set to OFF (see Fig. 9). This setting ensures

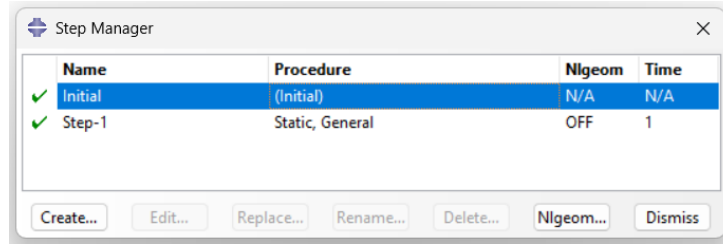


Figure 9: Temporal step definition.

that the numerical solution is based strictly on linear elasticity theory and small displacement assumptions. Incorporating geometric non-linearity would violate the fundamental assumptions of the plane-stress four-node (iQS4) iFEM shell element used for the shape sensing algorithm, which is inherently linear.

4.1.5. Interaction module

The definition of explicit interactions within the model is not strictly required for the iFEM algorithm to function correctly. Therefore, neither the 3PB nor the CP model requires modifications in the **Interaction module** when set up for the Displacement Control configuration.

However, a more complex approach is necessary for the LC configuration. To faithfully replicate the physical application of the load, two cases must be distinguished:

- (i) If the segment where the load is applied is an external edge of the instance, direct load application suffices (as detailed later in the **Load module**).
- (ii) If the load is to be applied on an internal partition line of the component, a Reference Point (RP) must be created at the midpoint of the transverse line where the load will act.

To realize this, one must select the RP option and choose the midpoint of the interested segment. Following this, it is necessary to select the Create Constraint tool and define a Coupling constraint. This coupling ties the movement of the internal line to the RP, allowing the load to be applied uniformly through the single RP. The setup configuration for this constraint is shown in Fig. 10.

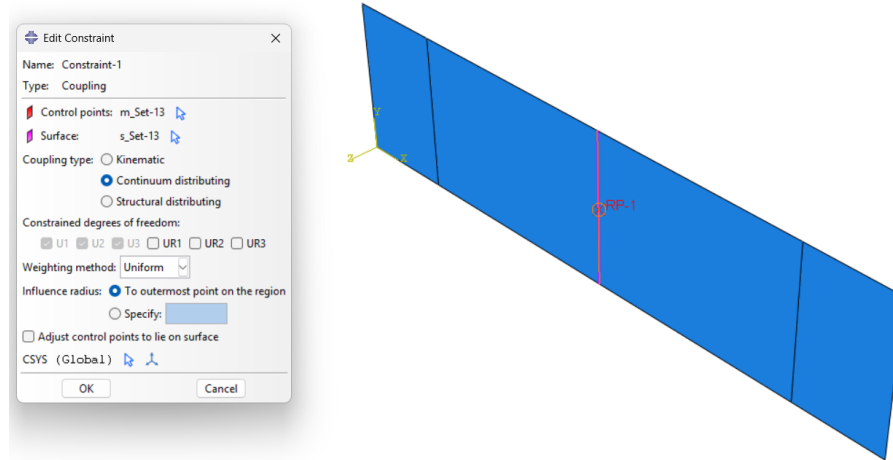


Figure 10: Constraint setting as continuum distributing coupling.

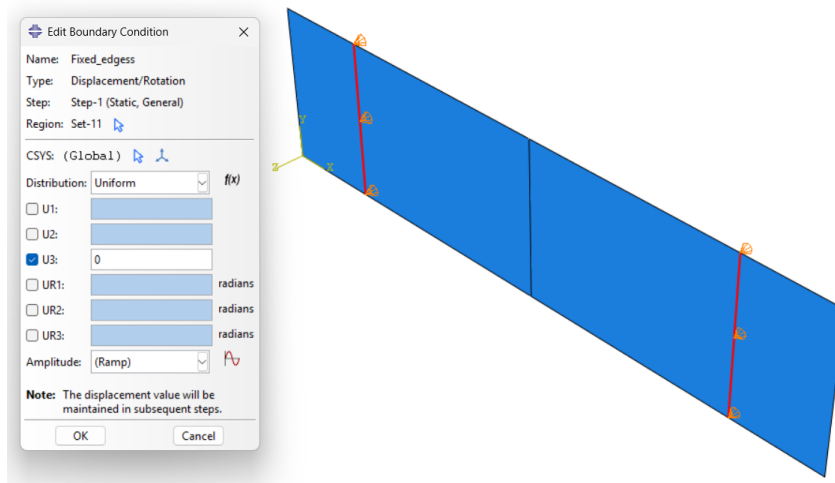
4.1.6. Load module

Here are defined boundary conditions and loads representing the condition under which the 3PB and CP will be executed.

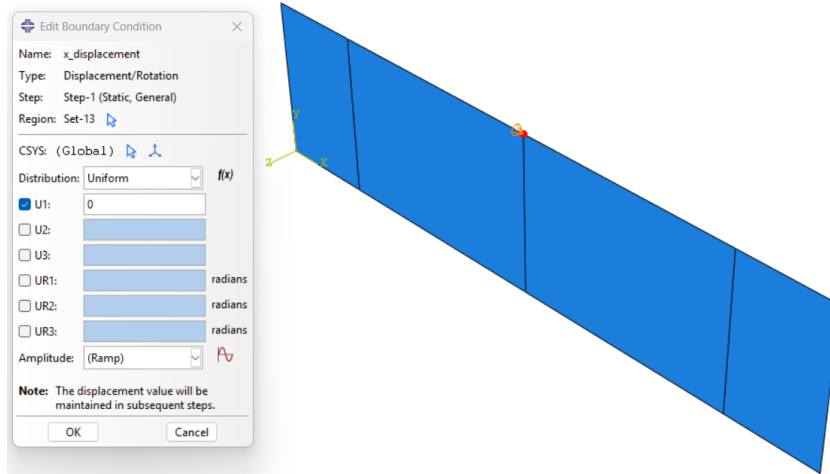
The code used to process the iFEM algorithm requires all parts present in the assembly to be modeled using shell elements. This introduces a limitation, as the loads and boundary conditions (BC) of the 3PB test are physically implemented using rigid, 3D solid components (rollers). Since these solid parts cannot be included in the model input for the iFEM algorithm, they must be replicated as accurately as possible through displacement and rotation constraints. The two sets of Boundary Conditions (BC) for the 3PB and CP configurations are therefore defined and will remain fixed, regardless of the type of applied load. All BC are meant to be created inside Step-1 temporal step.

3PB Boundary Condition

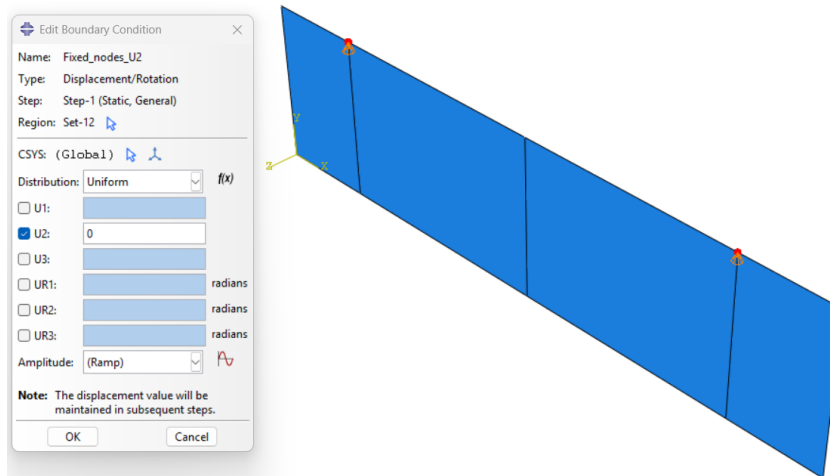
- B.C. 1: In correspondence of the fixed cylinders partition lines, translation along z direction is prevented by imposing U3 as null, by doing so also the rotation around x-axis is prevented (see Fig. 11a)
- B.C. 2: In correspondence of the moving cylinder partition line translation along x direction is prevented at a single point, in order to prevent rigid motion (and not applying it to the whole line since it would introduce virtual stiffness to the component)(see Fig. 11b)
- B.C. 3: In correspondence of the fixed cylinder partition lines translation along y direction is prevented at two separate points, in order to prevent rigid motion and also rigid rotation (and not applying it to the whole line since it would introduce virtual stiffness to the component)(see Fig. 11c)



(a)



(b)



(c)

Figure 11: Boundary conditions at the joints: (a) fixed-cylinder partition lines: translation along z constrained and rotation about x prevented; (b) moving-cylinder partition line: single-point constraint along x to avoid rigid-body motion; (c) fixed-cylinder partition lines: two-point constraint along y to prevent rigid-body motion.

CP Boundary Condition For the CP, a fully clamped boundary condition is implemented, where all degrees of freedom (DOFs) are restrained ($\mathbf{U} = \mathbf{UR} = 0$). This configuration is shown in Fig. 12. The regions subjected to this clamping constraint are those obtained through the partitioning system described in section 4.1.3. This specific partitioning was executed for two primary reasons:

- **Mesh Constraint Enforcement:** The defined partition constrains the mesh development, ensuring that the ordered and regular mesh described in the next module is obtained.
- **Realistic Clamping Regions:** The partition delineates specific nodal regions that can realistically be clamped during the experimental phase, as these areas are not intersected by the fiber optic sensor path.

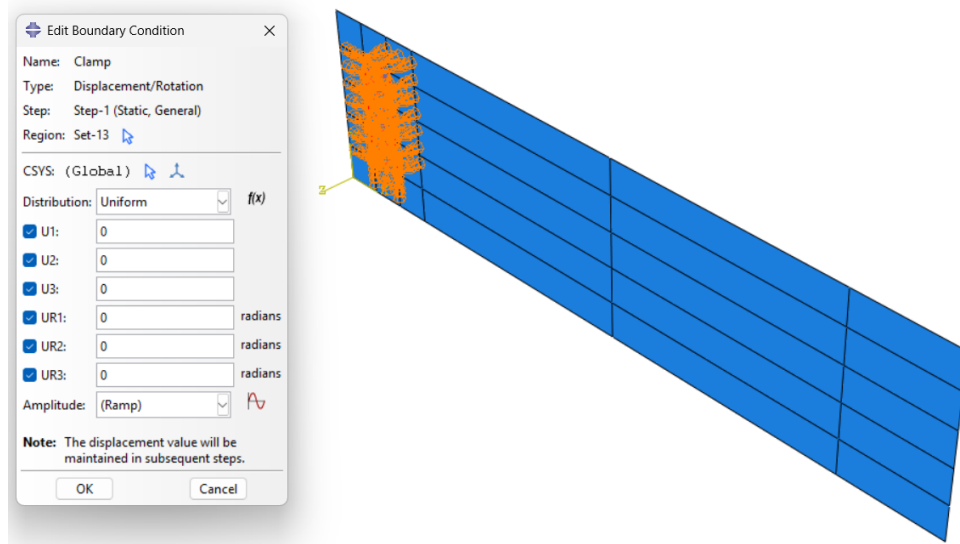


Figure 12: Clamp BC definition for the plate under CP configuration.

Consequently, two distinct loading configurations will be defined, both representative of the testing conditions to be replicated during the experimental campaign. These configurations fundamentally differ in the manner in which the load imposed by the testing machine is replicated. As mentioned before, they are divided in "Load Control" (LC) and "Displacement Control" (DC), and they're implemented this way.

Configuration A: Displacement Control This configuration is directly analogous to displacement-controlled testing, where the test machine enforces a specific, time-dependent kinematic constraint.

- **3PB Model:** The displacement is realized by setting $U3=0$ for all the nodes on the midline, which coincide with the position of the moving cylinder of the test machine (see Fig. 13a).
- **CP Model:** The displacement is realized by setting $U3=0$ for all the nodes on the external

edge, which coincide with the position at which the load will be applied (see Fig. 13b).

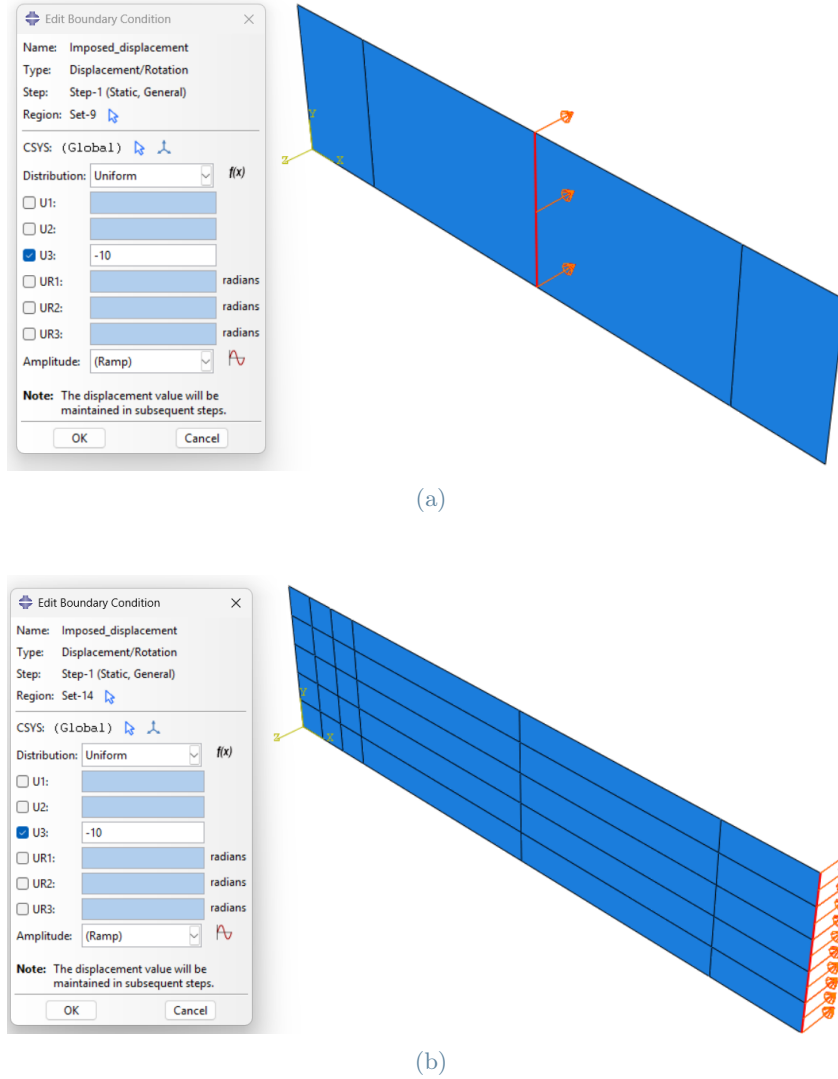


Figure 13: Applied displacement definition and application points for a) 3PB configuration b) CP configuration.

This setup ensures the model's deformed state is solely governed by the applied displacement, making it the primary method for generating strain data ($\mathbf{e}_{i,j}^\varepsilon$) to input into the linear iFEM algorithm.

Configuration B: Load Control This configuration is analogous to force-controlled testing, where the testing machine applies a specific external force. This simulation is primarily used to validate the structural response against known mechanical properties. It also aligned better to the known analytical solutions for strain field, as they're derived through force application.

- **3PB Model:** The load is applied through the exploitation of RP (as described in section 4.1.5), after which a concentrated force is applied at the RP (see Fig. 14a).
- **CP Model:** The load is realized by applying a shell edge load with uniform distribution

and transverse traction (see Fig. 14b and section 4.1.5).

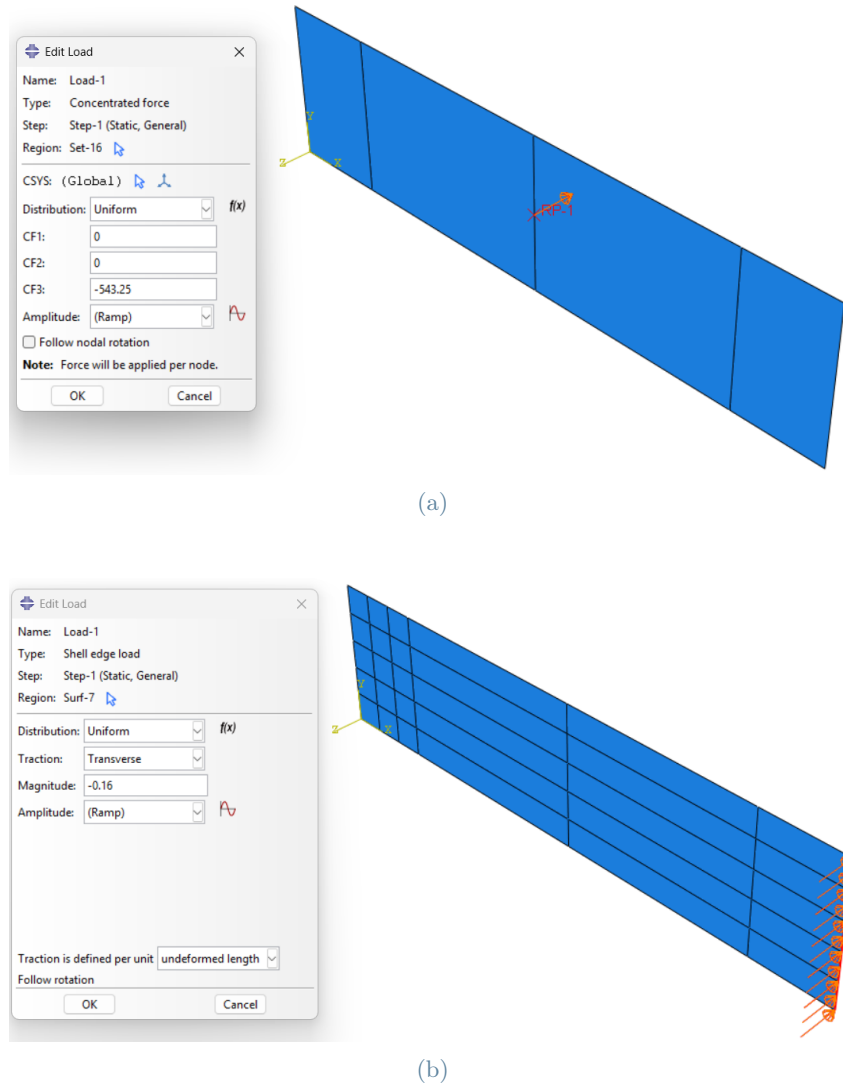


Figure 14: Applied load definition and application points for a) 3PB configuration b) CP configuration.

4.1.7. Mesh module

In this module, the mesh of the instance is defined. The mesh size can be set as desired, keeping in mind that the finer it is, the better the results and the iFEM input numerical data will be.

Below are the settings to be defined for mesh creation.

- **Mesh Dependency** (Mesh on Part): The mesh is generated directly on the part geometry rather than on the assembly instance. This approach, referred to as "Mesh on Part", creates a Dependent mesh instance within the assembly. This configuration is strictly required to facilitate the correct nodal and elemental data extraction process performed by the iFEM algorithm.

- Element Type and Integration Scheme:** The model is discretized using S4 elements (4-node, general-purpose, finite-membrane-strain shell). It is crucial to modify the default element controls by deselecting the *Reduced Integration* option. This enforces Full Integration for the shell elements. Although reduced integration is typically favored in direct FEM for computational efficiency, the mathematical formulation of the iFEM adopted in this work relies on full integration to ensure accurate strain field reconstruction and to avoid numerical instabilities such as hour-glassing.

To ensure the reliability of the numerical results, a mesh convergence analysis was performed for the 3PB in both DC and LC configurations. The objective of this study was to identify the optimal discretization level that guarantees a grid-independent solution while maintaining computational efficiency. The analysis involved monitoring the peak longitudinal strain (ε_{11}) along the region of interest as the mesh density was progressively increased. Seven different mesh configurations were tested, ranging from a coarse mesh size of 60 mm down to a refined mesh of 1.25 mm. As illustrated in the strain profile plots, coarsening the mesh results in a jagged representation of the strain field, which underestimates the peak gradients. Conversely, refining the element size leads to a smoother profile that progressively converges towards a stable distribution. This behavior is quantitatively summarized in the convergence curves, where the maximum strain value exhibits an asymptotic trend as the element size decreases. The convergence criterion was considered satisfied when the relative error between consecutive refinement steps fell below a negligible threshold (e.g., $< 1\%$), confirming that the chosen mesh size (typically the finest or the second finest) accurately captures the structural response without introducing significant discretization errors.

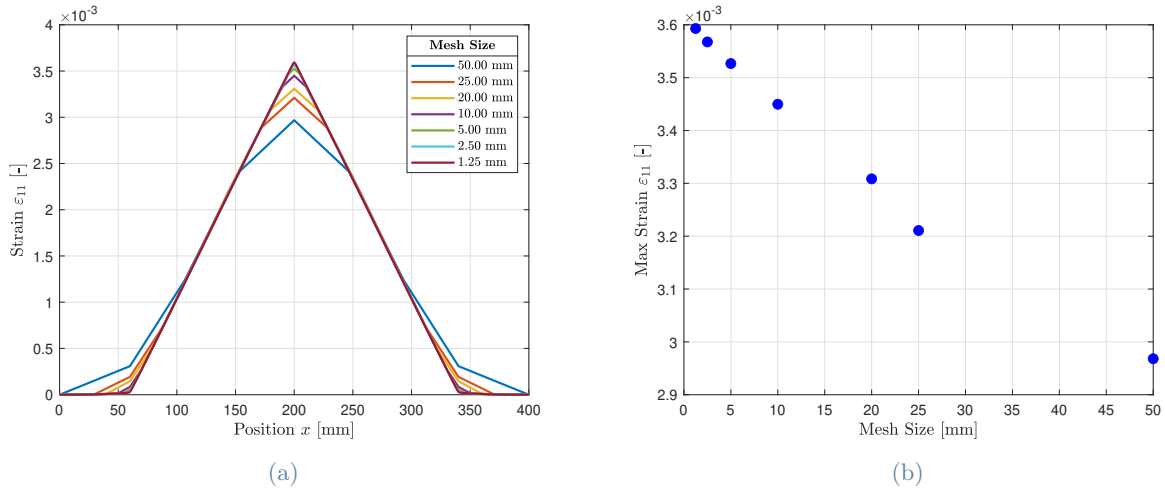


Figure 15: Convergence Analysis plots for 3PB model in LC configuration a) ε_{11} Strain profiles along the midline $y=50$ b) $\varepsilon_{11}max$ vs Mesh Size.

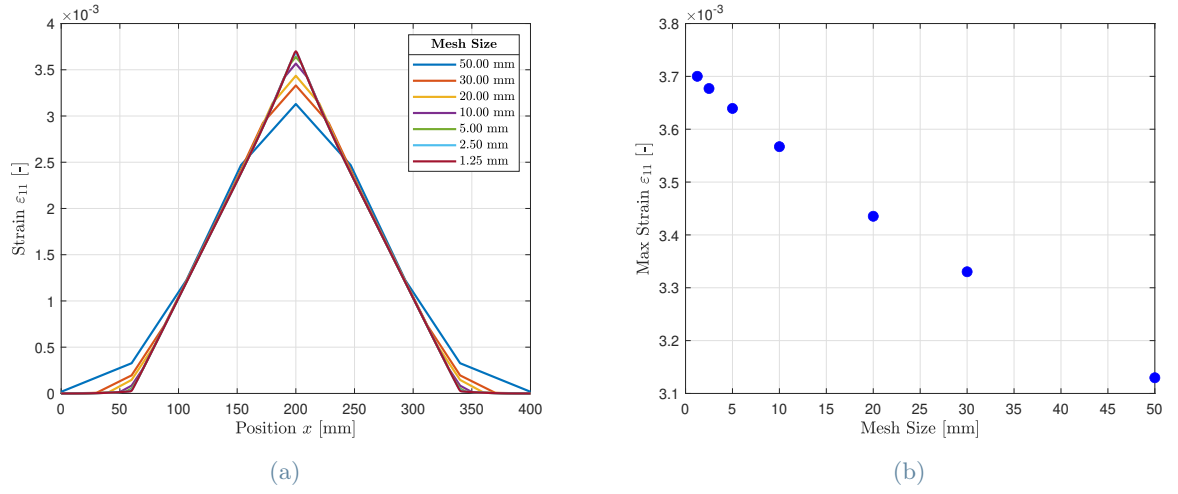


Figure 16: Convergence Analysis plots for 3PB model in DC configuration a) ε_{11} Strain profiles along the midline $y=50$ b) $\varepsilon_{11}max$ vs Mesh Size.

4.1.8. Job module

At this point, the simulation is ready to be performed through the creation of a job (no rule in the definition of the name). The simulation will give an `.odb` file as output, and the Matlab code will extract from it only the strain, theirs relative position and geometry information.

With the numerical models fully defined, the analyses are solved to capture the baseline mechanical behavior. The direct FEM solutions for the 3PB and CP load cases are depicted in Fig. 17 and Figure 18, respectively. The obtained structural responses demonstrate to be aligned with the theoretical predictions.

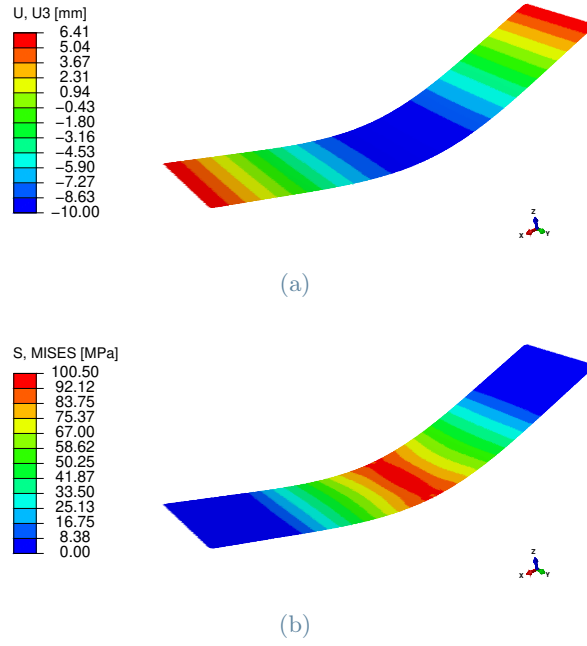


Figure 17: Direct FEM analysis results for 3PB model: (a) contour plot of the out-of-plane displacement field (U_3) and (b) Von Mises equivalent stress distribution.

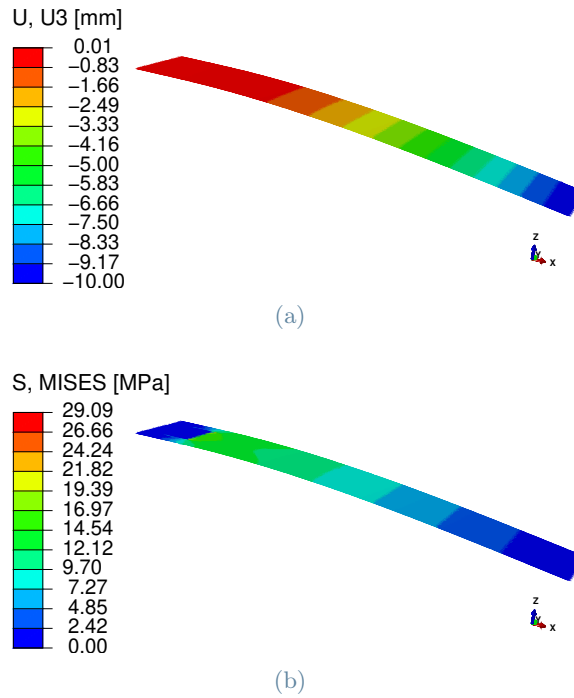


Figure 18: Direct FEM analysis results for CP model: (a) contour plot of the out-of-plane displacement field (U_3) and (b) Von Mises equivalent stress distribution.

4.2. iFEM Algorithm Implementation

4.2.1. Inverse Model Inputs

Following the definition of the Direct FEM model, the corresponding Inverse Finite Element (iFEM) model is developed within the same **Abaqus/CAE** environment. The creation of the inverse model is a procedural derivation of the direct one, modified to adhere to the load-independent nature of the iFEM formulation. The procedure involves the following key steps: first of all the newly created model have to be copied and assigned "iFEM" as name. After this a sequence of change have to be made to this new inverse model.

- Since the iFEM algorithm works independently of the load applied and reconstructs the displacement field solely based on strain measurements, all external loads defined in the **Load module** are suppressed or deleted, this is valid also for DC configuration, where even if the load is not involving force, the displacement applied act as one. The boundary conditions (BC) are retained but strictly defined using the *Displacement/Rotation* type.
- A crucial aspect of the iFEM implementation is the correct orientation of the strain components. From the Assembly module following "Create Datum CSYS" through three point → Rectangular Coordinate System Type → name it "SYS1", a local coordinate system is created at the origin of the model. This system defines the orientation of the measured strains. Even though it doesn't affect the algorithm, the direction chosen for these models are: z-direction tranverse to the plate, x-direction along the principal dimension of the plate, y-direction along the secondary dimension of the plate
- Subsequently, specific sets are defined to interface the **Abaqus/CAE** model with the Matlab post-processing code. From Assembly module, a geometry set named **SetOrientation1** is created containing the region of the part associated with the local coordinate system **SYS1** (so the generic SetOrientationN will be referring to SYSN)
- To identify sensor locations, element sets (composed by sensorized elements) are defined following a specific naming convention. The Matlab algorithm parses these names to determine what to measure, where to measure it, and how to assign it to the inverse model.

The general syntax structure is defined as follows:

SG_<Component>_<Surface>_<RefSys>_<Instance>

A detailed breakdown of each field is provided below:

- ▷ **SG Prefix:** A fixed identifier standing for "Strain Gauge", used to flag the set for processing.
- ▷ **<Component>** nomenclature: specifies which strain components are extracted with respect to the selected reference system.
 - **ALL:** Extracts all in-plane components ($\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}$).

- **XX**: Extracts only the longitudinal strain component (ε_{xx}).
 - **YY**: Extracts only the transverse strain component (ε_{yy}).
 - **XY**: Extracts only the shear strain component (γ_{xy}).
- ▷ **<Surface>** nomenclature: defines the mapping between the **.odb** source surface and the iFEM destination surface.
- **tt**: Extracts data from the **top** side of the **.odb** shell and assigns them to the **top** side of the iFEM model.
 - **bb**: Extracts data from the **bottom** side of the **.odb** shell and assigns them to the **bottom** side of the iFEM model.
 - **bt**: Extracts data from the **bottom** side of the **.odb** shell and assigns them to the **top** side of the iFEM model.
 - **direct**: Combined operation; extracts top and assigns to top, and extracts bottom and assigns to bottom.
 - **flip**: Inverted operation; extracts top and assigns to bottom, and extracts bottom and assigns to top.
- ▷ **<RefSys>** nomenclature: an integer linking the data to a specific **SetOrientation** (e.g., **SetOrientation=N** refers to the orientation defined by **SYSN**).
- ▷ **<Instance>** nomenclature (Optional): the name of the specific instance on which to perform the data extraction (e.g., **PART-1-1**). If omitted, the nearest element is searched across the entire **.odb** model. Providing this improves search performance.
- **Inverse Mesh Definition**: The mesh for the inverse model requires specific element attributes. The part is meshed using the same S4 shell elements. Regarding inverse mesh density, a general rule of thumb suggests a ratio of 3 or 5 direct elements for every single inverse element in each direction to ensure that the centroid of the inverse element aligns sufficiently with the underlying direct solution. However, given the fine discretization of the direct model in this study, a slightly coarser inverse mesh is also permissible.

Finally, a job named **iFEM** is run (to be defined within the iFEM model). This simulation does not produce physical results (as loads are absent) but generates the necessary **.odb** container structure that the Matlab code will populate with the reconstructed field.

The inverse model is now ready to generate the empty **.odb** file, which will serve as the baseline to be populated with the algorithm's reconstructions.

Once the creation of both the Direct and Inverse models in **Abaqus/CAE** is completed, the code is ready for execution. Upon launching the script, a sequence of simulation hyperparameters

must be defined. A list of these parameters is provided below, some of which will be discussed in greater detail in subsequent sections.

- **Scale factor:** scale factor on which the results will be visualized in the graphic output
- **alfa coefficient:** weight of the unsensorized elements (see section 3.4.1)
- **Polynomial Extrapolation:** boolean flag that enable the interpolation of polynomial type
- **Physics-based Extrapolation:** boolean flag that enable the interpolation of physics-based type
- **order_x_xx:** represent the order of x-variable in polynomial fitting for ε_{xx} strain component
- **order_y_xx:** represent the order of y-variable in polynomial fitting for ε_{xx} strain component
- **order_x_yy:** represent the order of x-variable in polynomial fitting for ε_{yy} strain component
- **order_y_yy:** represent the order of y-variable in polynomial fitting for ε_{yy} strain component
- **order_x_xy:** represent the order of x-variable in polynomial fitting for γ_{xy} strain component
- **order_y_xy:** represent the order of y-variable in polynomial fitting for γ_{xy} strain component
- **Reference.odt:** is the name of the empty .odt model which will be overwrite by iFEM reconstruction (previously named "iFEM")
- **noise:** A boolean flag that activates the synthetic noise injection strategy. When set to **true**, it simulates the inherent measurement uncertainty of physical sensors by adding random Gaussian noise to the numerical strain data (see section 5.3.3).
- **noise amp:** amplitude of the numerical noise (see section 5.3.3)
- **symmetry_type:** An integer parameter defining the data extraction and assignment regarding both surfaces of the part when available data are coming just from one of the two (see section 5.3.3 for further details), it is defined to assume specific values ,acting as flags, that defines the assignment strategy:
 - ▷ 1: the code takes the available data on one side and assign them to the corresponding position on the opposite side
 - ▷ -1: the code takes the available data on one side and assign them to the corresponding position on the opposite side after changing the strain's signs.
 - ▷ 0: not active, data available on both sides

- **CAE file:** name of the Abaqus/CAE direct CAE file from which to extract direct model feature
- **Model name:** name of the inverse model (namely "iFEM")
- **Measure file:** name of the .odb file from which extract numerical data

4.2.2. Inverse Model Output and Algorithm Execution

The shape sensing algorithm operates by processing the finite element geometry and sensor configuration data to reconstruct the structural deformation field. The reconstruction procedure integrates the measured strain data with the inverse finite element formulation, minimizing a weighted functional between measured and numerical strains to compute the displacement field. The algorithm then stores the reconstructed results directly into the Abaqus/CAE Output Database format as Field Outputs in a new analysis step and it generates a comprehensive set of results stored in a dedicated directory to ensure data traceability. For every execution, a new folder is created within the iFEM_Results directory, named according to the processed job (e.g., iFEM_Results/3PB_Load_Case_1/). The execution generates the following output files:

- **Mat Data Files (.mat):** Workspace files containing the reconstructed displacement field, strain field, and error metrics computed against the reference FEM solution.
- **Report:** A summary text file logging simulation parameters, execution time, and global error indices ($RMSE$, $\%Err_{max}$).
- **Reconstructed .odb:** A copy of the original .odb file populated with the iFEM results for post-processing in Visualization module in Abaqus/CAE.

The available output variables are defined as follows:

- **E_iFEM_bottom:** Strain from iFEM on the bottom surface of the plate (3 components). Centroidal strains derived from direct FEM are mapped onto the iFEM element centers. Integration is then executed without recourse to Gauss points, streamlining the inverse kinematic reconstruction
- **E_iFEM_bottom_GP:** Centroidal strains derived from direct FEM are mapped onto the integration points of the iFEM elements, assuming a constant strain distribution across the Gauss points. The numerical integration is then executed following the standard Gauss quadrature scheme
- **E_iFEM_top:** Strain from iFEM on the top surface of the plate (3 components).
- **E_iFEM_top_GP:** Same as above, at Gauss points (analogous reasoning).
- **E_iFEM_vm_bottom:** Von Mises strain value computed from iFEM strains on the bottom surface (scalar combining components).
- **E_iFEM_vm_top:** Von Mises strain value computed from iFEM strains on the top surface.

- **E_inp_bottom**: Input strains to iFEM corresponding to sensor-only values (not including pre-extrapolated strain).
- **E_inp_top**: Same as above for the top surface of the plate.
- **U**: Displacements obtained from the iFEM algorithm.
- **U_mag**: Magnitude of displacement components obtained from the iFEM (inverse FEM).
- **U_abaqus**: Displacements obtained from the direct FEM results.
- **U_abaqus_mag**: Magnitude of the displacement vector from the direct FEM results.
- **U_err**: Error (percentage) between direct FEM displacements and iFEM displacements for each component, normalized by the maximum displacement.
- **U_err_mag**: Same as **U_err** but using norms (magnitude) of displacement errors.
- **E_inp_fit_bottom**: Input strains to iFEM for elements without sensors where pre-extrapolation was imposed; contains also sensor values where available.
- **E_inp_fit_bottom_GP**: Same as above but evaluated at Gauss points.
- **E_inp_fit_top**: Same as **E_inp_fit_bottom** for the top surface.
- **E_inp_fit_top_GP**: Same as **E_inp_fit_top** evaluated at Gauss points.

5 | Sensor Network Design and Preliminary Numerical Assessment

5.1. Numerical Sensitivity Analysis

This section outlines the analyses, considerations, and numerical investigations that led to the definition of guidelines for determining the optimal SN in a bending-dominated loading scenario. The analysis is structured in multiple steps, aiming to investigate the algorithm's nature, flexibility, and adaptability with respect to its input parameters.

The logical workflow begins with a purely numerical approach regarding available data, initially based on the original iFEM formulation. It then progressively evolves step-by-step towards a realistic application of the algorithm, incorporating alternative formulations and realistic measurement conditions.

To evaluate the algorithm's performance, the percentage error U_{err} of the displacement magnitude at nodes was selected as the primary metric, specifically assessing both the maximum nodal value U_{err}^{max} and the mean nodal value U_{err}^{mean} . The percentage error is calculated using Equation 36, normalized with respect to maximum displacement, in order to avoid numerical overshooting in regions with close to zero displacement:

$$\mathbf{U}_{err} = \frac{|\mathbf{U}_{iFEM} - \mathbf{U}_{FEM}|}{|max(\mathbf{U}_{FEM})|} \times 100 \quad (36)$$

where $\mathbf{U}_{iFEM}$ represents the displacement magnitude reconstructed by the iFEM at a given node, and $\mathbf{U}_{FEM}$ is the displacement magnitude of the same node extracted from the Direct FEM model, which serves as the numerical reference source. This analysis was conducted on the 3PB model under both loading configurations.

Proof of Reconstruction Error Independence from Load Magnitude All the analysis presented are made under the same magnitude of load applied, this could be done exploiting the fact, under the assumption of linear elasticity, the percentage reconstruction error between the iFEM-reconstructed displacement field and the numerical reference field is independent of the magnitude of the applied load, if and only if the load vector is scaled homogeneously, which is what happens in the cases presented.

Let us consider a reference structural problem governed by linear elasticity. The relationship between the applied external load vector, $\mathbf{F}$, and the resulting global nodal displacement vector, $\mathbf{U}_{ref}$, in the direct Finite Element Model (FEM) is given by:

$$\mathbf{K}_{FEM}\mathbf{U}_{ref} = \mathbf{F} \quad (37)$$

where $\mathbf{K}_{FEM}$ is the constant global stiffness matrix of the direct model. Due to the linearity of the system, scaling the applied load by a scalar factor $\alpha \in \mathbb{R}$ results in a proportional scaling of the displacement field:

$$\mathbf{F}' = \alpha\mathbf{F} \implies \mathbf{U}'_{ref} = \alpha\mathbf{U}_{ref} \quad (38)$$

Consequently, the strain field $\boldsymbol{\varepsilon}$, which is derived from the displacements via the linear strain-displacement relations (Eq. 6), also scales linearly:

$$\boldsymbol{\varepsilon}'_{input} = \alpha\boldsymbol{\varepsilon}_{input} \quad (39)$$

where $\boldsymbol{\varepsilon}_{input}$ represents the experimental in-situ strain measurements (or numerically simulated strains) used as input for the iFEM algorithm.

Recall the global solving equation of the Inverse Finite Element Method derived in Eq. 18:

$$\mathbf{K}_{iFEM}\mathbf{U}_{iFEM} = \mathbf{F}_{iFEM}(\boldsymbol{\varepsilon}_{input}) \quad (40)$$

Here, $\mathbf{K}_{iFEM}$ (denoted as $\mathbf{K}$ or $\mathbf{K}_{FF}$ for the reduced matrix in Eq. 19) depends solely on the geometry and the shape functions, and is therefore independent of the load magnitude. The right-hand side vector $\mathbf{F}_{iFEM}$ (denoted as $\mathbf{F}$ in Eq. 17) is assembled from elemental contributions $\mathbf{f}^i$. As defined in Eq. 13, $\mathbf{f}^i$ is a linear function of the measured strains $\mathbf{e}_{i,j}^\varepsilon$, $\mathbf{k}_{i,j}^\varepsilon$, and $\mathbf{g}_{i,j}^\varepsilon$

$$\mathbf{f}^i = \frac{1}{n_s} \iint_{A_i} \sum_{j=1}^{n_s} \left(\mathbf{B}^{mT} \mathbf{e}_{i,j}^\varepsilon + (2h)^2 \mathbf{B}^{bT} \mathbf{k}_{i,j}^\varepsilon + \mathbf{B}^{sT} \mathbf{g}_{i,j}^\varepsilon \right) dx dy \quad (41)$$

Substituting the scaled input strains $\boldsymbol{\varepsilon}'_{input} = \alpha\boldsymbol{\varepsilon}_{input}$ into the definition of the force vector, and exploiting the linearity of the integral and summation operators, we obtain:

$$\mathbf{F}_{iFEM}(\alpha\boldsymbol{\varepsilon}_{input}) = \alpha\mathbf{F}_{iFEM}(\boldsymbol{\varepsilon}_{input}) \quad (42)$$

Substituting this into the global equilibrium equation for the scaled load case:

$$\mathbf{K}_{iFEM}\mathbf{U}'_{iFEM} = \mathbf{F}'_{iFEM} = \alpha\mathbf{F}_{iFEM} \quad (43)$$

Multiplying the original solution $\mathbf{U}_{iFEM} = \mathbf{K}_{iFEM}^{-1} \mathbf{F}_{iFEM}$ by α , we verify that:

$$\mathbf{U}'_{iFEM} = \alpha \mathbf{U}_{iFEM} \quad (44)$$

Thus, both the reference displacement field $\mathbf{U}_{ref}$ and the reconstructed displacement field $\mathbf{U}_{iFEM}$ scale linearly with the load factor α .

Finally, let us examine the percentage reconstruction error, $\mathbf{U}'_{err}$. for the scaled load case is:

$$\mathbf{U}'_{err}(\alpha) = \frac{|\mathbf{U}'_{iFEM} - \mathbf{U}'_{ref}|}{|max(\mathbf{U}'_{ref})|} \times 100 \quad (45)$$

Substituting the scaled variables:

$$\mathbf{U}'_{err}(\alpha) = \frac{|\alpha \mathbf{U}_{iFEM} - \alpha \mathbf{U}_{ref}|}{|max(\alpha \mathbf{U}_{ref})|} \times 100 = \frac{|\alpha| \cdot |\mathbf{U}_{iFEM} - \mathbf{U}_{ref}|}{|\alpha| \cdot |max(\mathbf{U}_{ref})|} \times 100 \quad (46)$$

Simplifying the scalar factor $|\alpha|$ (assuming $\alpha \neq 0$):

$$\mathbf{U}'_{err}(\alpha) = \frac{|\mathbf{U}_{iFEM} - \mathbf{U}_{ref}|}{|max(\mathbf{U}_{ref})|} \times 100 = \mathbf{U}_{err} \quad (47)$$

This concludes the proof that, within the linear elastic regime, the percentage reconstruction error of the iFEM algorithm is invariant with respect to the magnitude of the applied load.

Analogously, it could be demonstrated the independence of the reconstruction error from the load magnitude holds true also for displacement-controlled conditions.

Let us partition the global degrees of freedom (DOFs) of the reference FEM model into two sets: the free DOFs ($\mathbf{U}_f$) and the prescribed (imposed) DOFs ($\mathbf{U}_p$). The system equilibrium equation, in the absence of external forces on the free nodes, can be partitioned as follows:

$$\begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fp} \\ \mathbf{K}_{pf} & \mathbf{K}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{U}_f \\ \mathbf{U}_p \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{R}_p \end{bmatrix} \quad (48)$$

where $\mathbf{R}_p$ represents the reaction forces at the constrained nodes. The displacement of the free nodes is determined by the first row of the partitioned matrix:

$$\mathbf{K}_{ff} \mathbf{U}_f + \mathbf{K}_{fp} \mathbf{U}_p = \mathbf{0} \implies \mathbf{U}_f = -\mathbf{K}_{ff}^{-1} \mathbf{K}_{fp} \mathbf{U}_p \quad (49)$$

This equation establishes a linear relationship between the imposed displacements $\mathbf{U}_p$ and the structural response $\mathbf{U}_f$. Let us verify the scaling behavior. If the imposed displacement is scaled by a factor α (i.e., $\mathbf{U}'_p = \alpha \mathbf{U}_p$):

$$\mathbf{U}'_f = -\mathbf{K}_{ff}^{-1} \mathbf{K}_{fp} (\alpha \mathbf{U}_p) = \alpha \left(-\mathbf{K}_{ff}^{-1} \mathbf{K}_{fp} \mathbf{U}_p \right) = \alpha \mathbf{U}_f \quad (50)$$

Since the entire displacement field $\mathbf{U}_{ref}$ scales linearly with α , the strain field ϵ also scales by α . Following the same logic derived for the force-controlled case, the iFEM reconstruction $\mathbf{U}_{iFEM}$ is linear with respect to the input strains. Therefore:

$$\mathbf{U}'_{iFEM} = \alpha \mathbf{U}_{iFEM} \quad (51)$$

Finally, the percentage reconstruction error for the scaled displacement case is:

$$\mathbf{U}'_{err}(\alpha) = \frac{|\alpha \mathbf{U}_{iFEM} - \alpha \mathbf{U}_{ref}|}{|\max(\alpha \mathbf{U}_{ref})|} \times 100 = \frac{|\alpha| \cdot |\mathbf{U}_{iFEM} - \mathbf{U}_{ref}|}{|\alpha| \cdot |\max(\mathbf{U}_{ref})|} \times 100 = \mathbf{U}_{err} \quad (52)$$

This confirms that the accuracy of the iFEM shape sensing is intrinsic to the sensor configuration and the shape functions, and is invariant with respect to the magnitude of the stimulus, whether applied as a force or as an imposed displacement. These statements have been verified by numerical assessment, the mesh used is the same first mesh that will be used in the next section's analysis. In line with the analytical findings, the percentage errors do not vary with changes in load intensity, neither in the DC (Fig. 19a) nor in the LC (Fig. 19b) configuration, up until the fifth decimal place, but this difference is fully attributed to numerical approximation of the numerical solver.

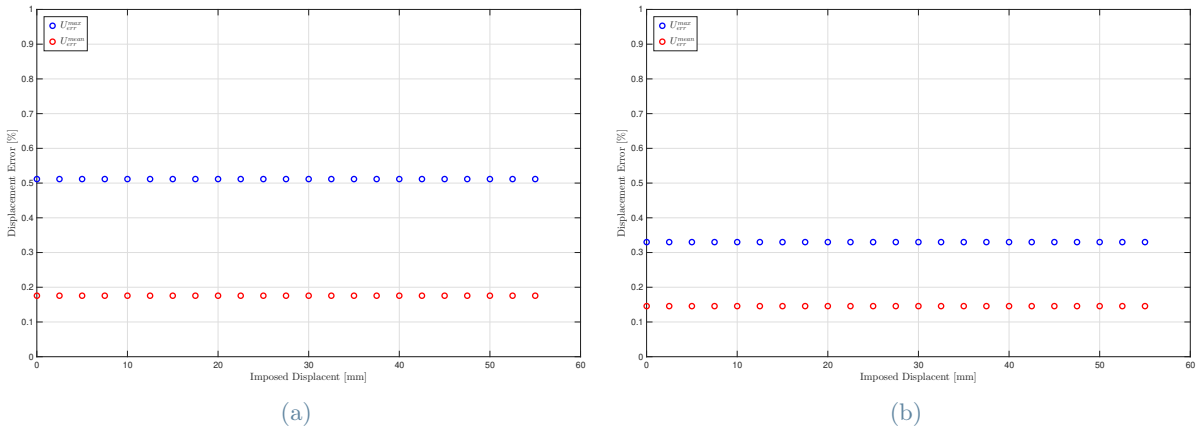


Figure 19: Analysis of independence of percentage error from load magnitude: a) Results for 3PB model under DC configuration, b) Results for 3PB model under LC configuration.

A note regarding the above statement is that for strain values close to zero, the effect of measurement uncertainty would begin to become non-negligible, as the uncertainty itself becomes of the same order of magnitude as the strain to be acquired. Let us model the experimental strain input, ϵ_{exp} , as the superposition of the true structural strain, ϵ_{true} , and a stochastic noise vector, δ_ϵ , introduced by the instrumentation (e.g., interrogator resolution, thermal noise, quantization error):

$$\epsilon_{exp} = \epsilon_{true} + \delta_\epsilon \quad (53)$$

The magnitude of the noise vector is bounded by the sensor accuracy and resolution.

Recalling the linear operator of the iFEM formulation, the reconstructed displacement field $\mathbf{U}_{iFEM}$ is:

$$\mathbf{U}_{iFEM} = \mathbf{K}_{iFEM}^{-1} \mathbf{F}_{iFEM}(\boldsymbol{\varepsilon}_{exp}) \quad (54)$$

Exploiting the linearity of the force vector assembly function, the reconstruction into a "physical" component and a "noise-induced" component can be decomposed in:

$$\mathbf{U}_{iFEM} = \mathbf{K}_{iFEM}^{-1} \mathbf{F}_{iFEM}(\boldsymbol{\varepsilon}_{true} + \boldsymbol{\delta}_{\varepsilon}) = \underbrace{\mathbf{K}_{iFEM}^{-1} \mathbf{F}_{iFEM}(\boldsymbol{\varepsilon}_{true})}_{\mathbf{U}_{physical}} + \underbrace{\mathbf{K}_{iFEM}^{-1} \mathbf{F}_{iFEM}(\boldsymbol{\delta}_{\varepsilon})}_{\mathbf{U}_{noise}} \quad (55)$$

Here, $\mathbf{U}_{noise}$ represents a fictitious displacement field generated solely by the measurement uncertainty. This term represents the "uncertainty floor" of the shape sensing algorithm.

For the reconstruction to be valid, the physical displacement must dominate the noise-induced displacement. We can define the validity condition as:

$$\|\mathbf{U}_{physical}\| \gg \|\mathbf{U}_{noise}\| \quad (56)$$

The minimum detectable deformation is defined as the structural state where the order of magnitude of the displacements is comparable to $\delta_{U,max}$. Below this threshold, the percentage error $E_{\%}$ diverges hyperbolically, as the denominator tends to zero while the numerator (absolute error) remains fixed at the noise level:

$$E_{\%} \approx \frac{\|\mathbf{U}_{noise}\|}{\|\mathbf{U}_{physical}\|} \rightarrow \infty \quad \text{as} \quad \|\mathbf{U}_{physical}\| \rightarrow 0 \quad (57)$$

Therefore, the instrumentation specifications (specifically the strain resolution ε_{min}) directly determine the smallest displacement the iFEM system can reliably reconstruct.

While this regards the lower bound of accuracy, the upper bound would be defined by the elastic limit of the material, since the whole formulation relies on elasticity assumption. That means that the error should start to grow approaching yield limit, but this can't be assessed numerically with the current model since the plastic behavior was not defined in **Abaqus/CAE** environment.

Since the percentage error is independent of the applied load, all subsequent analyses presented in this study are conducted using a common source of numerical measurements. Specifically, the same .odb output file from the direct FEM model is employed as the reference for extracting virtual measurements. The numerical baseline of the strain field of the 3PB model on configuration DC and LC are illustrated respectively in Fig. 20 and Fig. 21, for simplicity only the variable fields at the bottom surface are shown.

For a detailed discussion regarding the differences observed between Fig. 20b and Fig. 21b, the reader is referred to section 5.3.2.

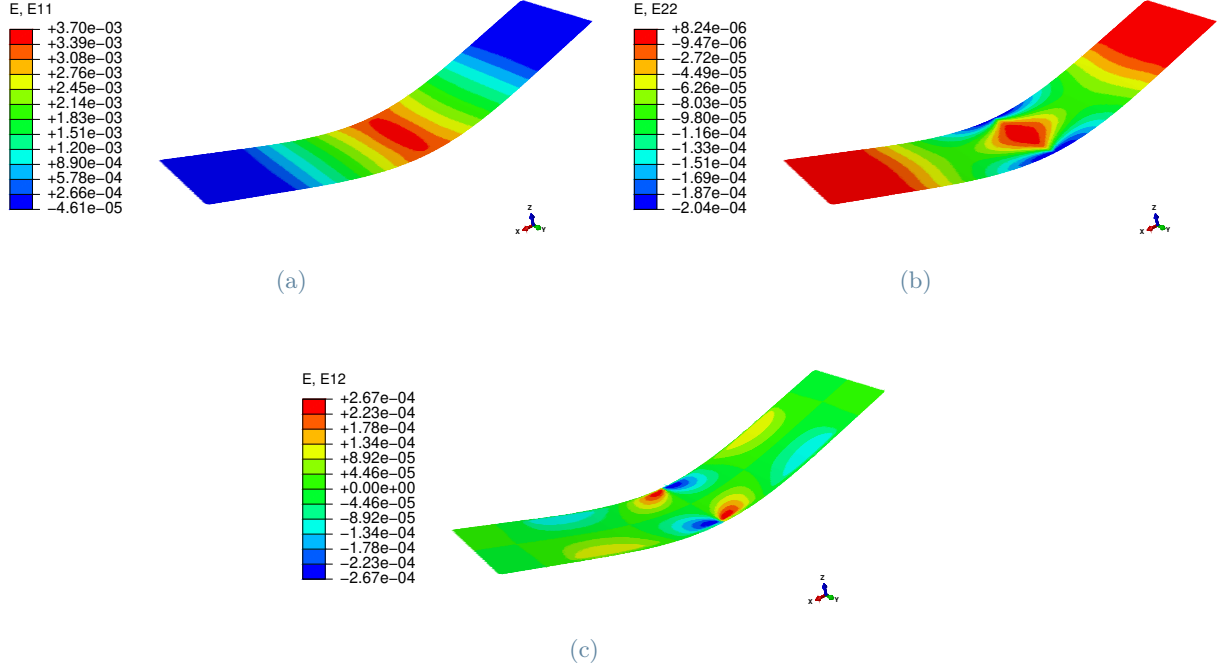


Figure 20: Strain field resulting from the bottom surface of the baseline 3PB direct FEM model used for iFEM input under LC: a) strain field ε_{xx} [mm/mm], b) strain field ε_{yy} [mm/mm], c) strain field ε_{xy} [mm/mm].

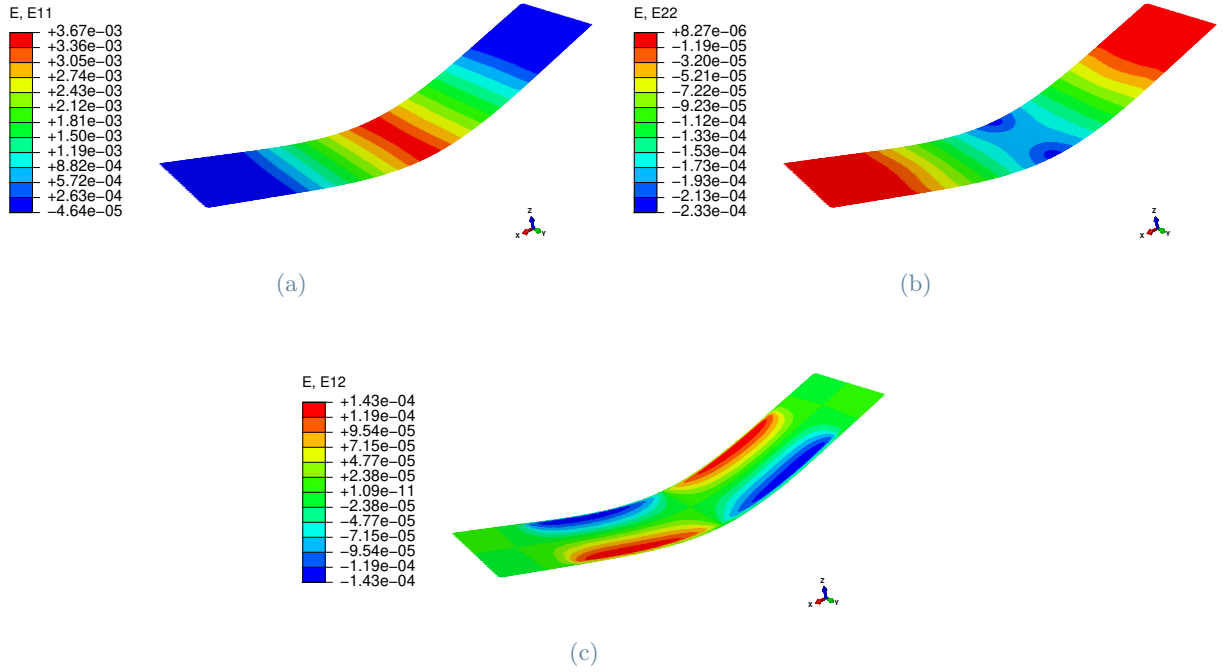


Figure 21: Strain field resulting from the bottom surface of the baseline 3PB direct FEM model used for iFEM input under LC: a) strain field ε_{xx} [mm/mm], b) strain field ε_{yy} [mm/mm], c) strain field ε_{xy} [mm/mm].

5.1.1. Convergence Analysis

The initial phase of the analysis investigates the influence of the inverse element size on the reconstruction quality. While maintaining the same input dataset, the analysis started with a coarse mesh, which was progressively refined until it matched the element size adopted for the direct model.

In this scenario, every inverse element is treated as fully sensorized, providing data on all three strain components ($\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}$). From a sensor perspective, this is equivalent to equipping the centroid of each element with a strain rosette.

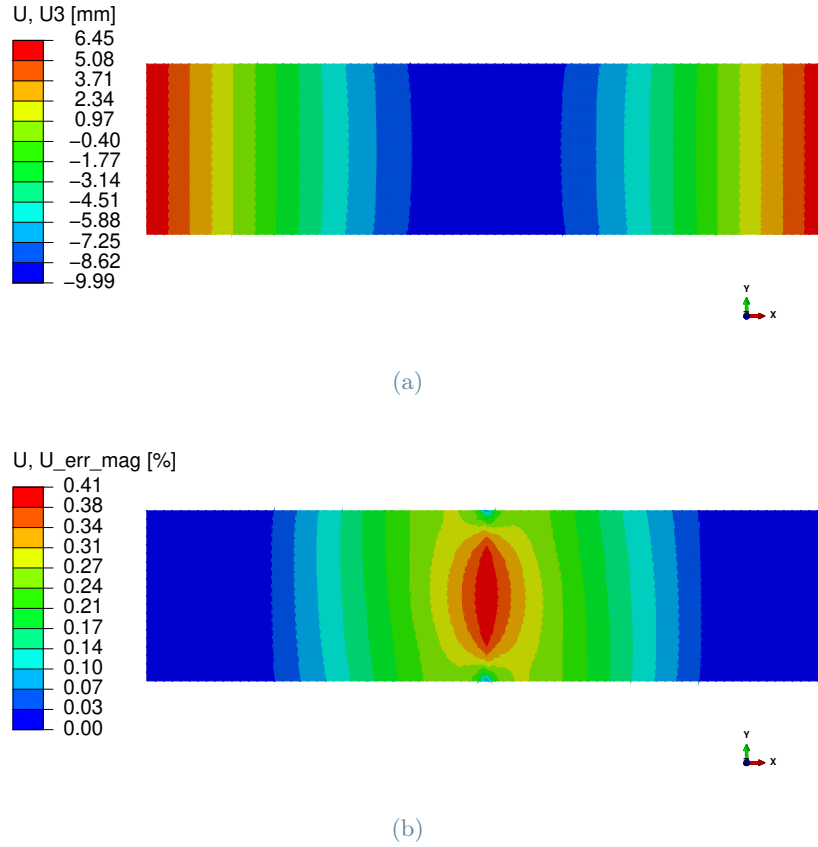


Figure 22: Contour plots of U3 displacement for 3PB model under DC configuration: (a) iFEM reconstruction, (b) percentage error U_{err} of the iFEM reconstruction.

Figs. 22b and 23b illustrates that, in the case of the finest mesh, the reconstruction exhibits excellent quality, with error magnitudes consistently below 1%. It can be observed that the error vanishes at the boundary constraints and the unloaded ends. Furthermore, the error reaches its minimum values as the strain increases toward the mid-span in a symmetric fashion, which is consistent with the symmetry of the input data provided by the sensors.



(a)



(b)

Figure 23: Contour plots of U3 displacement for 3PB model under LC configuration: (a) iFEM reconstruction, (b) percentage error U_{err} of the iFEM reconstruction.

As shown in Fig. 24a and Fig. 24b, a clear trend in the error relative to the mesh characteristics is observed. Specifically, as the inverse element size decreases, both error metrics decrease and stabilize at a constant value. This demonstrates that as the density of homogeneously distributed measurements increases, the input data available to the algorithm tends towards a continuous rather than discrete information field.

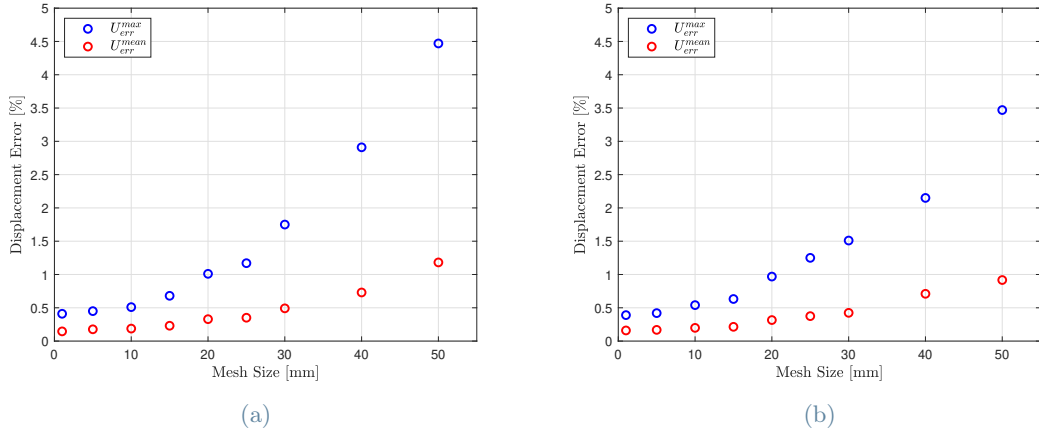


Figure 24: Convergence Analysis plots for 3PB model in a) DC configuration b) LC configuration.

5.1.2. Grid Analysis

The second step addresses the practical impossibility of equipping every inverse element with measurement instruments, due to feasibility limits in sensor installation. Therefore, the aim was to investigate the algorithm's response as the SN density decreases.

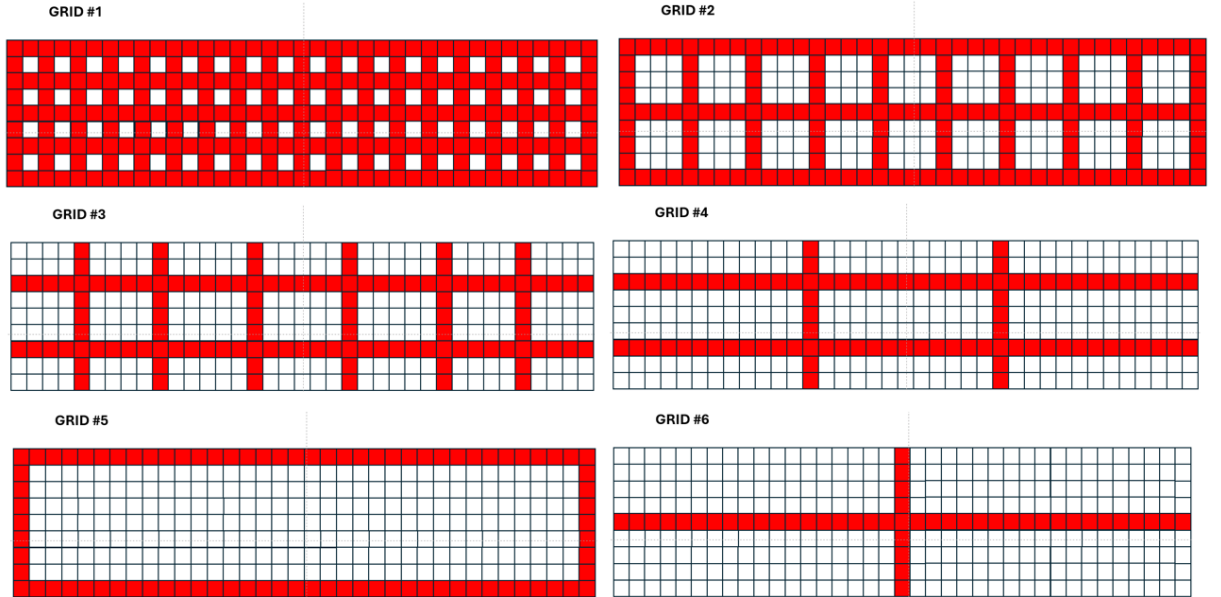


Figure 25: Six different grids used for the analysis.

For this simulation, six distinct sensor grids were identified, defined by the number of fully sensorized element rows and columns (see Fig. 25), with the number of rows and columns decreasing for subsequent grid indices. It is noted that Grid #4 and Grid #5 do not differ in the total number of sensorized elements but only in their positioning; this was designed to highlight a possible dependence on the localization of input measurements. For this step, all sensorized

elements are still provided with all three strain components. The inverse mesh remains fixed for all the grids under examination, and it's made by 37 elements in y direction and 9 elements in x direction.

As observed in Fig. 26a and Fig. 26b, as elements and rows of sensors are removed, the quality of the reconstruction diminishes. Therefore, it can be easily concluded that, ideally and without other design constraints, the densest possible SN will yield the best algorithm performance. A particular note arises from the comparison between Grid #4 and Grid #5. From a purely theoretical standpoint, given a closed domain of strain boundary values and the absence of internal loads, the strain field within the domain is uniquely determined by the analytical system [174]. Since the iFEM is founded precisely on this analytical framework, an input configuration such as that of Grid #5 should theoretically suffice for a high-quality reconstruction of the strain field.

However, the model under investigation (in both loading configurations) is subject to internal loads; consequently, the analytical solution is not uniquely defined. In particular, it is observed that inputs originating from the unloaded regions of the plate (specifically, the lateral columns of sensors positioned on the flanges external to the support rollers) lead the algorithm to an erroneous interpretation of the strain profile. Conversely, input data extracted from the internal region, as in the case of Grid #4, enables a more physically consistent reconstruction.

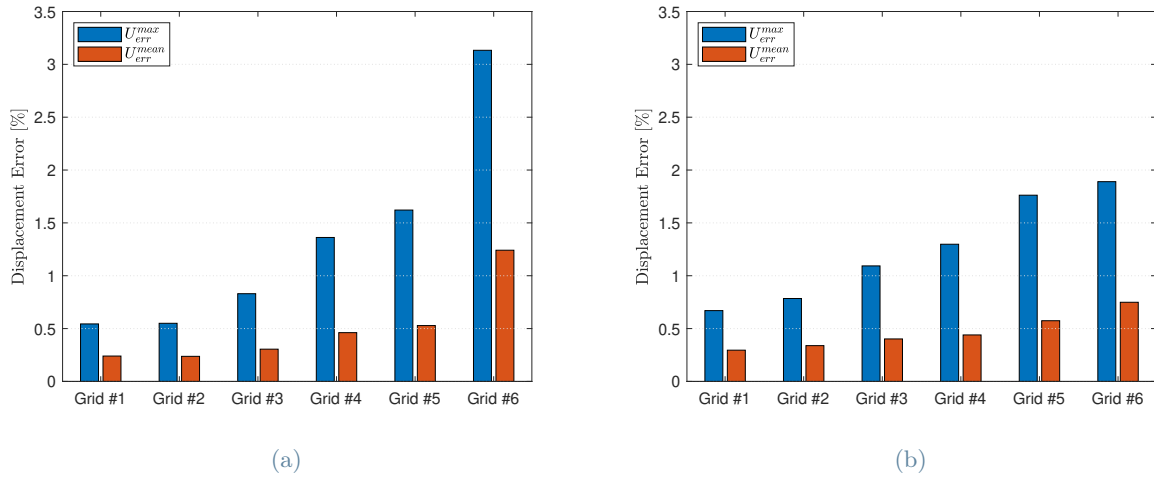


Figure 26: Grid Analysis plots for 3PB model in a) DC configuration b) LC configuration.

5.1.3. Strain Component Analysis

The subsequent phase of the analysis addresses the practical constraints associated with installing a large number of strain rosettes. Such an arrangement is often unfeasible due to the significant physical footprint and cabling complexity required. Consequently, in many operational scenarios, measurements obtained via fiber optic sensors are preferred.

The primary advantage of this technology lies in the ability to embed multiple discrete measurement points along a single fiber, distributed as needed. However, this introduces a limitation: at each measurement point, only the strain component parallel to the fiber's orientation can be acquired. This constraint necessitates an investigation into which strain components are most critical for enabling the algorithm to accurately reconstruct the strain field.

Based on a preliminary assessment of the strain distributions in this bending-dominated scenario, the longitudinal component aligned with the plate's principal dimension (ε_{xx}) is expected to be dominant, being directly generated by the bending moment. This is followed by the transverse component (ε_{yy}), which is expected to be one order of magnitude lower, and the shear component (γ_{xy}), two orders of magnitude lower, both primarily induced by Poisson's effects and boundary conditions.

To verify this hypothesis, a strain component analysis is conducted across all sensor grids examined in the previous section. The results obtained in the previous section (where all three strain components were acquired) serve as the benchmark (case 'ALL').

Starting from this baseline, analyses were performed first by omitting the shear component (γ_{xy}) from the input data (case ' $\varepsilon_{xx}-\varepsilon_{yy}$ '), subsequently by removing the transverse component (case ' ε_{xx} ') and the case in which only the transverse strain is acquired (case ' ε_{yy} ').

Finally, to further approximate a realistic application scenario, considering that the optical fiber must follow a continuous physical path along the plate, a specific case designated as 'MIX' is analyzed. In this setup, the horizontal rows of sensors acquire the longitudinal component (ε_{xx}), while the remaining elements, positioned along the vertical columns, acquire the transverse component (ε_{yy}). In the majority of instances for both loading methods, no significant difference is observed between the ALL case and the $\varepsilon_{xx}-\varepsilon_{yy}$ case; this confirms the low relevance of the shear component γ_{xy} within a bending-dominated strain field. It is generally observed that the first two cases yield better performance. This can be simply attributed to the fact that acquiring three and two components, respectively, provides a larger amount of information, thereby enhancing the optimization process. Furthermore, the ε_{xx} case consistently registers the worst performance across all scenarios. This stems from the fact that a strain field acquired in a single direction does not allow for a unique reconstruction of the full field. Although the ε_{xx} and MIX cases rely on the same number of input information information, the former proves to be insufficient.

In terms of SN design, this implies that, where possible, it is preferable to acquire components in both principal directions. This observation holds valid for U_{err}^{max} , whereas U_{err}^{mean} appears less sensitive to the different case configurations. This indicates that, from an average perspective, the resulting errors are comparable; however, higher-quality input leads to reduced propagation of displacement error. This propagation is notably driven by an integral accumulation of strain micro-errors, which results in a higher maximum error even in the absence of significant variations in the mean error.

Regarding Grid #5, unlike the other grids, a counter-intuitive trend is observed where the reconstruction accuracy improves or remains stable when limiting the input strain components (e.g., using only ε_{xx} or the 'MIX' configuration) compared to the full 'ALL' set. This behavior can be attributed to the specific topology of Grid #5, which has the vertical columns of sensors positioned in the theoretically unloaded regions (overhangs) of the 3PB specimen. In these zones, the physical strain components are negligible. However, enforcing all three strain components to zero (as in the 'ALL' case) introduces an excessive kinematic constraint (over-stiffening) on the inverse elements. Due to the continuity of the iFEM shape functions, this localized rigidity propagates into the active span, negatively affecting the global displacement reconstruction. Conversely, providing only the longitudinal strain ε_{xx} correctly enforces the null-bending condition along the principal direction while leaving the secondary degrees of freedom (associated with ε_{yy} and γ_{xy}) free to adjust. This reduced-constraint approach allows the algorithm to better accommodate the numerical continuity requirements without inducing artificial locking phenomena.

It is observed that this particular analysis did not reveal any significant difference between LC and DC configurations.

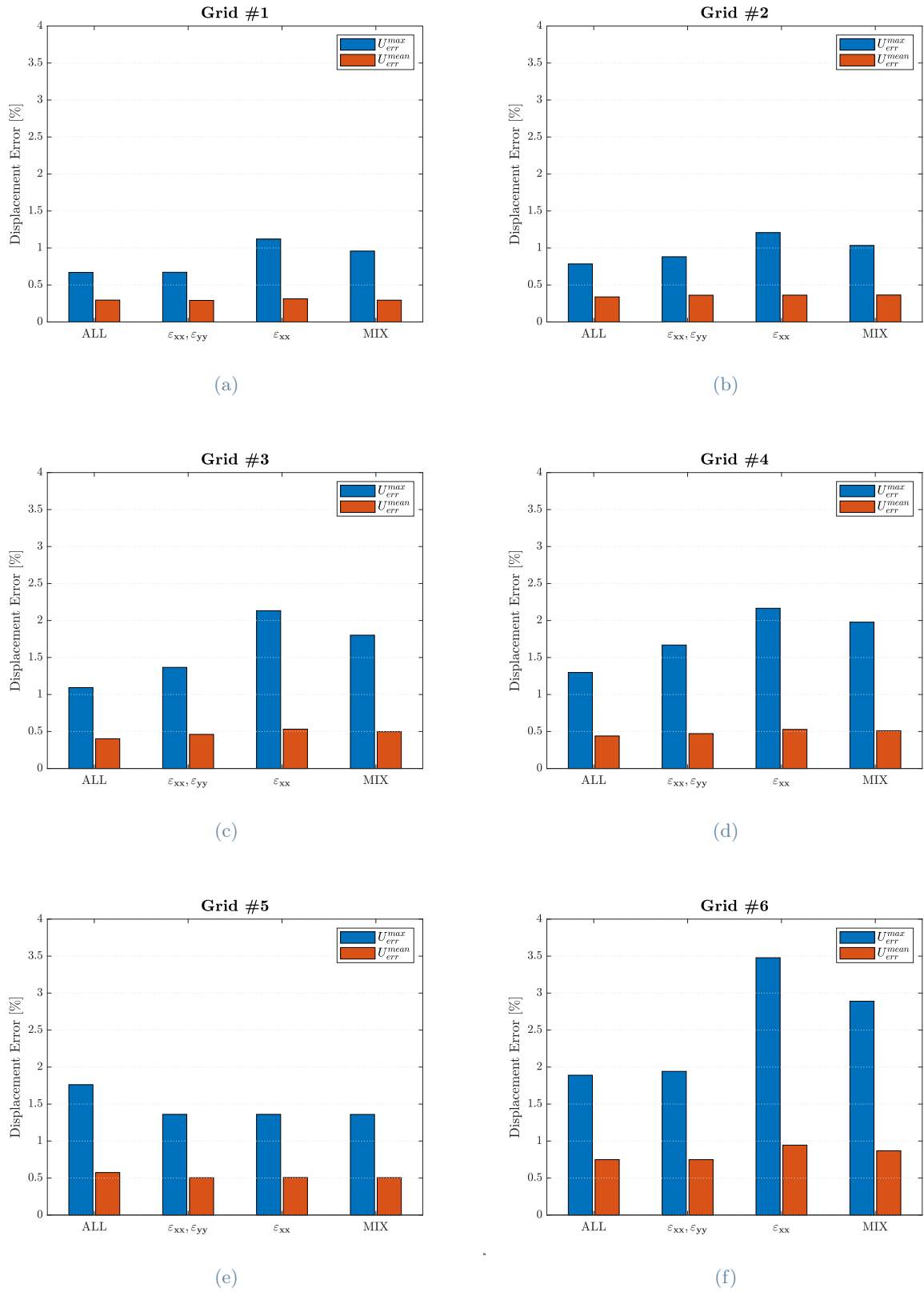


Figure 27: Strain Component analysis plots for 3PB model in DC configuration, a) to f) grid #1 to #6.

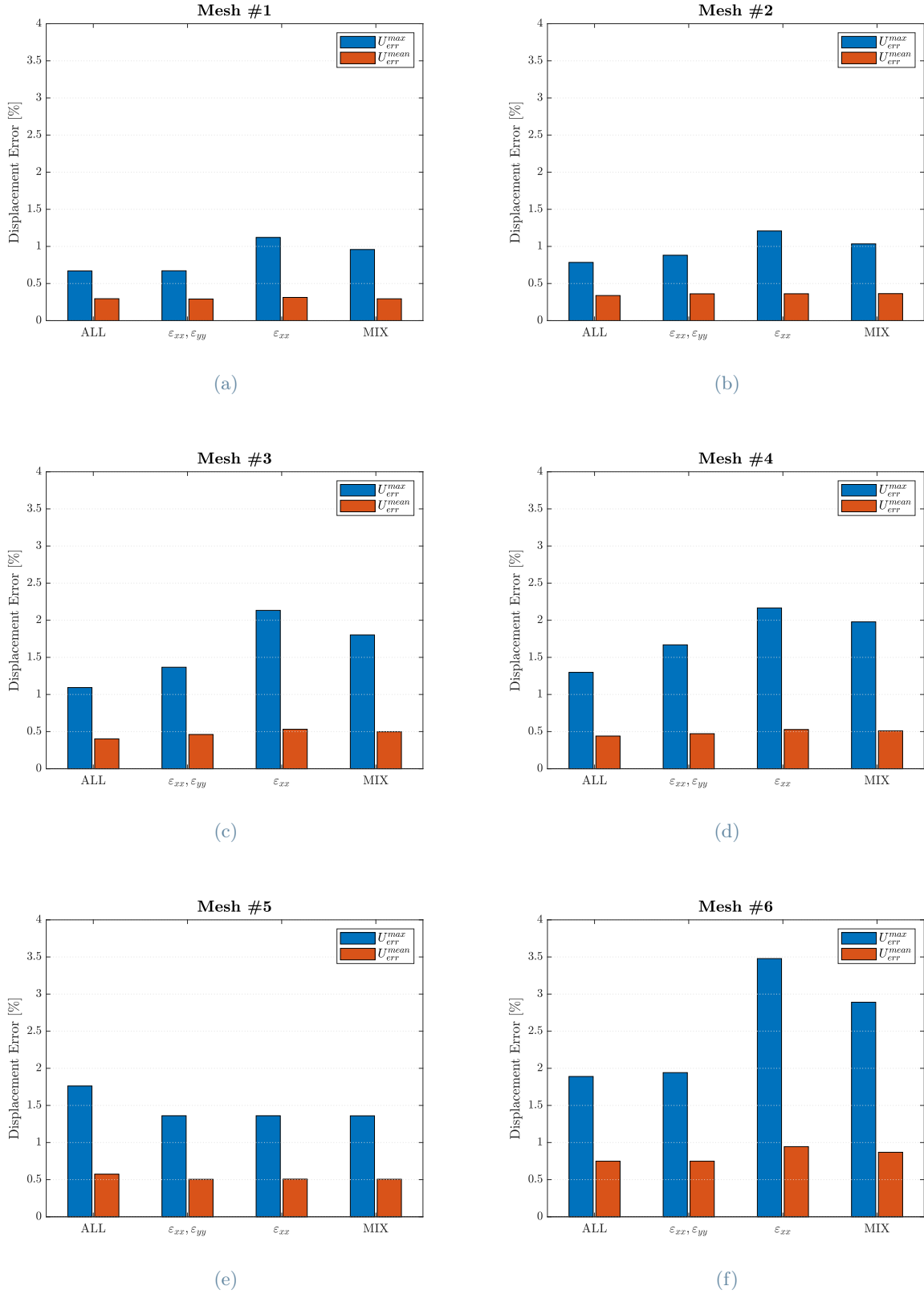


Figure 28: Strain Component analysis plots for 3PB model in LC configuration, a) to f) grid #1 to #6.

5.1.4. Pre-extrapolation Analysis

A preliminary investigation was conducted to assess the effect of performing strain pre-extrapolation on uninstrumented elements. At this stage, the configured pre-extrapolations were not intended to perfectly match the true strain profiles; rather, the objective was to gain a general understanding of the impact of introducing non-zero measurements, even if not physically exact, as opposed to providing the algorithm with null values.

Consequently, a slightly modified mesh configuration, shown in Fig. 29, was used. On this mesh, measurement elements (acquiring all three strain components) were selected and uniformly distributed across the entire plate surface.

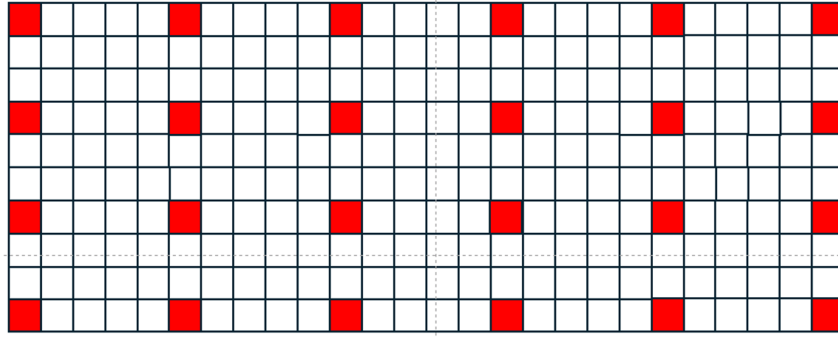


Figure 29: Grid and relative mesh used for extrapolation analysis.

A first round of simulations was executed without pre-extrapolation to serve as a baseline reference. Subsequent simulations involved three distinct levels of pre-extrapolation, defined by different order polynomial functions with the same degree in both directions of the reference system: linear (Fig. 30a), quadratic (Fig. 31a), and cubic (Fig. 31b).

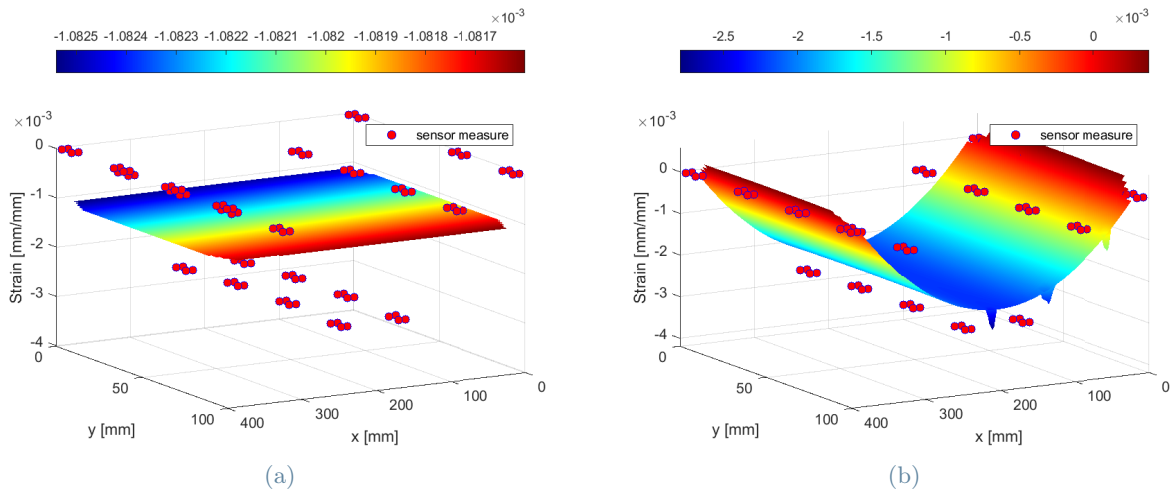


Figure 30: Strain polynomial fitting for 3PB model of (ε_{xx}) [mm/mm] on the top side of the panel: a) Bi-Linear fitting, b) Quadratic-Linear fitting.

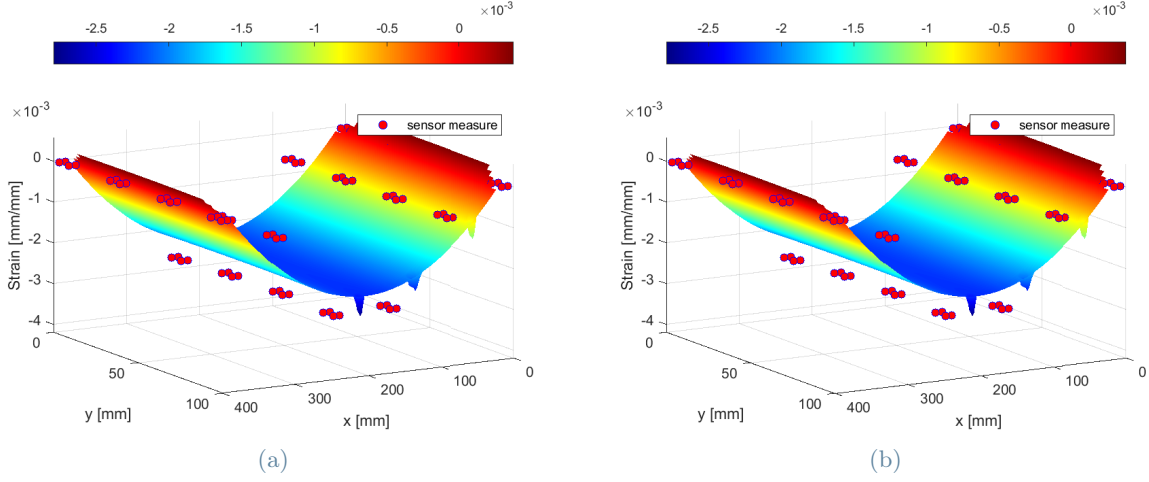


Figure 31: Strain polynomial fitting for 3PB model of (ε_{xx}) [mm/mm] on the top side of the panel: a) Bi-Quadratic fitting, b) Bi-cubic fitting.

Each pre-extrapolation level was repeated by varying the weight associated with the extrapolated measurements. This served as an initial step towards a more comprehensive analysis of the weighting strategy. Theoretically, these weights should be assigned proportionally to the quality (or confidence) of the pre-extrapolated data, thereby enabling the algorithm to weigh these inputs appropriately during the minimization of the error functional.

It is observed that, regardless of the assigned weight value (see Fig. 32 and Fig. 34), the trend across the various interpolation orders remains consistent. Specifically, the case without pre-extrapolation exhibits errors (both maximum around 80-85% and mean around 40-45%) that are double those of the worst pre-extrapolated case, i.e., the linear one (U_{err}^{max} approx. 40-45%, U_{err}^{mean} approx. 20-25%). Both quadratic and cubic cases record significantly lower errors (U_{err}^{max} approx. 15%, U_{err}^{mean} approx. 5%), suggesting that pre-extrapolation, even when not perfectly precise, leads to a substantial reduction in errors. This is due to the fact that the alternating constant/linear trend of strain components is better captured by a quadratic/cubic curve compared to a global linear trend over the entire plate.

The lack of significant difference between the quadratic and cubic orders (see Fig. 31) could be attributed to the fact that, in the direction of the principal component x , there are six measurement points. Since this number exceeds the points required for exact quadratic (3 points) and cubic (4 points) interpolation, the algorithm performs a least-squares interpolation, thereby smoothing out the differences between the two orders of variable monotonicity.

Having established the influence of interpolation orders, observing Fig. 33 and Fig. 35, it is noted that regardless of the interpolation order, identical maximum and mean errors are recorded as weights vary in the range 10^{-5} - 10^{-1} . This suggests that, although pre-extrapolations reduce errors, those performed in this manner do not yield high-quality strain fields, and the algorithm does not improve performance by assigning increasing weights.

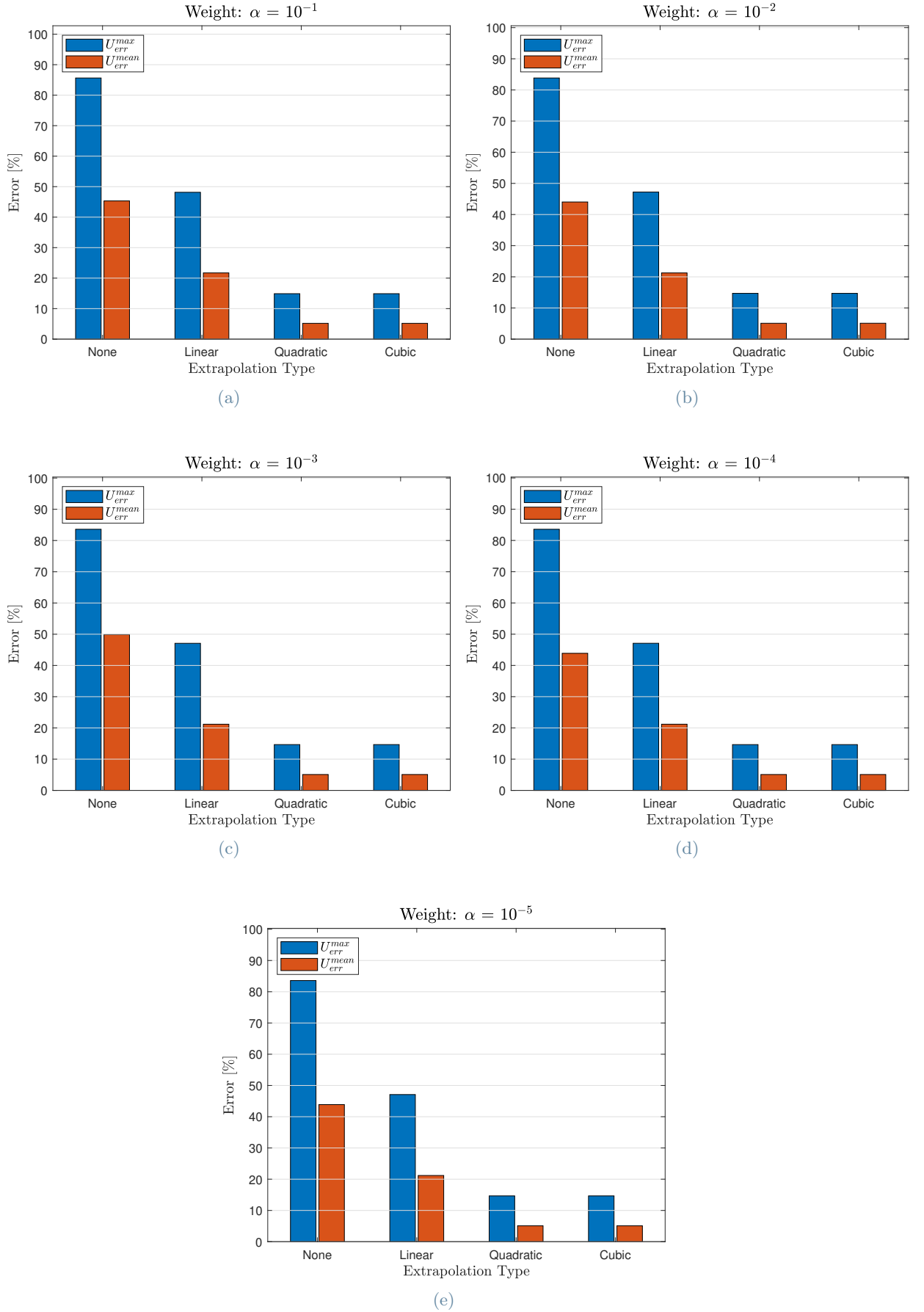


Figure 32: U_{err}^{max} , U_{err}^{mean} vs interpolation order, test repeated for α equal to a) 10^{-1} b) 10^{-2} , c) 10^{-3} , d) 10^{-4} , e) 10^{-5} , in LC configuration of the 3PB test.

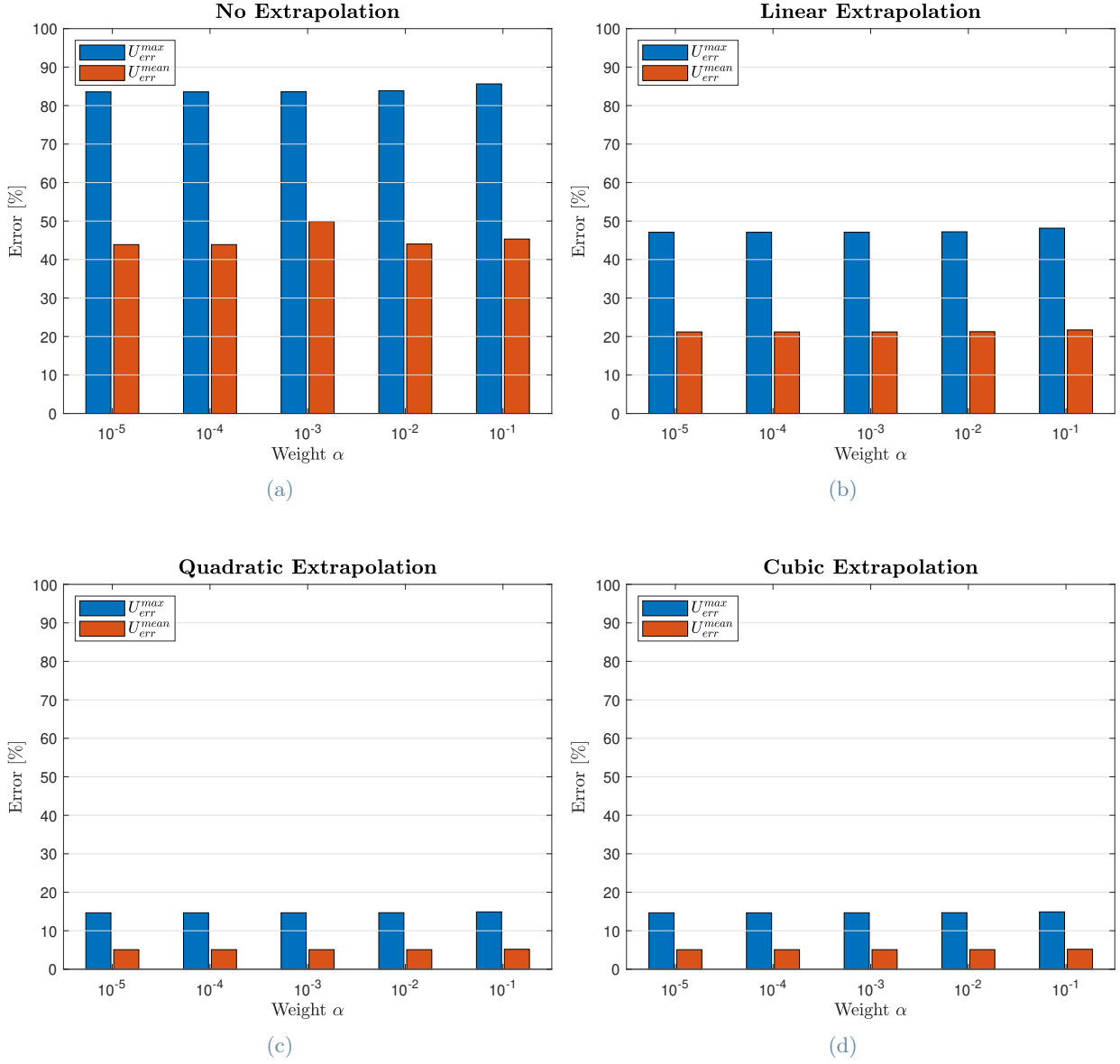


Figure 33: U_{err}^{max} , U_{err}^{mean} vs weight α , results collected for: a) no extrapolation active, b) linear extrapolation, c) quadratic extrapolation, d) cubic extrapolation, in LC configuration of the 3PB test.

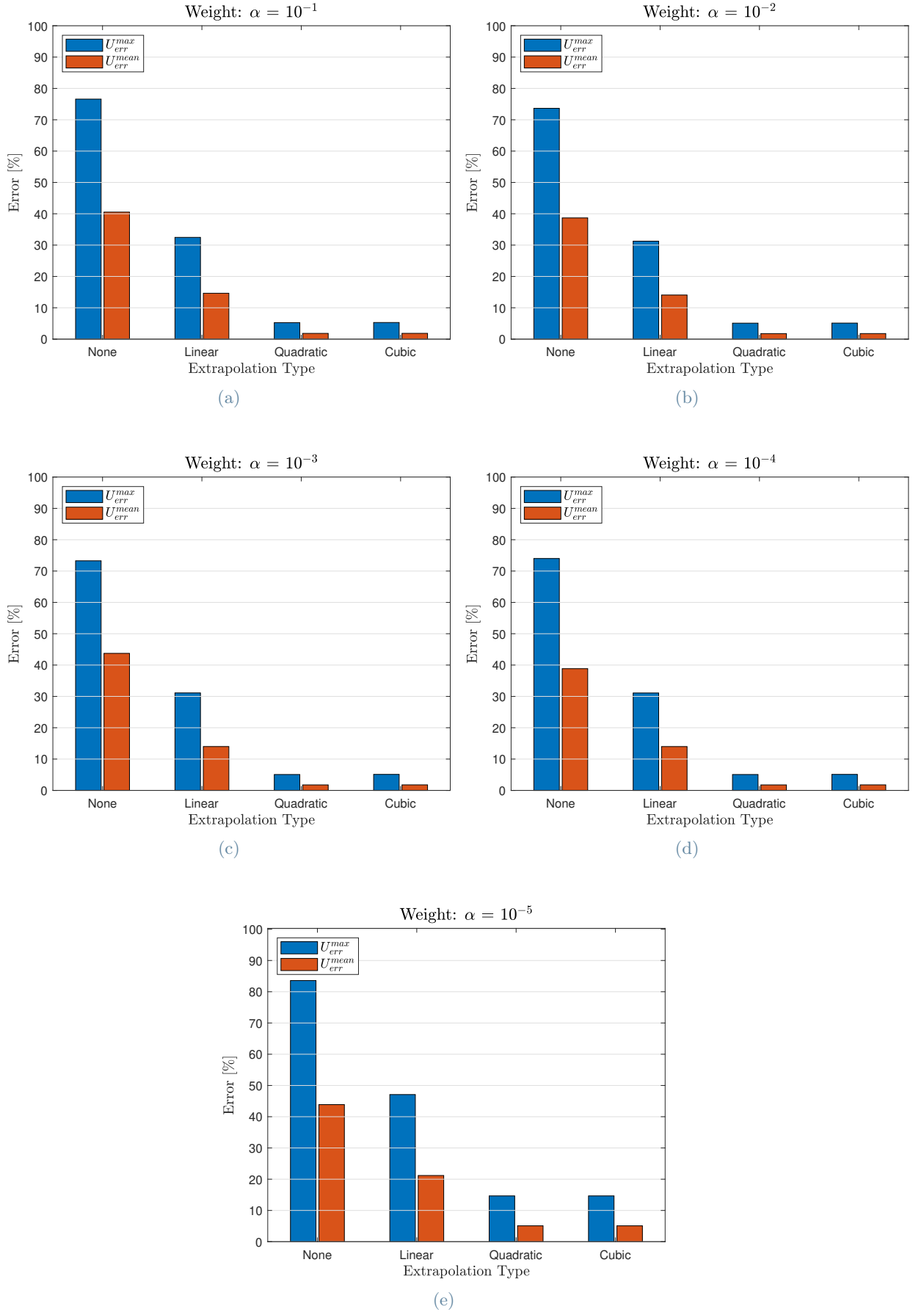


Figure 34: U_{err}^{max} , U_{err}^{mean} vs interpolation order, test repeated for α equal to a) 10^{-1} b) 10^{-2} , c) 10^{-3} , d) 10^{-4} , e) 10^{-5} , in DC configuration of the 3PB test.

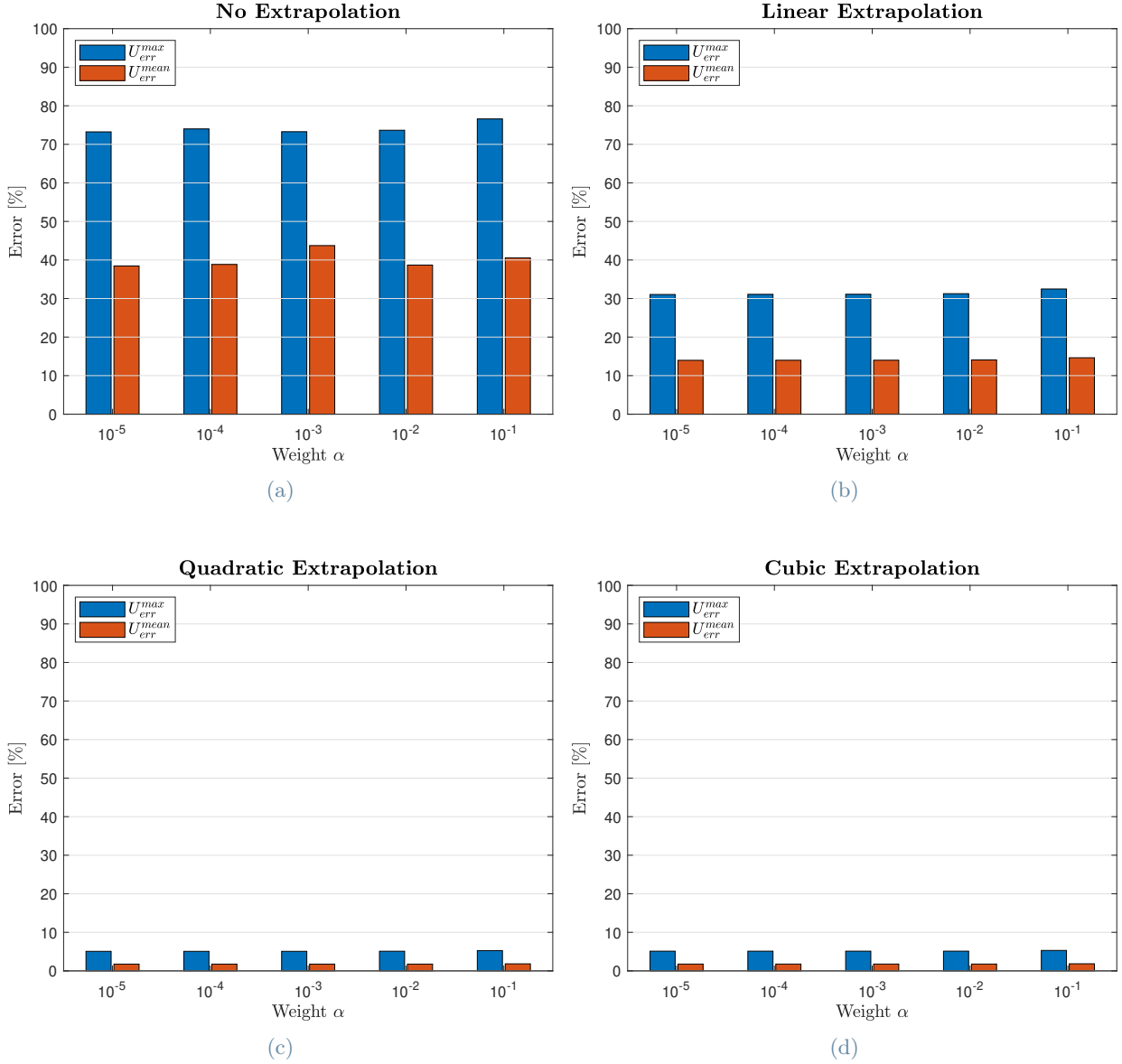


Figure 35: U_{err}^{max} , U_{err}^{mean} vs weight α , results collected for: a) no extrapolation active, b) linear extrapolation, c) quadratic extrapolation, d) cubic extrapolation, in DC configuration of the 3PB test.

5.2. Definition of the Final Sensor Network Topology

The culmination of the numerical sensitivity analyses, combined with the stringent physical requirements of the experimental setup, has led to the definition of the final SN topology. While the preliminary numerical studies highlighted the theoretical ideal of a high-density, multi-component sensor grid, the transition to a physical implementation necessitates the satisfaction of several conflicting constraints.

Based on the strain component analysis (section 5.1.3), the 'MIX' case was selected as the optimal measurement strategy. This approach aligns the fiber segments to acquire the longitudinal

strain (ε_{xx}) in the horizontal rows, where the bending moment induces the dominant deformation, and the transverse strain (ε_{yy}) in the vertical connection loops, effectively capturing Poisson's effects without requiring complex rosettes.

The considerations discussed during convergence analysis in section 5.1.1 lead to favoring an inverse mesh that is as fine as possible; however, this aspect must be balanced against a constraint of the optical fiber, which imposes a minimum distance of 15 mm between consecutive measurement points. This limitation, combined with the plate dimensions and the positions of the rollers (and consequently the boundary conditions), led to the selection of an inverse element size of 20 mm, ensuring regular square elements of identical dimensions and aspect ratio equal to one.

The same considerations also lead to the conclusion that a higher number of measurements improves performance. This was weighed against another technical limitation of the fiber, which allows for a maximum of 30 measurement points per cable. This latter aspect also falls within the performance-cost trade-off, as both the cost of the measurement instrument and its performance depend on the quantity of measurements to be acquired.

Although from a numerical standpoint the boundary conditions are applied along one-dimensional lines, in reality, the rollers are rigid three-dimensional bodies that occupy a non-negligible surface area when in contact with the plate. Therefore, this must be taken into account when defining the fiber path. Specifically, the external rollers span the full width of the plate, preventing the fiber from crossing over them to avoid crushing. Conversely, the central pressing roller has a reduced length, allowing the fiber to pass alongside it, resulting in a clearance between the fiber and the ends of the pressing cylinder.

Due to the inherent definition of the algorithm (section 3.2), the actual input required is the strain value on the neutral plane (see Eq. 8); this implies that if an inverse element is equipped with a measurement, a dual measurement on both sides of the plate is required. This constitutes another design constraint followed during the layout of the SN.

Since the strain field is symmetric with respect to the mid-span ($x = 200$), a symmetric distribution is not strictly necessary from the sole perspective of the algorithm, but it becomes fundamental for providing the correct data distribution for pre-extrapolations. Still from the perspective of pre-extrapolations, before determining the specific coupled orders (more thoroughly addressed in section 5.3.1), the aim is to guarantee a sufficient number of measurement points for each direction and variable, at least equal to the number of coefficients required to develop the interpolating polynomial.

A final design constraint for the SN arises from the maximum allowable curvature of the fiber along its path, which must maintain a bending radius of no less than 10 mm.

All things considered, three distinct categories of design constraints emerge, which can be summarized as follows:

- Physical constraints
- Constraints imposed by testing conditions
- Constraints related to the measurement instrumentation

The resulting configuration that satisfies these imposed constraints is illustrated in Fig. 36 for both the top and bottom surfaces. The performance of this SN is evaluated and reported in the following chapter.

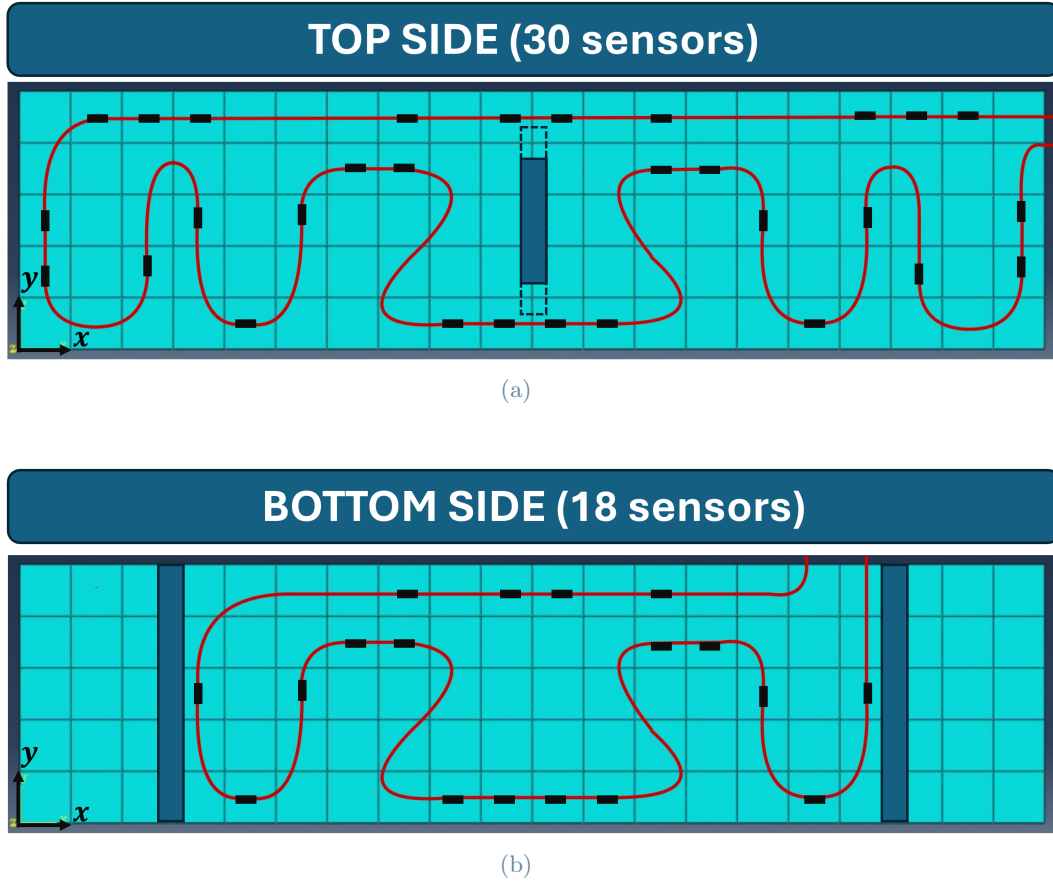


Figure 36: Sensor network design: fiber path (red line), sensor placement (black rectangles), test machines component placement (blue rectangles). In a) the dashed line represents the size of a second possible pressure roller available for performing the 3PB test.

To translate the theoretical SN design into a physical specification for manufacturing, a precise calculation of the curvilinear abscissa along the optical fiber is required. The manufacturer requires the exact distance between the centers of consecutive FBGs to inscribe them at the correct locations. Furthermore, the iFEM algorithm relies on the precise global coordinates (X, Y) of each sensing point.

Consequently, the continuous path of the optical fiber across the specimen surface has been discretized and schematized into a sequence of geometric primitives, therefore the path is modeled exclusively using linear segments and circular arcs.

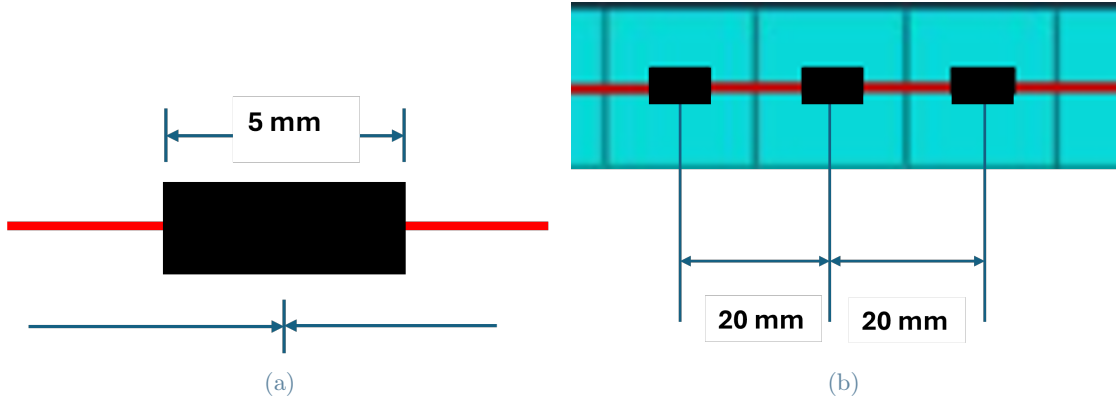


Figure 37: Schematic representation of FBG sensors: (a) physical dimensions of a single grating element with a characteristic length of 5 mm; (b) array configuration showing multiple sensors multiplexed along a single optical fiber with a center-to-center spatial separation of 20 mm.

Each FBG sensor is characterized by a grating length of 5 mm. For the purpose of coordinate definition and distance calculation, the measurement point is assumed to be located exactly at the geometric center of the grating (see Fig. 37a).

In regions where the fiber runs parallel to the principal axes of the mesh, the path is modeled as a linear segment (see Fig. 37b). In these sections, the optical fiber length ΔL_{linear} between two consecutive sensors corresponds directly to the Euclidean distance between the centroids of the instrumented inverse elements. As shown in the example in Fig. 37b, for a regular mesh size of 20×20 mm, the distance between consecutive sensors on a straight line is exactly 20 mm.

The routing of the fiber requires changes in direction to cover the desired area of the plate. These turns are critical due to the physical fragility of the optical fiber. As established in the design constraints, the bending radius ρ must not fall below a critical threshold to avoid signal attenuation or structural failure of the glass core.

Therefore, all turns are modeled as circular arcs with a constant radius of curvature ρ . Based on the technical specifications of the fiber and the experimental constraints, this radius is set to the minimum allowable value: $\rho = \rho_{min} = 10$ mm, to accommodate the requirement for rapid changes of direction and path reversals within the constrained footprint of the plate.

Figure Fig. 38a highlights this constraint. By fixing $\rho = 10$ mm, the geometry of the curves fits perfectly within the 20×20 mm grid of the inverse mesh, allowing for 90° turns to be executed within a single element quadrant or across element boundaries without violating spatial constraints.

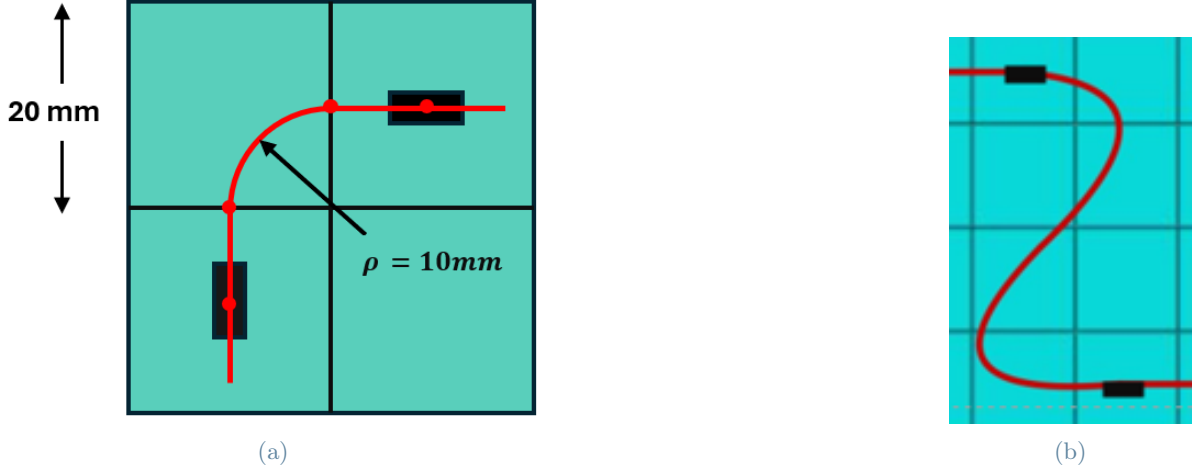


Figure 38: (a) Schematic representation of a standard 90° bend characterized by a constant radius of curvature $\rho = 10\text{ mm}$. (b) Experimental trace of a fiber segment featuring a non-circular path, representing the portion of the routing where the geometry deviates from simplified circumferential arc modeling.

The calculation of the fiber length consumed by these curves is derived from the arc length formula. As depicted in Fig. 39, two primary turning cases are identified:

1. **90° Turn:** Used to change direction from horizontal to vertical (or vice-versa). The length L_{90} is calculated as a quarter of a circle's circumference (Fig. 39a):

$$L_{90} = \frac{2\pi R}{4} = \frac{2 \cdot \pi \cdot 10}{4} \approx 15.71\text{ mm} \quad (58)$$

2. **180° Turn (U-turn):** Used to reverse direction between parallel paths. This is modeled as two consecutive 90° arcs, separated by a liner segment across element's centroid, since the fiber needs to be straight when passing through it.

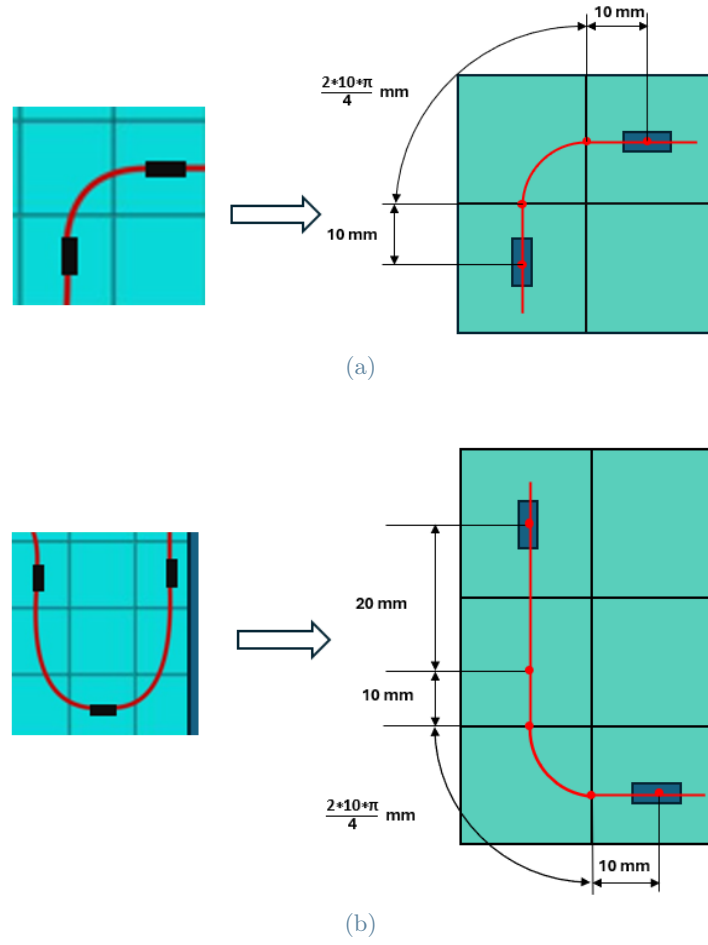


Figure 39: Geometric modeling and parameterization of fiber optic path segments. (a) Schematization of a 90° bend, where the transition is modeled as a circular arc with a radius of 10 mm connected to linear tangential segments. (b) Modeling of a 180° U-turn, highlighting the decomposition of the complex routing into a combination of a straight segment (20 mm) and a standard 90° curved arc.

Fig. 39 also demonstrates how the position of the sensors relative to the curve is determined. For a sensor located after a curve, its distance from the previous reference point is the sum of the linear approach, the arc length, and the linear departure.

While the majority of the SN can be accurately described using the linear and circular primitives described above, specific transition zones required a different approach. Specifically, the "S-shaped" loops used to connect distant sections of the network or to manage excess fiber length (as shown in Fig. 38b) exhibit a complex geometry that is difficult to schematize analytically with sufficient precision.

For these specific segments, an experimental approach was adopted. A dummy replica of the specimen was manufactured and instrumented with a wire to physically simulate the actual optical fiber routing. Subsequently, the cable was removed, and the segment connecting the two sensors was measured, yielding a value of approximately 115 mm. This empirically determined value was subsequently used in the total length calculation for the manufacturing order.

To verify the analytical predictions, the dummy specimen was fully sensorized to measure its actual total length. The experimental measurement was found to be in agreement with the theoretical value, with an absolute error of less than 2 mm over a total length of more than 2 m.

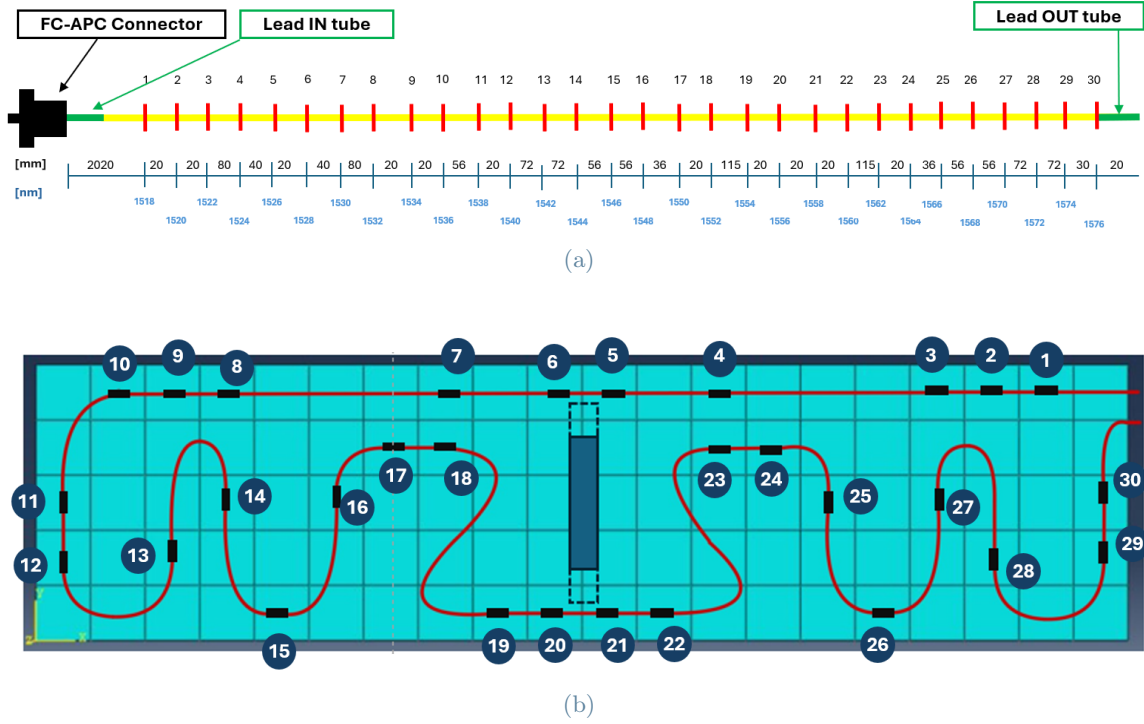


Figure 40: Schematic of the fiber optic cable system for "top" side: (a) longitudinal view showing the FC-APC connector, lead-in and lead-out tubes with 18 numbered sections and corresponding distances in millimeters and relative wavelength lengths in nanometers, (b) top view illustrating the cable path through the 18 positions.

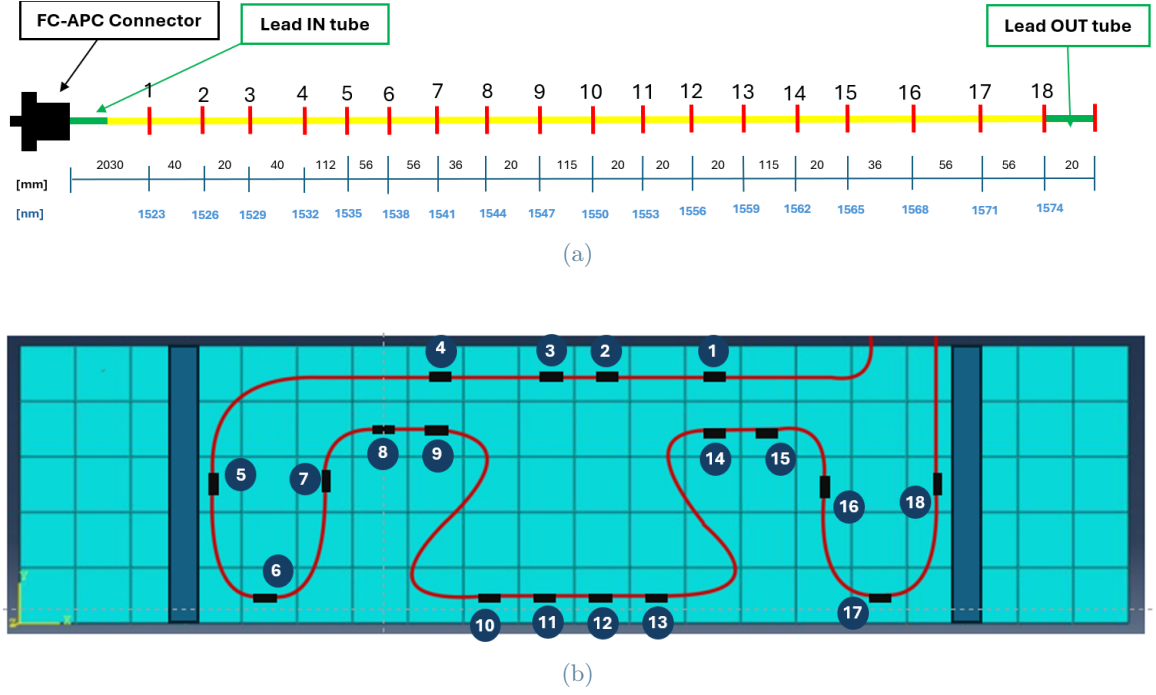


Figure 41: Schematic of the fiber optic cable system for "bottom" side: (a) longitudinal view showing the FC-APC connector, lead-in and lead-out tubes with 18 numbered sections and corresponding distances in millimeters and relative wavelength lengths in nanometers, (b) top view illustrating the cable path through the 18 positions.

Once the final Sensor Network (SN) has been defined, the performance of the baseline algorithm can be evaluated without further modifications on the framework itself. This configuration serves as a benchmark to assess the extent to which subsequent calibrations improve the reconstruction accuracy relative to this initial case. Figure Fig. 42 and Figure Fig. 43 present the error metrics for both models under the two loading conditions.

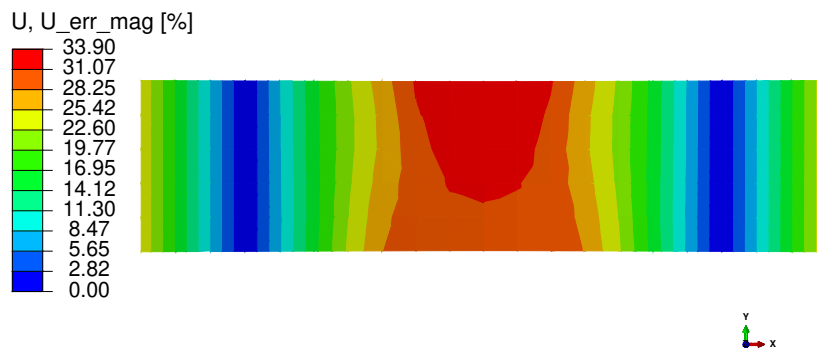
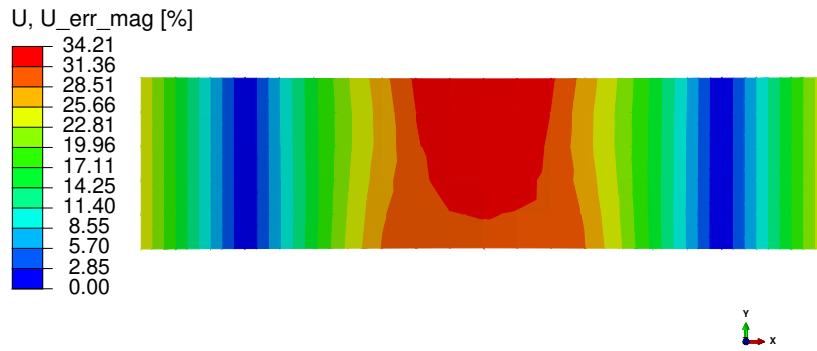


Figure 42: Contour plots of U_{err} displacement: (a) iFEM reconstruction, (b) percentage error U_{err} of the iFEM reconstruction.

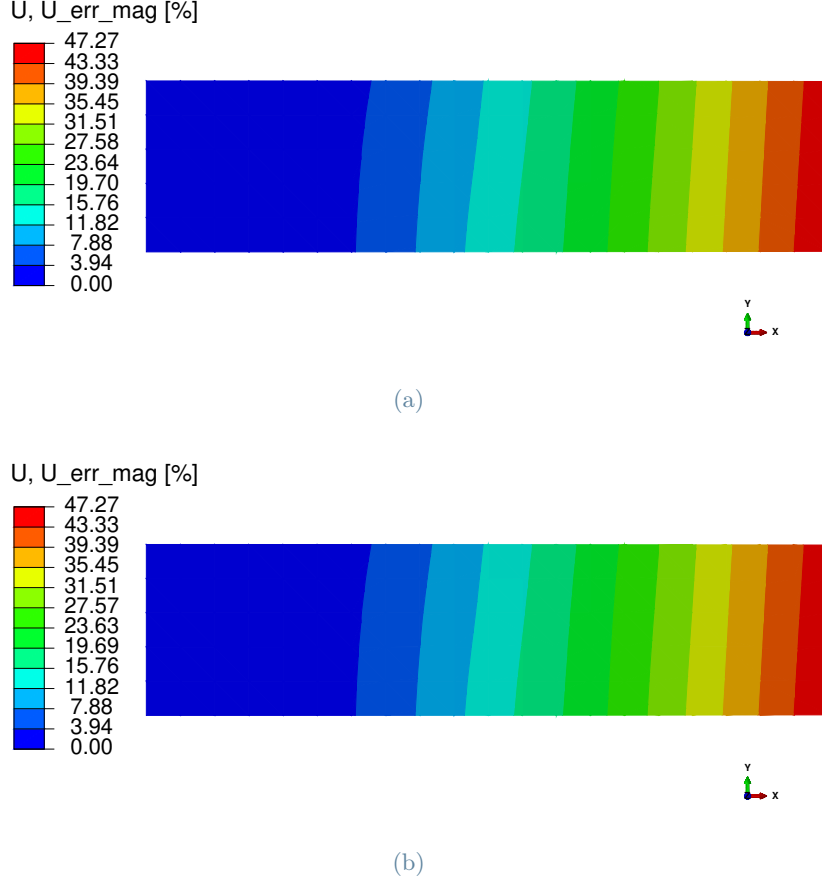


Figure 43: Contour plots of U_{err} displacement: (a) iFEM reconstruction, (b) percentage error U_{err} of the iFEM reconstruction.

A preliminary analysis of the contour plots reveals a difference in the highest error of the percentage error, U_{err} , depending on the nature of the applied load. The error remains bounded around 34% (3PB, Fig. 42) and 47% (CP, Fig. 43).

In particular, since the solution is enforced as a boundary condition, the error vanishes at the support rollers and the clamp. Furthermore, even a minor discrepancy between the optimal and target strain fields leads to an integral propagation of the error on the displacement field as the distance from the constraints increases. This behavior can be observed in both models, suggesting a common trend for bending strain fields. In the CP model, the distance between the constraint and the load application point is twice that of the 3PB model, which may have contributed to the substantial difference in error magnitude between the two configurations.

5.3. Parametric Sensitivity Analysis and Robustness Assessment

Following the definition of the optimal SN topology, a comprehensive campaign of parametric sensitivity analyses was conducted. The primary objective of this phase was to fine-tune the algorithm's hyperparameters and to assess its robustness against experimental uncertainties.

To ensure the general validity of the results, these analyses were performed across the complete set of boundary conditions defined in the experimental campaign. Specifically, the optimization was carried out for both the 3PB and the CP configurations, considering both LC and DC scenarios. In all cases, the input strain data were extracted from the numerical model using the final SN layout established in the previous section. The numerical source of strain measurements for the CP model is shown under DC (Fig. 44) and LC (Fig. 45).

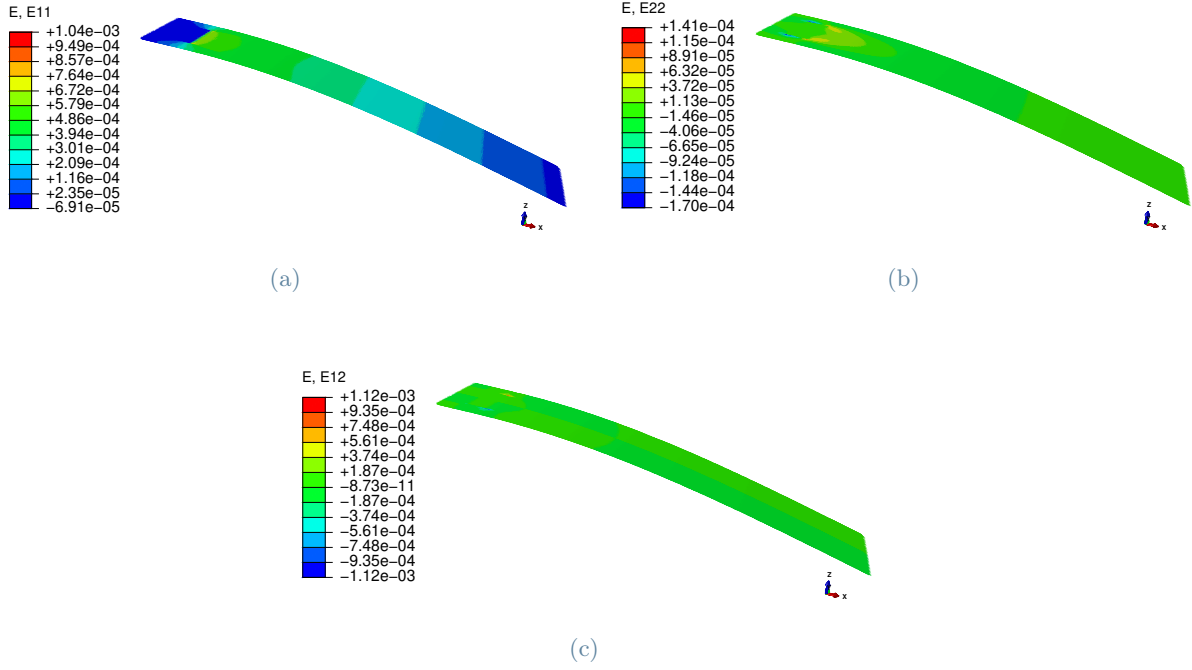


Figure 44: Strain field resulting from the bottom surface of the baseline CP direct FEM model used for iFEM input under DC: a) strain field ε_{xx} [mm/mm], b) strain field ε_{yy} [mm/mm], c) strain field ε_{xy} [mm/mm].

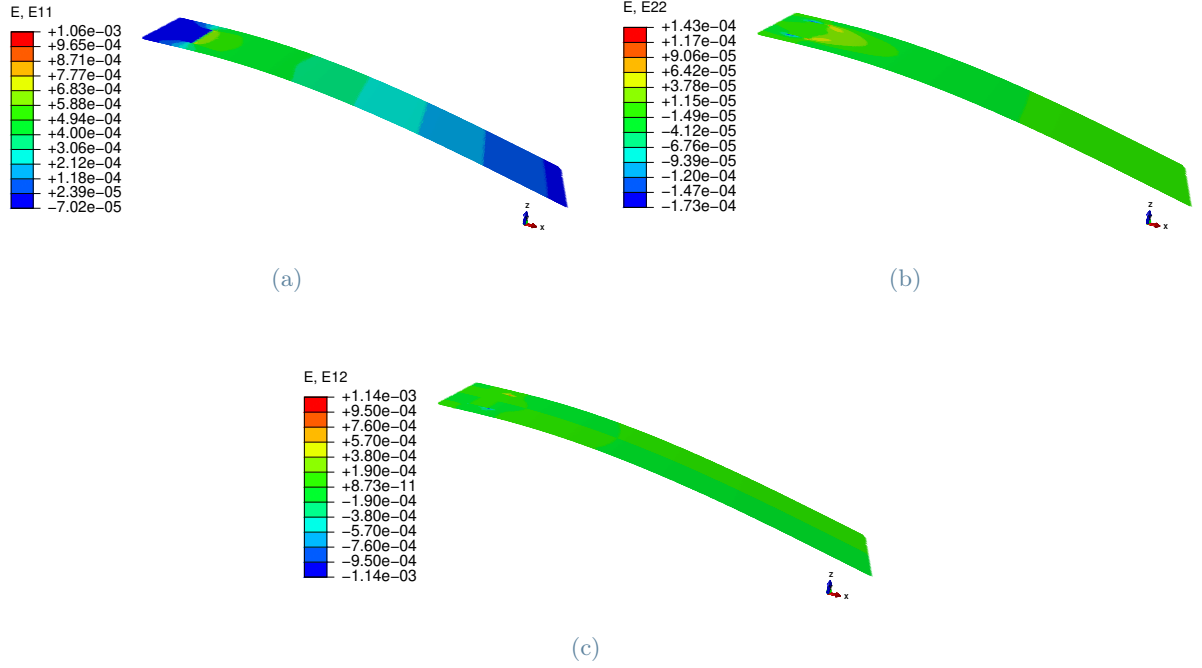


Figure 45: Strain field resulting from the bottom surface of the baseline CP direct FEM model used for iFEM input under LC: a) strain field ε_{xx} [mm/mm], b) strain field ε_{yy} [mm/mm], c) strain field ε_{xy} [mm/mm].

The investigation is structured into three distinct studies, corresponding to the critical parameters governing the reconstruction accuracy.

5.3.1. Optimization of the Data-Driven Interpolation Order

In the context of the Data-Driven pre-extrapolation strategy, the choice of the polynomial degree significantly influences the fidelity of the strain field approximation.

A more refined analysis, extending the work presented in section 5.1.3, moves beyond the assumption of globally homogeneous interpolation orders. i.e. same polynomial degree in both principal direction x and y . Instead, specific combinations of polynomial degrees are investigated to better adapt to the inherent strain distribution characteristics of each specific case. This study aims to identify the polynomial order that best captures the strain gradients of the composite plate while avoiding overfitting phenomena that could introduce spurious oscillations in the reconstructed displacement field. To this end, the strain distributions for both the top and bottom surfaces of the plate are analyzed across the various boundary and loading configurations.

The observed strain trends, combined with the general interpolation considerations discussed previously, allow for the exclusion of certain polynomial orders due to either insufficient interpolation capacity or poor representativeness of the physical phenomenon.

Regarding the 3PB model, for both LC and DC configurations:

Table 4: Reconstruction errors grouped by the same interpolation order class for ε_{yy} (3PB model in DC configuration).

Polynomial Order		$\mathbf{U}_{\text{err}}^{\text{max}} [\%]$		$\mathbf{U}_{\text{err}}^{\text{mean}} [\%]$	
X order	Y order	μ	σ	μ	σ
3	0	4.00	1.65	1.75	0.85
2	0	4.00	1.70	1.76	0.85
3	1	4.00	1.71	1.78	0.86
2	1	4.01	1.71	1.76	0.85

- For the longitudinal strain ε_{xx} along the x -direction, the distribution exhibits a bi-linear behavior, flanked by two constant zero-strain regions corresponding to the unloaded overhangs of the plate. Consequently, this profile cannot be accurately represented by constant or linear interpolation orders (which are therefore discarded), requiring instead quadratic or cubic polynomials.
- Along the y -direction, the trends are sufficiently constant to suggest constant, linear, or quadratic interpolation orders. Conversely, a cubic order would risk inducing overfitting in this direction. However, the implementation of a cubic order is practically precluded by the available data, which are provided at only three distinct y -coordinates; this constraint prevents the definition of a third-order polynomial, which would strictly require four data points.

Regarding the transverse strain ε_{yy} , the trends exhibit the same qualitative nature as ε_{xx} , scaled by the Poisson's ratio factor ($-\nu$). However, for this component in the y -direction, the quadratic interpolation order is discarded due to insufficient data, as the valid measurement points are distributed across only two distinct y -coordinates. Based on these premises, a set of combinations involving the interpolation order pairs for the x and y directions was generated for both the ε_{xx} and ε_{yy} strain components. For each distinct combination, the reconstruction algorithm was executed, and the performance metrics were collected.

Specifically, each fixed interpolation order selected for ε_{xx} was systematically combined with the entire set of admissible orders for ε_{yy} . To rigorously isolate the impact of specific interpolation classes of the single strain component ($\varepsilon_{xx}/\varepsilon_{yy}$), the mean values and standard deviations of both error metrics were computed by aggregating all simulations that shared a common interpolation order for one variable, while spanning all valid combinations for the other. This statistical approach facilitated a more concrete and robust assessment of the influence exerted by the different interpolation strategies.

The selection of polynomial orders for the CP model case follows an analogous reasoning:

- Starting from the polynomial order selection for the reconstruction of ε_{xx} , the longitudinal direction x is characterized by a bending moment that varies linearly from the clamped root to the free tip. Consequently, the strain distribution along this axis exhibits a linear

Table 5: Reconstruction errors grouped by the same interpolation order class for ε_{xx} (3PB model in DC configuration).

Polynomial Order		$\mathbf{U}_{\text{err}}^{\text{max}} [\%]$		$\mathbf{U}_{\text{err}}^{\text{mean}} [\%]$	
X order	Y order	μ	σ	μ	σ
3	0	4.16	0.10	2.53	0.03
3	1	3.39	0.01	1.26	0.03
3	2	6.75	0.02	1.18	0.01
2	2	1.45	0.01	0.56	<0.01
2	1	4.13	0.01	2.50	<0.01
2	0	4.13	0.01	2.51	0.004

Table 6: Reconstruction errors grouped by the same interpolation order class for ε_{yy} (3PB model in LC configuration).

Polynomial Order		$\mathbf{U}_{\text{err}}^{\text{max}} [\%]$		$\mathbf{U}_{\text{err}}^{\text{mean}} [\%]$	
X order	Y order	μ	σ	μ	σ
3	0	4.30	1.54	2.05	1.01
2	0	4.30	1.54	2.05	1.01
3	1	4.29	1.55	2.05	1.01
2	1	4.30	1.55	2.21	1.24

Table 7: Reconstruction errors grouped by the same interpolation order class for ε_{xx} (3PB model in LC configuration).

Polynomial Order		$\mathbf{U}_{\text{err}}^{\text{max}} [\%]$		$\mathbf{U}_{\text{err}}^{\text{mean}} [\%]$	
X order	Y order	μ	σ	μ	σ
3	0	4.62	0.01	2.94	<0.01
3	1	3.73	0.01	1.61	<0.01
3	2	6.47	0.01	1.11	0.01
2	2	1.73	0.01	0.76	<0.01
2	1	4.61	0.01	2.93	<0.01
2	0	4.62	0.01	3.19	0.50

Table 8: Reconstruction errors grouped by the same interpolation order class for ε_{yy} (CP model in DC configuration).

Polynomial Order		$\mathbf{U}_{\text{err}}^{\text{max}} [\%]$		$\mathbf{U}_{\text{err}}^{\text{mean}} [\%]$	
X order	Y order	μ	σ	μ	σ
1	0	7.88	1.53	3.35	0.60
1	1	7.90	1.51	3.36	0.60
2	0	7.90	1.50	3.35	0.60
2	1	7.90	1.52	3.36	0.60

Table 9: Reconstruction errors grouped by the same interpolation order class for ε_{xx} (CP model in DC configuration).

Polynomial Order		$\mathbf{U}_{\text{err}}^{\text{max}} [\%]$		$\mathbf{U}_{\text{err}}^{\text{mean}} [\%]$	
X order	Y order	μ	σ	μ	σ
1	0	9.15	0.01	3.86	<0.01
1	2	5.05	0.04	2.27	<0.01
1	1	8.22	0.01	3.56	<0.01
2	0	9.21	0.01	3.92	<0.01
2	1	7.85	0.01	3.26	<0.01
2	2	7.91	0.01	3.28	<0.01

trend. To capture this physical behavior accurately while minimizing computational cost, a linear or at most quadratic interpolation order is selected for the x -direction.

- Regarding the transverse direction y , the strain distribution across the width of the plate is assumed to be almost uniform, consistent with standard beam theory assumptions for the primary bending mode. Therefore, as for 3PB, constant, linear and quadratic are allowed while cubic is not included because of lack of input points.

For the ε_{yy} strain component, the deformation is primarily governed by the Poisson effect arising again from the longitudinal strain, so it follows the same spatial distribution as the longitudinal strain. Thus, the same interpolation orders are adopted, except for quadratic order in y direction for the same reason as before.

Table 10: Reconstruction errors grouped by the same interpolation order class for ε_{yy} (CP model in LC configuration).

Polynomial Order		$\mathbf{U}_{\text{err}}^{\text{max}} [\%]$		$\mathbf{U}_{\text{err}}^{\text{mean}} [\%]$	
X order	Y order	μ	σ	μ	σ
1	0	7.88	1.53	3.35	0.60
1	1	7.90	1.52	3.36	0.60
2	0	7.89	1.53	3.35	0.60
2	1	7.91	1.52	3.36	0.60

Table 11: Reconstruction errors grouped by the same interpolation order class for ε_{xx} (CP model in LC configuration).

Polynomial Order		$U_{\text{err}}^{\text{max}} [\%]$		$U_{\text{err}}^{\text{mean}} [\%]$	
X order	Y order	μ	σ	μ	σ
1	0	9.15	0.01	3.86	<0.01
1	2	5.03	0.02	2.27	<0.01
1	1	8.23	0.02	3.57	<0.01
2	0	9.21	0.01	3.91	<0.01
2	1	7.84	0.01	3.26	<0.01
2	2	7.91	0.01	3.28	<0.01

As can be observed in tables 4, 6, 8 and 10, when analyzing the performance of the interpolation order pairs regarding the ε_{yy} variable, the mean values of the metrics remain consistent. Moreover, they exhibit a high standard deviation, suggesting that the influence of the interpolation order on this specific variable is of limited relevance. Conversely, for the ε_{xx} variable, distinct changes are observed depending on the interpolation class, accompanied by very low standard deviations suggesting low variability inside the same class. This indicates that once the interpolation order for ε_{xx} is established, variations in the ε_{yy} order yield minimal changes in the results. On average, lower reconstruction errors are recorded for the 3PB model compared to the CP model, whereas no significant differences are observed regarding the type of applied load. Based on these analyses, the optimal interpolation class for each model and loading case is identified as follows:

- **3PB model:** orders (2, 2) for ε_{xx} and (3, 1) for ε_{yy} ; these are, respectively, the top performers both as individual pairs and as a general class (across both load cases).
- **CP model:** orders (1, 2) for ε_{xx} and (1, 0) for ε_{yy} ; these are, respectively, the top performers both as individual pairs and as a general class (across both load cases).

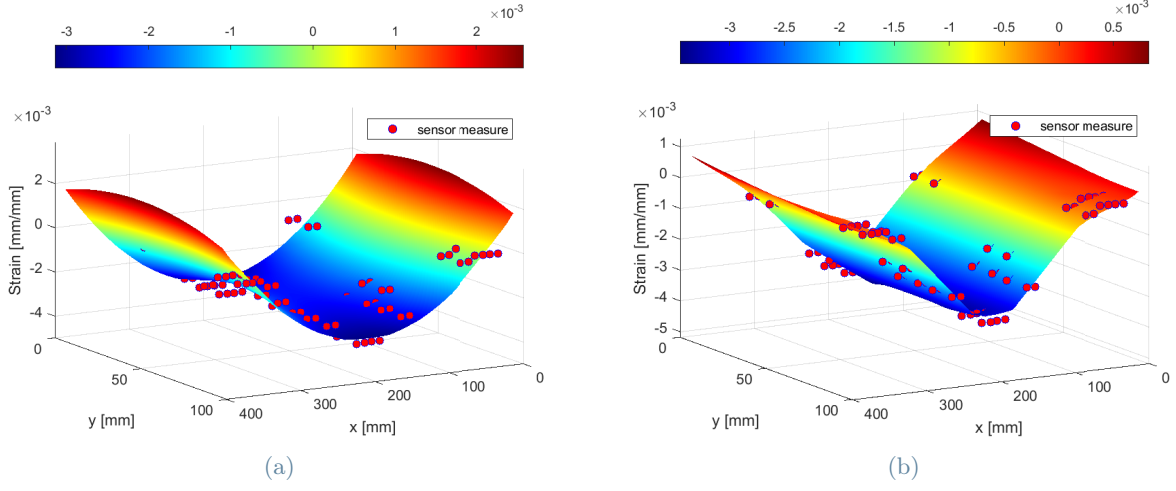


Figure 46: Strain polynomial fitting for 3PB model of (ϵ_{xx}) [mm/mm] on the top side of the panel: a) Polynomial fitting (Bi-quadratic), b) Physics-Based fitting

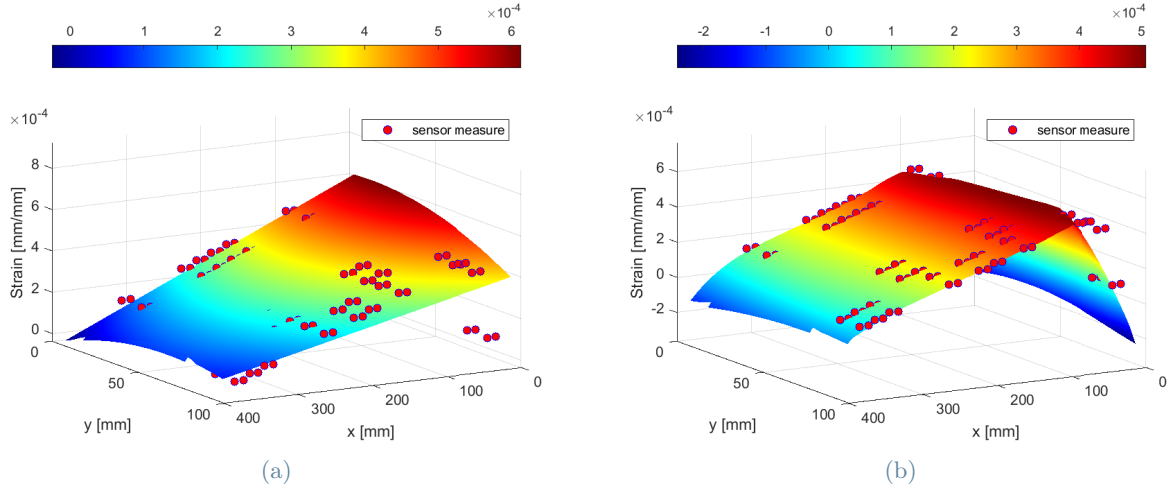


Figure 47: Strain polynomial fitting for CP model of (ϵ_{xx}) [mm/mm] on the top side of the panel: a) Polynomial fitting (Constant-Linear), b) Physics-Based fitting

The complete results for each combination are omitted for simplicity and are reported in full in appendix A.

The best performances achieved via polynomial pre-extrapolation are subsequently collected and compared with the performance obtained via physics-based pre-extrapolation.

Table 12: Comparison of reconstruction errors across different models and pre-extrapolation techniques ("Displ." and "Load" refer respectively to DC and LC configurations).

Model	Pre-extrapolation	$U_{\text{err}}^{\text{max}}[\%]$	$U_{\text{err}}^{\text{mean}}[\%]$
3PB Displ.	Polynomial	1.44	0.60
	Physics-based	1.31	0.50
3PB Load	Polynomial	1.72	0.76
	Physics-based	0.85	0.21
CP Displ.	Polynomial	5.01	2.27
	Physics-based	4.05	1.84
CP Load	Polynomial	5.01	2.27
	Physics-based	4.05	1.84

Although a more detailed performance analysis is presented in the next section, a preliminary assessment reveals that, for a fixed weighting factor ($\alpha = 10^{-4}$), the analysis of the error metrics highlights a consistent trend: the physics-based approach outperforms the polynomial fitting in all tested configurations. This superiority can be attributed to the underlying mathematical formulation of the two methods.

For the 3PB model, the structure behaves predominantly as a thin plate undergoing bending. As discussed in section 3.4.2.1, the physics-based method exploits the biharmonic nature of the transverse displacement field ($\nabla^4 w = 0$) in unloaded regions.

- In the LC 3PB model, the physics-based interpolation achieves a remarkable reduction in error, reducing the mean error from 0.76% (Polynomial) to 0.21% (Physics-based). This suggests that the biharmonic splines correctly capture the curvature gradients induced by the bending moment, whereas the polynomial regression tends to smooth out localized variations or introduce spurious oscillations between sensor points.
- In DC case, the improvement is less pronounced but still evident.

For the CP model, the specimen is subjected to pure bending with a clamped boundary condition at one end.

- The reconstruction errors are generally higher compared to the 3PB configurations (reaching up to 5% for the polynomial approach). This increase is primarily attributed to the challenge of reconstructing the steep strain gradients located near the fixed support (clamped root).
- Polynomial fitting is inherently purely mathematical and often suffers from extrapolation inaccuracies at the domain boundaries. In contrast, the physics-based interpolation reduces the maximum error by approximately 1% (from 5.01% to 4.05% in the displacement-controlled case).

- This improvement demonstrates that the biharmonic spline, by enforcing the minimization of the bending energy, provides a more mechanically consistent extrapolation towards the fixed constraint compared to standard polynomial regression.

While polynomial fitting provides a computationally efficient and purely Data-Driven solution, it lacks the physical insight required to accurately model regions with high strain gradients or complex boundary conditions. Conversely, the physics-based interpolation, by enforcing the minimization of the bending energy and adhering to the governing equations of the plate theory, ensures a more realistic strain field reconstruction.

5.3.2. Calibration of the Pre-Extrapolation Weight α

The analysis proposed in this section focuses on the calibration of the weighting factor α . This parameter plays a crucial role in the modified iFEM functional by defining the relative confidence assigned to the pre-extrapolated strain values compared to the direct sensor measurements. A range of orders of magnitude for α was investigated to identify the optimal trade-off. The goal is to allow the pre-extrapolated field to guide the reconstruction in unsensorized regions without overpowering the accurate local information provided by the physical sensors.

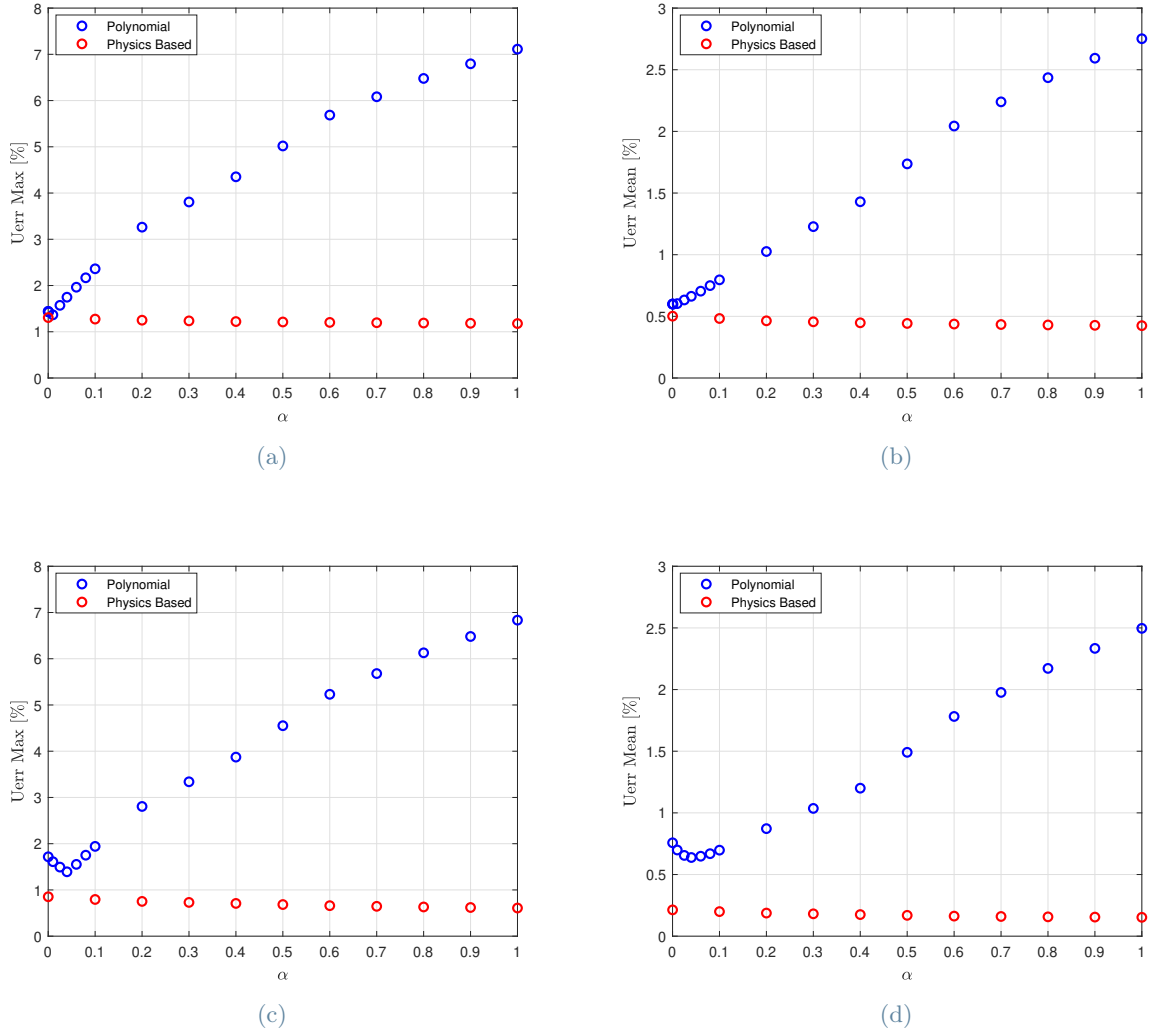


Figure 48: Calibration plots of the pre-extrapolation weights α : 3PB model in: a) and b) DC configuration (U_{err}^{max} and U_{err}^{mean}), c) and d) LC configuration (U_{err}^{max} and U_{err}^{mean}).

For the 3PB model under both load-imposed and displacement-imposed configurations, the results show definitive trends for both metrics and both pre-extrapolation types even at low weight values. Specifically, physics-based outperforms polynomial for all weight values (Fig. 48). Physics-based exhibits even at low values of weight an asymptotic trend, whereas polynomial

displays an increasing error trend with increasing pre-extrapolation weight, except in the region $\alpha \in (0, 0.1]$ (see Fig. 49).

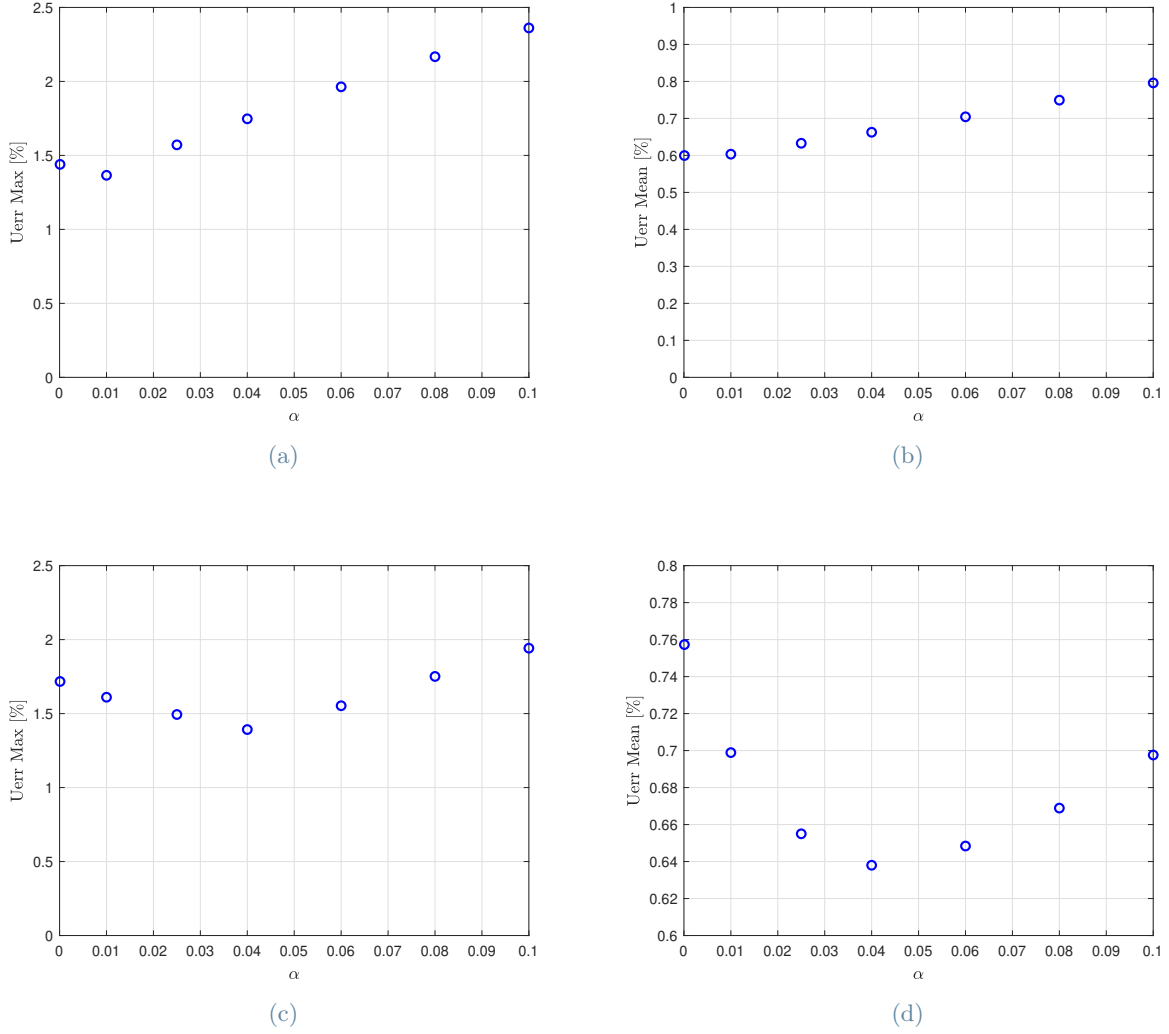


Figure 49: Enlarged view of Fig. 48 in the α range $(0, 0.1]$.

For the cantilever model, the trend is not showing early as for the 3PB. For $\alpha = 1$, neither method showed asymptotic behavior; thus, the analysis was extended to $\alpha = 10$ (Fig. 50). A plateau was then observed, with polynomial performing better than physics-based.

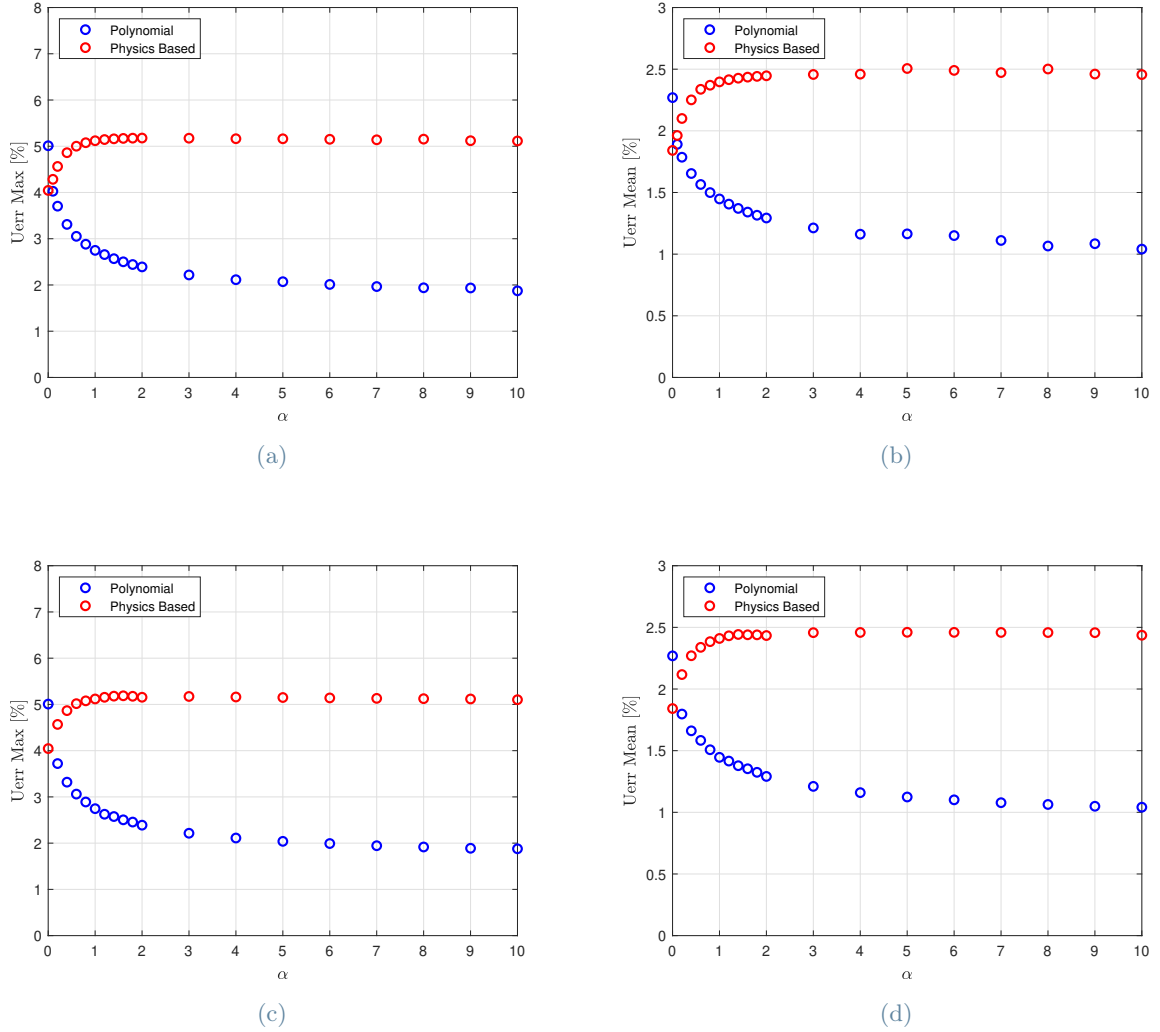


Figure 50: Calibration plots of the pre-extrapolation weights α : CP model in: a) and b) DC configuration, c) and d) LC configuration.

The sensitivity analysis of the weighting coefficient α reveals a distinct divergence in the optimal reconstruction strategy depending on the structural boundary conditions (3PB vs CP).

The most striking result is the inversion of the performance hierarchy between the Polynomial and Physics-Based methods across the two load cases.

- **Three-Point Bending:** In the 3PB configuration, the Physics-Based method consistently outperforms the Polynomial fit across the entire range of α . The Physics-Based error curve is asymptotic and stable even for low weights, indicating that the biharmonic spline provides an initial strain field guess that is inherently closer to the true physical deformation (which is dominated by smooth bending curvatures). Conversely, the Polynomial method exhibits a rising error trend as α increases. For this model, the polynomial approach is unable to capture the change in the strain trend at the support points, which exhibit cusps

or sharp changes. Even the best-performing class of interpolation results is quadratic, and therefore has a consistently negative concavity. This implies that, beyond the support points, it can never reverse the trend and will reconstruct negative strains in regions of zero strain, introducing a source of error that will then be propagated by iFEM. This also explains the continuous increase in error as the weight grows, as greater credit is given to a significantly erroneous set of strains. On the other hand, the physics-based method is perfectly interpolating, and is therefore capable of stopping and creating an interpolating surface that actually passes through all the provided points, yielding a high-quality basis on which the iFEM algorithm can then reconstruct the entire strain field. The plateau is reached quickly because the physics-based reconstruction is already very close to the exact strain field, so even for non-zero weights (e.g., 10^{-4} , 10^{-3}), the reconstruction error settles to a low level.

- **Cantilever Plate:** In the CP configuration, the trend reverses. The Polynomial method demonstrates superior performance, with the error decreasing monotonically as α increases, reaching a minimum at high weights ($\alpha = 10$). The Physics-Based method, while stable, plateaus at a higher error level. The cantilever case involves a complex boundary layer near the clamp ($x = 0$ to $x = 60$), where strain gradients are sharp and non-trivial, with a discontinuity of order 0 (instead of order 1 like 3PB model). The global biharmonic spline (Physics-Based) struggles to capture this local constraint effect accurately, creating a "smoothed" field that misses the clamp's rigidity. Moreover, the theoretical back ground of Physics-Based approach requires the application of concentrated forces, and differently from 3PB model, the clamp is realized as a region B.C. which partially violates the starting hypothesis. The Polynomial fit, of order 1 in direction x , inadvertently approximates this boundary behavior better by allowing a simpler, global curvature that aligns well with the cantilever's dominant mode shape.

A crucial observation in the Cantilever analysis is the interpretation of using weighting coefficients $\alpha \gg 1$ for the different pre-extrapolations. Mathematically, the iFEM functional defined by Eq. 22, where the experimental sensor weight is set to 1.

Assigning $\alpha > 1$ to the virtual (pre-extrapolated) sensors transforms them into quasi-hard constraints. It signals the solver that the fidelity to the pre-extrapolated surface is locally more important than the actual experimental/numerical data.

In the CP case, the Polynomial pre-extrapolation provides a kinematically consistent "guide" for the deformation. By setting $\alpha \rightarrow 10$, we effectively override the iFEM's tendency to "smooth out" the solution in the sparse regions, forcing the displacement field to lock onto this high-quality polynomial shape. The resulting plateau indicates that once the solver fully conforms to this shape, no further gain can be achieved; the error is bounded only by the intrinsic accuracy of the polynomial approximation itself.

The achievement of a plateau in the error curves signifies a saturation of information.

- **Plateau Reached:** The iFEM solution has converged to the shape dictated by the pre-extrapolation. Further increasing α does not change the result because the displacement field is already perfectly compliant with the virtual strain inputs.
- **No Plateau (Rising Trend):** In the case of Polynomial 3PB, the lack of a low-error plateau and the rising trend indicate a conflict. The pre-extrapolated surface contains geometric inaccuracies and as α increases, the solver is forced to adopt these inaccuracies, pulling the solution away from the true experimental optimum.

The analysis demonstrates that there is not a single best performing pre-extrapolation method. The Physics-Based approach is the robust choice for problems dominated by smooth, distributed bending (like 3PB). However, for problems with strong boundary constraints and global gradients (like the Cantilever), a well-chosen Polynomial model acts as a superior shape function. Crucially, in this latter case, treating the virtual data as high-confidence input ($\alpha > 1$) is a valid strategy to regularize the iFEM solution and minimize reconstruction error.

An analysis of the reconstruction results reveals a systematic, albeit slight, discrepancy between the two loading configurations simulated for the 3PB test. Specifically, the LC simulations consistently exhibit slightly higher reconstruction errors compared to the equivalent DC simulations.

While the macroscopic geometry of the deformation appears identical in both cases, this trend can be attributed to fundamental differences in how boundary conditions influence the local mechanics of the plate. The phenomenon is driven by the degree of freedom allowed to the structure, particularly regarding the anticlastic curvature.

The reasons for this discrepancy are threefold, ranging from kinematic constraints to local stress concentrations.

The primary factor differentiating the two strategies is the treatment of the transverse curvature induced by the Poisson effect.

- **Load Control (Free Deformation):** When a vertical force is applied (approximating a Neumann boundary condition), the plate is mechanically free to deform to minimize its strain energy. Due to the Poisson effect ($\nu > 0$), the longitudinal bending (κ_{xx}) induces a transverse anticlastic curvature (κ_{yy}). The resulting deformed surface creates a shallow "saddle" shape, where the vertical displacement $w(x, y)$ has a strong dependence on both x^2 and y^2 . This complex double-curvature surface is mathematically more challenging to approximate with lower-order polynomials.
- **Displacement Control (Constrained Deformation):** When the loading is applied via DC (approximating a Dirichlet boundary condition), the rigid roller enforces a specific kinematic constraint along the contact line (e.g., $w = \bar{w}$ for all y at $x = L/2$). This effectively acts as a "clamp" on the transverse curvature, locally inhibiting the anticlastic

effect by applying fictitiously a counter-moment that balance the curvature, this effect can be seen in when comparing Fig. 20b and Fig. 21b, where in the latest the transverse strain ε_{yy} correctly follow the negative relation through Poisson coefficient, while the former exhibit an anomalous trend, caused by the superposition of vertical action and counter-moment effect. Geometrically, this constraint "flattens" the saddle shape at the point of maximum deflection, rendering the deformed surface more cylindrical (dominated by x , with constant y). This simplified geometry is inherently easier for the shape functions and pre-extrapolation polynomials to fit, resulting in a lower reconstruction error.

Finally, the distribution of the reaction forces differs between the two cases. In a LC simulation with concentrated forces, sharp stress singularities and steep strain gradients may arise directly beneath the load application point. Conversely, in DC, the reaction force naturally distributes itself along the contact line to accommodate the contact stiffness. This distribution tends to produce slightly smoother strain gradients compared to the concentrated load scenario. Since polynomial pre-extrapolation acts as a low-pass filter, it reconstructs smoother fields more accurately than fields containing sharp gradients or peaks, further favoring DC results.

In conclusion, the lower error observed in DC configuration is an artifact of the simulation method. The DC acts as a physical regularizer, simplifying the surface geometry by suppressing the anticlastic curvature and smoothing local gradients. The LC configuration, by allowing the plate to assume its full, natural saddle shape, presents a mathematically more complex surface to reconstruct, resulting in a marginally higher—but physically more representative—reconstruction error.

In addition, although the real model of the 3PB is a simply supported plate, this constraint is numerically unattainable, and the support line must necessarily be fully constrained. This results in a stiffness consistent with the displacement-imposed condition but in contrast with the load-imposed one, a factor that would confirm what has just been stated.

At this stage, the parameters that optimize the defined SN have been fully established. Therefore, it is possible to present the results which, from a purely numerical standpoint, represent the best achievable outcomes for both pre-extrapolation methods.

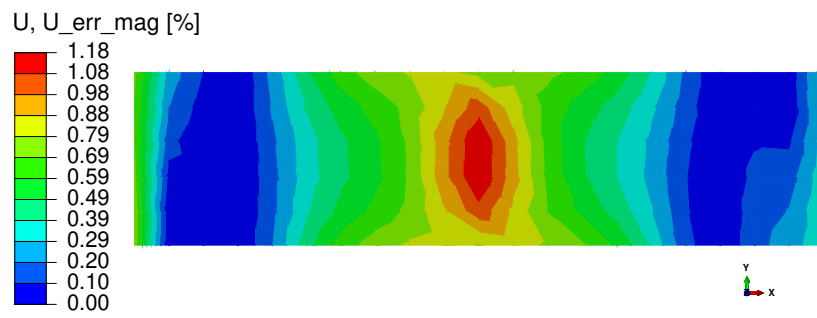
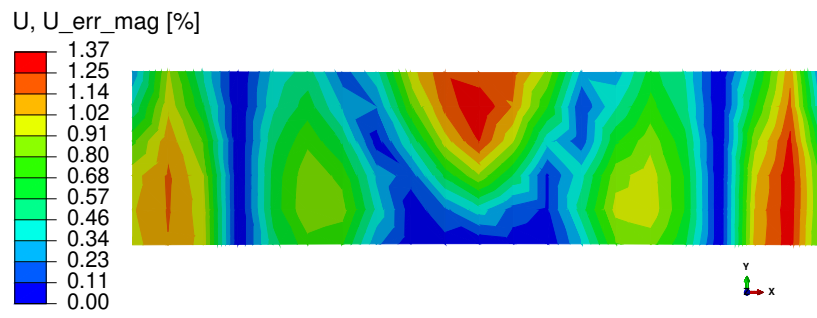


Figure 51: Contour plot of U_{err} of best reconstruction on model 3PB in DC configuration with: a) Polynomial approach, b) Physics-Based approach

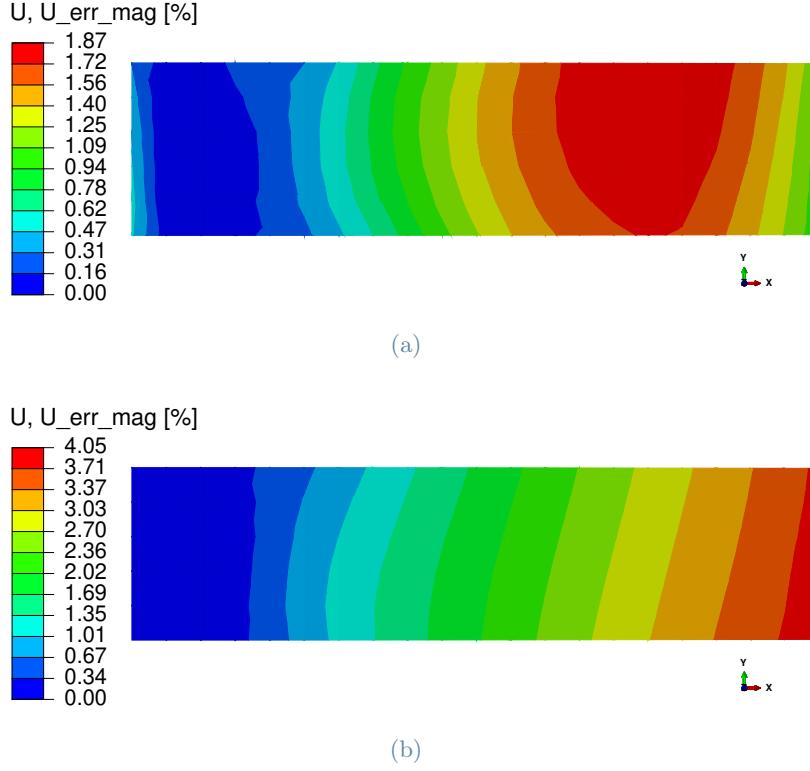


Figure 52: Contour plot of U_{err} of best reconstruction on model CP in DC configuration with: a) Polynomial approach, b) Physics-Based approach

5.3.3. Noise Robustness and Interpolation Strategy Comparison

This section presents a quantitative evaluation of the robustness of the proposed pre-extrapolation strategies, subjected to varying levels of measurement noise. The analysis aims to determine how the reconstruction accuracy, defined by the maximum (U_{err}^{max}) and mean (U_{err}^{mean}) displacement errors, degrades when the input strain data are corrupted by sensor noise.

To assess the robustness of the reconstruction algorithms under realistic operating conditions, a sensitivity analysis was performed by injecting synthetic noise into the numerical strain data. The noise is modeled as a Gaussian white noise sequence with zero mean and standard deviation σ_{noise} , representing the measurement uncertainty of the FBG sensors.

The injection strategy differs fundamentally depending on the assumed SN configuration. Two distinct noise modes were implemented: the *Double-Noise* strategy and the *Mirrored-Noise* strategy. Their operational differences and statistical implications are detailed below.

Double-Noise Strategy (Uncorrelated Error) This strategy represents the standard scenario where the structure is fully instrumented on both the top and bottom surfaces. Operatively, the procedure follows these steps:

1. **Extraction:** The numerical strain fields are extracted independently for the top surface (ε_{top}^{FEM}) and the bottom surface (ε_{bot}^{FEM}) from the direct FE model.

2. **Noise Generation:** Two statistically independent random noise vectors, η_{top} and η_{bot} , are generated from the same normal distribution $\mathcal{N}(0, \sigma_{noise})$.
3. **Application:** The noise is added separately to the respective surfaces:

$$\begin{aligned}\varepsilon_{top}^{meas} &= \varepsilon_{top}^{FEM} + \eta_{top} \\ \varepsilon_{bot}^{meas} &= \varepsilon_{bot}^{FEM} + \eta_{bot}\end{aligned}\tag{59}$$

In this configuration, the measurement errors on the two surfaces are uncorrelated. The probability of a sensor on the top surface overestimating the strain does not influence the error of the corresponding sensor on the bottom surface. This allows the least-squares minimization within the iFEM algorithm to effectively average out the random errors.

Mirrored-Noise Strategy (Correlated Error) This strategy simulates a reduced SN where only one surface (e.g., the top) is instrumented. The strain data for the opposite surface is derived virtually by exploiting the symmetric nature of the bending deformation (assuming $\varepsilon_{bot} \approx -\varepsilon_{top}$).

1. **Extraction:** Only the strain field for the instrumented surface (ε_{top}^{FEM}) is extracted.
2. **Noise Generation:** A single noise vector η_{top} is generated.
3. **Application and Mirroring:** The noise is added to the top signal, and this already corrupted signal is then negated to generate the virtual bottom data:

$$\begin{aligned}\varepsilon_{top}^{meas} &= \varepsilon_{top}^{FEM} + \eta_{top} \\ \varepsilon_{bot}^{virtual} &= -(\varepsilon_{top}^{meas}) = -\varepsilon_{top}^{FEM} - \eta_{top}\end{aligned}\tag{60}$$

In this scenario, the errors on the top and bottom surfaces are perfectly negatively correlated. If the top sensor reads a value higher than the true strain (positive error $+\eta$), the virtual bottom sensor will inevitably read a value lower than the true strain (negative error $-\eta$). This is particularly critical for the calculation of the section curvature κ_{xx} , which depends on the difference between the strains (see Eq. 8):

$$\kappa_{xx} \propto (\varepsilon_{top}^{meas} - \varepsilon_{bot}^{virtual}) \approx (\varepsilon_{top}^{FEM} + \eta) - (-\varepsilon_{top}^{FEM} - \eta) = 2\varepsilon_{top}^{FEM} + 2\eta\tag{61}$$

Instead of canceling out, the error is amplified. This systematic propagation makes the Mirrored-Noise strategy a more severe test for the reconstruction algorithm compared to the random Double-Noise scenario.

Table 13: Comparison of reconstruction accuracy for the 3PB model in DC configuration. The reported metrics evaluate the robustness of different interpolation strategies for the two noise injection strategies under varying noise levels.

σ_{noise} [$\mu\epsilon$]	Polynomial				Physics-Based			
	Double-Noise		Mirrored-Noise		Double-Noise		Mirrored-Noise	
	U_{err}^{max} [%]	U_{err}^{mean} [%]	U_{err}^{max} [%]	U_{err}^{mean} [%]	U_{err}^{max} [%]	U_{err}^{mean} [%]	U_{err}^{max} [%]	U_{err}^{mean} [%]
0	1.37	0.60	1.37	0.60	1.18	0.42	1.18	0.42
5	1.39	0.60	1.43	0.60	1.19	0.42	1.22	0.44
10	1.41	0.61	1.49	0.61	1.19	0.41	1.27	0.45
15	1.44	0.61	1.55	0.61	1.20	0.41	1.32	0.46
50	1.60	0.62	1.98	0.64	1.26	0.40	1.65	0.56
100	1.83	0.63	2.59	0.72	1.33	0.42	2.12	0.71
200	2.30	0.71	4.01	1.04	2.58	0.49	3.15	1.03

Table 14: Comparison of reconstruction accuracy for the 3PB model in LC configuration. The reported metrics evaluate the robustness of different interpolation strategies for the two noise injection strategies under varying noise levels.

σ_{noise} [$\mu\epsilon$]	Polynomial				Physics-Based			
	Double-Noise		Mirrored-Noise		Double-Noise		Mirrored-Noise	
	U_{err}^{max} [%]	U_{err}^{mean} [%]	U_{err}^{max} [%]	U_{err}^{mean} [%]	U_{err}^{max} [%]	U_{err}^{mean} [%]	U_{err}^{max} [%]	U_{err}^{mean} [%]
0	1.39	0.64	1.39	0.64	0.61	0.15	0.61	0.15
5	1.41	0.64	1.37	0.63	0.59	0.15	0.61	0.16
10	1.42	0.64	1.39	0.63	0.59	0.16	0.60	0.16
15	1.44	0.64	1.41	0.63	0.62	0.18	0.60	0.17
50	1.54	0.64	1.72	0.63	0.87	0.18	0.83	0.25
100	1.68	0.66	2.48	0.68	1.53	0.24	1.34	0.39
200	1.97	0.72	4.01	0.93	3.01	0.46	2.36	0.71

Table 15: Comparison of reconstruction accuracy for the CP model in DC configuration. The reported metrics evaluate the robustness of different interpolation strategies for the two noise injection strategies under varying noise levels.

σ_{noise} [$\mu\epsilon$]	Polynomial				Physics-Based			
	Double-Noise		Mirrored-Noise		Double-Noise		Mirrored-Noise	
	U_{err}^{max} [%]	U_{err}^{mean} [%]	U_{err}^{max} [%]	U_{err}^{mean} [%]	U_{err}^{max} [%]	U_{err}^{mean} [%]	U_{err}^{max} [%]	U_{err}^{mean} [%]
0	1.87	1.04	1.87	1.04	4.05	1.84	4.05	1.84
5	1.93	1.08	2.19	1.21	4.45	2.00	4.70	2.10
10	2.00	1.11	2.51	1.40	4.85	2.15	4.80	2.30
15	2.07	1.15	2.86	1.55	5.25	2.30	6.01	2.60
50	2.54	1.40	5.55	2.73	8.05	3.37	10.62	4.39
100	3.26	1.75	9.90	4.42	12.05	4.90	17.34	6.94
200	4.75	2.46	18.74	7.80	20.07	7.96	30.89	12.03

Table 16: Comparison of reconstruction accuracy for the CP model in LC configuration. The reported metrics evaluate the robustness of different interpolation strategies for the two noise injection strategies under varying noise levels.

σ_{noise} [$\mu\epsilon$]	Polynomial				Physics-Based			
	Double-Noise		Mirrored-Noise		Double-Noise		Mirrored-Noise	
	$U_{\text{err}}^{\text{max}}$ [%]	$U_{\text{err}}^{\text{mean}}$ [%]	$U_{\text{err}}^{\text{max}}$ [%]	$U_{\text{err}}^{\text{mean}}$ [%]	$U_{\text{err}}^{\text{max}}$ [%]	$U_{\text{err}}^{\text{mean}}$ [%]	$U_{\text{err}}^{\text{max}}$ [%]	$U_{\text{err}}^{\text{mean}}$ [%]
0	1.88	1.04	1.88	1.04	4.05	1.84	4.05	1.84
5	1.93	1.07	2.18	1.20	4.12	1.87	4.69	2.09
10	1.99	1.11	2.50	1.37	4.83	2.14	5.33	2.34
15	2.06	1.14	2.84	1.54	5.23	2.29	5.97	2.59
50	2.52	1.39	5.48	2.70	7.98	3.34	10.51	4.34
100	3.23	1.74	9.76	4.37	11.92	4.85	17.12	6.85
200	4.69	2.43	18.45	7.69	19.81	7.86	30.45	11.86

The results presented in tables 13 to 16 provide a quantitative assessment of the reconstruction algorithms' robustness against measurement uncertainty. By comparing the error metrics ($U_{\text{err}}^{\text{max}}$ and $U_{\text{err}}^{\text{mean}}$) across the 3PB and CP models, three critical trends emerge regarding the structural configuration, the noise injection strategy, and the pre-extrapolation method.

Load Case Sensitivity (3PB vs Cantilever): A cross-comparison between the structural models reveals a distinct hierarchy in reconstruction stability. The 3PB configuration demonstrates superior robustness compared to CP. In the 3PB case, regardless of the loading type (DC or LC), the maximum reconstruction error remains contained even under extreme noise conditions (200 $\mu\epsilon$).

Conversely, the CP model exhibits a significantly higher sensitivity to sensor noise. While the baseline errors at 0 $\mu\epsilon$ are comparable to the 3PB case, the degradation at high noise levels is much more severe. This discrepancy highlights the inherent difficulty in reconstructing the deformation field of a cantilevered structure, where the strain gradients near the clamped root are sharper and the error propagation from the fixed end to the free tip is more pronounced.

Table 17: Summary of percentage increase in maximum reconstruction error ($\Delta U_{\text{err}}^{\text{max}}$ and $\Delta U_{\text{err}}^{\text{mean}}$) relative to the noise-free baseline. The table details the error growth at the different noise levels for 3PB model, categorized by loading configuration (DC, LC), interpolation method (Polynomial, Physics-Based), and noise injection strategy (Double-Noise, Mirrored-Noise).

	σ_{noise} [$\mu\epsilon$]	Polynomial				Physics-Based			
		Double-Noise		Mirrored-Noise		Double-Noise		Mirrored-Noise	
		$\Delta U_{\text{err}}^{\text{max}}$	$\Delta U_{\text{err}}^{\text{mean}}$	$\Delta U_{\text{err}}^{\text{max}}$	$\Delta U_{\text{err}}^{\text{mean}}$	$\Delta U_{\text{err}}^{\text{max}}$	$\Delta U_{\text{err}}^{\text{mean}}$	$\Delta U_{\text{err}}^{\text{max}}$	$\Delta U_{\text{err}}^{\text{mean}}$
DC	0→15	+5.1%	+0.6%	+13.4%	+0.9%	+2.0%	+3.7%	+12.0%	+9.4%
	0→200	+68.7%	+18.5%	+193.3%	+71.9%	+119.4%	+15.2%	+167.3%	+142.8%
LC	0→15	+3.1%	+0.1%	+1.4%	-1.2%	+2.4%	+14.7%	-1.3%	+11.9%
	0→200	+41.7%	+12.9%	+187.7%	+45.4%	+393.9%	+197.8%	+287.4%	+364.7%

Table 18: Summary of percentage increase in maximum reconstruction error (ΔU_{err}^{max} and ΔU_{err}^{mean}) relative to the noise-free baseline. The table details the error growth at the different noise levels for CP model, categorized by loading configuration (DC, LC), interpolation method (Polynomial, Physics-Based), and noise injection strategy (Double-Noise, Mirrored-Noise).

	σ_{noise} [$\mu\epsilon$]	Polynomial				Physics-Based			
		Double-Noise		Mirrored-Noise		Double-Noise		Mirrored-Noise	
		ΔU_{err}^{max}	ΔU_{err}^{mean}	ΔU_{err}^{max}	ΔU_{err}^{mean}	ΔU_{err}^{max}	ΔU_{err}^{mean}	ΔU_{err}^{max}	ΔU_{err}^{mean}
DC	0→15	+10.3%	+10.2%	+52.6%	+48.8%	+29.7%	+24.9%	+48.4%	+41.4%
	0→200	+153.4%	+136.1%	+900.3%	+649.8%	+396.0%	+332.2%	+663.4%	+553.0%
LC	0→15	+9.7%	+9.8%	+51.1%	+47.6%	+29.2%	+24.5%	+47.7%	+40.7%
	0→200	+149.9%	+133.5%	+882.4%	+638.6%	+389.6%	+326.9%	+652.7%	+544.1%

Impact of Sensor Configuration (Double vs Mirrored-Noise): The analysis confirms the critical impact of the SN topology on error propagation. The Mirrored-Noise strategy, which simulates a single-sided sensor installation with virtually derived data for the opposite surface, consistently yields higher errors than the Double-Noise strategy.

This effect is particularly degrading for the CP model. Referring to table 16, shifting from Double-Noise to Mirrored-Noise at 200 $\mu\epsilon$ causes the Polynomial reconstruction error to surge from 4.69% to 18.45%. A similar trend is observed in the Physics-Based approach, where the error escalates from 19.81% to 30.45%. In the 3PB model, the impact is measurable but less catastrophic. For example, in LC configuration (table 14), the Polynomial error at 200 $\mu\epsilon$ increases from 1.97% (Double-Noise) to 4.01% (Mirrored-Noise). These results quantitatively validate the theoretical expectation that the correlation of errors in the Mirrored-Noise strategy amplifies the inaccuracy of the curvature estimation, whereas the uncorrelated errors in the Double-Noise strategy tend to cancel out during the least-squares minimization.

Polynomial vs Physics-Based Interpolation: The comparison between the two pre-extrapolation strategies reveals a clear distinction based on the boundary conditions of the problem.

For CP, the Data-Driven Polynomial approach is strictly superior. As shown in tables 15 and 16, the Polynomial method consistently maintains lower errors than the Physics-Based method across all noise levels. Notably, the Physics-Based method exhibits instability in this configuration, starting with a baseline error of $\approx 4.05\%$ and reaching unacceptable levels ($> 20\%$) at high noise. This failure suggests that the biharmonic assumption ($\nabla^4 w = 0$), which implies zero distributed load, does not adequately capture the complex boundary layer behavior near the clamp, whereas the low-order polynomial successfully approximates the global curvature of the beam-like deflection.

For 3PB, the scenario is more nuanced. In noise-free or low-noise conditions (0 – 15 $\mu\epsilon$), the Physics-Based method offers higher accuracy, particularly in the LC configuration where it achieves a remarkable baseline error of 0.61% compared to 1.39% for the Polynomial method.

However, their response to high noise differs. In the Double-Noise scenario, the Polynomial method proves slightly more robust at $200\ \mu\varepsilon$ (e.g., 1.97% vs 3.01% in LC). Interestingly, in the Mirrored-Noise scenario (correlated error), the Physics-Based method demonstrates unexpected resilience, outperforming the Polynomial method at high noise levels (e.g., 2.36% vs 4.01% in LC at $200\ \mu\varepsilon$). This suggests that while the Polynomial fit acts as a better stochastic filter for random noise, the rigid mechanical constraints of the Physics-Based spline may offer better stability against the systematic curvature bias introduced by the Mirrored strategy in bending-dominated plates.

6 | Conclusions and Future Developments

This thesis has presented a systematic approach to Structural Health Monitoring (SHM) for aerospace applications, specifically focusing on the deformation reconstruction of a composite radome panel using the inverse Finite Element Method (iFEM). The primary objective of the work was to develop and validate a robust numerical methodology capable of accurately estimating full-field displacements from discrete strain measurements, laying the groundwork for predictive maintenance strategies in advanced composite structures.

To achieve this, the study was structured around a rigorous numerical framework. Initially, high-fidelity Direct Finite Element Models were developed to simulate the mechanical response of a laminated Kevlar/epoxy plate under two distinct loading configurations: Three-Point Bending and Cantilever Plate. These models, evaluated under both DC and LC conditions, provided a reliable baseline and the synthetic strain data required to feed the inverse algorithm.

Subsequently, the iFEM algorithm was implemented, employing a quadrilateral inverse-shell element formulated on the First-Order Shear Deformation Theory. A comprehensive numerical sensitivity analysis was conducted to optimize the SN design. This investigation highlighted that while a higher density of multi-axial strain sensors naturally yields superior reconstruction accuracy, practical and physical constraints dictate a more strategic approach. The strain component analysis demonstrated that shear strain contributions are largely negligible in bending-dominated scenarios. This insight allowed for the definition of a more efficient "mixed" sensor topology that alternatively captures longitudinal and transverse strains without compromising the overall fidelity of the reconstructed displacement field.

A crucial aspect of the validation process involved assessing the algorithm's robustness against measurement uncertainties by artificially injecting noise into the simulated strain data. This noise analysis provided significant insights into the optimal sensor placement strategy. While instrumenting a single side of the plate presents obvious practical advantages—such as protecting the sensors from external environmental factors by placing them exclusively on the radome's inner surface—the results demonstrated that double-sided sensorization offers superior resilience to noise. By capturing data from both the top and bottom surfaces, the algorithm can more effectively decouple membrane and bending strain components, thereby inherently mitigating the propagation of random measurement errors.

Furthermore, the introduction of noise highlighted distinct behaviors in the strain pre-extrapolation techniques used to address the limitation of sparse sensor networks. The comparative analysis between Physics-Based methods (leveraging biharmonic spline interpolation) and Data-Driven approaches (utilizing polynomial surface fitting) revealed that no single technique is universally superior. In ideal conditions, the Physics-Based approach proved exceptionally accurate for smoothly distributed bending deformations (e.g., the 3PB scenario), while the Polynomial method demonstrated a better capability to act as a kinematic guide in the presence of strong localized boundary constraints (e.g., the CP configuration). However, under noisy conditions, the Data-Driven polynomial approach exhibited an inherent smoothing effect, effectively acting as a filter that reduced the impact of local signal perturbations. Conversely, the Physics-Based interpolation showed a higher sensitivity to noise, inadvertently propagating localized measurement errors due to its strict adherence to the data points. This emphasizes the necessity of selecting the interpolation strategy not only based on the expected deformation field but also on the anticipated noise levels in the operational environment.

Future Developments

While the numerical results obtained in this study firmly validate the theoretical framework and the proposed optimization strategies, several avenues for future research and development remain open to fully transition this technology into operational practice:

- **Experimental Validation:** The most immediate future development consists of transitioning from the virtual environment to physical testing. It is imperative to experimentally validate the numerically demonstrated findings by subjecting the instrumented composite plates to actual physical testing setups, utilizing Fiber Bragg Grating (FBG) sensor networks to acquire real strain data and assess the methodology against real-world measurement uncertainties.
- **Real-Time Data Acquisition and Processing:** The ultimate goal of the proposed SHM framework is its integration into a live operational setting. Future efforts will focus on optimizing the computational pipeline and the hardware interfaces to enable real-time data acquisition and instantaneous displacement reconstruction, which is a fundamental requirement for continuous, on-board structural monitoring.
- **Shape Sensing of the Full-Scale Radome Model:** The current methodology has been successfully tested on a simplified flat rectangular plate extracted from the radome structure. To fully realize the predictive maintenance objectives, the shape-sensing algorithm must be scaled and adapted to analyze the actual full-scale radome. This will involve addressing the geometric complexities of a doubly-curved shell, managing complex membrane-bending couplings, and accounting for the realistic boundary conditions of the operational aerospace component.

A | Appendix

Local Reference System and Transformation Matrix

The transformation matrix $\mathbf{T}^i$ and the local nodal coordinates of the element introduced in section 3.3 are derived hereafter.

By calling:

$$\mathbf{X}_q = \begin{bmatrix} X_q \\ Y_q \\ Z_q \end{bmatrix}, \quad (q = 1, 2, 3, 4) \quad (\text{A.1})$$

the global coordinates of the q^{th} element node, the three unitary vectors $\mathbf{l}, \mathbf{p}, \mathbf{n}$ defining the local reference system are calculated as:

$$\mathbf{n} = \frac{\mathbf{X}_{31} \times \mathbf{X}_{42}}{\|\mathbf{X}_{31} \times \mathbf{X}_{42}\|}, \quad \mathbf{p} = \frac{\mathbf{X}_{31} + \mathbf{X}_{42}}{\|\mathbf{X}_{31} + \mathbf{X}_{42}\|}, \quad \mathbf{l} = \mathbf{p} \times \mathbf{n} \quad (\text{A.2})$$

where

$$\mathbf{X}_{31} = \mathbf{X}_3 - \mathbf{X}_1, \quad \mathbf{X}_{42} = \mathbf{X}_4 - \mathbf{X}_2 \quad (\text{A.3})$$

is the diagonal vector pointing from node 1 to 3, while $\mathbf{X}_{42}$ is the one pointing from node 2 to 4.

Exploiting Eq. (A.2), the transformation matrix of the nodal d.o.f. of the i^{th} element from local to global coordinate system is defined as:

$$\mathbf{T}^i = \begin{bmatrix} \mathbf{T} & 0 & \cdots & 0 \\ 0 & \mathbf{T} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{T} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{l}^\top \\ \mathbf{p}^\top \\ \mathbf{n}^\top \end{bmatrix} \quad (\text{A.4})$$

Note that $\mathbf{T}$ is a 3×3 matrix (3 displacements/rotations for each node), while $\mathbf{T}^i$ is a 24×24 matrix, since the iQS4 element has 24 d.o.f.: 6 (i.e., 3 displacements and 3 rotations) for each node.

Then, in order to compute the local nodal coordinates of the i^{th} element with respect to a reference system centered in the element centroid, the global coordinates $\mathbf{C}$ of the centroid need to be calculated as:

$$\mathbf{C} = \frac{\sum_{\alpha=1}^4 \mathbf{c}_{\alpha} d_{\alpha}}{\sum_{\alpha=1}^4 d_{\alpha}}, \quad (\text{A.5})$$

where d_{α} is the edge length and $\mathbf{c}_{\alpha}$ the edge mid-point defined as:

$$\mathbf{c}_{\alpha} = \frac{\mathbf{X}_{\beta} + \mathbf{X}_{\alpha}}{2}, \quad \mathbf{d}_{\alpha} = \|\mathbf{X}_{\beta} - \mathbf{X}_{\alpha}\|, \quad (\alpha = 1, 2, 3, 4; \beta = 2, 3, 4, 1) \quad (\text{A.6})$$

The local coordinates of the element nodes are finally:

$$x_q = (\mathbf{X}_q - \mathbf{C}) \cdot \mathbf{l}, \quad y_q = (\mathbf{X}_q - \mathbf{C}) \cdot \mathbf{p} \quad (\text{A.7})$$

Shape Functions and B Matrices

The explicit form of the shape functions used for the definition of the iQS4 element are:

$$N_1 = \frac{(1-s)(1-t)}{4}, \quad N_2 = \frac{(1+s)(1-t)}{4}, \quad (\text{A.8a})$$

$$N_3 = \frac{(1+s)(1+t)}{4}, \quad N_4 = \frac{(1-s)(1+t)}{4}, \quad (\text{A.8b})$$

$$N_5 = \frac{(1-s^2)(1-t)}{16}, \quad N_6 = \frac{(1+s)(1-t^2)}{16}, \quad (\text{A.9a})$$

$$N_7 = \frac{(1-s^2)(1+t)}{16}, \quad N_8 = \frac{(1-s)(1-t^2)}{16}, \quad (\text{A.9b})$$

With these, we define the quantities:

$$L_1 = y_{14}N_8 - y_{21}N_5, \quad M_1 = x_{41}N_8 - x_{12}N_5 \quad (\text{A.10a})$$

$$L_2 = y_{21}N_5 - y_{32}N_6, \quad M_2 = x_{12}N_5 - x_{23}N_6 \quad (\text{A.10b})$$

$$L_3 = y_{32}N_6 - y_{43}N_7, \quad M_3 = x_{23}N_6 - x_{34}N_7 \quad (\text{A.10c})$$

$$L_4 = y_{43}N_7 - y_{14}N_8, \quad M_4 = x_{34}N_7 - x_{41}N_8 \quad (\text{A.10d})$$

with:

$$x_{\alpha\beta} = x_{\alpha} - x_{\beta}, \quad y_{\alpha\beta} = y_{\alpha} - y_{\beta} \quad (\alpha, \beta = 1, 2, 3, 4) \quad (\text{A.11})$$

By exploiting these shape functions, the matrices $\mathbf{B}^m$, $\mathbf{B}^b$, $\mathbf{B}^s$ can be obtained and their explicit form is:

$$\mathbf{B}^m = [\mathbf{B}_1^m \ \mathbf{B}_2^m \ \mathbf{B}_3^m \ \mathbf{B}_4^m], \quad \mathbf{B}^b = [\mathbf{B}_1^b \ \mathbf{B}_2^b \ \mathbf{B}_3^b \ \mathbf{B}_4^b], \quad \mathbf{B}^s = [\mathbf{B}_1^s \ \mathbf{B}_2^s \ \mathbf{B}_3^s \ \mathbf{B}_4^s] \quad (\text{A.12})$$

where

$$\left. \begin{aligned} \mathbf{B}_i^m &= \begin{bmatrix} N_{q,x} & 0 & 0 & 0 & 0 & L_{q,x} \\ 0 & N_{q,y} & 0 & 0 & 0 & M_{q,x} \\ N_{q,y} & N_{q,x} & 0 & 0 & 0 & L_{q,y} + M_{q,x} \end{bmatrix} \\ \mathbf{B}_i^b &= \begin{bmatrix} 0 & 0 & 0 & 0 & N_{q,x} & 0 \\ 0 & 0 & 0 & -N_{q,y} & 0 & 0 \\ 0 & 0 & 0 & -N_{q,x} & N_{q,y} & 0 \end{bmatrix} \\ \mathbf{B}_i^s &= \begin{bmatrix} 0 & 0 & N_{q,x} & -L_{q,x} & -M_{q,x} + N_i & 0 \\ 0 & 0 & N_{q,y} & -L_{q,y} - N_i & -M_{q,y} & 0 \end{bmatrix} \end{aligned} \right\} \quad (q = 1, 2, 3, 4) \quad (\text{A.13})$$

Note: The subscript $(\cdot)_{,x}$ denotes the partial derivative $\frac{\partial(\cdot)}{\partial x}$.

Jacobian Operator and Derivatives

The Jacobian operator is required to compute the partial derivatives of the shape functions with respect to x and y coordinates. It is computed as:

$$\mathbf{J} = \frac{\partial(x, y)}{\partial(s, t)} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} \quad (\text{A.14})$$

Where the partial derivatives of the x and y coordinates with respect to s and t are:

$$\frac{\partial x}{\partial s} = \sum_{q=1}^4 \frac{\partial N_q}{\partial s} x_q, \quad \frac{\partial x}{\partial t} = \sum_{q=1}^4 \frac{\partial N_q}{\partial t} x_q, \quad \frac{\partial y}{\partial s} = \sum_{q=1}^4 \frac{\partial N_q}{\partial s} y_q, \quad \frac{\partial y}{\partial t} = \sum_{q=1}^4 \frac{\partial N_q}{\partial t} y_q \quad (\text{A.15})$$

Finally, the partial derivatives of the generic $(\cdot)$ shape function can be written in matrix form as:

$$\begin{bmatrix} \frac{\partial(\cdot)}{\partial x} \\ \frac{\partial(\cdot)}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial(\cdot)}{\partial s} \\ \frac{\partial(\cdot)}{\partial t} \end{bmatrix} \quad (\text{A.16})$$

where $\mathbf{J}^{-1}$ is the inverse of the Jacobian matrix.

Results of the Optimization of the Data-Driven Interpolation Order

This section contains the full set of numerical results and graphical representations obtained from the analyses performed during the study.

All outcomes from the interpolation class combinations examined in section 5.3.1 are presented here, complementing the group means and standard deviations previously reported.

ε_{xx}		ε_{yy}		U_{err}^{max}			U_{err}^{mean}		
X order	Y order	X order	Y order	Val	μ	σ	Val	μ	σ
3	0	3	0	4.62%	4.62%	0.01%	2.94%	2.94%	0.00%
		2	0	4.62%			2.94%		
		3	1	4.63%			2.94%		
		2	1	4.63%			2.94%		
3	1	3	0	3.74%	3.73%	0.01%	1.61%	1.61%	0.00%
		2	0	3.74%			1.61%		
		3	1	3.71%			1.61%		
		2	1	3.73%			1.61%		
3	2	3	1	6.47%	6.46%	0.01%	1.12%	1.11%	0.01%
		2	1	6.47%			1.11%		
		3	0	6.46%			1.11%		
		2	0	6.46%			1.11%		
2	2	2	0	1.73%	1.72%	0.01%	0.76%	0.76%	0.00%
		3	0	1.73%			0.76%		
		3	1	1.72%			0.76%		
		2	1	1.72%			0.76%		
2	1	2	1	4.61%	4.61%	0.01%	2.93%	2.93%	0.00%
		2	0	4.62%			2.93%		
		3	0	4.62%			2.93%		
		3	1	4.60%			2.93%		
2	0	2	1	4.61%	4.62%	0.01%	3.94%	3.19%	0.50%
		2	0	4.62%			2.94%		
		3	0	4.62%			2.94%		
		3	1	4.61%			2.94%		

Table A.1: Statistical analysis of the maximum U_{err}^{max} and mean U_{err}^{mean} percentage errors for the 3PB model under LC configuration. The table reports the measured values, along with their corresponding means (μ) and standard deviations (σ), as a function of the polynomial degrees along the X and Y axes for the ε_{xx} and ε_{yy} components, grouped by common interpolation orders on ε_{xx}

ε_{yy}		ε_{xx}		U_{err}^{max}			U_{err}^{mean}		
X order	Y order	X order	Y order	Val	μ	σ	Val	μ	σ
3	0	2	0	4.62%	4.30%	1.54%	2.94%	2.05%	1.01%
		2	1	4.62%			2.93%		
		2	2	1.73%			0.76%		
		3	0	4.62%			2.94%		
		3	1	3.74%			1.61%		
		3	2	6.46%			1.11%		
2	0	2	0	4.62%	4.30%	1.54%	2.94%	2.05%	1.01%
		2	1	4.62%			2.93%		
		2	2	1.73%			0.76%		
		3	0	4.62%			2.94%		
		3	1	3.74%			1.61%		
		3	2	6.46%			1.11%		
3	1	2	0	4.61%	4.29%	1.55%	2.94%	2.05%	1.01%
		2	1	4.60%			2.93%		
		2	2	1.72%			0.76%		
		3	0	4.63%			2.94%		
		3	1	3.71%			1.61%		
		3	2	6.47%			1.12%		
2	1	2	0	4.61%	4.30%	1.55%	3.94%	2.21%	1.24%
		2	1	4.61%			2.93%		
		2	2	1.72%			0.76%		
		3	0	4.63%			2.94%		
		3	1	3.73%			1.61%		
		3	2	6.47%			1.11%		

Table A.2: Statistical analysis of the maximum U_{err}^{max} and mean U_{err}^{mean} percentage errors for the 3PB model under LC configuration. The table reports the measured values, along with their corresponding means (μ) and standard deviations (σ), as a function of the polynomial degrees along the X and Y axes for the ε_{xx} and ε_{yy} components, grouped by common interpolation orders on ε_{yy}

ε_{xx}		ε_{yy}		U_{err}^{max}			U_{err}^{mean}		
X order	Y order	X order	Y order	Val	μ	σ	Val	μ	σ
3	0	3	0	4.167%	4.158%	0.006%	2.510%	2.527%	0.035%
		2	0	4.156%			2.510%		
		3	1	4.155%			2.580%		
		2	1	4.153%			2.509%		
3	1	3	0	3.385%	3.390%	0.006%	1.239%	1.265%	0.030%
		2	0	3.384%			1.239%		
		3	1	3.397%			1.289%		
		2	1	3.393%			1.292%		
3	2	3	0	6.724%	6.750%	0.025%	1.179%	1.182%	0.006%
		2	0	6.740%			1.179%		
		3	1	6.753%			1.192%		
		2	1	6.784%			1.181%		
2	2	3	0	1.451%	1.447%	0.005%	0.599%	0.599%	0.001%
		2	0	1.451%			0.599%		
		3	1	1.439%			0.600%		
		2	1	1.447%			0.598%		
2	1	3	0	4.139%	4.134%	0.005%	2.502%	2.501%	0.001%
		2	0	4.139%			2.502%		
		3	1	4.129%			2.500%		
		2	1	4.130%			2.501%		
2	0	3	0	4.147%	4.135%	0.009%	2.501%	2.507%	0.004%
		2	0	4.137%			2.510%		
		3	1	4.128%			2.508%		
		2	1	4.128%			2.509%		

Table A.3: Statistical analysis of the maximum U_{err}^{max} and mean U_{err}^{mean} percentage errors for the 3PB model under DC configuration. The table reports the measured values, along with their corresponding means (μ) and standard deviations (σ), as a function of the polynomial degrees along the X and Y axes for the ε_{xx} and ε_{yy} components, grouped by common interpolation orders on ε_{xx}

ε_{yy}		ε_{xx}		U_{err}^{max}			U_{err}^{mean}		
X order	Y order	X order	Y order	Val	μ	σ	Val	μ	σ
3	0	2	0	4.15%	4.00%	1.70%	2.50%	1.76%	0.85%
		2	1	4.14%			2.50%		
		2	2	1.45%			0.60%		
		3	0	4.17%			2.51%		
		3	1	3.39%			1.24%		
		3	2	6.72%			1.18%		
2	0	2	0	4.14%	4.00%	1.70%	2.51%	1.76%	0.85%
		2	1	4.14%			2.50%		
		2	2	1.45%			0.60%		
		3	0	4.16%			2.51%		
		3	1	3.38%			1.24%		
		3	2	6.74%			1.18%		
3	1	2	0	4.13%	4.00%	1.71%	2.51%	1.78%	0.86%
		2	1	4.13%			2.50%		
		2	2	1.44%			0.60%		
		3	0	4.16%			2.58%		
		3	1	3.40%			1.29%		
		3	2	6.75%			1.19%		
2	1	2	0	4.13%	4.01%	1.72%	2.51%	1.77%	0.85%
		2	1	4.13%			2.50%		
		2	2	1.45%			0.60%		
		3	0	4.15%			2.51%		
		3	1	3.39%			1.29%		
		3	2	6.78%			1.18%		

Table A.4: Statistical analysis of the maximum U_{err}^{max} and mean U_{err}^{mean} percentage errors for the 3PB model under DC configuration. The table reports the measured values, along with their corresponding means (μ) and standard deviations (σ), as a function of the polynomial degrees along the X and Y axes for the ε_{xx} and ε_{yy} components, grouped by common interpolation orders on ε_{yy}

ε_{xx}		ε_{yy}		U_{err}^{max}			U_{err}^{mean}		
X order	Y order	X order	Y order	Val	μ	σ	Val	μ	σ
1	0	1	0	9.14%	9.15%	< 0.01%	3.85%	3.86%	< 0.01%
		1	1	9.15%			3.86%		
		2	0	9.14%			3.85%		
		2	1	9.16%			3.86%		
1	2	1	0	5.01%	5.03%	0.02%	2.27%	2.27%	< 0.01%
		1	1	5.03%			2.27%		
		2	0	5.01%			2.27%		
		2	1	5.06%			2.28%		
1	1	1	0	8.21%	8.23%	0.02%	3.56%	3.57%	< 0.01%
		1	1	8.23%			3.57%		
		2	0	8.21%			3.56%		
		2	1	8.26%			3.57%		
2	0	1	0	9.20%	9.21%	< 0.01%	3.91%	3.91%	< 0.01%
		1	1	9.21%			3.92%		
		2	0	9.20%			3.91%		
		2	1	9.22%			3.92%		
2	1	1	0	7.84%	7.84%	< 0.01%	3.25%	3.26%	< 0.01%
		1	1	7.85%			3.26%		
		2	0	7.84%			3.25%		
		2	1	7.86%			3.26%		
2	2	1	0	7.91%	7.91%	< 0.01%	3.27%	3.28%	< 0.01%
		1	1	7.91%			3.28%		
		2	0	7.91%			3.27%		
		2	1	7.92%			3.28%		

Table A.5: Statistical analysis of the maximum U_{err}^{max} and mean U_{err}^{mean} percentage errors for the CP model under LC configuration. The table reports the measured values, along with their corresponding means (μ) and standard deviations (σ), as a function of the polynomial degrees along the X and Y axes for the ε_{xx} and ε_{yy} components, grouped by common interpolation orders on ε_{xx}

ε_{yy}		ε_{xx}		U_{err}^{max}			U_{err}^{mean}		
X order	Y order	X order	Y order	Val	μ	σ	Val	μ	σ
1	0	1	0	9.14%	7.88%	1.53%	3.85%	3.35%	0.60%
		1	2	5.01%			2.27%		
		1	1	8.21%			3.56%		
		2	0	9.20%			3.91%		
		2	1	7.84%			3.25%		
		2	2	7.91%			3.27%		
1	1	1	0	9.15%	7.90%	1.52%	3.86%	3.36%	0.60%
		1	2	5.03%			2.27%		
		1	1	8.23%			3.57%		
		2	0	9.21%			3.92%		
		2	1	7.85%			3.26%		
		2	2	7.91%			3.28%		
2	0	1	0	9.14%	7.89%	1.53%	3.85%	3.35%	0.60%
		1	2	5.01%			2.27%		
		1	1	8.21%			3.56%		
		2	0	9.20%			3.91%		
		2	1	7.84%			3.25%		
		2	2	7.91%			3.27%		
2	1	1	0	9.16%	7.91%	1.52%	3.86%	3.36%	0.60%
		1	2	5.06%			2.28%		
		1	1	8.26%			3.57%		
		2	0	9.22%			3.92%		
		2	1	7.86%			3.26%		
		2	2	7.92%			3.28%		

Table A.6: Statistical analysis of the maximum U_{err}^{max} and mean U_{err}^{mean} percentage errors for the CP model under LC configuration. The table reports the measured values, along with their corresponding means (μ) and standard deviations (σ), as a function of the polynomial degrees along the X and Y axes for the ε_{xx} and ε_{yy} components, grouped by common interpolation orders on ε_{yy}

ε_{xx}		ε_{yy}		U_{err}^{max}			U_{err}^{mean}		
X order	Y order	X order	Y order	Val	μ	σ	Val	μ	σ
1	0	1	0	9.14%	9.15%	< 0.01%	3.85%	3.86%	< 0.01%
		1	1	9.15%			3.86%		
		2	0	9.14%			3.85%		
		2	1	9.16%			3.86%		
1	2	1	0	5.01%	5.05%	0.04%	2.27%	2.27%	< 0.01%
		1	1	5.06%			2.28%		
		2	0	5.10%			2.27%		
		2	1	5.04%			2.27%		
1	1	1	0	8.21%	8.22%	0.01%	3.56%	3.56%	< 0.01%
		1	1	8.24%			3.57%		
		2	0	8.21%			3.56%		
		2	1	8.20%			3.56%		
2	0	1	0	9.20%	9.21%	< 0.01%	3.91%	3.92%	< 0.01%
		1	1	9.21%			3.92%		
		2	0	9.21%			3.91%		
		2	1	9.22%			3.92%		
2	1	1	0	7.83%	7.85%	< 0.01%	3.25%	3.26%	< 0.01%
		1	1	7.84%			3.26%		
		2	0	7.85%			3.26%		
		2	1	7.85%			3.26%		
2	2	1	0	7.90%	7.91%	< 0.01%	3.27%	3.28%	< 0.01%
		1	1	7.91%			3.28%		
		2	0	7.90%			3.27%		
		2	1	7.92%			3.28%		

Table A.7: Statistical analysis of the maximum U_{err}^{max} and mean U_{err}^{mean} percentage errors for the CP model under DC configuration. The table reports the measured values, along with their corresponding means (μ) and standard deviations (σ), as a function of the polynomial degrees along the X and Y axes for the ε_{xx} and ε_{yy} components, grouped by common interpolation orders on ε_{xx}

ε_{yy}		ε_{xx}		U_{err}^{max}			U_{err}^{mean}		
X order	Y order	X order	Y order	Val	μ	σ	Val	μ	σ
1	0	1	0	9.14%	7.88%	1.53%	3.85%	3.35%	0.60%
		1	2	5.01%			2.27%		
		1	1	8.21%			3.56%		
		2	0	9.20%			3.91%		
		2	1	7.83%			3.25%		
		2	2	7.90%			3.27%		
1	1	1	0	9.15%	7.90%	1.51%	3.86%	3.36%	0.60%
		1	2	5.06%			2.28%		
		1	1	8.24%			3.57%		
		2	0	9.21%			3.92%		
		2	1	7.84%			3.26%		
		2	2	7.91%			3.28%		
2	0	1	0	9.14%	7.90%	1.50%	3.85%	3.35%	0.60%
		1	2	5.10%			2.27%		
		1	1	8.21%			3.56%		
		2	0	9.21%			3.91%		
		2	1	7.85%			3.26%		
		2	2	7.90%			3.27%		
2	1	1	0	9.16%	7.90%	1.52%	3.86%	3.36%	0.60%
		1	2	5.06%			2.28%		
		1	1	8.20%			3.56%		
		2	0	9.22%			3.92%		
		2	1	7.85%			3.26%		
		2	2	7.92%			3.28%		

Table A.8: Statistical analysis of the maximum U_{err}^{max} and mean U_{err}^{mean} percentage errors for the CP model under DC configuration. The table reports the measured values, along with their corresponding means (μ) and standard deviations (σ), as a function of the polynomial degrees along the X and Y axes for the ε_{xx} and ε_{yy} components, grouped by common interpolation orders on ε_{yy}

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