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Executive Summary of the Thesis

Three-dimensional wave propagation analysis in SPEED through the Domain Reduction Method

Laurea Magistrale in Mathematical Engineering - Ingegneria Matematica

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1. Introduction

Simulating earthquake ground motion in seismic regions to explore its effects on geological or man-made structures is a subject of great interest.
Traditional numerical methods typically take into account a single model that encompasses the entire geological structure in the domain, from the seismic source to the localized structure under study. However, this approach necessarily leads to some approximations in the maximum frequency of propagation and in the lowest propagation velocity. Moreover, the construction of a mesh that takes into account features that can be modeled with a variation of about three/four orders of magnitude is very complicated. In addition, these methods appear particularly disadvantageous when the seismic source is very far from the site of interest.

In light of this, the Domain Reduction Method, referred to hereinafter as DRM, represents an alternative and valuable approach (see [1]). More precisely, the DRM is a sub-structuring technique in which the problem is subdivided into two sequential parts. From the first one (*auxiliary problem*) it is computed the ground motion obtained in absence of the structure, the so-called *free-field motion*. After that, it is considered only a reduced part of the computational domain, referred to as *reduced problem*, in which the seismic source and most of the propagation path are not taken into consideration. Seismic excitation are introduced to this model in the form of localized effective forces, calculated thanks to the free-field motion obtained with the auxiliary domain.

We implement the DRM in SPEED (*SPectral Elements in Elastodynamics with Discontinuous Galerkin*), a high performance open-source numerical code (see [3]), under the assumption of either vertical or oblique incident plane waves. This last hypothesis turns out to be reasonable especially when the source is located far away from the site under study.
Moreover, suitable Matlab scripts have been developed for the creation of the input files necessary for running SPEED with the DRM.

Finally, three different three-dimensional geological models have been considered to validate the code.

2. The Domain Reduction Method

The Domain Reduction Method (DRM) is a two-step finite element (FE) methodology in which an equivalent seismic excitation is applied at a fictitious boundary that ideally divides the region of interest (e.g., the Near field) from an external region (e.g., the Far field). More precisely, the original problem (STEP 0) is divided into two numerical simpler sub-problems analysed sequentially:

- STEP 1, *auxiliary problem*: the auxiliary domain contains the seismic source and the propagation path, while the localized structure and the geological features of interest are neglected and replaced with a background structure having the same materials of the surrounding soil. In this step it is computed the *free-field ground motion*, i.e., the ground motion evaluated in absence of any type of structure.
- STEP 2, *reduced problem*: the initial domain is restricted in such a way as to delimit the presence of structures and geological features which are in turn examined. The seismic input is introduced as a collection of equivalent effective nodal forces acting within a single layer of elements at an ideal interface between the region of interest and an external domain.

2.1. Theoretical formulation of the DRM

We outline the three-dimensional theoretical formulation of the method described above as proposed by [1] with the contribution given by [4].

For STEP 0 (*original problem*) we consider a semi-infinite seismic region that contains localized geological features (e.g. basin, hill, cave) as well as man-made structures, henceforth simply called "structure", under earthquake excitation. Since the causative fault may be far from the structure, we want to define a new problem in which the seismic excitation is brought closer to the region of interest. To do so, we imagine to artificially divide by a surface Γ_i our original domain into one interior domain Ω_i and one exterior domain Ω_e (see Figure 1, top panel). $\boldsymbol{u_e}$, $\boldsymbol{u_i}$ and $\boldsymbol{u_b}$ represent the vector field of displacements in Ω_e, Ω_i and Γ_i, respectively, while the earthquake excitation (extended fault or plane wavefront) can be expressed in terms of a set of equivalent body forces $\boldsymbol{P_e}$ operating close to the fault. It is worth underline that the displacements $\boldsymbol{u_b}$ are continuous across Γ_i and $\boldsymbol{P_b}$ correspond to the forces transmitted from Ω_e onto Ω_i.

Then, it is possible to express the equations of motion in Ω_i and Ω_e in the following partitioned form after spatial discretization (for example by the use of spectral elements or discontinuous finite elements, without loss of generality):

$$\begin{bmatrix} \boldsymbol{M}_{ii}^{\boldsymbol{\Omega_i}} & \boldsymbol{M}_{ib}^{\boldsymbol{\Omega_i}} \\ \boldsymbol{M}_{bi}^{\boldsymbol{\Omega_i}} & \boldsymbol{M}_{bb}^{\boldsymbol{\Omega_i}} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{\boldsymbol{i}} \\ \ddot{\boldsymbol{u}}_{\boldsymbol{b}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_{ii}^{\boldsymbol{\Omega_i}} & \boldsymbol{C}_{ib}^{\boldsymbol{\Omega_i}} \\ \boldsymbol{C}_{bi}^{\boldsymbol{\Omega_i}} & \boldsymbol{C}_{bb}^{\boldsymbol{\Omega_i}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_{\boldsymbol{i}} \\ \dot{\boldsymbol{u}}_{\boldsymbol{b}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{ii}^{\boldsymbol{\Omega_i}} & \boldsymbol{K}_{ib}^{\boldsymbol{\Omega_i}} \\ \boldsymbol{K}_{bi}^{\boldsymbol{\Omega_i}} & \boldsymbol{K}_{bb}^{\boldsymbol{\Omega_i}} \end{bmatrix} \begin{bmatrix} \boldsymbol{u_i} \\ \boldsymbol{u_b} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{P_b} \end{bmatrix},$$

for the interior domain Ω_i.

$$\begin{bmatrix} \boldsymbol{M}_{bb}^{\boldsymbol{\Omega_e}} & \boldsymbol{M}_{be}^{\boldsymbol{\Omega_e}} \\ \boldsymbol{M}_{eb}^{\boldsymbol{\Omega_e}} & \boldsymbol{M}_{ee}^{\boldsymbol{\Omega_e}} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{\boldsymbol{b}} \\ \ddot{\boldsymbol{u}}_{\boldsymbol{e}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_{bb}^{\boldsymbol{\Omega_e}} & \boldsymbol{C}_{be}^{\boldsymbol{\Omega_e}} \\ \boldsymbol{C}_{eb}^{\boldsymbol{\Omega_e}} & \boldsymbol{C}_{ee}^{\boldsymbol{\Omega_e}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_{\boldsymbol{b}} \\ \dot{\boldsymbol{u}}_{\boldsymbol{e}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{bb}^{\boldsymbol{\Omega_e}} & \boldsymbol{K}_{be}^{\boldsymbol{\Omega_e}} \\ \boldsymbol{K}_{eb}^{\boldsymbol{\Omega_e}} & \boldsymbol{K}_{ee}^{\boldsymbol{\Omega_e}} \end{bmatrix} \begin{bmatrix} \boldsymbol{u_b} \\ \boldsymbol{u_e} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{P_b} \\ \boldsymbol{P_e} \end{bmatrix},$$

for the exterior domain Ω_e.

Here $\boldsymbol{M}$ denotes the mass matrix, $\boldsymbol{C}$ the damping matrix and $\boldsymbol{K}$ the stiffness matrix.

Then, we pass to STEP 1, in which the *auxiliary domain* is taken into consideration. This time the nodal displacements $\boldsymbol{u_e^0}$, $\boldsymbol{u_b^0}$, $\boldsymbol{u_i^0}$, referred to as *free-field motion*, and the forces $\boldsymbol{P_b^0}$ are related to the new interior domain Ω_i^0 (see Figure 1, bottom left panel).

After some calculations and substitutions, it is possible to replace the seismic forces $\boldsymbol{P_e}$ on the fault by the effective nodal forces $\boldsymbol{P^{eff}}$, given by:

$$\boldsymbol{P}^{eff} = \begin{bmatrix} \boldsymbol{P}_i^{eff} \\ \boldsymbol{P}_b^{eff} \\ \boldsymbol{P}_e^{eff} \end{bmatrix} = \begin{bmatrix} 0 \\ -\boldsymbol{M}_{be}^{\Omega_e}\ddot{\boldsymbol{u}}_e^0 - \boldsymbol{C}_{be}^{\Omega_e}\dot{\boldsymbol{u}}_e^0 - \boldsymbol{K}_{be}^{\Omega_e}\boldsymbol{u}_e^0 \\ \boldsymbol{M}_{eb}^{\Omega_e}\ddot{\boldsymbol{u}}_b^0 + \boldsymbol{C}_{eb}^{\Omega_e}\dot{\boldsymbol{u}}_b^0 + \boldsymbol{K}_{eb}^{\Omega_e}\boldsymbol{u}_b^0 \end{bmatrix}. \quad (1)$$

We note that in the previous computation are involved only the matrices $\boldsymbol{M_{be}}$, $\boldsymbol{M_{eb}}$, $\boldsymbol{C_{be}}$, $\boldsymbol{C_{eb}}$, $\boldsymbol{K_{be}}$ and $\boldsymbol{K_{eb}}$, that are defined on the exterior domain Ω_e. The latter entails that these effective forces do not vanish only on a single layer of elements in Ω_e and adjacent to Γ_i. The effective forces thus constitute the means by which seismic excitation is introduced into the

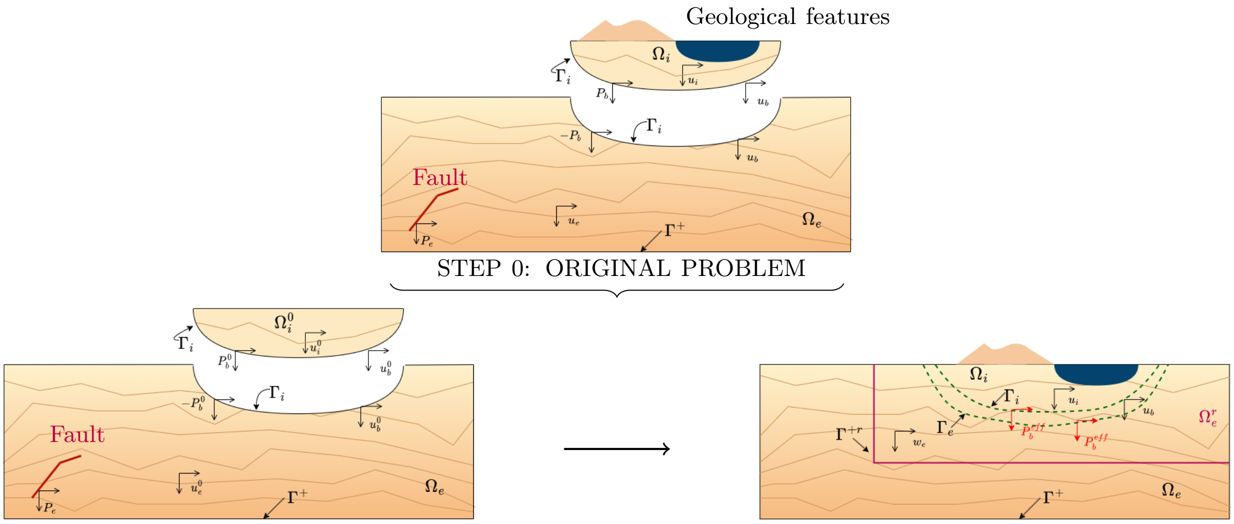

Figure 1: Domain Reduction Method (DRM) two-step procedure: the original problem is subdivided into two numerical submodels, namely the auxiliary problem (STEP 1) and the reduced problem (STEP 2). Adapted from [1] and [4].

reduced domain as an equivalent dynamic excitation (STEP 2).
Moreover, since in the outer domain Ω_e all the waves are outgoing, the external domain's size can be significantly reduced to get a smaller exterior domain Ω_e^r for solving equations of motion, provided suitable absorbing boundaries are used to reduce the occurrence of spurious waves (see Figure 1, bottom right panel). It is exactly for this reason that the method is called *Domain Reduction Method.*

3. Implementation of the DRM

We proceed with the implementation of the Domain Reduction Method within the code SPEED. This code, particularly suitable for simulating seismic wave propagation in visco-elastic heterogeneous three-dimensional media, is based on the spectral element method (SEM) coupled with the Discontinuous Galerkin (DG) technique (for further details see [3]).

By exploiting the SEM, it is possible to pass from the differential form of the wave equation to the following system of ordinary differential equations with respect to time:

$$\boldsymbol{M}\ddot{\boldsymbol{U}}(t) + \boldsymbol{K}\boldsymbol{U}(t) = \boldsymbol{F}^{ext}(t) + \boldsymbol{T}(t).$$

Here $\boldsymbol{U}$ is the vector of nodal displacements, $\boldsymbol{M}$ is the mass matrix and $\boldsymbol{K}$ is the stiffness matrix. Moreover, vectors $\boldsymbol{F}^{ext}$ and $\boldsymbol{T}$ represent the contributions of external forces and tractions, respectively. See [2] for more details.

Thanks to this formulation, one important simplification occurs in the computation of the effective forces, cf. (1):

$$\boldsymbol{P^{eff}} = \begin{bmatrix} \boldsymbol{P_i^{eff}} \\ \boldsymbol{P_b^{eff}} \\ \boldsymbol{P_e^{eff}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\boldsymbol{K}_{be}^{\Omega_e}\boldsymbol{u_e^0} \\ \boldsymbol{K}_{eb}^{\Omega_e}\boldsymbol{u_b^0} \end{bmatrix}. \qquad (2)$$

Indeed, the terms related to the extra-diagonal components of the mass matrix $\boldsymbol{M}$ are 0 because the SE mass matrix is diagonal. Moreover, without loss of generality, we assume the viscous terms C_{ij} in (2) to be understood.

Finally, looking at equation (2), we can state that the effective forces only depend on the stiffness matrix related to the DRM elements, namely the elements at the interface between Γ_i and Γ_e (see Section 2), and on the free-field displacements obtained from the auxiliary problem. The above stiffness matrix can be calculated for each element *el* by exploiting the following formula:

$$\boldsymbol{K_{el}} = \int_{V_{el}} \boldsymbol{B^T}\boldsymbol{E}\boldsymbol{B}dV. \qquad (3)$$

Here $\boldsymbol{B}$ is the matrix containing the derivatives of the shape functions used for the approximation of the solution in SEM, $\boldsymbol{E}$ is the elastic constitutive matrix and V_{el} indicates the volume of the element *el*. In particular, the application of the DRM in three dimensions requires overall the allocation of a matrix $\boldsymbol{K_{DRM}}$ of dimensions

$[3Nel_{DRM}(N+1)^3, 3(N+1)^3]$, where Nel_{DRM} is the number of elements composing the effective boundary.

For the computation of the *free-field motion*, instead, the DRM can be coupled with different semi-analytical solutions for pressure (P) or shear (S) plane waves propagating in horizontally layered media with arbitrary angle of incidence. In this work, the analytical solutions are computed through the Haskell-Thomson (H-T) propagation matrix method. This method allows to minimize the errors caused by spurious reflections emanating from the absorbing boundaries, making it particularly appropriate for three dimensional applications.

Finally, making reference to equation (2), we are able to compute the effective forces by means of the stiffness matrix (see equation (3)) and the free-field motion at the interface nodes.

3.1. DRM for plane wave propagation

The DRM was implemented in SPEED to deal with plane waves propagation problems. This was accomplished by adapting the implementation done in [4] for the 3D parallel version of GeoELSE (*GeoELastodynamics by Spectral Elements*).

SPEED has been provided with a set of subroutines that can compute the effective nodal forces from the analytical free-field solution computed as above.
More precisely, we depict in Figure 2 all the steps of the algorithm, that are:

1. Definition of the input parameters: time dependence of the imposed plane displacement wavefront, mechanical properties of the background geological model that constitute the auxiliary problem, type of the plane wave source, angle of incidence γ.
2. Computation of the effective boundary nodes in which the effective boundary forces will be evaluated (Legendre-Gauss-Lobatto nodes).
3. Definition of a one-dimensional reference soil profile made of N_f layers in the vertical direction. Each layer is delimited by two adjacent non-coinciding LGL nodes.
4. Computation of the free-field displacement time histories at the (N_f+1) interfaces by means of the (H-T) matrix method for the 1D soil profile determined at previous point.
5. Computation of the free-field displacements at the entire set of effective boundary nodes of the reduced problem.

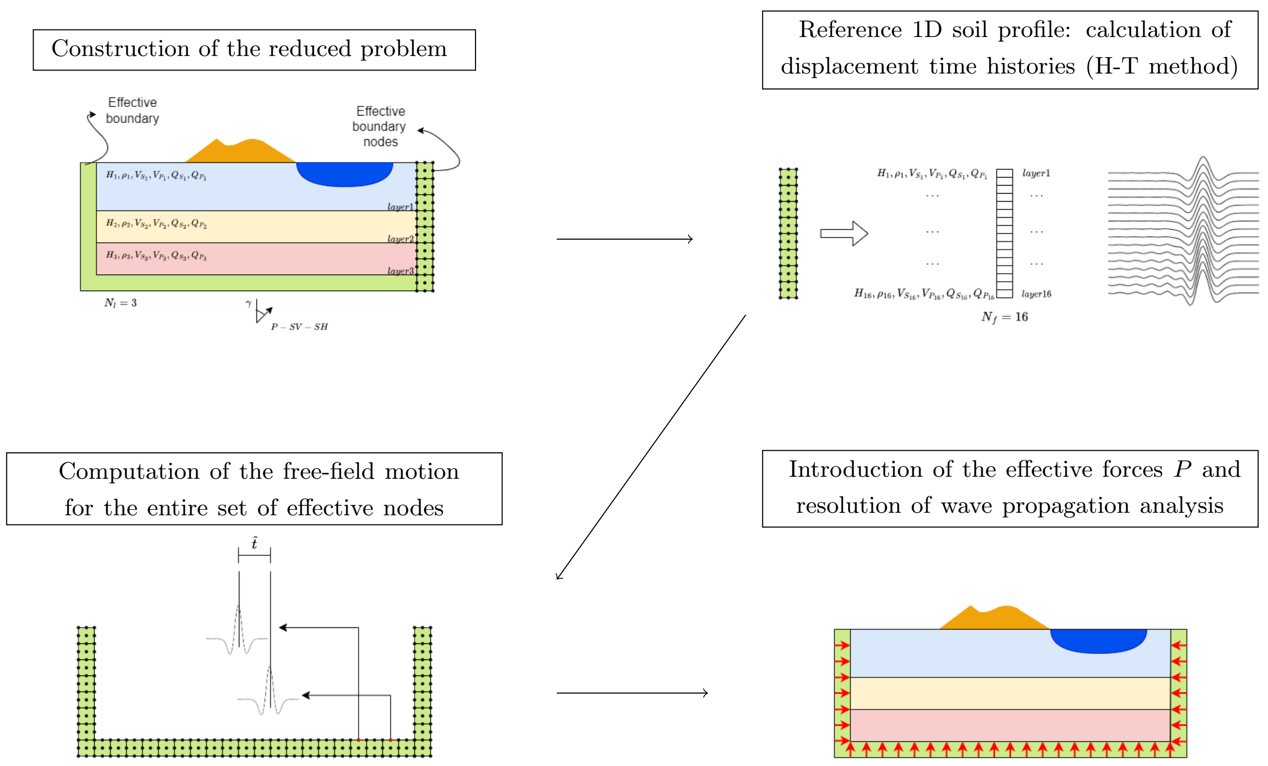

Figure 2: Plane wave propagation analyses in arbitrary complex media: sketch of the implementation. Adapted from [4].

6. Computation of the effective nodal forces $\boldsymbol{P}^{eff}$ exploiting the free-field displacements obtained at the previous step.

This approach for modeling plane wave propagation leads to several advantages with respect to the conventional approach, such as:

- Minimization of the spurious reflections due to the absorbing boundary conditions applied on the external boundary of the computational domain.
- Treatment of non orthogonal plane wave incidence to the free surface.

4. Input files generation

We focus on the generation of the input files necessary for the use of the SPEED library coupled with the Domain Reduction Method.

This process requires the following steps:

1. Creation of the spectral element (SE) reduced model using the software CUBIT (`https://cubit.sandia.gov/`) by means of quadrilateral and hexahedral elements. In order to apply the effective forces to the irregular structure of interest, the computational grid must include an appropriate strip of spectral elements, i.e. the effective DRM boundary. Fundamental, at this point, is the definition of the different blocks that characterise the domain under study. These blocks can be of different type: soil layers, DRM boundary elements, absorbing boundary conditions etc.
   Then, from the file generated with Cubit, it is possible to obtain the mesh file, that contains all the information on the grid composing the computational domain.
2. Generation of the Legendre-Gauss-Lobatto nodes, necessary for the computation of the effective forces. We recall that, in one dimension, the LGL nodes are defined as the roots of the first space derivatives of the Legendre polynomial of degree $N \geq 1$, beyond the two extremes ± 1 (see [2]). In order to compute the LGL nodes for the entire strip of DRM elements, the user has to run the Matlab script *read_grid_test.m*. The latter considers, in particular, a suitable tensorization process to compute the LGL nodes for each three-dimensional hexahedron.
3. Calculation of the PDRM nodes (effective DRM nodes) and FDRM functions (free-field displacements computed at PDRM). This is achieved by exploiting the Matlab scripts created in [5].
   Precisely, we make use of the Matlab interface program *input_4else_3D.m*. The latter considers as input, in addition to a file containing the input parameters of the model under study and a list of all the LGL coordinates, a file containing the free-field displacements at all interfaces of the 1D soil profile used for linear visco-elastic analyses. These are exactly the displacements calculated from the analytical free-field solution for *P-SV-SH* plane wave propagation obtained through the Haskell-Thomson (H-T) matrix method.
   Exploiting this, the solution at all effective nodes is obtained by applying a suitable time shift depending on the assumed direction of propagation. Note that for $\gamma = 0$ (vertical incident plane wave), time shift is equal to 0 for all nodes.

Given this computation, the following files are required to start the simulation:

- `File mesh`. It contains all the information on the computational grid of the model.
- `File mate`. It contains all the features characterising the blocks that compose the reduced domain in addition to the PDRM and FDRM as computed at point 3. The latter allows to evaluate at run time the effective nodal forces necessary to propagate the target plane wavefront.
- `SPEED.input`. It is the header file in which are fixed the fundamental parameters of the analysis, and the files and directories that are used for the simulation.
- `LS.input`. It is the file containing all the coordinates of the monitored points in which the numerical solution is calculated.

One last consideration must be made on the choice of the time step Δt used for the simulation in SPEED. Since the procedure adopted for

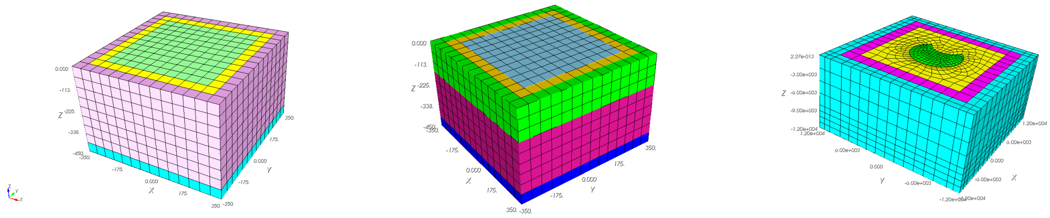

Figure 3: Computational domains used for the three tests: homogeneous model (left panel), heterogeneous model (center panel) and Croissant valley (right panel).

time discretization is the leap-frog scheme and, being this an explicit scheme, it is not unconditionally stable (see [3]). In particular, it must satisfy the Courant-Friedrichs-Levy (CFL) condition, that reads as follows:

$$\Delta t \leq C_{CFL} \frac{\Delta x_{min}}{V_{max}}. \tag{4}$$

In the expression above, C_{CFL} is a constant depending on the dimension, the order of the space discretization scheme, the mesh geometry and the polynomial degree. It takes value in between 0 and 1.
Δx_{min} represents the minimum distance between any couple of adjacent LGL nodes, while V_{max} is the maximum propagation velocity.

5. Tests and results

The implementation of the code SPEED coupled with the Domain Reduction Method (DRM) for plane wave propagation has been validated on three simple geological models to check the accuracy of the numerical results against semi-analytical solutions.

We report in Figure 3 the computational domains of the tests.

The first two tests are a homogeneous (single layer material) and a heterogeneous (double layers materials) models, respectively, and they are studied under the action of both a compressional P wave and a shear SV wave. We report here only the results relative to the P wave, since for the other wave type they are analogous.

We present in Figure 4 (top panel) the horizontal u_x and vertical u_z displacement time histories obtained due to the incidence of a vertical ($\gamma = 0°$) plane P wave relative to the first test. These displacements are evaluated in nine different monitored points, obtained starting from the center of the free surface and going down 25 m each time in direction z.

For a complete analysis, we make a comparison between the semi-analytical solution (computed with the Haskell-Thomson (HT) method and depicted as a blue continuous line), the numerical solution obtained with SPEED coupled with Domain Reduction Method (dashed green line), and the numerical solution obtained with SPEED without the implementation of DRM (dashed red line).

Then, in Figure 4 (bottom panel) we plot the results obtained with the same conditions, except for the angle of incidence γ, now set equal to 10°.
This time, the numerical solution obtained with SPEED without DRM is not taken into consideration, since it is not possible to compute it for the case of inclined wave.
Comparing the two cases (vertical and inclined plane wave), the results are very good in both

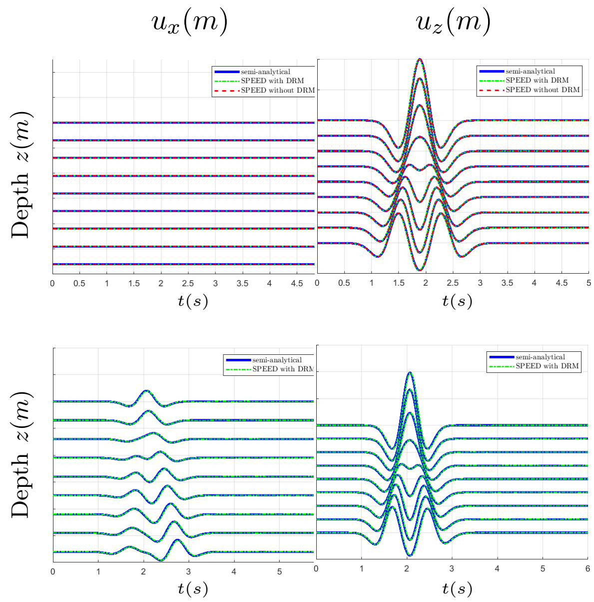

Figure 4: **Homogeneous model**: horizontal displacement u_x (left panel) and vertical displacement u_z (right panel) due to the vertical ($\gamma = 0°$) (bottom panel) and inclined ($\gamma = 10°$) incidence of a P plane wave.

cases, since the numerical solution obtained with the DRM is almost superimposed with the semi-analytical one. However, in the second case, some minimal spurious effects probably due to Absorbing Boundary Conditions (ABCs) arise.

Next, we consider in Figure 5 the results relative to the heterogeneous case (cf. Figure 3, middle panel). These are obtained analogously to the previous one, namely under the incidence of both a vertical and an inclined ($\gamma = 10°$) plane wave.

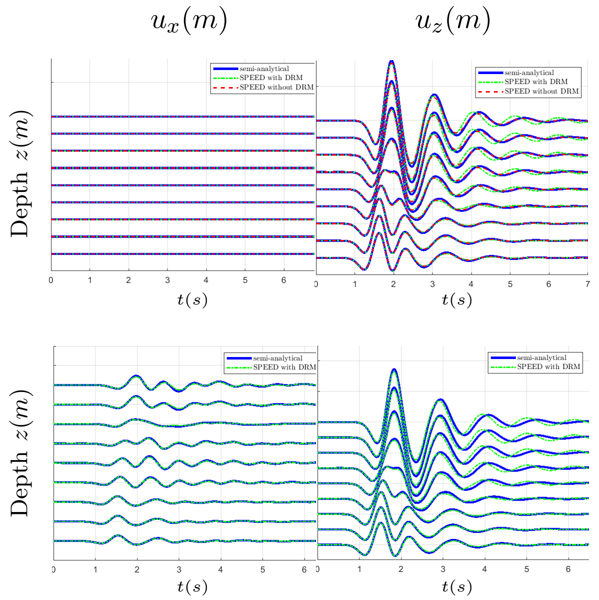

Figure 5: **Heterogeneous model**: horizontal displacement u_x (left panel) and vertical displacement u_z (right panel) due to the vertical ($\gamma = 0°$) (bottom panel) and inclined ($\gamma = 10°$) incidence of a P plane wave.

This time, in both cases, the solution obtained with SPEED coupled with DRM coincides with the semi-analytical solution mostly in the first part of the simulation (before $\approx 3.5 sec$), after that it tends to dampen more slowly.

In particular, this is more evident for the monitored points located closest to $z = 0$ and this is probably due to the change of values of the quality factors between the two physical layers of the model. However, in general, we can say that the comparison between the two solutions is quite good.

It is worth noticing that, in this case, the numerical solution obtained with the traditional implementation of SPEED, namely without DRM, is affected a lot from the spurious effects of the ABCs, differently from what happens for SPEED coupled with DRM. For this reason, in order to obtain the results shown in Figure 5, the model used for the simulation of SPEED without DRM was equipped with Dirichlet boundary conditions on the lateral surface.

The third test case concerns the analysis of a three-dimensional alluvial valley under the action of a vertical incident ($\gamma = 0°$) shear SH plane wave. We analyse the results in terms of x and y displacements components for a series of monitored points located on the free-surface of the model. Precisely, 98 points laying on the $x - y$ plane for $z = 0$ have been taken into considerations: 49 along the x-axis and 49 along the y-axis. Each point is distant from the adjacent one on the same axis by a distance of 320 m. In Figure 6 we report a comparison between the semi-analitycal solution obtained by indirect boundary element method (IBEM), the numerical solution computed by SPEED coupled with DRM and the one obtained with SPEED without DRM.

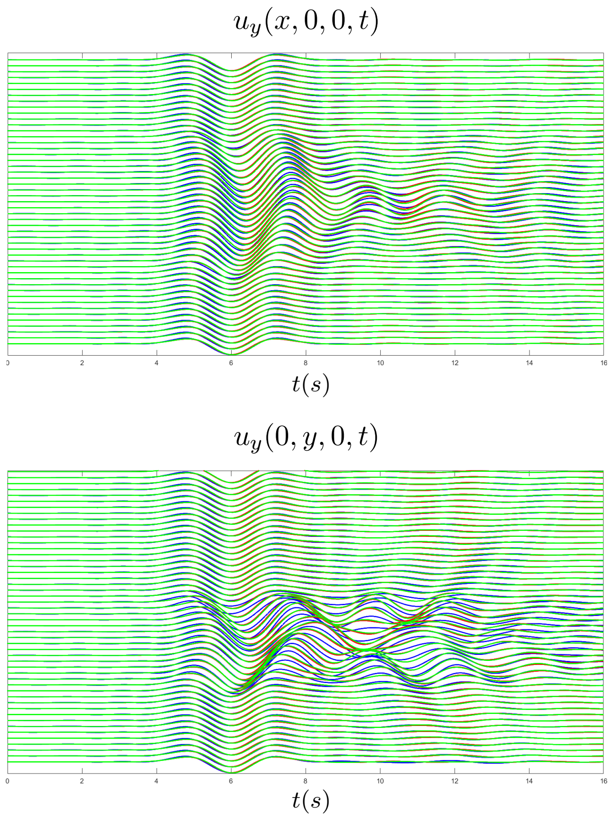

Figure 6: **Croissant valley**: x component (top panel) and y component (bottom panel) of the u_y displacement corresponding to the surface monitored points along x axis (top panel) and y axis (bottom panel) due to the incidence of a vertical ($\gamma = 0°$) plane SH wave.

Looking at the results, we can state that there is a good correspondence in the three solutions, specially for what concerns the x component of the displacement. Once again, the good fit between the two numerical solutions obtained with SPEED proves the capability of the DRM of not being affected from impurities caused by ABCs.

6. Conclusions

The analysis of the results leads to highlight some main advantages of using the DRM. First of all, DRM seems more robust with respect to numerical spurious effects due to ABCs. Moreover, DRM allows to reduce a lot the size of the computational domain, thus permitting to decrease the computational cost of the simulation. Moreover, this sub-structuring method allows to the possibility of performing simulations in which the plane wave is not vertical, in addition to the possibility to perform parametric analyses with respect to different input motions, including accelerograms derived from seismic hazard analysis.

Possible future developments include the following improvements. We consider to analyze further cases, for example with a different seismic input type or considering more complex geometry.
Moreover, in the light of the obtained results, some precautions need to be made in the implementation of the method. Indeed, especially if we look at the results obtained for the heterogeneous test, the numerical solution obtained with SPEED coupled with the DRM is not perfectly capable of modeling the transition from one material to another, and this produces spurious effects.
Another important future development regards the implementation of the three-dimensional parallel version of the code, currently implemented only in sequential version. This will lead, above all, to an advantageous reduction of the computational time.

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