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Executive Summary of the Thesis

Concentration phenomena for some spectral optimization problems

Laurea Magistrale in Mathematical Engineering - Ingegneria Matematica

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. We deal with problems of the kind

$$\begin{cases} -\Delta u = \lambda^1 m u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

The quantity λ^1 denotes the principal positive eigenvalue of (1) and $m(\mathbf{x})$ is a bang-bang weight, i.e. a function of the form

$$m(\mathbf{x}) = \overline{m}\mathcal{X}_E - \underline{m}\mathcal{X}_{\Omega\setminus E},$$

where $\overline{m}, \underline{m}$ are two positive constants and $E \subset \Omega$ is Lebesgue measurable.
Such problems arise in the context of the heterogeneous Fisher-KPP model in population dynamics, where Ω represents the habitat, the function u describes the population density, and the weight $m(\mathbf{x})$ describes the favourability of the habitat: there exist a region E which is favourable for the population, and vice-versa for $\Omega \setminus E$. The homogeneous Dirichlet boundary conditions model the case of a habitat surrounded by a completely hostile region.
It has been proved by Cantrell and Cosner in [1] that the lower is $\lambda^1 = \lambda^1(E, \Omega)$, the higher are the chances of population survival for long time. Moreover, there exist optimal favourability patterns of bang-bang type, i.e. that minimize λ^1. This is the main reason why we concentrate on bang-bang weights only.
The questions we address in our work are the *shape* and the *positioning* of the favourable regions inside of Ω. In particular, imposing a volume constraint on their measure, denoted with ε, we study such questions in the singularly perturbed asymptotic limit $\varepsilon \to 0^+$.
We study qualitatively the shape via blow-up techniques inspired by Mazzoleni, Pellacci and Verzini [4]. We are able to prove that, asymptotically, the favourable region is connected and its boundary is squeezed between two concentric spheres of collapsing radii. Namely, the shape of the favourable region is asymptotically a ball.
For what concerns the positioning, we observe that in the asymptotic regime the favourable regions concentrate at some point of Ω. Nonetheless, the study of their position is quite delicate. Due to a boundary effect, there is an interplay between the shape and the position of the favourable regions. This implies that, to obtain (sharp) results on the positioning, (sharp) quantitative estimates on the spherical asymmetry of the favourable regions are needed.
To tackle this problem, we combine projection and vanishing viscosity techniques developed by Ni and Wei in [5] with non-sharp quantitative isocapacitary estimates by Fusco, Maggi and Pratelli in [2]. Doing so, we are able to ob-

tain a non-sharp result on the positioning of the concentration points, which bounds from below their distance from $\partial\Omega$.
To get acquainted with the general problem, we carry out a preliminary analysis of the case in which the favourable regions are assumed *a priori* to be spherical. In this simpler case, exploiting analogous techniques, we are able to obtain that the positioning happens at a point at maximum distance from $\partial\Omega$. Further details on the analysis with spherical favourable regions is contained in the following section.

2. The case of spherical favorable regions

The study of the case with spherical favourable regions, can be seen as an intermediate step for the understanding of the original problem with generic weights. Notice that in this simplified version, the only question to be addressed is the positioning of the favourable regions.
The introduction of the asymptotic limit $\varepsilon \to 0^+$ allows us to perform the analysis via a blow-up technique. As usual in this procedure, it is fundamental to recognise a limit problem and study its properties. This work has been undertaken by Mazzoleni, Pellacci and Verzini in [4], and is summarized in the following

Theorem 2.1 ([4]). *Consider the class $\mathcal{M}'$ of bang-bang weights over $\mathbb{R}^N$ with positive part supported on a unitary measure set. The quantity*

$$\tilde{\lambda}_0 := \inf_{m\in\mathcal{M}'} \inf_{\substack{u\in H^1(\mathbb{R}^N)\\ \int_{\mathbb{R}^N} mu^2>0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\int_{\mathbb{R}^N} mu^2},$$

is a positive minimum, attained uniquely, up to a translation by a spherical bang-bang weight, and up to scaling by a function $w \in H^1(\mathbb{R}^N)$. Such function can be chosen positive, radially symmetric and strictly decreasing.

To proceed in our discussion, we denote with u_ε a family of eigenfunctions corresponding to the optimized eigenvalues

$$\lambda_\varepsilon := \inf_{B^\varepsilon(\mathbf{x})\in\Omega} \lambda^1(B^\varepsilon(\mathbf{x}), \Omega),$$

where $(B^\varepsilon(\mathbf{x})$ is the ball of measure ε and centered at $\mathbf{x}$). Then, we introduce the blow-up family of functions $\tilde{u}_\varepsilon$, correspondent to u_ε, and the blow-up family of eigenvalues $\tilde{\lambda}_\varepsilon$, correspondent to λ_ε.
The first result we prove is qualitative, and concerns the convergence of the blow-up family of eigenfunctions and eigenvalues to the ones of the limit problem. In particular, making use of the famous concentration-compactness principle of P.L. Lions [3], we prove the following

Proposition 2.1. *For any sequence ε_n there exists a subsequence, still denoted with ε_n, such that:*
- *(i)* $\tilde{\lambda}_{\varepsilon_n} \to \tilde{\lambda}_0$,
- *(ii)* $\tilde{u}_{\varepsilon_n} \to w$ *in* $C^{1,\alpha}(K)$ *for* $n \to +\infty$, *for any* $0 < \alpha < 1$ *and* $K \subset \mathbb{R}^N$ *compact set,*
- *(iii)* $\tilde{u}_{\varepsilon_n} \to w$ *in* $H^1(\mathbb{R}^N)$ *for* $n \to +\infty$.

We remark that the functions u_{ε_n} and w are chosen with the same $L^2(\mathbb{R}^N)$ normalization.
In order to obtain results on the positioning of the spherical favourable regions, however, such a qualitative result is not sufficient and quantitative results are needed. To this aim we make use of the projection and vanishing viscosity techniques developed by Ni and Wei in [5].
In order to state our next result, let us introduce the quantity $\beta_\varepsilon := \varepsilon^{-1/N}$, the centers of the spherical favourable regions $\mathbf{x}_\varepsilon \in \Omega$, the blow-up domains $\tilde{\Omega}_\varepsilon$, the function $P_{\tilde{\Omega}_\varepsilon} w$ which is the $H^1_0(\tilde{\Omega}_\varepsilon)$-projection of w and finally the function $\tilde{\Psi}_\varepsilon := -k_\varepsilon \log(w - P_{\tilde{\Omega}_\varepsilon} w)$. Then, we are able to prove the following

Theorem 2.2. *For any vanishing sequence ε_n there exists a subsequence still denoted with ε_n such that, for $n \to +\infty$:*
- *(i)* $d(\mathbf{x}_{\varepsilon_n}, \partial\Omega) \to \max_{\mathbf{p}\in\Omega} d(\mathbf{p}, \partial\Omega)$,
- *(ii)* $\tilde{\Psi}_{\varepsilon_n}(\mathbf{x}_{\varepsilon_n}) \to 2\sqrt{\tilde{\lambda}_0 \underline{m}} \max_{\mathbf{p}\in\Omega} d(\mathbf{p}, \partial\Omega)$,
- *(iii)* $\lambda_{\varepsilon_n} = \varepsilon_n^{-2/N}\left(\tilde{\lambda}_0 + (\Phi + o(1))\, e^{-\beta_{\varepsilon_n}\tilde{\Psi}_{\varepsilon_n}(\mathbf{x}_\varepsilon)}\right)$, *where Φ is a positive constant.*
- *(iv)* $\tilde{u}_{\varepsilon_n} = P_{\tilde{\Omega}_{\varepsilon_n}} w + e^{-\beta_{\varepsilon_n}\tilde{\Psi}_{\varepsilon_n}(\mathbf{x}_{\varepsilon_n})}\phi_{\varepsilon_n}$, *with* $\phi_{\varepsilon_n} \in H^2(\tilde{\Omega}_{\varepsilon_n})$ *and* $\|\phi_{\varepsilon_n}\|_{H^2(\tilde{\Omega}_{\varepsilon_n})} \leq C$ *uniformly for* $n \to +\infty$.

Without going deep into the details, point (i) answers the positioning question: in particular, it states that the favourable regions concentrate, in the asymptotic limit $\varepsilon \to 0^+$ at a point realizing the maximum distance prof $\partial\Omega$. Points (iii) and (iv) display asymptotic expansions, up to the second order, of the optimized eigenvalues and of the blow-up eigenfunctions. An important, but somehow more hidden information, is obtained combining points (i), (ii) and (iii), namely the information concerning the positioning of the favourable regions is contained in the second order of the asymptotic expansion of the eigenvalue.

3. The case of generic favorable regions

To study the case of generic favourable region, we follow a similar strategy to the case of spherical favourable regions, namely we proceed with a blow-up argument.
Notice however that, since the shape of the favourable regions is not fixed a priori, now there are three types of result to be obtained: the first type concerns the asymptotic behaviour of the eigenvalues, the second type concerns the asymptotic behaviour of the eigenfunctions while the third type concerns the behaviour of the optimal favourable regions.

Remark 3.1 (on notations)**.** *During this section, we introduce some quantities that also have their counterparts in the case of spherical weights (e.g. the blow-up points). Even though in many cases the notations are identical, in the course of this section they always refer to the case of generic favourable regions, unless otherwise explicitly stated.*

We introduce a family of eigenfunctions u_ε, corresponding to the optimized eigenvalues

$$\lambda_\varepsilon := \inf_{\{E\subset\Omega:\mathcal{L}(E)=\varepsilon\}} \lambda^1(E,\Omega),$$

attained by the optimal favourable regions E_ε.
Then, we denote with $\tilde{u}_\varepsilon$, $\tilde{\lambda}_\varepsilon$ and E_ε the blow-up families of eigenfunctions, eigenvalues and favourable regions, corresponding respectively to u_ε, λ_ε and E_ε.
The first result that we state is the counterpart of Proposition 2.1 for the case of generic weights.

Proposition 3.1. *For any vanishing sequence ε_n there exists a subsequence still denoted by ε_n, such that:*
- *(i)* $\lambda_{\varepsilon_n} \to \lambda_0$,
- *(ii)* $\tilde{u}_{\varepsilon_n} \to w$ *in* $C^{1,\alpha}(K)$ *for* $n \to +\infty$, *for any* $0<\alpha<1$ *and* $K \subset \mathbb{R}^N$ *compact set,*
- *(iii)* $\tilde{u}_{\varepsilon_n} \to w$ *in* $H^1(\mathbb{R}^N)$ *for* $n \to +\infty$.

Our second result concerns the qualitative asymptotic behaviour of the optimal favourable regions (in the blow-up scale). Let α_0 be the unique positive real number such that $\mathcal{L}(\{w > \alpha_0\}) = 1$. Then, exploiting techniques similar to those adopted by Mazzoleni, Pellacci and Verzini in [4], we obtain the following

Theorem 3.1. *For any sequence ε_n there exists a subsequence still denoted with ε_n such that, up to a set of zero Lebesgue measure:*
- *(i)* $\tilde{E}_{\varepsilon_n}$ *is a connected open set, for n sufficiently large,*
- *(ii) There exists a sequence of positive real numbers $\delta_n \to 0^+$ such that $\partial\tilde{E}_{\varepsilon_n}$ is contained in a δ_n-neighborhood of the α_0 level set of w.*

Hence, the optimal favourable regions are asymptotically connected and almost spherical. This result assesses qualitatively the question on the shape.
For what concerns the positioning of the optimal favourable regions, the matter is more delicate. Indeed, due to an interplay between shape and positioning caused by $\partial\Omega$, in order to obtain results on the positioning we need quantitative estimates on the spherical asymmetry of $\tilde{E}_{\varepsilon_n}$.
To this aim, we exploit the techniques developed by Fusco, Maggi and Pratelli in [2] for the isocapacitary case. In particular, introducing the Fraenkel asymmetry of the optimal favourable regions, defined as

$$\mathcal{A}(\tilde{E}_\varepsilon) := \inf_{\mathbf{x}\in\mathbb{R}^N} \mathcal{L}\left(\tilde{E}_\varepsilon \Delta(\mathbf{x}+B^1)\right),$$

the spherical symmetrization $\tilde{\Omega}_{\varepsilon,*}$ of $\tilde{\Omega}_\varepsilon$, i.e. a ball centered at the origin and having the same Lebesgue measure as $\tilde{\Omega}_\varepsilon$, and its associated optimized eigenvalue $\tilde{\mu}_\varepsilon$ defined as

$$\tilde{\mu}_\varepsilon := \inf_{\{E\subset\tilde{\Omega}_{\varepsilon,*}:\mathcal{L}(E)=\varepsilon\}} \lambda^1(E,\tilde{\Omega}_{\varepsilon,*}),$$

we prove the following

Theorem 3.2. *For any sequence $\varepsilon_n \to 0^+$ there exists a positive constant $C = C(N)$ independent of n, such that*

$$\mathcal{A}(E_{\varepsilon_n})^4 \le C(N)\left(\frac{\tilde{\lambda}_{\varepsilon_n}}{\tilde{\mu}_{\varepsilon_n}} - 1\right). \tag{2}$$

We remark that the right hand side of (2), can be estimated by the second order in the asymptotic expansion of $\tilde{\lambda}_{\varepsilon_n}$, which, as we have seen in the spherical analysis, brings the information

about the positioning of the favourable regions. In other words, inequality (2) is fundamental since it bounds the spherical asymmetry of the favourable regions with a quantity depending on their positioning. Morally, this implies the reduction of a problem depending on both shape and positioning, to a problem depending only on the positioning.
Hence, denoting with $\mathbf{x}_\varepsilon$ the blow-up points for generic weights (always chosen inside the favourable regions) and combining the techniques of Ni and Wei in [5] in a similar fashion to the spherical problem with the asymmetry estimate (2), we are able to obtain the following non-sharp result on the positioning.

Theorem 3.3. *For any sequence $\varepsilon_n \to 0^+$ there exists a subsequence, still denoted with ε_n, such that*

$$d(\mathbf{x}_{\varepsilon_n}, \partial\Omega) \geq \frac{1}{4} \max_{\mathbf{p}\in\Omega} d(\mathbf{p}, \partial\Omega) + \varepsilon_n^{1/N} O(1) \quad (3)$$

for $n \to +\infty$.

Notice that the factor 1/4 appearing in (3) is exactly the same appearing at exponent in (2). Theorem 3.3 basically states that the optimal favourable regions concentrate, in the asymptotic limit, at points with distance greater then the inradius of Ω by a factor at least 1/4. This gives a partial answer to the positioning question.

4. Conclusions

The main conceptual result arising from our work is the fact that, in the context of the heterogeneous Fisher-KPP model with bang-bang favourability patterns and completely hostile surrounding regions, when the optimal favourable regions are small they prefer to be connected, almost spherical and to stay away from the boundary of the habitat, at a distance which is proportional to the inradius by a factor at least 1/4 (see Theorem 3.3).
From the mathematical point of view, on the other hand, the main result of our work is the development of a strategy which allows to find asymptotic quantitative estimates on the positioning (and thus indirectly also on the shape) for the small optimal favourable regions when there is an interplay between the two, due to a boundary effect which is not negligible.
This is done combining the projection and vanishing viscosity techniques developed by Ni and Wei in [5] with the techniques developed for quantitative asymmetry estimates by Fusco, Maggi and Pratelli in [2].
Briefly, for what concerns future developments, a natural extension of our work is to obtain a sharp quantitative estimate on the spherical asymmetry of the favourable region.
Such a result and the techniques used to prove it, might pave the way for asymptotic *sharp* quantitative asymmetry estimates, at least when concentration phenomena occur. The main advantage is to obtain sharp asymmetry estimates (even though "only" asymptotic) even in the case of non-spherical ambient domains.

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