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Differential Problems with Stochastic Boundary Conditions

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Abstract

In this thesis a specific kind of stochastic problems is treated: partial differential equations with white noise boundary conditions. First results about this topic can be found [DZ93]. This topic was introduced recently, its literature is still very fragmented and wide possibilities of development are available. This kind of equations has a relevant modeling interest. In fact, in several physical cases, random perturbations come from the boundary of the domain and are not distributed inside it [DL06],[Kim06].

In the first part of the thesis, I will recall part of the literature available on this kind of problems, dwelling on the heat equation, proving that the solution of this problem belongs to proper functional spaces and has some regularity properties. The heat equation was the first problem where white noise boundary conditions were introduced, therefore its literature is more detailed. In addition, it was proven that different boundary conditions lead to different regularity for the solution of the heat equation with white noise boundary conditions.

Subsequently, I will analyze a fourth order linear evolution equation with several stochastic boundary conditions. No results are available on this kind of problems. However, it is possible to replicate what has been recalled for the heat equation, getting analogous results to the ones found in [DZ93],[AB02b] and improve, focusing on a specific case, the results proven in [Mas95].

Sommario

In questa tesi viene trattata una particolare classe di problemi stocastici: equazioni differenziali con condizioni al bordo stocastiche di tipo rumore bianco. I primi risultati su questo argomento si trovano in [DZ93]. Questo argomento è stato introdotto recentemente, la letteratura è ancora molto frammentata e sono possibili ampi margini di sviluppo. Questo tipo di equazioni ha un importante interesse modellistico. In diversi problemi di interesse fisico, le perturbazioni caotiche provengono dal bordo del sistema e non sono distribuite al suo interno [DL06],[Kim06].
Nella prima parte di questa tesi si ripercorrerà parte della letteratura presente su questo tipo di problemi, soffermandosi sull'equazione del calore, provando l'appartenenza della soluzione ad opportuni spazi funzionali e alcune sue proprietà di regolarità. L'equazione del calore è stata la prima su cui sono state introdotte condizioni al bordo stocastiche di tipo white noise e i risultati a riguardo sono meglio dettagliati. Inoltre, è noto per questa equazione che problemi al bordo diversi portano a risultati differenti nella regolarità della soluzione.
Successivamente si analizzerà un'equazione di evoluzione lineare del quarto ordine con vari tipi di condizioni al bordo stocastiche. Non sono disponibili risultati su questa tipologia di problemi. Tuttavia è possibile estendere i risultati noti per l'equazione del calore [DZ93],[AB02b] e migliorare, focalizzandosi su un caso specifico, i risultati di [Mas95].

Contents

Chapter 1

Introduction

Stochastic equations with white noise boundary conditions are a recent topic between mathematical analysis and stochastic calculus. They were introduced by Da Prato and Zabczyk in [DZ93]. In general stochastic differential equations describe all dynamical systems, in which random effects can be taken into account as perturbations. Stochastic boundary conditions have a clear model interpretation because in some applications the noise can effect the evolution of a system only through the boundary of a region. An easy example of what kind of problems is affected by this noise can be found in [DL06]. Imagine that rain falls on the surface of a lake, producing a sound wave that propagates under water. This noise is produced by a large number of small contributions (the rain droplets). After suitable rescaling, the noise can be considered to be spatially homogeneous on the surface of the lake, propagating through a three-dimensional medium. Hence, the noise is concentrated on the two dimensional boundary of a three-dimensional domain.

1.1. Plan of the work

The aim of this work is on the analysis of mild solution of partial differential equation with white noise boundary conditions. We will concentrate on existence, uniqueness and regularity of solutions, but this is not the only kind of issue related to these stochastic problems. Particularly relevant are also topics related to ergodicity properties and long time behavior of the solutions [DZ96],[AB02a], stochastic optimal control problems [FG09],[Mas10],[DFT07]. The discussion will be focused to give a clear explanation of the literature devoted to the heat equation with white noise boundary conditions of Dirichlet type or Neumann type [AB02b],[DZ96],[DZ93]. These two cases present relevant differences in the regularity of their own solution. Therefore, they are enough to understand some critical issues of this kind of problems. Then, our attention will be focused to expose some original results on a fourth order linear evolution equation with white noise boundary conditions.

- In chapter 2, some general notions and results useful for the analysis of stochastic partial differential equations are introduced and stated [LR15],[DZ93]. At the beginning of the chapter, we introduce the notion of mild solution for a general stochastic problem, in order to get a more flexible notion of solution that, actually, generalize the notion of mild solution for a deterministic problem. Then we move to a problem with stochastic boundary conditions. Let Γ be a regular domain in $\mathbb{R}^n$, $A_0 : D(A_0) \subset L^2(\Gamma) \to H = L^2(\Gamma)$ a differential operator, $\tau : D(\tau) \subset L^2(\Gamma) \to L^2(\partial\Gamma) = U_2$ an operator defining boundary conditions. We are interested in giving a meaning and, then, a proper definition of solution to the equation

$$\begin{cases} \frac{\partial X}{\partial t}(t) = A_0 X(t) + F(X(t)) + B\dot{W}^1(t) \quad t \in [0,T] \\ \tau X(t) = \dot{W}^2(t) \quad t \in]0,T] \\ X(0) = x \end{cases} \tag{1.1.1}$$

  where $F : D(F) \subset H \to H$ is a non linear operator, B is a linear operator from a Hilbert space U_1 with values in H and $\dot{W}^1(t)$, $\dot{W}^2(t)$ are two white noise processes in U_1 and U_2. In general we are not interested in the notion of solution introduced in classical stochastic calculus [Bal17], but to a different one, called mild solution. Under reasonable assumptions on A_0:

  1. its restriction to the kernel of τ generates a C_0 semigroup of operators $S(t)$ on H;
  2. the stationary problem

  $$(A_0 - \lambda)z = 0, \;\; \tau(z) = u$$

     has a unique solution for some $\lambda \in \mathbb{R}$, $z = \mathcal{D}u$ $\forall u \in U_2$.

  Then a mild solution of equation (1.1.1) can be defined at least in a formal way as a process satisfying:

$$\begin{aligned} X(t) = S(t)x + (\lambda - A)\int_0^t S(t-s)\mathcal{D}\dot{W}^2(s)ds + \\ \int_0^t S(t-s)B\dot{W}^1(s)ds + \int_0^t S(t-s)F(X(s))ds. \end{aligned} \tag{1.1.2}$$

  It can be noted that the first summand is the mild solution of the deterministic linear problem, the third one and the fourth one are given by variation

of constants formula and are due to the non-linearities of the equation, lastly the second summand takes care of stochastic boundary conditions and is obtained by detection of the boundary conditions. Then it is clear why the two above assumptions on A_0 are needed. Of course the above formula is not well defined in general since it has an implicit representation and the two stochastic integral are well posed under suitable assumptions. Hence, at the end of chapter 2, some abstract conditions for the well posedness of equation (1.1.2) are stated.
In addition, some results on partial differential equations with white noise boundary conditions are related to a notion of solution different from the mild one, called weak solution in analogy to the weak solution of deterministic problems [Sow94], [AB02b],[Brz+15]. We neglect results related to this notion of solution in this thesis.

- Chapter 3 is devoted to the application of the general theory to some specific equation in this stochastic framework. The first part (sections 3.1, 3.2) is focused on presenting some available results on the heat equation in a bounded interval. To be more precise in section 3.1(resp. 3.2) the heat equation in a closed interval with Neumann (Dirichlet) boundary conditions is analyzed, proving that formula (1.1.2) is well defined and it is a L^2 continuous function (a distribution) for each fixed time t. Further analysis on heat equation in a general Riemannian manifold with Neumann boundary conditions can be found in [Sow94]. We just present a direct proof of the fact that the solution of the problem in the half-line is a L^2 function almost everywhere in $\Omega \times [0, T]$ in section 3.1.
  The fact that the solution of the problem described in section 3.2 for a fixed time t is a distribution leads us to the question if this is due to some blow up in a region of Γ. This idea moves to the analysis introduced in [AB02b] and reported in section 3.3. Actually, they consider a more general heat equation with Dirichlet white noise boundary conditions

$$\begin{cases} \frac{\partial Z}{\partial t}(t,\xi) = \frac{\partial^2 Z}{\partial \xi^2}(t,\xi) + \sum_{j=1}^n [b_j(\xi)\frac{\partial Z}{\partial \xi}(t,\xi) + F_j(t,\xi,Z(t,\xi))]\dot{W}^j(t) \\ Z(t,0) = \dot{V}(t) \\ Z(0,\xi) = 0 \qquad (t,\xi) \in I^T := [0,T] \times \mathbb{R}^+ \end{cases} \tag{1.1.3}$$

  Where $W(t) = (W^1, ..., W^n)(t)$ is a real standard n-dimensional Wiener process and $V(t)$ is real standard Wiener process adapted to the filtration generated by $W(t)$. Equation (1.1.3) obviously includes the heat equation, taking $b_j = F_j \equiv 0$. In particular, it is proven that, under suitable assumptions on b_j and F_j, the solution belongs to some L^p weighted spaces, $M^p(\Omega \times [0, T]; L^p_\gamma)$, where L^p_γ is the space of real measurable functions such that

$$\int_0^{+\infty} |f(x)|^p (1 \wedge x^{p-1+\gamma}) dx < +\infty$$

and the degree of increasing close to the origin can be quantified $\forall T > 0,\ \alpha > 0$

$$\lim_{x \to 0} \sup_{0 \leq t \leq T} x^{1+\alpha} |u(t,x)| = 0 \ \ \mathbb{P} - a.s.$$

In addition, the solution of equation (1.1.3) is continuous far from the origin, namely $\forall \delta > 0$ it exists a modification of the solution continuous in $[\delta, +\infty)$. In conclusions, the authors prove that the solution is not only a distribution-valued process, but it belongs to some L^2 function space, provided an appropriate weight is introduced.

- In chapter 4, we are interested to a fourth order parabolic problem, considering different kinds of domain Γ and boundary conditions.

$$\frac{\partial Z}{\partial t}(t,\xi) + \frac{\partial^4 Z}{\partial \xi^4}(t,\xi) = 0 \ \ t \geq 0 \ \xi \in \Gamma. \tag{1.1.4}$$

1. White noise zero and second order boundary conditions (Navier Boundary conditions) in a bounded interval (P1).
2. White noise zero and second order boundary conditions (Navier Boundary conditions) in the half-line (P2).
3. White noise first and third order boundary conditions in a bounded interval (P3).
4. White noise first and third order boundary conditions in the half-line (P4).

This is the original part of the thesis. The results on these problems can be coupled. In fact problems P1 and P3 are as related as problems P2 and P4. Equation (1.1.4) has no direct modeling interpretation, but adding some terms it leads to a Cahn–Hilliard equation. For this reason, it can be the first step to deeper and much more relevant analysis. Concerning problems P1 and P3 we prove that the solution belongs to distributional spaces for each fixed time t, in addition boundary conditions of higher order derivative give us more regularity as in the Neumann-Dirichlet case previously described. Concerning problems P2 and P4, we find that, again as in section 3.3, the solution belongs to some L^p weighted spaces, $L^p(\Omega \times [0,T]; L^p_\gamma)$. In the case of P2, L^p_γ is the space of real measurable functions such that

$$\int_0^{+\infty} |f(x)|^p (1 \wedge x^{3p-1+\gamma}) dx < +\infty.$$

In addition, the solution of both problems is continuous away from the origin.

- All the analysis made in chapters 2, 3 and 4 needs some advanced mathematical tools which are reported in the appendices. Some results on sine and cosine transforms are presented in appendix A, then they are applied to some deterministic linear parabolic equations. In appendix B we introduce the notion of C_0 semigroup of linear operator on a Banach space and of mild solution for a deterministic problem. Then, in appendix D, we introduce the notion of Wiener process with value in a general Hilbert space and the notion of stochastic integral with respect of this kind of processes exploiting some operator theory tools recalled in appendix C. In appendix E some standard results of stochastic calculus are recalled.

All the original results are fully explained and their proofs are as complete as possible. Instead, concerning the rest of the statements, not all the proofs are presented. To be more precise, concerning the more technical statements, the idea behind them is often only described, then a reference for the complete proof is added.

1.2. New results

- The proof of proposition 3.1.6 is new. Actually, it is not an original result. In fact, in [Sow94] more general results in a Riemannian manifold are shown. The relevance of this proof is that, in this specific case, we can find an easier and more direct approach.

- Chapter 4 is completely new. In particular, the first part is an easy application of some abstract results presented in chapter 2. Instead, the second part, namely propositions 4.2.1, 4.2.3, 4.2.4, 4.2.5, improves, focusing on a specific case, the results available in [Mas95].
  To get all the results presented in chapter 4 some complications emerge. In fact, there are no available results for parabolic equation with white noise boundary conditions of order $2m$ with $m > 1$, except a few general results like proposition 2.2.3 and a sufficient condition for the belonging of the solution for a fixed time t to an L^2 space with respect to the Lebesgue measure [Mas95]. Of course this sufficient condition is not so useful in the applications, because, as proven in section 3.2, also the solution of an elementary problem, namely the heat equation with stochastic Dirichlet boundary conditions, is not a L^2 function for a fixed time t. Further analysis presented in section 3.3 and [Sow94] exploit the explicit elementary formula for the kernel of the differential operator or some useful estimates for the kernel. These

kinds of results are not completely available for the fourth order parabolic equation. For this reason it is necessary to work with some non elementary representation for the solution of problems P2 and P4.

1.3. Some notations

In general, all the notations are clearly introduced before being used. In this section we define a few basic mathematical objects, whose notation will be kept throughout the entire thesis.

- For a Banach space W, we denote by $L^p([0,T];W)$, where $p \in [1,+\infty]$, the L^p-space of W-valued, Bochner-integrable functions on $[0,T]$. We denote by $C([0,T];W)$ the set of W-valued continuous functions on $[0,T]$.
- Even if it is not explicitly done, we will consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the natural filtration $\mathcal{F}_t$ generated by the noises in the equation.
- For an a Hilbert space W (so for a Banach space too), we denote its dual space by W^*, its inner product by $\langle \cdot, \cdot \rangle_W$, its duality pairing by $\langle \cdot, \cdot \rangle_{W^*,W}$, and its norm by $\|\cdot\|_W$.
- $C_b^k(\mathbb{R})$ is the space of function with derivatives up to the order k continuous and bounded.
- When we refer to constant quantities whose exact value is irrelevant, we may share the same symbol for more than one of these objects.

More specialized functional and probabilistic tools will be specified when necessary.

Chapter 2

Systems Perturbed Through the Boundary

In this chapter we will introduce the main topics of this thesis: the concept of mild solution and its application to some partial differential equations with stochastic boundary conditions in a very abstract framework. A more detailed description about these topics can be found in [DZ93], [DZ96],[DZ14],[LR15].

2.1. Mild solution

Let $(U, \|\cdot\|_U)$ and $(H, \|\cdot\|_H)$ be separable Hilbert spaces. We fix a cylindrical Wiener process $W(t),\ t \geq 0$, in U on a probability space $(\Omega; \mathcal{F}; \mathbb{P})$ with a normal filtration $\mathcal{F}_t, t \geq 0$. Moreover, we fix $T > 0$ and consider the following type of stochastic differential equations in H

$$\begin{cases} dX(t) = [AX(t) + F(X(t))]dt + B(X(t))dW(t) & t \in [0, T] \\ X(0) = \xi \end{cases} \tag{2.1.1}$$

where

- $A : D(A) \to H$ is the infinitesimal generator of a C_0 semigroup $S(t),\ t \geq 0$, of linear operators on H,
- $F : H \to H$,
- $B : H \to L(U, H)$,

- ξ is an H valued, $\mathcal{F}_0$-measurable random variable.

First, we want to try to motivate the definition of mild solution. In fact, we note that only in very special cases can one find a solution to (2.1.1) such that $X \in D(A)$ $dt \otimes \mathbb{P} - a.s.$ Therefore, one reformulates the equation using the following heuristics: Consider the integral form of (2.1.1) and apply the (in general not-defined!) operator e^{-tA} for $t \in [0,T]$ to this equation. Applying Itô's product rule (again heuristically), we find

$$e^{-tA}X(t) = \xi + \int_0^t e^{-sA}(AX(s)+F(X(s)))ds + \int_0^t e^{-sA}B(X(s))dW(s) - \int_0^t e^{-sA}(AX(s))ds$$

that implies

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s) \quad \mathbb{P} - a.s.$$

This heuristic drives our definition of mild solution.

Definition 2.1.1 (mild solution)**.** An H-valued predictable process $X(t)$, $t \in [0,T]$, is called mild solution of problem (2.1.1) if

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s) \quad \mathbb{P} - a.s.$$

for each $t \in [0,T]$.

To be more clear the $\mathbb{P}$-zero set where the above equation does not hold may depend on t. If we introduce an enough common set of hypotheses we get existence, uniqueness and continuity of the mild solution in a proper space and that the solution map is Lipshitz continuous. To be more specific:
Hypothesis M.0

- $A : D(A) \to H$ is the infinitesimal generator of a C_0 semigroup $S(t),\ t \geq 0$, of linear operators on H,
- $F : H \to H$ is Lipschitz continuous,
- $B : H \to L(U,H)$ is strongly continuous, i.e. the map $x \to B(x)u$ is continuous from H to H $\forall u \in U$,
- $\forall t \in]0,T]$ and $x \in H$ we have that $S(t)B(x) \in L_2(U,H)$,
- there is a square integrable mapping $K : [0,T] \to [0,+\infty]$ such that

- $\|S(t)(B(x) - B(y))\|_{L_2(U,H)} \leq K(t)\|x - y\|_H$
- $\|S(t)B(x)\|_{L_2(U,H)} \leq K(t)(1 + \|x\|_H)$

$\forall t \in]0, T]$ and $x, y \in H$

If we also introduce for a fixed $T > 0$ and $p \geq 2$ the space $\mathcal{H}^p(T, H)$ as the space of all the H-valued predictable process Y such that

$$\|Y\|_{\mathcal{H}^p} := \sup_{t\in[0,T]} (\mathbb{E}[\|Y(t)\|_H^p])^{\frac{1}{p}} < +\infty$$

Then $(\mathcal{H}^p(T, H), \|\cdot\|_{\mathcal{H}^p})$ is Banach space (obviously considering the usual equivalence classes of processes).
Now it is possible to state the following result.

Theorem 2.1.2. *Under Hypothesis M.0 there exists a unique mild solution $X(\xi) \in \mathcal{H}^p(T, H)$ of problem (2.1.1) with initial condition $\xi \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H) =: L^p_0$. In addition, the mapping*
$X : L^p_0 \to \mathcal{H}^p(T, H)$ that associates to each initial datum its mild solution is Lipschitz continuous.

Proof. The proof of this theorem can be found in [LR15] and is based on Contraction Mapping Theorem and some results about continuity of the implicit function. □

Concerning the regularity of the solution, to prove theorem 2.1.2 as sub-result we get that $\int_0^t S(t-s)F(X(s))ds$ has a continuous version, hence to get the continuity of $X(t)$ it is enough to show that $\int_0^t S(t-s)B(X(s))dW(s)$ has a continuous version. To obtain this result we need to add a further hypothesis on $K(t)$ introduced in Hypothesis M.0.

Proposition 2.1.3. *Assume that A, B, F satisfy Hypothesis M.0, and let $p \geq 2$. If there exists $\alpha \in]\frac{1}{p}, +\infty[$ such that $\int_0^T s^{-2\alpha} K^2(s)ds < +\infty$ then the mild solution $X(\xi)$ of the problem (2.1.1) has a continuous version for all initial conditions $\xi \in L^p_0$.*

Proof. The proof of this fact can be found in [LR15]. □

2.2. Equations with non-homogeneous boundary conditions

Let Γ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Gamma$, let $A_0 : D(A_0) \subset L^2(\Gamma) \to H = L^2(\Gamma)$ a partial differential operator. Let moreover $\tau : D(\tau) \subset$

$L^2(\Gamma) \to L^2(\partial\Gamma) = U_2$ be a linear operator defining boundary conditions. We are interested about equations of the type

$$\begin{cases} \frac{\partial X}{\partial t}(t) = A_0 X(t) + F(X(t)) + B\dot{W}^1(t) & t \in [0,T] \\ \tau X(t) = \dot{W}^2(t) & t \in]0,T] \\ X(0) = x \end{cases} \tag{2.2.1}$$

Where $F : D(F) \subset H \to H$ is a nonlinear operator, B is a bounded linear operator from a Hilbert Space U_1 into H and $\dot{W}^1(t)$, $\dot{W}^2(t)$ represent white noise processes in $U_1,\ U_2$ respectively. The analysis of this equation can be done replacing the noises by arbitrary functions or exploiting the properties of semigroup method [Bal12]. This section will follow this approach to get some satisfactory results about the study of this equation.

First we consider two orthonormal bases of $\mathcal{U}_1 := L^2(0,T;U_1)$ and $\mathcal{U}_2 := L^2(0,T;U_2)$ made by smooth functions $\{\psi_n^1\}$ and $\{\psi_n^2\}$ and two sequences of standard and independent one dimensional normal random variables $\{\xi_n\}$ and $\{\eta_n\}$, then we consider the approximating form of (2.2.1)

$$\begin{cases} \frac{\partial X_N}{\partial t}(t) = A_0 X_N(t) + F(X_N(t)) + B\dot{W}_N^1(t) & t \in [0,T] \\ \tau X_N(t) = \dot{W}_N^2(t) & t \in]0,T] \\ X(0) = x \end{cases} \tag{2.2.2}$$

where $\dot{W}_N^1(t) = \sum_{n=1}^N \xi_n \psi_n^1(t),\ \dot{W}_N^2(t) = \sum_{n=1}^N \eta_n \psi_n^2(t)$. Now we solve this approximating equation thanks to semigroup method. Let $Ay = A_0 y,\ y \in D(A) = \{y \in D(A_0) \cap D(\tau); \tau y = 0\}$. Assume that A generates a C_0 semigroup of operators $S(t)$ on H and that the stationary boundary value problem

$$(A_0 - \lambda)z = 0,\ \tau(z) = u$$

has for some $\lambda \in \mathbb{R}$ a unique solution $z = \mathcal{D}u\ \forall u \in U_2$. With the operators A and $\mathcal{D}$ defined this way, problem (2.2.2) can be written as

$$X_N(t) = S(t)x + (\lambda - A)\int_0^t S(t-s)\mathcal{D}\dot{W}_N^2(s)ds +$$

$$\int_0^t S(t-s)B\dot{W}_N^1(s)ds + \int_0^t S(t-s)F(X_N(s))ds$$

In the sequel we will use the more compact notation $Z_N^2(t) = (\lambda - A)\int_0^t S(t-s)\mathcal{D}\dot{W}_N^2(s)ds$ and $Z_N^1(t) = \int_0^t S(t-s)B\dot{W}_N^1(s)ds$. The reason why we can consider this equivalent problem instead of equation (2.2.2) can be found in [DZ96], now we explain the formal reason why this is possible at least in case $F \equiv 0$ (case with F generic follows by variation of constants argument). Assuming $\dot{W}^1_N(t)$ and $\dot{W}^2_N(t)$ are two deterministic and regular functions $u_1(t)$ and $u_2(t)$ (at least C^2 functions, with values in U_1 and U_2 respectively) and $x - \mathcal{D}u_2(0) \in D(A)$, thanks to proposition B.4.6, there exists a strong solution of the equation

$$\begin{cases} z'(t) = Az(t) + \lambda \mathcal{D}u_2(t) - \mathcal{D}u_2'(t) + Bu_1(t) \quad t > 0 \\ z(0) = x - \mathcal{D}u(0) \end{cases} \tag{2.2.3}$$

In particular $\tau(z(t)) = 0, \forall t \geq 0$. If we define $y(t) = z(t) + \mathcal{D}u_2(t),\ t \geq 0$. Then we have

$$\begin{cases} \tau(y(t)) = \tau(\mathcal{D}u_2(t)) = u_2(t),\ t \geq 0 \\ \frac{d}{dt}(y(t) - \mathcal{D}u_2(t)) = (A_0 - \lambda)(y(t) - \mathcal{D}u_2(t)) + \lambda y(t) - \mathcal{D}(u_2'(t)) + B(u_1(t)) \end{cases}$$

Taking into account that $(A_0 - \lambda)y(t) = 0\ \forall t \geq 0$ we get that

$$\begin{cases} \frac{\partial y}{\partial t} = A_0 y(t) + Bu_1(t),\ t > 0 \\ \tau(y(t)) = u_2(t),\ t > 0 \\ y(0) = x \end{cases}$$

So $y(t)$ solves the deterministic version of equation (2.2.2) and is given by

$$y(t) = S(t)x + (\lambda - A)\int_0^t S(t-s)\mathcal{D}u_2(s)ds + \int_0^t S(t-s)Bu_1(s)ds$$

Now we come back to our approximating stochastic problem (2.2.2). We start considering the homogeneous case (namely $F \equiv 0$), so that $X_N(t) = S(t)x + Z^1_N(t) + Z^2_N(t)$. It is well known [Bal12] that $Z^1_N(t)$ converges in H in mean square if and only if $\int_0^t |S(r)B|^2_{L_2(U,H)}dr < +\infty$ and it holds

$$Z^1_N(t) \to Z^1(t) = \int_0^t S(t-s)B(s)dW^1(s).$$

hence we are interested in finding necessary and sufficient conditions so that

$$Z^2_N(t) \to Z^2(t) = \int_0^t (\lambda - A)S(t-s)\mathcal{D}dW^2(s) \tag{2.2.4}$$

in H in mean square $\forall t \in [0,T]$. For this reason, by linearity of the equation we can concentrate on the problem

$$\begin{cases} \frac{\partial Z_N}{\partial t}(t) = A_0 Z_N(t) \quad t \in [0,T] \\ \tau Z_N(t) = \dot{W}^2_N(t) \quad t \in]0,T] \\ Z(0) = 0 \end{cases} \tag{2.2.5}$$

The solution of this problem is called Ornstein-Uhlenbeck process in analogy to the classical case. Now it is possible to state some technical sufficient and necessary conditions, so that conditions (2.2.4) holds. These are useful to get some manageable conditions. They explain the possibility to extend the deterministic operator $\mathcal{L}_T u = (\lambda - A)\int_0^t S(t-s)\mathcal{D}u(s)ds$, which is linear from $D(\mathcal{L}_T) = W^{1,2}(0,T;U_2)$ into $L^2(0,T;H) = \mathcal{H}$, to a bigger domain which includes our boundary conditions, namely $\mathcal{U}_2$.

Theorem 2.2.1. *The operator $\mathcal{L}_T$ has an extension to a Hilbert-Schmidt operator from $\mathcal{U}_2$ to $\mathcal{H}$ if and only if for almost $r \in [0,T]$ the operators $\mathcal{D}^* S^*(r)A^* : D(A^*) \to H$ have Hilbert-Schmidt extensions and*

$$\int_0^T \left(\int_0^t \|\mathcal{D}^* S^*(r) A^*\|^2_{L_2(H;U_2)} dr \right) dt < +\infty \tag{2.2.6}$$

Note that if B and C are respectively bounded and densely defined unbounded linear operators and the operator BC has a bounded extension, then $ImB^* \subset D(C^*)$ and the operator C^*B^* is a bounded operator adjoint to BC. If we consider $B = \mathcal{D}^* S^*(r)$, $C = A^*$, since a Hilbert-Schmidt operator is bounded we get the following equivalent formulation of Theorem 2.2.1.

Theorem 2.2.2. *The operator $\mathcal{L}_T$ has an extension to an Hilbert-Schmidt operator from $\mathcal{U}_2$ to $\mathcal{H}$ if and only if*

$$\int_0^T \left(\int_0^t \|A(S(r)\mathcal{D})\|^2_{L_2(U_2;H)} dr \right) dt < +\infty$$

Proof. The proofs of these theorems are very technical and can be found in [DZ93]. □

Actually if just a weaker version of (2.2.6) holds

$$\int_0^T \|\mathcal{D}^* S^*(r) A^*\|^2_{L_2(H;U_2)} dr < +\infty \tag{2.2.7}$$

we get that the process

$$Z^2(t) = \int_0^t (\lambda - A) S(t-s) \mathcal{D} dW^2(s) \quad t \in [0,T]$$

is well defined and formally $\mathcal{L}_T(\dot{W}^2) = Z^2(\cdot)$ $\mathbb{P}-a.s.$ If a stronger version of (2.2.7) holds, namely

$$\exists \gamma > 0 : \int_0^t r^{-\gamma} \|AS(r)\mathcal{D}\|^2_{L_2(U_2;H)} dr < +\infty \tag{2.2.8}$$

then the processes $Z_N^2(t)$ converge to $Z^2(t)$ that is well defined and has a version H-continuous. In applications to nonlinear equations it is important that the space H is as small as possible. So let us introduce some intermediate versions of H. We assume that A generates an analytic semigroup on H and denote by H_α the fractional power spaces of A.
If $\alpha \geq 0$, then $H_\alpha = D((-A)^\alpha) \subset H$ with the graph norm induced by $(-A)^\alpha$. Instead, if $\alpha < 0$ then H_α is the completion of H with respect to the norm $\|x\|_\alpha = \|(-A)^\alpha x\|_H$. From now on we assume that A is self-adjoint and that there exists a complete orthonormal basis of eigenvectors $\{g_n\}$ of A corresponding to the sequence $\{-\lambda_n\} \downarrow -\infty$ of non positive eigenvalues. In particular we have the following result, really useful for the applications.

Proposition 2.2.3. *If for some $\gamma > 0$*

$$\sum_{n=1}^{+\infty} \lambda_n^{2\alpha+\gamma+1} \|\mathcal{D}^* g_n\|_{U_2}^2 < \infty$$

Then the process $Z^2(t)$ has an H_α-continuous version.

Proof. Note that

$$\int_0^t r^{-\gamma} \|AS(r)\mathcal{D}\|^2_{L_2(U_2;H^\alpha)} dr = \int_0^t r^{-\gamma} \|(-A)^{1+\alpha} S(r)\mathcal{D}\|^2_{L_2(U_2;H)} dr$$

But

$$\|(-A)^{1+\alpha} S(r)\mathcal{D}\|^2_{L_2(U_2;H)} = \sum_{n=1}^{+\infty} \|\mathcal{D}^* S(r)(-A)^{1+\alpha} g_n\|^2_{U_2} = \sum_{n=1}^{+\infty} e^{-2\lambda_n r} \lambda_n^{2(1+\alpha)} \|\mathcal{D}^* g_n\|^2_{U_2}$$

Therefore $\int_0^t r^{-\gamma} \|AS(r)\mathcal{D}\|^2_{L_2(U_2;H^\alpha)} dr < +\infty$ holds true if and only if

$$\sum_{n=1}^{+\infty} \lambda_n^{2(1+\alpha)} \|\mathcal{D}^* g_n\|^2_{U_2} \int_0^t r^{-\gamma} e^{-2\lambda_n r} dr < +\infty.$$

The last inequality holds if and only if $\sum_{n=1}^{+\infty} \lambda_n^{2\alpha+\gamma+1} \|\mathcal{D}^* g_n\|^2_{U_2} < \infty$, hence we get the continuity of the process by equation (2.2.8) and the thesis. □

Remark 2.2.4. The last proposition is a sort of summability regularity of Fourier coefficients of our solutions. As in the deterministic case, the bigger is the summability order of the Fourier series the more regular is the solution.

The last result is really useful in the applications because it gives us an easy way to compute the regularity of the solution provided that it is possible to find an orthonormal basis of H made by eigenvectors of A

2.3. Nonlinear case

Now it is possible to move to the nonlinear case, assuming that the solution of the linear problem belongs to a set of proper regular function (distributional solution are not allowed in this framework, because composition of function and distribution is not well defined).

$$\begin{cases} \frac{\partial X}{\partial t}(t) = A_0 X(t) + F(X(t)) + B\dot{W}^1(t) \quad t \in [0,T] \\ \gamma X(t) = \dot{W}^2(t) \quad t \in]0,T] \\ X(0) = x \end{cases} \tag{2.3.1}$$

To solve this problem we need to introduce several Hilbert space and formulate assumptions. Let us start considering $J \subset K \subset H$ Hilbert spaces and $F : K \to H$ a mapping such that

Hypothesis L.1

1. $F - \omega I$ is m-dissipative[1] on H for some $\omega \in \mathbb{R}$.
2. F maps J into K and the restriction of $F - \omega I$ to J is dissipative in K.
3. F transforms bounded sets in J into bounded sets in K and bounded sets in K into bounded sets in H.

A intermediate function $y(t)$, mild solution of a homogeneous problem, namely

$$y'(t) = Ay(t) + F(y(t) + \psi(t)) \quad y(0) = x \tag{2.3.2}$$

where $\psi \in C([0,T]; J)$, or equivalently

$$y(t) = S(t)x + \int_0^t S(t-s)F(y(s) + \psi(s))ds \quad t \in [0,T].$$

Then, if this solution y exists, then X solution of equation (2.3.1) can be written, thanks to linearity, as $X(t) = y(t) + Z^1(t) + Z^2(t) \;\; t \in [0,T]$, where $\psi(t)$ is exactly $Z^1(t) + Z^2(t)$. To guarantee that equation (2.3.2) has a proper solution, useful to solve problem (2.3.1) we need to introduce new assumptions on A, namely

Hypothesis L.2

[1] Let $(Y, \langle \cdot, \cdot \rangle_Y)$ be a Hilbert space. An unbounded linear operator $(A, D(A))$ on Y is m-dissipative if

- $\langle A(y), y \rangle_Y \leq 0 \quad \forall y \in D(A)$.
- $\forall f \in Y, \; \forall \lambda > 0, \quad \exists y \in D(A)$ such that $\lambda y - Ay = f$.

1. A generates a contraction semigroup $S(\cdot)$ on H.

2. The part of A in K generates a contraction semigroup $S(\cdot)$ on K.

Now we can state the following theorem and its obvious but crucial corollary.

Theorem 2.3.1. *Under hypothesis L.1 and L.2, for arbitrary $x \in K$ and $\psi \in C([0,T];J)$, equation (2.3.2) has a mild K-valued solution.*

Proof. the proof of this theorem can be found in [DZ93] and exploits F_α, i.e. the Yoshida approximations of F, then it is considered the approximated problem:

$$y'_\alpha(t) = Ay_\alpha(t) + F_\alpha(y_\alpha(t) + \psi(t)) \; y_\alpha(0) = x \tag{2.3.3}$$

Subsequently, it is shown that solution of problem (2.3.3) exists for each $\alpha > 0$ and $lim_{\alpha\to 0} y_\alpha(t), \; t \in [0,T]$ exists uniformly in $[0,T]$ and is the required solution of equation (2.3.2). □

Corollary 2.3.2. *Assume that conditions of theorem 2.3.1 are satisfied and that processes $Z^1(\cdot), Z^2(\cdot)$ have J-continuous versions, then the mild form of equation (2.3.3) has a K-valued and H-continuous solution.*

Remark 2.3.3. The continuity property in H holds also for $y(t)$ since an inspection of the proof of theorem 2.3.1 shows us that $y_\alpha(t) \to y(t)$ in H as $\alpha \to 0$, uniformly on $[0,T]$.

Chapter 3

The Heat Equation

There are several articles about the analysis of heat equation with white noise boundary conditions. To see some examples [BBT14],[Brz+15],[Sow94],[AB02b], [DZ93],[SV10]. In this chapter we will introduce some results related to one dimensional heat equation, showing the critical differences between Dirichlet and Neumann conditions. This is just a taste about the literature devoted to this topic, there is a lot of research about different notions of solutions, some ergodicity properties or control problems, see for example [DZ96],[AB02a],[Mas10],[FG09],[DFT07].

3.1. Neumann boundary conditions

Consider the problem

$$\begin{cases} \frac{\partial Z}{\partial t}(t,\xi) = \frac{\partial^2 Z}{\partial \xi^2}(t,\xi) \quad t \geq 0 \ \xi \in [0,\pi] \\ \frac{\partial Z}{\partial \xi}(t,0) = \dot{v_1}(t) \quad \frac{\partial Z}{\partial \xi}(t,\pi) = \dot{v_2}(t) \quad t \geq 0 \\ Z(0,\xi) = 0 \quad \xi \in [0,\pi] \end{cases} \tag{3.1.1}$$

where $v_1(t)$ and $v_2(t)$ are two independent real Wiener processes. In this case $H = L^2(0,\pi)$, $A = \frac{\partial^2}{\partial \xi^2}$, $D(A) = \{f \in H^2 : f'(0) = f'(\pi) = 0\}$, $U_2 = \mathbb{R}^2$. It is clear that A is self-adjoint, it generates an analytic semigroup on H (see for example [Lun12]) and there exists an orthonormal basis of H made by eigenvectors of A, namely $g_0(\xi) = \pi^{-\frac{1}{2}}$, $g_n(\xi) = (\frac{2}{\pi})^{\frac{1}{2}} \cos(n\xi)$, $\lambda_n = -n^2$, $n \in \mathbb{N}$. If we set $\lambda = 1$ then we can define the mapping $\mathcal{D} : \mathbb{R}^2 \to H$ introduced in section 2.2 as

$$\mathcal{D}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}(\xi) = \psi(\xi) = -\frac{cosh(\pi-\xi)}{sinh\pi}\alpha + \frac{\cosh \xi}{\sinh \pi}\beta,$$

so that $\psi - \frac{d^2\psi}{d\xi^2} = 0 \;\; \xi \in [0,\pi], \;\psi'(0) = \alpha, \;\psi'(\pi) = \beta$ and it is possible to find the adjoint operator of $\mathcal{D}$, $\mathcal{D}^* : H \to \mathbb{R}^2$ given by

$$\mathcal{D}^*(\psi) = \begin{pmatrix} \frac{-1}{sinh(\pi)} \int_0^\pi \cosh{(\pi - \xi)}\psi(\xi)d\xi \\ \frac{1}{sinh(\pi)} \int_0^\pi \cosh{(\xi)}\psi(\xi)d\xi \end{pmatrix}.$$

By easy computations it is possible to get:

$$|\mathcal{D}^* g_n|^2 = \frac{2}{\pi}(1 - cos(n\pi))^2(1+n^2)^{-2} \quad n \in \mathbb{N}_0.$$

Therefore it is possible to apply proposition 2.2.3 and we get that the inequality holds true if and only if $4\alpha + 2\gamma - 1 < 0$; we can formulate the following proposition about the regularity of the solution.

Proposition 3.1.1.

(i) The solution $Z(\cdot)$ of problem (3.1.1) is an H_α-valued process if and only if $\alpha < \frac{1}{4}$.

(ii) If $\alpha < \frac{1}{4}$, then $Z(\cdot)$ has an H_α continuous version.

(iii) For arbitrary $p > 1$ the process $Z(\cdot)$ has an $L^p(0,\pi)$-continuous version.

Proof. Part (i) and (ii) follow by previous computations and proposition 2.2.3. Part (iii) follows by part 2 and Sobolev embedding theorem.

□

Remark 3.1.2. Even if the argument was proven just in the case $\Gamma =]0,\pi[$, we can find similar result on the cube $]0,\pi[^n$, see [DZ96].

Remark 3.1.3. It is possible to show $Z(t)$, $t > 0$ is Markov and strong Feller on $H = L^2(0,\pi)$, see [DZ93].

Since the solution to the linear problem belongs to a proper space of functions, it is possible to find a solution of a nonlinear equation where f is, for example, a function of polynomial growth, thanks to the result described in section 2.3.

Example 3.1.4.

$$\begin{cases} \frac{\partial X}{\partial t}(t,\xi) = \frac{\partial^2 X}{\partial \xi^2}(t,\xi) + f(X(t,\xi)) \;\; t \geq 0 \; \xi \in [0,\pi] \\ \frac{\partial X}{\partial \xi}(t,0) = \dot{v}_1(t) \;\; \frac{\partial X}{\partial \xi}(t,\pi) = \dot{v}_2(t) \;\; t \geq 0 \\ X(0,\xi) = x \;\; \xi \in [0,\pi] \end{cases} \tag{3.1.2}$$

where $f(x) = -x^{2n+1} + \sum_{j=0}^{2n} a_j x^j$. Let now consider $K = L^{2(2n+1)}(0,\pi)$ and

$J = L^{2(2n+1)^2}(0,\pi)$. Then the mapping $F = f \circ x,\ x \in K$ satisfies the condition of theorem 2.3.1. Moreover

$$Z(t) = Z^1(t) + Z^2(t) = (I-A)\int_0^t S(t-s)\mathcal{D}\begin{pmatrix} dv_1(s) \\ dv_2(s) \end{pmatrix}.$$

By proposition 3.1.1 the process $Z(\cdot)$ has a J-continuous version. So, by corollary 2.3.2, equation (3.1.2) has a $L^2(0,\pi)$-continuous solution $X(\cdot)$.

Remark 3.1.5. Concerning the analysis of the heat equation with Neumann boundary conditions in a general Riemannian manifold see [Sow94]. In this section we just add a direct proof of the fact that the solution of the heat equation in the half-line with white noise Neumann boundary conditions is a proper L^p function.

Let us consider the following heat equation.

$$\begin{cases} \frac{\partial Z}{\partial t}(t,\xi) = \frac{\partial^2 Z}{\partial \xi^2}(t,\xi) \quad t \geq 0 \ \xi \in \mathbb{R}^+ \\ \frac{\partial Z}{\partial \xi}(t,0) = \dot{v_1}(t) \quad t \geq 0 \\ Z(0,\xi) = 0 \quad \xi \in \mathbb{R}^+ \end{cases} \tag{3.1.3}$$

where $v_1(t)$ is a real Wiener process in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with natural filtration generated by $v_1(t)$. We call $L^p(\Omega \times [0,T]; L^p)$ as the space of p-integrable random process with values in $L^p := L^p(0,+\infty)$. For the solution of equation (3.1.3), the following proposition holds.

Proposition 3.1.6. *The solution of the stochastic differential equation (3.1.3) is a process in $L^p(\Omega \times [0,T]; L^p)$.*

Proof. Thanks to example A.1.11, we have that a formal solution of equation (3.1.3) is given by

$$Z(t,\xi) = -\frac{1}{\sqrt{\pi}}\int_0^t dv_1(s)\frac{e^{\frac{-\xi^2}{4(t-s)}}}{\sqrt{t-s}}.$$

Since the integrand process is deterministic, the thesis guarantees the well posedness of the process $Z(t,\xi)$. If we call $\Phi(s,t,\xi) = -\frac{e^{\frac{-\xi^2}{4(t-s)}}}{\sqrt{t-s}\sqrt{\pi}}$, then $Z(t,\xi) = \int_0^t \Phi(s,t,\xi)dv_1(s)$.

$$Z(t,\xi) \in L^p(\Omega \times [0,T]; L^p) \iff$$

$$\int_0^T dt\ \mathbb{E}\left[\|Z\|_{L^p}^p\right] < +\infty$$

Thanks to Fubini-Tonelli theorem and Burkholder-Davis-Gundy inequality E.3.1, we get that

$$\int_0^T dt\ \mathbb{E}\left[\|Z\|^p_{L^p}\right] \le c\int_0^T dt \int_0^{+\infty} d\xi \left(\int_0^t ds|\Phi(s,t,\xi)|^2\right)^{\frac{p}{2}} =$$

$$c\int_0^T dt\int_0^1 d\xi\left(\int_0^t ds|\Phi(s,t,\xi)|^2\right)^{\frac{p}{2}} + c\int_0^T dt\int_1^{+\infty} d\xi\left(\int_0^t ds|\Phi(s,t,\xi)|^2\right)^{\frac{p}{2}}.$$

For matter of simplicity, we are interested to the innermost integral $\int_0^t ds|\Phi(s,t,\xi)|^2 = \int_0^t ds\frac{e^{\frac{-\xi^2}{2(t-s)}}}{t-s} = \int_0^t ds\frac{e^{\frac{-\xi^2}{2s}}}{s}$. We can get this useful estimate $\forall q\ge 1$:

$$\frac{e^{\frac{-\xi^2}{2s}}}{s} = e^{\frac{-\xi^2}{2s}}s^{-\frac{1}{q}}s^{-1+\frac{1}{q}} \le s^{-1+\frac{1}{q}}\max_{s\in[0,+\infty]} e^{\frac{-\xi^2}{2s}}s^{-\frac{1}{q}}.$$

Calling $g(s) = e^{\frac{-\xi^2}{2s}}s^{-\frac{1}{q}}$, it is true that $\hat{s} = argmax_{s\in[0,+\infty]}g(s) = \frac{q\xi^2}{2}$ and

$$g(\hat{s}) = e^{-\frac{1}{q}}\left(\frac{q}{2}\right)^{-\frac{1}{q}}\frac{1}{\xi^{\frac{2}{q}}} = C(q)\frac{1}{\xi^{\frac{2}{q}}}.$$

In conclusion

$$\forall q\ge 1\quad \left(\int_0^t ds|\Phi(s,t,\xi)|^2\right)^{\frac{p}{2}} \le C(q)\left(\int_0^t ds\frac{s^{-1+\frac{1}{q}}}{\xi^{\frac{2}{q}}}\right)^{\frac{p}{2}} = \frac{C(q)}{\xi^{\frac{p}{q}}}t^{\frac{p}{2q}}.$$

In particular $\forall p\ge 2$ there exist $q_1, q_2 \ge 1$ such that $\frac{p}{q_1} < 1$ e $\frac{p}{q_2} > 1$ and

$$\int_0^T dt\ \mathbb{E}\left[\|Z\|^p_{L^p}\right] \le c\int_0^T dt\int_0^1 d\xi\left(\int_0^t ds|\Phi(s,\xi)|^2\right)^{\frac{p}{2}} + c\int_0^T dt\int_1^{+\infty} d\xi\left(\int_0^t ds|\Phi(s,\xi)|^2\right)^{\frac{p}{2}} \le$$

$$C(q_1)\int_0^T dt\int_0^1 d\xi\ \frac{t^{\frac{p}{2q_1}}}{\xi^{\frac{p}{q_1}}} + C(q_2)\int_0^T dt\int_1^{+\infty} d\xi\ \frac{t^{\frac{p}{2q_2}}}{\xi^{\frac{p}{q_2}}} < +\infty.$$

□

3.2. Dirichlet boundary conditions

Consider the problem

$$\begin{cases} \frac{\partial Z}{\partial t}(t,\xi) = \frac{\partial^2 Z}{\partial \xi^2}(t,\xi) \quad t\ge 0\ \xi\in[0,\pi] \\ Z(t,0) = \dot{v}_1(t) \quad Z(t,\pi) = \dot{v}_2(t) \quad t\ge 0 \\ Z(0,\xi) = 0 \quad \xi\in[0,\pi] \end{cases} \tag{3.2.1}$$

where $v_1(t)$ and $v_2(t)$ are two independent real Wiener processes. In this case $H = L^2(0,\pi)$, $A = \frac{\partial^2}{\partial \xi^2}$, $D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$, $U_2 = \mathbb{R}^2$. It is clear that A is self-adjoint, it generates an analytic semigroup on H (see for example [Lun12]) and there exists an orthonormal basis of H made by eigenvectors of A, namely $g_n(\xi) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(n\xi)$, $\lambda_n = -n^2$, $n \in \mathbb{N}_0$. If we set $\lambda = 0$ then we can define the mapping $\mathcal{D} : \mathbb{R}^2 \to H$ introduced in section 2.2 as

$$\mathcal{D}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(\xi) = \psi(\xi) = \frac{\pi-\xi}{\pi}\alpha + \frac{\xi}{\pi}\beta,$$

so that $\frac{d^2\psi}{d\xi^2} = 0 \quad \xi \in [0,\pi]$, $\psi(0) = \alpha$, $\psi(\pi) = \beta$ and it is possible to find the adjoint operator of $\mathcal{D}$, $\mathcal{D}^* : H \to \mathbb{R}^2$ given by

$$\mathcal{D}^*(\psi) = \begin{pmatrix}\frac{1}{\pi}\int_0^\pi (\pi-\xi)\psi(\xi)d\xi\\ \frac{1}{\pi}\int_0^\pi \xi\psi(\xi)d\xi\end{pmatrix}.$$

By easy computations it is possible to get

$$|\mathcal{D}^* g_n|^2 = \frac{4}{\pi n^2} \quad n \in \mathbb{N}_0.$$

Therefore it is possible to apply proposition 2.2.3 and we get that the inequality holds true if and only if $4\alpha + 2\gamma + 1 < 0$; we can formulate the following proposition about the regularity of the solution.

Proposition 3.2.1.

(i) The solution $Z(\cdot)$ of problem (3.2.1) is an H_α-valued process if and only if $\alpha < -\frac{1}{4}$

(ii) If $\alpha < -\frac{1}{4}$, then $Z(\cdot)$ has an H_α continuous version

Proof. Part (i) and (ii) follow by previous computations and proposition 2.2.3. □

Remark 3.2.2. Note that problem (3.2.1) does not have a solution in the original space $H = H_0 = L^2(0,\pi)$.

Remark 3.2.3. Since the solution of problem (3.2.1), without deeper analysis made in the following, belongs to a space of distribution it is not possible to continue with the non-linear case as made in the previous section.

Remark 3.2.4. Note that this analysis reflects what is known in the deterministic case, namely the lesser regularity I have on the boundary, the lesser regularity I have on the solution. In fact the Wiener process is never differentiable, very irregular. Consequently, a solution of the heat equation with the white noise on the boundary cannot be very regular. Using Neumann boundary conditions we do not impose an irregular trace for the function, so a better regularity of the solution is recovered.

3.3. The heat equation in weighted spaces

In [AB02b] a crucial step in the analysis of heat equation with Dirichlet boundary conditions is done. Equation (3.2.1) is generalized, considering it in the half-line with several sources of randomness. In particular, the authors analyzed the problem

$$\begin{cases} \frac{\partial Z}{\partial t}(t,\xi) = \frac{\partial^2 Z}{\partial \xi^2}(t,\xi) + \sum_{j=1}^n [b_j(\xi)\frac{\partial Z}{\partial \xi}(t,\xi) + F_j(t,\xi,Z(t,\xi))]\dot{W}^j(t) \\ Z(t,0) = \dot{V}(t) \\ Z(0,\xi) = 0 \qquad (t,\xi) \in I^T := [0,T] \times \mathbb{R}^+ \end{cases} \tag{3.3.1}$$

Where $W(t) = (W^1, ..., W^n)(t)$ is a real standard n-dimensional Wiener process in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $V(t)$ is real standard Wiener process adapted to the filtration generated by $W(t)$. Object of this section is trying to find a weighted space where the solution of this equation can be properly defined, then studying its continuity and blow up properties. To be more precise let L^p_γ the space of the real-valued measurable functions such that

$$\int_0^{+\infty} |f(x)|^p (1 \wedge x^{p-1+\gamma}) dx < +\infty.$$

We define as $L^p(\Omega \times [0,T]; L^p_\gamma)$ the set of p-integrable random processes with value in L^p_γ and as $M^p(\Omega \times [0,T]; L^p_\gamma)$ the subspace of $L^p(\Omega \times [0,T]; L^p_\gamma)$ adapted to the filtration generated by $W(t)$. We introduce the following hypotheses on functions b_j and F_j.

Hypothesis H.1

1. The drift coefficients $b_j : \mathbb{R}^+ \to \mathbb{R}$ belongs to C^3_b and they satisfy the joint ellipticity condition
$$\Sigma(x) = 1 - \frac{1}{2}\sum_{j=1}^n b_j^2(x) \geq \epsilon > 0.$$
2. The nonlinear terms $F_j(t,\xi,u)$ are uniformly Lipschitz in u, namely $\exists L > 0$ such that for all $t \in [0,T],\ \xi \in \mathbb{R}$ and $j = 1, ..., n$
$$|F_j(t,\xi,u) - F_j(t,\xi,v)| \leq L|u - v|.$$

We also assume that condition $(F)_{p,\theta}$ holds, namely exists $p \geq 2,\ 0 < \theta < 1$ and $h(x) \in L^p_\theta$ such that for all $t \in [0,T]$ and $j = 1, ..., n$

$$|F_j(t,\xi,0)| \leq h(\xi) \quad \forall \xi \in \mathbb{R}^+.$$

Under these assumptions we can state the following theorem proved in subsection 3.3.3.

Theorem 3.3.1. *Assume that condition $(F)_{p,\theta}$ holds for some $p \geq 2,\ 0 < \theta < 1$. Then equation (3.3.1) has a unique solution $Z(t,\xi) \in M^p(\Omega \times [0,T]; L^p_\gamma)$ for any $\gamma \in (0,1)$. If moreover condition $(F)_{p,\theta}$ holds for some $p > 2,\ \theta \in (0,1)$, then the function $Z(t,\cdot)$ is continuous on $[\delta, +\infty)$ for every $\delta > 0$ and it satisfies that*

$$\xi^{1+\alpha} Z(t,\xi) \to 0 \quad a.s.$$

for every $\alpha > 0$.

All the details omitted in this section can be found in [AB02b] where another notion of solution that can be well defined for the process $Z(t,\xi)$ is also presented. Concerning the ergodicity properties of $Z(t,\xi)$ some results can be found in [AB02a].

3.3.1. The stochastic heat kernel

In this subsection we recall the definition and the basic properties of the stochastic kernel related with equation (3.3.1). This construction follows that in [NV00].
Let $B = \{B_t,\ t \in [0,T]\}$ be a Brownian motion with variance $2t$ defined in an additional probability space $(\mathcal{W}, \mathcal{G}, \mathbb{Q})$. Consider the following backward stochastic differential equation on the probability space $(\Omega \times \mathcal{W}, \mathcal{F} \times \mathcal{G}, \mathbb{P} \otimes \mathbb{Q})$:

$$\varphi_{t,s}(\xi) = \xi - \sum_{j=1}^{n} \int_s^t b_j(\varphi_{t,r}(\xi)) dW^j(r) + \int_s^t \sqrt{\Sigma(\varphi_{t,r}(\xi))} dB(r) \tag{3.3.2}$$

It can be proven, thanks to results present in [Kun97], that this equation has a solution $\varphi = \{\varphi_{t,s}(\xi), 0 \leq s \leq t \leq T, \xi \in \mathbb{R}\}$ continuous in all the three variables and verifying $\varphi_{r,s}(\varphi_{t,r}(\xi)) = \varphi_{t,s}(\xi)$ for all $s < r < t \ \ \xi \in \mathbb{R}$. The existence of the kernel for the operator $\frac{\partial^2}{\partial \xi^2} + \sum_{j=1}^n b_j(\xi) \frac{\partial}{\partial \xi} \dot{W}^j(t)$ was proven in [Nua06] .

Proposition 3.3.2. *Let $\varphi_{t,s}$ defined as in equation (3.3.2). Then there is a version of the marginal density $p(s,t,\upsilon,\xi) = \frac{Q[\varphi_{t,s}(\xi) \in d\upsilon]}{d\upsilon}$ which is $\mathcal{F}^t_s$-adapted and satisfies the semigroup property*

$$p(s,t,\upsilon,\xi) = \int_{\mathbb{R}} p(s,r,\upsilon,\zeta) p(r,t,\zeta,\xi) d\zeta$$

for all $\xi, \upsilon \in \mathbb{R}$ and $0 \leq s < r < t \leq T$.

Remark 3.3.3. A crucial side result of the proof of this proposition is that if we call the stochastic heat kernel as $p_D(s,t,\upsilon,\xi) = p(s,t,\upsilon,\xi) - p(s,t,-\upsilon,\xi)$, we have an implicit representation of $p_D(s,t,\upsilon,\xi)$:

$$p_D(s,t,\upsilon,\xi) = q_D(s,t,\upsilon,\xi) + \sum_{j=1}^{n} \int_s^t \left(\int_{\mathbb{R}^+} b_j(\zeta) q_D(r,t,\zeta,\xi) \frac{\partial p_D}{\partial z}(s,r,\upsilon,\zeta) d\zeta \right) dW^j_r$$

where $q_D(s,t,\upsilon,\xi)$ is the heat kernel on $\mathbb{R}^+$ with zero Dirichlet boundary conditions.

To prove theorem 3.3.1 we need some estimates on the L^p norms of p_D and its derivatives. They are given by the following proposition.

Proposition 3.3.4. *For all $s<t,\ \xi,\upsilon\in\mathbb{R}^+$ it holds that*

$$\|p_D(s,t,\upsilon,\xi)\|_{L^p(\Omega)} \leq C(t-s)^{-1/2} exp\left(-\frac{|\upsilon-\xi|^2}{c(t-s)}\right) \tag{3.3.3}$$

$$\|p_D(s,t,\upsilon,\xi)\|_{L^p(\Omega)} \leq C\upsilon^a (t-s)^{-1/2-a/2} exp\left(-\frac{|\upsilon-\xi|^2}{c(t-s)}\right) \tag{3.3.4}$$

$$\|\frac{\partial^{m+k} p_D}{\partial\upsilon^k\partial\xi^m}(s,t,\upsilon,\xi)\|_{L^p(\Omega)} \leq C(t-s)^{-(m+k+1)/2} exp\left(-\frac{|\upsilon-\xi|^2}{c(t-s)}\right) \tag{3.3.5}$$

for each $m=0,1,2,\ k=0,1,\ 0\leq a\leq 1$ and for some constant $C,c>0$.

3.3.2. The boundary term

All the computations presented in this subsection prove theorem 3.3.1 in the easier case $F_j\equiv 0$.
Let us consider now

$$\psi(t,\xi) := \int_0^t \frac{\partial p_D}{\partial\upsilon}(s,t,0,\xi)dV(s).$$

It is, at least formally, the mild solution of problem (3.3.1) when $F_j\equiv 0\ \forall j\in\{1,\cdots,n\}$. Now we want to prove that the thesis of theorem 3.3.1 holds in this easier case. In this way we may hope that it could be true in the general case. It is easy to prove that $\psi\in M^p(\Omega\times[0,T];L^p_\gamma)$ for every $p\geq 2,\ 0<\gamma<1$. In fact, we have using inequality (3.3.5) and Burkholder–Davis–Gundy Inequality E.3.1

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t \frac{\partial p_D}{\partial\upsilon}(s,t,0,\xi)\ dV(s)\right|^p\right] &\leq C\left|\int_0^t (t-s)^{-2} exp\left(-\frac{\xi^2}{c(t-s)}\right)\right|^{\frac{p}{2}} \\ &= C\left|\frac{1}{\xi^2}e^{-\xi^2/t}\right|^{\frac{p}{2}} = C\xi^{-p}e^{-\frac{p\xi^2}{2t}}.\end{aligned} \tag{3.3.6}$$

Hence, by Fubini-Tonelli theorem and the above inequality,

$$\begin{aligned}\|\psi\|_{M^p} &= \int_0^T dt\ \mathbb{E}\left[\int_0^{+\infty} d\xi(1\wedge\xi^{p-1+\gamma})\left|\int_0^t \frac{\partial p_D}{\partial\upsilon}(s,t,0,\xi)\ dV(s)\right|^p\right] \\ &\leq C\int_0^T dt\int_0^{+\infty} d\xi(1\wedge\xi^{p-1+\gamma})\xi^{-p}e^{-\frac{p\xi^2}{2t}} < +\infty.\end{aligned}$$

Lemma 3.3.5. *For any $\alpha > 0$ the following convergence holds for any $t \in [0, T]$, almost surely*

$$\xi^{1+\alpha}\psi(t, \xi) \to 0 \text{ as } \xi \to 0.$$

Proof. It is enough to prove that for some $\alpha' < \alpha$ the function $\xi^{1+\alpha'}\psi(t, \xi)$ is a.s. bounded in a neighborhood of 0, so a fortiori if we prove that exists a version of $\xi^{1+\alpha'}\psi(t, \xi)$ almost surely Hölder continuous in an interval $[0, K]$, $K > 0$ we get the thesis. By Kolmogorov's continuity theorem E.1.1 we are left to prove that $\forall p \geq 2$ and $\xi, \zeta \in (0, K), \xi < \zeta$ the following estimate holds:

$$\mathbb{E}[\left|\zeta^{1+\alpha}\psi(t, \zeta) - \xi^{1+\alpha}\psi(t, \xi)\right|^p] \leq C|\zeta - \xi|^{p\alpha}. \tag{3.3.7}$$

Adding and subtracting $\mathbb{E}[|\xi^{1+\alpha}\psi(t, \zeta)|^p]$ and exploiting convexity

$$\mathbb{E}[\left|\zeta^{1+\alpha}\psi(t, \zeta) - \xi^{1+\alpha}\psi(t, \xi)\right|^p] \leq A_1 + A_2,$$

where

$$A_1 = \xi^{p(1+\alpha)}\mathbb{E}[|\psi(t, \zeta) - \psi(t, \xi)|^p]$$
$$A_2 = (\zeta^{1+\alpha} - \xi^{1+\alpha})^p\mathbb{E}[|\psi(t, \zeta)|^p].$$

By inequality (3.3.6) we have proved that $\mathbb{E}[|\psi(t, \zeta)|^p] \leq C\zeta^{-p}$ so that

$$A_2 \leq C|\zeta - \xi|^{p\alpha}.$$

On the other hand, by Hölder and Burkholder–Davis–Gundy inequalities we can estimate the difference

$$\mathbb{E}\left[|\psi(t, \zeta) - \psi(t, \xi)|^p\right] \leq C\mathbb{E}\left[\left|\int_0^t (\frac{\partial p_D}{\partial \upsilon}(s, t, 0, \zeta) - \frac{\partial p_D}{\partial \upsilon}(s, t, 0, \xi))^2 \, ds\right|^{\frac{p}{2}}\right]$$
$$\leq C\left|\int_0^t \|\frac{\partial p_D}{\partial \upsilon}(s, t, 0, \zeta) - \frac{\partial p_D}{\partial \upsilon}(s, t, 0, \xi)\|^2_{L^p(\Omega)} ds\right|.$$

In the end by inequality (3.3.3) and (3.3.5) we can deduce (recalling that $\xi < \zeta$)

$$\mathbb{E}[|\psi(t, \zeta) - \psi(t, \xi)|^p] \leq C|\zeta - \xi|^{p\alpha}\left|\int_0^t (t - s)^{-2-\alpha} exp\left(-\frac{\xi^2}{c(t - s)}\right) ds\right|^{\frac{p}{2}} \leq$$
$$C|\zeta - \xi|^{p\alpha}\xi^{-p(1+\alpha)}.$$

Combining all these inequalities we get that inequality (3.3.7) holds, so Kolmogorov's continuity theorem can be applied and the lemma is proved. □

Remark 3.3.6. As a side result of this proof, we get that $\xi \to \psi(t, \xi)$ is continuous on $(0, +\infty)$ and uniformly Hölder continuous on $(\delta, +\infty)$. This result is really interesting. In particular, it explains how the solution is regular within the half-line and the degree of growth close to the boundary can be estimated.

3.3.3. Mild Solution

The proof of the general case follows by a classical fixed point argument. Two technical lemmas are needed to get the results. They are stated without proofs. Some basis of Malliavin calculus are required to completely understand them (see for example [Bel12],[KS84],[KS85],[Kus87],[Nua06]).

Lemma 3.3.7. *The application Λ defined by*

$$(\Lambda\Phi)(t,\xi) = \sum_{j=1}^{n} \int_0^t \left(\int_{\mathbb{R}^+} p_D(s,t,\upsilon,\xi)\Phi(s,\upsilon)d\upsilon \right) dW^j(s) \tag{3.3.8}$$

is a contraction from $M^p(\Omega\times[0,T];L^p_\gamma)$ to $M^p(\Omega\times[0,T];L^p(\mathbb{R}^+))$, for all $0<\gamma<1$ and $p\geq 2$.

Lemma 3.3.8. *Assume that $\Phi(t,\xi)\in M^p(\Omega\times[0,T];L^p_\gamma)$ for some $p>2$ and $0<\gamma<1$. Then for any $t\in[0,T]$ and $a>\frac{1+\gamma}{p}$ the mapping*

$$\xi \to \xi^a(\Lambda\Phi)(t,\xi)$$

is almost surely uniformly Hölder continuous on $[0,K]$, for all $K>0$, of order ρ, where

$$\rho < \frac{1}{2}(a-\frac{1+\gamma}{p}) \wedge (\frac{1}{2}-\frac{1}{p}).$$

Remark 3.3.9. In the proofs of this theorem is fully exploited the fact that $\gamma<1$, so it can be understood why in theorem 3.3.1 that hypothesis, never used in previous computations, is required.

Proof of theorem 3.3.1. Let us introduce the transformation $\mathcal{K}$ as

$$(\mathcal{K}\phi)(t,\xi) = \psi(t,\xi) + \sum_{j=1}^{n}\int_0^t\int_{\mathbb{R}^+} p_D(s,t,\upsilon,\xi)F_j(s,\upsilon,\phi(s,\upsilon))d\upsilon\; dW^j(s).$$

For any $\lambda\in(0,1)$ the application $\mathcal{K}$ maps $M^p(\Omega\times[0,T];L^p_\lambda)$ into itself. The boundary term belongs to $M^p(\Omega\times[0,T];L^p_\gamma)$ as it is shown in the previous subsection. Thanks to lemma 3.3.8 it remains to prove that for any $\phi\in M^p(\Omega\times[0,T];L^p_\lambda)$ the process $\Phi(t,\xi)=F(s,\upsilon,\phi(s,\upsilon))$ belongs to $M^p(\Omega\times[0,T];L^p_\lambda)$. In fact, adding and subtracting $F_j(s,\upsilon,0)$, exploiting hypothesis H.1 and condition $(F)_{p,\theta}$, we have

$$\mathbb{E}\left[\sum_{j=1}^{n}\int_0^T\int_{\mathbb{R}^+} |F_j(s,\upsilon,\phi(s,\upsilon))|^p(1\wedge\upsilon^{p-1+\lambda})d\upsilon ds\right]$$

$$= \mathbb{E}\left[\sum_{j=1}^{n}\int_0^T\int_{\mathbb{R}^+}|F_j(s,\upsilon,\phi(s,\upsilon)) - F_j(s,\upsilon,0) + F_j(s,\upsilon,0)|^p(1\wedge \upsilon^{p-1+\lambda})d\upsilon ds\right]$$

$$\leq 2^{p-1}\mathbb{E}\left[\sum_{j=1}^{n}\int_0^T\int_{\mathbb{R}^+}(|F_j(s,\upsilon,\phi(s,\upsilon)) - F_j(s,\upsilon,0)|^p + |F_j(s,\upsilon,0)|^p)(1\wedge \upsilon^{p-1+\lambda})d\upsilon ds\right]$$

$$\leq C\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^+}(|\phi(s,\upsilon)|^p + |h(\upsilon)|^p)(1\wedge \upsilon^{p-1+\lambda})d\upsilon ds\right].$$

Then, by lemma 3.3.8, $\Lambda\Phi(s,\upsilon)$ belongs to $M^p(\Omega\times[0,T];L^p(\mathbb{R}^+))$ and $L^p(\mathbb{R}^+)\hookrightarrow L^p_\gamma$ for any $\gamma\in(0,1)$. So, taking $\lambda=\gamma$, the mapping $\mathcal{K}$ is a contraction. Then we want to prove the boundary regularity statement, namely

$$\xi^{1+\alpha}\psi(t,\xi) + \xi^{1+\alpha}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{R}^+}p_D(s,t,\upsilon,\xi)F_j(s,\upsilon,\phi(s,\upsilon))d\upsilon\ dW^j(s)\to 0 \quad a.s.$$

The first term tends to 0 thanks to lemma 3.3.5 the second one does the same by lemma 3.3.9. Concerning the regularity, again, everything follows easily from lemma 3.3.5 and lemma 3.3.9. □

Remark 3.3.10. The solution $Z(t,\xi)$ of equation (3.3.1) is the solution of the fixed point problem

$$Z(t,\xi) = \int_0^t\frac{\partial p_D}{\partial\upsilon}(s,t,0,\xi)dV(s)+\sum_{j=1}^{n}\int_0^t\left(\int_{\mathbb{R}^+}p_D(s,t,\upsilon,\xi)F_j(s,\upsilon,Z(s,\upsilon))d\upsilon\right)dW^j(s).$$

Chapter 4

A Fourth Order Parabolic Problem

The aim of this chapter is trying to extend what was done in chapter 3 to a fourth order problem, a sort of fourth order heat equation. The literature about stochastic parabolic equations of order $2m$ is not so large. Some very general and not so easily applicable sufficient conditions for the L^2 solvability of this kind of problems can be found in [Mas95]. Although this really general theorem cannot be applied, a more direct approach will lead us to some results similar to the ones of section 3.3.

4.1. Equation in a bounded domain

Let us start considering the problem

$$\begin{cases} \frac{\partial Z}{\partial t}(t,\xi) + \frac{\partial^4 Z}{\partial \xi^4}(t,\xi) = 0 \quad t \geq 0 \ \xi \in [0,\pi] \\ Z(t,0) = \dot{v_1}(t) \quad Z(t,\pi) = \dot{v_2}(t) \quad \frac{\partial^2 Z}{\partial \xi^2}(t,0) = \dot{v_3}(t) \quad \frac{\partial^2 Z}{\partial \xi^2}(t,\pi) = \dot{v_4}(t) \quad t \geq 0 \\ Z(0,\xi) = 0 \quad \xi \in [0,\pi] \end{cases} \tag{4.1.1}$$

where $v_1(t)$, $v_2(t)$, $v_3(t)$ and $v_4(t)$ are four independent real Wiener processes. This kind of boundary conditions are called in literature as *Navier boundary conditions* and they will be called with this name in the following. In this case $H = L^2(0,\pi)$, $A = -\frac{\partial^4}{\partial \xi^4}$, $D(A) = \{f \in H^4 : f(0) = f(\pi) = f''(0) = f''(\pi) = 0\}$, $U_2 = \mathbb{R}^4$. It is clear that A is self-adjoint, it generates an analytic semigroup on H (see for example [Lun12]) and there exists an orthonormal basis of H made by eigenvectors of A, namely
$g_n(\xi) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(n\xi)$, $\lambda_n = -n^4$, $n \in \mathbb{N}$. If we set $\lambda = 0$ then we can define the

mapping $\mathcal{D} : \mathbb{R}^4 \to H$ introduced in section 2.2 as

$$\mathcal{D}\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}(\xi) = \psi(\xi) = \frac{\delta-\gamma}{6\pi}\xi^3 + \frac{\gamma}{2}x^2 + (\frac{\beta-\alpha}{\pi} - \frac{\pi}{6}\delta - \frac{\pi}{3}\gamma)\xi + \alpha,$$

so that $-\frac{d^4\psi}{d\xi^4} = 0 \quad \xi \in [0,\pi],\ \psi(0) = \alpha,\ \psi(\pi) = \beta\ \psi''(0) = \gamma,\ \psi''(\pi) = \delta$ and it is possible to find the adjoint operator of $\mathcal{D}$, $\mathcal{D}^* : H \to \mathbb{R}^4$ given by

$$\mathcal{D}^*(\psi) = (\int_0^\pi (1-\frac{\xi}{\pi})\psi(\xi)d\xi; \int_0^\pi \frac{\xi}{\pi}\psi(\xi)d\xi; \int_0^\pi (\frac{-\xi^3}{6\pi} + \frac{\xi^2}{2} - \frac{\pi\xi}{3})\psi(\xi)d\xi; \int_0^\pi (\frac{\xi^3}{6\pi} - \frac{\pi\xi}{6})\psi(\xi)d\xi)'.$$

By easy computations it is possible to get

$$\mathcal{D}^*(g_n) = (\frac{\sqrt{2}}{\sqrt{\pi}n}; \frac{\sqrt{2}(-1)^{n+1}}{\sqrt{\pi}n}; \frac{-\sqrt{2}}{\sqrt{\pi}n^3}; \frac{(-1)^n}{\sqrt{\pi}n^3})', \quad |\mathcal{D}^* g_n|^2 \approx \frac{1}{n^2} \quad n \in \mathbb{N}_0.$$

Therefore, it is possible to apply proposition 2.2.3 and we get that the inequality holds true if and only if $8\alpha+4\gamma+3 < 0$; we can formulate the following proposition about the regularity of the solution.

Proposition 4.1.1.

(i) The solution $Z(\cdot)$ of problem (4.1.1) is an H_α-valued process if and only if $\alpha < -\frac{3}{8}$

(ii) If $\alpha < -\frac{3}{8}$, then $Z(\cdot)$ has an H_α continuous version

Proof. Both parts follow from previous computations and proposition 2.2.3. □

Remark 4.1.2. This result is in a certain sense reasonable and predictable since, as in the second order case, we have little regularity on the trace of the solution, so we cannot expect that the solution can be a square integrable function. More surprising is the fact that, with a fourth order problem, not even considering some noises on the boundary conditions of the first and third order derivatives is enough to get the $L^2(0,\pi)$ regularity, as the following computations will show.

So, let us consider the problem

$$\begin{cases} \frac{\partial Z}{\partial t}(t,\xi) + \frac{\partial^4 Z}{\partial \xi^4}(t,\xi) = 0 \quad t \geq 0\ \xi \in [0,\pi] \\ \frac{\partial Z}{\partial \xi}(t,0) = \dot{v}_1(t) \quad \frac{\partial Z}{\partial \xi}(t,\pi) = \dot{v}_2(t) \quad \frac{\partial^3 Z}{\partial \xi^3}(t,0) = \dot{v}_3(t) \quad \frac{\partial^3 Z}{\partial \xi^3}(t,\pi) = \dot{v}_4(t) \quad t \geq 0 \\ Z(0,\xi) = 0 \quad \xi \in [0,\pi] \end{cases} \tag{4.1.2}$$

where $v_1(t)$, $v_2(t)$, $v_3(t)$ and $v_4(t)$ are again four independent real Wiener processes. Again, $H = L^2(0,\pi)$ and $A = -\frac{\partial^4}{\partial\xi^4}$, but now $D(A) = \{f \in H^4 : f'(0) = f'(\pi) = f'''(0) = f'''(\pi) = 0\}$, $U_2 = \mathbb{R}^4$. It is clear that A is self-adjoint, it generates an analytic semigroup on H (see for example [Lun12]) and there exists an orthonormal basis of H made by eigenvectors of A, namely $g_0(\xi) = \pi^{-\frac{1}{2}}$, $g_n(\xi) = (\frac{2}{\pi})^{\frac{1}{2}}\cos(n\xi)$, $\lambda_n = -n^4$, $n \in \mathbb{N}$. If we set $\lambda = 4$, then we can define the mapping $\mathcal{D} : \mathbb{R}^4 \to H$ introduced in section2.2 as

$$\mathcal{D}\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}(\xi) = \psi(\xi) = a\cos(\xi)e^{\xi} + b\sin(\xi)e^{\xi} + c\cos(\xi)e^{-\xi} + d\sin(\xi)e^{-\xi}$$

where $\begin{cases} a = -\frac{2\alpha-\gamma+2\beta e^{\pi}-\delta e^{\pi}}{4(e^{2\pi}-1)} \\ b = -\frac{2\alpha+\gamma+2\beta e^{\pi}+\delta e^{\pi}}{4(e^{2\pi}-1)} \\ c = -e^{\pi}\frac{2\beta-\delta+2\alpha e^{\pi}-\gamma e^{\pi}}{4(e^{2\pi}-1)} \\ d = e^{\pi}\frac{2\beta+\delta+2\alpha e^{\pi}+\gamma e^{\pi}}{4(e^{2\pi}-1)}. \end{cases}$

In this way $-\frac{d^4\psi}{d\xi^4} - \lambda\psi = 0 \ \ \xi \in [0,\pi],\ \psi'(0) = \alpha,\ \psi'(\pi) = \beta\ \psi'''(0) = \gamma,\ \psi'''(\pi) = \delta$ and it is possible to find the adjoint operator of $\mathcal{D}$, $\mathcal{D}^* : H \to \mathbb{R}^4$ given by

$$\mathcal{D}^*(\psi) = \begin{pmatrix} \int_0^{\pi} \frac{-2\cos(\xi)e^{\xi}-2\sin(\xi)e^{\xi}+2e^{2\pi}\cos(\xi)e^{-\xi}+2e^{2\pi}\sin(\xi)e^{-\xi}}{4(e^{2\pi}-1)}\psi(\xi)d\xi \\ \int_0^{\pi} e^{\pi}\frac{-2\cos(\xi)e^{\xi}-2\sin(\xi)e^{\xi}+2\cos(\xi)e^{-\xi}+2\sin(\xi)e^{-\xi}}{4(e^{2\pi}-1)}\psi(\xi)d\xi \\ \int_0^{\pi} \frac{\cos(\xi)e^{\xi}-\sin(\xi)e^{\xi}-e^{2\pi}\cos(\xi)e^{-\xi}+e^{2\pi}\sin(\xi)e^{-\xi}}{4(e^{2\pi}-1)}\psi(\xi)d\xi \\ \int_0^{\pi} e^{\pi}\frac{\cos(\xi)e^{\xi}-\sin(\xi)e^{\xi}-\cos(\xi)e^{-\xi}+\sin(\xi)e^{-\xi}}{4(e^{2\pi}-1)}\psi(\xi)d\xi \end{pmatrix}.$$

Substituting g_n in the above formula we obtain $|\mathcal{D}^* g_n|^2 \approx \frac{1}{n^4}\ \ n \in \mathbb{N}_0$. Therefore it is possible to apply proposition 2.2.3 and we get that the inequality holds true if and only if $8\alpha + 4\gamma + 1 < 0$; we can formulate the following proposition about the regularity of the solution.

Proposition 4.1.3.

(i) The solution $Z(\cdot)$ of problem (4.1.2) is an H_α-valued process if and only if $\alpha < -\frac{1}{8}$

(ii) If $\alpha < -\frac{1}{8}$, then $Z(\cdot)$ has an H_α continuous version

Proof. Both parts follow from previous computations and proposition 2.2.3.

□

Remark 4.1.4. Boundary conditions on higher derivatives allow us to gain more regularity. Opposite to the second order case, this time we do not reach enough regularity to get a proper $L^2(0,\pi)$ function. For both cases described in this subsection, we expect that this would be due to the blow up of the solution near to the boundary of the domain.

4.2. Equation in the half-line

Since for both the problems analyzed in the previous subsection we did not find a standard $L^2(0,\pi)$ solution, we move as in section 3.3 trying to follow the approach of [AB02b]. In particular we start considering the following fourth order problem with Navier boundary conditions.

$$\begin{cases} \frac{\partial Z}{\partial t} + \frac{\partial^4 Z}{\partial \xi^4} = 0 \\ Z(t,0) = \dot{v_1}(t) \ \frac{\partial^2 Z}{\partial \xi^2}(t,0) = \dot{v_2}(t) \\ Z(0,\xi) = 0 \quad \xi \in \mathbb{R}^+ \ t \in [0,T] \end{cases} \tag{4.2.1}$$

where obviously $v_1(t)$ and $v_2(t)$ are two independent real Wiener processes in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with natural filtration generated by $v_1(t)$ and $v_2(t)$. As is done in section 3.3, let $\gamma(\xi)$ be an almost everywhere positive measurable function. For $p \geq 2$ we introduce the vector spaces of measurable, real valued functions such that $\int_0^{+\infty} |f(x)|^p \gamma(x) dx < +\infty$. If we consider the norm

$$\|f\| = \left(\int_0^{+\infty} |f(x)|^p \gamma(x) dx \right)^{\frac{1}{p}}$$

these spaces are Banach spaces (obviously considering the equivalence classes of functions) and we call them L^p_γ. In a natural way we introduce the spaces $L^p(\Omega \times [0,T]; L^p_\gamma)$ as the space of $p-$integrable random processes adapted to the filtration generated by $v_1(t)$ and $v_2(t)$, with values in L^p_γ above defined. Then we can state and prove the following proposition about the summability of the solution of equation (4.2.1).

Proposition 4.2.1. *The solution of the stochastic differential equation (4.2.1) is a process in $L^p(\Omega \times [0,T]; L^p_\gamma)$, taking $\gamma(\xi) = 1 \wedge \xi^{3p-1+\delta}$, for $\delta \in (0,1)$.*

Proof. Thanks to the representation formula in example A.1.8, we have that a formal solution of equation (4.2.1) is given by

$$Z(t,\xi) = \frac{2}{\pi} \left[\int_0^t \int_0^{+\infty} w^3 sin(w\xi) e^{-w^4(t-s)} \, dw \, dV_1(s) - \int_0^t \int_0^{+\infty} w sin(w\xi) e^{-w^4(t-s)} \, dw \, dV_2(s) \right]$$

Since the integrand processes are deterministic, then the thesis guarantees the well posedness of the process $Z(t,\xi)$. So, if we call

$$\Psi_1(t,\xi) = \int_0^t \int_0^{+\infty} w^3 sin(w\xi) e^{-w^4(t-s)}\, dw\, dV_1(s)$$

and

$$\Psi_2(t,\xi) = \int_0^t \int_0^{+\infty} w sin(w\xi) e^{-w^4(t-s)}\, dw\, dV_2(s)],$$

it is enough to prove that for $i \in \{1,2\}$ $\int_0^T dt \mathbb{E}[\int_0^{+\infty} d\xi\ \gamma(\xi)|\Psi_i|^p(t,\xi)] < +\infty$. Let us start considering $\Psi_1(t,\xi)$. Thanks to Fubini-Tonelli theorem and Burkholder–Davis–Gundy inequality E.3.1, we get

$$I = \int_0^T dt \int_0^{+\infty} d\xi\gamma(\xi) \left(\int_0^t \left(\int_0^{+\infty} w^3 sin(w\xi) e^{-w^4(t-s)}\, dw \right)^2 ds \right)^{\frac{p}{2}} < +\infty \Longrightarrow$$

$$\Psi_1(t,\xi) \in L^p(\Omega \times [0,T]; L^p_\gamma).$$

Now we introduce the following change of variables $\begin{cases} l = (t-s) \\ v = \sqrt[4]{l} w \end{cases}$ and we get

$$I = \int_0^T dt \int_0^{+\infty} d\xi\gamma(\xi) \left(\int_0^t \frac{1}{l^2} \left(\int_0^{+\infty} v^3 sin(v\frac{\xi}{\sqrt[4]{l}}) e^{-v^4}\, dv \right)^2 dl \right)^{\frac{p}{2}}.$$

To simplify the notation, we consider just the innermost integral and we call $\alpha = \frac{\xi}{\sqrt[4]{l}}$.

$$I_1 = \int_0^{+\infty} v^3 sin(v\alpha) e^{-v^4}\, dv.$$

All the following steps are integration by parts, where the sinusoidal term is the derivative term.

$$I_1 = \left[-\frac{cos(v\alpha)}{\alpha} v^3 e^{-v^4} \right]_0^{+\infty} + \int_0^{+\infty} \frac{cos(v\alpha)}{\alpha}(-v^2)e^{-v^4}(4v^4-3)\, dv =$$

$$\left[\frac{sin(v\alpha)}{\alpha^2}(-v^2)e^{-v^4}(4v^4-3) \right]_0^{+\infty} - 2\int_0^{+\infty} \frac{sin(v\alpha)}{\alpha^2} v e^{-v^4}(8v^8 - 18v^4 + 3)\, dv =$$

$$\left[2\frac{cos(v\alpha)}{\alpha^3} v e^{-v^4}(8v^8 - 18v^4 + 3) \right]_0^{+\infty} + 4\int_0^{+\infty} \frac{cos(v\alpha)}{\alpha^3} e^{-v^4}(32v^{12} - 144v^8 + 102v^4 - 3)\, dv =$$

$$\frac{4}{\alpha^3}\int_0^{+\infty} cos(v\alpha) e^{-v^4}(32v^{12} - 144v^8 + 102v^4 - 3)\, dv \leq \frac{4}{\alpha^3}\int_0^{+\infty} e^{-v^4}(32v^{12} + 144v^8 + 102v^4 + 3)\, dv$$

$$\leq \frac{C}{\alpha^3}$$

Where C is a proper positive constant. The same estimates holds true also for $|I_1|$. Substituting the estimates of $|I_1|$ in the definition of I and taking $\gamma(\xi) = 1 \wedge \xi^{(3p-1)+\delta}$

$$I \leq C^p \int_0^T dt \int_0^{+\infty} d\xi \gamma(\xi) \left(\int_0^t \frac{1}{\sqrt{s}\xi^6} \, ds \right)^{\frac{p}{2}} = 2^{\frac{p}{2}} C^p \int_0^T dt \int_0^{+\infty} d\xi \gamma(\xi) \frac{\sqrt[4]{t^p}}{\xi^{3p}} < \infty.$$

The second summand is absolutely analogous to treat and easier to manage.

$$J = \int_0^T dt \int_0^{+\infty} d\xi \gamma(\xi) \left(\int_0^t \left(\int_0^{+\infty} w sin(w\xi) e^{-w^4(t-s)} \, dw \right)^2 ds \right)^{\frac{p}{2}} < +\infty \Longrightarrow$$

$$\Psi_2(t,\xi) \in L^p(\Omega \times [0,T]; L^p_\gamma).$$

Again introducing the change of variables $\begin{cases} l = (t-s) \\ v = \sqrt[4]{l} w \end{cases}$ we get

$$J = \int_0^T dt \int_0^{+\infty} d\xi \gamma(\xi) \left(\int_0^t \frac{1}{l} \left(\int_0^{+\infty} v sin(v \frac{\xi}{\sqrt[4]{l}}) e^{-v^4} \, dv \right)^2 dl \right)^{\frac{p}{2}}$$

Again for matter of simplicity, we take care the innermost integral and call $\alpha = \frac{\xi}{\sqrt[4]{l}}$.

$$J_1 = \int_0^{+\infty} v sin(v\alpha) e^{-v^4} \, dv = \left[-\frac{cos(v\alpha)}{\alpha} v e^{-v^4} \right]_0^{+\infty} + \int_0^{+\infty} \frac{cos(v\alpha)}{\alpha} e^{-v^4} (1-4v^4) \, dv \leq$$

$$\int_0^{+\infty} \frac{1}{\alpha} e^{-v^4} (1+4v^4) \, dv \leq \frac{D}{\alpha}$$

Where D is a proper positive constant. The same estimates holds true also for $|J_1|$. Substituting the estimates of $|J_1|$ in the definition of J and taking $\gamma(\xi) = 1 \wedge \xi^{(p-1)+\delta}$.

$$J \leq D^p \int_0^T dt \int_0^{+\infty} d\xi \gamma(\xi) \left(\int_0^t \frac{1}{\sqrt{s}\xi^2} \, ds \right)^{\frac{p}{2}} = 2^{\frac{p}{2}} D^p \int_0^T dt \int_0^{+\infty} d\xi \gamma(\xi) \frac{\sqrt[4]{t^p}}{\xi^p} < \infty$$

In conclusion, taking $\gamma(\xi) = 1 \wedge \xi^{(3p-1)+\delta}$, $\delta \in (0,1)$, then $\Psi_i(t,\xi) \in L^p(\Omega \times [0,T]; L^p_\gamma)$ for $i \in \{1,2\}$ and this completes the proof. □

Remark 4.2.2. Note that $\gamma(\xi)$ has exactly the same form of the one obtained in section 3.3, replacing p with $3p$.

Everything can be repeated for the equation with boundary conditions on the first and third order derivative. Let us consider the following equation

$$\begin{cases} \frac{\partial Z}{\partial t} + \frac{\partial^4 Z}{\partial \xi^4} = 0 \\ \frac{\partial Z}{\partial \xi}(t,0) = \dot{v}_1(t) \ \frac{\partial^3 Z}{\partial \xi^3}(t,0) = \dot{v}_2(t) \\ Z(0,\xi) = 0 \quad \xi \in \mathbb{R}^+ \ t \in [0,T] \end{cases} \tag{4.2.2}$$

Following all the notations given for equation (4.2.1) we can state and prove an analogous result about the solution of equation (4.2.2).

Proposition 4.2.3. *The solution of the stochastic differential equation (4.2.2) is a process in $L^p(\Omega \times [0,T]; L^p_\gamma)$, taking $\gamma(\xi) = 1 \wedge \xi^{2p-1+\delta}$, for $\delta \in (0,1)$.*

Proof. Thanks to the representation formula in example A.1.9, we have that a formal solution of equation (4.2.2) is given by

$$Z(t,\xi) = \frac{2}{\pi}\left[\int_0^t \int_0^{+\infty} -w^2 cos(w\xi)e^{-w^4(t-s)}\, dw\, dV_1(s) + \int_0^t \int_0^{+\infty} cos(w\xi)e^{-w^4(t-s)}\, dw\, dV_2(s)\right].$$

Since the integrand processes are deterministic, then the thesis guarantees the well posedness of the process $Z(t,\xi)$. If we call

$$\Psi_1(t,\xi) = \int_0^t \int_0^{+\infty} w^2 cos(w\xi)e^{-w^4(t-s)}\, dw\, dV_1(s)$$

and

$$\Psi_2(t,\xi) = \int_0^t \int_0^{+\infty} cos(w\xi)e^{-w^4(t-s)}\, dw\, dV_2(s)],$$

it is enough to prove that for $i \in \{1,2\}$ $\int_0^T dt\mathbb{E}\left[\int_0^{+\infty} d\xi\ \gamma(\xi)|\Psi_i|^p(t,\xi)\right] < +\infty$.
Let us start considering $\Psi_1(t,\xi)$. Thanks to Fubini-Tonelli theorem and Burkholder–Davis–Gundy inequality E.3.1, we get

$$I = \int_0^T dt \int_0^{+\infty} d\xi\gamma(\xi)\left(\int_0^t \left(\int_0^{+\infty} w^2 cos(w\xi)e^{-w^4(t-s)}\, dw\right)^2 ds\right)^{\frac{p}{2}} < +\infty \Longrightarrow$$

$$\Psi_1(t,\xi) \in L^p(\Omega \times [0,T]; L^p_\gamma).$$

Now we introduce the following change of variables $\begin{cases} l = (t-s) \\ v = \sqrt[4]{l}w \end{cases}$ so we get

$$I = \int_0^T dt \int_0^{+\infty} d\xi\gamma(\xi)\left(\int_0^t \frac{1}{\sqrt{l^3}}\left(\int_0^{+\infty} v^2 cos(v\frac{\xi}{\sqrt[4]{l}})e^{-v^4}\, dv\right)^2 dl\right)^{\frac{p}{2}}.$$

To simplify the notation we consider just the innermost integral and we call $\alpha = \frac{\xi}{\sqrt[4]{l}}$.

$$I_1 = \int_0^{+\infty} v^2 cos(v\alpha) e^{-v^4}\, dv.$$

All the following steps are integration by parts, where the sinusoidal term is the derivative term.

$$I_1 = \left[\frac{sin(v\alpha)}{\alpha} v^2 e^{-v^4}\right]_0^{+\infty} + \int_0^{+\infty} \frac{sin(v\alpha)}{\alpha}(2v)e^{-v^4}(2v^4-1)\, dv =$$

$$\left[-\frac{cos(v\alpha)}{\alpha^2}(2v)e^{-v^4}(2v^4-1)\right]_0^{+\infty} - \int_0^{+\infty} \frac{cos(v\alpha)}{\alpha^2} 2e^{-v^4}(8v^8 - 14v^4 + 1)\, dv =$$

$$-\int_0^{+\infty} \frac{cos(v\alpha)}{\alpha^2} 2e^{-v^4}(8v^8 - 14v^4 + 1)\, dv \le \frac{2}{\alpha^2}\int_0^{+\infty} e^{-v^4}(8v^8 + 14v^4 + 1)\, dv$$

$$\le \frac{C}{\alpha^2}$$

Where C is a proper positive constant. The same estimates holds true also for $|I_1|$. Substituting the estimates of $|I_1|$ in the definition of I and taking $\gamma(\xi) = 1 \wedge \xi^{(2p-1)+\delta}$

$$I \le C^p \int_0^T dt \int_0^{+\infty} d\xi \gamma(\xi) \left(\int_0^t \frac{1}{\sqrt{s}\xi^4}\, ds\right)^{\frac{p}{2}} = 2^{\frac{p}{2}} C^p \int_0^T dt \int_0^{+\infty} d\xi \gamma(\xi) \frac{\sqrt[4]{t^p}}{\xi^{2p}} < \infty.$$

The second summand is absolutely analogous to treat and easier to manage.

$$J = \int_0^T dt \int_0^{+\infty} d\xi \gamma(\xi) \left(\int_0^t \left(\int_0^{+\infty} cos(w\xi) e^{-w^4(t-s)}\, dw\right)^2 ds\right)^{\frac{p}{2}} < +\infty \Longrightarrow$$

$$\Psi_2(t,\xi) \in L^p(\Omega \times [0,T]; L^p_\gamma).$$

Again introducing the change of variables $\begin{cases} l = (t-s) \\ v = \sqrt[4]{l}w \end{cases}$ we get

$$J = \int_0^T dt \int_0^{+\infty} d\xi \gamma(\xi) \left(\int_0^t \frac{1}{\sqrt{l}} \left(\int_0^{+\infty} cos(v\frac{\xi}{\sqrt[4]{l}}) e^{-v^4}\, dv\right)^2 dl\right)^{\frac{p}{2}}$$

Again for matter of simplicity we take care the innermost integral and call $\alpha = \frac{\xi}{\sqrt[4]{l}}$

$$J_1 = \int_0^{+\infty} cos(v\alpha) e^{-v^4}\, dv = \left[\frac{sin(v\alpha)}{\alpha} e^{-v^4}\right]_0^{+\infty} + \int_0^{+\infty} \frac{sin(v\alpha)}{\alpha} e^{-v^4}(4v^3)\, dv \le$$

$$\int_0^{+\infty} \frac{1}{\alpha} e^{-v^4}(4v^3)\, dv \leq \frac{D}{\alpha}.$$

Where D is a proper positive constant. The same estimates holds true also for $|J_1|$. Substituting the estimates of $|J_1|$ in the definition of J and taking $\gamma(\xi) = 1 \wedge \xi^{(p-1)+\delta}$

$$J \leq D^p \int_0^T dt \int_0^{+\infty} d\xi\gamma(\xi)(\int_0^t \frac{1}{\xi^2}\, ds)^{\frac{p}{2}} = D^p \int_0^T dt \int_0^{+\infty} d\xi\gamma(\xi)\frac{\sqrt{t^p}}{\xi^p} < \infty$$

In conclusion, taking $\gamma(\xi) = 1 \wedge \xi^{(2p-1)+\delta}$, $\delta \in (0,1)$, then $\Psi_i(t,\xi) \in L^p(\Omega \times [0,T]; L^p_\gamma)$ for $i \in \{1,2\}$ and this completes the proof. □

Now we want to prove a continuity results for both the problems presented in this subsection. Actually, the form of the weight let us no hope to get continuity in $[0,+\infty)$, so we try to find a weaker result.

Proposition 4.2.4. *For each $\delta > 0$, the solution of problem (4.2.1) is continuous in $[\delta, +\infty)$ for each fixed t. More precisely, for each fixed $\delta > 0$, $t \in [0,T]$, there exists a continuous modification of the solution of problem (4.2.1) continuous in $[\delta, +\infty)$.*

Proof. We want to apply Kolmogorov's continuity theorem E.1.1. So we fix $t \in [0,T]$, $\delta > 0$ and the goal is trying to prove the following inequality:

$$\exists\, a > 0 \; s.t. \; \mathbb{E}[|Z(t,\xi) - Z(t,v)|^p] \leq c|v-\xi|^{1+a} \quad \delta < \xi, v < K < +\infty.$$

Recalling that the solution of equation (4.2.1) is

$$Z(t,\xi) = \frac{2}{\pi}[\int_0^t \int_0^{+\infty} w^3 sin(w\xi) e^{-w^4(t-s)}\, dw\, dV_1(s)$$

$$-\int_0^t \int_0^{+\infty} w sin(w\xi) e^{-w^4(t-s)}\, dw\, dV_2(s)] = \psi_1(t,\xi) + \psi_2(t,\xi)$$

and exploiting convexity

$$\mathbb{E}[|\psi_1(t,\xi)+\psi_2(t,\xi)-\psi_1(t,v)-\psi_2(t,v)|^p] \leq 2^{p-1}(\mathbb{E}[|\psi_1(t,\xi)-\psi_1(t,v)|^p]+\mathbb{E}[|\psi_2(t,\xi)-\psi_2(t,v)|^p])$$

we can consider the two summand separately. Thanks to Burkholder–Davis–Gundy inequality E.3.1 and the change of variable $\begin{cases} l = (t-s) \\ v = \sqrt[4]{l} w \end{cases}$ we get:

$$\mathbb{E}[|\psi_1(t,\xi)-\psi_1(t,v)|^p] \leq C_0 \left(\int_0^t ds \left(\int_0^{+\infty} w^3 (sin(wv) - sin(w\xi)) e^{-w^4(t-s)}\, dw \right)^2 \right)^{\frac{p}{2}}$$

$$= C_0 \left(\int_0^t \frac{1}{l^2} \left(\int_0^{+\infty} v^3 (sin(v\frac{\xi}{\sqrt[4]{l}}) - sin(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\, dv \right)^2 dl \right)^{\frac{p}{2}}$$

Considering the innermost integral $I = \int_0^{+\infty} v^3(sin(v\frac{\xi}{\sqrt[4]{l}}) - sin(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\, dv$ and integrating four times by parts, namely one more time than in the first part of the proof of proposition 4.2.1, we get

$$I = \int_0^{+\infty} v^3 (sin(v\frac{\xi}{\sqrt[4]{l}}) - sin(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\, dv$$

$$= 16l \int_0^{+\infty} dv\ (\frac{sin(\frac{v\xi}{\sqrt[4]{l}})}{\xi^4} - \frac{sin(\frac{v\upsilon}{\sqrt[4]{l}})}{\upsilon^4})v^3 e^{-v^4}(32v^{12} - 240v^8 + 390v^4 - 105)$$

If we call $f(x) = \frac{sin(\frac{vx}{\sqrt[4]{l}})}{x^4}$, $f'(x) = \frac{vcos(\frac{vx}{\sqrt[4]{l}})}{x^4\sqrt[4]{l}} - 4\frac{sin(\frac{vx}{\sqrt[4]{l}})}{x^5}$,
then $|f'(x)| \le \frac{v}{x^4\sqrt[4]{l}} + \frac{4}{x^5}$. So applying Lagrange's theorem for $|I|$ we get

$$|I| \le 16l \int_0^{+\infty} dv\ v^3 e^{-v^4}(32v^{12} + 240v^8 + 390v^4 + 105)(\frac{v}{\eta^4\sqrt[4]{l}} + \frac{4}{\eta^5})|\xi - \upsilon|$$

$$\le 16l \int_0^{+\infty} dv\ v^3 e^{-v^4}(32v^{12} + 240v^8 + 390v^4 + 105)(\frac{v}{\delta^4\sqrt[4]{l}} + \frac{4}{\delta^5})|\xi - \upsilon|$$

$$\le (C_1\frac{\sqrt[4]{l^3}}{\delta^4} + C_2\frac{l}{\delta^5})|\xi - \upsilon|.$$

Coming back to $\mathbb{E}[|\psi_1(t,\xi) - \psi_1(t,\upsilon)|^p]$ and exploiting $(a+b)^2 \le 2a^2 + 2b^2$ we reach the following useful estimate

$$\mathbb{E}[|\psi_1(t,\xi) - \psi_1(t,\upsilon)|^p] \le C_0(\int_0^t \frac{1}{l^2}(\frac{C_1^2\sqrt{l^3}}{\delta^8} + \frac{C_2^2 l^2}{\delta^{10}})\ dl)^{\frac{p}{2}}|\xi - \upsilon|^p$$

$$= C_0(\frac{C_1^2\sqrt{t}}{\delta^8} + \frac{C_2^2 t}{\delta^{10}})^{\frac{p}{2}}|\xi - \upsilon|^p = C(T,p,\delta)|\xi - \upsilon|^p.$$

Now we move to the second summand. Thanks to Burkholder–Davis–Gundy inequality and the change of variable $\begin{cases} l = (t-s) \\ v = \sqrt[4]{l}w \end{cases}$ we get:

$$\mathbb{E}[|\psi_2(t,\xi) - \psi_2(t,\upsilon)|^p] \le C_0 \left(\int_0^t ds \left(\int_0^{+\infty} w(sin(w\upsilon) - sin(w\xi))e^{-w^4(t-s)}\, dw \right)^2 \right)^{\frac{p}{2}}$$

$$= C_0 \left(\int_0^t \frac{1}{l} \left(\int_0^{+\infty} v(sin(v\frac{\xi}{\sqrt[4]{l}}) - sin(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\, dv \right)^2 dl \right)^{\frac{p}{2}}$$

Considering the innermost integral $J = \int_0^{+\infty} v(sin(v\frac{\xi}{\sqrt[4]{l}}) - sin(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\, dv$ and integrating by parts twice, namely one more time than in the second part of proposition 4.2.1, we get

$$J = \int_0^{+\infty} v(sin(v\frac{\xi}{\sqrt[4]{l}}) - sin(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\, dv$$

$$= 4\sqrt{l}\int_0^{+\infty}(\frac{sin(\frac{v\xi}{\sqrt{l}})}{\xi^2} - \frac{sin(\frac{v\upsilon}{\sqrt{l}})}{\upsilon^2})v^3e^{-v^4}(4v^4-5)\, dv.$$

If we call $g(x) = \frac{sin(\frac{vx}{\sqrt[4]{l}})}{x^2}$, $g'(x) = \frac{vcos(\frac{vx}{\sqrt[4]{l}})}{x^2\sqrt[4]{l}} - 2\frac{sin(\frac{vx}{\sqrt[4]{l}})}{x^3}$,
then $|g'(x)| \le \frac{v}{x^2\sqrt[4]{l}} + \frac{2}{x^3}$. So applying Lagrange's theorem for $|J|$ we get

$$|J| \le 4\sqrt{l}\int_0^{+\infty} dv\ v^3e^{-v^4}(4v^4+5)(\frac{v}{\eta^2\sqrt[4]{l}} + \frac{2}{\eta^3})|x-y|$$

$$\le 4\sqrt{l}\int_0^{+\infty} dv\ v^3e^{-v^4}(4v^4+5)(\frac{v}{\delta^2\sqrt[4]{l}} + \frac{2}{\delta^3})|\xi-\upsilon|$$

$$\le (D_1\frac{\sqrt[4]{l}}{\delta^2} + D_2\frac{\sqrt{l}}{\delta^3})|\xi-\upsilon|.$$

Coming back to $\mathbb{E}[|\psi_2(t,\xi)-\psi_2(t,\upsilon)|^p]$ and exploiting $(a+b)^2 \le 2a^2+2b^2$ we reach the last estimate

$$\mathbb{E}[|\psi_2(t,\xi)-\psi_2(t,\upsilon)|^p] \le C_0(\int_0^t \frac{1}{l}(\frac{D_1^2\sqrt{l}}{\delta^4} + \frac{D_2^2 l}{\delta^6})\ dl)^{\frac{p}{2}}|\xi-\upsilon|^p$$

$$= C_0(\frac{D_1^2\sqrt{t}}{\delta^4} + \frac{D_2^2 t}{\delta^6})^{\frac{p}{2}}|\xi-\upsilon|^p = D(T,p,\delta)|\xi-\upsilon|^p.$$

In conclusion

$$\mathbb{E}[|Z(t,\xi)-Z(t,\upsilon)|^p] \le (D(T,p,\delta)+C(T,p,\delta))|\xi-\upsilon|^p \quad \delta < x,y < K < +\infty.$$

Hence we can apply Kolmogorov's continuity theorem with $a = p-1$ and the proof is complete. □

Proposition 4.2.5. *For each $\delta > 0$, the solution of problem (4.2.2) is continuous in $[\delta,+\infty)$ for each fixed t. More precisely, for each fixed $\delta > 0$, $t \in [0,T]$, there exists a continuous modification of the solution of problem (4.2.2) continuous in $[\delta,+\infty)$.*

Proof. We want to apply Kolmogorov's continuity theorem E.1.1. So we fix $t \in [0,T]$, $\delta > 0$ and the goal is trying to prove the following inequality:

$$\exists a > 0 \ s.t. \ \mathbb{E}[|Z(t,\xi) - Z(t,\upsilon)|^p] \le c|\upsilon - \xi|^{1+a} \quad \delta < \xi, \upsilon < K < +\infty$$

Recalling that the solution of equation (4.2.2) is

$$Z(t,\xi) = \frac{2}{\pi}[\int_0^t \int_0^{+\infty} -w^2 cos(w\xi) e^{-w^4(t-s)}\, dw\, dV_1(s)$$

$$+ \int_0^t \int_0^{+\infty} cos(w\xi) e^{-w^4(t-s)}\, dw\, dV_2(s)] = \psi_1(t,\xi) + \psi_2(t,\xi)$$

and exploiting convexity

$$\mathbb{E}[|\psi_1(t,\xi) + \psi_2(t,\xi) - \psi_1(t,\upsilon) - \psi_2(t,\upsilon)|^p] \le 2^{p-1}(\mathbb{E}[|\psi_1(t,\xi) - \psi_1(t,\upsilon)|^p] + \mathbb{E}[|\psi_2(t,\xi) - \psi_2(t,\upsilon)|^p])$$

we can consider the two summand separately. Thanks to Burkholder–Davis–Gundy inequality E.3.1 and the change of variable $\begin{cases} l = (t-s) \\ v = \sqrt[4]{l} w \end{cases}$ we get:

$$\mathbb{E}[|\psi_1(t,\xi) - \psi_1(t,\upsilon)|^p] \le C_0 \left(\int_0^t ds \left(\int_0^{+\infty} w^2(cos(w\upsilon) - cos(w\xi)) e^{-w^4(t-s)}\, dw \right)^2 \right)^{\frac{p}{2}}$$

$$= C_0 \left(\int_0^t \frac{1}{\sqrt{l^3}} \left(\int_0^{+\infty} v^2 (cos(v\frac{\xi}{\sqrt[4]{l}}) - cos(v\frac{\upsilon}{\sqrt[4]{l}})) e^{-v^4}\, dv \right)^2 dl \right)^{\frac{p}{2}}.$$

Considering the innermost integral $I = \int_0^{+\infty} v^2(cos(v\frac{\xi}{\sqrt[4]{l}}) - cos(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\, dv$ and integrating three times by parts, namely one more time than in the first part of the proof of proposition 4.2.3, we get

$$I = \int_0^{+\infty} v^2(cos(v\frac{\xi}{\sqrt[4]{l}}) - cos(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\, dv$$

$$= -8\sqrt[4]{l^3} \int_0^{+\infty} (\frac{sin(\frac{v\xi}{\sqrt[4]{l}})}{\xi^3} - \frac{sin(\frac{v\upsilon}{\sqrt[4]{l}})}{\upsilon^3}) v^3 e^{-v^4}(8v^8 - 30v^4 + 15)\, dv.$$

If we call $f(x) = \frac{sin(\frac{vx}{\sqrt[4]{l}})}{x^3}$, $f'(x) = \frac{vcos(\frac{vx}{\sqrt[4]{l}})}{x^3\sqrt[4]{l}} - 3\frac{sin(\frac{vx}{\sqrt[4]{l}})}{x^4}$,
then $|f'(x)| \le \frac{v}{x^3\sqrt[4]{l}} + \frac{3}{x^4}$. So applying Lagrange's theorem for $|I|$ we get

$$|I| \le 8\sqrt[4]{l^3} \int_0^{+\infty} dv\ v^3 e^{-v^4}(8v^8 + 30v^4 + 15)(\frac{v}{\eta^3\sqrt[4]{l}} + \frac{3}{\eta^4})|\xi - \upsilon|$$

$$\leq 8\sqrt[4]{l^3}\int_0^{+\infty} dv\ v^3 e^{-v^4}(8v^8+30v^4+15)(\frac{v}{\delta^3\sqrt[4]{l}}+\frac{3}{\delta^4})|\xi-\upsilon|$$

$$\leq (C_1\frac{\sqrt{l}}{\delta^3}+C_2\frac{l}{\delta^4})|\xi-\upsilon|.$$

Coming back to $\mathbb{E}[|\psi_1(t,\xi)-\psi_1(t,\upsilon)|^p]$ and exploiting $(a+b)^2 \leq 2a^2+2b^2$ we reach the following useful estimate

$$\mathbb{E}[|\psi_1(t,\xi)-\psi_1(t,\upsilon)|^p] \leq C_0(\int_0^t \frac{1}{l^{\frac{3}{2}}}(\frac{C_1^2 l}{\delta^6}+\frac{C_2^2 l^{\frac{3}{2}}}{\delta^8})\ dl)^{\frac{p}{2}}|\xi-\upsilon|^p$$

$$= C_0(\frac{C_1^2\sqrt{t}}{\delta^6}+\frac{C_2^2 t}{\delta^8})^{\frac{p}{2}}|\xi-\upsilon|^p = C(T,p,\delta)|\xi-\upsilon|^p.$$

Now we move to the second summand. Thanks to Burkholder–Davis–Gundy inequality and the change of variable $\begin{cases} l=(t-s) \\ v=\sqrt[4]{l}w \end{cases}$ we get:

$$\mathbb{E}[|\psi_2(t,\xi)-\psi_2(t,\upsilon)|^p] \leq C_0\left(\int_0^t ds\left(\int_0^{+\infty}(cos(w\upsilon)-cos(w\xi))e^{-w^4(t-s)}\,dw\right)^2\right)^{\frac{p}{2}}$$

$$= C_0\left(\int_0^t \frac{1}{\sqrt{l}}\left(\int_0^{+\infty}(cos(v\frac{\xi}{\sqrt[4]{l}})-cos(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\,dv\right)^2 dl\right)^{\frac{p}{2}}.$$

Considering the innermost integral $J=\int_0^{+\infty} v(cos(v\frac{\xi}{\sqrt[4]{l}})-cos(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\,dv$ and integrating by parts, as it is done in the second part of proposition 4.2.3, we get

$$J = \int_0^{+\infty}(cos(v\frac{\xi}{\sqrt[4]{l}})-cos(v\frac{\upsilon}{\sqrt[4]{l}}))e^{-v^4}\,dv$$

$$= 4\sqrt[4]{l}\int_0^{+\infty}(\frac{sin(\frac{v\xi}{\sqrt[4]{l}})}{\xi}-\frac{sin(\frac{v\upsilon}{\sqrt[4]{l}})}{\upsilon})v^3e^{-v^4}dv.$$

If we call $g(x)=\frac{sin(\frac{vx}{\sqrt[4]{l}})}{x}$, $g'(x)=\frac{vcos(\frac{vx}{\sqrt[4]{l}})}{x\sqrt[4]{l}}-\frac{sin(\frac{vx}{\sqrt[4]{l}})}{x^2}$,
then $|g'(x)| \leq \frac{v}{x\sqrt[4]{l}}+\frac{1}{x^2}$. So applying Lagrange's theorem for $|J|$ we get

$$|J| \leq 4\sqrt[4]{l}\int_0^{+\infty} dv\ v^3e^{-v^4}(\frac{v}{\eta\sqrt[4]{l}}+\frac{1}{\eta^2})|\xi-\upsilon|$$

$$\leq 4\sqrt[4]{l}\int_0^{+\infty} dv\ v^3e^{-v^4}(\frac{v}{\delta\sqrt[4]{l}}+\frac{1}{\delta^2})|\xi-\upsilon|$$

$$\leq (\frac{D_1}{\delta} + D_2 \frac{\sqrt[4]{l}}{\delta^2})|\xi - v|.$$

Coming back to $\mathbb{E}[|\psi_2(t,\xi) - \psi_2(t,v)|^p]$ and exploiting $(a+b)^2 \leq 2a^2 + 2b^2$ we reach the last estimate

$$\mathbb{E}[|\psi_2(t,\xi) - \psi_2(t,v)|^p] \leq C_0 \left(\int_0^t \frac{1}{\sqrt{l}} (\frac{D_1^2}{\delta^2} + \frac{D_2^2 \sqrt{l}}{\delta^4}) \; dl \right)^{\frac{p}{2}} |\xi - v|^p$$

$$= C_0 (\frac{D_1^2 \sqrt{t}}{\delta^4} + \frac{D_2^2 t}{\delta^6})^{\frac{p}{2}} |\xi - v|^p = D(T,p,\delta)|\xi - v|^p.$$

In conclusion

$$\mathbb{E}[|Z(t,\xi) - Z(t,v)|^p] \leq (D(T,p,\delta) + C(T,p,\delta))|\xi - v|^p \quad \delta < \xi, v < K < +\infty.$$

Hence we can apply Kolmogorov's continuity theorem with $a = p-1$ and the proof is complete. $\square$

Chapter 5

Conclusions and Future Work

The aim of this thesis was to present some techniques used for the analysis of parabolic differential problems with stochastic boundary conditions, then apply some of these tools to a new problem with no results available in the literature. We started in an abstract framework, presenting a class of stochastic problem and some conditions for the well posedness of their solutions, then we moved to a concrete example deeply analyzed in the literature, lastly we proved some regularity results for a fourth order model problem. In particular, we proved that the solution of equations 4.1.1 and 4.1.2 belongs to proper distributional spaces. The analysis of equations 4.2.1 and 4.2.2 was deeper. It led us to a representation formula for the solutions via A.1.8 and A.1.9, then by exploiting these representation formulas we could prove the well posedness of the solutions in some proper weighted L^p spaces, namely $L^p(\Omega \times [0,T]; L^p_\gamma)$, and some continuity results (propositions 4.2.1, 4.2.3, 4.2.4, 4.2.5). This was done not only for its intrinsic mathematical interest, but also in order to become familiar with the methods in such a way to consider more relevant physical problems in the future.

Actually the analysis of the fourth order heat equation with stochastic boundary conditions is not completed. Several questions are already open. Among these the more relevant ones are:

- Quantify the degree of increase close to the origin for the solutions of equation 4.2.1 and 4.2.2. The definition of the spaces $L^p(\Omega \times [0,T]; L^p_\gamma)$ leads us to conjecture that the solution of problem 4.2.1 has a singularity of order three in the origin, instead the solution of problem 4.2.2 has a singularity of order two in the origin.

- Find stronger continuity results for the solutions of equation 4.2.1 and 4.2.2. The majority of the literature is devoted to find continuity results in the space variable for a fixed time, similar results were obtained also for our cases. It could be interesting find continuity results in space-time variables via Kolmogorov's continuity theorem.

- Generalize results to multidimensional domains. Of course differential equations have a more relevant impact on applications if results are available for multidimensional domains. As a first step, a possible goal is trying to replicate what is done in chapter 4 at least in hypercubes or half-spaces $\mathbb{R}^{n-1} \times \mathbb{R}^+$.

- Handling some nonlinear terms satisfying suitable assumptions. Since the solutions of equation 4.2.1 and 4.2.2 belong to some weighted L^p spaces, on suitable conditions on nonlinearities, the membership of the solution of the nonlinear problem to $L^p(\Omega \times [0,T]; L^p_\gamma)$ could be recovered.

- Introduce other notions of solution. In [Sow94],[AB02b], [Brz+15] other relevant notions of solution for the nonlinear problems are treated different from the mild one already described.

Among some physical problems where stochastic boundary conditions have an impact, the linear wave equation with bilaplacian in a multidimensional domain or some problems of stochastic fluid dynamics and material sciences can be taken as main examples. In fact, in several physical cases, random perturbations come from the boundary of the domain and are not distributed inside it. So far great part of the literature devoted to this kind of topics is related to Ginzburg-Landau equation and classical wave equation (see for example [DL06],[Kai19],[Kim06]). Anyhow fourth order wave equation has a mechanical interpretation as interesting as classical wave equation. In fact, it represents the vibrations of a plate in the presence of noise on the boundary.

For all the problems treated in this thesis our interest was on existence and regularity issues. Actually, these equations (and the same holds for all the other mentioned, namely wave equation, fourth order wave equation and so on) can lead us to other issues relevant in the applications, for example:

- Behavior of the solution for large times. The literature devoted to this topic is composed by few results, generally on heat equation [DZ96],[AB02a].

- Stochastic optimal control problems. Optimal control is a relevant topic in stochastic analysis. Nevertheless the results for differential problems with stochastic boundary conditions are related to one dimensional equations, see for example [FG09],[Mas10],[DFT07].

- Differential problems with dynamic boundary conditions. Even if this type of problems could look not related to the topic presented in this thesis, in case of the dynamic boundary conditions subject to some noise, the equations can be reformulated in the framework of differential problems with stochastic boundary conditions.

Appendix A

Some Applications of Sine and Cosine Transforms

A.1. Sine and Cosine Transforms

Integral transforms have a crucial role in several topics of engineering or applied mathematics as signal processing, probability or studying of differential equations in particular domains. A complete and clear treatment of this topic can be found in [Pou18]. In this appendix we will use some techniques related to sine and cosine transforms to solve parabolic equations in the half-line. In particular examples A.1.8, A.1.9 and A.1.11 give some explicit formulas for the formal solutions of problems 4.2.1, 4.2.2 and 3.1.3.

Definition A.1.1 (Fourier sine transform)**.** Let $f(t) : [0, +\infty) \to \mathbb{C}$ be a $L^1(0, +\infty)$ piecewice continuous function. Then its Fourier sine transform is defined as

$$F_s(w) = \mathcal{F}_s[f(t)] = \int_0^{+\infty} f(t) \sin(wt) dt \ w \geq 0.$$

Definition A.1.2 (Fourier cosine transform)**.** Let $f(t) : [0, +\infty) \to \mathbb{C}$ be a $L^1(0, +\infty)$ piecewice continuous function. Then its Fourier cosine transform is defined as

$$F_c(w) = \mathcal{F}_c[f(t)] = \int_0^{+\infty} f(t) \cos(wt) dt \ w \geq 0.$$

If also $F_s(w)$ or $F_c(w)$ are $L^1(0, +\infty)$ piecewice continuous function, then the inverse Fourier sine and cosine transform can be defined.

Definition A.1.3 (inverse Fourier sine transform)**.** Let $F_s(w) : [0, +\infty) \to \mathbb{C}$ be a $L^1(0, +\infty)$ piecewice continuous function. Then its inverse Fourier sine transform is defined as

$$\mathcal{F}_s^{-1}[F_s(w)] = \frac{2}{\pi}\int_0^{+\infty} F_s(w)\sin(wt)dw \ t \geq 0.$$

Definition A.1.4 (inverse Fourier cosine transform)**.** Let $F_c(w) : [0, +\infty) \to \mathbb{C}$ be a $L^1(0, +\infty)$ piecewice continuous function. Then its inverse Fourier cosine transform is defined as

$$\mathcal{F}_c^{-1}[F_c(w)] = \frac{2}{\pi}\int_0^{+\infty} F_c(w)\cos(wt)dw \ t \geq 0.$$

Now we state some results about the well posedness of these definitions and some useful properties for solving differential equations.

Proposition A.1.5. *Let $f(t) : [0, +\infty) \to \mathbb{C}$ be a $L^1(0, +\infty)$ piecewice continuous function, such that $f'(t)$ is piece-wise continuous in each bounded subinterval of $[0, +\infty)$. Then if f is continuous in t*

$$f(t) = \frac{2}{\pi}\int_0^{+\infty} F_c(w)\cos(wt)dw = \frac{2}{\pi}\int_0^{+\infty}\left[\int_0^{+\infty} f(\tau)\cos(w\tau)d\tau\right]\cos(wt)dw.$$

$$f(t) = \frac{2}{\pi}\int_0^{+\infty} F_s(w)\sin(wt)dw = \frac{2}{\pi}\int_0^{+\infty}\left[\int_0^{+\infty} f(\tau)\sin(w\tau)d\tau\right]\sin(wt)dw.$$

If f has a jump discontinuity in t_0 then

$$\frac{2}{\pi}\int_0^{+\infty}\left[\int_0^{+\infty} f(\tau)\cos(w\tau)d\tau\right]\cos(wt)dw = \frac{1}{2}\left(\lim_{t\to t_0^+} f(t) + \lim_{t\to t_0^-} f(t)\right).$$

$$\frac{2}{\pi}\int_0^{+\infty}\left[\int_0^{+\infty} f(\tau)\sin(w\tau)d\tau\right]\sin(wt)dw = \frac{1}{2}\left(\lim_{t\to t_0^+} f(t) + \lim_{t\to t_0^-} f(t)\right).$$

Proposition A.1.6. *Let $f(t) : [0, +\infty) \to \mathbb{C}$ be a continuous function such that $f'(t)$ is continuous in $[0, +\infty)$, $f(t)$ and $f'(t)$ vanish as $t \to +\infty$. Then*

$$\mathcal{F}_c[f''(t)] = -w^2 F_c(w) - f'(0).$$

$$\mathcal{F}_s[f''(t)] = -w^2 F_s(w) + wf(0).$$

Remark A.1.7. More complex formulas hold without assuming continuity of $f(t)$ and $f'(t)$.

Example A.1.8. Let us start considering $f_1 : [0, +\infty) \to \mathbb{R}$ and $f_2 : [0, +\infty) \to \mathbb{R}$, $f_1,\ f_2 \in L^\infty(0, +\infty)$ and the fourth order heat equation with Navier boundary conditions. Find a formal solution of the equation:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0 \\ u(t,0) = f_1(t)\ \frac{\partial^2 u}{\partial x^2}(t,0) = f_2(t) \\ u(0,x) = 0 \ \ x \in \mathbb{R}^+\ t \geq 0 \end{cases} \tag{A.1.1}$$

Applying the Fourier sine transform and proposition A.1.6 getting rid of Navier boundary conditions, we get the easier equation

$$\begin{cases} \frac{\partial U}{\partial t} + w^4 U = w^3 f_1 - w f_2 \\ U(0,w) = 0\ w \in \mathbb{R}^+\ t \geq 0 \end{cases} \tag{A.1.2}$$

Where $U(\cdot, w) = \mathcal{F}_s[u(\cdot, x)]$. Solving the ordinary differential equation, it is easy to obtain

$$U(t,w) = e^{-w^4 t} \int_0^t ds\ e^{w^4 s}(w^3 f_1(s) - w f_2(s)) \qquad w \in \mathbb{R}^+\ t \geq 0.$$

Taking the inverse transform to both side and applying Fubini-Tonelli theorem we get the solution of equation (A.1.1)

$$u(t,x) = \frac{2}{\pi} \int_0^t ds \int_0^{+\infty} dw\ sin(wx) e^{-w^4(t-s)}(w^3 f_1(s) - w f_2(s)) \qquad x \in \mathbb{R}^+\ t \geq 0.$$

Example A.1.9. Let us start considering $f_1 : [0, +\infty) \to \mathbb{R}$ and $f_2 : [0, +\infty) \to \mathbb{R}$, $f_1,\ f_2 \in L^\infty(0, +\infty)$ and the fourth order heat equation with boundary conditions on first and third derivative. Find a formal solution of the equation:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0 \\ \frac{\partial u}{\partial x}(t,0) = f_1(t)\ \frac{\partial^3 u}{\partial x^3}(t,0) = f_2(t) \\ u(0,x) = 0 \ \ x \in \mathbb{R}^+\ t \geq 0 \end{cases} \tag{A.1.3}$$

Applying the Fourier cosine transform and proposition A.1.6 getting rid of boundary conditions, we get the easier equation

$$\begin{cases} \frac{\partial U}{\partial t} + w^4 U = f_2 - w^2 f_1 \\ U(0,w) = 0\ w \in \mathbb{R}^+\ t \geq 0 \end{cases} \tag{A.1.4}$$

Where $U(\cdot, w) = \mathcal{F}_c[u(\cdot, x)]$. Solving the ordinary differential equation, it is easy to obtain

$$U(t,w) = e^{-w^4 t} \int_0^t ds\ e^{w^4 s}(f_2(s) - w^2 f_1(s)) \qquad w \in \mathbb{R}^+\ t \geq 0.$$

Taking the inverse transform to both side and applying Fubini-Tonelli theorem we get the solution of equation (A.1.3)

$$u(t,x) = \frac{2}{\pi}\int_0^t ds \int_0^{+\infty} dw\ cos(wx)e^{-w^4(t-s)}(f_2(s) - w^2 f_1(s)) \qquad x \in \mathbb{R}^+\ t \geq 0.$$

Remark A.1.10. In both examples Fubini-Tonelli theorem can be applied because, calling

$$C = max(\|f_1\|_{L^\infty}, \|f_2\|_{L^\infty}),$$

we have for $x \neq 0$

1.
$$\int_0^{+\infty} dw \int_0^t ds\ |sin(wx)|e^{-w^4(t-s)}|w^3 f_1(s) - w f_2(s)| \leq$$
$$C\int_0^{+\infty} dw\ \frac{1-e^{-w^4 t}}{w^4}(w^3 + w)|sin(wx)| < +\infty.$$
Since $\int_0^{+\infty} dw\ |sin(wx)|\frac{1-e^{-w^4 t}}{w^4}w \leq \int_0^{+\infty} dw\ \frac{1-e^{-w^4 t}}{w^4}w < \infty$ and
$$\int_0^{+\infty} dw\ \frac{1-e^{-w^4 t}}{w^4}w^3|sin(wx)| =$$
$$\int_0^{\frac{\pi}{x}} dw\ \frac{1-e^{-w^4 t}}{w}|sin(wx)| + \int_{\frac{\pi}{x}}^{+\infty} dw\ \frac{1-e^{-w^4 t}}{w}|sin(wx)| \leq$$
$$\leq \int_0^{\frac{\pi}{x}} dw\ \frac{1-e^{-w^4 t}}{w}|sin(wx)| + \int_{\frac{\pi}{x}}^{+\infty} dw\ \frac{1}{w}|sin(wx)|.$$
The first summand converges since $\frac{1-e^{-w^4 t}}{w}|sin(wx)| \sim w^4 t|x|$ for $w \to 0$. For what concern the second summand
$$\int_{\frac{\pi}{x}}^{+\infty} dw\ \frac{1}{w}|sin(wx)| = \sum_{k=0}^{+\infty}\left|\int_{\frac{1}{x}(\pi+k\pi)}^{\frac{1}{x}(\pi+(k+1)\pi)} \frac{1}{w}sin(wx)\ dw\right| =$$
$$\sum_{k=0}^{+\infty}\left|\left[-\frac{1}{wx}cos(wx)\right]_{\frac{1}{x}(\pi+k\pi)}^{\frac{1}{x}(\pi+(k+1)\pi)} - \int_{\frac{1}{x}(\pi+k\pi)}^{\frac{1}{x}(\pi+(k+1)\pi)} sin(wx)\frac{1}{w^2 x}dw\right| \leq$$
$$\sum_{k=0}^{+\infty}\left(\frac{1}{(k+1)\pi} - \frac{1}{(k+2)\pi}\right) + \int_{\frac{\pi}{x}}^{+\infty}\frac{1}{w^2 x}dw < +\infty.$$

2.

$$\int_0^{+\infty} dw \int_0^t ds\, |cos(wx)|e^{-w^4(t-s)}|-w^2 f_1(s)+f_2(s)| \le C \int_0^{+\infty} dw\, \frac{1-e^{-w^4 t}}{w^4}(w^2+1) < +\infty.$$

Since $\frac{1-e^{-w^4 t}}{w^4}w^2 \sim w^2 t$ and $\frac{1-e^{-w^4 t}}{w^4} \sim t$ for $w \to 0$.

Example A.1.11. Let us start considering $f_1 : [0, +\infty) \to \mathbb{R}$, $f_1 \in L^\infty(0, +\infty)$ and the heat equation with Neumann boundary conditions. Find a formal solution of the equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \\ \frac{\partial u}{\partial x}(t,0) = f_1(t) \\ u(0,x) = 0 \quad x \in \mathbb{R}^+ \; t \ge 0 \end{cases} \tag{A.1.5}$$

Applying the Fourier cosine transform and proposition A.1.6 getting rid of boundary conditions, we get the easier equation

$$\begin{cases} \frac{\partial U}{\partial t} + w^2 U = -f_1 \\ U(0,w) = 0 \; w \in \mathbb{R}^+ \; t \ge 0 \end{cases} \tag{A.1.6}$$

Where $U(\cdot, w) = \mathcal{F}_c[u(\cdot, x)]$. Solving the ordinary differential equation, it is easy to obtain

$$U(t,w) = -e^{-w^2 t} \int_0^t ds\; e^{w^2 s} f_1(s) \qquad w \in \mathbb{R}^+ \; t \ge 0.$$

Taking the inverse transform to both side and applying Fubini-Tonelli theorem we get the solution of equation (A.1.5)

$$u(t,x) = -\frac{2}{\pi} \int_0^t ds f_1(s) \int_0^{+\infty} dw\; cos(wx) e^{-w^2(t-s)} \qquad x \in \mathbb{R}^+ \; t \ge 0.$$

This solution can be written in an easy way. In fact

$$\int_0^{+\infty} dw\; cos(wx) e^{-w^2(t-s)} = \frac{1}{2}\int_{\mathbb{R}} dw\; cos(wx) e^{-w^2(t-s)} = \mathrm{Re}\left(\frac{1}{2}\int_{\mathbb{R}} dw\; e^{iwx} e^{-w^2(t-s)}\right)$$

$$= \frac{\sqrt{\pi}}{2} \frac{e^{\frac{-x^2}{4(t-s)}}}{\sqrt{t-s}}.$$

and

$$u(t,x) = -\frac{1}{\sqrt{\pi}} \int_0^t ds\; f_1(s) \frac{e^{\frac{-x^2}{4(t-s)}}}{\sqrt{t-s}} \qquad x \in \mathbb{R}^+ \; t \ge 0.$$

Remark A.1.12. In this example Fubini-Tonelli theorem can be applied because, calling

$$C = \|f_1\|_{L^\infty},$$

we have for $x \neq 0$

$$\int_0^{+\infty} dw \int_0^t ds \; |cos(wx)| e^{-w^2(t-s)} |f_1(s)| \leq C \int_0^{+\infty} dw \; \frac{1 - e^{-w^2 t}}{w^2} < +\infty.$$

Since $\frac{1-e^{-w^2 t}}{w^2} \sim t$ for $w \to 0$.

Appendix B

C_0 Semigroups of Linear Operators

C_0 semigroups of linear operators are a functional analysis topic useful to generalize some concepts about linear ordinary differential equations to a Banach space framework. This increase of generality is useful to associate a parabolic equation to a linear differential equation in a Banach space. A detailed discussion on this topic and its applications to the study of differential equations can be found in [Lun12],[EN01],[Paz12]. The proofs of all the results stated in this chapter can be found in these books.

B.1. Definition and basic properties

Definition B.1.1. Let $(E, \|\cdot\|_E)$ be a Banach space, a family of operators $\{S(t) : t \geq 0\}$ belonging to $L(E)$ is a C_0 semigroup of linear operator if

1. $S(0) = I$.
2. $S(t + s) = S(t)S(s) \quad \forall t, s \geq 0$.
3. $S(\cdot)x$ is continuous in $[0, +\infty), \quad \forall x \in E$.

Definition B.1.2 (infinitesimal generator)**.** Let $S(t)$ a C_0 semigroup of linear operator in E, the infinitesimal generator of $S(\cdot)$, A, is the linear operator defined as

$$\begin{cases} D(A) = \{x \in E : \exists \ \ lim_{t\to 0^+} \frac{S(t)x-x}{t}\} \\ Ax = lim_{t\to 0^+} \frac{S(t)x-x}{t}, \quad \forall x \in D(A). \end{cases}$$

Before going on some properties of C_0 semigroup of linear operator we recall some classical functional analysis definitions.

Definition B.1.3. Let $(E, \|\cdot\|_E)$ be a Banach space, a linear operator $L : D(L) \subset E \to E$ is closed if its graph is closed, i.e. if

$$\mathscr{G}_L = \{(x, y) \in E \times E : x \in D(L), y = Lx\}$$

is closed in $E \times E$ with respect to the product topology.

If L is closed we always endowed $D(L)$ with the graph norm

$$\|x\|_{D(L)} = \|x\|_E + \|Lx\|_E$$

such a way $(D(L), \|\cdot\|_{D(L)})$ is a Banach space too.

Definition B.1.4. Let $(E, \|\cdot\|_E)$ be a Banach space and L a linear operator $L : D(L) \subset E \to E$, we define the resolvent set and the sprectrum of L respectively as

$$\rho(L) = \{\lambda \in \mathbb{C} : \exists(\lambda I - A)^{-1} \in L(E)\}, \quad \sigma(L) = \mathbb{C} \setminus \rho(L).$$

Now we can state some properties of a C_0 semigroup that will explain its usefulness.

Proposition B.1.5. *Let $S(\cdot)$ be a C_0 semigroup in E and A its infinitesimal generator. Then A is closed and $D(A)$ is dense in E. Moreover, if $x \in D(A)$, then*

$$S(\cdot)x \in C^1([0, +\infty); E) \cap C([0, +\infty); D(A)])$$

and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax, \quad t \geq 0.$$

Remark B.1.6. Last proposition explain how the concept of C_0 semigroup generalize the exponential mapping for solution of ordinary linear differential equation. In fact, if we consider the Cauchy problem

$$\begin{cases} u'(t) = Au(t), \quad t \geq 0 \\ u(0) = x \in E. \end{cases} \tag{B.1.1}$$

Then $S(t)$ is the solution map of this problem, namely $S(t)x = u(t, x) \ \ \forall x \in D(A)$. Given its analogies with the exponential function, sometimes $S(t)$ is also denoted as e^{At}.

Theorem B.1.7 (Hille-Yosida). *Let $A : D(A) \subset E \to E$ be a linear closed operator on E. Then the following statements are equivalent.*

- *A is the infinitesimal generator of a C_0 semigroup $S(\cdot)$ such that*

$$\|S(t)\| \leq Me^{\omega t} \quad t \geq 0.$$

- *$D(A)$ is dense in E, $(\omega, +\infty) \subset \rho(A)$ and the following estimates hold*

$$\|R(\lambda, A)\| \leq \frac{M}{(\lambda - \omega)^k}, \quad \forall k \in \mathbb{N}.$$

Moreover if the two properties above hold then

$$R(\lambda, A)x = \int_0^{+\infty} e^{-\lambda t} S(t)x \, dt, \quad \forall x \in E, \ \lambda > \omega.$$

Finally

$$S(t)x = \lim_{n \to +\infty} e^{tA_n} x \quad \forall x \in E$$

where $A_n = nAR(n, A)$ and the following estimate holds

$$\|e^{tA_n}\| \leq Me^{\frac{\omega n t}{n - \omega}}, \quad \forall t \geq 0, \quad n > \omega.$$

Remark B.1.8. If $M = 1$, then $S(\cdot)$ is called a pseudo-contraction C_0 semigroup. If in addition $\omega \leq 0$, it is called a contraction C_0 semigroup. The number

$$\omega_0 = \liminf_{t \to +\infty} \frac{1}{t} log(\|S(t)\|),$$

is called the type of the semigroup $S(\cdot)$. Then, for any $\epsilon > 0$ there exists $M_\epsilon \geq 1$ such that

$$\|S(t)\| \leq M_\epsilon e^{(\omega_0 + \epsilon)t}.$$

Remark B.1.9. The operator A_n are called the Yoshida approximations of A and they inherently interesting. In fact the following proposition holds.

Proposition B.1.10. *Let $A : D(A) \subset E \to E$ be the infinitesimal generator of a C_0 semigroup. Then*

$$\begin{cases} \lim_{n \to +\infty} nR(n, A)x = x, & \forall x \in E, \\ \lim_{n \to +\infty} A_n x = Ax, & \forall x \in D(A). \end{cases}$$

B.2. Analytic generator

For any $\omega \in \mathbb{R}$ and $\theta \in (0, \pi)$ we denote by

$$S_{\omega,\theta} = \{\lambda \in \mathbb{C} \setminus \{\omega\} : |arg(\lambda - \omega)| \leq \theta\}.$$

Assume now that A is a linear closed operator such that the following hypothesis holds.

Hypothesis B.0

1. $\exists\, \omega \in \mathbb{R},\ \theta_0 \in (\frac{\pi}{2}, \pi)$ such that $\rho(A) \supset S_{\omega,\theta_0}$
2. $\exists\, M > 0$ such that

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|} \quad \forall \lambda \in S_{\omega,\theta_0}.$$

Then we can define a semigroup $S(\cdot)$ of bounded linear operators in E by setting $S(0) = I$ and

$$S(t) = \frac{1}{2\pi i} \int_{\gamma_{\epsilon,\theta}} e^{\lambda t} R(\lambda, A) d\lambda, \quad t > 0. \tag{B.2.1}$$

In equation (B.2.1) $\theta \in (\frac{\pi}{2}, \theta_0)$ and $\gamma_{\epsilon,\theta}$ is the following path in $\mathbb{C}$

$$\gamma_{\epsilon,\theta} = \gamma^+_{\epsilon,\theta} \cup \gamma^-_{\epsilon,\theta} \cup \gamma^0_{\epsilon,\theta},$$

$$\gamma^{\pm}_{\epsilon,\theta} = \{z \in \mathbb{C}; z = \omega + re^{\pm i\theta}, r \geq \epsilon\},$$
$$\gamma^0_{\epsilon,\theta} = \{z \in \mathbb{C}; z = \omega + \epsilon e^{\pm i\eta}, |\eta| \leq \theta\}.$$

Remark B.2.1. Equation (B.2.1) is well defined since $\theta \geq \frac{\pi}{2}$ and by Cauchy theorem on holomorphic functions ($\lambda \to e^{t\lambda} R(\lambda, A)$ is holomorphic in $S_{\omega,\theta}$), $S(t)$ does not depend on the choice of ϵ and θ. In fact let $\frac{\pi}{2} < \theta_1 \leq \theta_2 < \theta$,

$$\int_{\gamma_{\epsilon_1,\theta_1}} e^{\lambda t} R(\lambda, A) d\lambda - \int_{\gamma_{\epsilon_2,\theta_2}} e^{\lambda t} R(\lambda, A) d\lambda = \lim_{n \to +\infty} \int_{C_n} e^{\lambda t} R(\lambda, A) d\lambda = 0.$$

Where C_n is the closed curve obtained linking $\{\lambda \in \gamma_{\epsilon_1,\theta_1} : |\lambda| \leq n\} - \{\lambda \in \gamma_{\epsilon_2,\theta_2} : |\lambda| \leq n\}$ with two circumference arcs of center ω and radius n. The integrals over C_n are identically 0 since the integrand is an olomorphic function.

Remark B.2.2. We say that $S(t)$ is the semigroup generared by A, nevertheless $D(A)$ may be not dense in E and $S(t)$ may not satisfy the continuity property required in the definition of C_0 semigroup.

Theorem B.2.3. *Assume that A fulfills hypothesis B.0 and $S(t)$ defined by equation (B.2.1). Then the following statements hold.*

1. *The mapping $S : (0, +\infty) \to L(E),\ t \to S(t)$ is analytic. Moreover for any $x \in E,\ t > 0$ and $n \in \mathbb{N},\ S(t)x \in D(A^n)$ and*

$$S^n(t)x = A^n S(t)x.$$

2. *$S(t + s) = S(t)S(s) \quad \forall t, s \geq 0$.*

3. *$S(\cdot)x$ is continuous in 0 if and only if $x \in \overline{D(A)}$.*

4. *There exist $M, N > 0$ such that*

$$\|S(t)\| \leq Me^{\omega t}, \quad \|AS(t)\| \leq e^{\omega t}(\frac{N}{t} + \omega M) \quad \forall t \geq 0.$$

5. *$S(\cdot)$ can be extended to an analytic $L(E)$-valued function in $S_{0,\theta_0 - \frac{\pi}{2}}$.*

Because of property 5 the semigroup $S(t)$ is called analytic. If in equation (B.1.1) A satisfies hypothesis B.0 then the solution of the equation shares many properties with the solution of a classical parabolic equation. Therefore, in this case Cauchy problem (B.1.1) is called parabolic.

Remark B.2.4. It can be proven (see for example [Sin85]) that A fulfilling hypothesis B.0 generates a C_0 semigroup if and only if $D(A)$ is dense in E.

Remark B.2.5. In the general case let call A_F the part of A in $F = \overline{D(A)}$ defined by

$$\begin{cases} D(A_F) = \{y \in D(A) \cap F : \ Ay \in F\} \\ A_F y = Ay, \quad \forall y \in D(A_F). \end{cases}$$

It can be proven that $D(A_F)$ is dense in F and the restriction $S_F(\cdot)$ of $S(\cdot)$ to F is a C_0 semigroup.

Remark B.2.6. If A fulfills hypothesis B.0 and $D(A)$ is dense in E, then $\sup_{\lambda \in \sigma(A)} Re(\lambda)$ is the type of the C_0 semigroup $S(t)$ as defined in remark B.1.8.

B.3. Fractional powers and interpolation spaces

In this section we assume the hypothesis B.0 holds and also that $\omega < 0$, so that the semigroup $S(t)$ is of negative type. With these hypotheses we can introduce some scales of subspace of E useful to analyze the regularity of the solution of the abstract Cauchy problem (B.1.1). With the notations of previous section, we set for $\alpha \in (0,1)$

$$(-A)^{-\alpha}x = \frac{1}{2\pi i}\int_{\gamma_{\epsilon,\theta}} (-\lambda)^{\alpha} R(\lambda, A)x d\lambda, \quad t > 0 \;\; x \in E.$$

It can be proven that $(-A)^{-\alpha}$ is one to one and we denote as $(-A)^{\alpha}$ its inverse with domain $D((-A)^{\alpha})$. It also can be proven that

$$(-A)^{\alpha}(-A)^{\beta} = (-A)^{\alpha+\beta}, \quad \forall \alpha, \beta \in (0,1) \; : \; \alpha + \beta \leq 1.$$

The operator $(-A)^{\alpha}$ are called fractional powers of $-A$ and their domains form a first scale of subspace of E. Thanks to representation formula (B.2.1) we can find a representation formula for $(-A)^{\alpha}S(\cdot)$ too.

Proposition B.3.1. *Let A be a linear operator fulfilling Hypothesis B.0 with $\omega <$ 0. Then for any $\alpha \in (0,1)$ and $t > 0$ we have $S(t)x \in D((-A)^{\alpha})$, $\forall x \in E$ and*

$$(-A)^{\alpha}S(t)x = \frac{1}{2\pi i}\int_{\gamma_{\epsilon,\theta}} (-\lambda)^{\alpha} e^{\lambda t} R(\lambda, A)x d\lambda.$$

Moreover for any $\epsilon > 0$, there exists $N_{\alpha,\epsilon} > 0$ such that

$$\|(-A)^{\alpha}S(t)\| \leq N_{\alpha,\epsilon} t^{-\alpha} e^{(\omega+\epsilon)t}.$$

B.4. Cauchy problem for nonhomogeneous equations

Let us consider the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), \quad t \in [0,T], \\ u(0) = x \in E. \end{cases} \tag{B.4.1}$$

where A is the infinitesimal generator of a C_0 semigroup $S(\cdot)$ in E and $f \in L^p(0,T;E) \quad p \geq 1$. There were introduced several notions of solution of this problem

Definition B.4.1.

(i) A strict solution of problem (B.4.1) in $L^p(0,T;E)$, $p \in [1, +\infty]$, is a function u that belongs to $W^{1,p}(0,T;E) \cap L^p([0,T];D(A))$ and fulfils (B.4.1).

(ii) A strict solution of problem (B.4.1) in $C(0,T;E)$, is a function u that belongs to $C^1([0,T];E) \cap C^0([0,T];D(A))$ and fulfils (B.4.1).

(iii) A weak solution of problem (B.4.1) is a function $u \in C([0,T];E)$ such that

$$\psi(u(t)) = \psi(x) + \int_0^t (A^*\psi)(u(s)) + \int_0^t f(s)\ ds, \quad \forall \psi \in D(A^*).$$

Remark B.4.2. A strict solution is a weak solution, too. The converse is generally false.

Now we introduce some sufficient condition for the existence of these kinds of solution.

Proposition B.4.3. *Let A be the infinitesimal generator of a C_0 semigroup $S(\cdot)$ in E and $f \in L^1(0,T;E)$. Then there exists a unique weak solution u of equation (B.4.1) and is given by the variation of constants formula*

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\ ds \quad t \in [0,T]. \tag{B.4.2}$$

The function $u(\cdot)$ defined by equation (B.4.2) is called mild solution of problem (B.4.1).

Proposition B.4.4. *Let A be the infinitesimal generator of a C_0 semigroup $S(\cdot)$ in E.*

(i) If $x \in D(A)$ and $f \in W^{1,p}(0,T;E)$ with $p \geq 1$, then problem (B.4.1) has a unique strict solution in $C([0,T];E)$, given by formula (B.4.2) and moreover $u \in C^1([0,T];E) \cap C^0([0,T];D(A))$.

(ii) If $x \in D(A)$ and $f \in L^p(0,T;D(A))$ then problem (B.4.1) has a unique strict solution in $L^p(0,T;E)$, given by formula (B.4.2) and moreover $u \in W^{1,p}(0,T;E) \cap C^0([0,T];D(A))$.

Proof. The proof of this fact can be found in [DZ14] and exploits Yosida approximations of A defined in section B.1. □

We introduce one last kind notion of solution weaker than strict in $C(0,T;E)$.

Definition B.4.5. Let $f \in C^0([0,T];E)$ and $x \in E$. A strong solution of problem (B.4.1) is a function $u(\cdot)$: $u(0) = x$ and there exists a sequence $\{u_n\}_{n \in N} \subset C^1([0,T];E) \cap C^0([0,T];D(A))$ such that

$$u_n' - Au_n \to f \ in\ C([0,T];E)\ n \to +\infty,$$

$$u_n \to u\ in\ C([0,T];E)\ n \to +\infty.$$

Proposition B.4.6. *Let $x \in X$, $f \in C^0([0,T];X)$. Then problem (B.4.1) has a unique strong solution given by formula (B.4.2).*

Appendix C

Operator Theory Tools

In this chapter we will introduce some tools of functional analysis necessary for the definition of the stochastic integral in infinite dimensional spaces. A more complete treatment of these topics and all the proofs omitted can be found in [LR15],[DZ14] or in some classical books of operator theory[VS17],[Mor10]. In all this chapter $(U, \langle\cdot,\cdot\rangle_U)$ and $(H, \langle\cdot,\cdot\rangle)$ will be two separable Hilbert spaces. Also we will say that $L \in L(U)$ is nonnegative if $\langle Lu, u\rangle_U \geq 0$ for all $u \in U$.

C.1. Nuclear and Hilbert-Schmidt operators

Definition C.1.1 (Nuclear operator)**.** An element $T \in L(U, H)$ is said to be a nuclear operator if there exists a sequence $\{a_j\}_{j\in\mathbb{N}} \subset H$ and a sequence $\{b_j\}_{j\in\mathbb{N}} \subset U$ such that

$$Tx = \sum_{j=1}^{+\infty} a_j \langle b_j, x\rangle_U \quad \forall x \in U$$

and

$$\sum_{j=1}^{+\infty} \|a_j\|\|b_j\|_U < +\infty.$$

The space of nuclear operator from U to H is denoted by $L_1(U, H)$.

If $U = H$, $T \in L_1(U, H)$ and T is symmetric, then T is called trace class.

Proposition C.1.2. *The space $L_1(U, H)$ endowed with the norm*

$$\|T\|_{L_1(U,H)} := inf \left\{ \sum_{j=1}^{+\infty} \|a_j\|\|b_j\|_U \;\middle|\; Tx = \sum_{j=1}^{+\infty} a_j \langle b_j, x\rangle_U, \quad x \in U \right\}$$

is a Banach space.

Definition C.1.3. Let $T \in L(U)$ and $e_k,\ k \in \mathbb{N}$ an orthonormal basis of U. Then we define

$$trT := \sum_{j=1}^{+\infty} \langle Te_k, e_k \rangle_U$$

if the series is convergent.

Remark C.1.4. It can be proven (see for example [LR15]) that the definition does not depend of the choice of the orthonormal basis $e_k,\ k \in \mathbb{N}$. Moreover we have that $|trT| \leq \|T\|_{L_1(U,H)}$.

Definition C.1.5 (Hilbert-Schmidt operator)**.** $T \in L(U,H)$ is called an Hilbert-Schmidt operator if

$$\sum_{k=1}^{+\infty} \|Te_k\|^2 < +\infty$$

The space of Hilbert-Schmidt operator from U to H is denoted by $L_2(U,H)$.

Remark C.1.6.

(i) The definition of Hilbert-Schmidt operator and the number

$$\|T\|^2_{L_2(U,H)} := \sum_{k=1}^{+\infty} \|Te_k\|^2$$

do not depend of the choice of the orthonormal basis $e_k,\ k \in \mathbb{N}$. Moreover $\|T\|_{L_2(U,H)} = \|T^*\|_{L_2(H,U)}$.

(ii) $\|T\|_{L(U,H)} \leq \|T\|_{L_2(U,H)}$.

(iii) Let G be another Hilbert space, $S_1 \in L(H,G),\ S_2 \in L(G,U),\ T \in L_2(U,H)$. Then $S_1 T \in L_2(U,G),\ TS_2 \in L_2(G,H)$ and

$$\|S_1 T\|_{L_2(U,G)} \leq \|T\|_{L_2(U,H)} \|S_1\|_{L(H,G)},$$

$$\|TS_2\|_{L_2(G,H)} \leq \|T\|_{L_2(U,H)} \|S_2\|_{L(G,U)}.$$

Proof. (i) Let $e_k\ k \in U$ and $f_k\ k \in H$ be orthonormal basis of U and H respectively.

$$\|T\|^2_{L_2(U,H)} = \sum_{k=1}^{+\infty} \langle Te_k, Te_k \rangle = \sum_{k=1}^{+\infty} \langle T^*Te_k, e_k \rangle_U = tr(T^*T)$$

which does not depend of the choice of the orthonormal basis by remark C.1.4. By Parceval identity we get

$$\|T\|^2_{L_2(U,H)} = \sum_{k=1}^{+\infty} \langle Te_k, Te_k\rangle = \sum_{k=1}^{+\infty}\sum_{j=1}^{+\infty} |\langle Te_k, f_j\rangle|^2 = \sum_{j=1}^{+\infty} \|T^* f_j\|^2_U = \|T^*\|^2_{L_2(H,U)}.$$

(ii) Let $x \in U$ and f_k $k \in H$ be an orthonormal basis of H. Then we get

$$\|Tx\|^2 = \sum_{k=1}^{+\infty} \langle Tx, f_k\rangle^2 \leq \|x\|^2_U \sum_{k=1}^{+\infty} \|T^* f_k\|^2_U = \|T\|^2_{L_2(U,H)} \|x\|^2_U$$

which is the thesis.

(iii) Let e_k be an orthonormal basis of U. Then

$$\sum_{k=1}^{+\infty} \|S_1 T e_k\|^2_G \leq \|S_1\|^2_{L(H,G)} \|T\|^2_{L_2(U,H)}.$$

For what concern the second claim, since $(TS_2)^* = S_2^* T^*$ and $\|S_2^*\|_{L(U,G)} = \|S_2\|_{L(G,U)}$ we get

$$\|TS_2\|_{L_2(G,H)} = \|(TS_2)^*\|_{L_2(H,G)} = \|S_2^* T^*\|_{L_2(H,G)} \leq \|T\|_{L_2(U,H)} \|S_2\|_{L(G,U)}.$$

Hence $TS_2 \in L_2(G,H)$ and the thesis follows.

□

The following proposition can be proven with classical arguments (see for example [LR15]).

Proposition C.1.7. *Let $S, T \in L_2(U,H)$ and e_k an orthonormal basis of U. If we define*

$$\langle S, T\rangle_{L_2} := \sum_{k=1}^{+\infty} \langle Se_k, Te_k\rangle$$

we obtain that $(L_2(U,H), \langle\cdot,\cdot\rangle_{L_2})$ is a separable Hilbert space. Moreover if f_k is an orthonormal basis of H we get that $f_j \otimes e_k := f_j \langle e_k, \cdot\rangle_U$, $j,k \in \mathbb{N}$ is an orthonormal basis of $L_2(U,H)$.

Proposition C.1.8. *Let $(G, \langle\cdot,\cdot\rangle_G)$ be a further Hilbert space. If $T \in L_2(U,H)$ and $S \in L_2(H,G)$ then $ST \in L_1(U,G)$ and*

$$\|ST\|_{L_1(U,G)} \leq \|S\|_{L_2} \|T\|_{L_2}.$$

Proof. Let $f_k,\ k \in \mathbb{N}$ be an orthonormal basis of H. Then we have

$$STx = \sum_{k=1}^{+\infty} \langle Tx, f_k \rangle S f_k, \quad x \in U.$$

Therefore by definition of norm in $L_1(U, G)$ and Cauchy-Scwhartz inequality it is possible to get

$$\|ST\|_{L_1(U,G)} \leq \sum_{k=1}^{+\infty} \|T^* f_k\|_U \|S f_k\|_G \leq (\sum_{k=1}^{+\infty} \|T^* f_k\|_U^2)^{\frac{1}{2}} (\sum_{k=1}^{+\infty} \|S f_k\|_G^2)^{\frac{1}{2}} = \|S\|_{L_2} \|T\|_{L_2}.$$

□

Remark C.1.9. Let $e_k\ k \in \mathbb{N}$ an orthonormal basis of U. An immediate consequence of previous proposition is that if $T \in L(U)$ is symmetric, nonnegative and $\sum \langle Te_k, e_k \rangle < +\infty$, then $T \in L_1(U)$.

We end this section with another proposition which links Hilbert-Schmidt and nuclear operator. The proof can be found in [LR15] and it is an easy consequence of Proposition C.1.8 and remark C.1.6.

Proposition C.1.10. *Let $L \in L(H),\ B \in L_2(U, H)$. Then $LBB^* \in L_1(H), B^*LB \in L_1(U)$ and we have that*

$$trLBB^* = trB^*LB.$$

C.2. The pseudo inverse of linear operators

Definition C.2.1 (Pseudo Inverse)**.** Let $T \in L(U, H)$. The pseudo inverse of T is defined as

$$T^{-1} := (T|_{(KerT)^\perp})^{-1} : T(Ker(T)^\perp) = T(U) \to Ker(T)^\perp.$$

Remark C.2.2. If $T \in L(U, H)$ then $T^{-1} : T(U) \to Ker(T)^\perp$ is linear and bijective.

Proposition C.2.3. *Let $T \in L(U)$ and T^{-1} the pseudo inverse of T.*

(i) If we define an inner product on $T(U)$ by

$$\langle x, y \rangle_{T(U)} := \langle T^{-1}x, T^{-1}y \rangle_U \quad \forall x, y \in T(U),$$

then $(T(U), \langle \cdot, \cdot \rangle_{T(U)})$ is an Hilbert space.

(ii) Let $e_k,\ k \in \mathbb{N}$, be an orthonormal basis of $(KerT)^\perp$. Then Te_k is an orthonormal basis of $(T(U), \langle \cdot, \cdot \rangle_{T(U)})$.

Now it is possible to present some results about the image of linear operators.

Proposition C.2.4. *Let* $(U_1, \langle\cdot,\cdot\rangle_1)$, $(U_2, \langle\cdot,\cdot\rangle_2)$ *be two Hilbert spaces,* $T_1 \in L(U_1, H)$ *and* $T_2 \in L(U_2, H)$. *Then the following statements hold:*

(i) if there exists $c \geq 0$ *such that* $\|T_1^* x\|_{U_1} \leq \|T_2^* x\|_{U_2}$ *for all* $x \in H$, *then*

$$\{T_1 u \mid u \in U_1, \|u\|_1 \leq 1\} \subset \{T_2 v \mid v \in U_2, \|v\|_2 \leq c\}.$$

In particular, this implies $ImT_1 \subset ImT_2$.

(ii) if $\|T_1^* x\|_{U_1} = \|T_2^* x\|_{U_2}$ *for all* $x \in H$, *then* $ImT_1 = ImT_2$ *and* $\|T_1^{-1} x\|_{U_1} = \|T_2^{-1} x\|_{U_2}$ *for all* $x \in ImT_1$, *where* T_i^{-1} *is the pseudo inverse of* T_i $i \in \{1, 2\}$.

Proof. The proof of this fact can be found in [LR15] and it follows by an application of the geometric version of the Hanh-Banach theorem (see [Bre10]). □

Corollary C.2.5. *Let* $T \in L(U, H)$ *and set* $Q = TT^* \in L(H)$. *Then we have*

$$ImQ^{\frac{1}{2}} = ImT \quad \|Q^{-\frac{1}{2}} x\| = \|T^{-1} x\|_U \quad \forall x \in ImT,$$

where $Q^{-\frac{1}{2}}$ *is the pseudo inverse of* $Q^{\frac{1}{2}}$.

Proof. Since $Q = TT^*$ then Q is symmetric and nonnegative, so $Q^{\frac{1}{2}}$ exists and is symmetric. Moreover $\forall x \in H$

$$\|(Q^{\frac{1}{2}})^* x\|^2 = \|Q^{\frac{1}{2}} x\|^2 = \langle Qx, x\rangle = \langle T^* T x, x\rangle = \|T^* x\|_U^2.$$

Then it is possible to apply proposition C.2.4. and the thesis follows. □

Appendix D

The Stochastic Integral for Cylindrical Wiener Processes

In this chapter we will present the Cylindrical Wiener Process and the stochastic integral with respect of such kind of processes. A complete and deeper presentation of the topics analyzed in this appendix can be found in [LR15] and [DZ14]. In all this chapter $(U, \langle\cdot,\cdot\rangle_U)$ and $(H, \langle\cdot,\cdot\rangle)$ will be two separable Hilbert spaces and $(\Omega, \mathcal{F}, \mathbb{P})$ will be a probability space.

D.1. Cylindrical Wiener processes

Let $Q \in L(U)$ be nonnegative definite and symmetric. If Q is of finite trace then a Q-Wiener process can be represented as

$$W(t) = \sum_{k\in\mathbb{N}} \beta_k(t) e_k, \quad t \in [0,T] \tag{D.1.1}$$

where $e_k,\ k \in \mathbb{N}$, is an orthonormal basis of $U_0 := Q^{\frac{1}{2}}(U)$ (with inner product defined as in section C.2) and $\beta_k,\ k \in \mathbb{N}$, is a family of independent real-valued Brownian motion. In this case the series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0,T];U))$ because the inclusion $U_0 \subset U$ is a Hilbert-Schmidt embedding, hence, for a fixed t, in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$ too. In general, we can recover this convergence property introducing a further Hilbert space $(U_1, \langle\cdot,\cdot\rangle_1)$ and a Hilbert-Schmidt embedding $J : (U_0, \langle\ ,\ \rangle_0) \to (U_1, \langle\cdot,\cdot\rangle_1)$.

Proposition D.1.1. *Let $e_k,\ k \in \mathbb{N}$, be an orthonormal basis of U_0 and $\beta_k,\ k \in \mathbb{N}$, a family of independent real-valued Brownian motion. Define $Q_1 := JJ^*$. Then*

$Q_1 \in L(U_1)$, is nonnegative definite,symmetric, with finite trace and the series

$$W(t) = \sum_{k=1}^{+\infty} \beta_k(t) J e_k, \quad t \in [0, T]$$

converges in $\mathcal{M}_T^2(U_1)$[1] *and define a Q_1-Wiener process on U_1. Moreover we have that $Q_1^{\frac{1}{2}}(U_1) = J(U_0)$ and $\forall\ u_0 \in U_0$*

$$\|u_0\|_0 = \|Q^{-\frac{1}{2}} J u_0\|_1 = \|J u_0\|_{Q_1^{\frac{1}{2}}(U_1)}$$

i.e. $J : U_0 \to Q_1^{\frac{1}{2}}(U_1)$ is an isometry.

Proof. The proof of this fact can be found in [LR15]. The first claim is just a matter of patience, second claim follows by an application of corollary C.2.5. □

Definition D.1.2. Let $e_k,\ k \in \mathbb{N}$, be an orthonormal basis of U_0 and $\beta_k,\ k \in \mathbb{N}$, a family of independent real-valued Brownian motion. Define $Q_1 := JJ^*$. Then

$$W(t) = \sum_{k=1}^{+\infty} \beta_k(t) J e_k, \quad t \in [0, T]$$

is a cylindrical Q-Wiener process in U.

D.2. Stochastic integral for Wiener processes

Definition D.2.1 (Elementary process $\mathcal{E}$)**.** A process $\Phi(t)$, on $(\Omega, \mathcal{F}, \mathbb{P})$, with normal filtration $\mathcal{F}_t$, taking value in $L(U, H)$ is said to be elementary if there exists $0 = t_0 < \cdots < t_k = T,\ k \in \mathbb{N}$, such that

$$\Phi(t) = \sum_{m=0}^{k} \Phi_m 1_{]t_m, t_{m+1}]}(t), \quad t \in [0, T]$$

where:

[1] let E be a Banach space, for a fixed $0 < T < +\infty$ we denote by $\mathcal{M}_T^2(E)$ the space of all E-valued continuous square integrable martingales $M(t),\ t \in [0, T]$. It can be proven that $\mathcal{M}_T^2(E)$ with the norm

$$\|M\|_{\mathcal{M}_T^2(U_1)} := \sup_{t \in [0,T]} (\mathbb{E}(\|M(t)\|))^{\frac{1}{2}} = (\mathbb{E}(\|M(T)\|))^{\frac{1}{2}}$$

is a Banach space.

- $\Phi_m : \Omega \to L(U,H)$ is $\mathcal{F}_{t_m}$ measurable with respect to Borel σ-algebra on $L(U,H)$, $0 \le m \le k-1$.
- Φ_m takes only a finite number of values in $L(U,H)$.

For a standard Q-Wiener process $W(t)$ we define

$$Int(\Phi)(t) := \int_0^t \Phi(s)dW(s) := \sum_{m=0}^{k-1} \Phi_m(W(t_{m+1} \wedge t) - W(t_m \wedge t)), \quad t \in [0,T].$$

Remark D.2.2. It can be proven that $Int : \mathcal{E} \to \mathcal{M}_T^2$ and

$$\| \int_0^{\cdot} \Phi(s)dW(s)\|^2_{\mathcal{M}_T^2} = \mathbb{E}\left[\int_0^T \|\Phi(s) \circ Q^{\frac{1}{2}}\|^2_{L_2} ds\right] := \|\Phi\|_T^2.$$

The last equality is called Ito isometry. In particular

$$Int : (\mathcal{E}, \|\cdot\|_T) \to (\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2})$$

is an isometric transformation and there is a unique isometric extension of Int to $\bar{\mathcal{E}}$ (namely the closure of $\mathcal{E}$ with respect to $\| \; \|_T$).

An explicit representation of $\bar{\mathcal{E}}$ is available:
$\mathcal{N}_W^2(0,T;H) := \bar{\mathcal{E}} = \{\Phi : [0,T] \times \Omega \to L_2^0 := L_2(U_0,H) | \; \Phi$ is predictable and $\|\Phi\|_T < +\infty\} = L^2([0,T]\times\Omega, \mathcal{P}_T{}^2, dt \otimes \mathbb{P}; L_2^0)$. Via localization argument, actually, the definition of stochastic integral can be extended to a more general class of processes:

$$\mathcal{N}_W(0,T;H) := \left\{ \Phi : \Omega \times [0,T] \to L_2^0 \;\middle|\; \Phi \textit{ is predictable with } \mathbb{P}\left(\int_0^T \|\Phi(s)\|^2_{L_2^0} < +\infty\right) = 1 \right\}.$$

For $\Phi \in \mathcal{N}_W(0,T;H)$ we define $\tau_n := \{t \in [0,T] | \int_0^t \|\Phi(s)\|^2_{L_2^0} > n\} \wedge T$ and

$$\int_0^t \Phi(s)dW(s) := \int_0^t 1_{]0,\tau_n]}(s)\Phi(s)dW(s) \quad \omega \in \{\tau_n \ge t\}.$$

In the general case $Q \in L(U)$ nonnegative, symmetric but not necessarily of finite trace the idea is to integrate with respect to the standard U_1-valued Q_1-Wiener process given by proposition D.1.1. Thanks to proposition D.1.1. we also get

$$\Phi \in L_2^0 = L_2\left(Q^{\frac{1}{2}}(U), H\right) \iff \Phi \circ J^{-1} \in L_2\left(Q_1^{\frac{1}{2}}(U_1), H\right),$$

[2] $\mathcal{P}_T := \sigma(Y : \Omega \times [0,T] \to \mathbb{R} | \; Y$ is left continuous and adapted to $\mathcal{F}_t, \; t \in [0,T])$.

then we define

$$\int_0^t \Phi(s)dW(s) := \int_0^t \Phi \circ J^{-1}(s)dW(s) \quad t \in [0,T]. \tag{D.2.1}$$

So, in both cases of standard or cylindrical Wiener process, the class of integrable processes is given by

$$\mathcal{N}_W(0,T;H) = \left\{ \Phi : \Omega \times [0,T] \to L_2^0 \,\middle|\, \Phi \textit{ is predictable with } \mathbb{P}\left(\int_0^T \|\Phi(s)\|^2_{L_2^0} < +\infty \right) = 1 \right\}.$$

Remark D.2.3. The definition given by equation (D.2.1) does not depend on the choice of $(U_1, \langle\ ,\ \rangle_{U_1})$ and J. This is due to the fact that the definition is independent in the case of elementary processes.

D.3. Properties of stochastic integral

In this section we collect some properties of stochastic integral. First two representation formulas, then some miscellaneous results.

Lemma D.3.1. *Let $W(t)$, $t \in [0,T]$ be a standard Q-Wiener process and $\Phi \in \mathcal{N}_W^2(0,T;H)$. Then $\mathbb{P} - a.s.$*

$$\int_0^t \Phi(s)dW(s) = \sum_{k=1}^{+\infty} \sqrt{\lambda_k} \int_0^t \Phi(s)(f_k)\ d\beta_k(s), \quad t \in [0,T]$$

where $\lambda_k,\ f_k,\ \beta_k,\ k \in \mathbb{N}$ are as in the representation (D.1.1) (i.e. $\sqrt{\lambda_k} f_k = e_k$ is an orthonormal basis of U_0) and the sum on the right-hand side converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0,T];H))$.

More generally:

Proposition D.3.2. *Let $W(t)$, $t \in [0,T]$ be a cylindrical Q-Wiener process and $\Phi(t) \in \mathcal{N}_W^2(0,T;H)$. Then $\mathbb{P} - a.s.$*

$$\int_0^t \Phi(s)dW(s) = \sum_{k=1}^{+\infty} \int_0^t \Phi(s)(e_k)\ d\beta_k(s), \quad t \in [0,T]$$

where $e_k,\ \beta_k,\ k \in \mathbb{N}$ are as definition (D.1.2) and the sum on the right-hand side converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0,T];H))$.

Proof. By definition

$$\int_0^t \Phi(s)dW(s) := \int_0^t \Phi(s)J^{-1}dW(s).$$

For $\Phi(s)J^{-1}$ lemma D.3.1. holds (obviously in U_1). Given e_k orthonormal basis of U_0, by proposition D.1.1. we get that Je_k is an orthonormal basis of $Q^{\frac{1}{2}}(U_1)$, then

$$\int_0^t \Phi(s)J^{-1}dW(s) = \sum_{k=1}^{+\infty}\int_0^t \Phi(s)\circ J^{-1}(Je_k)d\beta_k(s) = \sum_{k=1}^{+\infty}\int_0^t \Phi(s)(e_k)\, d\beta_k(s), \quad \forall t \in [0,T]$$

and convergence properties hold. □

Theorem D.3.3. *Assume that $\Phi \in \mathcal{N}_W^2(0,T;H)$ then the stochastic integral $\int_0^t \Phi(s)dW(s)$ is a continuous square integrable martingale, and its quadratic variation is of the form*

$$\langle \int_0^t \Phi(s)dW(s)\rangle = \int_0^t Q_\Phi(s)\ ds$$

where $Q_\Phi(s) = (\Phi(s)Q^{\frac{1}{2}})(\Phi(s)Q^{\frac{1}{2}})^*$, $s,t \in [0,T]$. If $\Phi \in \mathcal{N}_W(0,T;H)$, then $\langle\int_0^t \Phi(s)dW(s)\rangle$ is a local martingale.

Proposition D.3.4. *Let Φ be a L_2^0-valued stochastically integrable process, $(\tilde{H}, \langle\ ,\ \rangle_{\tilde{H}})$ a further separable Hilbert space and $L \in L(H,\tilde{H})$. Then the process $L(\Phi(t))$, $t \in [0,T]$, is an element of $\mathcal{N}_W(0,T;\tilde{H})$ and*

$$L\left(\int_0^T \Phi(t)dW(t)\right) = \int_0^T L(\Phi(t))dW(t) \quad \mathbb{P}-a.s.$$

Proposition D.3.5. *Assume $\Phi_1,\ \Phi_2 \in \mathcal{N}_W^2(0,T;H)$. Then*

$$\mathbb{E}\left[\int_0^t \Phi_i(s)dW(s)\right] = 0, \quad \mathbb{E}\left[\|\int_0^t \Phi_i(s)dW(s)\|^2\right] < +\infty, \quad t \in [0,T],\ i \in \{1,2\}.$$

Moreover the correlation operators

$$V(t,s) = Cor\left(\int_0^t \Phi_1(r)dW(r), \int_0^s \Phi_2(r)dW(r)\right), \quad t,s \in [0,T]$$

are given by the formulas

$$V(t,s) = \mathbb{E}\left[\int_0^{t\wedge s}(\Phi_2(r)Q^{\frac{1}{2}})(\Phi_1(r)Q^{\frac{1}{2}})^* dr\right].$$

Remark D.3.6. All the missing proofs and some interesting results about stochastic integral can be found in [LR15], [DZ14].

Appendix E

Some Stochastic Tools

In this appendix we introduce some results used repeatedly in chapter 2 and chapter 3. For the proofs of these facts and a deeper discussion see [Bal17],[DZ14].

E.1. Kolmogorov's continuity theorem

Theorem E.1.1. *Let $D \subset \mathbb{R}^m$ be an open set and $(X_y)_{y\in D}$ a family of d-dimensional random variables such on $(\Omega, \mathcal{F}, \mathbb{P})$ such that there exist $\alpha > 0,\ \beta > 0,\ c > 0$ satisfying*

$$\mathbb{E}[|X_y - X_z|^\beta] \leq c|y-z|^{m+\alpha}.$$

Then there exists a family $(\tilde{X}_y)_{y\in D}$ of $\mathbb{R}^d$-valued random variables such that

$$X_y = \tilde{X}_y \quad a.s. \quad \forall y \in D.$$

(i.e. $\tilde{X}$ is a modification of X) and that, for every $\omega \in \Omega$, the map $y \to \tilde{X}_y(\omega)$ is continuous and even Hölder continuous with exponent γ for every $\gamma < \frac{\alpha}{\beta}$ on every compact subset of D.

Proof. The proof of this fact can be found in [Bal17]. □

E.2. The Itô Formula

Let $(U, \|\cdot\|_U)$ and $(H, \|\cdot\|_H)$ be separable Hilbert spaces. We fix a cylindrical Wiener process $W(t),\ t \geq 0$, in U on a probability space $(\Omega; \mathcal{F}; \mathbb{P})$ with a normal filtration $\mathcal{F}_t, t \geq 0$. Moreover, we fix $T > 0$. We assume that

- $\Phi \in \mathcal{N}_W(0, T; H)$

- $\varphi : \Omega \times [0,T] \to H$ is a predictable and $\mathbb{P}$-a.s. Bochner integrable process on $[0,T]$
- $X(0) : \Omega \to H$ is $\mathcal{F}_0$-measurable
- $F : [0,T] \times H \to \mathbb{R}$ is twice Fréchet differentiable with derivatives

$$\frac{\partial F}{\partial t} := D_1 F : [0,T] \times H \to \mathbb{R}$$
$$DF := D_2 F : [0,T] \times H \to L(H, \mathbb{R})$$
$$D^2 F := D_2^2 F : [0,T] \times H \to L(H)$$

which are uniformly continuous on bounded subsets of $[0,T] \times H$.
Under these assumptions the process

$$X(t) := X(0) + \int_0^t \varphi(s) ds + \int_0^t \Phi(s) dW(s), \quad t \in [0,T]$$

is well defined and the following Itô Formula holds.

Theorem E.2.1. *There exists a $\mathbb{P}$-null set $N \in \mathcal{F}$ such that the following formula is fulfilled on N^c for all $t \in [0,T]$:*

$$\begin{aligned} F(t, X(t)) = F(0, X(0)) &+ \int_0^t \langle DF(s, X(s)), \Phi(s) dW(s) \rangle^1 \\ &+ \int_0^t \frac{\partial F}{\partial t}(s, X(s)) + \langle DF(s, X(s)), \varphi(s) \rangle \\ &+ \frac{1}{2} tr \left[D^2 F(s, X(s)) (\Phi(s) Q^{\frac{1}{2}}) (\Phi(s) Q^{\frac{1}{2}})^* \right] ds \end{aligned}$$

Proof. The proof of this fact can be found in [DZ14]. □

E.3. Burkholder–Davis–Gundy Inequality

In this section we follow all the notation introduced in the previous one. Then we can state the following result.

Theorem E.3.1. *Let $p \geq 2$ and $\Phi \in \mathcal{N}_W(0,T;H)$. Then*

$$\left(\mathbb{E} \left[\sup_{t \in [0,T]} \| \int_0^t \Phi(s) dW(s) \|^p \right] \right)^{\frac{1}{p}} \leq p \left(\frac{p}{2(p-1)} \right)^{\frac{1}{2}} \left(\int_0^T (\mathbb{E}[\|\Phi(s)\|_{L_2^0}^p])^{\frac{2}{p}} ds \right)^{\frac{1}{2}}.$$

[1] $\int_0^T \langle f(t), \Phi(t) dW(t) \rangle := \int_0^T \tilde{\Phi}_f(t) \, dW(t)$ with $\tilde{\Phi}_f(t)(u) := \langle f(t), \Phi(t) u \rangle, \quad u \in U_0$

Proof. The proof of this fact can be found in [DZ14]. □

Remark E.3.2. If $\Phi \in \mathcal{N}_W^2(0,T;H)$ we get that $\int_0^t \Phi(s)dW(s)$ is a martingale and therefore

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|\int_0^t \Phi(s)dW(s)\|^2\right] = \mathbb{E}\left[\|\int_0^T \Phi(s)dW(s)\|^2\right].$$

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