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On existence of rotund Gâteaux smooth norms which are not midpoint locally uniformly rotund

Tesi di Laurea Magistrale in
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Abstract

The field of renorming theory in Banach spaces focuses on adjusting the norm while preserving topological properties. This theory is instrumental in unveiling the underlying geometric structures within Banach spaces and generating valuable mathematical insights. Traditionally, Banach spaces have been categorized based on their topological and geometrical properties.

In this study, we narrow our focus to investigate the relationship between three fundamental aspects: the norm's geometric properties, smoothness, and separability of a space. It was previously established that every separable Banach space could be renormed with locally uniformly rotund (LUR) norms that are also Gâteaux differentiable. Similarly, separable Banach spaces with separable duals could be renormed with LUR norms that are also Fréchet differentiable.

However, we tackle two open problems that question the existence of renormings for spaces possessing these topological properties but with weaker geometric norm properties. Our objectives are to prove the existence of norms that are rotund, Gâteaux differentiable, and not midpoint locally uniformly rotund (MLUR) for separable Banach spaces. Additionally, we aim to establish the existence of renormings that are weakly uniformly rotund, Fréchet differentiable, and not MLUR. This research delves into the nuanced relationship between topological and geometric properties of Banach spaces, contributing to the broader understanding of these spaces in functional analysis.

Abstract in lingua italiana

Il campo della teoria dei rinormamenti negli spazi di Banach si concentra sulla modifica della norma preservando le proprietà topologiche. Questa teoria è fondamentale per studiare le strutture geometriche degli spazi di Banach attraverso cui si possono classificare in maniera efficace.

In questa tesi, restringiamo la nostra attenzione sulla relazione tra tre aspetti fondamentali: le proprietà geometriche della norma, la regolarità e la separabilità di uno spazio. In precedenza, è stato dimostrato che ogni spazio di Banach separabile poteva essere rinormato con norme localmente uniformemente convesse (LUR) che sono anche Gâteaux differenziabili. Allo stesso modo, gli spazi di Banach separabili con duale separabile potevano essere ri-normati con norme localmente uniformemente convesse che sono anche Fréchet differenziabili.

I nostri obiettivi sono dunque dimostrare l'esistenza di norme che sono strettamente convesse, Gâteaux differenziabili e non MLUR per spazi di Banach separabili. Inoltre, si proverà a stabilire l'esistenza di rinormamenti che sono debolmente uniformemente convessi, Fréchet differenziabili e non MLUR in spazi di Banach con duale separabile. Questa ricerca approfondisce la complessa relazione tra le proprietà topologiche e geometriche degli spazi di Banach, contribuendo alla comprensione più ampia di questi spazi nell'ambito dell'analisi funzionale.

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1 | Introduction

1.1. Introduction

Banach spaces, a mathematical concept named after the early 20th-century mathematician Stefan Banach, have an historical lineage that continues to be relevant in contemporary mathematics.

Stefan Banach was born in Krakow on March 30, 1892. It is said that Banach spent some years of his childhood under the wings of his grandmother in Ostrowsko, the birthplace of his father. After his graduation in 1910, Banach enrolled at the Faculty of Engineering of the Polytechnical Institute in Lwow. Due to the outbreak of World War I, he was not able to finish his studies. A well-known story spread among mathematicians says that sometime in 1916, Steinhaus (then assistant at the University of Lwow) walked through a park in Krakow. Suddenly, he overheard the words Lebesgue integral; the youngsters who were discussing this unusual matter were Stefan Banach and Otton Nikodym. This encounter was the beginning of a lifelong collaboration and the big bang of the famous Lwow school. Thanks to this episode Stefan Banach was able to pursue a career in academia, where he managed to give substantial contributions to mathematics. To appreciate the extent of Banach's contributions it's enough to recall that Steinhaus always claimed that Banach was his greatest mathematical discovery [10]. The (arguably) greatest achievement he made is the formalization of what are now called Banach spaces, which serve as the playground for many branches of mathematics. Without this fundamental tool, it would have been substantially harder to develop fields such as partial differential equations, harmonic analysis, functional analysis, operator theory, and so on...

It is not surprising that research in the field grew wider as the years passed, given its versatility. Among the 100 years that have passed since the inception of the concept, many surprising facts have arisen. In particular, it became apparent how the understanding of the properties of Banach spaces was strongly connected to the geometrical properties of their norm functions. One of the first instances of this connection lies in the Milman-Pettis theorem, for which such geometrical properties are related to a topological characteristic of the Banach space. This thesis is set in this context, aiming to introduce new results relating some topological characteristics of a Banach space to the geometrical and smoothness properties of the norm function.

The main protagonist of our research are complete real normed spaces, or as Fréchet called them, Banach spaces. We will spend some pages introducing the main concepts and some basic properties, specifically it will be introduced the concept of renorming, which is the concept of changing a norm with an equivalent one. We say that two norms are equivalent if and only if they induce the same norm topology on the space X. This will be the base for the final result. After some characterization of the newly introduced concept, we end this first part by introducing the notion of basis in Banach spaces. We will mainly use the so called Markushevich basis :$\{e_n, e_n^*\}$ which are a total and fundamental biorthogonal system of the space X. This will allow us to work with "orthogonal" directions in a Banach space, a strongly non trivial task, and will be crucial when defining the final renormings. We then move on introducing most of the known geometrical properties of the norms. At first, it will be introduced the concept of rotundity which, intuitively, is just the strict adherence of the concept of convexity. If a shape is rotund, we can then expect not to find any straight line on its edge. We then strengthen this conditions introducing the concept of midpoint local uniform convexity, local uniform convexity, uniform convexity in both the strong and weak form. This will be followed up with some example and intuitive explanations to show their connections and meaning. Our final result will be tied mainly to three of the above properties: rotundity, midpoint local uniform convexity and weak uniform convexity. We will then move on to the more analytical concept of lattice norm, in which we wish to establish a relation between a partial order on a set and it's norm. We somehow wish to adapt the notion of monotonicity to norm functions, where domains are complex objects. Even the concept of derivative needs to be adjusted before it can be applied to such infinite dimensional domains. This is why we will be introducing two extensions of derivative, which are the most widely adopted in these scenarios: Gâteaux and Fréchet derivatives. Even if these properties seem to be more analytical than geometrical some astonishing results (such as Šmulyan's lemma) draw some very deep connections between such analytical properties and the previously mentioned geometric ones through duality. Lastly, we will adapt some results which are known to the state of the art to our purposes. We will be imposing multiple conditions to specific renormings of separable Banach space such as Gâteaux or Fréchet combined with rotundity conditions or introduce slices, which are a way to generate some "cuts" in a geometrical object on a space with a dual.

It is still not known (Section 52.3 [7] Question 3) in the state of the art whether one can change the norm on a separable Banach space, preserving its topology, with a norm that satisfies the condition of rotundity but not midpoint locally uniform rotundity. It is also not known (Section 52.3 [7] Question 6) whether one can change the norm on a Banach space with a separable dual, preserving its topology, with a norm that satisfies the condition of weak uniform rotundity but not midpoint locally uniform rotundity. A proposed proof of this two statements is the content of the last part of this work.

2 | Banach Spaces

Banach spaces, named after the renowned mathematician Stefan Banach, form a central pillar in functional analysis and modern mathematics. These mathematical structures provide a framework for studying the properties of vector spaces equipped with a norm, where convergence and continuity play pivotal roles. Understanding Banach spaces opens the gateway to investigating a wide range of problems, from the foundational principles to more advanced applications in various scientific disciplines.

2.1. Norms

Definition 2.1. *Let X be a real vector space. A* ***norm*** *is a function $\|\cdot\| : X \to \mathbb{R}^+$ such that:*

- $\|x\| = 0 \Leftrightarrow x = 0.$
- $\forall \alpha \in \mathbb{R} \quad \forall x \in X : \quad \|\alpha x\| = |\alpha| \, \|x\|.$
- $\forall x, y \in X : \quad \|x + y\| \leq \|x\| + \|y\|.$

The second property is also called *homogeneity* property and indicates the multiplicative scale behaviour of the norm function. The third property is the famous *triangle inequality*, which ensures that given $x, y \in X$ the space X with the norm $\|\cdot\|$ "the linear path" that goes from 0 to $x + y$ is always shorter than the path that goes from 0 to x and then from x to $x + y$.

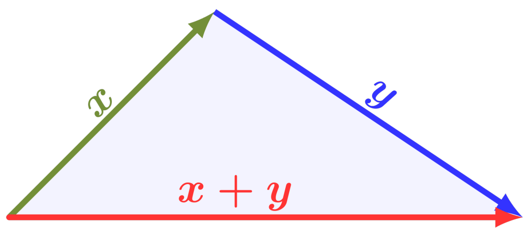


Figure 2.1: Triangle inequality.

It's easy seeing, using the definition, how every norm induces a distance over the same space X in the following way: $d(x, y) = \|x - y\|$. On the other hand not all the distances induce a norm (i.e. distance don't need to be homogeneus).

Definition 2.2. *The couple $(X, \|\cdot\|)$ is said to be a **normed space**.*

Example 2.1:

- $(\mathbb{R}, |\cdot|)$: The easiest example considering the line of the real number and the mono dimensional euclidean distance (the absolute value function).

- $(\mathbb{R}^n, \|\cdot\|_e)$: The n-dimensional euclidean space with its relative norm $\|x\|_e := (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$.

- $(\mathcal{C}^0([a, b]), \|\cdot\|_\infty)$: $\mathcal{C}^0([a, b])$ is the space of continuous functions on the interval $[a, b]$ and the infinite norm is $\|f\|_\infty := \max_{x\in[a,b]} |f(x)|$.

- $(L^1([a, b]), \|\cdot\|_1)$: $L^1([a, b])$ is the space of Lebesgue integrable functions on the interval $[a, b]$ and its norm $\|f\|_1 := \int_a^b |f| \, d\mu$.

Just to show the process of proving that a function is indeed a norm we focus on the canonical example $(\mathbb{R}^n, \|\cdot\|_e)$.

Definition 2.3. *If the normed space $(X, \|\cdot\|)$ is complete with respect to the metric induced by the norm then it is said to be a **Banach space**, namely if every Cauchy sequence in the metric space $(X, \|\cdot\|)$ is convergent.*

Example 2.2:

-$(\mathbb{R}^n, \|\cdot\|_e)$: The n-dimensional euclidean space with its relative norm $\|x\|_e := (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$.

- $(L^p, \|\cdot\|_p)$: It can be proven that all the Lebesgue Spaces L^p with their relative norms are Banach spaces for all $p \in [1, \infty]$.

- $(C^0[a, b], \|\cdot\|_\infty)$: The space of continuous function and the infinite norm.

A few example of *incomplete* normed spaces are:

- $(c_{00}, \|\cdot\|_\infty)$, the space of eventually zero sequences. If we now consider the sequence $\{x_n\}$ made by $x_n = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, 0, ...)$. This Cauchy sequence has no limit inside c_{00}.

- $(C^0[-1, 1], \|\cdot\|_1)$: The space of continuous function and the 1-norm, we can consider the sequence of functions $\{\frac{1}{1+e^{-nx}}\}_{n\in\mathbb{N}}$. The limit of the Cauchy sequence doesn't exist in $C^0[-1, 1]$.

Definition 2.4. *Given a normed space $(X, \|\cdot\|)$ we define the* ***unit sphere*** *and* ***unit ball*** *the following sets:*

- $S_{(X,\|\cdot\|)} := \{x \in X : \|x\| = 1\}$.
- $B_{(X,\|\cdot\|)} := \{x \in X : \|x\| \leq 1\}$.

Unit balls define univocally a norm since, by the homogeneity property of the norm, they also define all the level curves.

Example 2.3:

Here we can see the picture representing how the unit sphere of the *p-norm* $\|x\|_p := (\sum_{i=1}^n x_i^p)^{\frac{1}{p}}$ varies with respect to the parameter p in the space $\mathbb{R}^n$, for $p = \infty$ we define $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$.

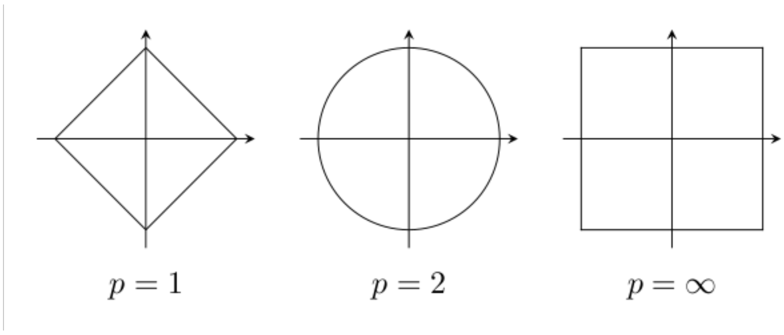


Figure 2.2: Unit spheres of $\mathbb{R}^2$

It's worth saying that if $p < 1$ the function $\|\cdot\|_p$ is not a norm. A norm cannot have a non-convex unitary ball, otherwise we would be able to violate triangle inequality.

2.2. Equivalent Norms

Definition 2.5. *Two norms defined on the same space X are said to be* ***equivalent*** *if the following condition holds $\exists\, m, M \in \mathbb{R}^+$:*

$$\forall x \in X \;\; m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1 .$$

This property induces an equivalence relation on the set of all norms and thus it is *symmetric*. One reason that underlines the importance of this notion is the fact that by changing the norm with an equivalent one, also known as *renorming*, a space will hold the same converging sequences, generating the same *topology of the open sets*. Two norms are equivalent if there exists two positive real constants such that $mB_{(X,\|\cdot\|)} \subset B_{(X,|\cdot|)} \subset MB_{(X,\|\cdot\|)}$, in this way we can think about this concept from a geometrical point of view.

Definition 2.6 ([6, page 53]). *Given a Banach space $(X, \|\cdot\|)$, we define* ***the metric space of all equivalent norms*** *(P, ρ) where the distance ρ for two norms $p, q \in P$ is defined as:*

$$\rho(p, q) = \sup\{|p(x) - q(x)|; x \in B_{(X,\|\cdot\|)}\}.$$

This concept expands upon the previously introduced notion, providing us with a valuable analytical tool to examine the characteristics of the functional space consisting of all norms that preserve the topology of the original Banach space. Additionally, it's noteworthy that this newly introduced space possesses numerous favorable properties, including its classification as a Baire space. Furthermore, it's important to highlight that space P is a subset of space Q, wherein Q encompasses all continuous seminorms defined on the same space $(X, \|\cdot\|)$, and Q itself constitutes a complete metric space using the same distance metric denoted as ρ.

Theorem 2.1. *If the Banach space $(X, \|\cdot\|)$ is finite dimensional, all norms are equivalent.*

Proof. To show that every norm is equivalent it will be enough to prove that every norm is equivalent to the 2-norm $\|\cdot\|_e$. This follows from the fact that every finite dimensional space is isomorphic to $\mathbb{R}^n$ and the notion of equivalence for norms is *transitive*. Since every normed space is also a vector space, and in our case $n = \dim(X) < \infty$,
it always exists a subset of the space which is also a basis $\{e_k\}_{k=1}^n$. We can then state the following:

$$\forall x \in X \;\; x = \sum_{k=1}^{n} x^{(k)} e_k.$$

Where $x^{(k)}$ are the projection along the k-th component of the basis. Let's rewrite the norm using the triangle inequality

$$\|x\| = \left\| \sum_{k=1}^{n} x^{(k)} e_k \right\| \leq \sum_{k=1}^{n} \left\| x^{(k)} e_k \right\| = \sum_{k=1}^{n} |x^{(k)}| \left\| e_k \right\| .$$

We can now use Cauchy-Schwarz inequality and setting $M^2 = C = \sum_{k=1}^{n} \|e_k\|^2$ we obtain the following

$$\|x\|^2 \leq (\sum_{k=1}^{n} |x^{(k)}| \, \|e_k\|)^2 \leq \sum_{k=1}^{n} |x^{(k)}|^2 \sum_{k=1}^{n} \|e_k\|^2 = C \, \|x\|_e^2 \implies \|x\| \leq M \, \|x\|_e \, .$$

To find the second inequality we shall focus on the fact that:

- X is finite dimensional $\Leftrightarrow$ $S_{(X,\|\cdot\|_e)}$ is a *compact set.*
- The norm function is continuous.

To prove the second point is enough to show that:

$$| \, \|x_n\| - \|x\| \, | \leq \|x_n - x\| \leq M \, \|x_n - x\|_e \, .$$

Which implies that for all $\{x_n\} \subset X$ for which $\exists \, x \in X : d(x_n, x) = \|x_n - x\|_e \to 0$ also the function $| \, \|x_n\| - \|x\| \, | \to 0$ and this is the notion of continuity for real functions.

We notice that these are the hypothesis of the Weierstrass theorem allowing us to infer that $\exists \; m \in \mathbb{R}^+ : \|x\| \geq m \; \forall \; x \in S_{(X,\|\cdot\|_e)}$ and there exists $x_0 \in S_{(X,\|\cdot\|_e)}$ such that $\|x_0\| = m$.

It follows that $m \geq 0$. If $m = 0$ then $\exists \; x_0 : \|x_0\| = m = 0$ which implies $x = 0$ but this is impossible since $x_0 \in S_{(X,\|\cdot\|_e)}$. The last step is just to consider the element $x \in X$, $x \neq 0$. Then $x/ \| \, x\|_e \in S_{(X,\|\cdot\|_e)}$ leading to:

$$\left\| \frac{x}{\| \, x\|_e} \right\| \geq m \implies \|x\| \geq m \, \|x\|_e \;\; \forall \; x \in X.$$

This last inequality combined with $\|x\| \leq M \, \|x\|_e$ for all $x \in X$ give us the thesis.

□

We will now introduce the concept of dual norm $\| \cdot \|^*$ on X^* which is the dual space of a starting Banach space $(X, \| \cdot \|)$.

Proposition 2.1. *Let $(X, \| \cdot \|)$ be a Banach, let X^* be its dual and let $T \in X^*$ we define the* ***operator norm on the dual*** *with the following: $\|T\|^* = \sup_{x \in S_{(X,\|\cdot\|)}} |Tx|$, this function is a norm on the dual space.*

It's worth noting that not every norm on the dual corresponds to a norm on its predual. It is now introduced a sufficient condition on a generic $\| \cdot \|^*$ to be a dual norm on some space X.

Proposition 2.2. *Given a Banach space $(X^*, \|\cdot\|^*)$ this admits a predual Banach space $(X, \|\cdot\|)$ if the norm $\|\cdot\|^*$ is weak*-lower-semicontinuous.*

Having introduced the dual space, we can now state and prove a fundamental theorem on the equivalence of the norm in infinite dimensional Banach spaces, which underlies once again the strange properties that we obtain from considering infinite dimensional spaces.

Proposition 2.3. *Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space then it exists $\|\cdot\|_2$ that is not equivalent to $\|\cdot\|$.*

Proof. This proof is based on the known result that given an infinite dimensional Banach space $(X, \|\cdot\|)$ it always exists, under the axiom of choice, a linear and unbounded operator $f : X \to \mathbb{R}$.

We can now consider the following norm:

$$\|x\|_2 = \|x\| + |f(x)| \ \forall x \in X.$$

By hypothesis is known that the operator f is unbounded, hence $f \notin (X^*, \|\cdot\|^*)$. Let's now check if $f \in (X^*, \|\cdot\|_2^*)$:

-*Linearity*:

We know f to be linear by hypothesis.

-*Boundedness*:

The boundeness condition is the following $\exists\ M \in \mathbb{R} : \ \forall x \in X$

$$\|f\|_2^* \leq M.$$

Given:

$$\|f\|_2^* = \sup_{x \in B_{(X, \|x\|_2)}} |f(x)|.$$

Let's take all the points $x \ \in B_{(X, \|x\|_2)}$:

$$\|x\|_2 = \|x\| + |f(x)| \leq 1 \implies \sup_{x \in B_{(X, \|x\|_2)}} |f(x)| \leq 1.$$

But this implies the boundedness of the operator and consequently $f \in (X^*, \|\cdot\|_2^*)$.

Let's now perform the last step assuming that $\exists\ C \in \mathbb{R} : \|x\|_2 \leq C\|x\|$ (necessary condition to be equivalent). It follows that for all $x \in B_{(X,\|x\|_2)}$:

$$|f(x)| \leq \|x\|_2 \leq C\|x\| \implies f \in (X^*, \|\cdot\|^*).$$

But this generates a contradiction with the initial hypothesis. Then the two norms are *not equivalent*.

$\square$

Proposition 2.4. *Given two Banach spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ where $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, the identity map τ is an isomorphism between the two spaces and we will write $(X^*, \|\cdot\|_1^*) \simeq (X^*, \|\cdot\|_2^*)$.*

Proof. To show this result, it's enough to prove that $\tau : (X^*, \|\cdot\|_1^*) \to (X^*, \|\cdot\|_2^*)$ such that it is *bounded*, *injective* and *surjective*, where τ is the previously mentioned identity map.

Let us recall that if the two norms are equivalent then we know:

$$\forall x \in X\ m \parallel x \parallel_1 \leq \parallel x \parallel_2 \leq M \parallel x \parallel_1.$$

We can now consider the identity map:

$$\tau(L) = L.$$

-*Well Posed*

Let's start from the fact that τ is linear by definition, we also know that

$$L \in (X^*, \|\cdot\|_1^*) \implies \|\tau(L)\ \|_1^* = \|L\ \|_1^* \leq C\|x\|_1 \leq \frac{C}{m}\|x\|_2\ \forall x \in X.$$

But this means that $\tau(L) \in (X^*, \|\cdot\|_2^*)$

-*Bounded*

$$\|\tau\|_* = \sup_{L \in \|L\|_1^*=1} \|\tau(L)\|_2^* = \sup_{\sup_{\|x\|_1=1}|L(x)|=1} |L(x)|\ \leq\ C.$$

-*Injective and Surjective*

Injectivity follows from the very definition of the map τ. To show surjectivity we can proceed as follows:

Let's suppose τ is not surjective, this means that

$$\nexists\ L \in (X^*, \|\cdot\|_1^*):\ \tau(L) = L_\delta \in (X^*, \|\cdot\|_2^*).$$

But since $L_\delta \in (X^*, \|\cdot\|_2^*)$ and the norms are equivalent we can conclude (as in the first point of the proof) that $L_\delta \in (X^*, \| \ \|_1^*)$.

But now if we take $L = L_\delta$ we get $\tau(L_\delta) = L_\delta$ and this generates a contradiction with the initial assumption. Thus it is surjective.

Every step applied above can also be applied to the inverse identity map, proving its boundednes (thus continuity) so, since we have a bijective bounded map with bounded inverse between two spaces they are *isomorphic*.

□

2.3. Basis in Banach spaces

In this chapter, we delve into the fundamental concepts of vector space bases, exploring the distinct characteristics of Hamel, Schauder, Auerbach, and Markushevich bases. These diverse types of bases play crucial roles in understanding the structures of different vector spaces, ranging from finite dimensions to infinite dimensions. By examining the properties and applications of these bases, we gain valuable insights into the representation and analysis of elements in various mathematical spaces.

Definition 2.7. *A **Hamel basis** is a subset B of a vector space V over the field $\mathbb{R}$ such that every element $v \in V$ can uniquely be written as*

$$v = \sum_{b \in B} \alpha_b b \quad \alpha_b \in \mathbb{R}\ b \in B.$$

Hamel bases are considered the most fundamental due to their simple requirement of unique representation in vector spaces. While every vector space has a Hamel basis, proven through the Axiom of Choice, they tend to become less of an asset (at least for our purposes) as we will see in the next theorem. Consequently, in infinite-dimensional settings, alternative bases like Schauder bases are explored, offering more convenient properties of the basis.

Proposition 2.5. *Let $(X, ||.||)$ be an infinite dimensional Banach space. Then, any Hamel basis in X is uncountable.*

This is still the main technical difficulty of the Hamel Basis. This proposition can be proved by the fact that every finite-dimensional subspace of an infinite-dimensional vector space X has empty interior, and is no-where dense in X. It then follows from the *Baire category theorem* that a countable union of bases of these finite-dimensional subspaces cannot serve as a basis.

Definition 2.8. *Let $(X, \|\cdot\|)$ be a Banach space. A **Schauder basis** is a sequence $\{b_n\}$ of elements of X such that for every element $x \in X$ there exists a unique sequence $\{\alpha_n\}$ of scalars in $\mathbb{R}$ so that:*

$$x = \sum_{n=0}^{\infty} \alpha_n b_n.$$

This type of basis is particularly useful in the study of functional analysis, where it enables the analysis of convergence and continuity properties of functions and operators. Schauder bases provide a powerful tool for representing functions and studying the behavior of functions in infinite-dimensional spaces. Unlike Hamel bases, Schauder bases allow for infinite linear combinations using limits instead of finite sums.

Definition 2.9 ([9, Definition 1.15]). *If X is a Banach space, a **biorthogonal system** $\{(x_n, e_n)\}_{n\in\mathbb{N}}$ for X is a subset of $X \times X^*$ such that $\langle x_i, e_j \rangle = \delta_{i,j}$ for all $i, j \in \mathbb{N}$, where $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$. It is called an **Auerbach system** if, moreover, $\|x_n\| = \|e_n\| = 1$ for all $n \in \mathbb{N}$.*

Example 2.4:

Consider the Banach space $X = c_0(\mathbb{N})$ of all real sequences that converge to zero, equipped with the supremum norm. Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of X, where e_n is the sequence whose n-th entry is 1 and all other entries are 0. Now, define the sequence of functionals $\{e_n^*\}_{n\in\mathbb{N}}$ in X^* as follows:

$$e_n^*(x) = x_n \quad \text{for } x = (x_1, x_2, \dots) \in X.$$

Definition 2.10 ([9, Definition 1.4-1.6]). *A biorthogonal system $\{e_n, e_n^*\}_{n\in\mathbb{N}}$ for a Banach space X is called **fundamental** if $\overline{span\{e_n : n \in \mathbb{N}\}} = X$. It is called **total** if $span\{e_n^* : n \in \mathbb{N}\}$ is w^*-dense in X^*.*

It can be shown that the previous example $\{e_n, e_n^*\}_{n\in\mathbb{N}}$ forms an Auerbach system in c_0. This system is both fundamental, as the span of $\{e_n\}_{n\in\mathbb{N}}$ is dense in X, and total, as the span of $\{e_n^*\}_{n\in\mathbb{N}}$ is weak*- dense in c_0^*.

Definition 2.11 ([9, Definition 1.7]). *A fundamental and total biorthogonal system for a Banach space X is said to be a Markushevich basis (for short, an M-basis). If the **M-basis** is, moreover, an Auerbach system, it is called an **Auerbach basis**.*

Example 2.5:

In the Hilbert space ℓ^2, the system $\{e_k\}_{k\in\mathbb{N}}$ and $\{e_k^*\}_{k\in\mathbb{N}}$ forms a Markushevich basis which is of course also a Schauder basis and a complete orthonormal system. Here, e_k represents the sequence whose kth entry is 1 and all other entries are 0, and e_k^* is the corresponding functional in ℓ^2 with $e_k^*(x) = x_k$ for $x \in \ell^2$.

Now, two definitions are going to be introduced to further distinct between the possible M-basis.

Definition 2.12. *An M-basis $\{e_n, e_n^*\}_{n\in\mathbb{N}}$ is said to be **bounded** if $\sup \|e_n\|\|e_n^*\| \leq \infty\ \forall n \in \mathbb{N}$. Also an M-basis is said to be **shrinking** if $\overline{\operatorname{span}\{e_n^* : n \in \mathbb{N}\}} = X^*$*

We now present two theorems that follow from the previously introduced definitions, one is key to tackling of the open questions, the other is a fun historical fact.

Theorem 2.2 ([9, Theorem 1.22],[11, Proposition 8.13]). *Every separable Banach space has a bounded M-basis such. If, moreover, X has a separable dual, the M-basis can be taken to be shrinking and bounded. In both cases the M-basis can be taken to also satisfy $\|e_1^*\|^* = 1$ and $\|e_n\| = 1\ \forall n \in \mathbb{N}$.*

The existence of bounded M-basis in every separable Banach space has been solved in the affirmative. It was later adjusted to produce an "almost" Auerbach (even norming) M-basis in every separable Banach space. What ultimately concerns us is the fact that separable Banach spaces admit bounded M-basis. If one also adds the separability of the dual the M-basis is shrinking. Now we take a few lines to talk about the famous "Scottish Book" (more information can be found by clicking at this link : http://www.math.lviv.ua/szkocka/index.php).

The "Scottish Book" is a renowned artifact in the history of mathematics. It originated at the Scottish Café in Lwów, where prominent mathematicians of the Lwów School gathered during the 1930s and 1940s to collaboratively explore problems, particularly in functional analysis and topology. The café's marble tabletops allowed them to write directly with pencils during discussions.

To preserve their findings, Stefan Banach's wife supplied them with a large notebook, which came to be known as the Scottish Book. This notebook served as a repository for solved, unsolved, and often seemingly unsolvable mathematical problems. Any guest of the café could borrow the book and attempt to solve these challenges. Rewards, including valuable items like fine brandy, were offered for successfully tackling the most difficult problems, a practice that persisted even during the Great Depression and on the eve of World War II.

The Scottish Book thus became a symbol of the collaborative spirit and intellectual camaraderie among mathematicians during that era, encapsulating their dedication to advancing the field.

The problem 153, proposed by Mazur, revolved around the assertion that every separable Banach space must have a Schauder basis, a statement that had defied proof and baffled mathematicians for an extended period, gaining notoriety for its formidable challenge. However, in the year 1972, a pivotal moment in mathematical history arrived when someone unveiled a solution to this long-standing enigma.

Theorem 2.3 ([4]). *Not every separable Banach space admits a Schauder basis.*

[1]

In 1972, Per Enflo made a groundbreaking mathematical discovery by constructing a separable Banach space without the approximation property and a Schauder basis. To celebrate this achievement, a memorable "goose reward" ceremony was held at the Stefan Banach Center in Warsaw, Poland, and broadcast nationwide, highlighting the profound impact of Enflo's work beyond the realm of mathematics.

[1] Image taken from Wikipedia. Link : https://en.wikipedia.org/wiki/Per_Enflo

3 | Geometrical Properties

Within this chapter, we will explore the diverse geometric attributes inherent in norms. As previously underscored, our principal objective is to establish a substantial correlation between these norm-derived geometric characteristics and distinct topological properties within the domain of Banach spaces. Our central focus will center on analyze the interplay between the norm and convexity.

3.1. Rotundity

We now come to the concept of rotundity. In this case, the norm's role goes beyond just meeting the criteria for convexity. Instead, it's required to go a step further, demonstrating a strict adherence to this condition.

Definition 3.1. *The norm $\|\cdot\|$ of the Banach space $(X, \|\cdot\|)$ is said to be* ***rotund*** *(or* ***strictly convex****), if its unit sphere contains no nondegenerate straight line segments.*

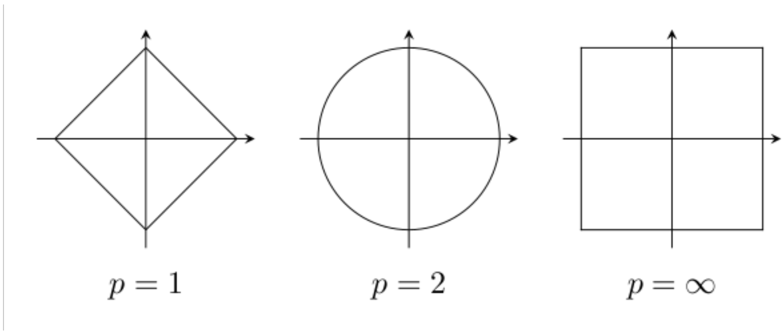


Figure 3.1: rotundity of the p-norm previously mentioned on the space $\mathbb{R}^2$

It can be shown that the p-norm is rotund for all $p \in (1, \infty)$, as it can be seen in the image above for 1 and ∞ the unitary sphere contains straight line segments while for $p = 2$ (but it is the same for every other p) all the points are extreme points.

Some *analytical characterizations* of this property can be found in the following lemma:

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a normed space. The following are equivalent:*

1. $\|\cdot\|$ *is rotund.*
2. $\|tx_1 + (1-t)x_2\| < 1 \ \forall \ x_1, x_2 \ \in \ S_{(X,\|\cdot\|)}$ *such that* $x_1 \neq x_2, \ t \in (0,1)$.
3. $\left\|\frac{x_1+x_2}{2}\right\| < 1 \ \forall \ x_1, x_2 \ \in \ S_{(X,\|\cdot\|)}$ *such that* $x_1 \neq x_2$.
4. *If* $x_1, x_2 \ \in \ S_{(X,\|\cdot\|)}$ *satisfy* $\|x_1 + x_2\| = 2$ *then* $x_1 = x_2$.
5. *If* $x_1, x_2 \in X$ $x_1 \neq 0, \ x_2 \neq 0$ *satisfy* $\|x_1 + x_2\| = \|x_1\| + \|x_2\|$ *then* $x_1 = \lambda x_2$ *for some* $\lambda > 0$.
6. *If* $x_1, x_2 \in X$ *satisfy:* $2\|x_1\|^2 + 2\|x_2\|^2 - \|x_1 + x_2\|^2 = 0$ *then* $x_1 = x_2$.

Proof. Let's prove the chain of implication.

-1 $\Leftrightarrow$ *2*

This is just the analytical characterization of strict convexity. We will also prove (in the Lemma 4.1) that the unit ball is convex, forcing the average of the two points to be at most 1.

-2 $\implies$ *3*

If the condition holds for every $t \in (0,1)$ then it's also true for $t = \frac{1}{2}$, for which we get

$$\left\|\frac{1}{2}x_1 + (1 - \frac{1}{2})x_2\right\| = \left\|\frac{1}{2}(x_1 + x_2)\right\|.$$

And thus 3 follows since:

$$\left\|\frac{1}{2}(x_1 + x_2)\right\| < 1 \ \forall \ x_1, x_2 \in S_{(X,\|\cdot\|)}.$$

-3 $\implies$ *4*

We derive 4 assuming 3 to be true, so for all $x_1, x_2 \ \in \ S_{(X,\|\cdot\|)}$ such that $x_1 \neq x_2$ we know that $\left\|\frac{1}{2}(x_1 + x_2)\right\| < 1$, and hence $\|x_1 + x_2\| < 2$. But then if we consider $x_1, x_2 \ \in \ S_{(X,\|\cdot\|)}$ such that $\|x_1 + x_2\| = 2$ then $x_1 \neq x_2$ cannot be true or we would contradict the hypothesis that 3 is true. Thus $x_1 = x_2$, the thesis.

-4 $\implies$ *5*

Let's suppose 4 is satisfied:

Let $\|x+y\| = \|x\| + \|y\|$ for some $x \neq 0$, $y \neq 0$. We may assume that $0 < \|x\| \leq \|y\|$. Then,

$$2 \geq \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \geq \left\| \frac{x}{\|x\|} + \frac{y}{\|x\|} \right\| - \left\| \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\|$$

$$= \frac{1}{\|x\|}\|x+y\| - \|y\| \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) = \frac{1}{\|x\|}(\|x\| + \|y\|) - \|y\| \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) = 2.$$

Thus $\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| = 2$, by 4 we get $\frac{x}{\|x\|} = \frac{y}{\|y\|}$ implies that $x = \lambda y$.

-*5* $\implies$ *2*

Suppose that 5 holds. Let z_1 and z_2 be different members of $S_{(X,\|\cdot\|)}$ and observe that $z_1 \neq \lambda z_2$ for every $\lambda > 0$, which implies that $\|tz_1 + (1-t)z_2\| < \|tz_1\| + \|(1-t)z_2\| = 1$. The space X is therefore rotund, so $5 \Rightarrow 2$.

We have now closed the loop of implications between 1 and 5, showing the equivalence. We now consider a separate case for 6, showing the equivalence to the point number 4.

-*4* $\Leftrightarrow$ *6*

Let's start from:

$$2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 \geq 2\|x\|^2 + 2\|y\|^2 - (\|x\| + \|y\|)^2 = (\|x\| - \|y\|)^2 \geq 0.$$

Thus, if $2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 0$, then $\|x\| = \|y\|$. Hence, we may assume that $x, y \in S_X$ (if they were not in the unit sphere we might rescale of the factor $\|x\|$); we get $\|x+y\| = 2$, and 4 implies $x = y$.

The converse implication is obvious considering $x, y \in S_{(X,\|\cdot\|)}$ then we get

$$\|x+y\| = 2 \implies 0 = 2 + 2 - \|x+y\|^2 \implies 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 0$$

Thus finishing the converse implication.

We have then proved the mutual implications of 1 to 5 and then the equivalence of 6 with 4, getting the final result for which every proposition is equivalent to every other proposition in the lemma.

□

Theorem 3.1. *Given a normed space $(X, \|\cdot\|)$ which is rotund, then it doesn't admit any non-rotund subspace $(Y, \|\cdot\|)$.*

Proof. Suppose $(X, \|\cdot\|)$ admits a non rotund subspace $(Y, \|\cdot\|)$, then $\exists x_1, x_2 \in S_{(Y,\|\cdot\|)}$ distinct points such that $\left\|\frac{1}{2}(x_1+x_2)\right\| = 1$. But $S_{(Y,\|\cdot\|)} \subseteq S_{(X,\|\cdot\|)}$ which implies $x_1, x_2 \in S_{(X,\|\cdot\|)}$ such that $\left\|\frac{1}{2}(x_1+x_2)\right\| = 1$, which contradicts the initial hypothesis (through Lemma 3.1).

□

Example 3.1:

The Banach space ℓ^2 with its relative norm $\|x\|_{\ell^2} := (\sum_{i=1}^{\infty} x_i^2)^{\frac{1}{2}}$ is *rotund.*

Let us first remember that in the space ℓ^2 we can define a inner product. It is well known that all the spaces with a norm that arises from the above mentioned inner product also satisfy the parallelogram identity for all $x, y \in X$:

$$\|x+y\|_{\ell^2}^2 + \|x-y\|_{\ell^2}^2 = 2\,\|x\|_{\ell^2}^2 + 2\,\|y\|_{\ell^2}^2\,.$$

So to retrieve our thesis it's enough to consider that if we take two points $x, y \in S_{(X,\|\cdot\|)}$ such that:

$$\|x+y\|_{\ell^2} = 2.$$

Then

$$\|x+y\|_{\ell^2}^2 = 4.$$

But if we apply the parallelogram law combined with the fact that $x, y \in S_{(X,\|\cdot\|)}$ we get:

$$4 + \|x-y\|_{\ell^2}^2 = 2 + 2 \Rightarrow \|x-y\|_{\ell^2}^2 = 0 \Leftrightarrow x = y.$$

Where the last step is justified by the first norm property 2.1.

But what we have written is exactly the fourth point of the lemma previously introduced, hence the space is *rotund.* This example also suggests, by applying Theorem 3.1 that every euclidean space $\mathbb{R}^n$ with the euclidean norm also satisfied the rotundity porperty.

Example 3.2:

To show how articulate it might get when trying to prove this property we will try to prove it for the following norm in the Hilbert space ℓ^2, we will use the standard notation $x = (x_n)_{n=1}^{\infty}$:

We will take the same norm used in the article [3], given the Banach space $(\ell^2, \|\cdot\|_{\ell^2})$:

$$\|x\|^2 = \|x\|_0^2 + \sum_{n=1}^{\infty} \frac{1}{4^n} |x_n|^2$$

where:

$$\|x\|_0 = \max(\frac{1}{2} \left\| x \right\|_{\ell^2}, \sup |x_n|).$$

We start by noticing that the second term of the norm is just the square of the euclidean norm of the vector $(\frac{1}{2}x_1, \frac{1}{4}x_2, ...)$, it's also fairly easy to check it is well posed since $\frac{1}{4^n}$ is always less than 1.

Let's start by applying condition 6 of Lemma 3.1 to see if we can prove the implication.

$$0 = 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 2\left(\|x\|_0^2 + \left\|(\frac{1}{2}x_1, \frac{1}{4}x_2, ...)\right\|_{\ell^2}^2\right) + \left(\|y\|_0^2 + \left\|(\frac{1}{2}y_1, \frac{1}{4}y_2, ...)\right\|_{\ell^2}^2\right)$$

$$-\left(\|x+y\|_0^2 + \left\|(\frac{1}{2}x_1, \frac{1}{4}x_2, ...) + (\frac{1}{2}y_1, \frac{1}{4}y_2, ...)\right\|_{\ell^2}^2\right) = 0$$

Rearranging the terms of the equality written above we get:

$$(2\|x\|_0^2 + \|y\|_0^2 - \|x+y\|_0^2) +$$

$$+\left(2\left\|(\frac{1}{2}x_1, \frac{1}{4}x_2, ...)\right\|_{\ell^2}^2 + 2\left\|(\frac{1}{2}y_1, \frac{1}{4}y_2, ...)\right\|_{\ell^2}^2 - \left\|\left(\frac{1}{2}x_1, \frac{1}{4}x_2, ...\right) + \left(\frac{1}{2}y_1, \frac{1}{4}y_2, ...\right)\right\|_{\ell^2}^2\right).$$

But now, if we apply triangular inequality we realize that each term above is non negative (we will explain better this passage with the next theorem), and since their sum is zero, it is implied that both terms need to be equal to zero. In particular

$$2\left\|(\frac{1}{2}x_1, \frac{1}{4}x_2, ...)\right\|_{\ell^2}^2 + 2\left\|(\frac{1}{2}y_1, \frac{1}{4}y_2, ...)\right\|_{\ell^2}^2 - \left\|\left(\frac{1}{2}x_1, \frac{1}{4}x_2, ...\right) + \left(\frac{1}{2}y_1, \frac{1}{4}y_2, ...\right)\right\|_{\ell^2}^2 = 0.$$

But we know the euclidean norm to be rotund, thus implying $\left(\frac{1}{2}x_1, \frac{1}{4}x_2, ...\right) = \left(\frac{1}{2}y_1, \frac{1}{4}y_2, ...\right)$, but these two vectors are equal only if $x = y$. Thus we retrieve the rotundity of $\|\cdot\|$ since implication 6 of Lemma 3.1.

The algebraic structure of the norm was not involved in the calculations, this suggests that we can extend this result to a much more general framework, as the next theorem suggests.

Theorem 3.2 ([7, Proposition 152][7, page 147]). *Assume that* $(X_i, \|\cdot\|_i)$ $i = 1,2$ *are Banach spaces such that* $(X_2, \|\cdot\|_2)$ *is strictly convex and there is a one-to-one continuous linear operator* $T : X_1 \longrightarrow X_2$*. Then:*

$$\|x\|_3 = \|x\|_1 + \|Tx\|_2 \,, \ x \in \ X_1$$

and

$$\|x\|_4^2 = \|x\|_1^2 + \|Tx\|_2^2 \,, \ x \in \ X_1$$

define two equivalent strictly convex norms on X_1*.*

Proof. We start by proving the first statement. Proving that $\| \cdot \|_3$ is a renorming we can write:

$$\|x\|_1 \leq \|x\|_1 + \|Tx\|_2 = \|x\|_3 \leq (K+1)\|x\|_1$$

The first inequality needs no explanation while the second follows from the continuity of the operator T for which we know $\exists K \in \mathbb{R} : \ \|Tx\|_2 \leq K\|x\|_1$ for all $x \ \in X_1$, thus it is bounded.

Now we take $x_1, y_1 \in X_1$ different from zero for which $\|x_1 + y_1\|_3 = \|x_1\|_3 + \|y_1\|_3$, specifically:

$$\|x_1 + y_1\|_1 + \|T(x_1 + y_1)\|_2 = \|x_1\|_1 + \|y_1\|_1 + \|T(x_1)\|_2 + \|T(y_1)\|_2$$

But we know by triangle inequality that $\|x_1 + y_1\|_1 \leq \|x_1\|_1 + \|y_1\|_1$ and $\|T(x_1 + y_1)\|_2 = \|T(x_1) + T(y_1)\|_2 \leq \|T(x_1)\|_2 + \|T(y_1)\|_2$ and thus both inequality need to be satisfied with strict equality. We get in particular that $\|T(x_1)\|_2 + \|T(y_1)\|_2 = \|T(x_1) + T(y_1)\|_2$. We know by injectivity of T that $Tx_1 \neq 0$ and $Ty_1 \neq 0$. By the rotundity of the norm $\| \cdot \|_2$ and the point 5 of Lemma 3.1 we get that $Tx_1 = \lambda Ty_1$ for some $\lambda > 0$, but then $Tx_1 = T(\lambda y_1)$ and by injectivity $x_1 = \lambda y_1$.

But this leads us to the conclusion that $\|x_1 + y_1\|_3 = \|x_1\|_3 + \|y_1\|_3 \implies x_1 = \lambda y_1$, which is a characterization of rotundity of the previously mentioned lemma.

Let's use the previous example to finalize a formal proof of the second part of the theorem.

The fact that this is an equivalent norm comes naturally since we can see $\|x\|_3 = |\|x\|_1| + |\|Tx\|_2| = \|(\|x\|_1, \|Tx\|_2)\|_{\ell^1}$ which is the standard norm of the space ℓ^1 along the first two dimensions, but since we are considering a subspace of finite dimensions all norms are equivalent, also the euclidean norm $\|x\|_4 = \sqrt{\|x\|_1^2 + \|Tx\|_2^2} = \|(\|x\|_1, \|Tx\|_2)\|_{\ell^2}$, thus $\|\cdot\|_4$ is equivalent to norm $\|\cdot\|_3$ which is equivalent to the initial norm $\|\cdot\|_1$.

We try retrieving the implication of point 6 of Lemma 3.1. Let's assume by hypothesis that:

$$0 = 2\|x\|_4^2 + 2\|y\|_4^2 - \|x+y\|_4^2 = 2(\|x\|_1^2 + \|Tx\|_2^2) + 2(\|y\|_1^2 + \|Ty\|_2^2) - (\|x+y\|_1^2 + \|Tx+Ty\|_2^2).$$

Rearranging:

$$(2\|x\|_1^2 + 2\|y\|_1^2 - \|x+y\|_1^2) + (2\|Tx\|_2^2 + 2\|Ty\|_2^2 - \|Tx+Ty\|_2^2) = 0.$$

By triangle inequality:

$$2\|x\|_1^2 + 2\|y\|_1^2 - \|x+y\|_1^2 \geq 2\|x\|_1^2 + 2\|y\|_1^2 - (\|x\|_1 + \|y\|_1)^2 = (\|x\|_1 - \|y\|_1)^2 \geq 0.$$

The same consideration can be made about the other term of the sum, but since we are summing two terms which are greater than zero and result is zero by hypothesis, then we must have both terms equal to zero, in particular:

$$2\|Tx\|_2^2 + 2\|Ty\|_2^2 - \|Tx+Ty\|_2^2 = 0$$

Which is a rotund norm, so by point 6 of Lemma 3.1 we get $Tx = Ty$, but now we use the hypothesis that T is one-to-one, leading to $x = y$, thus implying the rotundity of the norm $\|\cdot\|_4$. $\square$

We now investigate via examples if some another properties of rotundity can be formalized, like: is it enough to know that a norm $|\cdot|_2^2 = \|\cdot\|_1^2 + \|\cdot\|_2^2$ is rotund to conclude that $|\cdot| = \|\cdot\|_1 + \|\cdot\|_2$ is rotund too?

Example 3.3:

We start by assuming that $|\cdot|_2^2 = \|\cdot\|_1^2 + \|\cdot\|_2^2$ is a rotund norm. Let $(X_1, \|\cdot\|_1) = (\mathbb{R}^2, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2) = (\mathbb{R}^2, \|\cdot\|_e)$ where:

$$\|(x_1, x_2)\|_1 = 2|x_1|_e + |x_2|_e$$

$$\|(x_1, x_2)\|_2 = |x_1|_e + 2|x_2|_e$$

where $|\cdot|_e$ is the monodimensional euclidean norm.

$$|(x_1,x_2)| = \|(x_1,x_2)\|_1 + \|(x_1,x_2)\|_2 = 3(|x_1|_e + |x_2|_e)$$

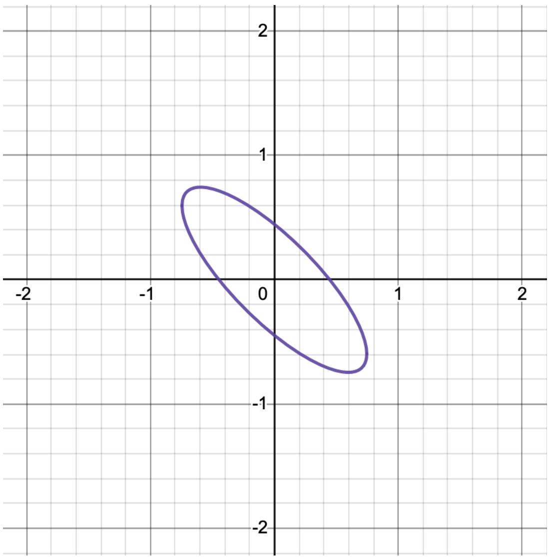
Which is clearly not rotund since we can take $x = (\frac{1}{3}, 0)$ and $(0, \frac{1}{3})$, those two points have midpoint contained in the unitary sphere even if they are different.

On the other hand:

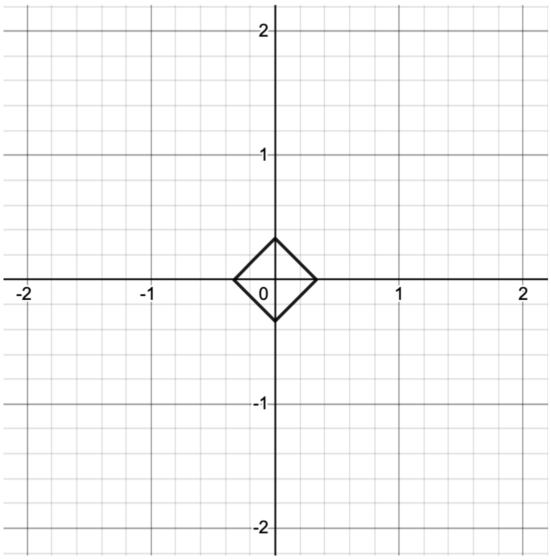
$$|(x_1,x_2)|_2^2 = \|(x_1,x_2)\|_1^2 + \|(x_1,x_2)\|_2^2 = (2|x_1|_e + |x_2|_e)^2 + (|x_1|_e + 2|x_2|_e)^2 =$$

$$= 5|x_1|^2 + 5|x_2|^2 + 8|x_1||x_2|.$$

We will show in the next chapter (Example 4.3) that this quantity is indeed a norm. Even though it is not formal, it can be shown that the unitary sphere of this norm is an ellipse in $\mathbb{R}^2$ which is well know to contain only extreme points and hence it is rotund.



Unit sphere norm $|\cdot|$



Unit sphere norm $|\cdot|_2$

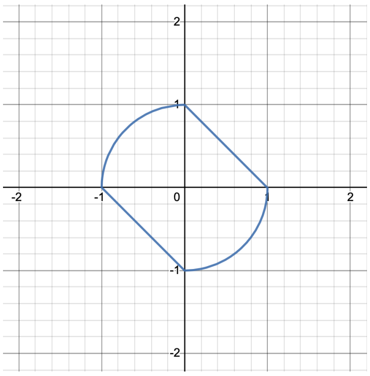
The last example revolves around showing that the rotundity of at least one of the two norms is not necessary condition for the rotundity of the "sum" norm, written formally: $\exists\ \|\cdot\|_1,\ \|\cdot\|_2$ which are not rotund but $|\cdot| = \|\cdot\|_1 + \|\cdot\|_2$ is.

We can, for example take two norms in $\mathbb{R}^2$ like this:

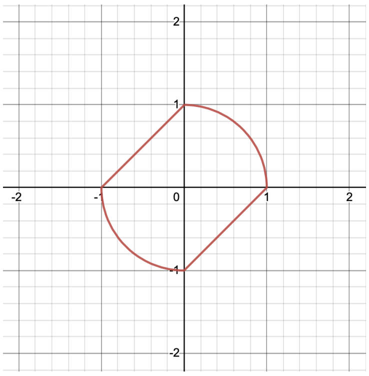
$$\|(x,y))\|_1 = \max(\|(x,y)\|_e, |x+y|)$$

$$\|(x,y))\|_2 = \max(\|(x,y)\|_e, |x-y|).$$

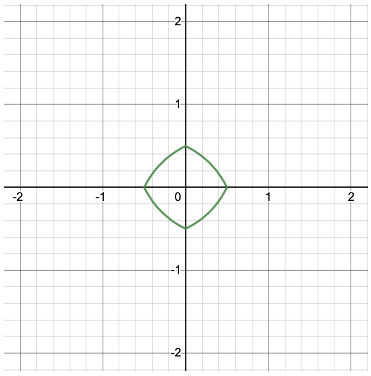
We will show in a very non rigorous way what happens to the unitary spheres in $\mathbb{R}^2$:



Unit sphere norm $\|\cdot\|_1$



Unit sphere norm $\|\cdot\|_2$



Unit sphere norm $|\cdot|$

From this we can clearly see that the first two norms contain some segments in the unit sphere so they can't be rotund, while the sum only contains extreme points.

Example 3.4:

Let's try to show the rotundity of the previously seen norm:

$$\|x\|^2 = \|x\|_0^2 + \sum_{n=1}^{\infty} \frac{1}{4^n} |x_n|^2$$

where:

$$\|x\|_0 = \max(\frac{1}{2}\|x\|_e, \sup|x_n|).$$

By applying Theorem 3.2 with:

$$(X_1, \|\cdot\|_1) = (\ell^2, \|\cdot\|_0), \ (X_2, \|\cdot\|_2) = (\ell^2, \|\cdot\|_{\ell^2})$$

$$T : \ell^2 \ \to \ell^2 \ (\subset \ell^\infty) \ : \ Tx^{(n)} = \frac{1}{2^n} x^{(n)}.$$

We need to establish if the space $(\ell^2, \|\cdot\|_0)$ is still a Banach space (as required by the hypothesis of the theorem), but this comes naturally considering the equivalence between the norms:

$$\|x\|_0 \leq \|x\|_\infty + \frac{1}{2}\|x\|_{\ell^2} \leq \|x\|_{\ell^2} + \frac{1}{2}\|x\|_{\ell^2} = \frac{3}{2}\|x\|_{\ell^2}.$$

While for the converse inequality:

$$\|x\|_0 \geq \frac{1}{2}\|x\|_{\ell^2}.$$

Being equivalent to the 2-norm of the space ℓ^2 we get the same convergent Cauchy sequences, hence the space is Banach. Let's check if the operator meets the other requirements of the Theorem 3.2.

-Linearity:

$$T(\alpha x_1 + \beta x_2)^{(n)} = \frac{1}{2^n}(\alpha x_1^{(n)} + \beta x_2^{(n)}) = \frac{1}{2^n}\alpha x_1^{(n)} + \frac{1}{2^n}\beta x_2^{(n)} = \alpha T x_1^{(n)} + \beta T x_2^{(n)}.$$

-One-to-one:

$$Tx_1 = Tx_2 \implies \frac{1}{2^n}x_1^{(n)} = \frac{1}{2^n}x_2^{(n)}\ \forall\ n \implies x_1^{(n)} = x_2^{(n)}\ \forall\ n \implies x_1 = x_2.$$

-Continuous:

We prove its continuity by showing its boundedness:

$$\forall\ x \in \ell^2\ \ \|Tx\|_{\ell^2}^2 = \sum_{n=1}^{\infty} \frac{1}{4^n}|x^{(n)}|^2 \leq \frac{1}{4}\|x\|_{\ell^2}^2 .$$

All the hypothesis are satisfied, thus $\|\cdot\|$ is rotund.

3.2. Midpoint Local Uniform Rotundity

Once foundational results about rotundity are in place, it's natural to consider its extensions. From strengthening the condition to refining the concept to be applicable to sequences, various variations come to light. The first one we present is the following:

Definition 3.2. *Given a normed space $(X, \|\cdot\|)$ and $\{x_n\}$, $\{y_n\} \subseteq S_{(X,\|\cdot\|)}$ such that every time $\|x_n\| \to 1$ $\|y_n\| \to 1$ and $\frac{1}{2}(x_n + y_n) \to z$ with $z \in S_{(X,\|\cdot\|)}$, it follows that $\|x_n - y_n\| \to 0$ we say that the norm is **midpoint locally uniformly rotund** or **MLUR** as abbreviation.*

This concept was historically introduced by K. Anderson in 1960 in his Phd thesis [1]. His motivation for such study was to find the correct notion of rotundity in a reflexive Banach space X that would result in a Fréchet smooth norm in the dual X^* (a concept that we will introduce later in Definition 4.6). Unfortunately his hope did not materialized he managed to introduce the correct notion of rotundity that paired with the Kadec property would guarantee Fréchet differentiability of the dual norm.

Lemma 3.2. *Let $(X,\|\cdot\|)$ be a normed space. The following are equivalent:*

1. $\|\cdot\|$ *is MLUR.*
2. *Whenever* $\{x_n\}$, $\{y_n\} \subseteq X$ *such that* $\|x_n\| \to 1$ $\|y_n\| \to 1$ *and* $\frac{1}{2}(x_n + y_n) \to z$ *with* $z \in S_{(X,\|\cdot\|)}$, *it follows that* $\|x_n - y_n\| \to 0$.
3. *Whenever* $x \in S_{(X,\|\cdot\|)}$ *and* $\{x_n\} \subseteq X$ *such that* $\|x + x_n\| \to 1$, $\|x - x_n\| \to 1$ *then* $x_n \to 0$.
4. *Whenever* $\{x_n\}$, $\{y_n\} \subseteq X$ *such that* $\|x_n\| \to \|x\|$, $\|y_n\| \to \|x\|$ *and* $\|\frac{1}{2}(x_n + y_n) - x\| \to 0$ *for some* $x \in X \setminus \{0\}$ *then* $\|y_n - x_n\| \to 0$.

Proof. We will prove the chain of implications even in this case.

-*1* $\Leftrightarrow$ *2*

Let's start by proving 1 $\implies$ 2.

Suppose $\|\cdot\|$ is MLUR and $\{x_n\}, \{y_n\} \subseteq X$ such that $\|x_n\| \to 1$ and $\|y_n\| \to 1$, for all $n \in \mathbb{N}$ $x_n \neq 0$, $y_n \neq 0$. Suppose also that $\frac{1}{2}(x_n + y_n) \to z \in S_{(X,\|\cdot\|)}$. So for all n:

$$0 \leq \left\| \frac{1}{2}(\|x_n\|^{-1}x_n + \|y_n\|^{-1}y_n) - z \right\| \leq \left\| \frac{1}{2}\left(\|x_n\|^{-1}x_n - x_n\right) \right\| + \left\| \frac{1}{2}(\|y_n\|^{-1}y_n - y_n) \right\| + \left\| \frac{1}{2}(x_n + y_n) - z \right\|.$$

It follows that $\frac{1}{2}(\|x_n\|^{-1}x_n + \|y_n\|^{-1}y_n) \to z$. Therefore $\|\|x_n\|^{-1}x_n - \|y_n\|^{-1}y_n\| \to 0$. This and the fact that

$$0 \leq \|x_n - y_n\| \leq \|x_n - \|x_n\|^{-1}x_n\| + \|\|x_n\|^{-1}x_n - \|y_n\|^{-1}y_n\| + \|\|y_n\|^{-1}y_n - y_n\|.$$

for each n together show that $\|x_n - y_n\| \to 0$, as required.

We now prove 2 $\implies$ 1

This is a far easier task, since if 2 is true and we take $\{x_n\}, \{y_n\} \subseteq S_{(X,\|\cdot\|)}$ such that $\frac{1}{2}(x_n + y_n) \to z \in S_{(X,\|\cdot\|)}$, then we can also state that the same $\|x_n\| \to 1$ and $\|y_n\| \to 1$ and applying condition 2 we get $\|x_n - y_n\| \to 0$, then condition 1 is satisfied.

-2 $\implies$ 3

Let's suppose implication 2 to be true and let us consider $x \in S_{(X,\|\cdot\|)}$ and $\{x_n\} \subseteq X$ such that $\|x + x_n\| \to 1$, $\|x - x_n\| \to 1$. We can now rename $x'_n = x + x_n$ and $y'_n = x - x_n$, if we now apply condition two we get $\|x'_n\| \to 1$ and $\|y'_n\| \to 1$, while $\frac{1}{2}(x'_n+y'_n) = \frac{1}{2}(x+x_n+x-x_n) = x \in S_{(X,\|\cdot\|)}$, but then we know $\|x'_n-y'_n\| = \|x+x_n-x+x_n\| = 2\|x_n\| \to 0 \Leftrightarrow x_n \to 0$. Forcing proposition 3 to be true.

-3 $\implies$ 4

Let's assume proposition 3 to be true and we also take $\{x_n\}$, $\{y_n\} \subseteq X$ such that $\|x_n\| \to \|x\|$, $\|y_n\| \to \|x\|$ and $\|\frac{1}{2}(x_n + y_n) - x\| \to 0$. We consider the point different from zero $\frac{x}{\|x\|} \in S_{(X,\|\cdot\|)}$, but then if we consider $\{\frac{x_n}{\|x\|}\}, \{\frac{y_n}{\|x\|}\} \subseteq X$ we see that $\frac{1}{2}(\frac{x_n}{\|x\|} + \frac{y_n}{\|x\|}) \to \frac{x}{\|x\|} \in S_{(X,\|\cdot\|)}$ since we know by hypothesis that $\|\frac{1}{2}(x_n+y_n)-x\| \to 0 \implies \frac{1}{\|x\|}\|\frac{1}{2}(x_n+y_n)-x\| \to 0 \implies \|\frac{1}{2}(\frac{x_n}{\|x\|} + \frac{y_n}{\|x\|}) - \frac{x}{\|x\|}\| \to 0$. So we have two converging sequences $\{\frac{x_n}{\|x\|}\}, \{\frac{y_n}{\|x\|}\} \subseteq X$ which tend to an element of the unit sphere $\frac{x}{\|x\|} \in S_{(X,\|\cdot\|)}$, so if we substitute $\frac{x_n}{\|x\|} = \frac{x}{\|x\|}+x'_n$ and $\frac{y_n}{\|x\|} = \frac{x}{\|x\|} - x'_n$ then we have two sequences $\|\frac{x}{\|x\|} + x'_n\| \to 1$, $\|\frac{x}{\|x\|} - x'_n\| \to 1$ with $\frac{x}{\|x\|} \in S_{(X,\|\cdot\|)}$ so we can applying proposition 3 getting the implication $x'_n \to 0$, but $x'_n = \frac{1}{2}(\frac{x_n}{\|x\|} - \frac{y_n}{\|x\|})$ so $\frac{1}{2}(\frac{x_n}{\|x\|} - \frac{y_n}{\|x\|}) \to 0 \implies \|\frac{1}{2}(\frac{x_n}{\|x\|} - \frac{y_n}{\|x\|})\| = \frac{1}{2\|x\|}\|x_n - y_n\| \to 0 \implies \|x_n - y_n\| \to 0$.

-4 $\implies$ 2

We finally finish proving this last statement. We assume 4 to be true and we also take $\{x_n\}$, $\{y_n\} \subseteq X$ such that $\|x_n\| \to 1$ $\|y_n\| \to 1$ and $\frac{1}{2}(x_n + y_n) \to z$ with $z \in S_{(X,\|\cdot\|)}$. The conclusion is quite obvious since we can just see $1 = \|z\|$, so since every hypothesis of 4 is satisfied it follows $\|x_n - y_n\| \to 0$ proving the implication.

□

Example 3.5:

A simple and recurrent example is the space $(\ell^2, \|\cdot\|_{\ell^2})$, given that it is very regular. We shall prove now that $(\ell^2, \|\cdot\|_{\ell^2})$ satisfies the MLUR condition.

To show this we can consider the fact that on this space $(\ell^2, \|\cdot\|_2)$ the parallelogram identity holds:

$$\|x + y\|^2_{\ell^2} + \|x - y\|^2_{\ell^2} = 2\|x\|^2_{\ell^2} + 2\|y\|^2_{\ell^2} \; \forall x, y \in X.$$

So, we can write considering sequences $\{x_n\}, \{y_n\}$ along $S_{(X,\|\cdot\|)}$:

$$\|x_n - y_n\|^2_{\ell^2} = 4\left(\frac{\|x_n\|^2_{\ell^2}}{2} + \frac{\|y_n\|^2_{\ell^2}}{2} - \left\|\frac{1}{2}(x_n + y_n)\right\|^2_{\ell^2}\right).$$

But now if we let $n \to \infty$

$$\|x_n - y_n\|^2_{\ell^2} = 4\left(\frac{\|x_n\|^2_{\ell^2}}{2} + \frac{\|y_n\|^2_{\ell^2}}{2} - \left\|\frac{1}{2}(x_n + y_n)\right\|^2_{\ell^2}\right) \to 4\left(\frac{1}{2} + \frac{1}{2} - 1\right) = 0 \ \forall x, y \in X.$$

By the second item of Lemma 3.2.1 the norm $\|\cdot\|_{\ell_2}$ is MLUR.

We now investigate the relation between the geometrical concepts that we have introduced with the following theorem.

Theorem 3.3. *A normed space $(X, \|\cdot\|)$ which is MLUR is also rotund.*

Proof. Let's prove this fact via contradiction: assume $\|\cdot\|$ to be MLUR but not rotund. If the space is not rotund then there must exist two points with such property $\exists\, x, y \in S_{(X,\|\cdot\|)}$ such that $x \neq y$ and $\left\|\frac{1}{2}(x+y)\right\| = 1$. But then if we take $x_n := x$ for all $n \in \mathbb{N}$ and $y_n := y$ for all $n \in \mathbb{N}$ we can state:

$$\|\frac{1}{2}(x_n + y_n)\| = \|\frac{1}{2}(x + y)\| = 1, \ \|x_n - y_n\| = \|x - y\| \neq 0 \ \forall n \in \mathbb{N}.$$

Leading to a contradiction.

□

Example 3.6:

The converse is not true, for example, if we take the previously introduced norm on the Hilbert space ℓ^2 by S.Draga in the article [3]:

$$\|x\|^2 = \|x\|_0^2 + \sum_{n=1}^{\infty} \frac{1}{4^n} |x^{(n)}|^2$$

where:

$$\|x\|_0 = \max(\frac{1}{2}\|x\|_{\ell^2}, \sup |x^{(n)}|).$$

We proved in Example 3.1 the rotundity of ℓ^2 and it can also be checked easily that it is not MLUR. If we consider two sequences in ℓ^2: $\{x_n\} = (\sqrt{\frac{4}{5}(1 - \frac{1}{4^{n+1}})}, 0, ..., -1/2, 0, ...)$ and $\{y_n\} = (\sqrt{\frac{4}{5}(1 - \frac{1}{4^{n+1}})}, 0, ..., 1/2, 0, ...)$ where the $\frac{1}{2}$ and $-\frac{1}{2}$ occupy the n-th position.

Since

$$\|x_n\|_0 = \max(\frac{1}{2}\|x_n\|_{\ell^2}, \sup |x_n^{(k)}|) = \sqrt{\frac{4}{5}}$$

for all $n \geq 1$ and

$$\sum_{k=1}^{\infty} \frac{1}{4^n}|x_n^{(k)}|^2 = \frac{1}{4} \cdot \frac{4}{5}(1 - \frac{1}{4^{n+1}}) + \frac{1}{4^n} \cdot \frac{1}{4}$$

we get if $n \to \infty$

$$\|x_n\|_0^2 = \|x_n\|_0^2 + \sum_{k=1}^{\infty} \frac{1}{4^n}|x_n^{(k)}|^2 = \frac{4}{5} + \frac{1}{4} \cdot \frac{4}{5}(1 - \frac{1}{4^{n+1}}) + \frac{1}{4^n} \cdot \frac{1}{4} \to 1.$$

The same holds for $\{y_n\}$ by applying the same calculations. While the term $\frac{1}{2}(x_n + y_n)$ as $n \to \infty$:

$$\frac{1}{2}(x_n + y_n) = (\sqrt{\frac{4}{5}(1 - \frac{1}{4^{n+1}})}, 0, ...) \to (\sqrt{\frac{4}{5}}, 0, 0, ...) \in S_{(X,\|\cdot\|)}.$$

However considering:

$$\|x_n - y_n\|^2 = \|(0, ..., 1, 0, ...)\|^2 = 1 + \frac{1}{4^{n+1}} \to 1 \neq 0.$$

So the norm is not MLUR.

In the study of Banach spaces, especially on those treating relations with topology, another notion might turn out to be useful. We should not only consider the strong convergence of the factors, but also the weak convergence. Leading us to the usage of an alternative concept:

Definition 3.3. *Let $(X, \|\cdot\|)$ be a Banach space. If, for every $z \in S_{(X,\|\cdot\|)}$ and every $\{x_n\}$, $\{y_n\} \subseteq S_{(X,\|\cdot\|)}$ such that $\frac{1}{2}(x_n + y_n) \to z$, it follows that $x_n - y_n \rightharpoonup 0$, we say that the norm is* ***weakly midpoint locally uniformly rotund****,* ***wMLUR*** *as abbreviation.*

There is no need to prove how the MLUR condition is stronger than wMLUR, after keeping in mind that the strong convergence implies the weak one. Whilst the converse is not true, we are still not able to prove it, but we will come back to this point later (Example 3.12).

Theorem 3.4. *A normed space $(X, \|\cdot\|)$ which is wMLUR is also rotund.*

Proof. Let's prove this fact via contradiction: Assume $\|\cdot\|$ to be wMLUR but not rotund.

If the space is not rotund then it must exist two points with such property $\exists\, x, y \in S_{(X,\|\cdot\|)}$ such that $x \neq y$ and $\left\|\frac{1}{2}(x + y)\right\| = 1$.

If we now take $x_n := x$ for all $n \in \mathbb{N}$ and $y_n := y$ for all $n \in \mathbb{N}$

$$\|\frac{1}{2}(x_n + y_n)\| = \|\frac{1}{2}(x + y)\| = 1, \ \|x_n - y_n\| = \|x - y\| \neq 0 \ \forall n \in \mathbb{N}.$$

By virtue of the first property of the norm in the Definition 2.1 we can conclude that $x \neq y$. We now wish to mention a famous corollary of the Hanh-Banach theorem for which given $x, y \in X$ such that $x^*(x) = x^*(y)$ for all $x^* \in X^* \implies x = y$. Since we know that $x \neq y$ we also know that there must exist $x^* \in X^*$ such that $x^*(x) \neq x^*(y)$, but if we take into account the definition of the two sequences we can write by linearity of the functional of the dual that $x^*(x_n - y_n) \neq 0$ which excludes the weak convergence of the two sequences.

□

A small remark shall be made, this was only possible because of the definition of the two sequences $x_n := x$ for all $n \in \mathbb{N}$ and $y_n := y$ for every $n \in \mathbb{N}$, since they are "stationary" in some sense, the corollary can be applied. If this condition is not satisfied by the sequences we cannot apply the same corollary concluding that $\exists x^* \in X^*$ such that $x^*(x_n - y_n)$ doesn't go to zero, or we would implicitly imply that not converging strongly implies not converging weakly, which is not true.

3.3. Local Uniform Rotundity

A similar concept to MLUR that extends the notion of convexity and presents several ties with the one above is the concept of *locally uniformly rotund*, which is based on the notion of LUR modulus.

Definition 3.4. *Suppose that X is a normed space. Define a function $\delta_X : [0,2] \times S_X \to [0,1]$ by the formula:*

$$\delta_X(\varepsilon, x) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : y \in S_{(X,\|\cdot\|)}, \|x - y\| \geq \varepsilon \right\}.$$

Then δ_X is the ***LUR modulus*** *of X. The space X is locally uniformly rotund or locally uniformly convex if $\delta_X(\varepsilon, x) > 0$ whenever $0 < \varepsilon \leq 2$ and $x \in S_{(X,\|\cdot\|)}$.*

Another definition for the LUR condition which is not based on the geometrical concept of the LUR modulus is the following, which will be mainly adopted in our thesis.

Definition 3.5. *Let $(X, \|\cdot\|)$ be a Banach space. Let $x_0 \in S_{(X,\|\cdot\|)}$. The norm $\|\cdot\|$ on X is* ***locally uniformly rotund (LUR)*** *at x_0 if for every $\varepsilon \in (0,2]$, $\exists \delta = \delta(x_0, \varepsilon) > 0$ such that $\|\frac{x_0+x}{2}\| \leq 1-\delta$ whenever $x \in S_X$ and $\|x_0 - x\| \geq \varepsilon$.*

If this property is satisfied at every point of $S_{(X,\|\cdot\|)}$ then we say that the norm is **LUR**. As someone might suspect at this point there is always a characterization lemma relative to every condition introduced

Here we can find a few of the most common examples of spaces that satisfy this property.

Example 3.7:

-All the $(\ell^p, \|\cdot\|_p)$ spaces are known to be LUR for $p \in (1, \infty)$.

-All the Hilbert spaces are LUR so also $(L^2, \|\cdot\|_{L^2})$.

-$(\mathbb{R}^n, \|\cdot\|_e)$ is a LUR space for every n (parallelogram identity holds).

Lemma 3.3. *Suppose that $(X, \|\cdot\|)$ is a normed space. Then the following are equivalent:*

1. *The space X is LUR.*
2. *When $x \in S_{(X,\|\cdot\|)}$ and $\{y_n\} \subseteq S_{(X,\|\cdot\|)}$ such that $\|\frac{1}{2}(x+y_n)\| \to 1$, then it follows $\|x - y_n\| \to 0$.*
3. *When $x \in S_{(X,\|\cdot\|)}$ and $\{y_n\} \subseteq B_{(X,\|\cdot\|)}$ such that $\|\frac{1}{2}(x+y_n)\| \to 1$, then it follows $\|x - y_n\| \to 0$.*
4. *When $x \in S_{(X,\|\cdot\|)}$ and $\{y_n\} \subseteq X$ such that both $\|y_n\|$, $\|\frac{1}{2}(x+y_n)\| \to 1$, then it follows $\|x - y_n\| \to 0$.*
5. *$\forall \{x_n\} \subseteq S_{(X,\|\cdot\|)}$, $x \in S_{(X,\|\cdot\|)}$ such that $\lim_n 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0$, it follows $x_n \to x$.*

Proof. Suppose that the second point holds and $x \in S_{(X,\|\cdot\|}$ and $\{y_n\} \subset X$ such that $\|y_n\|$, $\|\frac{1}{2}(x+y_n)\|$ both tend to 1. It will be shown that $\|x - y_n\| \to 0$. By discarding a finite number of terms we can assume $y_n \neq 0$ for all $n \in \mathbb{N}$. Then

$$1 \geq \left\| \frac{1}{2}(x + \|y_n\|^{-1} y_n) \right\| \geq \left\| \frac{1}{2}(x+y_n) \right\| - \left\| \frac{1}{2}(1 - \|y_n\|^{-1}) y_n \right\| \to 1$$

So $\left\|\frac{1}{2}(x + \|y_n\|^{-1} y_n)\right\| \to 1$. Since, by the second point $\|y_n\|^{-1} y_n \to x$, it then follows that

$y_n \to x$, which establishes that the second point implies the fourth. Now suppose that the fourth point holds and that $x \in S_{(X,\|\cdot\|)}$ and $y_n \in B_{(X,\|\cdot\|)}$ such that $\|\frac{1}{2}(x+y_n)\| \to 1$. Since

$$\left\|\frac{1}{2}(x+y_n)\right\| \leq \frac{1}{2}(1+\|y_n\|) \leq 1$$

for each n, it follows that $\|y_n\| \to 1$, so $\|x-y_n\| \to 0$ by the fourth point. Therefore the fourth point implies the third from which it follows $2 \Leftrightarrow 3 \Leftrightarrow 4$. Suppose X is LUR and $x \in S_{(X,\|\cdot\|)}$, $\{y_n\} \subset S_{(X,\|\cdot\|)}$ such that $\|\frac{1}{2}(x+y_n)\| \to 1$ but $\|x-y_n\|$ doesn't go to zero. Let δ_X be the LUR modulus of X. It follows that there is a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $\|x-y_{n_j}\| \geq \varepsilon$ for some positive ε and each j, which implies that $\|\frac{1}{2}(x+y_{n_j})\| \leq 1-\delta_X(\varepsilon,x)$ for each j, a contradiction. Therefore the first point implies the second.

Finally suppose that X is not LUR then there is $\varepsilon \in (0,2]$ and $x \in S_{(X,\|\cdot\|)}$ for which $\delta_X(\varepsilon,x)=0$. Therefore there is a sequence $\{y_n\} \in S_{(X,\|\cdot\|)}$ such that $\|x-y_n\| \geq \varepsilon$ for each n, but $\|\frac{1}{2}(x+y_n)\| \to 1$, so the second point doesn't hold and this shows that the second point implies the first one. While the converse implication is obvious.

In this last part of the proof we prove the equivalence between statement 2 and 5. We start by the first implication $2 \implies 5$, we assume 2 to be true and consider all the points $\{x_n\} \subseteq S_{(X,\|\cdot\|)}$ $x \in S_{(X,\|\cdot\|)}$ such that $\lim_n 2\|x\|^2+2\|x_n\|^2-\|x+x_n\|^2=0$. We know that $\lim_n \|x_n\|=1$ and $\|x\|=1$, so it implies:

$$\lim_n 2\|x\|^2+2\|x_n\|^2-\|x+x_n\|^2 = \lim_n 2+2-\|x+x_n\|^2 = 0.$$

We can conclude that $\lim_n \|x+x_n\|^2=4$, thus $\|\frac{1}{2}(x+x_n)\| \to 1$. We can now apply proposition two, concluding $\|x-x_n\| \to 0$.

Now the opposite, let's assume 5 to be true and let's also assume $x \in S_{(X,\|\cdot\|)}$ and $\{y_n\} \subseteq S_{(X,\|\cdot\|)}$ such that $\|\frac{1}{2}(x+y_n)\| \to 1$. We can immediately conclude that $\|x+y_n\|^2 \to 4$ but we can express $4 = 2\|x\|^2+2\|x_n\|^2$ since the sequence and the point are inside the unit sphere, leading to:

$$\lim_n 2\|x\|^2+2\|x_n\|^2-\|x+x_n\|^2=0.$$

From which we conclude $\|x-x_n\| \to 0$ and hence the thesis.

$\square$

Theorem 3.5. *A normed space $(X,\|\cdot\|)$ which is LUR is also MLUR.*

Proof. To prove we start by considering a Banach space $(X, \|\cdot\|)$ with a LUR norm, we will use characterization 2 of Lemma 3.3. Now we consider two sequences $\{x_n\}, \{y_n\} \subseteq S_{(X,\|\cdot\|)}$ with $x \in S_{(X,\|\cdot\|)}$ such that $\left\|\frac{1}{2}(x_n + y_n) - x\right\| \to 0$.

We start by seeing that:

$$\left\|\frac{1}{2}(x_n + x)\right\| \leq \left\|\frac{1}{2}x_n\right\| + \left\|\frac{1}{2}x\right\| = 1.$$

Also we can see that:

$$\left\|\frac{1}{2}(x_n + x)\right\| = \left\|\frac{1}{2}(x_n + x) + \frac{1}{2}(y_n - y_n) + x - x\right\| = \left\|\frac{1}{2}(x_n + y_n) - x + \frac{3}{2}x - \frac{y_n}{2}\right\|.$$

If we now apply inverse triangle inequality we get:

$$\left\|\frac{1}{2}(x_n + y_n) - x + \frac{3}{2}x - \frac{y_n}{2}\right\| \geq \frac{3}{2}\|x\| - \left\|\frac{1}{2}(x_n + y_n) - x - \frac{y_n}{2}\right\|.$$

By applying one last time triangle inequality:

$$\frac{3}{2}\|x\| - \left\|\frac{1}{2}(x_n + y_n) - x - \frac{y_n}{2}\right\| \geq \frac{3}{2}\|x\| - \left\|\frac{1}{2}(x_n + y_n) - x\right\| - \left\|\frac{y_n}{2}\right\| =$$

$$= \frac{3}{2} - \frac{1}{2} - \left\|\frac{1}{2}(x_n + y_n) - x\right\| \to 1.$$

But now we can apply the LUR characterization to retrieve that $\|x_n - x\| \to 0$. The same exact algebraic steps can be done with $\{y_n\}$ obtaining analogously that $\|y_n - x\| \to 0$. But now we also obtain with the following application of triangle inequality:

$$\|x_n - y_n\| \leq \|x_n - x\| + \|y_n - x\| \to 0.$$

Which show that the norm $\|\cdot\|$ is MLUR.

□

We will now provide two examples, of why the converse implication is false. We will start by introducing a definition needed to prove the properties of the following norms.

A Banach space $(X, \|\cdot\|)$ has the *Kadec property* if the norm and the weak topology coincide on $S_{(X,\|\cdot\|)}$.

Example 3.8:

We will now make use of an example taken by the article [12]. Consider the space ℓ^1 endowed with the following norm:

$$\|x\| = \sum_{i=1}^{\infty} |x_i| + (\sum_{i=1}^{\infty} 2^{-i+1} x_i^2)^{\frac{1}{2}}$$

We can apply Theorem 3.2 to obtain rotundity of the norm, this norm also satisfies the *Kadec property*, thus by Corollary 408, page 298 of the book [7] we also know it is MLUR. However if we consider the point $x = (\frac{1}{2}, 0, 0...)$ and the sequence $x_n = (0, \frac{1}{n}, ..., \frac{1}{n}, 0, ...)$ where $\frac{1}{n}$ is repeated n times then: $\|x\| = \frac{1}{2} + \frac{1}{2} = 1$, $\|x_n\| = n\frac{1}{n} + \sqrt{\frac{1}{n^2} \sum_{i=2}^{n+1} 2^{-i+1}} \to 1$ and $\|x + x_n\| = \frac{1}{2} + n\frac{1}{n} + \sqrt{\frac{1}{2} + \frac{1}{n^2} \sum_{i=2}^{n+1} 2^{-i+1}} \geq 1$ for every $n \in \mathbb{N}$. Thus the norm is not LUR.

Example 3.9:

We won't discuss in details this example since it requires some very involved argument, but we mention that the first person to exhibit a norm which was MLUR but not LUR was the very inventor of the MLUR definition, Kenneth Wayne Anderson in 1960, with his Phd Thesis [1] he managed to characterize this new concept. It also managed to show this property without the direct involvment of the Kadec property.

Theorem 3.6 ([5, Theorem 8.1])**.** *Every separable Banach space admits an equivalent LUR norm.*

Proof. To prove this fact we propose a norm that is general to every Banach space and satisfies the two properties. We take $\{e_n : n \in \mathbb{N}\} \subset S_{(X, \|\cdot\|)}$, also dense in $S_{(X, \|\cdot\|)}$. We also consider $\{e_n^* : n \in \mathbb{N}\} \subset S_{(X^*, \|\cdot\|^*)}$, a separating family for X. We now consider the sets $E_n = \text{span}(e_1, ..., e_n)$ and note that $\text{dist}(x, E_n) \to 0$ for all $x \in X$. We can now define a norm:

$$|\cdot|^2 = \|\cdot\|^2 + \sum_{n=1}^{\infty} \frac{1}{2^n} \text{dist}(\cdot, E_n)^2 + \sum_{n=1}^{\infty} \frac{1}{2^n} e_n^*(\cdot)^2$$

First of all this is an equivalent norm since the distance is a positive homogeneous subadditive function, thus the equivalence. We will now show that it is also LUR: assume $x \in X$ and $\{x_n\} \subseteq X$ such that $\lim_k 2|x|^2 + 2|x_k|^2 - |x + x_k|^2 = 0$. Proceeding as in the proof of Theorem 3.2 we can conclude:

$$\lim_k 2\|x\|^2 + 2\|x_k\|^2 - \|x + x_k\|^2 = 0$$

$$\lim_k 2\,\mathrm{dist}(x, E_n)^2 + 2\,\mathrm{dist}(x_k, E_n)^2 - \mathrm{dist}(x + x_k, E_n)^2 = 0\ \forall n \in \mathbb{N}$$

$$\lim_k 2e_n^*(x)^2 + 2e_n^*(x_k)^2 - e_n^*(x + x_k)^2 = 0\ \forall n \in \mathbb{N}$$

and also that, by the same calculation, using the fact that $\lim_k 2|x|^2 + 2|x_k|^2 - |x + x_k|^2 \geq \lim_k(|x| - |x_k|)^2 = 0$:

$$\lim_k |x_k| = |x| \tag{1}$$

$$\lim_k \mathrm{dist}(x_k, E_n) = \mathrm{dist}(x, E_n)\ \forall n \in \mathbb{N} \tag{2}$$

$$\lim_k e_n^*(x_k) = e_n^*(x)\ \forall n \in \mathbb{N}. \tag{3}$$

Since $\{e_n^*\}$ is a separating family of elements of the dual the topology of pointwise convergence, on $\{e_n^*\}$, is an Hausdorff topology on X (for every two distinct points in X one can always find two open sets containing respectively one point with empty intersection). We will show that $\overline{\{x_k\} \cup \{x\}}$ is compact with respect to the norm. Therefore on $\overline{\{x_k\} \cup \{x\}}$ the topology of pointwise convergence on $\{e_n^*\}$ is equivalent to the norm topology. Thus (3) would imply $\lim_k \|x_k - x\| = 0$ and the proof is complete. Using (1), choose $K > 0$ such that $\|x_k\| \leq K$ for every k. Let $\varepsilon \in (0, 1)$, choose $n \in \mathbb{N}$ such that $\mathrm{dist}(x, E_n) < \varepsilon$ and choose a finite ε-net E in $(K+1)B_{E_n}$. Using (2) choose k_0 such that $\mathrm{dist}(x_k, E_n) < \varepsilon$ for $k > k_0$. We claim that $\{x_1, x_2, ..., x_{k_0}\} \cup E$ is a 2ε-net for $\{x_k\}$. Indeed for every $k > k_0$ there is $x_k' \in E_n$ such that $\|x_k - x_k'\| < \varepsilon$. Since $\|x_k\| \leq K$ for every k and $\varepsilon < 1$ we have $\|x_k'\| < K + 1$. As E is an ε-net for $(K+1)B_{E_n}$, there is $x_k'' \in E$ such that $\|x_k' - x_k''\| < \varepsilon$. Thus the thesis.

□

Definition 3.6. *Let $(X, \|\cdot\|)$ be a Banach space. If, for every $x \in S_{(X,\|\cdot\|)}$ and $\{y_n\} \subseteq S_{(X,\|\cdot\|)}$ such that $\left\|\frac{1}{2}(x + y_n)\right\| \to 1$ it follows $y_n \rightharpoonup x$, we say that the norm is* ***weakly locally uniformly rotund, wLUR*** *as abbreviation.*

Clearly in the same fashion mentioned before for wMLUR also the LUR condition implies the wLUR condition, we will prove later that the converse is not true (Example 3.12).

Theorem 3.7. *A normed space $(X, \|\cdot\|)$ which is wLUR is also wMLUR.*

Proof. We start by considering a Banach space $(X, \|\cdot\|)$ with a wLUR norm, we will use the following characterization $x \in S_{(X,\|\cdot\|)}$ and $\{y_n\} \subseteq S_{(X,\|\cdot\|)}$ such that $\|\frac{1}{2}(x + y_n)\| \to 1$, then it follows $x - y_n \rightharpoonup 0$. Now we consider two sequences $\{x_n\}, \{y_n\} \subseteq S_{(X,\|\cdot\|)}$ with $x \in S_{(X,\|\cdot\|)}$ such that $\left\|\frac{1}{2}(x_n + y_n) - x\right\| \to 0$.

Let's notice that:

$$\left\|\frac{1}{2}(x_n + x)\right\| \leq \left\|\frac{1}{2}x_n\right\| + \left\|\frac{1}{2}x\right\| = 1.$$

We can also see that:

$$\left\|\frac{1}{2}(x_n + x)\right\| = \left\|\frac{1}{2}(x_n + x) + \frac{1}{2}(y_n - y_n) + x - x\right\| = \left\|\frac{1}{2}(x_n + y_n) - x + \frac{3}{2}x - \frac{y_n}{2}\right\|.$$

If we now apply inverse triangle inequality we get:

$$\left\|\frac{1}{2}(x_n + y_n) - x + \frac{3}{2}x - \frac{y_n}{2}\right\| \geq \frac{3}{2}\|x\| - \left\|\frac{1}{2}(x_n + y_n) - x - \frac{y_n}{2}\right\|.$$

By applying triangle inequality:

$$\frac{3}{2}\|x\| - \left\|\frac{1}{2}(x_n + y_n) - x - \frac{y_n}{2}\right\| \geq \frac{3}{2}\|x\| - \left\|\frac{1}{2}(x_n + y_n) - x\right\| - \left\|\frac{y_n}{2}\right\| =$$

$$= \frac{3}{2} - \frac{1}{2} - \left\|\frac{1}{2}(x_n + y_n) - x\right\| \to 1$$

But now we can apply the wLUR characterization to retrieve that $x_n - x \rightharpoonup 0$. The same exact algebraic steps can be done with $\{y_n\}$ obtaining analogously that $x - y_n \rightharpoonup 0$. But this means that for every $L \in X^*$ $L(x_n - x) \to 0$ and for all $L \in X^*$ $L(y_n - x) \to 0$, so also for every $L \in X^*$ $L(x_n - x) + L(x - y_n) \to 0$ which implies by linearity for all $L \in X^*$ $L(x_n - y_n) \to 0$ or, in other words $x_n - y_n \rightharpoonup 0$

□

3.4. Uniform Rotundity

The last geometrical concept that we will see is the one *Uniform rotundity*. This concept was one of the first geometrical concepts introduced in the study of Banach spaces. It was formalized by James A. Clarkson in 1936 because, as we will see, is strongly related with some topological properties. Intuitively we must have that the center of a line segment inside the unit ball must lie deep inside the unit ball unless the segment is short. We will introduce a similar concept as LUR modulus, but adapted to give an even stronger condition.

Definition 3.7. *The norm $\|\cdot\|$ of a Banach space X is called **uniformly rotund** or **UR** if $\delta_X(\varepsilon) > 0$ for each $0 < \varepsilon \leq 2$, where δ_X is the modulus of convexity, defined in the following way:*

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : x, y \in B_{(X,\|\cdot\|)},\ \|x-y\| \geq \varepsilon \right\}.$$

We introduce a characterization of the UR condition with the following lemma.

Lemma 3.4. *Suppose that $(X, \|\cdot\|)$ is a normed space. Then the following statements are equivalent:*

1. *The space X is UR.*
2. *Given $\{x_n\}, \{y_n\} \subseteq S_{(X,\|\cdot\|)}$ such that $\|x_n + y_n\| \to 2$ then $\|x_n - y_n\| \to 0$.*
3. *When $\{x_n\}, \{y_n\} \subseteq X$ such that $\|x_n\|, \|y_n\|, \|\frac{1}{2}(x_n + y_n)\| \to 1$, then it follows $\|x_n - y_n\| \to 0$.*

Proof. Let's prove the chain of implications.

-2 $\implies$ 3

Suppose that the second statement is true and that $\{x_n\}, \{y_n\} \subset X$ are such that $\|x_n\| \to 1$, $\|y_n\| \to 1$ and $\left\|\frac{1}{2}(x_n + y_n)\right\| \to 1$. It will be shown that $\|x_n - y_n\| \to 0$. We discard the same terms as in the previous proof, so no x_n nor y_n will be equal to zero. Then

$$1 \geq \left\| \frac{1}{2}(\|x_n\|^{-1}x_n + \|y_n\|^{-1}y_n) \right\| \geq \left\| \frac{1}{2}(x_n + y_n) \right\| + \left\| \frac{1}{2}(1 - \|x_n\|^{-1})x_n \right\| + \left\| \frac{1}{2}(1 - \|y_n\|^{-1})y_n \right\|.$$

The last term $\to 1$ implying that also $\left\|\frac{1}{2}(\|x_n\|^{-1}x_n + \|y_n\|^{-1}y_n)\right\| \to 1$. Since $\|\|x_n\|^{-1}x_n - \|y_n\|^{-1}y_n\| \to 0$ for the second condition it follows that:

$$0 \leq \|x_n - y_n\| \leq \|\|x_n\|^{-1}x_n - \|y_n\|^{-1}y_n\| - \left\|(1 - \|x_n\|^{-1})x_n\right\| - \left\|(1 - \|y_n\|^{-1})y_n\right\|.$$

Which goes to 0. Therefore $\|x_n - y_n\| \to 0$ which establishes that the second point implies the third.

-3 $\implies$ 2 This is obvious since if we assume 3 to be true and we select points satisfying all the hypothesis of 2 : $\{x_n\}, \{y_n\} \subseteq S_{(X,\|\cdot\|)}$ such that $\|x_n + y_n\| \to 2$ we automatically get the implication $\|x_n - y_n\| \to 0$ since those points also satisfy all the hypothesis of point 3.

-1 $\implies$ 2

Consider sequences in $S_{(X,\|\cdot\|)}$ such that $\|x_n + y_n\| \to 1$ but $\|x_n - y_n\|$ does not tend to 0. Let δ_X be the modulus of rotundity of X. It follows that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - y_{n_k}\| \geq \varepsilon$ for some positive ε and each k, which implies that $\frac{1}{2}\|x_{n_k} + y_{n_k}\| \leq 1 - \delta_X(\varepsilon)$ for each k, a contradiction. Therefore the first condition implies the second.

-2 $\implies$ 1

Finally, suppose that X is not uniformly rotund. Then there is an ε such that $0 < \varepsilon \leq 2$ and $\delta_X(\varepsilon) = 0$. Therefore there are sequences $\{x_n\}$ and $\{y_n\}$ in $S_{(X,\|\cdot\|)}$ such that $\|x_n - y_n\| \geq \varepsilon$ for each n and $\|x_n + y_n\| \to 1$, so the second condition does not hold. This shows that the second condition implies the first one.

□

Example 3.10:

-$(\ell^2, \|\cdot\|_2)$ through the use of parallelogram identity.

-$(L^p, \|\cdot\|_p)$, $p \in (1, \infty)$ are UR spaces. The proof makes use of Clarkson'inequalities:

If $p \in (1, 2)$

$$\left\|\frac{f+g}{2}\right\|_{L^p}^q + \left\|\frac{f-g}{2}\right\|_{L^p}^q \leq \left(\frac{1}{2}\|f\|_{L^p}^p + \frac{1}{2}\|g\|_{L^p}^p\right)^{\frac{q}{p}}.$$

If $p \in [2, \infty)$

$$\left\|\frac{f+g}{2}\right\|_{L^p}^p + \left\|\frac{f-g}{2}\right\|_{L^p}^p \leq \frac{1}{2}\left(\|f\|_{L^p}^p + \|g\|_{L^p}^p\right).$$

where p and q are conjugate numbers.

-$(\mathbb{R}^n, \|\cdot\|_2)$ is a UR space for all n (parallelogram identity holds).

Theorem 3.8. *A normed space $(X, \|\cdot\|)$ which is UR is also LUR.*

Proof. We take a Banach space $(X, \|\cdot\|)$ whose norm is assumed to be UR, specifically we'll use characterization 3 of Lemma 3.4, let us also consider points in the unit sphere. We now assume to have found $\{x_n\} \subseteq S_{(X,\|\cdot\|)}$, $x \in S_{(X,\|\cdot\|)}$ such that $\left\|\frac{x+x_n}{2}\right\| \to 1$, but now if we set $x_n = x_n$ $n \in \mathbb{N}$ and $y_n = x$ $n \in \mathbb{N}$ we can apply the UR condition since

$\|x_n\| = 1$ for every $n \in \mathbb{N}$, $\|y_n\| = \|x\| = 1$ for all $n \in \mathbb{N}$ and $\left\|\frac{y_n+x_n}{2}\right\| \to 1$ thus implying that $\|x_n - y_n\| = \|x_n - x\| \to 0$.

□

Example 3.11:

We now show an example of the fact that the converse is not true. We construct a space with a LUR norm which is not UR.

We take, for instance the functional space L^1. It can be proven that the polynomials with rational coefficients are dense in L^p $p \in [1, \infty)$ and thus follows the separability of the space. We also know that the space L^1 is not reflexive.

We have proven that every separable space admits a LUR renorming (Theorem 3.6), but this norm cannot also be UR or otherwise we could apply another theorem that we will see in the following section (Theorem 3.11) theorem and get the reflexivity of the space. Thus we created a norm on L^1 which is LUR but not UR.

Similarly to the twin concept introduced after the MLUR and LUR conditions, we can build another concept which will be more crucial to the focus of our thesis:

Definition 3.8. *Let $(X, \|\cdot\|)$ be a Banach space. If, for every $\{x_n\} \subseteq S_{(X,\|\cdot\|)}$ and $\{y_n\} \subseteq S_{(X,\|\cdot\|)}$ such that $\left\|\frac{1}{2}(x_n + y_n)\right\| \to 1$, it follows $x_n - y_n \rightharpoonup 0$, we say that the norm is **weakly uniformly rotund, wUR** as abbreviation.*

The following theorem investigates the relation with the UR property and the other geometrical properties.

Theorem 3.9. *A normed space $(X, \|\cdot\|)$ which is wUR is also wLUR.*

Proof. We take a Banach space $(X, \|\cdot\|)$ whose norm is assumed to be wUR. We now consider a sequence $\{x_n\} \subseteq S_{(X,\|\cdot\|)}$, $x \in S_{(X,\|\cdot\|)}$ such that $\left\|\frac{x+x_n}{2}\right\| \to 1$, but now if we set $x_n = x_n$ for every $n \in \mathbb{N}$ and $y_n = x$ for all $n \in \mathbb{N}$ we can apply the wUR condition since $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $\|y_n\| = \|x\| = 1$ for all $n \in \mathbb{N}$ and $\left\|\frac{y_n+x_n}{2}\right\| \to 1$ thus implying that $x_n - y_n = x_n - x \rightharpoonup 0$.

□

Example 3.12:

We now introduce an example based on the norm that can be found in the paper [3] used also in Example 3.6. This will clarify all the open question of this chapter.

Let $X = \ell^2$ endowed with the following norm:

$$\|x\|^2 = \|x\|_0^2 + \sum_{n=1}^{\infty} \frac{1}{4^n} |e^*(x)|^2.$$

where:

$$\|x\|_0 = \max(\frac{1}{2} \|x\|_{\ell^2}, \sup |e^*(x_n)|).$$

We start by taking $x_n, x'_n \in S_{(X,|\cdot|)}$ such that $|x_n + x'_n| \to 2$. We then set $z_n = (||x_n||_0, \frac{1}{2}e_1^*(x_n), ...)$ and $z'_n = (||x'_n||_0, \frac{1}{2}e_1^*(x'_n), ...)$. We can now calculate:

$$\|z_n+z'_n\|_2^2 = (||x_n||_0+||x'_n||_0)^2+\sum_{n=1}^{\infty} \frac{1}{4^m} e_m^*(x_n+x'_n)^2 \geq (||x_n+x'_n||_0)^2+\sum_{n=1}^{\infty} \frac{1}{4^m} e_m^*(x_n+x'_n)^2.$$

The inequality is given by the triangle inequality applied to $||| \cdot |||$. But then:

$$(||x_n + x'_n||_0)^2 + \sum_{n=1}^{\infty} \frac{1}{4^m} e_m^*(x_n + x'_n)^2 = |x_n + x'_n|^2 \to 2^2 = 4.$$

Since $\|z_n\|_2 = |x_n| = 1$ and $\|z'_n\|_2 = |x'_n| = 1$, we can apply the trinagular ineuqality

$$\|z_n + z'_n\|_2 \leq \|z_n\|_2 + \|z'_n\|_2 = |x_n| + |x'_n| = 2.$$

To prove that also $\|z_n + z'_n\| \to 2$. We also know that the norm $\| \cdot \|_2$ of the space ℓ^2 satisfies the UR condition, but then by the characterization lemma of UR we also know that $\|z_n - z'_n\|_2 \to 0$. This means:

$$\|z_n - z'_n\|_2^2 = (||x_n||_0 - ||x'_n||_0)^2 + \sum_{n=1}^{\infty} \frac{1}{4^m} (e_m^*(x_n) - e_m^*(x'_n))^2 \to 0.$$

Which implies our final result $\lim_{n\to\infty} e_m^*(x_n) = \lim_{n\to\infty} e_m^*(x'_n)$. We now use the key assumption that $\overline{\text{span}\{e_n^*\}} = X^*$, this allows us to infer that every operator in the dual when applied to x_n or x'_n mantains it's convergence, or, written in other terms $x_n - x'_n \rightharpoonup 0$. This proves that this norm is wUR.

Let us also recall that in Example 3.6 we proved that the norm is not MLUR. Given the chain of implications that we have established throughout the chapters, specifically Theorem 3.8, Theorem 3.9, Theorem 3.7 and Theorem 3.5, we have simultaneously found a norm which is wMLUR but not MLUR, wLUR but not LUR and wUR but not UR.

3.5. Comparison

We have finished introducing and characterizing many different notions of convexity. We now wish, stating a theorem, to shed some light on why we might need all these different concepts.

Theorem 3.10. *A finite dimensional normed space $(X, \|\cdot\|)$ is UR if and only if is rotund.*

This result underlies how in a finite dimensional space all the previous definitions can be used interchangeably, those distinctions start to make sense once we consider the infinite dimensional case.

$$\begin{tikzcd}
\boxed{\text{UR}} \arrow[r] & \boxed{\text{LUR}} \arrow[r] & \boxed{\text{MLUR}} \arrow[r] & \boxed{\text{R}} \arrow[lll, bend left]
\end{tikzcd}$$

Figure 3.2: Scheme of the cascade of implications in finite dimension

In the infinite dimensional case we also lose track of some topological properties, one example is the fact that every finite dimensional Banach space is also separable or the fact that all the finite dimensional spaces are reflexive. Once we start dealing with infinite dimensional Banach spaces these properties not only emerge creating separate kind of Banach spaces, but they also have a strong bound with the geometrical properties of the norm thus showing the need for several characterizations.

One example of this connection is:

Theorem 3.11 ([5, Theorem 9.11], Milman-Pettis)**.** *Every UR Banach space is Reflexive.*

We now know that UR spaces are also reflexive, but not many spaces are "regular enough" to be UR, so by introducing weaker notions of rotundity we let go of some of this regularity, gaining generalization. The next example is the LUR property, it has been shown that every Banach space which is weakly compactly generated admits a LUR renorming. MLUR condition was introduced in the 60' in the article [1]. His motivation for such a study was to find the correct rotundity notion in a reflexive Banach space X that would be in duality to the smoothness notion of Fréchet differentiability of the norm in the conjugate space $X*$. We also know that every rotund space every extreme point of the closed unit ball is an exposed point. Different topological notions can be tied to different geometrical properties giving us the need to distinguish them.

Just to sum up everything we need to know about the relationship between those concept we propose the follwing scheme:

$$\begin{tikzcd}
\boxed{\text{UR}} \arrow[r] \arrow[d] & \boxed{\text{LUR}} \arrow[r] \arrow[d] & \boxed{\text{MLUR}} \arrow[r] \arrow[d] & \boxed{\text{R}} \\
\boxed{\text{wUR}} \arrow[r] & \boxed{\text{wLUR}} \arrow[r] & \boxed{\text{wMLUR}} \arrow[ur] &
\end{tikzcd}$$

Figure 3.3: Scheme of all implications related to rotundity

In the open problems that we will try to tackle a topological hypothesis is made at the beginning of the statement, we will specifically deal with renormings of *separable* Banach spaces.

4 | Analytical Properties

This section is devoted to formalizing some analytical properties that will be crucial to the solution of the open problems that we will be solving. We will be introducing concepts related to the monotonicity of the norms and smoothness.

4.1. Lattice Norms

We will spend this few pages introducing and characterizing the condition of monotonicity for norms to obtain a very useful renorming technique.

Definition 4.1 ([13, Section 1.8 page 5]). *Given a set A, a* ***relation*** *R is defined as any subset of the space $A \times A$. When the couple $(a, b) \in R$ we use the notation aRb*

After having introduced this general concept, we lose some generalization with this next definition to retrieve another related concept that will help us model some notion of "order" in an arbitrary set.

Definition 4.2 ([13, Section 1.8 page 5]). *A relation R satisfying the following three properties is called* ***partial order****:*

- $\forall\ a \in A\ aRa$ *(reflexive)*
- aRb *and* $bRa \implies a = b$ *(antisymmetric)*
- aRb *and* $bRc \implies aRc$ *(Transitive)*

When aRb we will use the notation $a \preceq b$.

We now have a formal way to refer to the order inside a certain set, which is a crucial concept when introducing the concept of lattice.

Example 4.1:

The relation $\leq$ on the space $\mathbb{R}^+$ is an order relation. In fact for all $x \in \mathbb{R} x \leq x$, if we consider $x, y \in \mathbb{R}$ such that both $x \leq y$ and $y \leq x$ then $x = y$. Given $x, y, z \in \mathbb{R}$ if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 4.3. *Let $(X, L(\cdot))$ a lattice, where $L(x) = x \vee -x$, let us also assume $(X, \|\cdot\|)$ to be a Banach space. The norm $\|\cdot\|$ is said to be* ***lattice*** *whenever the following proposition holds:*

$$L(x) \leq L(y) \implies \|x\| \leq \|y\| \; \forall x, y \in X$$

Example 4.2:

-Starting from the last analogy we can easily generate a simple example for $(\mathbb{R}, |\cdot|)$ Considering the homogeneity property of the norms the order relation corresponds to the output of the norm function. In some sense it can be stated that the norm function and the order relation coincide.

-$(L^p, \|\cdot\|_p)$ are Banach lattices with respect to the order that comes from the pointwise almost everywhere order of functions.

- Some other spaces like ℓ^p or c_0 with their canonical norms are lattice with respect to the component-wise ordering.

Definition 4.4. *The* ***canonical order*** *on $\mathbb{R}^n$ is the following:*

$$x \preceq y \Leftrightarrow (x_1 \leq y_1) \wedge ... \wedge (x_n \leq y_n).$$

The need to specify this in a formal way arises from the fact that when we deal with multidimensional euclidean spaces it's not very easy to extend the "canonical" partial order without losing some crucial properties.

One might think to use the (euclidean) distance with respect to the origin, but this doesn't generate a partial order since, if we consider $x = (1, 0)$ and $y = (0, 1)$ in $\mathbb{R}^2$ we can write $x \preceq y$ and $y \preceq x$ but the implication required for the antisymmetrical property of the partial order fails: $x \neq y$.

On the other hand the price of introducing such order on $\mathbb{R}^2$ is losing the property that the order is **total**, formally not every couple $\in \mathbb{R}^2$ belongs to the relation R. One example of this is the couple $x = (2, 1)$ and $y = (1, 2)$, for which we cannot write xRy nor yRx.

We might also think about a partial order defined similarly to the one above that takes into account this problem

$$x \preceq y \Leftrightarrow (x_1 \leq y_1) \vee (x_1 = y_1 \wedge x_2 \leq y_2).$$

This allows us to compare every couple of the 2-dimensional plane, but it also makes "canonical" norms such as the euclidean one non lattice. To show this it is enough to think about $x = (1, 0)$ and $y = (0, 2)$ for which $y \preceq x$ but $\|x\| = 1$ while $\|y\| = \sqrt{2}$.

We can now show some *counterexamples* of norms which are not lattice with respect to the order that we have just introduced. We start from the following norm in $\mathbb{R}^2$

$$\|(x, y)\| = \sqrt{x^2 + (x - y)^2}$$

can be proven to be a norm, while for the lattice condition it is sufficient to consider the points $(\|(1, 0)\| = \sqrt{2}$ and $(1, 1) = 1$ while we have $(1, 0) \leq (1, 1))$.

Another similar example is given by the following norm

$$\|(x, y)\| = |x| + |(x - y)|.$$

For which the same example shows the fact that the norm is not lattice.

To introduce the next lemma we still need to define one concept concerning the space $\mathbb{R}^n$. We will now introduce the *canonical projection operator* $P_i : \mathbb{R}^n \to \mathbb{R}^n$ which is defined for $x = (x_1, x_2, ...)$ as $P_i(x) = (0, ..., x_i, 0, ...)$.

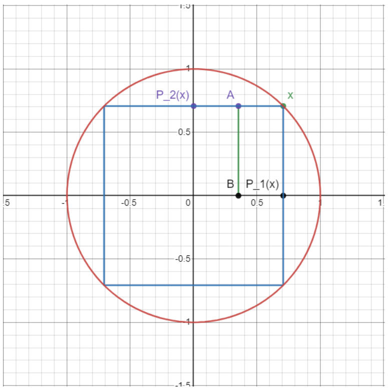
Lemma 4.1. *Given the finite dimensional normed space* $(\mathbb{R}^n, \|\cdot\|)$*, the norm is lattice if:*

$$\|P_i(x)\| \leq 1,\ \forall\ i \in\ \{1, ..., n\},\ \forall\ x \in S_{(\mathbb{R}^n, \|\cdot\|)}.$$

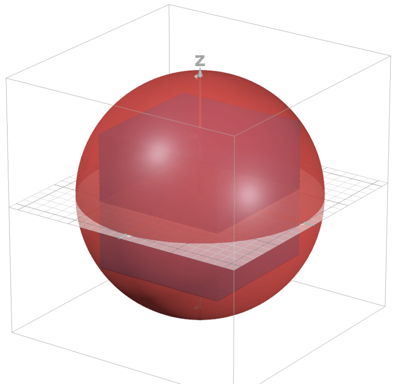
Proof. We shall start from remembering that every unit ball is convex since, from the triangle inequality, it follows the definition of convex function:

$$\|\lambda x + (1 - \lambda)y\| \leq \|\lambda x\| + \|(1 - \lambda)y\| = \lambda\|x\| + (1 - \lambda)\|y\|,\ \lambda \in [0, 1].$$

We now take a vector belonging to the unit sphere $x \in S_{(\mathbb{R}^n, \|\cdot\|)}$ if by hypothesis $\|P_i(x)\| \leq 1$, then $P_i(x) \in B_{(\mathbb{R}^n, \|\cdot\|)}$ for all $i \in \{1, ..., n\}$. This implies that every element of the form $P_i(x) + \lambda_i(x - P_i(x)) \in B_{(\mathbb{R}^n, \|\cdot\|)}$ given $\lambda_i \in [0, 1]$ since it is just the line connecting two points inside a convex set. A similar argument can be applied to all the points of the shape $\lambda_i P_i(x)$ with $\lambda_i \leq 1$ given the fact that the norm is homogeneous, what we are effectively doing is that we are showing that the set $\operatorname{conv}(\{0, x, P_1(x), \ldots, P_n(x)\})$ lies inside the unit ball. To give an idea we add the two and three dimensional cases with some graphic visualization. In the case of $\mathbb{R}^2$ the square can be built taking linear combinations of points on the edge that we have formally proven to be inside the ball and the same stands for $\mathbb{R}^3$.



$\mathbb{R}^2$ with the euclidean norm



$\mathbb{R}^3$ with the euclidean norm

We have, then, found the thesis since we have written for all $(y_1, y_2, y_3, ...) \in \mathbb{R}^n$ such that $0 \leq y_i \leq x_i$ the following inequality holds:

$$\|(y_1, y_2, y_3, ...)\| \leq 1 = \|(x_1, x_2, x_3, ...)\|$$

Hence the definition of lattice.

□

Corollary 4.1. *The Banach space $(\mathbb{R}^2, \|\cdot\|_e)$ is lattice when considering the euclidean norm*

Proof. To show this we just see that if $x \in S_{(\mathbb{R}^2, \|\cdot\|)}$ it means that:

$$x_1^2 + x_2^2 = 1 \implies \|P_1(x)\|_e^2 = |x_1|^2 \|(1,0)\|_e^2 \leq 1, \;\; \|P_2(x)\|_e^2 = |x_2|^2 \|(0,1)\|_e^2 \leq 1.$$

Otherwise we would get $x_1^2 + x_2^2 > 1$ contradicting the initial hypothesis. If we now apply the previous lemma we get the thesis.

□

Example 4.3:

We introduce a way to generate norms. Given $Q \in \mathbb{R}^{2\times 2}$ we can show, under some conditions, that $\|x\|_Q = \sqrt{x^T Q x}$, $x \in \mathbb{R}^2$ is a norm. Clearly not every Q induces a norm, for example if we take the matrix:

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

the vector $(-1, 1)$ has negative output, so that function is not a norm. Let's impose the three norm properties:

-*Positivity*:

By imposing positivity we obtain that Q needs to be **positive-semidefinite**. To ensure this condition we can ask for every eigenvalue λ_i of Q to be non negative, from which it follows that $\det Q \geq 0$ and $\operatorname{Tr} Q \geq 0$.

-*homogeneity*:

This property doesn't introduce any new constraint on the matrix Q, since it is already satisfied:

$$\|\alpha x\|_Q = (\alpha x)^T Q(\alpha x) = |\alpha|^2 x^T Qx = |\alpha|^2 \|x\|_Q$$

-*Triangle Inequality*

$\|x + y\|_Q = \sqrt{(x + y)^T Q(x + y)} = \sqrt{x^T Qx + y^T Qy + 2x^T Qy}.$

Since the matrix Q is positive semidefinite we can always decompose as $Q = (Q^{\frac{1}{2}})^T Q^{\frac{1}{2}}$. If we let $u = (xQ^{\frac{1}{2}})^T$ and $v = Q^{\frac{1}{2}} y$ we can apply Cauchy-Schwartz inequality to the scalar product $\langle u, v\rangle^2 \leq \|u\|^2 \|v\|^2$, where the norms $\|u\|^2 = \langle u, u\rangle = \langle (xQ^{\frac{1}{2}})^T, (xQ^{\frac{1}{2}})^T\rangle = x^T Qx$ and similarly for v: $\|v\|^2 = y^T Qy$.

We can now write:

$$\|x + y\|_Q = \sqrt{(x + y)^T Q(x + y)} = \sqrt{x^T Qx + y^T Qy + 2x^T Qy} \leq$$

$$\leq \sqrt{x^T Qx + y^T Qy + 2(x^T Qx)(y^T Qy)} = \sqrt{(x^T Qx + y^T Qy)^2} = \|x\|_Q + \|y\|_Q.$$

We can conclude that asking for the matrix to be positive semidefinite is enough to obtain a norm from the formula $\|x\|_Q = \sqrt{x^T Qx}$. Now we can see how the norm in Example 3.3 on $\mathbb{R}^2$ defined as $|(x_1, x_2)|_2^2 = 5|x_1|^2 + 5|x_2|^2 + 8|x_1||x_2|$ is actually a norm, since it corresponds to a quadratic form with:

$$Q = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

Which has $\det Q = 9$ and $\operatorname{Tr} Q = 10$. We state that this norm is not lattice. If we calculate the Q-norm of the vector $x = (-1, 1)$ we get $\|x\|_Q = \sqrt{2}$. We now notice how given $y = (-1, 0)$, we get $y \preceq x$ on the partial order previously introduced, but if we calculate the norms we get $\|y\|_Q = \sqrt{5}$ breaking the lattice condition. We could also potentially create a number of infinite norms which are not lattice imposing the condition $(1, 0)Q(1, 0) \geq (1, 1)Q(1, 1)$ and then solve for the values in the matrix.

4.2. Smoothness

This section is devoted to treating another analytical property: smoothness of the norm. We introduce two new concepts related to derivatives that can be applied to a very general framework. As an instance of this, one of the extensions introduced here can be also intended as an extension of the directional derivative to a functional setting.

Definition 4.5. *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|_2)$ be Banach spaces and $f : U \to Y$ a function, where $U \subseteq X$ is open. We say that f is **Gâteaux differentiable** at $x \in X$ if for each $h \in X$ the following limit exists and defines a linear and continuous mapping $f' : X \to Y$ such that:*

$$f'(x)(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

Given a Banach space $(X, \|\cdot\|)$ we say $\|\cdot\|$ is Gâteaux differentiable if the norm satisfies this condition for all $x \in S_{(X,\|\cdot\|)}$.

We now want to characterize a different notion of smoothness and we can achieve it by imposing a stronger notion of convergence. By taking the uniform convergence of $h \in X$ we obtain the so called Fréchet derivative.

Definition 4.6. *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|_2)$ be Banach spaces and $f : X \to Y$ a function. We say that f is **Fréchet differentiable** at $x \in X$ if the following limit exists, defines a linear and continuous mapping and it is uniform for each $h \in S_{(X,\|\cdot\|)}$:*

$$f'(x)(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

Given a Banach space $(X, \|\cdot\|)$ we say $\|\cdot\|$ is Fréchet differentiable if the norm satisfies this condition for all $x \in S_{(X,\|\cdot\|)}$.

Example 4.4:

To show how one of its applications we now calculate the derivative of a functional $T : C([0,1]) \to \mathbb{R}$ equipped with the supremum norm $\|\cdot\|_\infty$ defined in the following way:

$$T : u \to \int_0^1 u(x)^2 \sin(\pi x) dx$$

We can also see the Fréchet derivative as that functional λ such that as for all $h \in X$ such that $h \to 0$ the following approximation holds $T(u+h) = T(u) + \lambda h + \psi(h)$.

$$T(u+h) = \int_0^1 (u(x)+h(x))^2 \sin(\pi x)\, dx = T(u) + 2\int_0^1 u(x)h(x) \sin(\pi x)\, dx + \int_0^1 h(x)^2 \sin(\pi x) dx.$$

Let's focus on the third term, clearly if $h \to 0$:

$$0 \leq \int_0^1 h(x)^2 \sin(\pi x) dx \leq \|h\|_\infty^2 \int_0^1 \sin(\pi x) dx = 2\|h\|_\infty^2 \to 0.$$

What we call λ is the Fréchet derivative of the operator, in our case $T'(u) : C([0,1]) \to \mathbb{R}$ is defined as:

$$T'(u)(h) := 2\int_0^1 u(x)h(x) \sin(\pi x) dx.$$

In the second definition we are imposing the uniform convergence which is, of course, a stronger condition. From this fact it follows the simple implication that if f is Fréchet differentiable at x then it is also Gâteaux differentiable.

Lemma 4.2 ([6, Lemma 1.2]). *Given a Banach space $(X, \|\cdot\|)$ the following statements are equivalent:*

1. *The norm $\|\cdot\|$ is Gâteaux differentiable.*
2. $\lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t}$ *exists for every $x \in X \setminus \{0\}$ and $h \in X$.*
3. $\lim_{t\to 0} \frac{\|x+th\|+\|x-th\|-2\|x\|}{t} = 0$ *for every $x \in X \setminus \{0\}$ and $h \in X$.*

Proof. - $1 \implies 2$

This is just an easy consequence of homogeneity, since if 1 is true then for all $x \in X$ one can find $x' \in S_{(X,\|\cdot\|)}$ such that $\|x\|x' = x$ for which it exists its Gâteaux derivative $\|x'\|'(h)$. But since the Gâteaux derivative is a continuous linear mapping with respect to h one can write $\|x\|\|x'\|'(h) = \|\|x\|x'\|'(h) = \|x\|'(h)$.

- 2 $\implies$ 1

To prove this implication we just use the subadditivity of the norm function and note that $\lim_{t\to 0+} \frac{\|x+th\|-\|x\|}{t}$ is subadditive and less or equal than $\|h\|$. While $\lim_{t\to 0-} \frac{\|x+th\|-\|x\|}{t}$ is superadditive in h.

- 2 $\implies$ 3

If we assume that 2 is true then we need to remember since the limit exists both right and left limits need to exist and coincide.

$$\lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t} = \lim_{t\to 0^+} \frac{\|x+th\|-\|x\|}{t} = \lim_{t\to 0^-} \frac{\|x+th\|-\|x\|}{t}.$$

But we can also write

$$\lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t} = \lim_{t\to 0^-} \frac{\|x-th\|-\|x\|}{t}.$$

But if we now notice that

$$\frac{\|x+th\|-\|x\|}{t} - \frac{\|x-th\|-\|x\|}{-t} = \frac{\|x+th\|+\|x-th\|-2\|x\|}{t}.$$

We obtain

$$\lim_{t\to 0} \frac{\|x+th\|+\|x-th\|-2\|x\|}{t} = 0$$

To end the proof of this implication one would just need to substitute $y = th$.

- 3 $\implies$ 2 If we apply the standard convexity argument that can be found at Lemma 4.1 then the quotient $\frac{\|x+th\|-\|x\|}{t}$ is a monotone function in t. Thus the one sided limits

$$\lim_{t\to 0^+} \frac{\|x+th\|-\|x\|}{t}$$

$$\lim_{t\to 0^-} \frac{\|x+th\|-\|x\|}{t}$$

always exist, but if 3 holds and we apply once again the formula

$$\frac{\|x+th\|-\|x\|}{t} - \frac{\|x-th\|-\|x\|}{-t} = \frac{\|x+th\|+\|x-th\|-2\|x\|}{t}$$

we get that the two limits are the same, thus implying 2.

$\square$

While the equivalent for Fréchet differentiation is:

Lemma 4.3 ([6, Lemma 1.3]). *Given a Banach space $(X, \|\cdot\|)$ the following statements are equivalent:*

- *The norm $\|\cdot\|$ is Fréchet differentiable.*
- $\lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t}$ *exists for every $h \in X$ and is uniform in $h \in S_{(X,\|\cdot\|)}$.*

Proof. The proof for this is analogous to the one for Gâteaux differentiability, since nothing specific about the properties of those two concept was used.

- 1 $\implies$ 2

This is just an easy consequence of homogeneity, since if 1 is true then for all $x \in X$ one can find $x' \in S_{(X,\|\cdot\|)}$ such that $\|x\|x' = x$ for which it exists its Fréchet derivative $\|x'\|'(h)$. But since the Fréchet derivative is a continuous linear mapping one can write $\|x\|\|x'\|'(h) = \|\|x\|x'\|'(h) = \|x\|'(h)$.

- 2 $\implies$ 1

To prove this implication we just use the subadditivity of the norm function and note that $\lim_{t\to 0+} \frac{\|x+th\|-\|x\|}{t}$ is subadditive and $\leq \|h\|$. While $\lim_{t\to 0-} \frac{\|x+th\|-\|x\|}{t}$ is superadditive in h.

□

Many of the rules that apply to standard differentiation also apply to the new introduced concepts, in fact for both Gâteaux and Fréchet differentiation we can still make use of *differential of a constant* which is equal to zero, *sum rule* from which we can easily calculate the derivative of sum of functions, the *product rule and quotient rule* and finally the *chain rule*. These properties will be of strong value during our proofs.

Theorem 4.1 ([8, Theorem 70]). *Let X, Y, and Z be normed linear spaces, $A \subset X$, and let $g : A \to Y$ be Gâteaux differentiable at $a \in A$. Suppose further that $g(A) \subset V$, where $V \subset Y$ is an open set, and $f : V \to Z$. If either f is Fréchet differentiable at $g(a)$, or f is Gâteaux differentiable at $g(a)$ and f is Lipschitz on V, then $f \circ g : A \to Z$ is Gâteaux differentiable at a and*

$$D(f(g(a))) = Df(g(a))D(g(a)).$$

We will also introduce another version for Frèchet differentiability of this theorem that will allow us to deal with the same kind of situations.

Theorem 4.2 ([8, Theorem 69]). *Given three Banach spaces $(X, \|\cdot\|_X)$,$(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$, we take $U \subset X$ open and $g : U \to Y$ be Fréchet differentiable at $a \in U$. Suppose further that $g(U) \subset V$, $V \subset Y$ open and $f : V \to Z$ is Fréchet differentiable at $g(a)$. Then:*

$$D(f(g(a))) = Df(g(a))D(g(a)).$$

4.3. Geometrical Connections

We will use this section to draw a connection between this seemingly unrelated analytical properties and the geometrical properties of the norms. This bridge will be built using a well known concept in functional analysis: the dual.

Theorem 4.3 ([6, Theorem 1.4],Šmulyan). *Given a Banach space $(X, \|\cdot\|)$ and its dual Banach space $(X^*, \|\cdot\|^*)$ then:*

- *The norm $\|\cdot\|$ is Fréchet differentiable at $x \in S_{(X,\|\cdot\|)}$ if and only if whenever $f_n, g_n \in S_{(X^*,\|\cdot\|^*)}$, $f_n(x) \to 1$, and $g_n(x) \to 1$, then $\|f_n - g_n\|^* \to 0$.*
- *The norm $\|\cdot\|^*$ is Fréchet differentiable at $f \in S_{(X^*,\|\cdot\|^*)}$ if and only if whenever $x_n, y_n \in S_{(X,\|\cdot\|)}$, $f(x_n) \to 1$, and $f(y_n) \to 1$, then $\|x_n - y_n\| \to 0$.*
- *The norm $\|\cdot\|$ is Gâteaux differentiable at $x \in S_{(X,\|\cdot\|)}$ if and only if whenever $f_n, g_n \in S_{(X^*,\|\cdot\|^*)}$, $f_n(x) \to 1$, and $g_n(x) \to 1$, then $f_n - g_n \xrightarrow{w^*} 0$.*
- *The norm $\|\cdot\|^*$ is Gâteaux differentiable at $f \in S_{(X^*,\|\cdot\|^*)}$ if and only if whenever $x_n, y_n \in S_{(X,\|\cdot\|)}$, $f(x_n) \to 1$, and $f(y_n) \to 1$, then $x_n - y_n \xrightarrow{w} 0$.*

Proof. Only the first point will be proven, since the others are held in a similar way.

Assume that the norm $\|\cdot\|$ of X is Fréchet differentiable at $x \in S_{(X,\|\cdot\|)}$. By characterization of Fréchet differentiability at Lemma 4.3, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|x + h\| + \|x - h\| \leq 2 + \varepsilon\|h\| \quad \text{whenever } \|h\| < \delta. \tag{1}$$

If $f_n, g_n \in S_{(X^*,\|\cdot\|)}$, we have for $h \in X$

$$f_n(x + h) + g_n(x - h) \leq \|x + h\| + \|x - h\|. \tag{2}$$

If, moreover, $f_n(x) \to 1$ and $g_n(x) \to 1$, then (1) and (2) show that there is an n_0 such that for every $n \geq n_0$ and every $\|h\| < \delta$, we have

$$(f_n - g_n)(h) \leq 2 - f_n(x) - g_n(x) + \varepsilon\|h\| \leq 2\varepsilon\delta.$$

Hence

$$\|f_n - g_n\|^* \leq 2\varepsilon \quad \text{for } n \geq n_0.$$

This shows that $\lim_{n\to\infty} \|f_n - g_n\|^* = 0$.

Conversely, if $\|\cdot\|$ is not Fréchet differentiable at $x \in S_{(X,\|\cdot\|)}$, then by Lemma 4.3, there are $\varepsilon > 0$ and $h_n \to 0$, $h_n \neq 0$, such that

$$\|x + h_n\| + \|x - h_n\| \geq 2 + \varepsilon\|h_n\|. \tag{3}$$

Choose $f_n, g_n \in S_{(X^*,\|\cdot\|^*)}$ such that

$$f_n(x + h_n) \geq \|x + h_n\| - \frac{1}{n}\|h_n\| \quad \text{and} \quad g_n(x - h_n) \geq \|x - h_n\| - \frac{1}{n}\|h_n\|.$$

First, note that $f_n(x) = f_n(x + h_n) - f_n(h_n) \to 1$ and similarly $g_n(x) \to 1$. On the other hand, from (3) and from the choice of f_n and g_n, we have

$$f_n(x + h_n) + g_n(x - h_n) \geq 2 + \left(\varepsilon - \frac{2}{n}\right)\|h_n\|.$$

Thus

$$(f_n - g_n)(h_n) \geq \left(\varepsilon - \frac{2}{n}\right)\|h_n\| + 2 - f_n(x) - g_n(x) \geq \left(\varepsilon - \frac{2}{n}\right)\|h_n\|.$$

And there is thus an n_0 such that

$$\|f_n - g_n\|^* \geq \frac{\varepsilon}{2} \quad \text{for } n \geq n_0.$$

Therefore, leading to a contradiction.

□

This is a very important theorem because it builds the bridge mentioned above, starting from analytical property of the norms we landed on notions of the dual space which have some relation with all the geometric concepts introduced earlier. Based on this bridge there will introduce a couple of results that are of fundamental importance.

Theorem 4.4 ([6, Corollary 1.5]). *Let X be a Banach space and $\|\cdot\|$ be a norm on X.*

- *The norm $\|\cdot\|$ is Gâteaux differentiable at $x \in S_{(X,\|\cdot\|)}$ if and only if there is a unique $f \in S_{(X^*,\|\cdot\|^*)}$ such that $f(x) = 1$. We say that f is* **exposed** *in $B_{(X^*,\|\cdot\|^*)}$ by x or that x exposes f in $B_{(X^*,\|\cdot\|^*)}$.*
- *The norm $\|\cdot\|$ is Fréchet differentiable at $x \in S_{(X,\|\cdot\|)}$ if and only if there is a unique $f \in S_{(X^*,\|\cdot\|^*)}$ exposed by x satisfying: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $g \in B_{(X^*,\|\cdot\|^*)}$ and $g(x) > 1 - \delta$, we have $\|g - f\| < \varepsilon$. We say that f is strongly exposed in $B_{(X^*,\|\cdot\|^*)}$ by x or that x* **strongly exposes** *f in $B_{(X^*,\|\cdot\|^*)}$.*

Proof. We show only the first implication since the second follows. If $\|\cdot\|$ is Gâteaux differentiable at $\hat{x} \in S_{(X,\|\cdot\|)}$, from point three of Lemma 4.3, we have the uniqueness of $f \in S_{(X^*,\|\cdot\|^*)}$ with $f(x) = 1$. Conversely, if $\|\cdot\|$ is not Gâteaux differentiable at $\hat{x} \in S_{(X,\|\cdot\|)}$, from the same theorem, we have that there are $f_n, g_n \in S_{(X^*,\|\cdot\|^*)}$ such that $f_n(x) \to 1$, $g_n(x) \to 1$, and $\liminf(f_n - g_n)(h) > 0$ for some $h \in X$. If f and g are weak* limit points of (f_n) and (g_n) respectively, then $f(x) = g(x) = 1$, $f \in S_{(X^*,\|\cdot\|^*)}$, $g \in S_{(X^*,\|\cdot\|^*)}$, and $(f - g)(h) \neq 0$.

□

We now have all the basis to establish such connections in an explicit manner. We will start with a relation between smoothness and rotundity.

Theorem 4.5 ([6, Corollary 1.5]). *Let $(X, \|\cdot\|)$ be a Banach space then:*

- *If $\|\cdot\|^*$ is rotund then $\|\cdot\|$ is Gâteaux differentiable.*
- *If $\|\cdot\|^*$ is Gâteaux differentiable then $\|\cdot\|$ is rotund.*

Proof. We prove the first implication: let's suppose that X is not Gâteaux smooth. Then by the Theorem 4.4 on exposed point it exists a point $x \in S_{(X,\|\cdot\|)}$ for which there exists $f, g \in S_{(X^*,\|\cdot\|^*)}$ such that $f(x) = g(x) = 1$, which implies $\frac{1}{2}(f(x) + g(x)) = 1$, but then this implies that $\|f\|^* = \|g\|^* = \|\frac{1}{2}(f + g)\|^* = 1$ from which follows the fact that the dual is not rotund.

We now prove the second point: let's assume $\|\cdot\|^*$ is Gâteaux smooth but $\|\cdot\|$ is not rotund. Then there must exist two distinct points $x, y \in S_{(X,\|\cdot\|)}$ such that $\|\frac{x+y}{2}\| = 1$. We can now choose $f \in X^*$ such that $f\left(\frac{x+y}{2}\right) = 1$ and since $f(x) \leq 1$ and $f(y) \leq 1$ then by linearity we get $f(x) = f(y) = 1$. We can now apply Theorem 4.4 for which we get that if we have $f(x) = f(y) = 1$ then the only option is that $x = y$ since the norm is Gâteaux smooth, but this is a contradiction.

□

The same result doesn't hold for Fréchet differentiability. It holds another geometric relation between Uniform Fréchet (that we will not treat) and Uniform rotundity. We now provide two other examples of relations between smoothness and geometric porperty of the norms:

Theorem 4.6 ([6, Proposition 1.5])**.** *Let $(X, \|\cdot\|)$ be a Banach space. If $\|\cdot\|^*$ on X^* is LUR then $\|\cdot\|$ on X is Fréchet differentiable.*

Proof. Let $x \in S_{(X,\|\cdot\|}$, $\{f_n\} \subset S_{(X^*,\|\cdot\|^*)}$. Then there exists $f \in S_{(X^*,\|\cdot\|^*)}$ such that $f(x) = 1$ and $f_n(x) \to 1$. It follows that

$$2 \geq \|f_n + f\|^* \geq (f_n + f)(x) \to 2 \text{ as } n \to \infty \Rightarrow \|f_n + f\|^* \to 2.$$

So,

$$\lim_{n\to\infty} \left(2\|f_n\|^{*2} + 2\|f\|^{*2} - \|f_n + f\|^{*2}\right) = 0.$$

Since X^* is LUR, $\lim_{n\to\infty} \|f_n - f\|^* = 0$. We can apply the same process to another $\{g_n\} \subset S_{(X^*,\|\cdot\|^*)}$ such that $g_n(x) \to 1$ concluding that also $\lim_{n\to\infty} \|g_n - f\|^* = 0$.

The last step is to apply triangle inequality to the following terms:

$$\|f_n - g_n\|^* \leq \|f_n - f\|^* + \|f - g_n\|^*.$$

Then, by Lemma 4.3, $\|\cdot\|$ is Fréchet differentiable at $x \in S_{(X,\|\cdot\|)}$.

□

Theorem 4.7. *For any space X, the dual norm of X^* is uniformly convex if and only if its predual norm is uniformly Fréchet differentiable. Also, the dual norm of X^* is uniformly Fréchet differentiable if and only if its predual norm is uniformly convex.*

We won't provide the proof for this theorem. It is however very important to recognize the impact of this theorem, it apparently connects only the geometric view and the analytical one, but that's not the end. In fact as stated in the previous chapter we can also use Milman-Pettis Theorem 3.11 to also prove the reflexivity of the space. Several connections have been made also in this regard underling how all the concepts are somehow related, but this goes beyond the aim of this dissertation. We will conclude mentioning another historical theorem published in 1981 by Yost ([14]) which proved that every reflexive Banach space can be renormed with a norm which is not LUR, but its dual norm is Fréchet smooth.

5 | Tools for Renormings

This chapter will be entirely devoted to some useful renorming techniques such as slices, minkowski functional and some powerful renorming theorems.

5.1. Norm Construction

Given $(Y_1, \|\cdot\|_1)$,$(Y_2, \|\cdot\|_2)$ normed spaces, let us consider the map $v : Y_1 \oplus Y_2 \to \mathbb{R}^2$ defined by $v(y_1, y_2) = (\|y_1\|_1, \|y_2\|_2)$.

Theorem 5.1. *Let us consider* $(Y_1, \|\cdot\|_1)$,$(Y_2, \|\cdot\|_2)$ *two Banach spaces and the Banach space* $\mathbb{R}^2$ *endowed with a lattice norm* $|\cdot|$. *Then the function* $|||\cdot||| : Y_1 \oplus Y_2 \to \mathbb{R}$ *defined as* $|||(y_1, y_2)||| = |v(y_1, y_2)|$ *is a norm on* $Y_1 \oplus Y_2$*, where* v *is defined as above.*

Proof. We start by checking all the properties of Definition 2.1.

-*Positivity and Zero*:

The function $|||\cdot|||$ is defined as the output of a norm on $\mathbb{R}^2$ and hence, non negative.

On the other hand if $y = (y_1, y_2) \in Y_1 \oplus Y_2$ we have:

$$0 = |||y||| = |v(y_1, y_2)| = |(\|y_1\|_1, \|y_2\|_2)| \Leftrightarrow (\|y_1\|_1 = 0) \wedge (\|y_2\|_2 = 0).$$

And so we get to the conclusion that $y_1 = 0$ and $y_2 = 0$, showing the desired implication.

-*Homogeneity*:

Let $a \in \mathbb{R}$ and $y = (y_1, y_2) \in Y_1 \oplus Y_2$

$$|||ay||| = |(\|ay_1\|_1, \|ay_2\|_2)| = |av(y_1, y_2)| = |a||||y|||.$$

-*Triangle Inequality*:

Take $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $Y_1 \oplus Y_2$, then

$$|||x+y||| = |v(x_1+y_1, x_2+y_2)| = |(\|x_1+y_1\|_1, \|x_2+y_2\|_2)|.$$

Here we are in a crucial step of the proof, by triangle inequality of the norm $|\cdot|$ and the fact that $|\cdot|$ is lattice, we obtain:

$$\begin{aligned} |(\|x_1+y_1\|_1, \|x_2+y_2\|_2)| &\leq |(\|x_1\|_1+\|y_1\|_1, \|x_2\|_2+\|y_2\|_2)| \\ &= |(\|x_1\|_1, \|x_2\|_2)+(\|y_1\|_1, \|y_2\|_2)| \\ &\leq |(\|x_1\|_1, \|x_2\|_2)|+|(\|y_1\|_1, \|y_2\|_2)| \\ &= |v(x_1,x_2)|+|v(y_1,y_2)| = |||x|||+|||y|||. \end{aligned}$$

Since all three hypothesis are satisfied we can conclude that the function $|||\cdot|||$ is a norm.

□

Theorem 5.2. *If the Banach spaces $(Y_1, \|\cdot\|_1)$, $(Y_2, \|\cdot\|_2)$ and $(\mathbb{R}^2, |\cdot|)$ also have all Gâteaux (Fréchet) differentiable norms then the norm $|||\cdot|||$ is Gâteaux (Fréchet) differentiabile.*

Proof. Any step of this proof can be repeated for Fréchet differentiability with little to no variations. We will make use of the chain rule treated in Theorem 4.1 for Gâteaux differentiability to prove the property. By hypothesis the norm $|\cdot|$ is Gâteaux differentiable, if we now consider the function v and apply the definition we get:

$$\lim_{t\to 0}\frac{v(y+th)-v(y)}{t} = \lim_{t\to 0}\frac{1}{t}(\begin{pmatrix}\|y_1+th_1\|_1\\ \|y_2+th_2\|_2\end{pmatrix} - \begin{pmatrix}\|y_1\|_1\\ \|y_2\|_2\end{pmatrix}) = \lim_{t\to 0}\frac{1}{t}\begin{pmatrix}\|y_1+th_1\|_1-\|y_1\|_1\\ \|y_2+th_2\|_2-\|y_2\|_2\end{pmatrix}$$

$$= \begin{pmatrix}\lim_{t\to 0}\frac{\|y_1+th_1\|_1-\|y_1\|_1}{t}\\ \lim_{t\to 0}\frac{\|y_2+th_2\|_2-\|y_2\|_2}{t}\end{pmatrix} = \begin{pmatrix}\mathrm{D}_{h_1}(\|y_1\|_1)\\ \mathrm{D}_{h_2}(\|y_2\|_2)\end{pmatrix} = D_h(v(y)).$$

The existence of the Gâteaux derivative of v is guaranteed by the existence of the derivative of the two norms, which happens by hypothesis. The fact that the function $|\cdot|$ is Lipschitz is retrieved applying inverse triangle inequality.

□

Some needed clarifications on the lattice condition hypothesis of the norm defined on $\mathbb{R}^2$. If we drop this hypothesis our resulting function $|||\cdot||| = |v(\cdot,\cdot)|$ might not satisfy triangle inequality and therefore it wouldn't be a norm, to which purpose we introduce the following example.

Example 5.1:

We define $Y_1 = (\mathbb{R}, |\cdot|)$ and $Y_2 = (\mathbb{R}, |\cdot|)$ while the norm on $\mathbb{R}^2$ assumes the following values: $|(1,1)| = 1$, $|(1,0)| = \alpha$ and $|(0,1)| = \alpha$ where $\alpha > 1$ from which we retrieve the non lattice condition. We consider the above introduced function $|||\cdot||| = |v(\cdot,\cdot)| : Y_1 \oplus Y_2 \to \mathbb{R}$.

Let's now see how this breaks the triangle inequality $x = (1,1) \in Y_1 \oplus Y_2$, $y = (1,-1) \in Y_1 \oplus Y_2$, $x + y = (2,0)$. We also see that $v(x) = (1,1)$, $v(y) = (1,1)$ and $v(x+y) = (2,0)$ so the triangle equality for these three points:

$$|v(x+y)| \le |v(x)| + |v(y)| \implies |(2,0)| \le |(1,1)| + |(1,1)| \implies 2|(1,0)| \le 2|(1,1)|$$

from which it follows

$$|(1,0)| \le |(1,1)| \implies \alpha \le 1$$

So, to satisfy the triangle inequality holds only if $\alpha \le 1$, which is false.

We can now question if the lattice condition on the norm $|\cdot|$ is necessary in order for $|||\cdot||| = |v(\cdot,\cdot)| : Y_1 \oplus Y_2 \to \mathbb{R}$ to be a norm. To which we introduce the following proposition. It's worth noting that the function $|||\cdot|||$ can be defined both on the direct sum of two spaces $Y_1 \oplus Y_2$ or on a third space X provided that $X = Y_1 \oplus Y_2$.

Proposition 5.1. *Let $(X, \|\cdot\|)$ be a normed space and let $X = Y_1 \oplus Y_2$, where Y_1 and Y_2 are nontrivial subspaces. The function $|||\cdot||| = |v(\cdot,\cdot)| : Y_1 \oplus Y_2 \to \mathbb{R}$ is a norm if and only if $|\cdot| : \mathbb{R}^2 \to \mathbb{R}$ is lattice.*

Proof. We already proved the implication that if the norm $|\cdot|$ is lattice then $|||\cdot|||$ is a norm. So it is enough to show the converse implication. Since $|\cdot|$ is not a lattice norm by Lemma 4.1 there must exist $(a,b) \in S_{(\mathbb{R}^2,|\cdot|)}$ such that $|(a,0)| > 1$ or $|(0,b)| > 1$. We now suppose $|(a,0)| > 1$ since the other case might be held similarly. Since the two spaces are non trivial it exist $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $\|y_1\| = a$ and $\|y_2\| = b$. Let us now consider the points $x_1 = (y_1, y_2) \in X$ and $x_2 = (y_1, -y_2) \in X$. Since $|(a,b)| = 1$, we conclude that $|||x_1||| = 1$ and $|||x_2||| = 1$. We now have:

$$|||x_1+x_2||| = |v(y_1+y_1, y_2-y_2)| = 2|v(y_1,0)| = 2|(a,0)| > 2 = |(a,b)|+|(a,b)| = |||x_1|||+|||x_2|||$$

Which is a violation of triangle inequality and hence the norm $|||\cdot|||$ is not a norm.

□

5.2. Minkowski Functional

This section is devoted to the so called Minkowski functional, also known as the Minkowski gauge or Minkowski functional operator which is a mathematical tool used in convex geometry and functional analysis. It is particularly useful for defining and analyzing properties of convex sets in a vector space. Given some certain set C in a vector spaces, the Minkowski functional of C at a given point x is a real-valued function that quantifies how "far" x is from the boundary of C.

Let's now start by introducing the conditions that need to be satisfied by the set:

Definition 5.1. *A subspace B of a vector space X is said to be **absorbing** if:*

$$\forall\, x \in X, \exists \lambda > 0 : x \in \mu B \;\forall\, \mu \in \mathbb{R} \quad \text{with} \quad |\mu| \geq \lambda,$$

where the notation μB denotes the set $\mu B = \{\mu b : b \in B\}$.

If the set we are considering satisfies this condition then it can "be fed" into the **Minkowski functional** that we will introduce below:

Definition 5.2 ([7, Section 3.6.1]). *Let B be an absorbing subset of a vector space X. For each x in X, let $P_B(x) = \inf\{t : t > 0, x \in tB\}$. Then P_B is the Minkowski functional of B.*

The next theorem characterizes those sets B such that the corresponding Minkowski functional P_B is an equivalent norm on X.

Theorem 5.3. *Let $(X, \|\cdot\|)$ be a Banach space and $B \subseteq X$ a convex set such that $aB \subseteq B$, whenever $a \in \mathbb{R}$ satisfies $|a| \leq 1$. Then P_B is an equivalent norm on X if and only if B is bounded and has a nonempty topological interior.*

Example 5.2:

Let us consider the subset of $\mathbb{R}^2$ (see Figure 5.1):

$$M := \{\{\sqrt{x^2 + (y-1)^2} \leq 1\} \cup \{\{x^2 \leq 1\} \cap \{y^2 \leq 1\}\} \cup \{\sqrt{x^2 + (y+1)^2} \leq 1\}\}$$

We will now show that the set M generates a norm with the Minkowski functional.

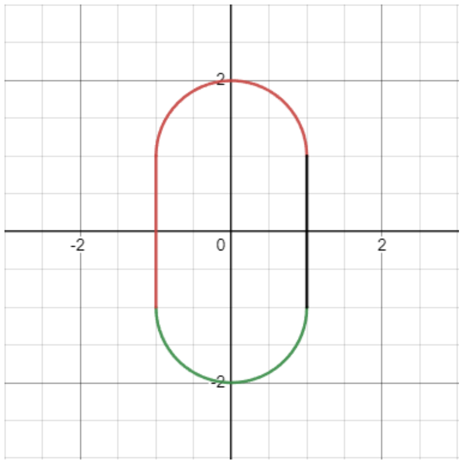


Figure 5.1: Boundary of the set M.

Proof. Checking the conditions reported in Theorem 5.3 is the same as checking the following conditions:

-Bounded:

It's easy seeing that the whole set is contained in the set $B := \{x \in \mathbb{R}^2 : \|x\|_e < 3\}$.

-Non empty interior:

Clearly $\mathrm{Int}(M) \neq \emptyset$ since the set $B = \{x \in \mathbb{R}^2 : \|x\|_e < 1/2\} \subseteq M$.

-Closed and convex:

Since all the inequalities defining M are not strict we can establish that all the boundary points are also included in M, hence: $M \cup \partial M = M$.

Convexity can be handled because it's the composition of three different convex shapes (two semi circle and a square with side 1), so for all the cases in which we select two points inside the same geometrical object we know that the condition is valid, we just need to check the case in which we select a point inside the semi circle and the square:

Here we can see an illustration of the case we need to be considering in order to prove the convexity of the set M. But also this comes along as quite obvious when considering the geometry of it.

-*Origin symmetry*

Even this feature is obvious when considering the geometry of M. We see that every point on the straight edge would get mapped to another point of the opposite straight edge, while the circumferences are symmetric when considering the y axes. They are also

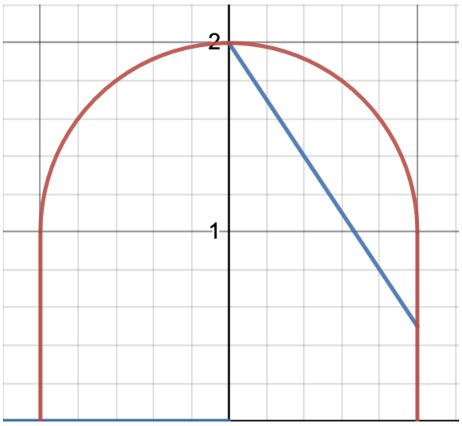


Figure 5.2: Line example on the set M.

symmetric when considering the x axis because their centers are symmetric and they have the same radius.

Since all these conditions are satisfied we obtain a norm.

□

5.3. Renorming Theorems

We now wish to take a few lines to describe a few very powerful theorems that will be crucial to the solution of the open questions. These theorems relate the separability of a Banach space and its dual to the existence of renormings satisfying additional smoothness and rotundity conditions. In some sense, they are just an extension of Theorem 3.6, generalized with differentiability and other properties.

Theorem 5.4 ([5, Theorem 8.2]). *Every separable Banach space X admits an equivalent Gâteaux differentiable, LUR norm.*

Proof. We will make use of a dense set $\{e_n : n \in \mathbb{N}\} \subset S_{(X,\|\cdot\|)}$ whose existence is given by the separability of X. We can also renorm X with a LUR norm $\|\cdot\|$ according to Theorem 3.6. We now define a new dual norm in the following way:

$$\|f\|_n^{*2} = \|f\|^{*2} + \frac{1}{n}\left(\sum_{i=1}^{\infty} 2^{-i} f^2(e_i)\right)$$

notice that $\|\cdot\|^*$ is a dual norm and each term of the series is w^*-continuous. Then $\|\cdot\|_n^*$ are rotund norms as a consequence of Theorem 3.2 and their predual norms converge to $\|\cdot\|$ uniformly on bounded sets. We define a new norm on X in the following way $\|x\|_0^2 = \sum_{n=1}^{\infty} 2^{-n}\|x\|_n^2$. This norm is Gâteaux differentiable since it is sum of norms with rotund dual (see Theorem 4.5) and we can apply the sum rule for differentiation, keeping in mind that the derivative of the norms $\|\cdot\|_n$ are uniformly bounded.

We now need to prove that the norm $\|\cdot\|_0$ is LUR. Let us consider a sequence $\{x_k\}$ such that $\lim_{k\to\infty} 2\|x_k\|_0^2 + 2\|x\|_0^2 - \|x + x_k\|_0^2 = 0$. We can conclude the boundedness of the sequence $\{x_k\}$ by noticing that by triangle inequality $2\|x_k\|_0^2+2\|x\|_0^2-\|x+x_k\|_0^2 \geq 2\|x_k\|_0^2+2\|x\|_0^2-(\|x\|_0+\|x_k\|_0)^2 = (\|x_k\|_0-\|x\|_0)^2$, so if $\lim_{k\to\infty} 2\|x_k\|_0^2+2\|x\|_0^2-\|x+x_k\|_0^2 = 0$ also $\lim_{k\to\infty}(\|x_k\|_0 - \|x\|_0)^2 = 0$. On the other hand the relation $\lim_{k\to\infty} 2\|x\|_n^2 + 2\|x_k\|_n^2 - \|x + x_k\|_n^2 = 0$ holds for every $n \in \mathbb{N}$ by virtue of the same calculations made in the proof of Theorem 3.2 for the quadratic case. Since the norm converges uniformly on a bounded set, as our $\{x_k\}$ is, we can conclude that also the limit of the predual norms $\lim_n \|\cdot\|_n = \|\cdot\|$ will satisfy $\lim_{k\to\infty} 2\|x\|^2 + 2\|x_k\|^2 - \|x + x_k\|^2 = 0$. If we now use the fact that $\|\cdot\|$ is a LUR renorming we get the thesis $\|x - x_k\| \to 0$ and since $\|\cdot\|_0$ is equivalent to $\|\cdot\|$ then we also conclude that $\|x - x_k\|_0 \to 0$. This is condition 5 of Lemma 3.3 that is a characterization of the LUR condition for the norm $\|\cdot\|_0$.

□

Theorem 5.5 ([5, Theorem 8.6]). *Every separable Banach space X with separable dual X^* admits an equivalent Fréchet differentiable and LUR norm.*

Proof. We will make use of the dense set $\{e_n^* : n \in \mathbb{N}\} \subset S_{(X^*,\|\cdot\|^*)}$ whose existence is given by the separability of the dual space X^*. We can also renorm X with a LUR norm $\|\cdot\|$ according to Theorem 3.6.

We shall now need a norm whose dual norm is LUR. We might think to apply Theorem 3.6 to retrieve the LUR condition on the dual norm. It's however unfortunate that not all the dual norms have a corresponding predual norm. We need to prove that this new norm $\|\cdot\|^*$ is w^*-lower semicontinuous.

Let's consider the norm introduced in Theorem 3.6 but on the dual space X^*. We build the sets $E_n = \text{span}\{e_1^*, e_2^*, ..., e_n^*\}$, we notice that $E_n + B_{(X^*,\|\cdot\|^*)}$ is w^*-closed as E_n is w^*-closed and $B_{(X^*,\|\cdot\|^*)}$ is w^*-compact by Alaoglu's theorem. Since $\{f \in X^* : \text{dist}(f, E_n) \leq 1\} = E_n + B_{(X^*,\|\cdot\|^*)}$ the functions $\text{dist}(f, E_n)$ are w^*-lower semicontinuous and so the supremum of their weighted sum and thus the norm of Theorem 3.6 applied to the dual has indeed a predual norm. So we can renorm X^* with a dual norm that we will call $\|\cdot\|_1^*$.

Setting this aside we can continue by renorming the dual $(X^*, \|\cdot\|^*)$ where $\|\cdot\|^*$ is the dual norm of the LUR norm $\|\cdot\|$ with the following norm:

$$\|f\|_n^{*2} = \|f\|^{*2} + \frac{1}{n}\|f\|_1^{*2} \quad \forall\, f \in X^*$$

Clearly, this norm is LUR in the dual and this can be seen by applying the same argument applied in Theorem 3.2 for the quadratic case. Applying Theorem 4.6 we also get the Fréchet differentiability of the predual norms $\|\cdot\|_n$, moreover the sequence $\{\|\cdot\|_n\}$ converges uniformly to $\|\cdot\|$ on a bounded set as stated in the previous theorem.

We now apply one last renorming similar to the previous theorem to show that this new norm is indeed the one we are looking for:

$$\|x\|_0^2 = \sum_{n=1}^{\infty} 2^{-n}\|x\|_n^2 \quad \forall\, x \in X$$

Being the sum of Fréchet differentiable norms, we instantly retrieve the differentiability. We now need to prove that the norm $\|\cdot\|_0$ is LUR. Let us consider a sequence $\{x_k\}$ such that $\lim_{k\to\infty} 2\|x_k\|_0^2 + 2\|x\|_0^2 - \|x + x_k\|_0^2 = 0$. We can conclude the boundedness of the sequence $\{x_k\}$ by noticing that by triangle inequality $2\|x_k\|_0^2 + 2\|x\|_0^2 - \|x+x_k\|_0^2 \geq 2\|x_k\|_0^2 + 2\|x\|_0^2 - (\|x\|_0 + \|x_k\|_0)^2 = (\|x_k\|_0 - \|x\|_0)^2$, so if $\lim_{k\to\infty} 2\|x_k\|_0^2 + 2\|x\|_0^2 - \|x + x_k\|_0^2 = 0$ also $\lim_{k\to\infty}(\|x_k\|_0 - \|x\|_0)^2 = 0$. On the other hand the relation $\lim_{k\to\infty} 2\|x\|_n^2 + 2\|x_k\|_n^2 - \|x + x_k\|_n^2 = 0$ holds for every $n \in \mathbb{N}$ by virtue of the same calculations made in the proof of Theorem 3.2 for the quadratic case. Since the norm converges uniformly to $\|\cdot\|$ on a bounded set, as our $\{x_k\}$ is, we can conclude that also the limit of the predual norms $\lim_n \|\cdot\|_n = \|\cdot\|$ will satisfy $\lim_{k\to\infty} 2\|x\|^2 + 2\|x_k\|^2 - \|x + x_k\|^2 = 0$. If we now use the fact that $\|\cdot\|$ is a LUR renorming we get that $\|x - x_k\| \to 0$ and since $\|\cdot\|_0$ is equivalent to $\|\cdot\|$ then we also conclude that $\|x - x_k\|_0 \to 0$. This is condition 5 of Lemma 3.3 that is a characterization of the LUR condition for the norm $\|\cdot\|_0$.

□

We now introduce a renorming theorem based on the one introduced in the article "Rotund Gâteaux smooth norms which are not locally uniformly rotund" [2, Theorem 3.1]. This result will be helpful in determining the equivalence of the first term of the final renorming with the initial one. In this case we can see how to retrieve the failure of the MLUR condition: we choose a specific direction (e_1) and make it flat enough for some sequences to violate such condition.

Theorem 5.6 (cf. [2, Theorem 3.1]). *Let $(X, \|\cdot\|)$ be a LUR and Gâteaux differentiable (Fréchet differentiable respectively) Banach space and $\{e_n, e_n^*\}$ an M-basis on X. Then $\|\cdot\|_1$ defined by:*

$$\|x\|_1^2 = \|x - e_1^*(x)e_1\|^2 + \|e_1^*(x)e_1\|^2.$$

is LUR and Gâteaux differentiable (Fréchet differentiable respectively).

Proof. Considering the expression of the norm:

$$\|x\|_1^2 = \|x - e_1^*(x)e_1\|^2 + \|e_1^*(x)e_1\|^2.$$

we can conclude that this new norm is Gâteaux differentiable (Fréchet differentiable) since we can apply the sum rule to the two terms, which are both Gâteaux (Fréchet). Let's now show it is LUR by verifying condition 5 of Lemma 3.3. Let us suppose that:

$$\lim_n 2\|x\|_1^2 + 2\|x_n\|_1^2 - \|x + x_n\|_1^2 = 0, \tag{5.1}$$

which implies:

$$\lim_n 2\|x - e_1^*(x)e_1\|^2 + 2\|e_1^*(x)e_1\|^2 + 2\|x_n - e_1^*(x_n)e_1\|^2 + 2\|e_1^*(x_n)e_1\|^2$$

$$-\|x + x_n - e_1^*(x + x_n)e_1\|^2 - \|e_1^*(x + x_n)e_1\|^2 = 0.$$

But now we observe that:

$$2\|x - e_1^*(x)e_1\|^2 + 2\|x_n - e_1^*(x_n)e_1\|^2 - \|x + x_n - e_1^*(x + x_n)e_1\|^2 \geq$$

$$2\|x - e_1^*(x)e_1\|^2 + 2\|x_n - e_1^*(x_n)e_1\|^2 - (\|x - e_1^*(x)e_1\| + \|x_n - e_1^*(x_n)e_1\|)^2 =$$

$$(\|x - e_1^*(x)e_1\|^2 - \|x_n - e_1^*(x_n)e_1\|)^2 \geq 0.$$

A similar argument can be applied to the other terms leading to:

$$2\|e_1^*(x)e_1\|^2 + 2\|e_1^*(x_n)e_1\|^2 - \|e_1^*(x + x_n)e_1\|^2 \geq 0.$$

If we coonsider 5.1, the only possibility left is that both inequalities are met with equality.

But this leads to:

$$\lim_n 2\|x - e_1^*(x)e_1\|^2 + 2\|x_n - e_1^*(x_n)e_1\|^2 - \|x + x_n - e_1^*(x + x_n)e_1\|^2 = 0$$

and

$$\lim_n 2\|e_1^*(x)e_1\|^2 + 2\|e_1^*(x_n)e_1\|^2 - \|e_1^*(x + x_n)e_1\|^2 = 0.$$

Since the norm $\|\cdot\|$ is LUR, we have

$$\|x - x_n + e_1^*(x - x_n)e_1\| \to 0 \quad \text{and} \quad \|e_1^*(x_n)e_1 - e_1^*(x)e_1\| \to 0,$$

then by triangle inequality $\|x - x_n\| \leq \|x - x_n + e_1^*(x - x_n)e_1\| + \|e_1^*(x_n)e_1 - e_1^*(x)e_1\|$ from which we conclude that $\|x - x_n\| \to 0$. Being $\|\cdot\|$ equivalent to $\|\cdot\|_1$ we also get the conclusion that $\|x - x_n\|_1 \to 0$

□

We conclude this section with a result about density of norms satisfying smoothness or rotundity conditions, in the metric space of all equivalent norms on a given Banach space X.

Theorem 5.7 ([6, Theorem 4.1]). *Given a separable Banach space X the norms which are Gâteaux and LUR are dense in the metric space of all its equivalent norms. If, moreover, X^* is also separable we conclude that the norms which are Fréchet and LUR are dense in the metric space of all its equivalent norms.*

5.4. Slices

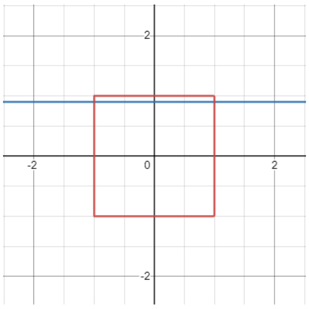
Slices are a powerful tool invented to apply "cuts" to an arbitrary subset of the initial space selecting specific elements belonging to the dual, which are metaphorically the "knives". The concept is really simple as we use a dual element to separate points on the original space, hence the name slice.

Definition 5.3 ([7, Section 5.3.2]). *Given $x^* \in X^* \setminus \{0\}$ and $\delta > 0$, the **slice** defined by x^* and δ of the ball $B_{(X,\|\cdot\|)}$ is the set $S(B_{(X,\|\cdot\|)}, x^*, \delta)$:*

$$S(B_{(X,\|\cdot\|)}, x^*, \delta) = \{x \in B_{(X,\|\cdot\|)} : x^*(x) > \|x^*\| - \delta\}.$$

A natural question one might ask is how "big" is the part of the unit ball left out for every cut we apply. It's clear that depending on the shape of the unit ball and on the direction of the cut we get different results. There are, however, results that guarantee that some norms allow arbitrarily small cuts when the quantity δ tends to zero.

This fact is not obvious at all, if for instance we consider the Banach space $(\ell^\infty, \|\cdot\|_\infty)$ and its unit ball, we can see that if we consider cuts parallel to the faces, even if δ is small we cannot ensure the fact that the part we are cutting is small enough.



We will now show a simple example of the projection in 2 dimension of this case.The blue line is an hyperplane generated by an element of the dual through the formula above, while the red object is just a 2d projection of the unitary ball of the above mentioned space. We can imagine, when considering the infinite dimensional space, with the infinite dimensional hyperplane that even if δ is very small we are cutting out a consistent chunk of the unit sphere.

Theorem 5.8. *Let $(E, \|\cdot\|)$ be a LUR Banach space and $x \in S_{(E,\|\cdot\|)}$. Let $x^* \in S_{(E^*,\|\cdot\|^*)}$ be such that $x^*(x) = 1$ then $\operatorname{diam} S(B_{(E,\|\cdot\|)}, x^*, \delta) \to 0$ as $\delta \to 0^+$.*

Example 5.3:

We will now illustrate an example in which we apply an infinite amount of cuts to the unit sphere and still obtain an equivalent norm $\|\cdot\|_M$. We start by renorming the separable Banach space X with a LUR norm $\|\cdot\|$, by virtue of Theorem 5.4. We then consider its unit ball $B_{(X,\|\cdot\|)}$, by Theorem 5.8 we can take arbitrarily small slices of this set using linear functionals belonging to the dual. We can now define the slices in the following way: we use the bounded M-basis $\{e_n, e_n^*\}$ with $\|e_n^*\|^* = 1,\ n \in \mathbb{N}$ whose existence is guaranteed by Theorem 2.2 to select the linear functionals. We set $\delta_n = \frac{\varepsilon}{2^n}$ where $\varepsilon \in [0, 1]$ and define:

$$SL_n(B_{(X,\|\cdot\|)}, e_n^*, \delta_n) := \{x \in B_{(X,\|\cdot\|)} : |e_n^*(x)| > 1 - \delta_n\}.$$

We deviated a little from the formal definition of slice, introducing the absolute value, to generate a symmetric subset of X with respect to the origin. We now apply the infinite amount of slices, one for each component of the M-basis and we consider the set $M = B_{(X,\|\cdot\|)} \setminus \bigcup_{n=1}^\infty SL_n(B_{(X,\|\cdot\|)}, e_n^*, \delta_n)$. We now apply the Minkowski functional P_M to M and this generates a norm since:

-Bounded:

The set M is a subset of the bounded set $S_{(X,\|\cdot\|)}$, hence it is bounded.

-Non empty interior:

The set M contains the set $\{x \in X : \|x\| \leq \frac{1}{2}\}$ since the factor $\varepsilon \in [0,1]$ and so every $\delta < \frac{1}{2}$.

-Symmetry with respect to the origin:

We applied symmetrical slices on a symmetrical set to retrieve this property.

-Closed and convex:

We started from a closed set and we applied the slices where the criteria needs to be met with a strict inequality, leaving the new boundaries inside the new set M and thus closure follows. While the convexity follows from the fact the new set M can be seen as the intersection of an infinite number of convex sets (we can see the slices as separating, since they are linear, two convex part of the space X), which ensures convexity of M.

We have now generated the new norm $\|\cdot\|_M$ and we also notice that this norm is equivalent to $\|\cdot\|$ since: $(1-\frac{\varepsilon}{2})B_{(X,\|\cdot\|)} \subseteq B_{(X,\|\cdot\|_M)} \subseteq B_{(X,\|\cdot\|)}$ (this follows from the fact that the deepest cut is the one applied to the face e_1).

Theorem 5.9. *Let $(X, \|\cdot\|)$ be a separable Banach space and $\{e_n, e_n^*\}$ a bounded M-basis such that $\|e_1^*\|^* = 1$ and $\|e_n\| = 1$. Then, for every $\varepsilon \in (0,1)$, there exist an equivalent norm $\|\cdot\|_M$ and $\eta > 0$ such that*

1. $(1-\varepsilon)B_{(X,\|\cdot\|)} \subseteq B_{(X,\|\cdot\|_M)} \subseteq B_{(X,\|\cdot\|)}$
2. $(1-\epsilon)e_1, (1-\epsilon)e_1 \pm \eta e_n \in S_{(X,\|\cdot\|_M)}$ *whenever* $n \in \mathbb{N}$ *and* $n > 1$
3. $\|e_n\|_M = 1$ *for all* $n > 1$

Proof. Let us consider a bounded M-basis $\{e_n, e_n^*\}$ with $\|e_1^*\|^* = 1$ and $\|e_n\| = 1$, whenever $n \in \mathbb{N}$ whose existence can be guaranteed by Theorem 2.2. We set $\varepsilon \in (0,1)$ an let:

$$SL(B_{(X,\|\cdot\|)}, e_1^*, \varepsilon) := \{x \in B_{(X,\|\cdot\|)} : |e_1^*(x)| > 1-\varepsilon\}.$$

This is a simplification of Example 5.3, where we apply one cut instead of infinite, so we conclude that the Minkowski functional corresponding to $M = B_{(X,\|\cdot\|)} \setminus SL(B_{(X,\|\cdot\|)}, e_1^*, \varepsilon)$ gives us a norm an equivalent norm $\|\cdot\|_M$ for which the following holds: $(1-\varepsilon)B_{(X,\|\cdot\|)} \subseteq B_{(X,\|\cdot\|_M)} \subseteq B_{(X,\|\cdot\|)}$. Define $\delta = 1-\varepsilon$, then we have $\|\delta e_1\|_M = \|(1-\varepsilon)e_1\|_M = 1$. We notice that $\delta e_1 \in \mathrm{Int}(B_{(X,\|\cdot\|)})$ since $\delta = 1-\varepsilon < 1$, thus it follows that there must exist some $\eta \in \mathbb{R}^+$ such that $\delta e_1 + \eta B_{(X,\|\cdot\|)} \subseteq B_{(X,\|\cdot\|)}$. Hence also $\delta e_1 + \eta e_n \in B_{(X,\|\cdot\|)}$ whenever

$n \in \mathbb{N}$, since $\|e_n\| = 1$. Observing that $e_n \in \ker(e_1^*)$ for all $n > 1$, we have that:

$$e_1^*(\delta e_1 + \eta e_n) = 1 - \varepsilon,$$

and then we deduce that $\delta e_1 + \eta e_n \in S_{(X,\|\cdot\|_M)}$ $(n > 1)$. Similarly, we obtain that $\delta e_1 - \eta e_n \in S_{(X,\|\cdot\|_M)}$ $(n > 1)$. We conclude by noticing that condition 3 is satisfied since $e_n \notin SL(B_{(X,\|\cdot\|)}, e_1^*, \varepsilon)$.

□

6 | Results

We will now state and solve two open problems that can be found on the book [7] section 52.3.

6.1. Problem 1

The first problem we present is number 3 of section 52.3 of [7], for which we ask the following question: *Can every infinite-dimensional separable space be renormed to be rotund but not MLUR?*

Theorem 6.1. *Every separable Banach space can be renormed with a norm which is Gâteaux differentiable, rotund but not MLUR.*

Proof. Let X be a separable Banach space. By Theorem 5.4 and Theorem 2.2 there exist a *Gâteaux differentiable* renorming $\|\cdot\|_G$ and a bounded M-basis $\{e_n, e_n^*\}$ such that $\|e_n\|_G = 1$ for all $n \in \mathbb{N}$. We can also renorm once again with $\|\cdot\|_{G'}$ using Theorem 5.6, obtaining the following properties:

$$\|x\|_{G'}^2 = \|x - e_1^*(x)e_1\|_G^2 + \|e_1^*(x)e_1\|_G^2.$$

We now use the construction introduced in Theorem 5.1 using:

- $Y_1 = \operatorname{span}\{e_1\}$.
- $Y_2 = \overline{\operatorname{span}\{e_n\}_{n\geq 2}}$.
- The norm on $\mathbb{R}^2$ defined as the Minkowski of the set M given in Example 5.2, that we will denote as $\|\cdot\|_M$.

By the choice of Y_1, Y_2 we can see every element of X as a decomposition on the two subspaces Y_1 and Y_2 since one of the two spaces is finite dimensional. In our case, for every $x \in X$ we write $x = y_1 + y_2$, where $y_1 = e_1^*(x)e_1$ and $y_2 = x - e_1^*(x)e_1$. For both subspaces we use the initial norm $\|\cdot\|_G$. We then define

$$|||x||| = \|(\|y_1\|_G, \|y_2\|_G)\|_M.$$

Observe that the unit sphere of $\|\cdot\|_M$ satisfies the hypothesis of Lemma 4.1, indeed every canonical projection of $S_{(X,\|\cdot\|_M)}$ is contained in $B_{(X,\|\cdot\|_M)}$. Applying Theorem 5.1, we conclude that $|||\cdot|||$ is a norm. The two spaces $(Y_1, \|\cdot\|_G)$, $(Y_2, \|\cdot\|_G)$ are Gâteaux differentiable space since they are subspaces of $(X, \|\cdot\|_G)$, a Gâteaux differentiable Banach space. If the Gâteaux condition is satisfied at every point, at every direction then it must also be satisfied for the specific direction generated by $\text{span}\{e_n\}_{n\geq 2}$ and e_1 on the two relative subspaces. So, as stated in Theorem 5.2 we get the Gâteaux differentiability of the norm $|||\cdot|||$. The norm $|||\cdot|||$ is also a renorming of $\|\cdot\|_{G'}$. To see why we let $\|\cdot\|_e$ be the euclidean norm on $\mathbb{R}^2$ and use the fact that all norms in finite dimensional spaces are equivalent (Theorem 2.1), in particular there exist $k, K \in \mathbb{R}^+$ such that $k\|\cdot\|_e \leq \|\cdot\|_M \leq K\|\cdot\|_e$. In our case:

$$\|x\|_{G'} = \|y_1 + y_2\|_{G'} = \|(\|y_1\|_G, \|y_2\|_G)\|_e \leq \frac{1}{k}||(\|y_1\|_G, \|y_2\|_G)||_M = \frac{1}{k}|||x|||.$$

While, for the second inequality:

$$|||x||| = \|(\|y_1\|_G, \|y_2\|_G)\|_M \leq K\|(\|y_1\|_G, \|y_2\|_G)\|_e = K\sqrt{\|y_1\|_G^2 + \|y_2\|_G^2}.$$

$$K\sqrt{\|y_1\|_G^2 + \|y_2\|_G^2} = K\sqrt{\|e_1^*(x)e_1\|_G^2 + \|x - e_1^*(x)e_1\|_G^2} = K\sqrt{\|x\|_{G'}^2} = K\|x\|_{G'}.$$

Obtaining the conclusion:

$$k\|x\|_{G'} \leq |||x||| \leq K\|x\|_{G'}.$$

We renorm once again with a new norm on X:

$$|x|^2 = |||x|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m}|e_m^*(x)|^2.$$

We claim that this renorming is *rotund*, *Gâteaux differentiable* and *not MLUR*.

-*Renorming and rotund*

We can now see this norm as $|x|^2 = |||x|||^2 + \|Tx\|_{\ell^2}^2$ where $\|\cdot\|_{\ell^2}$ is the norm on ℓ^2. We can define $T : X \to \ell^2$ in the following way $Tx^{(n)} = \frac{1}{2^n}e_n^*(x)$. It is one-to-one, given that if $Tx = Ty$ then $Tx^{(n)} = Ty^{(n)} \Leftrightarrow e_n^*(x) = e_n^*(y)$ for all n and since the M-basis separates points we get injectivity. The bonundedness of the operator T follows from the fact that the M-basis is bounded, in fact we can write for every $x \in S_{(X,\|\cdot\|_2)}$ it exists $M \in \mathbb{R} : \sup \|e_n\|\|e_n^*\| \leq M$ for all $n \in \mathbb{N}$, to which it follows that:

$$\|Tx\|_{\ell^2} \leq \|M'(\frac{1}{2}, \frac{1}{4}, ...)\|_{\ell^2} = M''$$

We can now apply Theorem 3.2 to show that $|\cdot|$ is renorming and rotund.

-*Gâteaux differentiable*

This property follows from the fact that the norm $|\cdot|$ is the composition of two Gateaux differentiable functions $|||\cdot|||$ and $\|Tx\|_{\ell^2}$ with the euclidean distance in $\mathbb{R}^2$ (which is Lipschitz) so all the hypothesis are satisfied in order to apply Theorem 4.1.

$$|x| = \sqrt{|||x|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(x)^2}.$$

-*Not MLUR*

To prove this we shall consider two sequences: $x_n = \sqrt{\frac{4}{5}}(e_1 - e_n)$ and $y_n = \sqrt{\frac{4}{5}}(e_1 + e_n)$.

$$|\sqrt{\frac{4}{5}}(e_1 - e_n)|^2 = \frac{4}{5}|(e_1 - e_n)|^2 = \frac{4}{5}(|||e_1 - e_n|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(e_1 - e_n)^2).$$

We know $|||e_1 - e_n||| = \|(\|e_1\|_G, \|e_n\|_G)\|_M = \|(1,1)\|_M = 1$ since $(1,1) \in S_{(\mathbb{R}^2,\|\cdot\|_M)}$ as it can be seen in Example 5.2.

$$\frac{4}{5}(|||e_1 - e_n|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(e_1 - e_n)^2) = \frac{4}{5}(1 + \frac{1}{4}e_1^*(e_1) + \frac{1}{4^n}e_n^*(e_n)) =$$

$$= \frac{4}{5}(1 + \frac{1}{4} + \frac{1}{4^n}) = \frac{4}{5}(1 + \frac{1}{4} + \frac{1}{4^n}) \to 1.$$

From which we conclude $|x_n| \to 1$. The same holds for y_n, since:

$$\frac{4}{5}(|||e_1 + e_n|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(e_1 + e_n)^2) = \frac{4}{5}(1 + \frac{1}{4}e_1^*(e_1)^2 + \frac{1}{4^n}e_n^*(e_n)^2) = \frac{4}{5}(1 + \frac{1}{4} + \frac{1}{4^n}).$$

We now notice $\frac{1}{2}(x_n + y_n) = \frac{1}{2}\sqrt{\frac{4}{5}}(e_1 + e_n + e_1 - e_n) = \sqrt{\frac{4}{5}}e_1 \in S_{(X,|\cdot|)}$, in fact:

$$|\sqrt{\frac{4}{5}}e_1|^2 = \frac{4}{5}(|||e_1|||^2 + \frac{1}{4}) = \frac{4}{5}(1 + \frac{1}{4}) = 1.$$

On the other hand: $|x_n - y_n| = \sqrt{\frac{4}{5}}|2e_n|$

$$\frac{4}{5}|2e_n|^2 = \frac{16}{5}|e_n|^2 = \frac{16}{5}(||(0, \|e_n\|_G)||_M^2 + \frac{1}{4^n}) = \frac{16}{5}(||(0,1)||_M^2 + \frac{1}{4^n}).$$

But we know that $||(0,1)||_M^2 = ||\frac{1}{2}(0,2)||_M^2 = \frac{1}{4}$ since $(0,2) \in S_{(\mathbb{R}^2, \|\cdot\|_M)}$ as it can be seen in Example 5.2:

$$|x_n - y_n| = \sqrt{\frac{16}{5}(\frac{1}{4} + \frac{1}{4^n})} \to \sqrt{\frac{4}{5}} \neq 0.$$

By using the second characterization of the MLUR condition found in Lemma 3.2 we can conclude that $|\cdot|$ is not MLUR.

□

6.2. Problem 2

The second open problem present is number 6 of the section 52.3 of [7], for which we ask the following question: *Can every infinite-dimensional space with separable dual be renormed by wUR and not MLUR?*

Theorem 6.2. *Every Infinite dimensional Banach space with separable dual can be renormed with a norm which is Fréchet, wUR and not MLUR.*

Proof. Let X be a Banach space with separable dual. By Theorem 5.5 and Theorem 2.2 there exist a *Fréchet differentiable* renorming $\|\cdot\|_F$ and a bounded, shrinking M-basis $\{e_n, e_n^*\}$ such that $\|e_n\|_F = 1$ for all $n \in \mathbb{N}$. We can also renorm once again with $\|\cdot\|_{F'}$ using Theorem 5.6, obtaining the following properties:

$$\|x\|_{F'}^2 = \|x - e_1^*(x)e_1\|_F^2 + \|e_1^*(x)e_1\|_F^2.$$

We now use the construction introduced in Theorem 5.1 using:

- $Y_1 = \text{span}\{e_1\}$.
- $Y_2 = \overline{\text{span}\{e_n\}_{n\geq 2}}$.
- The norm on $\mathbb{R}^2$ defined as the Minkowski of the set M given in Example 5.2, that we will denote as $\|\cdot\|_M$.

By the choice of Y_1, Y_2 we can see every element of X as a decomposition on the two subspaces Y_1 and Y_2 since one of the two spaces is finite dimensional. In our case, for every $x \in X$ we write $x = y_1 + y_2$, where $y_1 = e_1^*(x)e_1$ and $y_2 = x - e_1^*(x)e_1$. For both subspaces we use the initial norm $\|\cdot\|_F$. We then define

$$|||x||| = \|(\|y_1\|_F, \|y_2\|_F)\|_M.$$

Observe that the unit sphere of $\|\cdot\|_M$ satisfies the hypothesis of Lemma 4.1, indeed every canonical projection of $S_{(X,\|\cdot\|_M)}$ is contained in $B_{(X,\|\cdot\|_M)}$. Applying Theorem 5.1, we conclude that $|||\cdot|||$ is a norm. The two spaces $(Y_1, \|\cdot\|_F)$, $(Y_2, \|\cdot\|_F)$ are Fréchet differentiable space since they are subspaces of $(X, \|\cdot\|_F)$, a Fréchet differentiable Banach space. If the Fréchet condition is satisfied at every point, at every direction then it must also be satisfied for the specific direction generated by $\text{span}\{e_n\}_{n\geq 2}$ and e_1 on the two relative subspaces. So, as stated in Theorem 5.2 we get the Fréchet differentiability of the norm $|||\cdot|||$. The norm $|||\cdot|||$ is also a renorming of $\|\cdot\|_{F'}$. To see why we let $\|\cdot\|_e$ be the euclidean norm on $\mathbb{R}^2$ and use the fact that all norms in finite dimensional spaces are equivalent (Theorem 2.1), in particular there exist $k, K \in \mathbb{R}^+$ such that $k\|\cdot\|_e \leq \|\cdot\|_M \leq K\|\cdot\|_e$. In our case:

$$\|x\|_{F'} = \|y_1 + y_2\|_{F'} = \|(\|y_1\|_F, \|y_2\|_F)\|_e \leq \frac{1}{k}||(\|y_1\|_F, \|y_2\|_F)||_M = \frac{1}{k}|||x|||.$$

While, for the second inequality:

$$|||x||| = \|(\|y_1\|_F, \|y_2\|_F)\|_M \leq K\|(\|y_1\|_F, \|y_2\|_F)\|_e = K\sqrt{\|y_1\|_F^2 + \|y_2\|_F^2}.$$

$$K\sqrt{\|y_1\|_F^2 + \|y_2\|_F^2} = K\sqrt{\|e_1^*(x)e_1\|_F^2 + \|x - e_1^*(x)e_1\|_F^2} = K\sqrt{\|x\|_{F'}^2} = K\|x\|_{F'}.$$

Obtaining the conclusion:

$$k\|x\|_{F'} \leq |||x||| \leq K\|x\|_{F'}.$$

We renorm once again with a new norm on X:

$$|x|^2 = |||x|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m}|e_m^*(x)|^2.$$

We claim that this renorming is *wUR*, *Fréchet differentiable* and *not MLUR*.

-*Renorming and rotundity*

It is enough to apply Theorem 3.2 to retrieve the renomring and rotundity properties (see first step of Theorem 6.1).

-*Fréchet differentiability*

The first norm $|||\cdot|||$ is Fréchet differentiable as stated in Theorem 5.2, the second term of the norm can be seen as $\|Tx\|_{\ell^2}$ which is Fréchet differentiable. Now we combine these two functions with the euclidean norm on $\mathbb{R}^2$.

$$|x| = \sqrt{|||x|||^2 + \sum_{n=1}^{\infty} \frac{1}{4^m} e_m^*(x)^2}.$$

If we now apply Theorem 4.2 we can conclude the Fréchet differentiability of $|\cdot|$.

-wUR

We start by taking $x_n, x_n' \in S_{(X,|\cdot|)}$ such that $|x_n + x_n'| \to 2$. We then set $z_n = (|||x_n|||, \frac{1}{2}e_1^*(x_n), ..., \frac{1}{2^k}e^*(x_n), ...)$ and $z_n' = (|||x_n'|||, \frac{1}{2}e_1^*(x_n'), ..., \frac{1}{2^k}e^*(x_n'), ...)$. We can now calculate:

$$\|z_n + z_n'\|_{\ell^2}^2 = (|||x_n||| + |||x_n'|||)^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(x_n + x_n')^2 \geq (|||x_n + x_n'|||)^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(x_n + x_n')^2 =$$

$$= (|||x_n + x_n'|||)^2 + \sum_{n=1}^{\infty} \frac{1}{4^m} e_m^*(x_n + x_n')^2 = |x_n + x_n'|^2 \to 2^2 = 4.$$

Since $\|z_n\|_{\ell^2} = |x_n| = 1$ and $\|z_n'\|_{\ell^2} = |x_n'| = 1$, we can apply the triangle inequality

$$\|z_n + z_n'\|_{\ell^2} \leq \|z_n\|_{\ell^2} + \|z_n'\|_{\ell^2} = |x_n| + |x_n'| = 2.$$

This proves that also $\|z_n + z_n'\|_{\ell^2} \to 2$. We know that the norm $\|\cdot\|_{\ell^2}$ of the space ℓ^2 satisfies the UR condition, but then by characterization 3 of Lemma 3.4 we also know that $\|z_n - z_n'\|_{\ell^2} \to 0$. Written analytically:

$$\|z_n - z_n'\|_{\ell^2}^2 = (|||x_n||| - |||x_n'|||)^2 + \sum_{n=1}^{\infty} \frac{1}{4^m} (e_m^*(x_n) - e_m^*(x_n'))^2 \to 0.$$

Which implies our final result $\lim_{n\to\infty} e_m^*(x_n) = \lim_{n\to\infty} e_m^*(x_n')$. We now use the key assumption that $\overline{\text{span}\{e_n^*\}} = X^*$, this allows us to infer that every operator in the dual when applied to x_n or x_n' mantains it's convergence, or, written in other terms $x_n - x_n' \rightharpoonup 0$. Since we started by $x_n, x_n' \in S_{(X,|\cdot|)}$ such that $|x_n + x_n'| \to 2$ and obtained $x_n - x_n' \rightharpoonup 0$ we notice that this is the the definition of wUR (Definition 3.8)

-not MLUR

The calculations are conducted in the same way as Theorem 6.1. We start by the sequences

$x_n = \sqrt{\frac{4}{5}}(e_1 - e_n)$ and $y_n = \sqrt{\frac{4}{5}}(e_1 + e_n)$.

$$|\sqrt{\frac{4}{5}}(e_1 - e_n)|^2 = \frac{4}{5}|(e_1 - e_n)|^2 = \frac{4}{5}(|||e_1 - e_n|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(e_1 - e_n)^2).$$

We know $|||e_1 - e_n||| = \|(\|e_1\|_F, \|e_n\|_F)\|_M = \|(1,1)\|_M = 1$, this follows from looking at the unit sphere of $\|\cdot\|_M$ in Example 5.3 and seeing that $(1,1)$ belongs to it.

$$\frac{4}{5}(|||e_1 - e_n|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(e_1 - e_n)^2) = \frac{4}{5}(1 + \frac{1}{4}e_1^*(e_1) + \frac{1}{4^n}e_n^*(e_n)) = \frac{4}{5}(1 + \frac{1}{4} + \frac{1}{4^n}).$$

$$\frac{4}{5}(1 + \frac{1}{4} + \frac{1}{4^n}) \to 1$$

From which we conclude $|x_n| \to 1$.The same holds for y_n, since:

$$\frac{4}{5}(|||e_1 + e_n|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(e_1 + e_n)^2) = \frac{4}{5}(1 + \frac{1}{4}e_1^*(e_1)^2 + \frac{1}{4^n}e_n^*(e_n)^2) = \frac{4}{5}(1 + \frac{1}{4} + \frac{1}{4^n}).$$

We now notice $\frac{1}{2}(x_n + y_n) = \frac{1}{2}\sqrt{\frac{4}{5}}(e_1 + e_n + e_1 - e_n) = \sqrt{\frac{4}{5}}e_1 \in S_{(X,|\cdot|)}$, in fact:

$$|\sqrt{\frac{4}{5}}e_1|^2 = \frac{4}{5}(|||e_1|||^2 + \frac{1}{4}) = \frac{4}{5}(1 + \frac{1}{4}) = 1.$$

On the other hand: $|x_n - y_n| = \sqrt{\frac{4}{5}}|2e_n|$

$$\frac{4}{5}|2e_n|^2 = \frac{16}{5}|e_n|^2 = \frac{16}{5}(||(0, \|e_n\|_F)||_M^2 + \frac{1}{4^n}) = \frac{16}{5}(||(0,1)||_M^2 + \frac{1}{4^n}).$$

If we look at the unit sphere of $\|\cdot\|_M$ in Example 5.2 we can conclude by homogeneity of the norm that that $||(0,1)||_M^2 = ||\frac{1}{2}(0,2)||_M^2 = \frac{1}{4}$, so:

$$|x_n - y_n| = \sqrt{\frac{16}{5}(\frac{1}{4} + \frac{1}{4^n})} \to \sqrt{\frac{4}{5}} \neq 0.$$

By using the second characterization of the MLUR condition found in Lemma 3.2 we can conclude that $|\cdot|$ is not MLUR.

□

6.3. Expanding problem 1

Now that we have discussed the two open problems posed by the book showing that not only those two norms exist, but they can also satisfy stronger conditions we wish to take the two initial questions and try to further extend some specific characteristics about them.

Let's start from a separable Banach space X and consider (P, ρ), the metric space of all equivalent norms on X endowed with the ρ metric (see Definition 2.6). We now consider the space $P' \subseteq P$ of all norms which are rotund and not MLUR and we show that the following condition holds: $\overline{P'} = P$ or, in other words, that P' is dense in the metric space of equivalent renormings of X. This can be also translated to an analytical property for which, given any equivalent norm $\|\cdot\|$ on X and any $\epsilon > 0$, there exists a rotund not MLUR renorming $\|\cdot\|_1$ such that for every $x \in X$:

$$\|x\|_1 \leq \|x\| \leq (1+\varepsilon)\|x\|_1.$$

Theorem 6.3. *Given a separable Banach space X, the metric space (P', ρ) of all rotund and not MLUR equivalent norms on X is a dense subset of the metric space (P, ρ) of all equivalent norms on X.*

Proof. Let X be a separable Banach space and let $\|\cdot\|$ be an equivalent norm on X. By Theorem 2.2 there exist a bounded M-basis $\{e_n, e_n^*\}$ such that $\|e_n\| = 1$ for all $n \in \mathbb{N}$ and $\|e_1^*\|^* = 1$. We define $T : (X, \|\cdot\|) \to (\ell^2, \|\cdot\|_{\ell^2})$ as $Tx = (\frac{1}{2}e_1^*(x), \frac{1}{4}e_2^*(x), \ldots)$. Given the boundedness of the M-basis we conclude that T is bounded, meaning that there exists $C \in \mathbb{R}^+$ such that $\|Tx\|_{\ell^2} \leq C\|x\|$ $(x \in X)$.

Let us fix $\varepsilon \in (0,1)$ and let us apply Theorem 5.9 from which we obtain $\eta > 0$ and an equivalent norm $\|\cdot\|_D$ with the following properties:

1. $(1-\varepsilon)\|\cdot\|_D \leq \|\cdot\| \leq \|\cdot\|_D$;
2. $(1-\epsilon)e_1 + \eta e_n \in S_{(X,\|\cdot\|_D)}$, whenever $n \in \mathbb{N}$;
3. $(1-\epsilon)e_1 \in S_{(X,\|\cdot\|_D)}$;
4. $\|e_n\|_D = 1$, whenever $n > 1$.

Then we consider the norm defined as follows:

$$\|x\|_2^2 = \|x\|_D^2 + \varepsilon^2 \sum_{m=1}^{\infty} \frac{1}{4^m} |e_m^*(x)|^2.$$

We know by Theorem 3.2 that $\|\cdot\|_2$ is an equivalent, rotund norm. Moreover, it is clear that $\|x\| \leq \|x\|_D \leq \|x\|_2$. Since $\|x\|_D \leq \frac{1}{(1-\varepsilon)}\|x\|$ for all $x \in X$ and $\|Tx\|_{\ell^2} \leq C\|x\|$, we can conclude that:

$$\|x\|_2^2 \leq \left(\frac{1}{(1-\varepsilon)}\|x\|\right)^2 + (\varepsilon C\|x\|)^2 \implies \|x\|_2 \leq \left(\sqrt{\frac{1}{(1-\varepsilon)^2} + \varepsilon^2 C^2}\right)\|x\|.$$

Combining the two inequalities above we conclude:

$$\|x\| \leq \|x\|_2 \leq \left(\sqrt{\frac{1}{(1-\varepsilon)^2} + \varepsilon^2 C^2}\right)\|x\|.$$

Once we prove that $\|x\|_2$ is not MLUR, by the arbitrariness of $\|\cdot\|$ and $\varepsilon \in (0,1)$ and the fact that $\sqrt{\frac{1}{(1-\varepsilon)^2} + \varepsilon^2 C^2} \to 1$ as $\varepsilon \to 0^+$, we get that P' is dense in P.

It remains to prove that $\|x\|_2$ is not MLUR. We now set $k^2 = \frac{4}{4+\varepsilon^2(1-\varepsilon)^2}$ and select two sequences $x_n = k((1-\varepsilon)e_1 - \eta e_n)$ and $y_n = k((1-\varepsilon)e_1 + \eta e_n)$. We now calculate $\|(1-\varepsilon)e_1+\eta e_n\|_2^2 = \|(1-\varepsilon)e_1+\eta e_n\|_D^2+\varepsilon^2(\frac{(1-\varepsilon)^2}{4}+\frac{\eta^2}{4^n}) = 1+\varepsilon^2(\frac{(1-\varepsilon)^2}{4}+\frac{\eta^2}{4^n}) \to 1+\frac{\varepsilon^2(1-\varepsilon)^2}{4}$, from this we deduce that $k^2\|(1-\varepsilon)e_1 + \eta e_n\|_2^2 \to 1$. The same calculations hold for $k^2\|(1-\varepsilon)e_1 - \eta e_n\|_2 \to 1$. If we now calculate $\frac{1}{2}(x_n + y_n) = k(1-\varepsilon)e_1 \in S_{(X,\|\cdot\|_2)}$ since $\|(1-\varepsilon)e_1\|_2^2 = \|(1-\varepsilon)e_1\|_D^2 + \frac{\varepsilon^2(1-\varepsilon)^2}{4} = \frac{1}{k^2}$. On the other hand, we have

$$\|x_n - y_n\|_2^2 = \|2\eta e_n\|_D^2 + \frac{\varepsilon^2}{4^n} \to 4\eta^2 \neq 0,$$

and hence the norm $\|x\|_2$ is not MLUR. □

6.4. Expanding problem 2

Theorem 6.4. *Given a Banach space X with separable dual, the metric space (Q,ρ) of all wUR and not MLUR equivalent norms on X is a dense subset of the metric space (P,ρ) of all equivalent norms on X.*

Proof. Let X be Banach space with separable dual and let $\|\cdot\|$. By Theorem 2.2, there exist a bounded and shrinking M-basis $\{e_n, e_n^*\}$ such that $\|e_n\| = 1$ for all $n \in \mathbb{N}$ and

$\|e_1^*\|^* = 1$. We define $T : (X, \|\cdot\|) \to (\ell^2, \|\cdot\|_{\ell^2})$ as $Tx = (\frac{1}{2}e_1^*(x), \frac{1}{4}e_2^*(x), ...)$. Given the boundedness of the M-basis we conclude that T is bounded, meaning that there exists $C \in \mathbb{R}^+$ such that $\|Tx\|_{\ell^2} \leq C\|x\|$ $(x \in X)$.

Let us fix $\varepsilon \in (0, 1)$ and let us apply Theorem 5.9 from which we obtain $\eta > 0$ and an equivalent norm $\|\cdot\|_D$ with the following properties:

1. $(1-\varepsilon)\|\cdot\|_D \leq \|\cdot\| \leq \|\cdot\|_D$;
2. $(1-\varepsilon)e_1 + \eta e_n \in S_{(X,\|\cdot\|_D)}$ $\forall n \in \mathbb{N}$;
3. $(1-\varepsilon)e_1 \in S_{(X,\|\cdot\|_D)}$;
4. $\|e_n\|_D = 1$ $n > 1$.

Then we consider the norm defined in the following way:

$$\|x\|_2^2 = \|x\|_D^2 + \varepsilon^2 \sum_{m=1}^{\infty} \frac{1}{4^m}|e_m^*(x)|^2.$$

We know by Theorem 3.2 that this is an equivalent, rotund norm. By the same calculations held in Theorem 6.3 we also conclude that

$$\|x\| \leq \|x\|_2 \leq \left(\sqrt{\frac{1}{(1-\varepsilon)^2} + \varepsilon^2 C^2}\right)\|x\|.$$

Once we prove that $\|x\|_2$ is wUR and not MLUR, by the arbitrariness of $\|\cdot\|$ and $\varepsilon \in (0, 1)$ and the fact that $\sqrt{\frac{1}{(1-\varepsilon)^2} + \varepsilon^2 C^2} \to 1$ as $\varepsilon \to 0^+$, we get that Q is dense in P.

The fact that $\|x\|_2$ is not MLUR follows as in the proof of Theorem 6.3, it remains to prove that it is wUR. We start by taking $x_n, x_n' \in S_{(X,\|\cdot\|_2)}$ such that $\|x_n + x_n'\|_2 \to 2$. We then set $z_n = (||x_n||_D, \varepsilon\frac{1}{2}e_1^*(x_n), ..., \varepsilon\frac{1}{2^k}e_k^*(x_n), ...)$ and $z_n' = (||x_n'||_D, \varepsilon\frac{1}{2}e_1^*(x_n'), ..., \varepsilon\frac{1}{2^k}e_k^*(x_n'), ...)$. We now have:

$$\|z_n + z_n'\|_{\ell^2}^2 = (||x_n||_D + ||x_n'||_D)^2 + \varepsilon^2 \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(x_n + x_n')^2$$

$$\geq (||x_n + x_n'||_D)^2 + \varepsilon^2 \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(x_n + x_n')^2,$$

where the inequality is given by the triangle inequality applied to $\|\cdot\|_D$. Then:

$$(||x_n + x_n'||_D)^2 + \varepsilon^2 \sum_{m=1}^{\infty} \frac{1}{4^m} e_m^*(x_n + x_n')^2 = \|x_n + x_n'\|_2^2 \to 2^2 = 4.$$

Since $\|z_n\|_{\ell^2} = \|x_n\|_2 = 1$ and $\|z'_n\|_{\ell^2} = \|x'_n\|_2 = 1$, when we apply the triangle inequality we obtain:

$$\|z_n + z'_n\|_{\ell^2} \leq \|z_n\|_{\ell^2} + \|z'_n\|_{\ell^2} = \|x_n\|_2 + \|x'_n\|_2 = 2$$

this proves that also $\|z_n + z'_n\|_{\ell^2} \to 2$. We also know that the norm $\|\cdot\|_{\ell^2}$ of the space ℓ^2 satisfies the UR condition for which we can conclude, applying characterization 2 of Lemma 3.4, that $\|z_n - z'_n\|_{\ell^2} \to 0$. This means:

$$\|z_n - z'_n\|^2_{\ell^2} = (||x_n||_D - ||x'_n||_D)^2 + \varepsilon^2 \sum_{m=1}^{\infty} \frac{1}{4^m}(e^*_m(x_n) - e^*_m(x'_n))^2 \to 0.$$

Which implies our final result $\lim_{n\to\infty} e^*_m(x_n - x'_n) = 0$. We now use the key assumption that $\overline{\text{span}\{e^*_n\}} = X^*$, this allows us to infer that $x_n - x'_n \rightharpoonup 0$. So, since by taking $x_n, x'_n \in S_{(X,\|\cdot\|_2)}$ such that $\|x_n + x'_n\|_2 \to 2$ we get $x_n - x'_n \rightharpoonup 0$, we can conclude that the norm is wUR.

□

6.5. Conclusion

In conclusion we were able to answer the questions of whether a separable Banach space can be renormed with a norm which is rotund but not MLUR in the affirmative, adding an extra property of the norm: Gâteaux differentiability. We also proved that if the smoothness condition of Gâteaux differentiability is dropped such norms are a dense subset of all renormings. The second questions answered in the affirmative is whether a Banach space with separable dual can be renormed with a norm which is weakly uniformly rotund but not MLUR, adding the extra property of the norm: Fréchet differentiability. Even in this case by dropping the smoothness condition it was shown that one could retrieve the density property. Naturally, in mathematics, when some questions are answered new ones arise. Indeed, a continuation of this work lies in answering if both the smoothness condition and the density can be simultaneously kept, specifically:

Given a separable Banach space X, is it true that the set of all equivalent Gâteaux smooth, rotund and not MLUR norms is dense in (P, ρ)?.

Given a Banach space X with separable dual, is it true that the set of all equivalent Fréchet smooth, weakly uniformly rotund and not MLUR norms is dense in (P, ρ)?.

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