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On Kolmogorov-Fokker-Planck operators with linear drift and time dependent measurable coefficients

Tesi di Laurea Magistrale in
Mathematical Engineering - Ingegneria Matematica

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Abstract

The main objective of this thesis is to prove the well-posedness of a Cauchy problem for a particular class of Kolmogorov-Fokker-Planck nonhomogeneous equations with measurable time dependent coefficients. The class of operators we study could be viewed as an intermediate step to the case with coefficients which vary also in space. Actually, existence and uniqueness have been proved when the coefficients depend also on the space variable but under a quite restrictive condition on their regularity (see for instance [7]). Since we deal with measurable coefficients, even though we can employ an explicit fundamental solution (see [5]), the proof of the well-posedness needs a refined technique and is based on various results found in the paper [1].

Keywords: Kolmogorov-Fokker-Planck operators, nonhomogeneous Cauchy Problem, representation formulas, fundamental solution, Schauder estimates, compactness, Banach Alaoglu Bourbaki Theorem, Ascoli Arzelà Theorem

Abstract in lingua italiana

L'obbiettivo principale di questa tesi è di dimostrare la buona posizione di un problema di Cauchy per una classe di equazioni di Kolmogorov Fokker Planck non omogenee e con coefficienti misurabili dipendenti dal tempo. La classe di operatori studiati potrebbe essere vista come un passo intermedio per arrivare al caso in cui i coefficienti dipendono anche dalla variabile spaziale. In realtà, l'esistenza e l'unicità sono già state dimostrate per coefficienti dipendenti anche dalla variabile spaziale ma con assunzioni piuttosto restrittive sulla loro regolarità (per esempio si veda [7]). Siccome in questo caso si considerano coefficienti misurabili, anche se si ha a disposizione una soluzione fondamentale (si veda [5]), la dimostrazione della buona posizione necessita di una tecnica dimostrativa raffinata ed è basata sui vari risultati presenti nell'articolo [1].

Parole chiave: Operatori di Kolmogorov-Fokker-Planck, Problema di Cauchy non omogeneo, formule di rappresentazione, soluzione fondamentale, stime di Schauder, compattezza, Teorema di Banach-Alaoglu-Bourbaki

Contents

Introduction

The Kolmogorov Fokker Planck operator with linear drift is a differential operator of the form:

$$\mathcal{L} = \sum_{i,j=1}^{N} a_{ij}\partial_{ij} + \sum_{i,j=1}^{N} b_{ij}x_j\partial_i - \partial_t \tag{1}$$

where N is the number of space variables, $\mathbb{A} = \{a_{ij}\}_{i,j=1}^{N}$ is a matrix which may depend on (x,t) and is positive semidefinite while $\mathbb{B} = \{b_{ij}\}_{i,j=1}^{N}$ is a constant matrix.

Operators of this kind arise in the context of stochastic systems. Following [3] page 18, if we consider a stochastic system:

$$d\boldsymbol{x}(\tau) = \boldsymbol{b}(\boldsymbol{x}(\tau),\tau)d\tau + \boldsymbol{B}(\boldsymbol{x}(\tau),\tau)d\boldsymbol{w}(\tau); \quad \boldsymbol{x}(t) = \boldsymbol{x} \tag{2}$$

where $\boldsymbol{b}$ and $\boldsymbol{B}$ are a deterministic vector and matrix functions respectively and $\boldsymbol{w}$ represents an n-dimensional white noise, then the *transition probability density* $p = p(\boldsymbol{x},t,\boldsymbol{y},s)$ which is defined by

$$\mathbf{P}(\boldsymbol{x}(s) \in A | \boldsymbol{x}(t) = \boldsymbol{x}) = \int_A p(\boldsymbol{x},t,\boldsymbol{y},s)d\boldsymbol{y} \ ,$$

satisfies two partial differential equations. Indeed, if we define $\{a_{ij}\}_{i,j=1}^{N} = \boldsymbol{B}\boldsymbol{B}^T$, then $p(x,t;\cdot)$ satisfies the forward Kolmogorov equation (also called Fokker Planck equation), which is:

$$\partial_s p + \nabla_{\boldsymbol{y}} \cdot (\boldsymbol{b}p) - \frac{1}{2}\sum_{i,j=1}^{N} \partial^2_{y_i,y_j}(a_{ij}p) = 0 \tag{3}$$

while $p(\cdot;y,s)$ satisfies the backward Kolmogorov equation, which is:

$$\partial_t p + \boldsymbol{b} \cdot \nabla_{\boldsymbol{x}} p + \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}\partial^2_{x_i,x_j} p = 0 \ . \tag{4}$$

It is apparent that when the function $\boldsymbol{b}$ depends only on x in a linear way the equations (3) and (4) can be easily reduced to (1). Moreover, in many cases the matrix $\boldsymbol{B}\boldsymbol{B}^T$ has diagonal block structure with only the first block not identically zero and positive definite. This particular structure of the matrix is related to systems in which the white noise term appears only in a subset of the equations, for instance in the physical Brownian motion (Example 20 [3]) the white noise enters only the equations corresponding to the velocity of the particle. We refer to the survey book [3] (chapter 2) and the initial sections of the papers [16] and [7] for many other possible applications.

As a starting point for the study of the class of operators (1) it is interesting to consider the case of constant coefficients a_{ij}. Under this assumption the operator has many properties related to the structure of homogeneous group in $\mathbb{R}^N$ (see [4] chapter 3) which were first explained in the work by Lanconelli and Polidoro [17]. Moreover, still in the case of constant coefficients $\{a_{ij}\}_{i,j=1}^N$, the operator may show regular solutions whenever the datum is regular according to whether a certain condition between $\mathbb{A}$ and $\mathbb{B}$ is satisfied (Proposition A.1. [17]). A differential operator having this property is called hypoelliptic. Under the assumption of constant coefficients the hypoellipticity of (1) could be proved also through a smooth fundamental solution whose construction is sketched in the paper ([12]). Actually a smooth fundamental solution for the particular case $\partial_t + x\partial_y - \partial_x^2$ has been known since 1934 ([13], [3] page 6). After this first step ([17]) an extensive literature has been developed in the study of more general operators of this kind (see [3] section 5.1 and [16] section 1). As an example we mention the paper [7] by Di Francesco and Pascucci in which a fundamental solution is constructed (using the Levi parametrix method) for the operator:

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}(x,t)\partial_{ij}u + \sum_{i=1}^{q} a_i(x,t)\partial_i u + \sum_{i,j=1}^{N} b_{ij}x_j\partial_i u + c(x,t)u - \partial_t \tag{5}$$

under the assumption that $a_{ij}(x,t)$, $a_i(x,t)$ and $c(x,t)$ are Hölder continuous with respect to a certain quasi-distance on the homogeneous group (introduced in [17]). We remark also that a fundamental solution has been computed, under quite restrictive assumptions on the regularity of coefficients, for operators related to (1), also in [15], [14], [19], [23] and moreover a method for constructing a fundamental solution when $\{a_{ij}\}_{i,j}$ are constant is sketched in ([12]) and has been generalized, in the paper [5] by Bramanti and Polidoro, to the case of coefficients dependent on time in a nonsmooth way.

In this thesis we will focus on a particular case of (1) where the coefficients $\{a_{ij}\}_{i,j=1}^N$ depend on t in a nonsmooth way. There are several reasons for these assumptions. A first reason comes from the applications to stochastic systems, indeed, as remarked in [5] it is natural to assume the matrix $\boldsymbol{B}$ in (2) to be only measurable with respect to time. Another reason (see [18] Example 1.3 and [5]) is that the theory developed in [7] for the operator (5) assumes conditions which may be very restrictive. Moreover, the existence result of that paper requires the datum to be Hölder in space and L^∞ in time, therefore, it seems natural that a similar result could be proved for coefficients which are Hölder in space but only measurable in time. Hence, we may think to our assumptions as an intermediate step to this more general case. Actually, under the assumption of Hölder coefficients in space and measurable in time, in the article by Bramanti and Biagi [1], Schauder a priori estimates have been proved. However, existence is remained an open problem. In this thesis we shall prove the well-posedness of a nonhomogeneous Cauchy problem (Theorem 2.6), the existence for the nonhomogeneous Cauchy problem, some local estimates (Theorem 3.1), existence for unbounded datum satisfying a Gaussian bound (Theorem 3.6), a result which gives regularity of solutions (Theorem 3.3) and a uniqueness result for our definition (Theorem 3.4). We note that actually a result of existence and uniqueness for the homogeneous Cauchy problem has been proved in [5], hence it seems that there is no reason for the last two sections (sections 3.4 and 3.5), however we decided to add also these sections since they show a possible application of Theorem 3.1 and also since in [5] the definition of solution is given using classical derivatives while in this thesis we shall consider weak derivatives, therefore due to this difference we cannot apply the uniqueness result of [5].

The thesis is structured as follows:
Chapter 1 introduces the known results on the operator which we are interested in. Section 1.1 is a preliminary section which serves as an introduction to the context, in this section we shall recall the notion of hypoellittic operators and Hörmander's Theorem. Then in section 1.2 we deal with the case of constant coefficients presenting some material from [17]. Finally the chapter is ended by section 1.3 in which, following the article [5], we start the study of the operator with time depending coefficients a_{ij}. Chapter 2 is concerned with the Cauchy problem in which we are interested. It begins with section 2.1 in which we introduce the definition of solution and some preliminary results from [1] and [5], then section 2.2 deals with the existence for the case of regular datum while in section 2.3 with the help of some compactness results (the Banach-Alaoglu-Bourbaki Theorem A.2) and the Schauder estimates from [1], we obtain existence under minimal regularity assumptions on the datum. Finally in section 2.4 we combine the existence results from [5] to the one obtained in the thesis in order to obtain existence for the general Cauchy Problem with nontrivial initial datum. Chapter 3 introduces some extensions of the previous results. We begin with local estimates in section 3.1, then we pass to the existence for unbounded datum in section 3.2. Next, we consider the regularity of solutions and uniqueness in

section 3.3 and 3.4, respectively. Finally we conclude with well-posedness of the general Cauchy problem in section 3.5.

1 | The KFP operator with linear drift

Since in this thesis we are interested in proving the well posedness of a Cauchy problem when coefficients a_{ij} are L^∞ functions of time, after a preliminary discussion on the case of constant coefficients (see [17]), the last part of this chapter is devoted to the explanation of some results concerning the particular operator we are interested in. The references for these results are [5] and [1].

1.1. Hypoelliptic operators and Hörmander's Theorem

This section is devoted to hypoelliptic operators, in particular, we will introduce some ideas related to hypoelliptic operators and Hörmander's Theorem [12].

1.1.1. Hypoellipticity

Before defining the notion of hypoellipticity we shall define the singular support of a distribution.

Definition 1.1. *Let Ω be an open set of $\mathbb{R}^N$ and let $v \in D'(\Omega)$. The singular support of v, singsupp(v), is defined by:*

$$\text{singsupp}(v) = \Omega \setminus \{x \in \Omega : \exists U \in \mathcal{N}_x \, \exists w \in C^\infty(U) \quad v_{|U} = w \quad in \quad D'(\Omega)\}$$

($\mathcal{N}_x$ denotes the neighborhood system of x in Ω)

With this definition we can easily define hypoelliptic operators.

Definition 1.2. *A differential operator P, with possibly complex, $C^\infty(\Omega)$ coefficients, is* *said hypoelliptic if for every $u \in D'(\Omega)$:*

$$\operatorname{singsupp}(u) = \operatorname{singsupp}(Pu) \ .$$

That is, an operator P is hypoelliptic whenever for any given distribution u, if Pu is C^∞ in an open set, then u is C^∞ in the same open set. For operators with constant coefficients a simple criterion for the hypoellipticity is the following:

Theorem 1.1 (Theorem 1.2 [21]). *Let $\mathcal{L}$ be a differential operator with, possibly complex, constant coefficients. If there exists a fundamental solution which is $C^\infty(\mathbb{R}^N \setminus \{0\})$, then $\mathcal{L}$ is hypoelliptic.*

In the case of a differential operator L with constant coefficients, by fundamental solution we mean (see for instance [10]) a distribution $u \in D(\mathbb{R}^N)$ which satisfies:

$$Lu = \delta_0 \qquad D'(\mathbb{R}^N) \ .$$

We remark also that a similar notion could be defined also when the coefficients are of class C^∞. By definition any fundamental solution of an hypoelliptic operator is C^∞ outside the pole. Therefore, thanks to the Malgrange–Ehrenpreis Theorem, which states that any nontrivial operator with constant coefficients has a fundamental solution, the condition expressed by the previous theorem is also necessary for constant coefficient operators.

From now on we shall always consider real coefficients although hypoellipticity is defined for operators with possibly complex coefficients.

1.1.2. Hörmander's Theorem

In order to state the general result by Hörmander we need some preliminary definitions. Within the next sections capital letters represent C^∞ vector fields defined in Ω which denotes an open set in $\mathbb{R}^N$. A smooth vector field could be thought as C^∞ section of the tangent bundle or as derivation on the set of C^∞ functions. In other words we are thinking to the vector field $X : \Omega \to T(\Omega) \cong \mathbb{R}^N : x \mapsto (a_1(x), a_2(x), \ldots, a_N(x))$ as the first order differential operators with C^∞ coefficients

$$X = a_1\partial_1 + a_2\partial_2 + \cdots + a_N\partial_N \ .$$

This definition is particularly useful since, given any two vector fields X and Y, it is easy to define XY as the composition of the two derivations, that is, the differential operator which to any $f \in C^\infty(\Omega)$ associates $X(Y(f))$. Then, the commutator is easily defined by the standard formula:

$$[X, Y] = XY - YX \ . \tag{1.1}$$

Notice that the commutator is a vector field. Finally let $\mathcal{L}(X_0, X_1, \dots, X_q)$ represent the Lie algebra generated by the vectors X_0,. . . ,X_q which is the space generated by

$$X_{i_1}, [X_{i_1}, X_{i_2}], [X_{i_1}, [X_{i_2}, X_{i_3}]], \dots$$
$$\dots, [X_{i_1}, [X_{i_2}, \dots, [X_{i_{k-1}}, X_{i_k}]], \dots \qquad \text{with} \quad i_j \in \{1, \dots, q\}, \quad j \geq 1 \ .$$

We are now ready to state the theorem:

Theorem 1.2 (Hormander, 1967, [12]). *Let X_0,. . . ,X_q be (C^∞) vector fields on Ω, $c \in C^\infty$ and let $\mathcal{L}$ be the operator defined by*

$$\mathcal{L} = \sum_{i=1}^{q} X_i^2 + X_0 + c \ .$$

If the space $\mathcal{L}(X_0, X_1, \dots, X_q)$ has dimension N at any given point $x \in \Omega$, then $\mathcal{L}$ is hypoelliptic.

We remark, as it is done in [3] (page 11), that the operator $(\partial_y + 2x\partial_t)^2 + (\partial_x - 2y\partial_t)^2 - 4i\partial_t$ satisfies the Hörmander condition, but it is not hypoelliptic, hence Hörmander's Theorem cannot be extended in a trivial way to the case of complex coefficients. Moreover, it is interesting to notice that, in the case of real coefficients, the condition is almost necessary, indeed following the argument given in the original paper [12] whenever in a neighborhood of a point the space $\mathcal{L}(X_0, X_1, \dots, X_q)$ has constant dimension strictly less than N, owing to Frobenius Theorem, whenever there exists a nontrivial solution to the operator, a discontinuous solution could be constructed contradicting the hypoellipticity.

1.2. The KFP operator with constant coefficients

Now we turn our attention to the case of Kolmogorov-Fokker-Planck operators with linear drift and constant matrix $\mathbb{A}$.

1.2.1. Assumptions on the coefficients

As remarked in [17], Hormander's Theorem give necessary and sufficient conditions on $\mathbb{A}$ and $\mathbb{B}$ for the hypoellipticity of the operator (1) with constant coefficients. Assuming that $\mathbb{A}$ is positive semidefinite, which is a necessary condition thanks to a result still in

[12], denoting the square root of $\mathbb{A}$ by $\mathbb{A}^{\frac{1}{2}} = \{\tilde{a}_{ij}\}_{i,j=1}^{N}$ consider the vector fields:

$$\tilde{X}_i = \sum_{j=1}^{N} \tilde{a}_{ij}\partial_j, \qquad Y = \sum_{i,j=1}^{N} b_{ij}x_j\partial_i - \partial_t = \mathbb{B}x \cdot \nabla - \partial_t \ .$$

It is easily prooved that the operator (1), when coefficients a_{ij} are constant, is equal to

$$\sum_i \tilde{X}_j^2 + Y$$

so the condition on the vector fields $\tilde{X}_j$ and Y gives the conditons on the matrices $\mathbb{A}$ and $\mathbb{B}$ for the hypoellipticity of $\mathcal{L}$. Now let $X_i = \sum_{1\leq j\leq N} a_{ij}\partial_{ij}$, then it easily seen that $\mathcal{L}(X_1, \ldots, X_N, Y) = \mathcal{L}(\tilde{X}_1, \ldots, \tilde{X}_N, Y)$, hence as remarked in [17] the condition for the hypoellipticity could be expressed as:

$$dim(\mathcal{L}(X_1, \ldots, X_N, Y)) = N \ . \tag{H}$$

Moreover, the condition above is proved to be equivalent to each of the two following conditions (see [Proposition A.1 [17]]):

$$Ker(\mathbb{A}) \text{ does not contain any nontrivial subspace which is invariant for } \mathbb{B} \tag{H'}$$

and

$$C(t) > 0 \qquad \forall t > 0 \ , \tag{H''}$$

where the matrix C is defined by:

$$C(t) = \int_0^t E(s)\mathbb{A}E(s)^T ds \qquad , \qquad E(t) = e^{-t\mathbb{B}} \ . \tag{1.2}$$

The matrix C plays an essential role also in the case of nonconstant coefficients, since it enters the explicit formula for the fundamental solution. These conditions lead to a particular form of the matrices $\mathbb{A}$ and $\mathbb{B}$, more precisely by section 2 of [17], whenever the condition (H) is satisfied there exists an ortonormal base in $\mathbb{R}^N$ such that after a change of variables the matrices of the operator assume the form:

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_0 & \mathbb{O} & \dots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} \end{pmatrix} \qquad \mathbb{B} = \begin{pmatrix} * & * & \dots & * & * \\ \mathbb{B}_1 & * & \dots & * & * \\ \mathbb{O} & \mathbb{B}_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & \mathbb{B}_r & * \end{pmatrix} \tag{1.3}$$

where $\mathbb{A}_0$ is a symmetric positive definite matrix of order q and there exists a sequence of integers $q = m_0 \geq \cdots \geq m_j \geq \cdots \geq m_\kappa \geq 1$ with sum equal to N such that $\mathbb{B}_j$ is a $m_j \times m_{j-1}$ matrix of maximal rank ($r(\mathbb{B}_j) = m_{j-1}$) while the $*$ blocks represent arbitrary matrices. In the remaining part of this section the matrices $\mathbb{A}$ and $\mathbb{B}$ are assumed to satisfy these conditions.

1.2.2. The fundamental solution for the constant coefficients operator

An explicit expression for the fundamental solution with pole $(y, s) \in \mathbb{R}^{N+1}$ for this particular case has been known since the articles by Kuptsov [15], [14]. The fundamental solution we consider is the following:

$$\Gamma(x, t; y, s) = \frac{e^{-\frac{1}{4}(x-E(t-s)y)^t C(t-s)^{-1}(x-E(t-s)y)} e^{-(t-s)tr(\mathbb{B})}}{\sqrt{(4\pi)^N det(C(t-s))}}) \, . \tag{1.4}$$

In this case, in order to define the notion of fundamental solution we can exploit the distributional framework as for the case of operators with constant coefficients: $\Gamma(\cdot; \xi)$ is a fundamental solution with pole $\xi \in \mathbb{R}^{N+1}$ if

$$\mathcal{L}\Gamma(\cdot; \xi) = -\delta_\xi \quad \text{in } D'(\mathbb{R}^{N+1}) \, .$$

This approach is no more available when the coefficients are not constant and only measurable.

The properties of the fundamental solution (1.4) are essential in order to study the case with nonconstant coefficients, but before showing some properties of Γ we need to introduce some definitions related to homogeneous groups on $\mathbb{R}^N$, we refer to [4] chapter 3 for a complete treatment. Following the article by Lanconelli and Polidoro [17] we define the group law $\circ$ on $\mathbb{R}^N$ as follows:

$$(x,t) \circ (y,s) = (y + E(s)x, t+s) \ . \tag{1.5}$$

It is apparent that $\circ$ is a noncommutative group law and that:

$$(y,s)^{-1} = (-E(-s)y, -s) \quad \text{and} \quad (y,s)^{-1} \circ (x,t) = (x - E(t-s)y, t-s)$$

therefore $\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$ is a noncommutative group which is actually a Lie group. The operator $\mathcal{L}$ is invariant with respect to the left translation:

$$\forall \xi \in \mathbb{R}^{N+1} \quad L_\xi \mathcal{L} = \mathcal{L} L_\xi$$

where $L_\xi u(\eta) = u(\xi \circ \eta)$. We observe also that the group law allows us to define a convolution (see [4] ch. 3 section 3.4):

$$u * v(x,t) = \int u(y,s) v((y,s)^{-1} \circ (x,t)) dyds$$

which is such that for any $f \in C_c^\infty(\mathbb{R}^N)$ and any $v \in L^1(\mathbb{R}^N)$ (for a better explanation and sharp result see [4] Proposition 3.46)

$$\mathcal{L}(v * f) = v * (\mathcal{L} f) \ .$$

That is $\mathcal{L}$ is left invariant w.r.t. the convolution. Thanks to the fact that $\mathcal{L}$ is invariant with respect to the left translations, it is easily seen that:

$$\Gamma(x,t;y,s) = \Gamma((y,s)^{-1} \circ (x,t); 0, 0) = \gamma((y,s)^{-1} \circ (x,t))$$

where $\gamma(\cdot) = \Gamma(\cdot; (0,0))$ is the fundamental solution with pole at the origin.
Another important concept is that of dilations on the group $\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$:

$$D(\lambda) = diag(\lambda I_{m_0}, \lambda^3 I_{m_1}, \dots, \lambda^{2\kappa+1} I_{m_\kappa}, \lambda^2)$$

where I_m represents the identity matrix of order m. Notice that the function $D(\lambda)$ is also an automorphism of $\mathbb{G}$:

$$D(\lambda)((x,t) \circ (y,s)) = D(\lambda)(x,t) \circ D(\lambda)(y,s) \ . \tag{1.6}$$

For convenience we define also:

$$D_0(\lambda) = diag(\lambda I_{m_0}, \lambda^3 I_{m_1}, \dots, \lambda^{2\kappa+1} I_{m_\kappa}) \ .$$

We say (according to [17], see also [4] chapter 3) that the operator $\mathcal{L}$ is invariant with respect to a group of dilations $\mathcal{G} = \{\lambda^{\mathbb{M}}\}_{\lambda>0}$ with $\mathbb{M}$ symmetric and positive defined if for any $u \in D(\mathbb{R}^{N+1})$ and any $(x,t) \in \mathbb{R}^{N+1}$

$$\forall \lambda \geq 0 \quad \mathcal{L}u(\lambda^{\mathbb{M}}(x,t)) = \lambda^2 (\mathcal{L}u)(\lambda^{\mathbb{M}}(x,t)) \ .$$

Then, it is proved in [17] that the operator $\mathcal{L}$ is invariant with respect a group of transformations $\mathcal{G} = \{\lambda^{\mathbb{M}}\}_{\lambda>0}$ with $\mathbb{M}$ symmetric and positive defined if and only if

$$\lambda^{\mathbb{M}} = D(\lambda) \qquad \text{and} \qquad \mathbb{B} = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{B}_1 & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{B}_2 & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & \mathbb{B}_r & \mathbb{O} \end{pmatrix} . \tag{1.7}$$

Moreover when (1.7) is satisfied, the matrix C assumes the following form (see Proposition 2.3 [17], [15], [14]):

$$C(t) = D_0(\sqrt{t}) C(1) D_0(\sqrt{t}) \tag{1.8}$$

and hence, defining the *homogeneous dimension* as

$$Q := m_0 + 3m_1 + \dots + (2\kappa+1)m_\kappa \ , \tag{1.9}$$

we have:

$$\gamma(z,\tau) = \frac{1}{\sqrt{(4\pi)^N det(C(1))} t^{Q/2}} e^{-\frac{1}{4}(z^T D_0(\frac{1}{\sqrt{\tau}}) C(1)^{-1} D_0(\frac{1}{\sqrt{\tau}}) z)} \ . \tag{1.10}$$

Notice that this formula resembles the fundamental solution for the heat equation. Indeed looking at

$$H(x,t) = \frac{1}{\sqrt{(4\pi)^N)} \tau^{N/2}} e^{-\frac{1}{4}\frac{|z|^2}{\tau}} \tag{1.11}$$

which is the fundamental solution of the heat operator:

$$\mathcal{H} = \triangle_x - \partial_t \tag{1.12}$$

we see that it is a particular case of (1.10). Finally, we remark, as it is done in [17] section

3, that the matrix C is approximated by $C_0(t) := D_0(\sqrt{t})C(1)D_0(\sqrt{t})$ which is the matrix obtained by annihilating the $*$ terms in $\mathbb{B}$ (see [17], section 3). More precisely:

$$x^T C(t)^{-1} x = x^T C_0(t)^{-1} x(1 + tO(1)) \qquad \text{as} \quad t \to 0 \tag{1.13}$$

$$x^T C(t) x = x^T C_0(t) x(1 + tO(1)) \qquad \text{as} \quad t \to 0 \tag{1.14}$$

$$det(C(t)) = det(C_0(t))(1 + tO(1)) \qquad \text{as} \quad t \to 0\ . \tag{1.15}$$

The symbol $O(1)$ represents a bounded function of t.

1.3. KFP operators with rough coefficients

We end this chapter introducing the fundamental solution of the operator with rough time dependent coefficients (see [5]). We consider the KFP operator with linear drift (1) when the matrix $\mathbb{A}$ depends on time, more precisely the matrices are assumed to satisfy conditions (1.3) and the coefficients of $\mathbb{A}_0$ are L^∞ functions of time satisfying:

$$\nu|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(t)\xi_i\xi_j \leq \frac{1}{\nu}|\xi|^2 \qquad \forall \xi \in \mathbb{R}^q \quad \text{a.e. } t$$

for some $\nu > 0$.
With these conditions an explicit fundamental solution (see section 2 of [5]) is given by:

$$\Gamma(x, t; y, s) = \frac{e^{-\frac{1}{4}(x-E(t-s)y)^T C(t,s)^{-1}(x-E(t-s)y)} e^{-(t-s)tr(\mathbb{B})}}{\sqrt{(4\pi)^N det(C(t, s))}}) \tag{1.16}$$

where the matrix C is defined as

$$C(t, s) = \int_s^t E(t - \sigma)\mathbb{A}(\sigma)E(t - \sigma)^T d\sigma\ . \tag{1.17}$$

Notice that in this case the matrix C depends on both t and s, not only on their difference. It is known that the fundamental solution (2.4) enjoys some regularity properties (see [5]). Let $\mathbb{R}^{2N+2}_*$ be the region (as defined in [5])

$$\mathbb{R}^{2N+2}_* = \{(x, t; y, s) \in \mathbb{R}^{2N+2} : (x, t) \neq (y, s)\}$$

from Theorem 1.4 in [5], we have that:
a) Γ is jointly continuous in $(x, t; y, s)$ and of class C^∞ in the x and y variables in $\mathbb{R}^{2N+2}_*$;
b) For any multi-indexes $\alpha, \beta \in \mathbb{N}^N$, $\partial_x^\alpha \partial_y^\beta \Gamma$ is jointly continuous in the variables $(x, t; y, s)$

in $\mathbb{R}^{2N+2}_*$;
c) For any multi-indexes $\alpha, \beta \in \mathbb{N}^N$ (possibly equal to zero), $\partial_x^\alpha \partial_y^\beta \Gamma$ is Lipschitz continuous in any set of the kind $\{(x,t;y,s) : K \leq s+\delta \leq t \leq H\}$ for fixed constants $H, K \in \mathbb{R}$ and $\delta > 0$ satisfying $H \leq K$.
Once we have recalled these regularity property we can state the theorem which asserts that Γ is actually a solution:

Theorem 1.3 ([5] Theorem 4.4). *For every fixed $(y,s) \in \mathbb{R}^{N+1}$,*

$$\mathcal{L}_{(x,t)}\Gamma(x,t;y,s) = 0 \quad \textit{for a.e. } t > s \textit{ and every } x \in \mathbb{R}^N.$$

We remark also (see Proposition 4.5 [5]) that:

$$\int_{\mathbb{R}^N} \Gamma(x,t;y,s)dy = 1 \ . \tag{1.18}$$

Now let Γ_ν be the fundamental solution of the operator with

$$\mathbb{A}_0 = \lambda I_q \quad \text{and} \quad I_q = diag(\underbrace{1,\dots,1}_{q},0,\dots,0) \ .$$

Since we shall repeatedly use its explicit expression, we introduce some notation in order to simplify the computations, let $C_0(1)$ be the matrix

$$C_0(1) = \int_0^1 E(\sigma)I_qE(\sigma)^T d\sigma \quad \text{and} \quad c_0(1) := \det(C_0(1))$$

then, let $|\cdot|_0$ be the norm induced by $C_0(1)^{-1}$:

$$|x|_0 := \sqrt{C_0(1)^{-1}x \cdot x} \ . \tag{1.19}$$

In this way the fundamental solution Γ_ν is written as:

$$\Gamma_\nu(x,t;y,s) = \frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}(t-s)^{Q/2}} \exp(-\frac{1}{4\nu}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2) \ .$$

Now we shall state an important estimate.

Theorem 1.4 (Theorem 1.7 [5]). *For every $t > s$ and $x, y \in \mathbb{R}^N$,*

$$\nu^N \Gamma_\nu(x,t;y,s) \leq \Gamma(x,t;y,s) \leq \frac{1}{\nu^N}\Gamma_{\frac{1}{\nu}}(x,t;y,s).$$

This is a fundamental estimate since it lets us to estimate the fundamental solution of our operator in terms of the much simpler fundamental solution of the constant coefficients operator.

Some explicit examples of fundamental solution

We conclude with two simple examples, where we compute the fundamental solution. In the following computations s and t denote two real numbers satisfying $t - s > 0$.
The first example could be found in [5] (example 1.8). Let $a : \mathbb{R} \to \mathbb{R}$ be a measurable function such that:

$$\nu \leq a(t) \leq \frac{1}{\nu} \quad \text{a.e. } t \in \mathbb{R} \tag{1.20}$$

for some constant $\nu > 0$ and let $\mathcal{L}$ be the operator:

$$\mathcal{L} = a(t)\partial^2_{xx} + x\partial_y - \partial_t \ ,$$

we want to compute its fundamental solution. The operator $\mathcal{L}$ satisfies all the assumptions of this section with matrices $\mathbb{A}$ and $\mathbb{B}$ as follows:

$$\mathbb{A}(t) = \begin{pmatrix} a(t) & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \mathbb{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ .$$

In order to write the fundamental solution we first compute $E(s)$:

$$E(s) = \exp(-s\mathbb{B}) = \sum_{k=0}^{+\infty} \frac{(-s)^k}{k!}\mathbb{B}^k = I - s\mathbb{B} = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$$

then we can easily compute $C(t,s)$ as follows:

$$C(t,s) = \int_s^t E(t-\sigma)\mathbb{A}(\sigma)E(t-\sigma)^T d\sigma = \dots$$

$$\dots = \int_s^t \begin{pmatrix} 1 & 0 \\ \sigma - t & 1 \end{pmatrix} \begin{pmatrix} a(\sigma) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \sigma - t \\ 0 & 1 \end{pmatrix} d\sigma = \int_s^t \begin{pmatrix} a(\sigma) & (\sigma - t)a(\sigma) \\ (\sigma - t)a(\sigma) & (\sigma - t)^2 a(\sigma) \end{pmatrix} d\sigma$$

after integrating by parts a sufficient number of times we could easily see that the entries of C can be written in term of primitives of a (see [5] Example 1.8). Moreover, we can see that $C(t,s)$ is always positive definite and satisfies:

$$\nu C_0(t-s) \leq C(t,s) \leq \frac{1}{\nu} C_0(t-s) \tag{1.21}$$

where C_0 is the corresponding matrix of:

$$\mathcal{L}_0 = \partial^2_{xx} + x\partial_y - \partial_t \ .$$

Indeed, if $\xi \in \mathbb{R}^2$, after some computations, we find:

$$\xi^T C(t,s)\xi = \int_s^t a(\sigma)(\xi_1 + (\sigma - t)\xi_2)^2 d\sigma$$

and by the same computations we obtain also:

$$\xi^T C_0(t-s)\xi = \int_s^t (\xi_1 + (\sigma - t)\xi_2)^2 d\sigma$$

which gives (1.21). Now, denoting with $c_{ij}(t,s)$ the entries of $C(s,t)$, by the formula giving the inverse of a matrix, we find:

$$C(t,s)^{-1} = \frac{1}{c_{11}c_{22} - c_{12}^2} \int_s^t \begin{pmatrix} (\sigma-t)^2 a(\sigma) & -(\sigma-t)a(\sigma) \\ -(\sigma-t)a(\sigma) & a(\sigma) \end{pmatrix} d\sigma$$

hence, for any $\xi \in \mathbb{R}^2$

$$\xi^T C(t,s)^{-1}\xi = \frac{\int_s^t \{(\sigma-t)^2 a(\sigma)\xi_1^2 - 2(\sigma-t)a(\sigma)\xi_1\xi_2 + a(\sigma)\xi_2^2\} d\sigma}{\det(C(t,s))} = $$
$$= \frac{\int_s^t a(\sigma)\big((t-\sigma)\xi_1 + \xi_2\big)^2 d\sigma}{\det(C(t,s))} \ .$$

Finally, taking $y, x \in \mathbb{R}^2$, we have:

$$(x - E(t-s)y) = (x-y) - (t-s)\mathbb{B}y = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 + (t-s)y_1 \end{bmatrix}$$

hence

$$C(t,s)^{-1}(x - E(t-s)y)\cdot(x - E(t-s)y) = \frac{\int_s^t a(\sigma)\big((t-\sigma)x_1 - (s-\sigma)y_1 + x_2 - y_2\big)^2 d\sigma}{\det(C(t,s))}$$

so we can compute $\Gamma(x,t;y,s)$ as:

$$\Gamma(x,t;y,s) =$$
$$= \frac{1}{\sqrt{(4\pi)^N det(C(t,s))}} \exp\Big(-\frac{1}{4\det(C(t,s))}\int_s^t a(\sigma)\big((t-\sigma)x_1 - (s-\sigma)y_1 + x_2 - y_2\big)^2 d\sigma\Big)$$

while the fundamental solution for the model operator is much simpler: (we assume $s = 0$ and $y = 0$ since the general case could be easily recovered)

$$\Gamma_1(x,t;0,0) = \\ = \frac{1}{\sqrt{(4\pi)^2 c_0(1) t^2}} \exp\Big(- \frac{1}{4c_0(1)t^4} \int_0^t \big((t-\sigma)x_1 + x_2\big)^2 d\sigma \Big) \ .$$

Actually, it could be further simplified by computing the integral (see [3] Example 81):

$$\Gamma_1(x,t;0,0) = \frac{\sqrt{3}}{\sqrt{2\pi t^2}} \exp\Big(- (\frac{x_1^2}{t} + \frac{3x_2^2}{t^3} + \frac{3x_1x_2}{t^2}) \Big) \ .$$

Notice that the operator we have just considered satisfies the conditions (1.7) hence we shall consider another example in which this condition is not fulfilled (see [3] Example 81 for the case of constant coefficients). Let $\mathcal{L}$ be the operator

$$\mathcal{L} = a(t)\partial^2_{xx} + x\partial_y + y\partial_x - \partial_t$$

where a satisfies the same conditions of the previous example (1.20). The matrices $\mathbb{A}$ and $\mathbb{B}$ are:

$$\mathbb{A}(t) = \begin{pmatrix} a(t) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

As before we compute $E(s)$. Since $\mathbb{B}^2 = I$, we easily obtain:

$$E(s) = \sum_{k\geq 0} \frac{(-s)^k}{k!}\mathbb{B}^k = \sum_{k\geq 0} \frac{(-s)^{2k}}{2k!}\mathbb{B}^{2k} + \sum_{k\geq 0} \frac{(-s)^{2k+1}}{(2k+1)!}\mathbb{B}^{2k+1} = \cosh(s)I - \sinh(s)\mathbb{B} \ .$$

Then:

$$E(t-\sigma)\mathbb{A}(\sigma)E(t-\sigma)^T = \\ = \begin{pmatrix} \cosh(t-\sigma) & -\sinh(t-\sigma) \\ -\sinh(t-\sigma) & \cosh(t-\sigma) \end{pmatrix} \begin{pmatrix} a(\sigma) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh(t-\sigma) & -\sinh(t-\sigma) \\ -\sinh(t-\sigma) & \cosh(t-\sigma) \end{pmatrix} = \\ = a(\sigma) \begin{pmatrix} (\cosh(t-\sigma))^2 & -\sinh(t-\sigma)\cosh(t-\sigma) \\ -\sinh(t-\sigma)\cosh(t-\sigma) & (\sinh(t-\sigma))^2 \end{pmatrix}$$

therefore we easily compute $C(t,s)$:

$$C(t,s) = \int_s^t a(\sigma) \begin{pmatrix} (\cosh(t-\sigma))^2 & -\sinh(t-\sigma)\cosh(t-\sigma) \\ -\sinh(t-\sigma)\cosh(t-\sigma) & (\sinh(t-\sigma))^2 \end{pmatrix} d\sigma \ .$$

Inverting the matrix we obtain $C^{-1}(s,t)$

$$C(t,s)^{-1} = \frac{1}{\det(C(t,s))}\int_s^t a(\sigma)\begin{pmatrix} (\sinh(t-\sigma))^2 & \sinh(t-\sigma)\cosh(t-\sigma) \\ \sinh(t-\sigma)\cosh(t-\sigma) & (\cosh(t-\sigma))^2 \end{pmatrix} d\sigma$$

hence, if $\xi \in \mathbb{R}^N$, then:

$$\xi^T C(t,s)^{-1}\xi = \frac{\int_s^t a(\sigma)\big(\sinh(t-\sigma)\xi_1 + \cosh(t-\sigma)\xi_2\big)^2}{\det(C(t,s))} d\sigma \ . \tag{1.22}$$

Now, let $x, y \in \mathbb{R}^N$, we compute $(x - E(t-s)y)$ as follows:

$$(x - E(t-s)y) = x - \cosh(t-s)y + \sinh(t-s)\mathbb{B}y = \begin{bmatrix} x_1 - \cosh(t-s)y_1 + \sinh(t-s)y_2 \\ x_2 - \cosh(t-s)y_2 + \sinh(t-s)y_1 \end{bmatrix}$$

therefore, taking $\xi = (x - E(t-s)y)$, the integrand in (1.22) is

$$\begin{aligned} &\sinh(t-\sigma)\xi_1 + \cosh(t-\sigma)\xi_2 = \\ &\qquad = \sinh(t-\sigma)(x_1 - \cosh(t-s)y_1 + \sinh(t-s)y_2) + \\ &\qquad + \cosh(t-\sigma)(x_2 - \cosh(t-s)y_2 + \sinh(t-s)y_1) = \\ &\qquad = \sinh(t-\sigma)x_1 + \cosh(t-\sigma)x_2 + \\ &\qquad +(\cosh(t-\sigma)\sinh(t-s) - \sinh(t-\sigma)\cosh(t-s))y_1 + \\ &\qquad +(\sinh(t-\sigma)\sinh(t-s)y_2 - \cosh(t-\sigma)\cosh(t-s))y_2 = \dots \end{aligned}$$

by the properties of hyperbolic sine and hyperbolic cosine we obtain:

$$\dots = \sinh(t-\sigma)x_1 + \cosh(t-\sigma)x_2 - \sinh(s-\sigma)y_1 - \cosh(s-\sigma)y_2$$

hence:

$$\begin{aligned} &C(t,s)^{-1}(x - E(t-s)y)\cdot(x - E(t-s)y) = \\ &\qquad = \frac{\int_s^t a(\sigma)\big(\sinh(t-\sigma)x_1 - \sinh(s-\sigma)y_1 + \cosh(t-\sigma)x_2 - \cosh(s-\sigma)y_2\big)^2 d\sigma}{\det(C(t,s))} \end{aligned}$$

which let us to compute explicitly the fundamental solution:

$$\begin{aligned} &\Gamma(x,t;y,s) = \frac{1}{\sqrt{(4\pi)^N det(C(t,s))}} \cdot \\ &\cdot \exp\Big(- \frac{\int_s^t a(\sigma)\big(\sinh(t-\sigma)x_1 - \sinh(s-\sigma)y_1 + \cosh(t-\sigma)x_2 - \cosh(s-\sigma)y_2\big)^2 d\sigma}{4\det(C(t,s))}\Big). \end{aligned}$$

The fundamental solution of the corresponding model operator $\partial^2_{xx} + x\partial_y + y\partial_x - \partial_t$ could be computed in the same way taking $a \equiv 1$:

$$\Gamma(x,t;0,0) = \frac{1}{\sqrt{(4\pi)^N det(C(t)}} \exp\Big(- \frac{\int_0^t \big(\sinh(t-\sigma)x_1 + \cosh(t-\sigma)x_2\big)^2 d\sigma}{4\det(C(t))}\Big).$$

Computing the integral we can obtain (see [3] Example 81):

$$\Gamma_1(x,t;0,0) = \frac{1}{2\pi\sqrt{\sinh(t)^2 - t^2}} \cdot$$
$$\cdot \exp\Big(- \frac{((\sinh(t)\cosh(t) - t)^2 x_1^2 + (\sinh(t)\cosh(t) + t)^2 x_2^2 + 2(\sinh(t))^2 x_1 x_2)}{2(\sinh(t)^2 - t^2)}\Big) .$$

These computations show that, even for the most simple operator one could consider, the explicit fundamental solution is very complicated. This is the reason why it is very important to know sharp estimates on the fundamental solution and its derivatives in terms of simpler expression.

2 | Well posedness of the Chauchy problem

In this chapter we shall study a class of operators of Kolmogorov-Fokker-Planck which belongs the class studied by Bramanti and Polidoro in [5]. What we want to prove is the well-posedness of the nonhomogeneous Cauchy problem with null initial data and the existence of a solution in the general case. Concerning the uniqueness of the solution for the general Cauchy problem a result is contained in [5] but we cannot exploit it due to the differences between our definition of solution and the one given in that article.

The hypothesis on $\mathcal{L}$ are the following:

$$\mathcal{L} = \sum_{i,j=1}^{N} a_{ij}(t)\partial_{ij} + \sum_{i,j=1}^{N} b_{ij}x_j\partial_i - \partial_t \tag{2.1}$$

the coefficients $a_{ij}(t)$ are measurable functions of time and as usual we consider $\mathbb{A} = \{a_{ij}\}_{i,j=1}^{N}$ and $\mathbb{B} = \{b_{ij}\}_{i,j=1}^{N}$. Moreover, we now assume that these matrices are in the form:

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_0 & \mathbb{O} & \dots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} \end{pmatrix} \qquad \mathbb{B} = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{B}_1 & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{B}_2 & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & \mathbb{B}_\kappa & \mathbb{O} \end{pmatrix} \tag{2.2}$$

where for any t $\mathbb{A}_0(t)$ is a symmetric positive definite matrix of order q and there exists a sequence of integers $q = m_0 \geq \dots \geq m_j \geq \dots \geq m_\kappa \geq 1$ with sum equal to N such that $\mathbb{B}_j$ is a $m_j \times m_{j-1}$ matrix of maximal rank ($r(\mathbb{B}_j) = m_{j-1}$). Finally, as in chapter 1, there exists $\nu > 0$ such that the following inequalities hold:

$$\nu|\xi|^2 \leq \xi^T \mathbb{A}_0(t)\xi \leq \frac{1}{\nu}|\xi|^2 \qquad \forall \xi \in \mathbb{R}^q \quad \text{a.e. } t \; . \tag{2.3}$$

Notice that the assumptions on $\mathbb{B}$ are stronger than the one of section 1.3 and that they are the same conditions that $\mathbb{B}$ would satisfy if the operator had constant coefficients and was invariant with respect to the dilations $\{D(\lambda)\}_{\lambda>0}$. Since the trace of $\mathbb{B}$ is now zero the fundamental solution of $\mathcal{L}$ assumes the following form:

$$\Gamma(x,t;y,s) = \frac{e^{-\frac{1}{4}(x-E(t-s)y)^T C(t,s)^{-1}(x-E(t-s)y)}}{\sqrt{(4\pi)^N det(C(t,s))}}) \ . \tag{2.4}$$

The matrix C is still defined by (1.17).

The Cauchy problem we want to solve is the following:

$$\begin{cases} \mathcal{L}u = f & \mathbb{R}^N \times (-\infty, T) \\ u(\cdot,t) = g & t = 0 \end{cases} . \tag{2.5}$$

The definition of solution is not straightforward due to the low regularity of the matrix $\mathbb{A}$. Hence, we decided to adopt the most natural definition of solution suggested by the Schauder estimates [Theorem 4.7 [1]] which will be introduced in the next section.

Before proceeding further, we want to explain the main difficulty which is involved in the proof of our existence result. We begin with a quick description of this proof for the simple case of the heat equation:

$$\mathcal{H}u = f$$

where $\mathcal{H}$ is the heat operator (1.11). Starting with the representation formula for the solution of the Cauchy problem for the homogeneous heat equation

$$\begin{cases} \mathcal{H}u = 0 \ t > 0, x \in \mathbb{R}^N \\ u\left(0,x\right) = g\left(x\right) \end{cases}$$

which reads as

$$u\left(x,t\right) = -\int_{\mathbb{R}^N} H\left(x-y,t\right) g\left(y\right) dy,$$

the *Duhamel principle* suggests that the solution to the nonhomogeneous Cauchy problem

$$\begin{cases} \mathcal{H}u = f \text{ for } t > 0, x \in \mathbb{R}^N \\ u\left(0,\cdot\right) = 0 \text{ in } \mathbb{R}^N \end{cases} \tag{2.6}$$

should be given by

$$u\left(x,t\right) = -\int_0^t \int_{\mathbb{R}^n} H\left(x-y,t-s\right) f\left(y,s\right) dyds. \tag{2.7}$$

This is the candidate representation formula for the solution. Proving that the formula actually assignes the solution to the problem (2.6) requires to compute the derivatives of the integral. This can be done in two ways. The first is simpler but requires stronger assumptions on f, the second one is more delicate but works under weaker assumptions on f. Let us describe the simpler approach, assuming that $f \in C_c^2\left(\mathbb{R}^N \times (0,\infty)\right)$.

Consider the function:

$$u(x,t) = -\int_0^t \int_{\mathbb{R}^N} H(x-y,t-s) f(y,s) dyds \quad (x,t) \in \mathbb{R}^{N+1}$$

which is nothing but minus the convolution of f with $-H$. Then, making the change of variables $\{(z,\tau) = (x-y,t-s)\}$, we obtain:

$$u(x,t) = -\int_0^t \int_{\mathbb{R}^N} H(z,\tau) f(x-z,t-\tau) dzd\tau \quad (x,t) \in \mathbb{R}^{N+1} \ .$$

Now, we can compute the (classical) derivatives, for $(x,t) \in \mathbb{R}^{N+1}$

$$\partial_t u(x,t) = -\int_0^t \int_{\mathbb{R}^N} H(z,\tau)(\partial_t f)(x-z,t-\tau) dzd\tau - \int_{\mathbb{R}^N} H(z,t) f(x-z,0) dz$$

$$\partial_{ij} u(x,t) = -\int_0^t \int_{\mathbb{R}^N} H(z,\tau)(\partial_{ij} f)(x-z,t-\tau) dzd\tau \ .$$

Let $\mathcal{H}^*$ be the adjoint operator of $\mathcal{H}$:

$$\mathcal{H}^* = \triangle_x + \partial_t \ .$$

Computing $\mathcal{H}u(x,t)$ we obtain

$$\begin{aligned}\mathcal{H}u(x,t) &= -\int_0^t \int_{\mathbb{R}^N} H(z,\tau)(\mathcal{H}f)(x-z,t-\tau) dzd\tau + \int_{\mathbb{R}^N} H(z,t) f(x-z,0) dz = \\ &= -\int_0^t \int_{\mathbb{R}^N} H(z,\tau)\mathcal{H}^*_{(z,\tau)} f(x-z,t-\tau) dzd\tau + \int_{\mathbb{R}^N} H(z,t) f(x-z,0) dz \ .\end{aligned}$$

The notation $\mathcal{H}_{(z,\tau)}$ means that the operator is computed with respect to the variables (z,τ). Then, following [9], we split the integral in three parts:

$$\begin{aligned}\mathcal{H}u(x,t) = &-\int_\varepsilon^t \int_{\mathbb{R}^N} H(z,\tau)\mathcal{H}^*_{(z,\tau)} f(x-z,t-\tau) dzd\tau - \\ &-\int_0^\varepsilon \int_{\mathbb{R}^N} H(z,\tau)\mathcal{H}^*_{(z,\tau)} f(x-z,t-\tau) dzd\tau - \\ &+\int_{\mathbb{R}^N} H(z,t) f(x-z,0) dz =: I_\varepsilon + J_\varepsilon + K \ .\end{aligned}$$

It is easy to see that

$$|J_\varepsilon| \to 0 \quad \text{as } \varepsilon \to 0^+$$

while, integrating by parts in I_ε, we obtain:

$$\begin{aligned} I_\varepsilon = &- \int_\varepsilon^t \int_{\mathbb{R}^N} \mathcal{H}H(z,\tau) f(x-z, t-\tau) dz d\tau - \\ &- \int_{\mathbb{R}^N} H(z,t) f(x-z,0) dz d\tau + \int_{\mathbb{R}^N} H(z,\varepsilon) f(x-z, t-\varepsilon) dz d\tau = \dots \end{aligned}$$

since $\mathcal{H}H(z,\tau) = 0$ if $\tau > 0$

$$\dots = -K + \int_{\mathbb{R}^N} H(z,\varepsilon) f(x-z, t-\varepsilon) dz d\tau \ .$$

Therefore we have:

$$\begin{aligned} \mathcal{H}u(x,t) &= \int_{\mathbb{R}^N} H(z,\varepsilon) f(x-z, t-\varepsilon) dz d\tau = \\ &= \int_{\mathbb{R}^N} H(z,\varepsilon) f(x-z,t) dz d\tau + \int_{\mathbb{R}^N} H(z,\varepsilon)(f(x-z, t-\varepsilon) - f(x,z,t)) dz d\tau \ . \end{aligned}$$

It is easy to see that taking the limit as $\varepsilon \to 0^+$ we obtain:

$$\mathcal{H}u(x,t) = f(x,t) \quad \forall (x,t) \in \mathbb{R}^{N+1}$$

and this completes the description of the method. Notice that the above argument relies on the two following facts:

$$\mathcal{H}H(x,t) = 0 \quad \text{if } t > 0$$

and for any compactly supported continuous function g and any $x \in \mathbb{R}^N$

$$\int_{\mathbb{R}^N} H(x-z,t) g(z) dz \to g(x) \quad \text{as } t \to 0^+ \ .$$

These two properties could be taken (up to suitable modifications) as an alternative approach to the definition of fundamental solution (see for instance [8] or [11]). Moreover, this seems to be the only possible approach for the operator (2.1) since due to the low regularity of coefficients we cannot exploit the distributional framework. We notice that these conditions are proved in [5] for (1.16) (see Theorem 1.3 and Theorem 2.9).

A second proof is possible, requiring less regularity to the function f. The idea is to compute the derivatives of u on the formula (2.7) differentiating the fundamental solution

H, instead of f. This, however, is troublesome because the derivatives

$$H_{x_i x_i}(x-y,t-s), H_t(x-y,t-s)$$

are not locally integrable, so some refined idea is needed to make the integral converge.

In our present situation, our first attempt was to follow the simpler approach, requiring the due regularity on f. So, let $f \in C_c^2\left(\mathbb{R}^N \times (0,\infty)\right)$ then let u be defined by:

$$u(x,t) = -\int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;y,s) f(y,s) dy ds \ .$$

The first thing to do is the change of variable and since we want to eliminate the dependence on t from Γ we have chosen the following change of variables:

$$\{C(t,s)^{-\frac{1}{2}}(x-E(t-s)y = z), dy = \sqrt{\det(C(t,s))}dz\} \ .$$

Hence

$$\begin{aligned} u(x,t) &= -\int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;y,s) f(y,s) dy ds = \\ &= -\int_0^t \int_{\mathbb{R}^N} \frac{e^{-\frac{1}{4}(x-E(t-s)y)^T C(t,s)^{-1}(x-E(t-s)y)}}{\sqrt{(4\pi)^N det(C(t,s))}} f(y,s) dy ds = \\ &= \{C(t,s)^{-\frac{1}{2}}(x-E(t-s)y = z), dy = \sqrt{\det(C(t,s))}dz\} = \\ &= -\int_0^t \int_{\mathbb{R}^N} \frac{e^{-\frac{1}{4}|z|^2}}{\sqrt{(4\pi)^N}} f(C(t,s)^{-\frac{1}{2}} E(s-t)(x-z), s) dy ds \end{aligned}$$

but this immediately gives some problems. Indeed, due to the presence of $C(t,s)^{-\frac{1}{2}}$, whose t-derivative is not easily computed nor estimated, we cannot proceed in the argument.

If we look at the constant coefficient case we see that the previous approach could be applied with some modifications. Indeed, the function u could be interpreted as a convolution in the homogeneous group, therefore the most natural change of variable is $\{(z,\tau) = (y,s)^{-1} \circ (x,t)\}$ which leads to:

$$u(x,t) = \int_0^t \int_{\mathbb{R}^N} \gamma(z,\tau) f((x,t) \circ (z,\tau)^{-1}) dz d\tau \ .$$

Hence, we could say that the method does not work since the operator has varying coefficients.

So, we tried the second approach, which is harder but does not involves the troublesome change of variables inside the integral. Since this approach does not require to compute the derivatives of f, hopefully this should work under weaker assumptions on f, giving in the end a better result than the first one we expected.

In turn, to implement the second approach in our situation it is useful to split the proof in two steps. First (section 2.2) we prove the result assuming f smooth and compactly supported. We stress that this assumption is made for some technical reason, but the computation is performed computing the derivatives of Γ, not of f. In the second step (section 2.3) the result is established in the general case by a suitable approximation result.

2.1. Known results and definition of solution

In this section we introduce some definitions and notation from the article [1] which will be used extensively in the following sections. First we need some geometric notions. Let $d : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \to [0, \infty)$ be the quasi-distance (see [4] ch. 3) defined by:

$$d((x,t),(y,s)) = ||x - E(t-s)y|| + \sqrt{|t-s|} \tag{2.8}$$

where $||\cdot||$ is defined as follows:

$$||x|| := \sum_{i=1}^{N} |x_i|^{\frac{1}{q_i}}$$

with constants q_i defined by:

$$(q_1, \ldots, q_N) := (\underbrace{1, \ldots, 1}_{m_0}, \ldots, \underbrace{2i+1, \ldots, 2i+1}_{m_i}, \ldots, \underbrace{2\kappa+1, \ldots, 2\kappa+1}_{m_\kappa}) \; .$$

Notice that $D_0(\lambda) = \lambda^{diag(q_1,\ldots,q_N)}$. Moreover, if we define the following homogeneous norm:

$$\rho(x,t) = ||x|| + \sqrt{|t|} \; ,$$

we obtain:

$$d(\xi,\eta) = \rho(\eta^{-1} \circ \xi) \quad \forall \xi, \eta \in \mathbb{R}^{N+1}$$

therefore, d has the following property:

$$d(\xi,\eta) = d(\eta^{-1} \circ \xi, 0) = d(\chi \circ \xi, \chi \circ \eta) \quad \forall \xi, \chi, \eta \in \mathbb{R}^{N+1} \; .$$

With this quasi-distance the balls $B_r(\xi) = \{\eta \in \mathbb{R}^{N+1} : \quad d(\eta, \xi) < r\}$ satisfy the following condition:

$$B_r(\xi) = \xi \circ D(r) B_1(0)$$

and thanks to the fact that the Lebesgue measure is invariant with respect to the left (and the right) action of the group on itself, it follows that:

$$\mathcal{L}^{N+1}(B_r(\xi)) = r^Q \mathcal{L}^{N+1}(B_1(0))$$

for some Q. It is easily seen that Q is the homogeneous dimension (1.9).
Now we move to the definition of the spaces of functions $C^\alpha_x(S_T)$, $S^0(S_T)$ and $S^\alpha(S_T)$ still from [1] which will be used later.

Definition 2.1 (See Definition 1.2 [1]). *Let $\Omega = D \times I$ where I is an open interval and D is an open subset of $\mathbb{R}^N$ moreover, let $f : \Omega \to \mathbb{R}$ and $\alpha \in (0,1)$. We define:*

$$|f|_{C^\alpha(\Omega)} := \sup_{\substack{\xi,\eta \in \Omega \\ \xi \neq \eta}} \frac{|f(\xi) - f(\eta)|}{d(\xi,\eta)^\alpha} \ , \qquad ||f||_{C^\alpha(\Omega)} := |f|_{C^\alpha(\Omega)} + ||f||_{L^\infty(\Omega)}$$

$$|f|_{C^\alpha_x(\Omega)} := \operatorname*{ess\,sup}_{t \in I} \sup_{x \neq y} \frac{|f(x,t) - f(y,t)|}{d((x,t),(y,t))^\alpha} \ , \qquad ||f||_{C^\alpha_x(\Omega)} := |f|_{C^\alpha_x(\Omega)} + ||f||_{L^\infty(\Omega)}$$

and

$$C^\alpha(\Omega) := \left\{ f \in C(\Omega) : \ ||f||_{C^\alpha(\Omega)} < +\infty \right\}$$
$$C^\alpha_x(\Omega) := \left\{ f \in L^\infty(\Omega) : \ ||f||_{C^\alpha_x(\Omega)} < +\infty \right\} \ .$$

Notice that the functions in C^α_x do not need to be continuous and that $(C^\alpha(\Omega), ||\cdot||_{C^\alpha(\Omega)})$ and $(C^\alpha_x(\Omega), ||\cdot||_{C^\alpha_x(\Omega)})$ are Banach spaces.
Let T be a real number, then, once we define $S_T := \mathbb{R}^N \times (-\infty, T)$, the spaces $S^0(S_T)$ and $S^\alpha(S_T)$ are defined as follows:

$$S^0(S_T) := \{u \in C(\overline{S_T}) \cap L^\infty(S_T) : \forall i,j \in \{1,\dots,q\}\ \partial_{ij} u \in L^\infty(S_T), Yu \in L^\infty(S_T)\} \tag{2.9}$$

$$S^\alpha(S_T) := \{u \in S^0(S_T) : \ \forall i,j \in \{1,\dots,q\} \quad \partial_{ij} u \in C^\alpha_x(S_T), Yu \in C^\alpha_x(S_T)\} \tag{2.10}$$

where derivatives are considered as distributional.
Now we can pass to the necessary results still from [1]. In the following we need essentially four kinds of results: estimates on the fundamental solution, representation formulas, the Schauder estimates and a regularity result which let us obtain further regularity properties

of solutions.
We start the list of the needed results with the estimates on the fundamental solution. In order to state the theorem we need some notation still from section 3.2 of [1]. Let $\boldsymbol{l} = (l_1, \dots, l_{2N}) \in \mathbb{N}^{2N}$ be a multi-index the partial derivative $D^{\boldsymbol{l}}_{(x,y)}$ is defined as

$$D^{\boldsymbol{l}}_{(x,y)} = (\partial_{x_1})^{l_1} \dots (\partial_{x_N})^{l_N} (\partial_{y_1})^{l_{N+1}} \dots (\partial_{y_N})^{l_{2N}} \ .$$

Moreover the order and the length of $\boldsymbol{l}$ are defined as:

$$\omega(\boldsymbol{l}) := \sum_{j=1}^{N} q_j l_j + \sum_{j=N+1}^{2N} q_{j-N} l_j, \qquad |\boldsymbol{l}| = \sum_{i=1}^{2N} l_i \ .$$

Notice that for any sufficiently smooth function u, for any $\lambda > 0$ and multindex $\alpha \in \mathbb{N}^N$:

$$D^\alpha_x u(D(\lambda)(x,t)) = \lambda^{\omega(\alpha)} (D^\alpha_x u)(D(\lambda)(x,t)) \quad \forall (x,t) \ .$$

We are ready to state the theorem containing the sharp estimates on Γ.

Theorem 2.1 (Theorem 3.5 [1]). *Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in \mathbb{N}^{2N}$ be a fixed multi-index. Then, there exist $c = c(\nu, \boldsymbol{\alpha}) > 0$ and a constant $c_1 > 0$, independent if ν and $\boldsymbol{\alpha}$, such that*

$$\begin{aligned} \left| D^{\boldsymbol{\alpha}}_{(x,y)} \Gamma(\xi;\eta) \right| &= |D^{\boldsymbol{\alpha}_1}_x D^{\boldsymbol{\alpha}_2}_y \Gamma(\xi;\eta)| \\ &\le \frac{c}{(t-s)^{\omega(\alpha)/2}} \Gamma_{c_1 \nu^{-1}}(\xi;\eta) \\ &\le \frac{c}{d(\xi,\eta)^{Q+\omega(\boldsymbol{\alpha})}} \end{aligned}$$

for every $\xi, \eta \in \mathbb{R}^{N+1}$ with $t \neq s$. The resulting inequality

$$\left| D^{\boldsymbol{\alpha}}_{(x,y)} \Gamma(\xi;\eta) \right| \le \frac{c}{d(\xi,\eta)^{Q+\omega(\boldsymbol{\alpha})}}$$

actually holds for every $\xi, \eta \in \mathbb{R}^{N+1}$, $\xi \neq \eta$.

The next set of results in our list are the one related to representation formulas (see section 3.3 of [1]). The first result that we will recall is Lemma 3.8 of [1] which is not surprising if we look at the estimates above and at (1.18).

Lemma 2.1 (Lemma 3.8 [1]). *Let $\boldsymbol{\alpha} \in \mathbb{N}^N$ be a fixed nonzero multi-index. Then, we have*

$$\int_{\mathbb{R}^N} D^\alpha_x \Gamma(x,t;y,s) dy = 0 \qquad \text{for every } x \in \mathbb{R}^N \text{ and every } s < t.$$

This lemma is useful since lets us add a null term to integrals which makes them convergent. For this purpose we exploit also the following proposition:

Proposition 2.1 (Proposition 3.13 [1]). *Let $\alpha \in (0,1)$ be fixed, and let $1 \leq i,j \leq q$. Then, there exists a constant $c = c(\alpha) > 0$ such that, for every $x \in \mathbb{R}^N$ and every $\tau < t$, one has*

$$\int_{\mathbb{R}^N \times (\tau,t)} |\partial^2_{ij}\Gamma(x,t;y,s)|\ ||E(s-t)x-y||^\alpha dyds \leq c(t-\tau)^{\alpha/2}\ .$$

As a consequence, we have

$$\int_{\mathbb{R}^N \times (t-\varepsilon,t)} |\partial^2_{ij}\Gamma(x,t;y,s)|\ ||E(s-t)x-y||^\alpha dyds \to 0$$

uniformly w.r.t. $(x,t) \in \mathbb{R}^{N+1}$ as $\varepsilon \to 0$.

The actual representation formula contained in [1] is the following:

Theorem 2.2 (Theorem 3.11 [1]). *Let $T \in \mathbb{R}$ be fixed, and let $\tau < T$. Moreover, let $u \in S^0(\tau;T)$. Then, we have the following representation formula*

$$u(x,t) = -\int_{\mathbb{R}^N \times (\tau,t)} \Gamma(x,t;y,s)\mathcal{L}u(y,s)dyds\ ,$$

for every point $(x,t) \in S_T$.

The notation $S^0(\tau;T)$ means

$$S^0(\tau;T) = \{u \in S^0(S_T) :\ u(x,t) = 0 \text{ if } t \leq \tau\}\ .$$

We remark that other representation formulas hold for the x-derivatives (see Corollary 3.12 [1]] and [Theorem 3.14 [1]] but we do not explicitly need these results, although, in the proof of Theorem 2.5 we exploit some arguments which recall their proofs.
Now we come to the Schauder estimates which entail also the regularity of solution.

Theorem 2.3 (Theorem 3.15 [1]). *Let $T > \tau > -\infty$ and $\alpha \in (0,1)$. Then, there exists $c > 0$, only depending on $(T-\tau)$, α, ν, $\mathbb{B}$, such that*

$$\sum_{i,j=1}^{q} ||\partial^2_{i,j}u||_{C^\alpha_x(S_T)} \leq c|\mathcal{L}u|_{C^\alpha_x(S_T)}$$

$$||Yu||_{C^\alpha_x(S_T)} \leq c||\mathcal{L}u||_{C^\alpha_x(S_T)}\ ,$$

for every $u \in S^0(\tau;T)$ with $\mathcal{L}u \in C^\alpha_x(S_T)$.

Notice that this result lets us to obtain $u \in S^\alpha(S_T)$ starting form $u \in S^0(S_T)$ and $\mathcal{L}u \in C^\alpha_x(S_T)$. This property will be very useful in chapter 2. We remark also that this result is called *Global Schauder Estimates* in [1]. Finally, since we do not need the general version of the Schauder Estimates [Theorem 4.7 [1]] we shall state a simpler version of this theorem which directly follows from the general result.

Theorem 2.4 (Global Schauder Estimates (see Theorem 4.7 [1])). *Let $T > \tau > -\infty$ and $\alpha \in (0,1)$. Then, there exists $c > 0$, only depending on $(T-\tau)$, α, ν, $\mathbb{B}$, such that $\forall u \in S^\alpha(S_T)$*

$$\sum_{h,k=1}^{q} ||\partial^2_{i,j}u||_{C^\alpha_x(S_T)} + ||Yu||_{C^\alpha_x(S_T)} + \sum_{k=1}^{q} ||\partial^2_k u||_{C^\alpha(S_T)} + ||u||_{C^\alpha(S_T)} \\ \leq c\left\{||\mathcal{L}u||_{C^\alpha_x(S_T)} + ||u||_{L^\infty(S_T)}\right\} .$$

With all these definitions and theorems the following definition of solution seems to be natural.

Definition 2.2. *We say that $u \in S^0(S_T)$ is a solution of*

$$\begin{cases} \mathcal{L}u = f & \mathbb{R}^N \times (-\infty, T) \\ u(\cdot,t) = 0 & \forall t \leq 0 \end{cases} \tag{CP0}$$

if $u \equiv 0$ when $t \leq 0$ and for almost every $(x,t) \in S_T$:

$$\sum_{i,j=1}^{q} a_{ij}(t)\partial_{ij}u(x,t) + Yu(x,t) = f(x,t) \ . \tag{2.11}$$

We stress that in (2.11) the derivatives are considered in a weak sense. For instance the term Yu represents the L^1_{loc} function (which actually is L^∞) such that for any $\phi \in D(\mathbb{R}^{N+1})$:

$$\int_{\mathbb{R}^{N+1}} Yu\,\phi = \int_{\mathbb{R}^{N+1}} u\,Y^*\phi \ .$$

2.2. The Cauchy Problem with regular datum

This section is devoted to the solution of the Cauchy problem (CP0) when f is a C^∞ function with compact support contained in $\mathbb{R}^N \times (-\infty, T)$. We note that even though we are assuming $f \in C^\infty_c$ we need to employ a refined technique and differentiate Γ instead of f. In particular, we want to prove the following:

Theorem 2.5. *If $f \in C_c^\infty(S_T)$ then, $u : S_T \to \mathbb{R}$ defined by*

$$u(x,t) = -\int_{-\infty}^{t}\int_{\mathbf{R}^N} \Gamma(x,t;y,s)f(y,s)dyds$$

is such that $u \in S^\alpha(S_T)$ for any $\alpha \in (0,1)$ and it satisfies $\mathcal{L}u = f$.

Before proving the theorem we need some preliminary results.

Remark 2.1. *Let $w \in C(\Omega)$ where Ω is an open set of $\mathbb{R}^N$. Assume that*

$$\sup_{h\in(0,h_0)} \left|\left|\frac{w(\cdot + he_i) - w(\cdot)}{h}\right|\right|_\infty < +\infty \quad \text{and} \quad \frac{w(\cdot + he_i) - w(\cdot)}{h} \to \partial_i w(\cdot) \quad a.e. \ .$$

Then, thanks to the dominated convergence theorem we obtain:

$$\frac{u(\cdot + he_i) - w(\cdot)}{h} \xrightarrow[h]{*} \partial_i w(\cdot) \qquad in \quad L^\infty(\Omega)$$

and since

$$\frac{w(\cdot + he_i) - w(\cdot)}{h} \xrightarrow[h]{D'(\Omega)} D_i w(\cdot)$$

the classical derivative will be also a weak derivative[1]*.*

Lemma 2.2. *Let $f \in C_c^\infty(S_T)$ and let $\varepsilon > 0$. Moreover, let $u_\varepsilon : S_T \to \mathbb{R}$ be defined by:*

$$u_\varepsilon(x,t) = -\int_{-\infty}^{t-\varepsilon}\int_{\mathbf{R}^N} \Gamma(x,t;y,s)f(y,s)dyds \ .$$

Then, for any $\alpha \in (0,1)$

$$u_\varepsilon \in S^\alpha(S_T), \qquad \mathcal{L}u_\varepsilon(x,t) = \int_{\mathbf{R}^N} \Gamma(x,t;y,t-\varepsilon)f(y,t-\varepsilon)dy \ .$$

Proof. First we observe that Γ, $\partial_i\Gamma(x,t;y,s)$ and $\partial_{ij}\Gamma(x,t;y,s)$ are uniformly bounded on $\{(x,t,y,s) \in S_T \times S_T : t - s > \varepsilon/2\}$, indeed, thanks to the estimates in Theorem 2.1 it follows that $\forall(x,t;y,s) \in S_T \times S_T$, $t - s > \varepsilon$ entails that

$$\Gamma(x,t;y,s) \le c\frac{1}{\sqrt{(4\pi\nu c_1)^N c_0(1)}(t-s)^{Q/2}} e^{-\nu c_1/4|D(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} \le \frac{c}{\varepsilon^{Q/2}}$$

$$|\partial_i\Gamma(x,t;y,s)| \le \frac{c}{\sqrt{(4\pi\nu c_1)^N c_0(1)}(t-s)^{(\omega(e_i)+Q)/2}} e^{-\nu c_1/4|D(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} \le \frac{c}{\varepsilon^a}$$

[1] In order to distinguish between the weak and classical derivatives, the weak one is denoted with D.

$$|\partial_{i,j}\Gamma(x,t;y,s)| \le \frac{c}{\sqrt{(4\pi\nu c_1)^N c_0(1)}(t-s)^{\omega((e_i+e_j)+Q)/2}} e^{-\nu c_1/4|D(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} \le \frac{c}{\varepsilon^{a'}}$$

for some fixed constants $a, a' > 0$. Now we claim that a similar bound on $\partial_t\Gamma(x,t;y,s)$ holds in every set of the kind

$$\{(x,t;y,s) \in K \times (-\infty,T) \times \mathbb{R}^N \times (-\infty,T) : \ t-s > \varepsilon\}$$

with $K \subset\subset \mathbb{R}^N$. Actually, since for almost any $(x,t;y,s) \in K \times (-\infty,T) \times \mathbb{R}^N \times (-\infty,T)$ such that $t-s > \varepsilon$ we have $\mathcal{L}\Gamma(x,t;y,s) = 0$, taking a and a' as before:

$$|\partial_t\Gamma(x,t;y,s)| = \Big| \sum_{i,j=1}^{q} a_{ij}(t)\partial_{ij}\Gamma(x,t;y,s) + \sum_{i,j=1}^{N} x_i b_{ij}\partial_j\Gamma(x,t;y,s) \Big| \le$$

$$\le c(\nu) \sum_{i,j=1}^{q} |\partial_{ij}\Gamma(x,t;y,s)| + \sum_{i,j=1}^{N} |x_i b_{ij}\partial_j\Gamma(x,t;y,s)| \le N^2 \frac{c}{\varepsilon^{a'}} + \sup_{x\in K}(\sum_{i,j=1}^{N} |x_i b_{ij}|)\frac{c}{\varepsilon^{a}} \ .$$

Done these preliminary observations, we can proceed with the computation of the classical derivative of Γ with respect the variable t. For $|h| \in (0, \varepsilon/2)$ we compute the incremental ratio:

$$-\frac{1}{h}[u_\varepsilon(x,t+h) - u_\varepsilon(x,t)] = \int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^N} \left[\frac{1}{h}\int_0^h \partial_t\Gamma(x,t+\theta;y,s)d\theta\right] f(y,s)dyds +$$
$$+ \int_{t-\varepsilon}^{t+h-\varepsilon} \int_{\mathbb{R}^N} \left[\frac{1}{h}\int_0^h \partial_t\Gamma(x,t+\theta;y,s)d\theta\right] f(y,s)dyds +$$
$$+ \frac{1}{h}\int_{t-\varepsilon}^{t+h-\varepsilon} \int_{\mathbb{R}^N} \Gamma(x,t;y,s)f(y,s)dyds \equiv A_h + B_h + C_h \ .$$

We want to prove that for a.e. $(x,t) \in S_T$:

$$A_h \to \int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^N} \partial_t\Gamma(x,t;y,s)f(y,s)dyds$$
$$B_h \to 0$$
$$C_h \to \int_{\mathbb{R}^N} \Gamma(x,t;y,t-\varepsilon)f(y,t-\varepsilon)dy$$

as $h \to 0^+$. First we consider B_h. Thanks to the estimates on $\partial_t\Gamma$ we have:

$$|B_h| \le \Big| \int_{t-\varepsilon}^{t+h-\varepsilon} \int_{\mathbb{R}^N} \left[\frac{1}{h}\int_0^h |\partial_t\Gamma(x,t+\theta;y,s)|d\theta\right] |f(y,s)|dyds \Big| \le$$
$$\le (N^2\frac{c}{\varepsilon^{a'}} + \sup_{x\in K}(\sum_{i,j=1}^{N} |x_i b_{ij}|)\frac{c}{\varepsilon^{a}}) \Big| \int_{t-\varepsilon}^{t+h-\varepsilon} \int_{\mathbb{R}^N} |f(y,s)|dyds \Big| \ .$$

which tends to zero as $h \to 0$. The convergence of C_h is easily obtained. Indeed, owing to the mean value theorem it follows that for any h there exists $\delta = \delta(h) \in (0,1)$ such that

$$C_h = \int_{\mathbb{R}^N} \Gamma(x, t+h, y, t-\varepsilon+\delta h) f(y, t-\varepsilon+\delta h) dy \ .$$

Hence, taking the limit as $h \to 0$, by dominated convergence, we obtain:

$$C_h \to \int_{\mathbb{R}^N} \Gamma(x, t; y, t-\varepsilon) f(y, t-\varepsilon) dy \text{ for any } (x,t) \in S_T \ .$$

It is left to prove:

$$A_h \to \int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^N} \partial_t \Gamma(x,t;y,s) f(y,s) dy ds \qquad \text{a.e.} \quad (x,t) \in S_T \ .$$

Since the derivative $\partial_t \Gamma$ exists a.e. then, for a.e. $(x,t) \in S_T$:

$$\chi_{(\varepsilon,+\infty)}(t-s) \frac{1}{h} \int_0^h \partial_t \Gamma(x, t+\theta; y, s) d\theta \underset{h}{\to} \chi_{(\varepsilon,+\infty)}(t-s) \partial_t \Gamma(x,t;y,s) \quad \text{a.e. } (y,s) \in S_T$$

and moreover, thanks to the estimate on $\partial_t \Gamma$ for a.e. $(x,t) \in S_T$:

$$|\chi_{(\varepsilon,+\infty)}(t-s) f(y,s) \frac{1}{h} \int_0^h \partial_t \Gamma(x, t+\theta; y, s) d\theta| \leq (N^2 \frac{c}{\varepsilon^{a'}} + \sum_{i,j=1}^N |x_i b_{ij}| \frac{c}{\varepsilon^a}) |f(y,s)| \in L^1(S_T).$$

Therefore, applying the dominated convergence, we finally obtain that for a.e. $(x,t) \in S_T$:

$$A_h \to \int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^N} \partial_t \Gamma(x,t;y,s) f(y,s) dy ds \ .$$

This proves that for a.e. $(x,t) \in S_T$

$$\frac{u_\varepsilon(x,t+h) - u_\varepsilon(x,t)}{h} \xrightarrow[h \to 0]{} - \int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^N} \partial_t \Gamma(x,t;y,s) f(y,s) dy ds - \\ - \int_{\mathbb{R}^N} \Gamma(x,t;y,t-\varepsilon) f(y,t-\varepsilon) dy \ .$$

Now we shall obtain that the classical derivative (which is defined almost everywhere) is also a weak derivative by observing that the incremental ratio is locally bounded.
Actually, let K be a fixed compact subset of $\mathbb{R}^N$, then by the estimates obtained at the

beginning of the proof, for any $(x,t) \in K \times (-\infty, T)$ we have the following estimates:

$$|A_h| \leq \int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^N} \left[\frac{1}{h} \int_0^h |\partial_t \Gamma(x, t+\theta; y, s)| d\theta \right] |f(y,s)| dyds \leq$$

$$\leq \left(\frac{c}{\varepsilon^{a'}} N^2 + \frac{c}{\varepsilon^a} \sup_{x \in K} \sum_{i,j=1}^{N} |x_i b_{ij}|\right) \int_{S_T} |f| dyds$$

$$|B_h| \leq \left(N^2 \frac{c}{\varepsilon^{a'}} + \sup_{x\in K} \left(\sum_{i,j} |x_i b_{ij}|\right) \frac{c}{\varepsilon^a}\right) \int_{S_T} |f|$$

and

$$|C_h| \leq \frac{1}{h} \int_{t-\varepsilon}^{t+h-\varepsilon} \int_{\mathbb{R}^N} \Gamma(x,t;y,s) |f(y,s)| dyds \leq ||f||_{L^\infty(S_T)} \ .$$

The last inequality follows by the fact that $\int_{\mathbb{R}^N} \Gamma(x,t;y,s) dy \equiv 1$ for any $(x,t,s) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ such that $t > s$, see (1.18). This lets us conclude that for almost any (x,t) the partial derivative with respect to t exists:

$$\partial_t u_\varepsilon(x,t) = -\int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^N} \partial_t \Gamma(x,t;y,s) f(y,s) dyds - \int_{\mathbb{R}^N} \Gamma(x,t;y,t-\varepsilon) f(y,t-\varepsilon) dy \tag{2.12}$$

and moreover it is also a weak derivative.

For the derivatives with respect to the variables x_i for $i \in \{1, \dots, N\}$ we can apply the standard theorem of differentiation under the integral. Indeed for any fixed $t \in (-\infty, T)$ the function

$$h_t : \mathbb{R}^N \times \mathbb{R}^N \times (-\infty, t-\varepsilon) \to \mathbb{R} : (x,y,s) \mapsto \Gamma(x,t;y,s) f(y,s)$$

is of class C^2 with respect to x and moreover it and its x-derivatives are uniformly bounded by an L^1 function:

$\forall (x,y,s) \in \mathbb{R}^N \times \mathbb{R}^N \times (-\infty, t-\varepsilon)$

$$|h_t(x,y,s)| \leq \frac{c}{\varepsilon^{Q/2}} |f(y,s)| \in L^1(\mathbb{R}^{N \times (-\infty, t-\varepsilon)})$$

$$|\partial_i h_t(x,y,s)| \leq \frac{c}{\varepsilon^a} |f(y,s)| \in L^1(\mathbb{R}^N \times (-\infty, t-\varepsilon))$$

$$|\partial_{ij} h_t(x,y,s)| \leq \frac{c}{\varepsilon^{a'}} |f(y,s)| \in L^1(\mathbb{R}^N \times (-\infty, t-\varepsilon)) \ .$$

Therefore applying the standard theorem of differentiation under the integral sign we get that the classical derivatives of u_ε exist for all $(x,t) \in S_T$ and moreover:

$$\partial_i u_\varepsilon(x,t) = -\int_{-\infty}^{t-\varepsilon}\int_{\mathbb{R}^N} \partial_i\Gamma(x,t;y,s)f(y,s)dyds \tag{2.13}$$

$$\partial_{ij} u_\varepsilon(x,t) = -\int_{-\infty}^{t-\varepsilon}\int_{\mathbb{R}^N} \partial_{ij}\Gamma(x,t;y,s)f(y,s)dyds\ . \tag{2.14}$$

Since the integrands are continuous and uniformly bounded by an L^1 function by the dominated convergence theorem it follows that $\partial_i u_\varepsilon$ and $\partial_{ij} u_\varepsilon$ are continuous, hence these derivatives are also weak derivatives. Finally, exploiting (2.12), (2.13) and (2.14), we get that for almost every $(x,t) \in S_T$

$$\mathcal{L}u_\varepsilon(x,t) = -\int_{-\infty}^{t-\varepsilon}\int_{\mathbb{R}^N} \mathcal{L}\Gamma(x,t;y,s)f(y,s)dyds + \int_{\mathbb{R}^N}\Gamma(x,t;y,t-\varepsilon)f(y,t-\varepsilon)dy\ .$$

Hence, applying Theorem 1.3, it follows

$$\mathcal{L}u_\varepsilon(x,t) = \int_{\mathbb{R}^N}\Gamma(x,t;y,t-\varepsilon)f(y,t-\varepsilon)dy \qquad \text{a.e.}(x,t) \in S_T\ .$$

□

The following lemma is a refinement of Proposition 3.10 in [1] where pointwise convergence is proved.

Lemma 2.3. *If $f \in C_c^\infty(S_T)$ then, for every $K \subset\subset \mathbb{R}^N$*

$$\int_{\mathbb{R}^N}\Gamma(\cdot;y,t-\varepsilon)f(y,t-\varepsilon)dyds \xrightarrow[\varepsilon\to 0]{} f(\cdot) \text{ uniformly on } K\times(-\infty,T)\ .$$

Proof. We proceed as in the first part of the proof of Proposition 3.10 in [1]. Owing to

Theorem 1.4 and to (1.18):

$$
\begin{aligned}
|\int_{\mathbb{R}^N}\Gamma(x,t;y,t-\varepsilon)f(y,t-\varepsilon)dy - f(x,t)| &\le \int_{\mathbb{R}^N}\Gamma(x,t;y,t-\varepsilon)|f(y,t-\varepsilon)-f(x,t)|dy \le \\
&\le \frac{1}{\nu^N}\int_{\mathbb{R}^N}\Gamma_{\frac{1}{\nu}}(x,t;y,t-\varepsilon)|f(y,t-\varepsilon)-f(x,t)|dy = \\
&= \int_{\mathbb{R}^N}\frac{\exp(-\frac{\nu}{4}|D_0(\frac{1}{\sqrt{\varepsilon}})(x-E(\varepsilon)y|_0^2)}{\sqrt{\nu\, 4\pi)^N\varepsilon^Q c_0(1)}}|f(y,t-\varepsilon)-f(x,t)|dy = \\
&= \{z = D_0(\frac{1}{\sqrt{\varepsilon}})(x-E(\varepsilon)y), dz = \frac{1}{\varepsilon^{\frac{Q}{2}}}dy\} = \\
&= \int_{\mathbb{R}^N}\frac{\exp(-\frac{\nu}{4}|z|_0^2)}{\sqrt{\nu\, 4\pi)^N c_0(1)}}|f(E(\varepsilon)(x-D_0(\sqrt{\varepsilon})z),t-\varepsilon)-f(x,t)|dz \le \\
&\le \int_{\mathbb{R}^N}\frac{\exp(-\frac{\nu}{4}|z|_0^2)}{\sqrt{\nu\, 4\pi)^N c_0(1)}}||\nabla f||_\infty|(E(\varepsilon)x-x-E(\varepsilon)D_0(\sqrt{\varepsilon})z,-\varepsilon)|dz \le \\
&\le \int_{\mathbb{R}^N}\frac{\exp(-\frac{\nu}{4}|z|_0^2)}{\sqrt{\nu\, 4\pi)^N c_0(1)}}||\nabla f||_\infty\{|E(\varepsilon)x-x|+|E(\varepsilon)D_0(\sqrt{\varepsilon})z|+\varepsilon\}dz \le \\
&\le \int_{\mathbb{R}^N}\frac{\exp(-\frac{\nu}{4}|z|_0^2)}{\sqrt{\nu\, 4\pi)^N c_0(1)}}||\nabla f||_\infty\{|E(\varepsilon)x-x|+||E(\varepsilon)D_0(\sqrt{\varepsilon})|||z|+\varepsilon\}dz \le \\
&\le C||\nabla f||_\infty\{||E(\varepsilon)-I|||x|+||E(\varepsilon)D_0(\sqrt{\varepsilon})||+\varepsilon\}
\end{aligned}
$$

which, for x varying in a compact set, vanishes uniformly as $\varepsilon \to 0$. □

Now we can prove Theorem 2.5.

Proof of Theorem 2.5. In order to prove the existence theorem we exploit the uniform convergence of u_ε and its derivatives. We begin with the convergence of u_ε.
By (1.18) we easily get:

$$
|-\int_0^t\int_{\mathbb{R}^N}\Gamma(x,t;y,s)f(y,s)dyds - u_\varepsilon(x,t)| \le \int_{t-\varepsilon}^t\int_{\mathbb{R}^N}\Gamma(x,t;y,s)|f(y,s)|dyds \le \varepsilon||f||_\infty\ .
$$

Then for the first derivatives we proceed in the same way as in Corollary 3.12 in [1]. For any $i \in \{1,\dots,q\}$

$$
\begin{aligned}
&|-\int_0^t\int_{\mathbb{R}^N}\partial_{x_i}\Gamma(x,t;y,s)f(y,s)dyds - \partial_{x_i}u_\varepsilon| = |\int_{t-\varepsilon}^t\int_{\mathbb{R}^N}\partial_{x_i}\Gamma(x,t;y,s)f(y,s)dyds| \le \\
&\le \int_{t-\varepsilon}^t\int_{\mathbb{R}^N}|\partial_{x_i}\Gamma(x,t;y,s)|dyds||f||_\infty \le c\int_{t-\varepsilon}^t\frac{1}{\sqrt{t-s}}\left(\int_{\mathbb{R}^N}\Gamma_{c_1\nu^{-1}}(x,t,y,s)dy\right)ds||f||_\infty = \\
&= c\int_{t-\varepsilon}^t\frac{1}{\sqrt{t-s}}ds||f||_\infty = 2c\sqrt{\varepsilon}||f||_\infty\ .
\end{aligned}
$$

Finally, for the second derivatives we exploit Proposition 2.1. We claim that the integral

$$-\int_0^t \int_{\mathbb{R}^N} \partial^2_{x_j x_i}\Gamma(x,t;y,s)[f(E(s-t)x,s)-f(y,s)]dyds$$

is absolutely convergent and that $\partial_{x_i x_j} u_\varepsilon$ uniformly converge to it. Indeed, for any $(x,t) \in S_T$, since f is $C_c^\infty(\mathbb{R}^{N+1})$ (therefore also $C_x^\alpha(\mathbb{R}^{N+1})$ for any $\alpha \in (0,1)$), by Proposition 2.1 we get:

$$\begin{aligned}
&\int_{-\infty}^t \int_{\mathbb{R}^N} |\partial^2_{x_j x_i}\Gamma(x,t;y,s)[f(E(s-t)x,s)-f(y,s)]|dyds \leq \\
\leq &\mathbf{c}\int_{-\infty}^t \int_{\mathbb{R}^N} |\partial^2_{x_j x_i}\Gamma(x,t;y,s)|\,||E(s-t)x-y||^\alpha dyds \leq \mathbf{c}(t-\tau)^{\alpha/2} < +\infty
\end{aligned}$$

then, owing to Lemma 2.1:

$$\partial^2_{x_i x_j} u_\varepsilon(x,t) = -\int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^N} \partial^2_{x_j x_i}\Gamma(x,t;y,s)[f(E(s-t)x,s)-f(y,s)]dyds\ .$$

Therefore, by Proposition 2.1, we obtain:

$$\begin{aligned}
&|-\int_{-\infty}^t \int_{\mathbb{R}^N} \partial^2_{x_j x_i}\Gamma(x,t;y,s)[f(E(s-t)x,s)-f(y,s)]dyds - \partial_{ij}u_\varepsilon(x,t)| \leq \\
\leq &\int_{t-\varepsilon}^t \int_{\mathbb{R}^N} |\partial^2_{x_j x_i}\Gamma(x,t;y,s)[f(E(s-t)x,s)-f(y,s)]|dyds \leq \\
\leq &\mathbf{c}\int_{t-\varepsilon}^t \int_{\mathbb{R}^N} |\partial^2_{x_j x_i}\Gamma(x,t;y,s)|\,||E(s-t)x-y||^\alpha dyds \leq \mathbf{c}\varepsilon^{\alpha/2}\ .
\end{aligned}$$

We have proved that u has continuous derivatives up to the second order with respect to the variables x_i for $i \in \{1,\ldots,q\}$. Applying Lemma 2.3 we obtain that $\mathcal{L}u_\varepsilon \to f$ in L^∞_{loc} hence thanks to the formerly proved limits we get

$$Yu_\varepsilon \xrightarrow[\varepsilon]{L^\infty_{loc}} f - \sum_{i,j=1}^q a_{ij}\partial_{ij}u\ .$$

The convergence is also in $D'(S_T)$ so $Yu = f - \sum_{i,j=1}^q a_{ij}\partial_{ij}u$ in $D'(S_T)$. Therefore $Yu \in L^\infty(S_T)$ and $\mathcal{L}u = f$. Thus $u \in S^0(S_T)$ and by Theorem 2.3 it follows that $u \in S^\alpha(S_T)$. This concludes the proof. □

2.3. The Cauchy Problem with minimal regularity assumptions

Now we want to extend the existence result of the previous section, more precisely, we will prove the existence of solutions for functions f in C^α_x and then we will obtain the well posedness of the Cauchy Problem (CP0). The theorem we want to prove is the following:

Theorem 2.6. *Let $f \in C^\alpha_x(S_T)$ be such that $supp(f) \subset \mathbb{R}^N \times [0,T]$.*
Then, $\exists! u \in S^0(S_T)$ solution of

$$\begin{cases} \mathcal{L}u = f & \mathbb{R}^N \times (-\infty, T) \\ u(\cdot, 0) = 0 & \mathbb{R}^N \end{cases} .$$

Moreover $u \in S^\alpha(S_T)$ and there exists a constant c depending only on ν, T and α such that the following stability estimate holds:

$$\sum_{i,j=1}^{q} ||\partial_{ij}u||_{C^\alpha_x(S_T)} + ||Yu||_{C^\alpha_x(S_T)} + \sum_{i=1}^{q} ||\partial_i u||_{C^\alpha(S_T)} + ||u||_{C^\alpha(S_T)} \leq c||f||_{C^\alpha_x(S_T)} .$$

First we prove the existence theorem for compactly supported functions in $C^\alpha_x(S_T)$.

Theorem 2.7. *If $f \in C^\alpha_x(S_T)$ and is compactly supported, then, the function $u : S_T \to \mathbb{R}$ defined by*

$$u(x,t) = -\int_{-\infty}^{t} \int_{\mathbf{R}^N} \Gamma(x,t;y,s) f(y,s) dyds$$

is such that $u \in S^\alpha(S_T)$ and $\mathcal{L}u = f$.

Proof. In this proof the symbol $*$ denotes the standard convolution.
The proof exploits the compactness entailed by the Schauder estimates of the article [1]. Let $f_\varepsilon = f * \varphi_\varepsilon$ be the convolution of f with a mollifier and let $u_\varepsilon \in S^\alpha(S_T)$ be the solution given by the existence Theorem 2.5:

$$u_\varepsilon(x,t) = -\int_{-\infty}^{t} \int_{\mathbf{R}^N} \Gamma(x,t;y,s) f_\varepsilon(y,s) dyds .$$

We want to prove that $u_\varepsilon \xrightharpoonup[\varepsilon]{*} u$ in $L^\infty(S_T)$. First we prove that:

$$f_\varepsilon \xrightharpoonup[\varepsilon\to 0^+]{*} f \text{ in } L^\infty \ . \tag{2.15}$$

Indeed, since for any $\phi \in L^1(\mathbb{R}^{N+1})$, $\phi * \bar{\varphi}_\varepsilon \xrightarrow[\varepsilon]{L^1(S_T)} \phi$ where $\varphi_\varepsilon(-x,-t) = \bar{\varphi}_\varepsilon(x,t)$, then we easily have:

$$_{L^\infty}\langle f_\varepsilon, \phi\rangle_{L^1} = _{L^\infty}\langle f, \phi * \bar{\varphi}_\varepsilon\rangle_{L^1} \to \ _{L^\infty}\langle f, \phi\rangle_{L^1} \quad \forall \phi \in L^1(S_T) \ .$$

Now, since f has compact support, there exist $T_1 < T$ such that $supp(f), supp(f_\varepsilon) \subset \mathbb{R}^N \times (T_1, T)$ for any ε sufficiently small. Then, observing that $\Gamma(x,t;\cdot) \in L^1(\mathbb{R}^N \times (T_1,T))$ for any (x,t), we obtain:

$$\forall (x,t) \in S_T \qquad \int_{S_T} \Gamma(x,t;y,s) f_\varepsilon(y,s)dyds \to \int_{S_T} \Gamma(x,t;y,s) f(y,s)dyds$$

which means that $u_\varepsilon(x,t) \to u(x,t)$ for any $(x,t) \in S_T$.
Now, let $\phi \in L^1(S_T)$, since $u_\varepsilon \to u$ pointwise and $|\phi(u_\varepsilon - u)| \leq |\phi| 2 ||f||_\infty$, applying again the dominated convergence theorem we get:

$$\int_{S_T} \phi(x,t) \int_{S_T} \Gamma(x,t;y,s) f_\varepsilon(y,s)dydsdxdt \to \int_{S_T} \phi(x,t) \int_{S_T} \Gamma(x,t;y,s) f(y,s)dydsdxdt$$

hence

$$_{L^\infty}\langle u_\varepsilon, \phi\rangle_{L^1} \to _{L^\infty}\langle u, \phi\rangle_{L^1} \quad \forall \phi \in L^1(S_T) \ . \tag{2.16}$$

Then, notice that $||f_\varepsilon||_{C^\alpha_x(S_T)} \leq ||f||_{C^\alpha_x(S_T)}$ and $||u_\varepsilon|| \leq ||f||_\infty$. therefore, by the Schauder estimates (Theorem 2.4), it follows that, for some fixed constant $c > 0$:

$$\begin{aligned} ||u_\varepsilon||_{C^\alpha(S_T)} + \sum_{i=1}^q ||\partial_i u_\varepsilon||_{C^\alpha(S_T)} + \sum_{i,j=1}^q ||\partial_{ij} u_\varepsilon||_{C^\alpha_x(S_T)} + ||Y u_\varepsilon||_{C^\alpha_x(S_T)} \leq \\ \leq c\{||u_\varepsilon||_{L^\infty(S_T)} + ||f_\varepsilon||_{C^\alpha_x(S_T)}\} \leq c\{||u||_{L^\infty(S_T)} + ||f||_{C^\alpha_x(S_T)}\} \end{aligned}$$

for any $\varepsilon > 0$ sufficiently small. Hence the $L^\infty(S_T)$ norms of $\partial_i u_\varepsilon$, $\partial_{ij} u_\varepsilon$ $(i,j \in \{1, \ldots q\})$ and $Y u_\varepsilon$ are uniformly bounded (w.r.t. ε) therefore, applying the Banach-Alaoglu-Bourbaki Theorem (see the appendix A) we can obtain a subsequence converging in the weak* topology $\sigma(L^\infty(S_T), L^1(S_T))$. Moreover, noticing that the weak* convergence in $L^\infty(S_T)$ entails convergence in the sense of distributions, thanks to the uniqueness of the limit in the sense of distributions, we obtain that (for $i,j \in \{1,\dots,q\}$):

$$\exists \varepsilon_j \to 0 \quad s.t. \qquad \begin{aligned} &\partial_i u_{\varepsilon_i} \underset{\varepsilon_j}{\overset{*}{\rightharpoonup}} \partial_i u \\ &\partial_{ij} u_{\varepsilon_i} \underset{\varepsilon_j}{\overset{*}{\rightharpoonup}} \partial_{ij} u \\ &Y u_{\varepsilon_i} \underset{\varepsilon_j}{\overset{*}{\rightharpoonup}} Y u \ . \end{aligned}$$

Notice that actually it is not necessary to take a subsequence since the limit is unique. Finally, since u is continuous with weak derivatives Yu, $\partial_{ij}u$ in $L^\infty(S_T)$, it follows that $u \in S^0(S_T)$.
In order to complete the proof it is left to show that $\mathcal{L}u = f$, but, thanks to (2.15) we only need to prove that $\mathcal{L}u_\varepsilon \underset{\varepsilon}{\overset{*}{\rightharpoonup}} \mathcal{L}u$. Let $\phi \in L^1(S_T)$ and $i,j \in \{1,\dots,q\}$ then:

$$_{L^\infty}\langle a_{i,j}\partial_{i,j}u_\varepsilon, \phi\rangle_{L^1} = \ _{L^\infty}\langle \partial_{i,j}u_\varepsilon, a_{i,j}\phi\rangle_{L^1} \to \ _{L^\infty}\langle \partial_{i,j}u, a_{i,j}\phi\rangle_{L^1} = \ _{L^\infty}\langle a_{i,j}\partial_{i,j}u_\varepsilon, \phi\rangle_{L^1} \ .$$

Thus $u \in S^0(S_T)$ and $\mathcal{L}u = f \in C^\alpha_x(S_T)$. By Theorem 2.3 it follows that $u \in S^\alpha(S_T)$. □

Theorem 2.8. *Let $f \in C^\alpha_x(S_T)$ such that $supp(f) \subset \mathbb{R}^N \times (\tau, T)$ for some $\tau \in (-\infty, T)$ and let $u : S_T \to \mathbb{R}$ be defined by*

$$u(x,t) = -\int_{-\infty}^{t}\int_{\mathbf{R}^N} \Gamma(x,t;y,s)f(y,s)dyds$$

then $u \in S^\alpha(S_T)$ and $\mathcal{L}u = f$.

Proof. The proof of this theorem is similar to the previous one but this time f is approximated in a different way. Let $\{\phi_i\}_i \subset D(\mathbb{R}^N)$ be such that $0 \le \phi_i \uparrow 1$ (as $i \to +\infty$) and $\sup_i ||\phi_i||_{C^\alpha(\mathbb{R}^N)} < +\infty$, define $f_i = f\phi_i$ and u_i to be the solution of $\mathcal{L}u_i = f_i$ given in Theorem 2.7. Notice that u_i admits the representation formula:

$$u_i(x,t) = -\int_{-\infty}^{t}\int_{\mathbf{R}^N} \Gamma(x,t;y,s)f_i(y,s)dyds = -\int_{-\infty}^{t}\int_{\mathbf{R}^N} \Gamma(x,t;y,s)f(y,s)\phi_i(y)dyds \ .$$

Since $\Gamma(x,t;\cdot)$ is integrable for any fixed (x,t) then, thanks to dominated convergence, $u_i \to u$ poinwise and moreover $||u_i - u||_\infty \le ||f||_\infty$. Hence taking any $\psi \in L^1(S_T)$, by dominated convergence, we obtain:

$$_{L^\infty}\langle u - u_i, \psi\rangle_{L^1} = \int_{S_T} (u - u_i)\psi dxdt \to 0$$

thus $u_i \overset{*}{\rightharpoonup} u$ in $L^\infty(S_T)$.
Now we observe that $f_i \underset{i}{\overset{*}{\rightharpoonup}} f$ in $L^\infty(S_T)$, indeed $||f_i - f||_\infty \leq ||f_\infty||$ hence applying again the dominated convergence theorem, for any $\psi \in L^1(S_T)$ we obtain:

$$_{L^\infty}\langle f - f_i, \psi\rangle_{L^1} = \int_{S_T} f(1-\phi_i)\psi dxdt \to 0 \text{ for } i \to +\infty \ .$$

Finally, employing the Banach Alaoglu Bourbaki Theorem and the Schauder estimates [Theorem 2.4] as we did in the proof of Theorem 2.7, we find that, for $k, j \in \{1, \dots, q\}$:

$$\partial_k u_i \underset{i}{\overset{*}{\rightharpoonup}} \partial_i u \qquad \partial_{kj} u_i \underset{i}{\overset{*}{\rightharpoonup}} \partial_{kj} u \qquad Y u_i \underset{i}{\overset{*}{\rightharpoonup}} Y u \qquad \mathcal{L} u_i \underset{i}{\overset{*}{\rightharpoonup}} \mathcal{L} u \ .$$

This entails that $u \in S^0(S_T)$ and $\mathcal{L}u = f$. Therefore, applying again Theorem 2.3, we get $u \in S^\alpha(S_T)$. □

Now we can finally prove the Theorem 2.6.

Proof of Theorem 2.6. If u is a solution then, by Theorem 2.2, it is in the form $u(\cdot) = \int_{S_T} \Gamma(\cdot; y, s) f(y, s) dyds$, hence u is equal to the solution given by the Theorem 2.8. Moreover, it is also $S^\alpha(S_T)$ and by the Schauder estimates [Theorem 2.4] we get:

$$\sum_{i,j=1}^{q} ||\partial_{ij} u||_{C_x^\alpha(S_T)} + ||Yu||_{C_x^\alpha(S_T)} + \sum_{i,j=1}^{q} ||\partial_i u||_{C^\alpha(S_T)} + ||u||_{C^\alpha(S_T)} \leq$$
$$\leq c(||f||_{C_x^\alpha(S_T)} + ||u||_\infty) \leq 2c||f||_{C_x^\alpha(S_T)} \ .$$

□

2.4. On the general Cauchy Problem

This section concerns the general Cauchy problem:

$$\begin{cases} \mathcal{L}u = f & \mathbb{R}^N \times (-\infty, T) \\ u(\cdot, 0) = g & \mathbb{R}^N \end{cases} \tag{CP}$$

where $\mathcal{L}$ is satisfies the same conditions of sections 2.1 and 2.2.
First we recall the definition of solution of $\mathcal{L}u = 0$ and of (CP) with $f = 0$ from the article [5]. We remark that the article [5] requires only (1.3) instead of (2.2), but we still assume (2.2) to be satisfied.
We begin with the definition of solution from [5].

Definition 2.3 (Definition 1.2 [5]). *Given an interval I, a function $u : \mathbb{R}^N \times I \to \mathbb{R}$ is a solution of $\mathcal{L}u = 0$ in $\mathbb{R}^N \times I$ if:*

$u \in C(\mathbb{R}^N \times I)$;

for every $t \in I$, $u(\cdot, t) \in C^2(\mathbb{R}^N)$;

for every $x \in \mathbb{R}^N$, $u(x, \cdot)$ *is absolutely continuous and* $\frac{\partial u}{\partial t} \in L^\infty_{loc}(I)$;

for a.e. $t \in I$ *and every* $x \in \mathbb{R}^N$, $\mathcal{L}u(x, t) = 0$.

Definition 2.4 (Definition 1.3 [5]). *We say that u is a solution to the Cauchy problem:*

$$\begin{cases} \mathcal{L}u = 0 & \mathbb{R}^N \times (t_0, T) \\ u(\cdot, t_0) = g & \mathbb{R}^N \end{cases} \tag{2.17}$$

for some $T \in (-\infty, +\infty]$, $t_0 \in (-\infty, T)$, *where f is continuous in* $\mathbb{R}^N$ *of belongs to* $L^p(\mathbb{R}^N)$ *for some* $p \in [1, +\infty)$ *if:*
(a) u is a solution to the equation $\mathcal{L}u = 0$ *in* $\mathbb{R}^N \times (t_0, T)$;
(b_1) if $g \in C(\mathbb{R}^N)$ *then* $u(x, t) \to g(x_0)$ *as* $(x, t) \to (x_0, t_0^+)$ *for every* $x_0 \in \mathbb{R}^N$;
(b_2) if $g \in L^p(\mathbb{R}^N)$ *for some* $p \in [1, +\infty)$ *then* $u(\cdot, t) \in L^p(\mathbb{R}^N)$ *for every* $t \in (t_0, T)$ *and* $||u(\cdot, t) - g||_{L^p(\mathbb{R}^N)} \to 0$ *as* $t \to t_0$.

Now we recall the existence theorem from the article [5].

Theorem 2.9 (Theorem 4.11 [5]). *Let*

$$u(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) g(y) dy \tag{2.18}$$

Then:
(a) if $g \in L^p(\mathbb{R}^N)$ *for some* $p \in [1, +\infty]$ *or* $g \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *then u solves the equation* $\mathcal{L}u = 0$ *in* $\mathbb{R}^N \times (0, +\infty)$ *and* $u(\cdot, t) \in C^\infty(\mathbb{R}^N)$ *for any* $t > 0$.
(b) if $g \in C(\mathbb{R}^N)$ *and there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}^N} |g(x)| e^{-C|x|^2} dx < +\infty \ . \tag{2.19}$$

then there exists $T > 0$ *such that u solves the equation* $\mathcal{L}u = 0$ *in* $\mathbb{R}^N \times (0, T)$ *and* $u(\cdot, t) \in C^\infty(\mathbb{R}^N)$ *for any* $t > 0$.
The initial condition g is attained in the following senses:

(i) For every $p \in [1, +\infty)$, if $g \in L^p(\mathbb{R}^N)$ we have $u(\cdot, t) \in L^p(\mathbb{R}^N)$ for every $t > 0$, and

$$||u(\cdot, t) - g||_{L^p(\mathbb{R}^N)} \to 0 \; as \, t \to 0^+ \; .$$

(ii) If $g \in L^\infty(\mathbb{R}^N)$ and g is continuous at some point $x_0 \in \mathbb{R}^N$ then

$$u(x, t) \to g(x_0) \; as \, (x, t) \to (x_0, t_0^+) \; .$$

(iii) If $g \in C_*(\mathbb{R}^N)$ (i.e. vanishing at infinity) then

$$\sup_{x \in \mathbb{R}^N} |u(x, t) - f(x)| \to 0 \; as \, t \to 0 \; .$$

(iv) If $f \in C(\mathbb{R}^N)$ and satisfies (2.19), then

$$u(x, t) \to g(x_0) \; as \, (x, t) \to (x_0, 0^+) \; .$$

We remark that the article [5] contains also a uniqueness result which however we shall not recall here because in this section we limit ourselves to existence of a solution.
In order to obtain a solution we wish to add the one given by Theorem 2.9 to the one of Theorem 2.8 and obtain a solution of (CP). Before doing so we shall try to understand better the Definition 2.3.
In particular we want to show that if u is a function satisfying:
1) $u \in C(\mathbb{R}^N \times (0, T))$;
2) $u(\cdot, t) \in C^2(\mathbb{R}^N)$ for any $t \in (0, T)$;
3) $u(x, \cdot)$ is absolutely continuous for any $x \in \mathbb{R}^N$ and $\frac{\partial u}{\partial t} \in L^\infty_{loc}((0, T))$,
it is not guaranteed that classical x-derivatives of u are L^1_{loc} weak derivatives.
Consider the function:

$$u(x, t) =: \begin{cases} t^3 \cos(\frac{x}{t^2}) & t > 0, x \in \mathbb{R} \\ 0 & t \leq 0, x \in \mathbb{R} \end{cases} . \tag{2.20}$$

As we will see, this function satisfies all the properties above but its classical second derivative in the x direction is not an L^1_{loc} function on $\mathbb{R}^2$. Points 2) and 3) are immediate by the definition of u. Moreover, for any $(x, t) \in \mathbb{R}^2$ with $t > 0$ the classical derivative of any order at the point (x, t) exists and

$$\partial_x u(x, t) = -t \sin(\frac{x}{t^2}), \qquad \partial^2_{xx} u(x, t) = -\frac{1}{t} \cos(\frac{x}{t^2}),$$

$$\partial_t u(x,t) = 3t^2 \cos(\frac{x}{t^2}) - 2x \sin(\frac{x}{t^2}) \ .$$

In order to prove point 3) we compute the weak derivative of $t \mapsto u(x,t)$ for fixed $x \in \mathbb{R}$. Let $\phi \in D(\mathbb{R})$ and let x be a fixed point in $\mathbb{R}$, then for any $\varepsilon > 0$:

$$-\langle u(x,\cdot), \frac{d\phi}{dt}\rangle = -\int_{\{t<\varepsilon\}} u(x,t)\frac{d\phi}{dt}(t)dt - \int_{\{t>\varepsilon\}} u(x,t)\frac{d\phi}{dt}(t)dt = \dots$$

hence integrating by parts in the last integral

$$\begin{aligned}\dots = &-\int_{\{t<\varepsilon\}} u(x,t)\frac{d\phi}{dt}(t)dt - u(x,\varepsilon)\frac{d\phi}{dt}(\varepsilon) + \int_{\{t>\varepsilon\}} \partial_t u(x,t)\phi(t)dt = \\ =&o(\varepsilon) + \int_{\{t>\varepsilon\}} \left[3t^2\cos(\frac{x}{t^2}) - 2x\sin(\frac{x}{t^2})\right]\phi(t)dt\end{aligned}$$

and since the integrand is bounded we can take the limit, hence:

$$-\langle u(x,\cdot), \frac{d\phi}{dt}\rangle = \int_0^{+\infty} \left[3t^2\cos(\frac{x}{t^2}) - 2x\sin(\frac{x}{t^2})\right]\phi(t)dt$$

which proves that u satisfies point 3). Now we claim that the classical second order x-derivative is not in $L^1_{loc}(\mathbb{R}^2)$. Indeed, if it were L^1_{loc} then, by Fubini Tonelli theorem, for almost every $x \in (0,1)$ the function

$$t \mapsto \partial^2_{xx} u(x,t) = -\frac{1}{t}\cos(\frac{x}{t^2})$$

would be $L^1((0,1))$. But this is not true as the following computations show: fix $x \in (0,1)$ and $T > 0$

$$\int_{\frac{1}{T^2}}^1 |\frac{1}{t}\cos(\frac{x}{t^2})|dt = \{y = \frac{1}{t^2}, dy = -2\frac{dt}{t^3}\} = \int_1^T \frac{1}{y}|\cos(xy)|dy$$

hence taking the limit as $T \to +\infty$, by monotone convergence we obtain

$$\int_0^1 |\frac{1}{t}\cos(\frac{x}{t^2})|dt = \int_1^{+\infty} \frac{1}{y}|\cos(xy)|dy = +\infty \ .$$

Noticing that away from the line $t = 0$ the classical second x-derivative is of class C^∞ we obtain that the weak derivatives are not L^1_{loc} functions. Moreover we could give an explicit formula for the weak derivative: let $\phi \in D(\mathbb{R}^2)$ and $\varepsilon > 0$, then

$$\langle \partial^2_{xx} u, \phi \rangle = \int_{\mathbb{R}^2} u(x,t) \partial^2_{xx} \phi(x,t) dxdt =$$

$$= \int_{\mathbb{R}^2 \cap \{t > \varepsilon\}} u(x,t) \partial^2_{xx} \phi(x,t) dxdt + \int_{\mathbb{R}^2 \cap \{t < \varepsilon\}} u(x,t) \partial^2_{xx} \phi(x,t) dxdt = ...$$

integrating by parts in the first integral

$$= \int_{\mathbb{R}^2 \cap \{t > \varepsilon\}} \partial^2_{xx} u(x,t) \phi(x,t) dxdt - \int_{\mathbb{R}^2 \cap \{t < \varepsilon\}} u(x,t) \partial^2_{xx} \phi(x,t) dxdt$$

taking the limit as $\varepsilon \to 0^+$ we finally obtain:

$$\langle \partial^2_{xx} u, \phi \rangle = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2 \cap \{t > \varepsilon\}} \partial^2_{xx} u(x,t) \phi(x,t) dxdt \ .$$

This shows that, if true, in order to obtain that any solution u (in the sense of 2.3) has L^1_{loc} weak derivatives, it is necessary a nontrivial argument which takes into account also the fact that $\mathcal{L}u = 0$.

On the other hand, we are interested only in proving the existence of a solution so we can employ the explicit expression of u given by Theorem 2.9 and this leads to a solution with $\partial_i u, \partial_{ij} u \in L^\infty_{loc}(\mathbb{R}^N \times (0, +\infty))$ for any $i, j \in \{1, \ldots, N\}$ and $Yu \in L^\infty_{loc}(\mathbb{R}^N \times (0, +\infty))$. For simplicity we restrict ourselves to the case $g \in L^\infty(\mathbb{R}^N)$.

Proposition 2.2. *Let $g \in L^\infty(\mathbb{R}^N)$ and let u be defined by (2.18). Then:*
1) The weak derivatives $\partial_i u, \partial_{ij} u$ are $L^\infty_{loc}(\mathbb{R}^N \times (0, +\infty))$ for every $i, j \in \{1, \ldots, N\}$;
2) The weak derivative Yu is $L^\infty_{loc}(\mathbb{R}^N \times (0, +\infty))$;
3) $\mathcal{L}u = 0$ a.e. in $\mathbb{R}^N \times (0, +\infty)$.

Proof. Let u be defined by (2.18). As in the proof of Theorem 2.9 in [5], we can say that by the standard theorem of differentiation under the integral sign and the estimates on Γ, the classical x-derivatives of u can be taken inside the integral:

$$\partial_i u(x,t) = \int_{\mathbb{R}^N} \partial_i \Gamma(x,t;y,0) g(y) dy \tag{2.21}$$

$$\partial_{ij} u(x,t) = \int_{\mathbb{R}^N} \partial_{ij} \Gamma(x,t;y,0) g(y) dy \tag{2.22}$$

for any $i, j \in \{1, \ldots, N\}$.
Now we observe that for $(x,t) \in K \subset\subset \mathbb{R}^N \times (0,T)$ and $y \in \mathbb{R}^N$, applying Theorem 2.1,

we have:

$$|D_x^{\boldsymbol{\alpha}}\Gamma(x,t;y,0)| \leq \frac{c}{t^{\omega(\boldsymbol{\alpha})/2}}\Gamma_{\nu c_1} \leq c_{K,\alpha}\Gamma_{c_1\nu}(x,t;y,0) \ . \tag{2.23}$$

This implies (by (1.18)) that for any $(x,t) \in K$

$$|\partial_i u(x,t)| \leq \int_{\mathbb{R}^N} |\partial_i\Gamma(x,t;y,0)|\,|g(y)|dy \leq c_{K,i}||g||_{L^\infty} \tag{2.24}$$

$$|\partial_{ij} u(x,t)| \leq \int_{\mathbb{R}^N} |\partial_{ij}\Gamma(x,t;y,0)|\,|g(y)|dy \leq c_{K,i,j}||g||_{L^\infty} \tag{2.25}$$

therefore, the classical x-derivatives are bounded. Now, let $\phi \in D(\mathbb{R}^N \times (0,T))$, by Fubini-Tonelli theorem:

$$\int_{\mathbb{R}^N\times(0,T)} u(x,t)\partial_i\phi(x,t)dxdt = \int_0^T\int_{\mathbb{R}^N} u(x,t)\partial_i\phi(x,t)dxdt$$
$$\int_{\mathbb{R}^N\times(0,T)} u(x,t)\partial_{ij}\phi(x,t)dxdt = \int_0^T\int_{\mathbb{R}^N} u(x,t)\partial_{ij}\phi(x,t)dxdt$$

hence, integrating by parts in the inner integrals, we obtain

$$\int_{\mathbb{R}^N\times(0,T)} u(x,t)\partial_i\phi(x,t)dxdt = -\int_0^T\int_{\mathbb{R}^N} \partial_i u(x,t)\phi(x,t)dxdt$$
$$\int_{\mathbb{R}^N\times(0,T)} u(x,t)\partial_{ij}\phi(x,t)dxdt = \int_0^T\int_{\mathbb{R}^N} \partial_{ij} u(x,t)\phi(x,t)dxdt \ ,$$

applying again the Fubini-Tonelli theorem, we get

$$\int_{\mathbb{R}^N\times(0,T)} u(x,t)\partial_i\phi(x,t)dxdt = -\int_{\mathbb{R}^N\times(0,T)} \partial_i u(x,t)\phi(x,t)dxdt$$
$$\int_{\mathbb{R}^N\times(0,T)} u(x,t)\partial_{ij}\phi(x,t)dxdt = \int_{\mathbb{R}^N\times(0,T)} \partial_{ij} u(x,t)\phi(x,t)dxdt$$

and this proves the first point. The second point follows in the same way if we observe that the classical derivative $\partial_t u$ is equal to $\sum_{i,j=1}^q a_{ij}\partial_{ij}u - \sum_{i=1}^N b_{ij}x_j\partial_i u$ hence it is locally bounded. Therefore applying the same procedure for $\phi \in D(\mathbb{R}^N \times (0,T))$:

$$-\int_{\mathbb{R}^N\times(0,T)} u(x,t)\partial_t\phi(x,t)dxdt = -\int_{\mathbb{R}^N}\int_0^T u(x,t)\partial_t\phi(x,t)dxdt =$$
$$= \int_{\mathbb{R}^N}\int_0^T \partial_t u(x,t)\phi(x,t)dxdt = \int_{\mathbb{R}^N\times(0,T)} \partial_t u(x,t)\phi(x,t)dxdt$$

we get point two. The integration by parts is justified since $u(x,\cdot)$ is absolutely continuous.

This proves that the classical derivatives are also weak and hence the third point directly follows by the Theorem 2.9. □

Once we have done these observations, we can state an existence Theorem for (CP).

Theorem 2.10. *Let $g \in L^\infty(\mathbb{R}^N)$ and let $f \in C^\alpha_x(\mathbb{R}^N \times (0,+\infty))$. Moreover, let u be the function defined by*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0)g(y)dy - \int_{\mathbb{R}^N\times(0,T)} \Gamma(x,t;y,s)f(y,s)dyds \ . \tag{2.26}$$

Then it satisfies:
1) $u \in C(\mathbb{R}^N \times (0,+\infty))$;
2) $\partial_i u, \partial_{ij} u \in L^\infty_{loc}(\mathbb{R}^N \times (0,+\infty))$ for any $i,j \in \{1,\dots,q\}$;
3) $Yu \in L^\infty_{loc}(\mathbb{R}^N \times (0,+\infty))$;
4) $\mathcal{L}u = f$ where the derivative are considered as weak derivatives;
5) If $x_0 \in \mathbb{R}^N$ is a point of continuity for g,

$$u(x,t) \to g(x_0) \text{ as } (x,t) \to (x_0, 0^+)$$

and for any $p \in [1,+\infty)$

$$u(\cdot,t) \to g \text{ in } L^p_{loc}(\mathbb{R}^N) \text{ as } t \to 0^+ \ .$$

Proof. The first four points follow directly by Theorem 2.8, Theorem 2.9 and the Proposition 2.2. While the last point follows observing that if we take $R > 0$ and a nonnegative function $\phi \in C_c(\mathbb{R}^N)$ such that $\phi \equiv 1$ in $\{x : |x| < R\}$, writing

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0)\phi(y)g(y)dy + \int_{\mathbb{R}^N} \Gamma(x,t;y,0)(1-\phi(y))g(y)dy =: u_\phi(x,t) + u_{(1-\phi)}(x,t)$$

by Theorem 2.9, we obtain:

$$u_\phi(\cdot,t) \to g \text{ in } L^p(\{x : |x| < R\}) \text{ as } t \to 0^+$$

while

$$u_{(1-\phi)}(\cdot,t) \to g \text{ uniformly on } \{x : |x| < R\} \text{ as } t \to 0^+ \ .$$

□

Notice that in point 1) of Proposition 2.2 we have that the weak x-derivatives with respect to all the variables are locally essentially bounded while in point 2) of Theorem 2.10 we do not.

3 | Some extensions

In this chapter we shall prove some extensions of previous results. The first section contains some local estimates which allow us to prove the existence of a solution for unbounded datum f satisfying a growth condition at infinity (in Section 3.2). Moreover, section 3.3 contains a regularity result, together with a representation formula which allows us to prove uniqueness for the general Cauchy problem under weak assumptions on the solution (section 3.4). Finally, we conclude with the well-posedness of the general Cauchy problem (section 3.5).

The local estimates of section 3.1 and the results of section 3.3 aim to show that local regularity of a solution u depends only on the local regularity of $\mathcal{L}u$. Moreover, as we shall see, the local estimates (Theorem 3.1) let us prove some compactness properties like the one used in the proof of Theorems 2.7 and 2.8. These results let us to prove existence since if the datum is regular in some region we only need to care about the convergence of the integral which defines the solution. Indeed, if the function

$$u(x,t) = -\int_{\mathbb{R}^{N+1}} \Gamma(x,t;y,s) f(y,s)\,dyds \tag{3.1}$$

is well defined and L^∞_{loc} then, by approximating the datum f with C^∞_c functions and exploiting some suitable compactness properties we can prove that the function (3.1) is a solution. Actually, we could push forward this argument and consider a datum which may be singular in some region. The most natural data of this kind are measures in the form $g \otimes \delta_0$.

This kind of datum is natural since if we look at the heat equation we have the equivalence[1] between the two following problems:

$$\begin{cases} \mathcal{H}u = 0 & \mathbb{R}^N \times (0,T) \\ u(x,0) = g(x) & x \in \mathbb{R}^N \end{cases} \tag{3.2}$$

[1]Under suitable assumptions on the solution u, for instance $u \in C^{2,1}(\mathbb{R}^N \times [0,+\infty))$.

and

$$\mathcal{H}u = -g(\cdot) \otimes \delta_0 \qquad D'(\mathbb{R}^N \times (-\infty, T)) \ . \tag{3.3}$$

Moreover, proceeding formally, taking $f = -g \otimes \delta$ in (3.1) we obtain:

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0)g(y)dy$$

which is nothing but (2.18).

Actually, in the following g is assumed to be only a measure instead of a function (for instance L^1_{loc}) and in this case u assumes the form:

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0)g(dy) \ .$$

We decided to consider this weaker assumption on g since, if g were L^1_{loc} then, $g \otimes \delta$ would still be a measure on $\mathbb{R}^{N+1}$ and therefore the proof of Proposition 3.4, which is achieved by approximating $g \otimes \delta$ with test functions on $\mathbb{R}^{N+1}$, is the same if we assume g to be a measure. Another reason for this choice is that the uniqueness result (Theorem 3.4) assumes the initial datum to be satisfied in a very weak sense so that we still obtain uniqueness if g is a measure. Therefore, the only difference is a more general statement since any L^1_{loc} function could be viewed as an absolutely continuous[2] Radon measure.

3.1. Local estimates

Before stating the main result we introduce some notation.
In what follows D, D' and I, I' shall denote bounded open sets of $\mathbb{R}^N$ and of $\mathbb{R}$ respectively.
First we define the local version of C^α and C^α_x.

Definition 3.1. *Let $\alpha \in (0,1)$, we define:*

$$C^\alpha_{loc}(D' \times I') := \left\{ f \in C(D' \times I') : \ ||f||_{C^\alpha(D \times I)} < +\infty \ \forall D \times I \subset\subset D' \times I' \right\}$$

$$C^\alpha_{x,loc}(D' \times I') := \left\{ f \in L^\infty_{loc}(D' \times I') : \ ||f||_{C^\alpha_x(D \times I)} < +\infty \ \forall D \times I \subset\subset D' \times I' \right\}$$

where the norms are the same of Definition 1.17.

Then we define three spaces $X^\alpha(D' \times I')$, $X^\alpha_{loc}(D' \times I')$ and $S^\alpha_{loc}(D' \times I')$ as follows:

[2]With respect to dx.

Definition 3.2. *Let $\alpha \in (0,1)$, consider the following spaces:*

$$X^\alpha(D' \times I') := \{u \in C^\alpha(D' \times I') : \forall i,j \le q$$
$$\partial_i u \in C^\alpha(D' \times I'), \partial_{ij} u \in C^\alpha_x(D' \times I'),\ Yu \in C^\alpha_x(D' \times I')\}$$
$$X^\alpha_{loc}(D' \times I') := \{u \in C^\alpha_{loc}(D' \times I') :\ \forall i,j \le q$$
$$\partial_i u \in C^\alpha_{loc}(D' \times I'), \partial_{ij} u \in C^\alpha_{x,loc}(D' \times I'),\ Yu \in C^\alpha_{x,loc}(D' \times I')\}$$

$$S^\alpha_{loc}(D' \times I') := \{u \in C(D' \times I') :\ \forall i,j \le q \quad \partial_{ij} u \in C^\alpha_{x,loc}(D' \times I'),\ Yu \in C^\alpha_{x,loc}(D' \times I')\}\ .$$

Moreover, we define the norm:

$$||u||_{X^\alpha(D'\times I')} := \sum_{i,j=1}^{q} ||\partial_{ij} u||_{C^\alpha_x(D'\times I')} + \sum_{i,j=1}^{q} ||\partial_i u||_{C^\alpha(D'\times I')} + ||Yu||_{C^\alpha_x(D'\times I')} + ||u||_{C^\alpha(D'\times I')}\ .$$

We remark that the space $X^\alpha(D' \times I')$ with the norm $||\cdot||_{X^\alpha(D'\times I')}$ is a Banach space while $X^\alpha_{loc}(D' \times I')$ when endowed with the family of seminorms $\{||\cdot||_{X^\alpha(D\times I)}\}_{D\times I \subset\subset D'\times I'}$ is a Fréchet space.

The main result of this section is the following.

Theorem 3.1. *For every $D \times I$ and $D' \times I'$ nonempty open sets satisfying $D \times I \subset\subset D' \times I' \subset\subset \mathbb{R}^{N+1}$, there exists $c > 0$, depending only on $\mathcal{L}$, $D \times I$ and $D' \times I'$, such that:*

$$\forall u \in X^\alpha_{loc}(\mathbb{R}^{N+1}) \qquad ||u||_{X^\alpha(D\times I)} \le c(||u||_{L^\infty(D'\times I')} + ||\mathcal{L}u||_{C^\alpha_x(D'\times I')})\ .$$

This result let us to prove many interesting facts about the spaces previously defined. For instance, we can prove that $X^\alpha_{loc}(D \times I) = S^\alpha_{loc}(D \times I)$. Before proving Theorem 3.1 we need a preliminary lemma:

Lemma 3.1. *Let $\varepsilon > 0$ and let $\eta_\varepsilon \in D(B_\varepsilon)$ such that $\eta_\varepsilon \equiv 1$ on $B_{\varepsilon/2}$. Moreover, let $h \in L^1(\mathbb{R}^N)$ be a compactly supported function. Define $\xi_\varepsilon(x,t;y,s) = \eta_\varepsilon((y,s)^{-1} \circ (x,t))$ and the functions*

$$v : (x,t) \mapsto \int \Gamma(x,t;y,s)(1 - \xi_\varepsilon(x,t;y,s))h(y,s)dyds$$

and ($i \in \{1,\dots,N\}$)

$$w : (x,t) \mapsto \int \partial_{y_i}\Gamma(x,t;y,s)(1 - \xi_\varepsilon(x,t;y,s))h(y,s)dyds\ .$$

Then, for any β multi-index and for any $\alpha \in (0,1)$:

$$D_x^\beta v, D_x^\beta w \in C(\mathbb{R}^{N+1}), \qquad D_t D_x^\beta v, D_t D_x^\beta w \in C^\alpha_{x,loc}(\mathbb{R}^{N+1}) \ .$$

Notice that the functions v and w above have locally Lipschitz derivatives of any order with respect to x and $C^\alpha_{x,loc}$ derivatives with respect to time, therefore, they are also $X^\alpha_{loc}(\mathbb{R}^{N+1})$.

Proof. postponed. □

Proof of Theorem 3.1. In order to simplify the notation, within this proof we adopt the following conventions:
- we avoid writing the variables inside the integrals,
- since during the proof Γ is always evaluated at some point (x,t) and integrated with respect the last two variables, inside the integrals we shall write Γ without writing the variables,
- like in the previous point, inside an integral, $\nabla_y\Gamma$ stands for $\nabla_y\Gamma(\cdot;y,s)$ where (y,s) represent the integration variables,
- when the domain of integration is missing it is assumed to be $\mathbb{R}^{N+1}$.

For instance, if f is an L^∞ function on $\mathbb{R}^{N+1}$, we shall write:

$$u = \int \nabla_y \Gamma f \quad \text{in } \mathbb{R}^{N+1}$$

instead of

$$u(x,t) = \int_{\mathbb{R}^{N+1}} \nabla_y\Gamma(x,t;y,s) f(y,s) dyds \quad \forall (x,t)\mathbb{R}^{N+1} \ .$$

Let $F : X^\alpha_{loc}(D' \times I') \to C^\alpha_{x,loc}(D' \times I') \times C(D' \times I')$ be defined by $F(u) = (\mathcal{L}u, u)$ and let $\{(f_j, u_j)\}$ be a sequence in $Im(F)$ (the image of F) such that:

$$(f_j, u_j) \xrightarrow{C^\alpha_{x,loc} \times C} (f,u) \in \overline{Im(F)} \ .$$

We want to prove that $Im(F)$ is closed, equivalently that $(f,u) \in Im(F)$, so that we can apply the Open Mapping Theorem. Notice that $C^\alpha_{x,loc}(D' \times I') \times C(D' \times I')$ is a Fréchet space when endowed with the seminorms

$$\{||\cdot||_{L^\infty(D'\times I')} + ||\cdot||_{C^\alpha_x(D'\times I')}\}_{D'\times I' \subset\subset D \times I} \ .$$

Let $(x_0,t_0) \in D' \times I'$ and $U = \{(x,t) : |x - x_0| < r \quad |t - t_0| < \delta\} \subset\subset D' \times I'$, consider

$\phi \in D(D' \times I')$ such that $\phi \equiv 1$ on U. Then by the definition of X^{α}_{loc} it follows that $\phi u_j \in S^{\alpha}(\mathbb{R}^{N+1})$ and

$$\mathcal{L}(\phi u_j) = \mathcal{L}u_j\phi + [(\mathbb{A}\nabla\phi)\nabla u_j + \mathcal{L}\phi u_j]$$

hence owing to the representation formulas of Theorem 2.2 we obtain:

$$\phi u_j = -\int \Gamma(\mathcal{L}u)_j\phi - \int \Gamma[(\mathbb{A}\nabla\phi)\cdot\nabla u_j] - \int \Gamma(\mathcal{L}\phi)u_j \ .$$

The first integral converges in $X^{\alpha}(\mathbb{R}^{N+1})$ (as $j \to \infty$) thanks to Theorem 2.4.
While the second integral, when $(x,t) \in U$, can be integrated by parts since the singularity is excluded by the integration:

$$\int \Gamma[(\mathbb{A}\nabla\phi)\nabla u_j = \int [(-\mathbb{A}\partial^2\phi)u_j]\Gamma + \int [(-\mathbb{A}\nabla\phi)u_j]\cdot\nabla_y\Gamma \ .$$

Hence, we obtain that $\forall (x,t) \in U$

$$\begin{aligned}\phi u_j(x,t) = u_j(x,t) = -\int [\mathcal{L}\phi - \mathbb{A}\partial^2\phi]u_j\Gamma - \int [-(\mathbb{A}\nabla\phi)u_j]\cdot\nabla_y\Gamma - \\ -\int \phi f_j\Gamma = -\int (Y\phi)\Gamma u_j + \int \mathbb{A}\nabla\phi\cdot\nabla_y\Gamma u_j - \int \phi f_j\Gamma \ .\end{aligned}$$

Taking the limit as $j \to +\infty$, we obtain the following representation formula:

$$\forall (x,t) \in U \qquad u(x,t) = -\int (Y\phi)\Gamma u + \int \mathbb{A}\nabla\phi\cdot\nabla_y\Gamma u - \int \phi f\Gamma \ .$$

Now let $U_\varepsilon = \{(x,t) \in U : \ d((x,t);\partial U) > \varepsilon\}$ and let $\eta_\varepsilon \in D(B_\varepsilon)$ be such that $\eta_\varepsilon \equiv 1$ in $B_{\varepsilon/2}$. Define $\xi_\varepsilon(x,t;y,s) := \eta_\varepsilon((y,s)^{-1}\circ(x,t))$, then:
$\forall (x,t) \in U$

$$\int (Y\phi)\Gamma u = \int (Y\phi)\xi_\varepsilon(x,t;y,s)\Gamma u + \int (Y\phi)(1-\xi_\varepsilon(x,t;y,s))\Gamma u$$

$$\int \mathbb{A}\nabla\phi\cdot\nabla_y\Gamma u = \int \mathbb{A}\nabla\phi\cdot\nabla_y\Gamma\xi_\varepsilon(x,t;y,s)u + \int \mathbb{A}\nabla\phi\cdot\nabla_y\Gamma(1-\xi_\varepsilon(x,t;y,s))u \ .$$

Notice that when $(x,t) \in U_\varepsilon$ the two following equality holds:

$$\int [Y\phi u]\xi_\varepsilon(x,t;y,s)\Gamma = 0$$

$$\int [(-\mathbb{A}\nabla\phi)u]\xi_\varepsilon(x,t;y,s)\nabla_y\Gamma = 0$$

since, if $(x,t) \in U_\varepsilon$, the support of $(y,s) \mapsto \xi_\varepsilon(x,t;y,s)$ is contained in $\overline{U}$ while $Y\phi$ and $\nabla\phi$ are null in $\overline{U}$. Hence, we are left with terms which do not have any singularity, therefore, by Lemma 3.1, we obtain that $u \in X^\alpha_{loc}(D' \times I')$.
Lastly, thanks to the regularity properties of u_j and of u, the following step is justified. Let $\varphi \in D(D' \times I')$, integrating by parts, we have:

$$\begin{array}{ccccc} \int(\mathcal{L}u_j)\varphi & = & \int u_j\mathcal{L}^*\varphi & & \\ \downarrow & & \downarrow & & \text{as } j \to +\infty \ . \\ \int f\varphi & & \int u\mathcal{L}^*\varphi & = & \int(\mathcal{L}u)\varphi \end{array}$$

Hence we obtain that $\mathcal{L}u = f$, therefore, $F(u) = (f,u)$.
What we have just shown proves that $Im(F)$ is closed, hence, it is a Fréchet space (with the subspace topology) and since F is one-to-one and onto from $X^\alpha_{loc}(D' \times I')$ to $Im(F)$, by the Open Mapping Theorem (see the Appendix B), F is open (from $X^\alpha_{loc}(D' \times I')$ to $Im(F)$). Hence, for any $D \times I$ there exist a constant $c > 0$ and an open set $D'' \times I'' \subset\subset D' \times I'$, depending only on $\mathcal{L}$, $D \times I$ and $D' \times I'$, such that

$$\forall u \in X^\alpha_{loc}(D' \times I') \qquad ||u||_{X^\alpha(D\times I)} \le c(||u||_{L^\infty(D''\times I'')} + ||\mathcal{L}u||_{C^\alpha_x(D''\times I'')}) \ .$$

This follows in the same way as in (B.2). Indeed, taking a sequence of nonempty open subset of $\{D_n \times I_n\}_{n\ge 0}$ such that:

$$\bigcup_n D_n \times I_n = D' \times I'$$

and

$$D_n \times I_n \subset D_{n+1} \times I_{n+1} \subset\subset D' \times I' \quad \forall n \ge 0$$

we can apply the same argument used to prove (B.2) with $f = F$ and

$$\begin{aligned} E = X^\alpha_{loc}(D' \times I'), \qquad & p_j(\cdot) = ||\cdot||_{X^\alpha_{loc}(D_j\times I_j)} \\ F = Im(F), \qquad & q_i(\cdot) = ||\cdot||_{L^\infty(D_i\times I_i)} + ||\cdot||_{C^\alpha_x(D_i\times I_i)} \ . \end{aligned}$$

The thesis follows since any $u \in X^\alpha_{loc}(\mathbb{R}^{N+1})$ is also $X^\alpha_{loc}(D' \times I')$:

$$\begin{aligned} &\forall u \in X^\alpha_{loc}(\mathbb{R}^{N+1}) \\ &\quad ||u||_{X^\alpha(D\times I)} \le c(||u||_{L^\infty(D''\times I'')} + ||\mathcal{L}u||_{C^\alpha_x(D''\times I'')}) \le c(||u||_{L^\infty(D'\times I')} + ||\mathcal{L}u||_{C^\alpha_x(D'\times I')}) \ . \end{aligned}$$

□

Comparing the above proof with the one of Theorem 1.1 in [21] we immediately notice that the poof above exploits the same argument with some additional technicalities.

We are left to prove Lemma 3.1 and to do so, we exploit another technical lemma.

Lemma 3.2. *Let $K : \mathbb{R}^{2N+2} \to \mathbb{R}$ be such that for any multi-index β, $D_x^\beta K \in Lip_{loc}(\mathbb{R}^{2N+2})$. Then, given $h \in L^1(\mathbb{R}^{N+1})$ compactly supported, the function u defined by*

$$u(x,t) = \int_{\mathbb{R}^{N+1}} K(x,t;y,s)h(y,s)dyds \quad (x,t) \in \mathbb{R}^{N+1}$$

is such that for any multi-index β and for any $\alpha \in (0,1)$

$$D_x^\beta u \in C(\mathbb{R}^{N+1}), \qquad D_t D_x^\beta u \in C^\alpha_{loc}(\mathbb{R}^{N+1}) \ .$$

Proof. We begin by differentiating with respect to x under the integral sign. By the standard argument, we obtain:

$$D_x^\beta u(x,t) = \int_{\mathbb{R}^{N+1}} D_x^\beta K(x,t;y,s)h(y,s)dyds \ .$$

Moreover, since (by dominated convergence) these derivatives are continuous, then, they are also weak. Now we want to differentiate with respect to t inside the integral. Following the same argument as in the proof of Theorem 2.2, for any $(x,t) \in \mathbb{R}^{N+1}$

$$\frac{1}{\delta}[D_x^\beta u(x,t+\delta) - D_x^\beta u(x,t)] = \int_{\mathbb{R}^{N+1}} \frac{1}{\delta} \int_0^\delta D_t D_x^\beta K(x,t+\theta;y,s)d\theta h(y,s)dyds \ .$$

Since, for almost any $(y,s) \in \mathbb{R}^{N+1}$, we have:

$$\frac{1}{\delta}\int_0^\delta D_t D_x^\beta K(x,t+\theta;y,s)d\theta \to D_t D_x^\beta K(x,t;y,s)$$

and

$$\Big|\frac{1}{\delta}\int_0^\delta D_t D_x^\beta K(x,t+\theta;y,s)d\theta h(y,s)\Big| \leq |D_t D_x^\beta K(x,t;y,s)h(y,s)| \leq M|h(y,s)| \ ,$$

by dominated convergence, it follows that for any fixed $(x,t) \in \mathbb{R}^{N+1}$

$$\frac{1}{\delta}[D_x^\beta u(x,t+\delta) - D_x^\beta u(x,t)] \to \int_{\mathbb{R}^{N+1}} D_t D_x^\beta K(x,t;y,s)h(y,s)dyds \quad \text{as } \delta \to 0 \ .$$

Since the incremental ratio is locally uniformly bounded

$$\left|\frac{1}{\delta}[D_x^\beta u(x,t+\delta) - D_x^\beta u(x,t)]\right| \le M \int_{\mathbb{R}^{N+1}} |h(y,s)|dyds$$

the derivative is distributional (by Remark 2.1). Finally, once we obtained these formulas, denoting with u_ε the standard convolution of u with a mollifier, we obtain

$$|D_t D_x^\beta u_\varepsilon(x,t) - D_t D_x^\beta u_\varepsilon(x',t)| \le ||D_x D_t D_x^\alpha u_\varepsilon||_\infty |x-x'| \le ||D_t D_x D_x^\beta K||_\infty \int_{\mathbb{R}^{N+1}} |h||x-x'|$$

hence taking the limit almost everywhere as $\varepsilon \to 0^+$ we obtain that for almost any $(x,t) \in \mathbb{R}^{N+1}$

$$|D_t D_x^\beta u(x,t) - D_t D_x^\beta u(x',t)| \le ||D_t D_x D_x^\beta K||_\infty \int |h||x-x'| \ .$$

Since, for any $K \subset\subset \mathbb{R}^{n+1}$, there exists $c > 0$ such that if $(x,t), (x',t) \in K$

$$|x - x'| \le c\, d((x,t),(x',t)) = c\, ||x - x'||$$

then, there exists a representative of $D_t D_x^\beta u$ which belongs to $C^\alpha_{x,loc}(\mathbb{R}^{N+1})$. □

Now we an easily prove the Lemma 3.1.

Proof of Lemma 3.1. We want to apply the Lemma 3.2 for $K = \Gamma(1-\xi_\varepsilon)$ and $K = \nabla\Gamma(1-\xi_\varepsilon)$. Since $W^{1,\infty}_{loc}(\mathbb{R}^N) = Lip_{loc}(\mathbb{R}^N)$ it is sufficient to verify that $D_x^\beta K, D_t D_x^\beta K \in L^\infty_{loc}(\mathbb{R}^{2N+2})$. First we remark that the following Leibniz formula for partial derivatives holds (see, for instance, Theorem 2.5.2 of [10]):
If $f \in D'(\mathbb{R}^N)$ and $g \in C^\infty(\mathbb{R}^N)$ then for any multi-index α

$$D^\alpha fg = \sum_{\gamma+\beta=\alpha} \frac{\alpha!}{\gamma!\beta!} D^\gamma f D^\beta g \ .$$

Now, let Ψ denotes either Γ or $\partial_i\Gamma$ and let α be a multi-index, by the above formula:

$$D_x^\alpha K = \sum_{\gamma+\beta=\alpha} \frac{\alpha!}{\gamma!\beta!} D_x^\gamma \Psi D_x^\beta (1-\xi_\varepsilon)$$

hence applying again the Leibniz rule:

$$D_t D_x^\alpha K = \sum_{\gamma+\beta=\alpha} \frac{\alpha!}{\gamma!\beta!} D_t D_x^\gamma \Psi D_x^\beta (1-\xi_\varepsilon) + \sum_{\gamma+\beta=\alpha} \frac{\alpha!}{\gamma!\beta!} D_x^\gamma \Psi D_t D_x^\beta (1-\xi_\varepsilon) \ .$$

Therefore, we only need to prove that $D_t D_x^\gamma \Psi$ e $D_x^\gamma \Psi$ are bounded in $\{(x,t;y,s) : \varepsilon/2 < d((x,t),(y,s))\}$ since $1-\xi_\varepsilon$ is C^∞ and is equal to zero in $\{(x,t;y,s) : \varepsilon/2 > d((x,t),(y,s))\}$. Thanks to Theorem 2.1 this is immediate for $D_x^\gamma \Psi$ while for $D_t D_x^\gamma \Psi$ it is enough to observe that in $\{(x,t;y,s) : \varepsilon/2 < d((x,t),(y,s))\}$, denoting with $\mathcal{L}_x$ the operator $\sum_{i,j=1}^q a_{ij}(t) D_{ij} + \sum_{i,j=1}^N b_{ij} x_j D_i = \mathcal{L} + D_t$, we have:

$$D_t D_x^\gamma \Gamma = D_x^\gamma D_t \Gamma = D_x^\gamma \mathcal{L}_x \Gamma = \mathcal{L}_x D_x^\gamma \Gamma$$

and

$$D_t D_x^\gamma \partial_{y_i} \Gamma = D_x^\gamma \partial_i D_{y_i} D_t \Gamma = D_x^\gamma D_{y_i} \mathcal{L}_x \Gamma = \mathcal{L}_x D_x^\gamma D_{y_i} \Gamma \ .$$

This concludes the proof. □

3.2. Existence for unbounded datum

In this section we shall prove the existence of a solution to $\mathcal{L}u = f$ when f satisfies a suitable bound, but before stating the theorem we prove a compactness result for the space X^α_{loc} which exploits almost the same argument used in Theorems 2.7 and 2.8.

Proposition 3.1 (Compactness). *Let $\Omega = D \times I$ be an open set of $\mathbb{R}^{N+1}$ and let $\{u_j\}_j$ be a sequence in $X^\alpha_{loc}(\Omega)$ such that for any $D' \times I' \subset\subset \Omega$*

$$\sup_{j \geq 0} ||u_j||_{X^\alpha(D' \times I')} < +\infty \ .$$

Then, there exists a subsequence $\{u_{j_k}\}_k$ and a function u such that:
i) $u_{j_k} \to u$ locally uniformly;
ii) $\partial_i u_{j_k} \to \partial_i u$ locally uniformly;
iii) $u \in C^\alpha_{loc}(\Omega)$;
iv) $\partial_i u \in C^\alpha_{loc}(\Omega)$ for any $i \in \{1, \ldots, q\}$.
Moreover, for any fixed $D' \times I' \subset\subset \Omega$:
v) $Y u_{j_k |D' \times I'} \overset{*}{\rightharpoonup} Y u_{|D' \times I'}$ in $L^\infty(D' \times I')$;
vi) $\partial_{il} u_{j_k |D' \times I'} \overset{*}{\rightharpoonup} \partial_{il} u_{|D' \times I'}$ in $L^\infty(D' \times I')$ for any $i, l \in \{1, \ldots, q\}$;
vii) $\mathcal{L} u_{j_k |D' \times I'} \overset{*}{\rightharpoonup} \mathcal{L} u_{|D' \times I'}$ in $L^\infty(D' \times I')$.

Proof. Within the proof the indexes i and l are arbitrary integers in $\{1, \ldots, q\}$.
Let $D' \times I' \subset\subset \Omega$ be a nonempty open set, then, the norms $C^\alpha(D' \times I')$ of u_j and $\partial_i u_j$ are uniformly bounded hence, the functions $\{u_j\}_j$ and $\{\partial_l u_j\}_j$ are equicontinuous, therefore, by Ascoli Arzelà Theorem, it follows that there exists a subsequence given by j_k and a

function $v \in C(D' \times I')$ such that $\partial_i v \in C(D' \times I')$, $u_{j_k} \to v$ and $\partial_i u_{j_k} \to \partial_i v$.
Moreover, since the $C^\alpha(D' \times I')$ norms of u_{j_k} and $\partial_i u_{j_k}$ are uniformly bounded, then we obtain that $v, \partial_i v \in C^\alpha(D \times I)$. Indeed for v we proceed as follows: let $(x,t),(x',t') \in D' \times I'$, then

$$\frac{|v(x,t)-v(x',t')|}{d((x,t),(x',t'))} \le \frac{|v(x,t)-u_{j_k}(x,t)|}{d((x,t),(x',t'))} + \frac{|u_{j_k}(x,t)-u_{j_k}(x',t')|}{d((x,t),(x',t'))} + \frac{|v(x',t')-u_{j_k}(x',t')|}{d((x,t),(x',t'))}$$

taking the limit inferior, we obtain that

$$\frac{|v(x,t)-v(x',t')|}{d((x,t),(x',t'))} \le \liminf_{k\to+\infty} ||u_{j_k}||_{X^\alpha(\overline{D'\times I'})} \ .$$

We omit the proof for $\partial_i v$ since it is analogous.
Now, by the hypotheses, we see that the $L^\infty(D' \times I')$ norms of Yu_{j_k} and $\partial_{il} u_{j_k}$ are uniformly bounded (w.r.t. j), hence, by the Banach Alaoglu Bourbaki Theorem it follows that there exists a subsequence (which is still denoted with j_k) such that

$$Yu_{j_k|K} \overset{*}{\rightharpoonup} Yv, \qquad \partial_{il} u_{j_k|K} \overset{*}{\rightharpoonup} \partial_{il} v \ .$$

Now taking a sequence of subset $D_n \times I_n$ which is increasing and whose union is equal to Ω, by diagonalization, we can find a subsequence $\{u_{j_k^*}\}_k$ and a function u such that the conditions from i) to vi) are satisfied. It remains the condition vii). Let $\phi \in L^1(D' \times I')$ and let $H = D' \times I' \subset\subset \Omega$ then

$${}_{L^\infty}\langle a_{i,l}\partial_{i,l} u_{j_k^*|H}, \phi\rangle_{L^1} = {}_{L^\infty}\langle \partial_{i,l} u_{j_k^*|H}, a_{i,l}\phi\rangle_{L^1} \to {}_{L^\infty}\langle \partial_{i,l} u_{|H}, a_{i,l}\phi\rangle_{L^1} = {}_{L^\infty}\langle a_{i,l}\partial_{i,l} u_{|H}, \phi\rangle_{L^1} \ .$$

□

Now we sate the existence theorem.

Theorem 3.2. *Let $f \in C^\alpha_{x,loc}(\mathbb{R}^{N+1})$ t.c. $f \equiv 0 \quad t<0$. If there exists $0 < c$ such that*

$$|f(x,t)| \le const.e^{c|x|_0^2} \quad \forall (x,t) \ .$$

Then, there exists $T > 0$ such that the function

$$u(x,t) = -\int_{S_T} \Gamma(x,t;y,s) f(y,s) dyds$$

is well defined in S_T and is $S^\alpha_{loc}(S_T)$.

Moreover, it satisfies the following Cauchy Problem:

$$\begin{cases} \mathcal{L}u = f & S_T \\ u \equiv 0 & t \le 0 \end{cases} . \tag{3.4}$$

Notice that we used the norm $|\cdot|_0$ (1.19), but we could have stated it by using $|\cdot|$. However, we decided to use this norm since we shall use the estimates of Theorem 1.4, hence, with this notation the computations are a little simpler.

In order to prove the theorem we need a preliminary lemma.

Lemma 3.3. *Let $f \in L^\infty_{loc}(S_T)$ be such that $f \equiv 0$ in $\{(x,t) \in \mathbb{R}^{N+1} : t < 0\}$. Assume that there exists a positive constant c satisfying*

$$2c|E(-\sigma)D_0(\sqrt{\sigma})z|_0^2 < \frac{\nu}{4}|z|_0^2 \quad \forall \sigma \in (0,T]\, \forall z \neq 0 \tag{3.5}$$

and such that the following bound on f holds

$$|f(x,t)| \le const.e^{c|x|_0^2} \qquad a.e. \quad (x,t) \in S_T \ .$$

Then, the function

$$u(x,t) = \int_{S_T} \Gamma(x,t;y,s) f(y,s) dyds$$

is well defined for any $(x,t) \in S_T$ and there exists $c' > 0$ depending on c, T, $\mathbb{B}$ and ν such that:

$$|u(x,t)| \le cost.e^{c'|x|_0^2} \qquad \forall (x,t) \in S_T \ .$$

Remark 3.1. *Notice that the condition (3.5) could be seen as a relation between c and a suitable operator norm of $E(-\sigma)D_0(\sqrt{\sigma})$. However, we prefer to keep it in this way since it is directly applicable. Actually, condition of this kind arise naturally also in the next sections see: (3.12), (3.15), (3.26), (3.28) and (3.30).*

Proof. We can assume that

$$|f(x,t)| \le e^{c|x|_0^2} \qquad q.o. \quad (x,t) \in S_T \ .$$

Thanks to the estimates in Theorem 1.4:

$$|u(x,t)| \le \frac{1}{\nu^N}\int_0^t\int_{\mathbb{R}^N}\Gamma_{\frac{1}{\nu}}(x,t;y,s)e^{c|y|_0^2}dyds = \int_0^t\int_{\mathbb{R}^N}\frac{e^{-\frac{\nu}{4}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2}}{\sqrt{(4\pi\nu)^N(t-s)^Q c_0(1)}}e^{\mu|y|_0^2}dyds =$$

$$= \{w=E(t-s)y, t-s=\sigma\} = \int_0^t\int_{\mathbb{R}^N}\frac{e^{-\frac{\nu}{4}|D_0(\frac{1}{\sqrt{\sigma}})(x-w)|_0^2}}{\sqrt{(4\pi\nu)^N\sigma^Q c_0(1)}}e^{\mu|E(-\sigma)w|_0^2}dwd\sigma =$$

$$= \{x-w=z\} = \int_0^t\int_{\mathbb{R}^N}\frac{e^{-\frac{\nu}{4}|D_0(\frac{1}{\sqrt{\sigma}})z|_0^2}}{\sqrt{(4\pi\nu)^N\sigma^Q c_0(1)}}e^{\mu|E(-\sigma)(x-z)|_0^2}dzd\sigma \le$$

$$\le \int_0^t\int_{\mathbb{R}^N}\frac{e^{-\frac{\nu}{4}|D_0(\frac{1}{\sqrt{\sigma}})z|_0^2}}{\sqrt{(4\pi\nu)^N\sigma^Q c_0(1)}}e^{2c|E(-\sigma)z|_0^2}dze^{2c|E(-\sigma)x|_0^2}d\sigma = \{D(\sqrt{\sigma})\xi = z\} =$$

$$= \int_0^t\Big[\frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}}\int_{\mathbb{R}^N}e^{-\frac{\nu}{4}|z|_0^2}e^{2c|E(-\sigma)D(\sqrt{\sigma})z|_0^2}dz\Big]e^{2c|E(-\sigma)x|_0^2}d\sigma =: \star\ .$$

Exploiting the hypotheses we easily see that the integral between square bracket is uniformly bounded by a constant $M > 0$. Hence, there exists a constant $c' > 0$ such that

$$\star \le M\int_0^t e^{2c|E(-\sigma)x|_0^2}d\sigma \le Mte^{c'|x|_0^2}\ .$$

□

Now we can easily prove Theorem 3.2.

Proof of Theorem 3.2. Let $T > 0$ be such that $\forall t \in (0,T]$, $\forall w \neq 0$

$$2c|E(-t)D_0(\sqrt{t})w|_0^2 < \frac{\nu}{4}|w|_0^2\ .$$

Notice that this condition is nothing but (3.5). Hence, by Lemma 3.3, we obtain that the integral which defines u is absolutely convergent and that $u \in L^\infty_{loc}(S_T)$. Now take a partition of unity $\{\phi_n\}$ of $\mathbb{R}^{N+1}$. Let $u_n \in S^\alpha(S_T)$ be the solution to $\mathcal{L}u_n = \phi_n f$ as in Theorem 2.7, then, by the local estimate of Theorem 3.1, taken $D\times I \subset\subset D'\times I' \subset\subset \mathbb{R}^N\times(-\infty,T)$, we obtain

$$||\sum_{j\le n\le i}u_n||_{X^\alpha(D\times I)} \le c(||\sum_{j\le n\le i}u_n||_{L^\infty(D'\times I')} + ||\mathcal{L}\sum_{j\le n\le i}u_n||_{C^\alpha_x(D'\times I')})\ .$$

Since the series $\sum_{n=1}^{+\infty}u_n$ converges to u in $L^\infty_{loc}(\mathbb{R}^N\times(-\infty))$, the right-hand-side tends to zero as $i,j\to+\infty$. Hence, we obtain convergence of the series in $X^\alpha_{loc}(\mathbb{R}\times(-\infty,T))$ and therefore we have:

$$\mathcal{L}u = \mathcal{L}\sum_{n=1}^{+\infty} u_n = \sum_{n=1}^{+\infty} \mathcal{L}u_n = \sum_{n=1}^{+\infty} \phi_n f = f\ .$$

□

3.3. Regularity of solutions

In this section we prove a theorem which asserts that any solution with weak derivatives in a suitable L^p space belongs to X^α_{loc}. As a corollary we obtain the equivalence between X^α_{loc} and S^α_{loc}. Within this section a denotes a real number in $[1,+\infty)$ such that

$$\forall (x,t) \qquad \Gamma(x,t;\cdot) \in L^a(\mathbb{R}^N \times (\tau, T))$$

and a^* denotes its conjugate: $a^* = \frac{a}{a-1}$.
For instance we can consider $a \in [1, 1+\frac{2}{Q})$: (by Theorem 1.4)

$$\begin{aligned}
&\int_{t-T}^{t+T} [\Gamma(x,t;y,s)]^a dyds \leq \\
&\leq \int_{t-T}^{t}\int_{\mathbb{R}^{N+1}} \frac{1}{\sqrt{(4\pi\nu)^{Na} c_0(1)^a}} \frac{1}{(t-s)^{Q/2a}} e^{-\frac{\nu a}{4}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} dyds = \\
&= \{w = x - E(t-s)y, \sigma = t-s\} = \\
&= \frac{1}{\sqrt{(4\pi\nu)^{Na} c_0(1)^a}} \int_0^T \int_{\mathbb{R}^{N+1}} \frac{1}{\sigma^{aQ/2}} e^{-\frac{\nu a}{4}|D_0(\frac{1}{\sqrt{\sigma}})w|_0^2} dwd\sigma = \\
&= \{z = D_0(\frac{1}{\sqrt{\sigma}})w\} = \frac{1}{\sqrt{(4\pi\nu)^{Na} c_0(1)^a}} \int_0^T \int_{\mathbb{R}^{N+1}} \frac{1}{\sigma^{(a-1)Q/2}} e^{-\frac{\nu a}{4}|z|_0^2} dwd\sigma = \\
&= \frac{1}{\sqrt{(4\pi\nu)^{Na} c_0(1)^a}} \int_{\mathbb{R}^{N+1}} e^{-\frac{\nu a}{4}|z|_0^2} dz \int_0^T \frac{1}{\sigma^{(a-1)Q/2}} d\sigma < +\infty\ .
\end{aligned}$$

Theorem 3.3. *Let Ω be a nonempty open set and let u be such that:*
1) $u \in L^1_{loc}(\Omega)$;
2) $\partial_{ij}u \in L^{a^*}_{loc}(\Omega)$ for any $i,j \in \{1,\dots,q\}$;
3) $Yu \in L^{a^*}_{loc}(\Omega)$.
If for some $\alpha \in (0,1)$ $\mathcal{L}u \in C^\alpha_{x,loc}(\Omega)$ then $u \in X^\alpha_{loc}(\Omega)$.

Before proving the theorem we state a proposition which contains a representation formula.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^{N+1}$ and let u be such that*
1) $u \in L^1_{loc}(\Omega)$;
2) $\partial_{ij} u \in L^{a*}_{loc}(\Omega)$ for any $i, j \in \{1, \ldots, q\}$;
3) $Yu \in L^{a*}_{loc}(\Omega)$.
Then, for any $U \subset\subset \Omega$ open subset and for any $\phi \in D(\Omega)$ such that $\phi \equiv 1$ in U:

$$u(x,t) = \\ -\int_{\mathbb{R}^{N+1}} \Gamma(x,t;u,s)(\mathcal{L}u)(y,s)\phi(y,s)dyds - \int_{\mathbb{R}^{N+1}} \Gamma(x,t;y,s)Y\phi(y,s)u(y,s)dyds + \\ + \int_{\mathbb{R}^{N+1}} \nabla_y \Gamma(x,t;y,s) \cdot \mathbb{A}(s)\nabla\phi(y,s)u(y,s)dyds \quad \textit{for a.e. } (x,t) \in U \ .$$

If u is continuous then the equality holds for every $(x,t) \in U$.

Proof. Within this proof we adopt the same conventions we used in the proof of Theorem 3.1 but with the following additional notation:
• whenever w is a function w_ε indicates the mollified function (in $\mathbb{G}$).
With this notation the representation formula in the statement is rewritten as

$$u = -\int \Gamma(\mathcal{L}u)\phi - \int \Gamma Y\phi u + \int \nabla_y \Gamma \cdot \mathbb{A}\nabla\phi u \quad \text{a.e. in } U \ .$$

Notice also that $\mathcal{L}u_\varepsilon$ is the function obtained applying $\mathcal{L}$ to u_ε while $(\mathcal{L}u)_\varepsilon$ represents the mollified of $\mathcal{L}u$.

Let U and $\phi \in D(\Omega)$ be as in the statement of the theorem.
Since $u_\varepsilon \in X^\alpha_{loc}(\mathbb{R}^{N+1})$, by the properties of the convolution in $\mathbb{G}$, we have

$$\mathcal{L}(\phi u_\varepsilon) = (\mathcal{L}u)_\varepsilon \phi + [(\mathbb{A}\nabla\phi)\nabla u_\varepsilon + \mathcal{L}\phi u_\varepsilon] + [\mathbb{A}\partial^2 u_\varepsilon - (\mathbb{A}\partial^2 u)_\varepsilon]\phi$$

moreover, since $u_\varepsilon \phi \in X^\alpha(\mathbb{R}^{N+1})$ by Theorem 2.2 we obtain:

$$\phi u_\varepsilon = -\int \Gamma(\mathcal{L}u)_\varepsilon \phi - \int \Gamma[(\mathbb{A}\nabla\phi)\nabla u_\varepsilon + \mathcal{L}\phi u_\varepsilon] - \int \Gamma[\mathbb{A}\partial^2 u_\varepsilon - (\mathbb{A}\partial^2 u)_\varepsilon]\phi \text{ in } \mathbb{R}^{N+1} \ .$$

Integrating by parts for $(x,t) \in U$ we obtain:

$$\phi u_\varepsilon(x,t) = -\int \Gamma(\mathcal{L}u)_\varepsilon \phi - \int \Gamma[-(\mathbb{A}\partial^2\phi) + \mathcal{L}\phi]u_\varepsilon - \\ -\int \nabla_y \Gamma \cdot [-(\mathbb{A}\nabla\phi)u_\varepsilon] - \int \Gamma[\mathbb{A}\partial^2 u_\varepsilon - (\mathbb{A}\partial^2 u)_\varepsilon]\phi =: A_\varepsilon + B_\varepsilon + C_\varepsilon + D_\varepsilon$$

(the right hand side is evaluated at (x,t) according to the initial remarks on the notation).

Now we claim that:

$$A_\varepsilon \underset{\varepsilon}{\rightarrow} u_{\phi f} \text{ pointwise in } U \tag{3.6}$$

and

$$D_\varepsilon \underset{\varepsilon}{\rightarrow} 0 \text{ pointwise in } U \tag{3.7}$$

where $u_{\phi f} \in S^\alpha(\mathbb{R}^{N+1})$ is the solution of $\mathcal{L}u_{\phi f} = f\phi$ given by the existence Theorem 2.7. Actually, since for any $p \in (1, +\infty]$ and any $v \in L^p(\Omega)$ we have

$$v_\varepsilon \overset{*}{\underset{\varepsilon}{\rightharpoonup}} v \quad \text{in } L^p(\Omega)$$

(for $p \in (0, +\infty)$ it follows directly by the strong convergence, while for $p = +\infty$ it follows by a dominated convergence argument) we find:

$$(\mathcal{L}u)_\varepsilon \phi \overset{*}{\underset{\varepsilon}{\rightharpoonup}} (\mathcal{L}u)\phi \quad \text{in } L^{a^*}$$

$$[\mathbb{A}\partial^2 u_\varepsilon - (\mathbb{A}\partial^2 u)_\varepsilon]\phi \overset{*}{\underset{\varepsilon}{\rightharpoonup}} 0 \quad \text{in } L^{a^*} \ .$$

Therefore, by $\Gamma(x, t; \cdot) \in L^a$, we obtain (3.7) and (3.6).
Concerning B_ε and C_ε we want to prove that:

$$B_\varepsilon \underset{\varepsilon}{\rightarrow} -\int \Gamma[Y\phi u] \quad \text{pointwise in } U \tag{3.8}$$

and

$$C_\varepsilon \underset{\varepsilon}{\rightarrow} \int \nabla_y \Gamma \cdot (\mathbb{A}\nabla\phi)u \quad \text{pointwise in } U \ . \tag{3.9}$$

Notice that if $(x, t) \in U$ the functions $\Gamma(x, t; \cdot)Y\phi(\cdot)$ and $\nabla_y\Gamma(x, t; \cdot) \cdot (\mathbb{A}(\cdot)\nabla\phi(\cdot))$ are uniformly bounded and compactly supported, hence by the L^1_{loc} convergence of u_ε to u we obtain (3.8) and (3.9). This completes the proof since if u is continuous then u_ε converges to u pointwise, otherwise if u is only L^1_{loc}, u_ε converges up to a subsequence almost everywhere to u. □

Proof of Theorem 3.3. We shall adopt the same convention of the previous proof.
Let $(x_0, t_0) \in S_T$ and let $U \subset\subset \Omega$ be an open neighborhood of (x_0, t_0), let $\phi \in D(\Omega)$ be such that $\phi \equiv 1$ on U. Moreover, let $U_\varepsilon := \{(x, t) \in U : d((x, t); \partial U) > \varepsilon\}$ and $\eta_\varepsilon \in D(B_\varepsilon)$ be such that $\eta_\varepsilon \equiv 1$ in $B_{\varepsilon/2}$.

Define $\xi_\varepsilon(x,t;y,s) := \eta_\varepsilon((y,s)^{-1} \circ (x,t))$, then for every $(x,t) \in U$

$$\begin{aligned}&\int_{\mathbb{R}^{N+1}} \Gamma(x,t;y,s) Y\phi(y,s) u(y,s) dyds = \\ &= \int_{\mathbb{R}^{N+1}} \Gamma(x,t;y,s)\xi_\varepsilon(x,t;y,s) Y\phi(y,s) u(y,s) dyds + \\ &+ \int_{\mathbb{R}^{N+1}} \Gamma(x,t;y,s)(1-\xi_\varepsilon(x,t;y,s)) Y\phi(y,s) u(y,s) dyds\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^{N+1}} \nabla_y \Gamma(x,t;y,s) \cdot \mathbb{A}(s) \nabla\phi(y,s) u(y,s) dyds = \\ &= \int_{\mathbb{R}^{N+1}} \xi_\varepsilon(x,t;y,s) \nabla_y \Gamma(x,t;y,s) \cdot \mathbb{A}(s) \nabla\phi(y,s) u(y,s) dyds + \\ &+ \int_{\mathbb{R}^{N+1}} (1-\xi_\varepsilon(x,t;y,s)) \nabla_y \Gamma(x,t;y,s) \cdot \mathbb{A}(s) \nabla\phi(y,s) u(y,s) dyds \ .\end{aligned}$$

Since $\mathrm{supp}(\xi_\varepsilon(x,t;\cdot)) \subset B_\varepsilon(x,t)$ and $\phi \equiv 0$ on U then for any $(x,t) \in U_\varepsilon$:

$$\int_{\mathbb{R}^{N+1}} \Gamma(x,t;y,s)\xi_\varepsilon(x,t;y,s) Y\phi(y,s) u(y,s) dyds = 0$$

and

$$\int_{\mathbb{R}^{N+1}} \xi_\varepsilon(x,t;y,s) \nabla_y \Gamma(x,t;y,s) \cdot \mathbb{A}(s) \nabla\phi(y,s) u(y,s) dyds = 0 \ .$$

Hence, owing to the representation formula of Proposition 3.2, we can see that: (adopting for ξ_ε a convention similar to the one for Γ)

$$u = -\int \Gamma(\mathcal{L}u)\phi - \int \Gamma(1-\xi_\varepsilon) Y\phi u + \int (1-\xi_\varepsilon) \nabla_y \Gamma \cdot \mathbb{A} \nabla\phi u \quad \text{in } U \ .$$

Therefore since $\phi\mathcal{L}u \in C^\alpha_x(\mathbb{R}^{N+1})$ by the existence Theorem 2.7 and the Lemma 3.1 we find that $u \in X^\alpha_{loc}(U_\varepsilon)$. This concludes the proof since if ε is sufficiently small, $(x_0,t_0) \in U_\varepsilon$. □

From this it is easily proved the following Corollary.

Corollary 3.1. *Let Ω be an open set and let $\alpha \in (0,1)$, then*

$$X^\alpha_{loc}(\Omega) = S^\alpha_{loc}(\Omega) \ .$$

Proof. The hypotheses of Theorem 3.3 are satisfied for any $u \in S^\alpha_{loc}(\Omega)$. □

3.4. Uniqueness

In this section we aim to obtain uniqueness also for the case of nontrivial initial data. Actually, a uniqueness theorem for the Cauchy problem (CP) is present in the article [5] but the solution given by our existence theorem does not satisfy its hypotheses, hence it cannot be applied.

Let a and a^* as in section 3.3 then we have:

Theorem 3.4 (Uniqueness)**.** *Let u be a function such that:*
1) $u \in C(\mathbb{R}^N \times (0,T))$;
2) $\partial_{ij} u \in L^{a*}_{loc}(\mathbb{R}^N \times (0,T))$ for any $i,j \in \{1,\dots,q\}$;
3) $Yu \in L^{a*}_{loc}(\mathbb{R}^N \times (0,T))$;
4) $\mathcal{L}u = 0$ in $\mathbb{R} \times (0,T)$;
5) For any $\phi \in C_c(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} u(x,t)\phi(x)dx \xrightarrow[t\to 0^+]{} 0$$

(i.e. in the sense of zero order distributions);
6) There exists $c > 0$ such that:

$$\int_0^T \int_{\mathbb{R}^N} |u(x,t)| e^{-c|x|_0^2} dxdt < +\infty \ . \tag{3.10}$$

Then, $u \equiv 0$.

Before proving the theorem we point out the validity of a useful inequality:

$$\frac{1}{2}|a|_0^2 - |b|_0^2 \le |a-b|_0^2 \quad \forall\, a,b \in \mathbb{R}^N \ . \tag{3.11}$$

Indeed, since

$$\Big(\frac{|c+b|_0}{2}\Big)^2 \le \frac{|c|_0^2 + |b|_0^2}{2}$$

letting $c = a - b$ we obtain the inequality.

Proof. Consider $\overline{\nu} := \min\{\nu, \frac{\nu}{c_1}\}$ where c_1 is the constant of Theorem 2.1, then, let $\Delta \in (0,1)$ be such that:

- For any $\sigma \in (0,\Delta]$ and any $w \neq 0$, $\forall y \neq 0$

$$2(c+1)\frac{|y|_0^2}{|E(\sigma)y|_0^2} \le \frac{\overline{\nu}}{8}\frac{|D_0(\frac{1}{\sqrt{\sigma}})w|_0^2}{|w|_0^2} \ . \tag{3.12}$$

Notice that such a Δ exists since we can write the condition (3.12) as:
for any $\sigma \in (0, \Delta]$

$$2(c+1)||E(\sigma)^{-1}||_{Op} = 2(c+1) \sup_{y \neq 0} \frac{|y|_0^2}{|E(\sigma)y|_0^2} \leq \frac{\overline{\nu}}{8} \inf_{w \neq 0} \frac{|D_0(\frac{1}{\sqrt{\sigma}})w|_0^2}{|w|_0^2} = \frac{\overline{\nu}}{8} \frac{1}{||D_0(\sigma)||_{Op}} \tag{3.13}$$

where $||\cdot||_{Op}$ is a suitable operator norm. Therefore, the existence of Δ follows by the fact that $\sigma \mapsto ||E(\sigma)^{-1}||_{Op}$ and $\sigma \mapsto ||D_0(\sigma)||_{Op}$ are continuous and that the first is bounded in a neighborhood of 0 while the second tends to zero as $\sigma \to 0$.
Moreover, notice also that Δ depends only on $\overline{\nu}$, c, $\mathbb{B}$ and that $\forall \sigma \in (0, \Delta)$, $\forall z \in \mathbb{R}^N$

$$(c+1)|z|_0^2 \leq \frac{\overline{\nu}}{16} |D_0(\frac{1}{\sqrt{\sigma}})E(\sigma)z|_0^2 \leq \frac{\overline{\nu}}{8} |D_0(\frac{1}{\sqrt{\sigma}})E(\sigma)z|_0^2 \; . \tag{3.14}$$

From the definition of Δ it follows that there exists $\tilde{c} > 0$ such that:
• For any $\sigma \in (0, \Delta]$ and any $w \in \mathbb{R}^N$

$$\tilde{c}|w|_0^2 \leq \frac{\overline{\nu}}{8} |D_0(\frac{1}{\sqrt{\sigma}})w|_0^2, \qquad (c+1)|w|_0^2 \leq \frac{\tilde{c}}{2} |E(\sigma)w|_0^2 \; . \tag{3.15}$$

Fix $(x,t) \in \mathbb{R}^N \times (0, \Delta)$, we shall derive some estimates on $\Gamma(x,t;y,s)$ and on $\partial_i \Gamma(x,t;y,s)$ $(i \in \{1, \dots, q\})$ for (y,s) varying in $\mathbb{R}^N \times (0,t) \setminus \{(y,s) : \; x \neq E(t-s)y\}$.

Let $s \in (0,t)$ and $y \in \mathbb{R}^N$ be such that $x \neq E(t-s)y$, by Theorem 1.4:

$$\Gamma(x,t;y,s) \leq \frac{1}{\sqrt{(4\pi\overline{\nu})^N c_0(1)}(t-s)^{Q/2}} e^{-\frac{\overline{\nu}}{4}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} =: A$$

if $i \in \{1, \dots, q\}$ by Theorem 2.1 (c' is a fixed positive constant):

$$|\partial_{y_i}\Gamma(x,t;y,s)| \leq \frac{c'}{\sqrt{(4\pi\overline{\nu})^N c_0(1)}} \frac{1}{(t-s)^{(Q+1)/2}} e^{-\frac{\overline{\nu}}{4}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} =: B$$

Since $0 < t-s < \Delta < 1$ defining $m := \frac{\min\{1,c'\}}{\sqrt{(4\pi\overline{\nu})^N c_0(1)}}$ we obtain:

$$A, B \leq \frac{m}{\sqrt{(t-s)^{Q+1}}} e^{-\frac{\overline{\nu}}{4}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} =: \star$$

multiplying and dividing $\star$ by $|x-E(t-s)y|^{Q+1}$, since $t-s<\Delta<1$ we have:

$$\star = m\frac{|\frac{1}{\sqrt{t-s}}(x-E(t-s)y)|^{Q+1}}{|x-E(t-s)y|^{Q+1}}e^{-\frac{\overline{\nu}}{4}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} \leq$$

$$\leq m\frac{|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|^{Q+1}}{|x-E(t-s)y|^{Q+1}}e^{-\frac{\overline{\nu}}{4}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} \leq \dots$$

since every norm is equivalent in finite dimensional vector spaces, letting $h>0$ be a constant such that

$$|z| \leq h|z|_0 \quad \forall z \in \mathbb{R}^N$$

we obtain:

$$\dots \leq m\Big(\frac{8}{\overline{\nu}}h\Big)^{\frac{Q+1}{2}}\Big[\frac{\overline{\nu}}{8}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2\Big]^{\frac{Q+1}{2}}e^{-\frac{\overline{\nu}}{8}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2}$$

$$\cdot\frac{1}{|x-E(t-s)y|^{Q+1}}e^{-\frac{\overline{\nu}}{8}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} \leq$$

$$\leq M_{Q,\overline{\nu}}\frac{1}{|x-E(t-s)y|^{Q+1}}e^{-\frac{\overline{\nu}}{8}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2} =: \star\star$$

where $M_{Q,\overline{\nu}} := m\Big(\frac{8}{\overline{\nu}}h\Big)^{\frac{Q+1}{2}}\sup_{x>0}(x^{\frac{Q+1}{2}}e^{-x})$. Finally, by (3.15), we find:

$$\star\star \leq \frac{M_{Q,\overline{\nu}}}{|x-E(t-s)y|^{Q+1}}e^{-\tilde{c}|(x-E(t-s)y)|_0^2} \leq \dots$$

hence, exploiting the inequality (3.11) for $a=E(t-s)y$ and $b=x$ together with (3.11), we obtain:

$$\dots \leq \frac{M_{Q,\overline{\nu}}e^{\tilde{c}|x|_0^2}}{|x-E(t-s)y|^{Q+1}}e^{-\frac{\tilde{c}}{2}|E(t-s)y|_0^2} \leq \frac{M_{Q,\overline{\nu}}e^{\tilde{c}|x|_0^2}}{|x-E(t-s)y|^{Q+1}}e^{-(c+1)|y|_0^2}$$

Therefore, we proved that for any fixed $(x,t)\in\mathbb{R}^N\times(0,\Delta)$, and for any $(y,s)\in\mathbb{R}^N\times(0,t)$ such that $x\neq E(t-s)y$ ($i\in\{1,\dots,q\}$):

$$\Gamma(x,t;y,s), |\partial_{y_i}\Gamma(x,t;y,s)| \leq \frac{M_{Q,\overline{\nu}}e^{\tilde{c}|x|_0^2}}{|x-E(t-s)y|^{Q+1}}e^{-(c+1)|y|_0^2}\ . \tag{3.16}$$

Now fix $(x,t)\in\mathbb{R}^N\times(0,\Delta)$ then, there exists $R_0>0$ such that for any $y\in\mathbb{R}^N$ and any $\sigma\in(0,\Delta)$, if $|y|\geq R_0$

$$|x-E(\sigma)y| \geq 1\ .$$

Let $\rho\in D(0,1)$ be such that $\rho\geq 0$ and $\int_0^1\rho=1$, moreover, let $\phi(s) := \int_0^s\rho$. For any

$\tau \in \mathbb{R}$ $\varepsilon > 0$ and $R > 0$ define:

$$\varphi_R(y) := \phi(|y| - R), \qquad \phi_\varepsilon^\tau(y) := \phi(\frac{y-\tau}{\varepsilon})$$

It is apparent that φ_R has $C^1(\mathbb{R}^N)$ norm uniformly bounded (w.r.t. $R > 0$) and that

$$\frac{d}{ds}\phi_\varepsilon^\tau(s) \to \delta_\tau(s) \quad \text{in } (D^0(\mathbb{R}))' \; . \tag{3.17}$$

Taken $\tau \in (0,T)$, $\varepsilon \in (0, t-\tau)$ and $R > R_0$ by Proposition 3.2 (and Fubini Tonelli Theorem) it follows that:

$$\begin{aligned} u(x,t) = -\int_0^t \int_{\mathbb{R}^N} \mathbb{B}y \cdot \nabla\varphi_R(y)\phi_\varepsilon^\tau(s)u(y,s)\Gamma(x,t;y,s)dyds+ \\ +\int_0^t \partial_s\phi_\varepsilon^\tau(s)\int_{\mathbb{R}^N} \varphi_R(y)u(y,s)\Gamma(x,t;y,s)dyds- \\ -\int_0^t\int_{\mathbb{R}^N} \mathbb{A}(s)\nabla\varphi_R(y)\cdot\nabla_y\Gamma(x,t;y,s)u(y,s)\phi_\varepsilon^\tau(s)dyds \; . \end{aligned}$$

Then, we take the limit as $\varepsilon \to 0^+$, the first and the last integrals converge by dominated convergence while the second one converges since (3.17) and the function

$$s \mapsto \int_{\mathbb{R}^N} \varphi_R(y)\Gamma(x,t;y,s)u(y,s)dy$$

is continuous. Hence

$$\begin{aligned} u(x,t) = -\int_\tau^t \int_{\mathbb{R}^N} \mathbb{B}y \cdot \nabla\varphi_R(y)u(y,s)\Gamma(x,t;y,s)dyds+ \\ +\int_{\mathbb{R}^N} \varphi_R(y)u(y,\tau)\Gamma(x,t;y,\tau)dy- \\ -\int_\tau^t\int_{\mathbb{R}^N} \mathbb{A}(s)\nabla\varphi_R(y)\cdot\nabla_y\Gamma(x,t;y,s)u(y,s)dyds \end{aligned}$$

Now we take the limit as $\tau \to 0$, the first and the third integrals converge (still by dominated convergence) while the second integral tends to zero thanks to the hypothesis 5) and the fact that since

$$(y,\tau) \to \Gamma(x,t;y,\tau)\phi_R(y)$$

is uniformly continuous then

$$\Gamma(x,t;\cdot,\tau)\phi_R(\cdot) \to \Gamma(x,t;\cdot,0)\phi_R(\cdot) \quad \text{uniformly as } \tau \to 0 \; .$$

More precisely, due to the fact that, for any $R > 0$, $u(\cdot, t)$ could be viewed as an element of the dual of the space of continuous functions which are null at the boundary:

$$(C_{\#}(\overline{B_R(0)}), ||\cdot||_\infty)$$

we can write the second integral as:

$$\langle u(\cdot,\tau), \varphi_R(\cdot)\Gamma(x,t;\cdot,\tau)\rangle = \int_{\mathbb{R}^N} \varphi_R(y)u(y,\tau)\Gamma(x,t;y,\tau)dy$$

therefore by the well known fact that, given a Banach space X, if $x_j \underset{j}{\to} x$ in X and $L_j \overset{*}{\underset{j}{\rightharpoonup}} 0$ in X^* then $\langle L_j, x_j\rangle \to 0$, it follows that:

$$\langle u(\cdot,\tau), \varphi_R(\cdot)\Gamma(x,t;\cdot,\tau)\rangle = \int_{\mathbb{R}^N} \varphi_R(y)u(y,\tau)\Gamma(x,t;y,\tau)dy \to 0 \quad \text{as } \tau \to 0^+ \ .$$

Therefore, we obtain the following representation formula:

$$\begin{aligned} u(x,t) = &- \int_0^t \int_{\mathbb{R}^N} \mathbb{B}y \cdot \nabla\varphi_R(y)u(y,s)\Gamma(x,t;y,s)dyds + \\ &- \int_0^t \int_{\mathbb{R}^N} \mathbb{A}(s)\nabla\varphi_R(y)\cdot\nabla_y\Gamma(x,t;y,s)u(y,s)dyds \ . \end{aligned}$$

Finally, thanks to (3.16) and the hypothesis on R_0, as $R \to +\infty$ we have:

$$\begin{aligned} |u(x,t)| \leq &\int_0^t \int_{|y|>R} |\mathbb{B}y| \, ||\nabla\varphi_R||_\infty |u(y,s)|\Gamma(x,t;y,s)dyds + \\ &+ \int_0^t \int_{|y|>R} ||\mathbb{A}||_\infty ||\nabla\varphi_R||_\infty \sum_{j=1}^q |\partial_{y_j}\Gamma(x,t;y,s)| \, |u(y,s)|dyds \leq \\ &\leq M_{Q,\overline{\nu}} ||\nabla\varphi_R||_\infty \max\{1, ||\mathbb{A}||_\infty\} \\ &\left\{ \int_0^t \int_{|y|>R} (|\mathbb{B}y|+1)e^{-(c+1)|y|_0^2}|u(y,s)|dyds \right\} \leq \dots \\ \dots \leq &M_{Q,\overline{\nu}} ||\nabla\varphi_R||_\infty \max\{1, ||\mathbb{A}||_\infty\} \\ &\int_0^t \int_{\mathbb{R}^N} e^{-c|y|_0^2}|u(y,s)|dyds \sup_{|w|>R} \left[(|\mathbb{B}w|+1)e^{-|w|_0^2} \right] \to 0 \ . \end{aligned}$$

The proof follows by induction since, if $\Delta < T$, the hypotheses are verified for $u(x, t + \Delta/2)$. □

Here we state the uniqueness theorem from [5]. We remark that this theorem actually holds for the more general class of operators with matrix $\mathbb{B}$ whose $*$-entries in (1.3) may be nonnull.

Theorem 3.5 (Uniqueness, Theorem 4.13 [5]). *Let $T \in (0, +\infty]$, and let either $f \in C(\mathbb{R}^N)$, or $f \in L^p(\mathbb{R}^N)$ with $p \in [1, +\infty)$. If u_1 and u_2 are two solutions to the same Cauchy problem*

$$\begin{cases} \mathcal{L}u = 0 & \mathbb{R}^N \times (t_0, T), \\ u(\cdot, t_0) = g, \end{cases} \tag{3.18}$$

in the sense of Definition 2.4 and satisfy

$$\int_0^T \int_{\mathbb{R}^N} |u_j(x,t)| e^{-c|x|_0^2} dx dt < +\infty \quad j = 1, 2$$

for some $c > 0$, then $u_1 \equiv u_2$ in $\mathbb{R}^N \times (0, T)$.

3.5. More on the general Cauchy Problem

This section aims to develop further the remarks done in section 2.4, in particular we shall prove the existence of a solution (in S^α_{loc} for any $\alpha \in (0,1)$) under weaker assumptions on the initial datum. As remarked in the introduction to this chapter, the idea underlying the proof of the existence of a solution to the homogeneous problem is to approximate it with a sequence of solutions of suitable nonhomogeneous problems. In the following we shall consider initial data which are measures hence, if g as a measure, we shall replace (2.18) with:

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0) g(dy) = \int_{\mathbb{R}^{N+1}} \Gamma(x,t;y,s) g \otimes \delta(d(x,t)) \ .$$

It is easily seen that the case of L^1_{loc} functions is included in this one, since any L^1_{loc} function is simply an absolutely continuous Radon measure (absolutely continuous w.r.t. the Lebesgue measure). Moreover, due to the form of u whenever the initial datum is L^p or continuous we can exploit the Theorem 1.3 which let us to obtain stronger convergence to the initial datum.

Seen these preliminary observations and the formerly proved Theorems, we can state our last definition of solution:

Definition 3.3. *Let $f \in L^\infty_{loc}(\mathbb{R}^N \times (0, T))$, let g be a zero order distribution. Then, a* *function u is said to be a solution to the Cauchy Problem:*

$$\begin{cases} \mathcal{L}u = f & \mathbb{R}^N \times (0,T) \\ u(\cdot,0) = g & \mathbb{R}^N \end{cases} \tag{3.19}$$

if:
1) $u \in C(\mathbb{R}^N \times (0,T))$;
2) $Yu, \partial_{ij}u \in L^\infty_{loc}(\mathbb{R}^N \times (0,T))$ for any $i,j \in \{1,\dots,q\}$;
3) $\mathcal{L}u = f$ in $\mathbb{R}^N \times (0,T)$;
4) For any $\phi \in C_c(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} u(x,t)\phi(x)dx \xrightarrow[t\to 0^+]{} 0 \ .$$

Notice that, thanks to Theorem 2.9, we could say that $\Gamma(\cdot;y,0)$ is a solution of:

$$\begin{cases} \mathcal{L}\Gamma(\cdot;y,s) = 0 & \mathbb{R}^N \times (s,+\infty) \\ \Gamma(x,s) = \delta_y & \mathbb{R}^N \end{cases} \tag{3.20}$$

and actually it is unique among those which satisfy a certain condition (3.4).

Now we can state a theorem concerning existence and uniqueness:

Theorem 3.6. *Let g be a zero order distribution and let $f \in C^\alpha_{x,loc}(\mathbb{R}^N \times (0,+\infty))$. If there exists $c > 0$ such that:*

$$\sup\{\langle g(x), \varphi(x)e^{-c|x|_0^2}\rangle : \varphi \in C_c(\mathbb{R}^N),\ ||\varphi||_\infty \le 1\} < +\infty \tag{3.21}$$

$$\operatorname*{ess\,sup}_{t>0} \sup_{x\in\mathbb{R}^N} |f(x,t)|e^{-c|x|_0^2} < +\infty \tag{3.22}$$

then, there exists $T > 0$ and a unique solution of (3.19) satisfying

$$\int_s^{s+T} \int_{\mathbb{R}^N} |\Gamma(x,t;y,s)|e^{-c'|x|^2} dxdt$$

for some $c' > 0$. Moreover, the unique solution u assumes the form:

$$u(x,t) = -\int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;y,s)f(y,s)dyds + \int_{\mathbb{R}^N} \Gamma(x,t;y,0)g(dy)$$

and satisfies the two following conditions:
1) $u \in S^\alpha_{loc}(\mathbb{R}^N \times (0,T))$,
2) for any $D \times I$ and $D' \times I'$ satisfying $D \times I \subset\subset D' \times I' \subset\subset \mathbb{R}^N \times (0,T)$ there exist a

constant $C > 0$*, depending only on* c*,* $\mathcal{L}$*,* $D \times I$ *and* $D' \times I'$*, such that:*

$$\sum_{i,j=1}^{q} ||\partial_{ij} u||_{C_x^\alpha(D\times I)} + \sum_{i,j=1}^{q} ||\partial_i u||_{C^\alpha(D\times I)} + ||Yu||_{C_x^\alpha(D\times I)} + ||u||_{C^\alpha(D\times I)} \leq$$
$$\leq C\Big(\operatorname*{ess\,sup}_{t>0} \sup_{x\in\mathbb{R}^N} |f(x,t)| e^{-c|x|_0^2} + \sup_{\substack{\varphi\in C_c(\mathbb{R}^N)\\ ||\varphi||_\infty \leq 1}} \langle g(x), \varphi(x) e^{-c|x|_0^2}\rangle + ||f||_{C_x^\alpha(D'\times I')}\Big) \ .$$

Notice that, when g is a continuous function on $\mathbb{R}^N$, we may think to it as the zero order distribution defined by:

$$\langle g, \phi\rangle = \int_{\mathbb{R}^N} g(x)\phi(x)dx \quad \phi \in D(\mathbb{R}^N)$$

and the condition (3.21) reduces to:

$$\int_{\mathbb{R}^N} |g(x)| e^{-c|x|_0^2} dx < +\infty$$

hence, u is in the form:

$$u(x,t) = -\int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;y,s) f(y,s) dy ds + \int_{\mathbb{R}^N} \Gamma(x,t;y,0) g(y) dy$$

and the datum is attained locally uniformly. Moreover, if in addition g satisfies

$$\sup_{x\in\mathbb{R}^N} |g(x)| e^{-c|x|_0^2} < +\infty \tag{3.23}$$

then the stability estimate reduces to:

$$\sum_{i,j=1}^{q} ||\partial_{ij} u||_{C_x^\alpha(D\times I)} + \sum_{i,j=1}^{q} ||\partial_i u||_{C^\alpha(D\times I)} + ||Yu||_{C_x^\alpha(D\times I)} + ||u||_{C^\alpha(D\times I)} \leq$$
$$\leq C\Big(\operatorname*{ess\,sup}_{t>0} \sup_{x\in\mathbb{R}^N} |f(x,t)| e^{-c|x|_0^2} + \sup_{x\in\mathbb{R}^N} |g(x)| e^{-c|x|_0^2} + ||f||_{C_x^\alpha(D'\times I')}\Big) \ .$$

The proof of the theorem is achieved through some intermediate results and is given at the end of the section.

We begin with some notation which will be used.
Let c be a real constant and let $||\cdot||_c : C_c(\mathbb{R}^N) \to [0, +\infty)$ be defined by:

$$||\varphi||_c := \sup_{x\in\mathbb{R}^N} e^{c|x|_0^2} |\varphi(x)| \ .$$

It is easily seen that this defines a norm on $C_c(\mathbb{R}^N)$ (the space of compactly supported continuous functions) and that the completion of $(C_c(\mathbb{R}, ||\cdot||_c)$ could be seen as the set of functions $v \in C(\mathbb{R}^N)$ such that $x \mapsto v(x)e^{c|x|^2}$ belongs to $C_{\#}(\mathbb{R}^N)$ (the set of continuous function vanishing as $|x| \to +\infty$). We shall denote the completion of $(C_c(\mathbb{R}^N), ||\cdot||_c)$ with E_c. With this definition we can easily see that the condition on g is nothing but the requirement that g is an element of the dual of $(C_c(\mathbb{R}^N, ||\cdot||_c)$ (hence it can be defined on E_c). Moreover, since any zero order distribution on $\mathbb{R}^N$ could be seen as a Radon measure, if $\phi \in E_c$ then, extending g to the whole E_c, we write $g(\phi)$ as

$$\int_{\mathbb{R}^N} \phi(x) g(dx) \ .$$

We introduce also the following norm:

$$||g||_{c,*} := \sup\{\langle g(x), \varphi(x)e^{-c|x|_0^2}\rangle : \varphi \in C_c(\mathbb{R}^N),\ ||\varphi||_\infty \leq 1\} \ .$$

We can start with some preliminary results:

Lemma 3.4. *Let u_j be a sequence of continuous functions on $\mathbb{R}^N$ such that there exist $c \in \mathbb{R}$:*

$$\sup_{j\geq 0} ||u_j||_c < +\infty \tag{3.24}$$

and which converges locally uniformly as $j \to +\infty$ to a function u. Then u is continuous and such that

$$||u||_c \leq \liminf_j ||u_j||_c$$

moreover, for any $c' < c$:

$$u_j \xrightarrow[j\to+\infty]{||\cdot||_c} u \ .$$

Proof. The fact that u is continuous is immediate and by pointwise convergence we obtain also that $||u||_c < +\infty$. Indeed, for any $x \in \mathbb{R}^N$

$$|u(x)|e^{c|x|_0^2} = \liminf_j |u_j(x)|e^{c|x|_0^2} \leq \liminf_j ||u_j||_c \ .$$

Now let $R > 0$, we have:

$$\sup_{x\in\mathbb{R}^N} |u_j(x) - u(x)|e^{-c'|x|_0^2} \leq \sup_{|x|_0\leq R} |u_j - u|(x)e^{-c'|x|^2} + \sup_{|x|_0\geq R} |u_j - u|(x)e^{-c'|x|^2} \leq$$
$$\leq \sup_{|x|_0\leq R} |u_j - u|(x)e^{-c'R^2} + (\sup_j ||u_j||_c + ||u||_c)e^{-(c-c')R^2} \ .$$

Hence, taking the limit as $j \to +\infty$, we obtain

$$\lim_j ||u_j - u||_{c'} \leq (\sup_j ||u_j||_c + ||u||_c)e^{-(c-c')R^2}$$

therefore, by the fact that $c > c'$ we obtain our claim. □

We need another preliminary lemma:

Lemma 3.5. *The two following properties holds:*
1) For any fixed $c \geq 0$, for any $t > 0$ and $y \in \mathbb{R}^N$

$$\int_{\mathbb{R}^N} \Gamma(x,t;y,0)e^{-c|x|_0^2}dx \leq \frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}}\Big[\int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|w|_0^2}e^{+c|D_0(\sqrt{t})w|_0^2}dw\Big]e^{-\frac{c}{2}|E(t)y|_0^2} \ ;$$

2) If $f \in C_c(\mathbb{R}^N)$, then for any $c \geq 0$:

$$\int_{\mathbb{R}^N} \Gamma(x,t;\cdot,0)f(x)dx \xrightarrow{||\cdot||_c} f(\cdot) \quad as\ t \to 0 \ .$$

Proof. 1) Thanks to Theorem 1.4:

$$\int_{\mathbb{R}^N} \Gamma(x,t;y,0)e^{-c|x|_0^2}dx \leq \frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}}t^{-\frac{Q}{2}}\int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|D_0(\frac{1}{\sqrt{t}})(x-E(t)y)|_0^2}e^{-c|x|_0^2}dx =$$
$$= \{w = D_0(\frac{1}{\sqrt{t}})(x-E(t)y)\} = \frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}}\int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|w|_0^2}e^{-c|D_0(\sqrt{t})w+E(t)y|_0^2}dw =: \star$$

hence, applying the inequality (3.11) with $a = E(t)y$ and $b = D_0(\sqrt{t})w$, we obtain:

$$\star \leq \frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}}\Big[\int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|w|_0^2}e^{+c|D_0(\sqrt{t})w|_0^2}dw\Big]e^{-\frac{c}{2}|E(t)y|_0^2} \ .$$

2) In order to prove the second point we proceed in a similar way to the point (iii) of Theorem (2.9) (see [5]). Let $\phi \in D(\mathbb{R}^N)$ then

$$|\int_{\mathbb{R}^N} \Gamma(x,t;y,0)f(x)dx - f(y)| = |\int_{\mathbb{R}^N} \Gamma(x,t;y,0)[f(x)-f(y)]dx| \leq$$
$$\leq |\int_{\mathbb{R}^N} \Gamma(x,t;y,0)[\phi(x)-\phi(y)]dx| + 2||\phi - f||_\infty \int_{\mathbb{R}^N} \Gamma(x,t;y,0)dx =$$
$$= |\int_{\mathbb{R}^N} \Gamma(x,t;y,0)[\phi(x)-\phi(y)]dx| + 2||\phi - f||_\infty \ .$$

Consider the first integral:

$$\Big|\int_{\mathbb{R}^N}\Gamma(x,t;y,0)[\phi(x)-\phi(y)]dx\Big| \leq \int_{\mathbb{R}^N}\Gamma(x,t;y,0)|\phi(x)-\phi(y)|dx \leq$$

$$\leq \frac{t^{-\frac{Q}{2}}}{\sqrt{(4\pi\nu)^N c_0(1)}}\int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|D_0(\frac{1}{\sqrt{t}})(x-E(t)y)|_0^2}|\phi(x)-\phi(y)|dx = \{w = D_0(\frac{1}{\sqrt{t}})(x-E(t)y)\} =$$

$$= \frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}}\int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|w|_0^2}|\phi(D_0(\sqrt{t})w+E(t)y)-\phi(y)|dw \leq$$

$$\leq \frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}}$$

$$\Big\{\int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|w|_0^2}|\phi(D_0(\sqrt{t})w+E(t)y)-\phi(E(t)y)| + \int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|w|_0^2}dw|\phi(E(t)y)-\phi(y)|\Big\} \leq$$

$$\leq \frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}}\Big\{\int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|w|_0^2}|D_0(\sqrt{t})w|dw||\nabla\phi||_\infty + \int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|w|_0^2}dw|\phi(E(t)y)-\phi(y)|\Big\}$$

we see that, as $t \to 0^+$, it converges to zero uniformly with respect to y. Therefore we obtain:

$$\lim_{t\to 0^+}\sup_{y\in\mathbb{R}^N}|\int_{\mathbb{R}^N}\Gamma(x,t;y,0)f(x)dx - f(y)| \leq 2||f-\phi||_\infty$$

which proves the uniform convergence. Owing to point 1), for any $c \geq 0$ let $c' > 2c$ then:

$$\Big|\int_{\mathbb{R}^N}\Gamma(x,t;y,0)f(y)dy\Big|e^{c|y|_0^2} \leq \int_{\mathbb{R}^N}\Gamma(x,t;y,0)e^{-c'|y|_0^2}e^{c'|y|_0^2}|f(y)|dy e^{c|y|_0^2} \leq$$

$$\leq \Big[\sup_{y\in\mathbb{R}^N} e^{c'|y|_0^2}|f(y)|\Big]\int_{\mathbb{R}^N}\Gamma(x,t;y,0)e^{-c'|x|_0^2}dy e^{c|y|_0^2} \leq$$

$$\leq \frac{||f||_{E_{c'}}}{\sqrt{(4\pi\nu)^N c_0(1)}}\Big[\int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|w|_0^2}e^{+c'|D_0(\sqrt{t})w|_0^2}dw\Big]e^{-\frac{c'}{2}|E(t)y|_0^2+c|y|_0^2}\ .$$

This proves that, for t sufficiently small, the quantity

$$\sup_{y\in\mathbb{R}^N}\Big|\int_{\mathbb{R}^N}\Gamma(x,t;y,0)f(y)dy\Big|e^{c|y|_0^2}$$

is uniformly bounded, hence, by Lemma 3.4,the proof is concluded. □

Now we are in position to prove one of the two results which let us to prove Theorem 3.6.

Proposition 3.3. *If g is a zero order distribution and satisfies (3.21) for some $c \geq 0$.*

Then, the function

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0)g(dy) \quad (x,t) \in \mathbb{R}^N \times (0,T) \tag{3.25}$$

is well defined for some $T > 0$ and satisfies the following conditions:
1) There exists a constant $T > 0$ such that for any $t \in (0,T)$ and $x \in \mathbb{R}^N$

$$|u(x,t)| \leq \frac{1}{\sqrt{(4\pi\nu)^N c_0(1) t^Q}} e^{\frac{\nu}{4}|D_0(\frac{1}{\sqrt{t}})x|_0^2} ||g||_{c,*} \ ;$$

2) There exists $c' \geq c$ and a constant $M > 0$ depending only on T, ν, $\mathbb{B}$ and c such that for any $t \in (0,T)$ the following inequality holds

$$||u(\cdot,t)||_{c',*} \leq M||g||_{c,*} \ ;$$

3) For any $\phi \in C_c(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} u(x,t)\phi(x)dx \xrightarrow[t \to 0^+]{} 0 \ .$$

Proof. Let $T > 0$ be such that $\forall t \in (0,T]$, $\forall y \neq 0$, $\forall w \neq 0$

$$\frac{|y|_0^2}{|E(t)y|_0^2} 2c < \frac{\nu}{4} \frac{|w|_0^2}{|D_0(\sqrt{t})w|_0^2} \ . \tag{3.26}$$

The existence of such a T is proved as the existence of Δ in (3.12). Notice that in this case we are assuming a strict inequality since in what follows, we shall need the inequality to be strict.
Notice also that from the previous inequality it follows that $\forall t \in (0,T]$, $\forall w \in \mathbb{R}^N$

$$c|w|_0^2 \leq \frac{\nu}{8} |D_0(\frac{1}{\sqrt{t}})E(t)w|_0^2 \ . \tag{3.27}$$

which is nothing but (3.5).
1) Let $(x,t) \in \mathbb{R}^N \times (0,+\infty)$, thanks to Theorem 1.4:

$$|u(x,t)| \leq \int_{\mathbb{R}^N} \Gamma(x,t;y,0)d|g|(y) \leq \frac{1}{\sqrt{(4\pi\nu)^N c_0(1) t^Q}} \int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|D_0(\frac{1}{\sqrt{t}})(x-E(t)y)|_0^2} d|g|(y) \ .$$

Now, by (3.11), taking $c = D_0(\frac{1}{\sqrt{t}})(E(t)y)$ e $b = D_0(\frac{1}{\sqrt{t}})x$, we obtain:

$$|u(x,t)| \leq \frac{1}{\sqrt{(4\pi\nu)^N c_0(1) t^Q}} e^{\frac{\nu}{4}|D_0(\frac{1}{\sqrt{t}})x|_0^2} \int_{\mathbb{R}^N} e^{-\frac{\nu}{8}|D_0(\frac{1}{\sqrt{t}})(E(t)y)|_0^2} d|g|(y)$$

Since $\frac{\nu}{8}|D_0(\frac{1}{\sqrt{t}})(E(t)y)|_0^2 \geq c|y|_0^2$ for any $0 < t < T$ and any $y \in \mathbb{R}$ then, for any $0 < t < T$ and $x \in \mathbb{R}^N$, we have:

$$|u(x,t)| \leq \frac{1}{\sqrt{(4\pi\nu)^N c_0(1) t^Q}} e^{\frac{\nu}{4}|D_0(\frac{1}{\sqrt{t}})x|_0^2} \int_{\mathbb{R}^N} e^{-c|y|_0^2} d|g|(y) \ .$$

2) Let $c' > 0$ be such that $\forall t \in (0,T]$, $\forall y \neq 0$, $\forall w \neq 0$

$$\frac{|y|_0^2}{|E(t)y|_0^2} 2c < c' < \frac{\nu}{4} \frac{|w|_0^2}{|D_0(\sqrt{t})w|_0^2} \tag{3.28}$$

that is $\forall t \in [0,T]$, $\forall y \neq 0$, $\forall w \neq 0$

$$|y|_0^2 c < c'/2 |E(t)y|_0^2, \qquad |D_0(\sqrt{t})w|_0^2 c' < \frac{\nu}{4}|w|_0^2 \ . \tag{3.29}$$

It exists thanks to the definition of T, notice also the strict inequalities and the fact that t is assumed to be in $[0,T]$ (which is compact).
Then:

$$\int_{\mathbb{R}^N} |u(x,t)| e^{-c'|x|_0^2} dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{-c'|x|_0^2} \Gamma(x,t;y,0) d|g|(y) dx = \star \ ,$$

by Tonelli Theorem and point 1) of Lemma 3.5, we have:

$$\star \leq \int_{\mathbb{R}^N} \Big[\frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}} \int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|w|_0^2} e^{+c'|D_0(\sqrt{t})w|_0^2} dw \Big] e^{-\frac{c'}{2}|E(t)y|_0^2} d|g|(y)$$

therefore, by the definition of c', there exists a constant $M > 0$ such that for any $t \in (0,T)$ the following inequality holds:

$$\int_{\mathbb{R}^N} |u(x,t)| e^{-c'|x|_0^2} dx \leq M \int_{\mathbb{R}^N} e^{-c|y|_0^2} d|g|(y) \ .$$

The constant M exists since the inequalities (3.29) could be written as:

$$\max_{t \in [0,T]} ||E(t)^{-1}||_{Op}^2 < c'/2 \ , \quad c' < \frac{\nu}{4} \Big[\max_{t \in [0,T]} ||D_0(\sqrt{t})||_{Op}^2 \Big]^{-1}$$

where $||\cdot||_{Op}$ represents a suitable operator norm. Hence, we can find $\varepsilon > 0$ such that

$$|D_0(\sqrt{t})w|_0^2 c' - \frac{\nu}{4}|w|_0^2 \leq -\varepsilon|w|_0^2 \quad \forall w \in \mathbb{R}^N$$

and this proves that M exist:

$$M = \frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}} \int_{\mathbb{R}^N} e^{-\varepsilon|w|_0^2} dw \ .$$

3) Let $\phi \in C_c(\mathbb{R}^N)$ then, by point 2) of Lemma 3.5, we have:

$$\int_{\mathbb{R}^N} \Gamma(x,t;\cdot,0)\phi(x)dx \to \phi(\cdot) \quad \text{ in } E_c \text{ as } t \to 0 \ .$$

Therefore:

$$\Big| \int_{\mathbb{R}^N} u(x,t)\phi(x)dx - \int_{\mathbb{R}^N} \phi(y)dg(y) \Big| =$$
$$= \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} \Gamma(x,t;y,0)\phi(x)dx - \phi(y) \Big| d|g|(y) \to 0 \quad \text{ as } t \to 0^+ \ .$$

and this concludes the proof. □

Before moving to the next result we make two remarks concerning how the initial datum is attained.

Proposition 3.4. *Let g be a zero order distribution such that (3.21) holds. Then the function u defined by (3.25) belongs to $S^\alpha_{loc}(\mathbb{R}^N \times (0,T))$ for any $\alpha \in (0,1)$ and moreover $\mathcal{L}u = 0$.*

Proof. The proof is divided in two parts, the first part deals with compactly supported g while the second part deals with the general case.
The symbol $*$ denotes the standard convolution and the symbol $B_r(p)$ denotes the Euclidean ball or $\mathbb{R}^{N+1}$ centered at p with radius r.
1) Let g be a compactly supported zero order distribution, let $\rho_\varepsilon \in D(B_\varepsilon(0))$ be a mollifier ($\varepsilon > 0$) and let $g_\varepsilon = (g \otimes \delta) * \rho_\varepsilon$. Consider the function:

$$u_\varepsilon(x,t) = \int_{\mathbb{R}^{N+1}} \Gamma(x,t;y,s)g_\varepsilon(y,s)dyds$$

then, by Theorem 2.7, we obtain that $u_\varepsilon \in S^\alpha(\mathbb{R}^{N+1})$ and $\mathcal{L}u_\varepsilon = -g_\varepsilon$.
Now consider $D' \times I' \subset\subset \mathbb{R}^N \times (0,+\infty)$, let $\delta > 0$ be such that $D' \times I' \subset\subset \mathbb{R}^N \times (\delta,+\infty)$.

Then, for any $\varepsilon \in (0, \delta/2)$ and any $(x, t) \in D' \times I'$, owing to Theorem 1.4, we obtain:

$$\begin{aligned} |u_\varepsilon(x,t)| &\leq \int_{\mathbb{R}^N \times (-\varepsilon,\varepsilon)} \Gamma(x,t;y,s)|g_\varepsilon|(y,s)dyds \leq \\ &\leq \int_{\mathbb{R}^N \times (-\varepsilon,\varepsilon))} \frac{e^{-\frac{\nu}{4}|D_0(\frac{1}{\sqrt{t-s}})(x-E(t-s)y)|_0^2}}{\sqrt{(4\pi\nu)^N c_0(1)}(t-s)^{\frac{Q}{2}}}|g_\varepsilon|(y,s)dyds \leq \\ &\leq \frac{||g_\varepsilon||_{L^1(\mathbb{R}^{N+1})}}{(4\pi\nu)^{\frac{N}{2}}(\delta-\varepsilon)^{\frac{Q}{2}}} = \frac{|g|(\mathbb{R}^{N+1})}{(4\pi\nu)^{\frac{N}{2}}(\delta-\varepsilon)^{\frac{Q}{2}}} \leq \frac{2^{\frac{Q}{2}}|g|(\mathbb{R}^{N+1})}{(4\pi\nu)^{\frac{N}{2}}\delta^{\frac{Q}{2}}} \ . \end{aligned}$$

Now let $D \times I \subset\subset D' \times I'$, by Theorem 3.1, we obtain that there exists $\overline{c}$ such that for any $\varepsilon > 0$

$$||u_\varepsilon||_{X^\alpha(\overline{D\times I})} \leq \overline{c}(||u_\varepsilon||_{L^\infty(D'\times I')} + ||\mathcal{L}u_\varepsilon||_{C^\alpha_x(\overline{D'}\times I)})$$

and since $\mathcal{L}u_\varepsilon = -g_\varepsilon$ then, thanks to $\varepsilon \in (0, \delta/2)$, it follows that:

$$||u_\varepsilon||_{X^\alpha(\overline{D\times I})} \leq \overline{c}||u_\varepsilon||_{L^\infty(D\times I)} \leq \overline{c}\frac{2^{\frac{Q}{2}}|g|(\mathbb{R}^{N+1})}{\sqrt{(4\pi\nu)^N c_0(1)}\delta^{\frac{Q}{2}}} \ .$$

Therefore, we obtain convergence to a function v according to Proposition 3.1. Finally, since for any $\varepsilon > 0$

$$||g_\varepsilon||_{L^1(\mathbb{R}^{N+1})} = |g|(\mathbb{R}^N)$$

therefore since $D(\mathbb{R}^{N+1})$ is dense in $C_\#(\mathbb{R}^{N+1})$ and since $g_\varepsilon \to g \otimes \delta$ in $D'(\mathbb{R}^{N+1})$, we have for $\phi \in C_\#(\mathbb{R}^{N+1})$ and $\varphi \in D(\mathbb{R}^{N+1})$

$$\lim_\varepsilon |\langle g_\varepsilon - g\otimes\delta, \phi\rangle| \leq \lim_\varepsilon |\langle g_\varepsilon - g\otimes\delta, \phi-\varphi\rangle| + \lim_\varepsilon |\langle g_\varepsilon - g\otimes\delta, \varphi\rangle| \leq |g|(\mathbb{R}^N)||\phi-\varphi||_\infty$$

therefore

$$g_\varepsilon \overset{*}{\rightharpoonup} g\otimes\delta \quad \text{in } (C_\#(\mathbb{R}^N))^* \ .$$

This proves the pointwise convergence of u_ε to u in $\mathbb{R}^N \times (0, +\infty)$, hence, $u = v$.
It is easily proved that $\mathcal{L}u = 0$, actually, we have $\mathcal{L}u_\varepsilon \to 0$ in $L^\infty_{loc}(\mathbb{R}^N \times (0, T))$ and by Proposition 3.1 we have $\mathcal{L}u_{\varepsilon|K} \overset{*}{\rightharpoonup} \mathcal{L}u_{|K}$ in $L^\infty(K)$ for any $K \subset\subset \mathbb{R}^N \times (0, +\infty)$. Notice that by the previous section we know that u is S^α_{loc}, since $\mathcal{L}u = 0$.
2) Let g be as in the statement. Consider $R > 0$ and the measure $g_R = \chi_{\{|x|\leq R\}}g$. Let u_R be the function obtained as in the previous part.
Moreover, let $T > 0$ be as in Proposition 3.3 and let $D' \times I' \subset\subset \mathbb{R}^N \times (0, T)$, then, for any $(x, t) \in D' \times I'$ we have

$$|u_R(x,t)| \leq \int_{\mathbb{R}^N} \Gamma(x,t;y,0)d|g_R|(y) \leq \int_{\mathbb{R}^N} \Gamma(x,t;y,0)d|g|(y) \ .$$

Hence, by Proposition 3.3 we obtain that $||u_R||_{L^\infty(D'\times I')}$ is uniformly bounded with respect to $R>0$. Take $D\times I \subset\subset D'\times I'$, by Theorem 3.1 there exists a suitable constant $\overline{c}$ such that for any $R>0$

$$||u_R||_{X^\alpha(\overline{D\times I})} \le \overline{c}||u_R||_{L^\infty(D'\times I')} .$$

Therefore, by Proposition 3.1, it follows that u_R converges (in a suitable sense and up to a subsequence) to a function $v\in S^\alpha_{loc}(\mathbb{R}^N\times(0,T))$ which is such that $\mathcal{L}v=0$.
In order to conclude we need only to observe that u_R converges to u pointwise. Since $||g_R||_{c,*}\le ||g||_{c,*}$ and for fixed $(x,t)\in\mathbb{R}^N\times(0,T)$ the function $\Gamma(x,t;y,0)$ belongs to E_c, applying again the same argument as before we obtain our claim. □

We can finally prove Theorem 3.6.

Proof of Theorem 3.6. Let $T>0$ be such that $\forall t\in(0,T]$, $\forall y\neq 0$, $\forall w\neq 0$

$$\frac{|y|_0^2}{|E(t)y|_0^2}2c < \frac{\nu}{4}\frac{|w|_0^2}{|D_0(\sqrt{t})w|_0^2} . \tag{3.30}$$

Notice that T is the same of (3.26) and therefore it satisfies also the condition (3.5). The function

$$u(x,t) = -\int_0^t\int_{\mathbb{R}^N}\Gamma(x,t;y,s)f(y,s)dyds + \int_{\mathbb{R}^N}\Gamma(x,t;y,0)g(dy) =$$
$$=: u_f(x,t)+u_g(x,t) \quad (x,t)\in\mathbb{R}^N\times(0,T)$$

is well defined thanks to Lemma 3.3 and Prop 3.3. Moreover, by Proposition 3.4 and Theorem 3.2, we can say that $u\in S^\alpha_{loc}(\mathbb{R}^N\times(0,T))$ and $\mathcal{L}u=f$ in $\mathbb{R}^N\times(0,T)$. By point 3) of Proposition 3.3 we obtain that the initial datum is achieved, hence u is a solution of (3.19). Then by point 2) of Proposition 3.3, there exists $c'>0$ such that

$$\sup_{t\in(0,T)}\int_{\mathbb{R}^N}|u_g(x,t)|e^{-c'|x|_0^2}dx < +\infty$$

hence by the other estimate on u_f in Lemma 3.3 we obtain that the conditions of Theorem 3.4 are satisfied, therefore, the solution in unique among the functions which satisfy (3.10) for some c'. It is left only the stability estimate, in order to prove it we notice that looking at point 1) of Proposition 3.3 and to Lemma 3.3, we see that:

$$
\begin{aligned}
&|u(x,t)| \leq \\
&\leq \int_0^t \Big[\frac{1}{\sqrt{(4\pi\nu)^N c_0(1)}} \int_{\mathbb{R}^N} e^{-\frac{\nu}{4}|z|_0^2} e^{2c|E(-\sigma)D(\sqrt{\sigma})z|_0^2} dz \Big] e^{2c|E(-\sigma)x|_0^2} d\sigma \cdot \\
&\cdot \Big[\operatorname*{ess\,sup}_{t\in(0,T)} ||f(\cdot,t)||_{-c} \Big] + \frac{1}{\sqrt{(4\pi\nu)^N c_0(1) t^Q}} e^{\frac{\nu}{4}|D_0(\frac{1}{\sqrt{t}})x|_0^2} ||g||_{c,*} \ .
\end{aligned}
$$

Hence, for any pair of sets $D \times I$, $D' \times I'$ satisfying $D \times I \subset\subset \mathbb{R}^N \times (0,T)$ there exists a constant $M > 0$ depending only on $D' \times I'$, $\mathbb{B}$, ν and c such that:

$$
||u||_{L^\infty(D'\times I')} \leq M\big(\operatorname*{ess\,sup}_{t\in(0,T)} ||f(\cdot,t)||_{-c} + ||g||_{c,*} \big)
$$

this lets us to conclude since by 3.1 there exists $\tilde{c} > 0$ depending only on the two sets and the operator such that:

$$
||u||_{X^\alpha(D\times I)} \leq c(||u||_{L^\infty(D'\times I')} + ||f||_{C_x^\alpha(D'\times I')}) \ .
$$

Therefore letting $C := \max\{M, 1\}\tilde{c}$ we obtain:

$$
\begin{aligned}
&\sum_{i,j=1}^q ||\partial_{ij} u||_{C_x^\alpha(D\times I)} + \sum_{i,j=1}^q ||\partial_i u||_{C^\alpha(D\times I)} + ||Yu||_{C_x^\alpha(D\times I)} + ||u||_{C^\alpha(D\times I)} \leq \\
&\qquad \leq c\big(||g||_{c,*} + \operatorname*{ess\,sup}_{t\in(0,T)} ||f(\cdot,t)||_{-c} + ||f||_{C_x^\alpha(D'\times I')} \big) \ .
\end{aligned}
$$

□

A | Appendix A - The Banach-Alaoglu-Bourbaki Theorem

This appendix contains some results of Functional Analysis which have been used in the thesis.

Theorem A.1 (Theorem 6.7.1 [2]). *Let X be a separable normed space. Then every bounded sequence of linear functionals on X contains a weakly* convergent subsequence.*

In Theorems 2.7, 2.8 and Proposition 3.1 we apply the previous theorem to the case $X^* = L^\infty(\Omega)$ where Ω is a measurable sets of $\mathbb{R}^N$. Hence, we shall make some remarks which let us to apply the above theorem.
It is well known that $L^\infty(\Omega)$ is isometrically isomorphic to $L^1(\Omega)^*$ and the following function is an isometric isomorphism (see for instance Theorem 6.48 [24]):

$$\Phi : L^\infty(\Omega) \to L^1(\Omega)^* : g \mapsto \Lambda_g$$

$$\Lambda_g(f) = {}_{L^\infty}\langle g, f\rangle_{L^1}$$

where the notation ${}_{L^\infty}\langle g, f\rangle_{L^1}$ means:

$${}_{L^\infty}\langle g, f\rangle_{L^1} = \int_\Omega g(x)f(x)dx \ .$$

The weak* convergence is defined as:

$$g_j \xrightarrow[j\to+\infty]{*} g \text{ in } L^\infty(\Omega)$$

if for any $f \in L^1(A)$

$${}_{L^\infty}\langle g_j, f\rangle_{L^1} \to {}_{L^\infty}\langle g, f\rangle_{L^1} \ .$$

Since $L^1(\Omega)$ is separable we have sequential compactness.

The Theorem A.1 and the above argument are sufficient for the thesis. However, we decided to add a further explanation in terms of weak topologies since it may be useful in some parts of this thesis. We begin with the definition of weak topology:

Definition A.1 (Definition 8.9 [20]). *Let X be a set and X_α a topological space with $f_\alpha : X \to X_\alpha$, for each $\alpha \in A$. The weak topology induced on X by the collection of functions $\{f_\alpha : \alpha \in A\}$ is the smallest topology on X making each f_α continuous.*

From the previous definition, denoting with τ_α the topology of X_α, we can easily deduce that the family

$$\mathcal{S} = \{f_\alpha^{-1}(U_\alpha) : \ \alpha \in A, \ U_\alpha \in \tau_\alpha\}$$

is a subbase for the weak topology. That is, the finite intersections of subset of $\mathcal{S}$ form a base for the topology. Notice also that a sequence[1] $\{x_i\}_i$ converges to x in X iff for any $\alpha \in A$, for any $U_\alpha \in \tau_\alpha$ containing $f_\alpha(x)$, there exists $m \geq 0$ such that

$$\{x_i : i \geq m\} \subset f_\alpha^{-1}(U_\alpha)$$

which is equivalent to

$$f_\alpha(x_i) \xrightarrow[i \to +\infty]{} f_\alpha(x) \quad \forall \alpha \in A \ .$$

Now let E be a Banach space and let E^* be its topological dual, the weak* topology $\sigma(E^*, E)$ is the coarsest topology induced by the functions $\{\pi_x : x \in E\}$, where the function π_x is defined by:

$$\pi_x(L) = L(x) \ .$$

We could notice that the weak* topology on E^* is nothing but the subspace topology induced by the product topology on $\mathbb{R}^E$.

Theorem A.2 (Banach-Alaoglu-Bourbaki (Theorem 3.16 [6])). *The closed unit ball*

$$B_{E^*} = \{f \in E^* : ||f||_{E^*} \leq 1\}$$

is compact in the weak topology $\sigma(E^*, E)$.*

The last result we mention is a simple characterization of metrizability for the balls in E^* (with the topology induced by $\sigma(E^*, E)$) which could be employed in order to get Theorem A.1.

Theorem A.3 (Theorem 3.28 [6]). *Let E be a separable Banach space. Then B_{E^*} is*

[1] Actually this holds also for nets and filters.

metrizable in the weak topology* $\sigma(E^*, E)$.
Conversely, if B_{E^*} *is metrizable in* $\sigma(E^*, E)$*, then* E *is separable.*

Whenever E is separable, applying Theorem A.2 and A.3 we get A.1 but actually Theorem A.1 could be proved without involving the notion of weak topology.

B | Appendix B - The Open Mapping Theorem and Fréchet spaces

In this appendix we shall recall the Open Mapping Theorem for Fréchet spaces on $\mathbb{R}$. Actually, the Open Mapping Theorem holds in more general spaces but we do not need such a generality. Moreover, in the same spirit we shall give a characterization of Fréchet space avoiding any reference to locally convex topological vector space (briefly TVS) as well as to the notion of metrizability of a TVS ([22]).

We first give the standard definition of seminorm on a real vector space (actually it could be defined in a much more general context).

Definition B.1. *A seminorm on a real vector space E is a function $p : E \to [0, +\infty)$ such that:*
1) $P(0) = 0$,
2) for any x and y in E

$$p(x + y) \leq p(x) + p(y) \ ,$$

3) for any $\lambda \in \mathbb{R}$ and any $x \in E$

$$p(\lambda\, x) \leq |\lambda| p(x) \ .$$

Now we can give an equivalent characterization of Fréchet space (on $\mathbb{R}$) which is defined as a locally convex, complete and metrizable topological vector space (see [22]). Let E be a real vector space, and let $P = \{p_i\}_i$ be a countable set of seminorms on E, if:
1)for any x,

$$(\forall i \quad p_i(x) = 0) \implies x = 0 \ ,$$

2) if $\{x_i\}_i$ is a sequence in E such that for any i

$$p_i(x_k - x_m) \to 0 \quad \text{as } k, m \to +\infty$$

then there exists $x \in E$ such that for any i

$$p_i(x - x_k) \to 0 \quad \text{as } k \to +\infty \ .$$

then the space E with the topology induced by the seminorms P is a Fréchet space. Notice that with condition 1) we require that the following function on $E \times E$ is a metric:

$$d(x, y) = \sum_{i \in \mathbb{N}} \frac{1}{2^i} \frac{p_i(x - y)}{p_i(x - y) + 1} \tag{B.1}$$

and by point 2) we require that (E, d) is complete.

We can finally state the Open Mapping theorem in our particular case. For a more general statement see for instance [22] chapter 17.

Theorem B.1 (Open Mapping Theorem). *Let E and F be two Fréchet spaces and let $f : E \to F$ be a one-to-one and onto continuous linear map. Then its inverse is continuous.*

In this thesis the above theorem is applied in order to get local estimates. Hence, in the remaining part of this appendix we show how it has been applied.

First we observe that if E is a Fréchet space with seminorms $\{p_i\}_i$, we may assume the seminorms to be ordered. Indeed, taking

$$\overline{p_i} := \sum_{j \leq i} p_j$$

we obtain by (B.1) a metric which induces the same topology on E. This leads to the following remark. Assume that the norms are ordered, then, sets of the kind:

$$\{x \in E : p_i(x) < \varepsilon\} \quad \varepsilon > 0$$

form a neighbourhood base at 0.

Now let $P = \{p_i\}_i$ and $Q := \{q_i\}_i$ be the two families of seminorms on E and F respectively which make them Fréchet spaces, moreover, assume without loss of generality that they are ordered as before.
If f is a linear mapping like the one of Theorem B.1, then, for any p_j there exists q_i and

$c > 0$ such that the following estimate holds: for any $x \in E$

$$p_j(x) \leq cq_i(f(x)) \ . \tag{B.2}$$

Actually, this follows by a general property of continuous mappings between locally convex TVS (see Proposition 7.7 [22]), however, here we give a direct proof.
Since f is open, the set

$$f(\{x \in E : p_j(x) < 1\}) = \{f(x) \in F : x \in E\, p_j(x) < 1\}$$

is an open neighbourhood of 0 in F, therefore, there exists $\varepsilon > 0$ and q_i such that:

$$\{y \in F : q_i(y) < \varepsilon\} \subset \{f(x) \in F : x \in E\, p_j(x) < 1\} \ .$$

Thanks to the fact that f is one-to-one, for any $x \in E$

$$q_i(f(x)) < \varepsilon \implies p_j(x) < 1 \ . \tag{B.3}$$

If $q_i(f(x)) \neq 0$ taking $z := \varepsilon/(2q_i(f(x))x$ we have $q_i(f(z)) = \frac{\varepsilon}{2} < \varepsilon$, hence (by (B.3))

$$p_j(z) < 1$$

which means that

$$p_j(x) < \frac{2}{\varepsilon} q_i(f(x)) \ .$$

Else, if $q_i(f(x)) = 0$, for any $\lambda \in (0, +\infty)$

$$q_i(f(\lambda x)) = \lambda q_i(f(x)) = 0 < \varepsilon$$

therefore (by (B.3))

$$\lambda p_j(x) < 1$$

which entails that $p_j(x) = 0$. This proves that:

$$\forall x \in E \quad p_j(x) \leq \frac{2}{\varepsilon} q_i(f(x)) \ .$$

The inequality (B.2) is used in Theorem 3.1 in order to obtain local estimates.

Bibliography

[1] S. Biagi and M. Bramanti. Schauder estimates for Kolmogorov-Fokker-Planck operators with coefficients measurable in time and Hölder continuous in space. *arXiv*, 2022. doi: 10.48550/ARXIV.2205.10270. URL `https://arxiv.org/abs/2205.10270`.

[2] I. V. Bogachev and O. G. Smolyanov. *Real and Functional Analysis*. Moscow Lectures. Springer Cham, first edition. doi: 10.1007/978-3-030-38219-3. URL `https://doi.org/10.1007/978-3-030-38219-3`.

[3] M. Bramanti. *An Invitation to Hypoelliptic Operators and Hörmander's Vector Fields*. SprngerBriefs in Mathematics. Springer Cham, 01 2014. ISBN 978-3-319-02086-0. doi: 10.1007/978-3-319-02087-7. URL `https://doi.org/10.1007/978-3-319-02087-7`.

[4] M. Bramanti and L. Brandolini. *Hörmander Operators*. World Scientific, 2022.

[5] M. Bramanti and S. Polidoro. Fundamental solutions for Kolmogorov-Fokker-Planck operators with time-depending measurable coefficients. *Mathematics in Engineering*, 2(4):734–771, 2020. ISSN 2640-3501. doi: 10.3934/mine.2020035. URL `https://www.aimspress.com/article/doi/10.3934/mine.2020035`.

[6] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer New York, NY, 2011. doi: 10.1007/978-0-387-70914-7. URL `https://doi.org/10.1007/978-0-387-70914-7`.

[7] M. Di Francesco and A. Pascucci. On a class of degenerate parabolic equations of Kolmogorov type. *Applied Mathematics Research eXpress*, 2005(3):77–116, 05 2005. ISSN 1687-1200. doi: 10.1155/AMRX.2005.77. URL `https://doi.org/10.1155/AMRX.2005.77`.

[8] F. Dressel. The fundamental solution of the parabolic equation, ii. *Duke Mathematical Journal*, 13:186–203, 1940. doi: 10.1215/S0012-7094-40-00711-6. URL `https://doi.org/10.1215/S0012-7094-40-00711-6`.

[9] L. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010. ISBN 9780821849743.

[10] G. Friedlander and M. Joshi. *Introduction to the Theory of Distributions*. Cambridge University Press, 2nd edition edition, 1998.

[11] A. Friedman. *Partial Differential Equations of Parabolic Type*. Dover Publications, 1983.

[12] L. Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 119:147–171, 1967. doi: 10.1007/BF02392081. URL `https://doi.org/10.1007/BF02392081`.

[13] A. Kolmogoroff. Zufallige Bewegungen (zur Theorie der Brownschen Bewegung). *Annals of Mathematics*, 35:116, 1934.

[14] L. Kuptsov. The fundamental solution of a certain class of elliptic-parabolic second order equations. *Differentsial'nye Uravneniya*, 8(9):1649–1660, 1972.

[15] L. Kuptsov. Fundamental solutions of certain degenerate second-order parabolic equations. *Mathematical Notes of the Academy of Sciences of the USSR*, 31:283–289, 1982. doi: 10.1007/BF01138938. URL `https://doi.org/10.1007/BF01138938`.

[16] A. Lanconelli, A. Pascucci, and S. Polidoro. Gaussian lower bounds for non-homogeneous kolmogorov equations with measurable coefficients. *Journal of Evolution Equations*, 20:1399–1417, 2020. doi: 10.1007/s00028-020-00560-7. URL `https://doi.org/10.1007/s00028-020-00560-7`.

[17] E. Lanconelli and S. Polidoro. On a class of hypoelliptic evolution operators. *Rend. Sem. Mat. Univ. Politec. Torino*, 52(1):29–63, 1994.

[18] A. Pascucci and A. Pesce. On stochastic Langevin and Fokker-Planck equations: the two-dimensional case. *arXiv*, 2019. doi: 10.48550/ARXIV.1910.05301. URL `https://arxiv.org/abs/1910.05301`.

[19] I. Sonin. On a class of degenerate diffusion processes. *Theory of Probability and Its Applications*, 12:490–496, 1967. doi: 10.1137/1112059. URL `https://doi.org/10.1137/1112059`.

[20] W. Stephen. *General Topology*. Addison-Wesley series in mathematics. Dover Publications, 2004.

[21] F. Trèves. *Basic Linear Partial Differential Equations*. Academic Press, New York, 1975.

[22] F. Trèves. *Topological Vector Spaces, Distributions and Kernels*. Dover books on mathematics. Dover Publications, 2006.

[23] M. Weber. The fundamental solution of a degenerate partial differential equation of parabolic type. *Transactions of the American Mathematical Society*, 71:24–37, 1951. doi: 10.1090/S0002-9947-1951-0042035-0. URL `https://doi.org/10.1090/S0002-9947-1951-0042035-0`.

[24] W. P. Ziemer. *Modern Real Analysis*. Springer Cham, 2nd edition, 2017. ISBN 978-3-319-64628-2. doi: 10.1007/978-3-319-64629-9. URL `https://doi.org/10.1007/978-3-319-64629-9`.

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