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STABILITY IN NONLINEAR MODELS FOR SUSPENSION BRIDGES

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Abstract

IN this thesis the stability of some different models for suspension bridges is studied. In particular, the main concern of this research is the distribution of the energy among the fundamental vibrations of the structures considered. Our work is motivated by the fact that many bridges suffered unexpected oscillations, sometimes leading to collapses, under moderate external stimulation due to the wind or the crowd load. The most famous example of such phenomena, both because of the intriguing video testimony and because of the huge interest it generated among engineers and mathematicians, is represented by the failure of the Tacoma Narrow Bridge, occurred in 1940. So far, the scientific community has not given a unanimous explanation of such accidents, also because of the large variety of different physical phenomena involved in the dynamics of suspension bridges, including both external forces and internal structural interactions. In this work, we analyze some of these factors.

We introduce an abstract nonlinear nonlocal evolution equation modeling the dynamics of real-world structures subjected to an external load. This model turns out to be suitable to describe plates undergoing large deflections and suspension bridges with multiple intermediate piers. Some rigorous finite-dimensional approximations of the problem are studied. More precisely, we prove that our equation may be asymptotically approximated by a finite-dimensional system of ordinary differential equations under rather general hypotheses on the external load. In the case of antiperiodic in time forcing terms, we refine our results and we exploit them to analyze the distribution of the energy among the longitudinal fundamental modes of a suspension bridge as the position of the piers varies. We show that, according to the model considered, asymmetric suspension bridges appear to be more stable than suspension bridges with piers symmetric with respect to the center of the deck.

In order to analyze the appearance of the torsional motion in suspension bridges, we examine a degenerate plate model also described by two coupled nonlinear nonlocal evolution equations. This system represents the interaction between longitudinal and torsional motions generated by the presence of the sustaining cables under the hypothesis of rigid hangers. The action of the wind along the deck of the bridge is not considered as an explicit external force. Instead, in order to focus on the role of the structural nonlinearities, the aerodynamic contribution to the dynamics is introduced through the initial conditions. Since we are interested in the torsional motion triggered by the internal resonances of the bridge, we consider the linearization of the model in a neighbourhood of a purely longitudinal motion. The mathematical analysis of these equations strongly depends on the boundary conditions. First, we study the case with boundary conditions describing a

partially hinged plate model and we use some classical methods for the stability of the Hill equation to our problem. Next, we consider partially clamped boundary conditions and we employ a KAM reducibility scheme to study the stability of this system. In both cases, the torsional dynamics is proven to be stable for a large measure set of longitudinal initial data.

Further developments may follow from this thesis. Our study of asymptotic approximations of plate models undergoing an external load, though easily generalizable to large families of nonlinearities, does not cover the cases involving nonlinear nonlocal damping terms. Moreover, our analysis of the torsional stability of degenerate plates does not apply to suspension bridges with multiple intermediate piers because of the presence of weaker second order Melnikov's conditions in this case. Therefore, the extension of our results to more general models represents a concrete challenge, to which future works might be devoted.

Contents

CHAPTER 1

Introduction

Estrema temerità mi è parsa sempre quella di coloro che voglion far la capacità umana misura di quanto possa e sappia operar la natura, dove che, all'incontro, e' non è effetto alcuno in natura, per minimo che e' sia, all'intera cognizion del quale possano arrivare i piú specolativi ingegni. Questa cosí vana prosunzione d'intendere il tutto non può aver principio da altro che dal non avere inteso mai nulla, perché, quando altri avesse esperimentato una volta sola a intender perfettamente una sola cosa ed avesse gustato veramente come è fatto il sapere, conoscerebbe come dell'infinità dell'altre conclusioni niuna ne intende.

Galileo Galilei, in *Dialogo sopra i due massimi sistemi*, Giornata I

One of the main characteristics commonly considered as defining of a living being is the ability to respond to external stimuli. This capacity of interacting with the surrounding environment manifests itself in the tendency of moving. For examples, bacteria tend to move towards or away from chemicals (*chemotaxis*) or light (*phototaxis*), plants climb fences and walls and even more complex and somewhat surprising behaviours might be observed in a lot of different species. In particular, in the quest for resources and wellness, many animals, from ants to swallows, continuously overcome a separation between two places, let it be due to the presence of a river, an obstacle or simply the distance itself. Nonetheless, as observed by the german philosopher Georg Simmel, *"path-building [...] is a specifically human achievement"* since the animal *"do not accomplish the miracle of a road: freezing movement into a solid structure that commences from it and in which it terminates. This achievement reaches its zenith in the construction of a bridge."* [147].

The construction of a bridge represents a concrete challenge from many different aspects, both economical and engineering. Because of its affordability, its ability to cover long spans and its elegance, from the beginning of the XIX century until today, many suspension bridges have been built. A suspension bridge is composed by four high towers sustaining two parallel cables which in turn sustain the hangers and, lastly, the hangers are anchored to the roadway and sustain it from above. Often, the roadway is supported from below by a girder composed by stiffening trusses in order to improve the solidity as well as the stability of the structure.

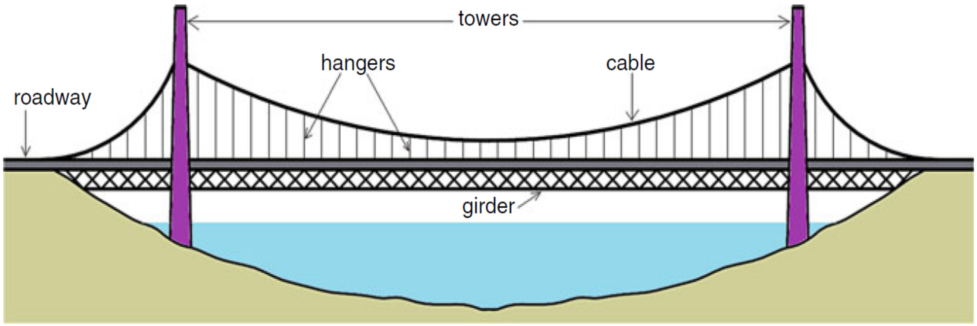

Figure 1.1: *Skecth of a suspension bridge, from [82].*

1.1 A short history of suspension bridges

The first design of a modern suspension bridge (Pons Ferreus) was a compromise between a suspension bridge and a cable-stayed bridge and it is attributed [82, 111, 137] to Fausto Veranzio (1551-1617) in the year 1595 [152]. This bridge, together with a similar project (Pons Canabeus), was never built and the first suspension bridges were erected only at the beginning of XIX century in Pennsylvania and, later, in England and France. Nonetheless, as noticed by Waddel in [153], at the end of the XIX century *"the field of bridgework [...] was neither a science nor an art, but merely a business or trade - and a poor one at that!"* and some pages later we found written that *"The designs of railway bridges in general were simply atrocious! They were absolutely unscientific!"*. Despite this lack of a rigorous science of construction, suspension bridges proved to be in many cases the most economic and elegant option and, for this reason, they kept being built.

Unfortunately, due to a large variety of different factors, many bridges suffered unexpected oscillations, sometimes leading to collapses, see e.g. [4, 49, 105]. In particular, the stability of such structures with respect to the action of the wind had been questioned quite early [137, p.161]. The failure of the Tacoma Narrow Bridge (TNB), occurred in 1940, raised a particular attention on the topic and the sudden change from a vertical to a torsional mode of oscillation was considered crucial by the board of engineers appointed by the Federal Works Agency to investigate about the accident [6]. Torsional oscillations were also considered the main culprit for the collapse of other suspension bridges, such as the Brighton Chain Pier (1836) [142], the Wheeling Suspension Bridge (1854) [106] and the Matukiki Suspension Footbridge (1977) [107, Ex. 4.6, p. 180] (see [82] for more details). The sudden appearance of the torsional motion was first attributed by Von Kármán to the vortex shedding [6, p.31], but this explanation was proven by Scanlan [144] to be incompatible with the phenomenon observed at the TNB by Farquharson [28, p.120] and the many attempts to provide a purely aeroelastic explanation of the failure of the TNB gave unsatisfactory results. We refer to [9] and the references therein for a detailed discussion of the related controversy. More recently, some attempts [7, 8, 22–24, 78, 116] were made to provide a qualitative explanation of the torsional motion in terms of internal resonances and structural instability. Nowadays, a complete comprehension of the reasons and the mechanics involved in the oscillations (and the collapse) of the TNB as well as of many other suspension bridges is not entirely achieved a lot of work is yet to be done.

1.2 The challenges of an adequate mathematical modelling

The elaboration of a rigorous mathematical model of the considered structures represents an important step towards the understanding of the complex and different phenomena involved in the instability of suspension bridges. Nevertheless, a fully reliable model appears out of reach, as we can observe for example in the work of Abdel-Ghaffar [1]. There, a complete mechanical description of an idealized suspension bridge is given and via variational principles a fairly general nonlinear system of equations describing the dynamics of the structure is obtained. However, even this complicated equations do not take into account the fluid-structure interaction between the bridge and the wind, the slackening of the cables and other complex as well as important phenomena. Moreover, after such huge effort, in order to make the problem analytically treatable, the author linearises the equation obtained. In this case, the attempt of giving a precise and complete description and motivation of the instability of a suspension bridge has been frustrated both by mathematical and engineering issues. Despite this problematic being quite pervasive in the modelling of complex physical systems, before and after the work of Abdel-Ghaffar a quite extensive literature around the topic and many different models have been developed.

1.2.1 A first model: the Melan's Equation

The various models developed to describe suspension bridges differentiate themselves by the degree of approximation of the model and by the focus on some particular phenomena in spite of others. After the fundamental report of Navier [137], that for several decades constituted the only mathematical treatise of the topic, and the work of Rankine [141], the first attempts to model suspension bridges were to view the roadway as a beam. This approximation was justified by the fact that, from a mechanical point of view, the roadway may naturally be considered as a plate whose width is much smaller than its length. We find this model in a milestone theoretical contribution to suspension bridges [132] by the Austrian engineer Joseph Melan, where the Castigliano Theorem [41, 42], is repeatedly applied to study the deflection of the structure. Biot-von Karman [29] call the Melan equation, that is

$$\begin{cases} EIw''''(x) - (H + h(w))w''(x) + \dfrac{q}{H}h(w) = p(x), \quad \forall x \in (0, L), \\ w(0) = w(L) = w''(0) = w''(L) = 0, \end{cases}$$

the fundamental equation of the theory of suspension bridges. Here, E is the elastic modulus; I is the second moment of area of the beams's cross section; q and $p(x)$ are the dead and live loads per unit length applied to the beam; H is the horizontal tension in the cable, when subject to the dead load q only and $h = h(w)$ is the additional tension in the cable produced by the live load p. We remark that the term $h(w)$, which represents the additional tension of the sustaining cables due to live loads, introduces a nonlocal nonlinearity in the equation and, for this reason, it is often considered as a constant in the engineering literature. However, the nonlinear behaviour of suspension bridges is by now well established, see e.g. [37, 60, 81, 116], and many different versions of $h(w)$ have been studied both from a theoretical and a numerical point of view [59, 83, 85, 120, 145, 146, 156]. The Melan equation and, more in general, monodimensional models, see e.g. [119, 131, 135], play an important role in the study of suspension bridges nowadays, in spite of failing to describe the longitudinal dynamics of the structure, that is one of the main menaces to the survival of suspension bridges, as we mentioned above.

1.2.2 Some plate models for suspension bridges

In order to study the torsional instability of the roadway, many authors got back taking into account the previously neglected spacial dimension of the structure considered, thus obtaining a plate model. Among the different plate models involved in the study of suspension bridges, we distinguish two main families of descriptions: the continuous partially-hinged plate model and the degenerate plates models.

The continuous partially-hinged plate describes the shape of the roadway with a certain function $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, where $\Omega = [-\pi, \pi] \times [-\ell, \ell]$ with $0 < \ell \ll \pi$. The main focus of those models is on the movements of the structure due to the action of the wind, whose force exerted on the deck of the bridge is usually considered in a suitable approximated form. Such models were firstly introduced by Ferrero and Gazzola in [71] and they were later studied, both from an engineering and a mathematical point of view, in a large variety of different analytical settings taking in account dissipative and nonlinear effects and different approximations of the action of the wind (see for example [5, 18, 33, 34, 70, 88]).

The degenerate plate models consider the cross section of the roadway as a rod having two degrees of freedom. The main advantage of such models is represented by the explicit separation between longitudinal and torsional dynamics in the equations. On the other hand, as suggested by Irvine [107], in those models the aerodynamic contribution on the dynamics is not explicitly described by a forcing term and it is here introduced through the initial conditions on the longitudinal dynamics. The attention in those models is focused on the nonlinear interaction between the different components of the bridge, that is, on the *structural instability* of the bridge, and less relevance is given to the interaction between the bridge and the external forces. Indeed, the developing literature regarding these descriptions of suspension bridges effectively shows how the phenomenon of internal resonance, that is the sudden transfer of energy from the longitudinal to the torsional dynamics, appears to be originated by the structure of the bridge itself rather than by the action of external forces. We distinguish two main different approaches to such idea: the fish-bone model and the Fermi-Pasta-Ulam-like (FPU-like) models.

The fish-bone model, firstly introduced in [23], is composed by a beam representing the midline of the plate, whose displacement from the rest position at a certain time t is given by the function $u(x, t)$, and by cross sections that are free to rotate around the beam, whose angle with respect to the horizontal position is described by the function $\theta(x, t)$ (see Figure 1.2). The fish-bone models

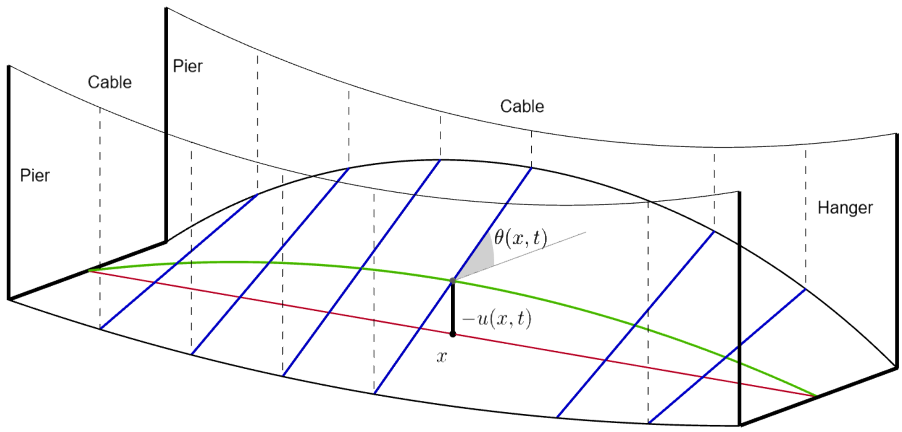

Figure 1.2: *The fish-bone model.*

has been analysed in a variety of different frameworks as the boundary conditions, the damping effects and the nonlinearities involved in the dynamics varied (see for example [3, 23, 78, 82, 86]).

In the FPU-like models we consider the bridge as finitely many cross sections (modelled by rods) linked by linear forces. Moreover, in order to model the action of the hangers on the structure, each of the rods is linked at its endpoints to two hangers that apply a certain restoring force to the rod. To the author's knowledge, this model was first suggested by Rocard [143, p. 121]. The parallelism with the celebrated FPU discrete system of nearest-neighbor coupled oscillators [69], which is the reason why we decided to call such models "FPU-like", is evident. This discretized model was first employed in the analysis of suspension bridges by McKenna [127] and it was later improved by

McKenna and Tuama [129]. As the hypotheses on the restoring force exterted by the hangers on the structures vary, both numerical and theoretical results proved that the nonlinear behaviour of such structures plays a fundamental role in the sudden change from longitudinal to torsional dynamics in suspension bridge (see, for example, [8, 63, 108, 118, 128, 130])

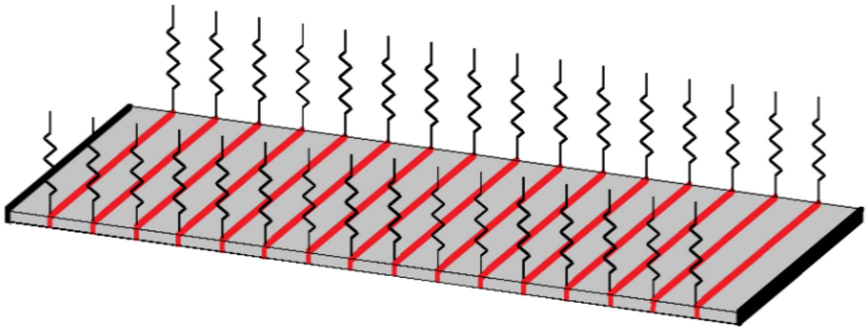

Figure 1.3: *Skecth of the FPU-like model, from [8].*

Many other things could be said regarding the modelling of suspension bridges. Recently, a vast interest has been devoted to a fluidodynamic approach to the problem of instability [36, 76, 84, 87, 88] and to a detailed discussion of the many different mechanical issues of these structures, such as for example the complex behaviour of the cables [31,32,40,57,62] and the possibility of composing the deck of the bridge with different materials [18–21]. Unfortunately, a detailed discussion of those interesting and challenging topics would go beyond the scopes of this introduction and of this thesis.

1.3 Motivations and structure of the thesis

As we noticed so far, it is a common practice in the study of suspension bridges, as well as in science more in general, to simplify the problems under examination through some more or less justified assumptions in order to make them treatable from an analytic point of view. This need of simplification is quite prominent for suspension bridges, as shown by the already cited work of Abdel-Ghaffar as well as in the majority of the articles we mentioned above. In particular, otherwise overwhelmingly complex infinite-dimensional dynamical systems are often simplified into a finite dimensional problems and the nonlinear terms are sometimes neglected. These approximations are a widespread practice often performed without a rigorous mathematical justification. This lack of mathematical rigour if often partially compensated with a physical interpretation of the procedure and by referring to the work of Galerkin [77] (we refer to Chapter 2 for a more detailed discussion). Nonetheless, as we explained above, the stability of suspension bridges is deeply related with the distribution of the energy among the fundamental modes of oscillation of the roadway and with the delicate role played by the nonlinear interaction among the different components of the bridge. Therefore, in this thesis, we aimed both to provide a rigorous justification of one of the most common approximation procedures and to study some of the systems that were previously studied in an approximated form only.

As we mentioned above, a widespread practice in the engineering approach to complex infinte-dimensional systems consists in neglecting the larger modes of the system. Motivated by this, in the Chapter 2, we aim to rigorously prove that some infinite-dimensional models for suspension bridges might be approximated via a finite number of ordinary differential equations. To this end, we consider some general elastic systems modelled by the equation

$$u_{tt} + A^2u + \delta u_t + \gamma \|A^{\theta/2}u\|^2 A^\theta u = g \tag{1.1}$$

where A^2 is a diagonal, self-adjoint and positive-definite operator and $\theta \in [0, 1]$. As the operator A^2 and the value of θ vary, a large family of different physical models, both mono-dimensional and multi-dimensional, turns out to be described by (1.1). We refer to Section 2.1 for a detailed discussion of the many applications of equation (1.1). In order to study the possibility of approximating the models described by (1.1) with a finite-dimensional system of ODEs, we proceed as follows. First, by exploiting an adaptation to our framework of some abstract recent results of Haraux [98], we analyze the dynamics in the case when the forcing term g is finite-dimensional. Next, we prove the continuous dependence of u by the forcing term g and we employ such result to estimate the error we commit by neglecting the modes larger than a certain N. We then prove, for a particular class of forcing terms, a theoretical result allowing to study the distribution of the energy among the modes and, with this background, we refine the previous results. Some applications to the study of the stability of suspension bridges are given. More precisely we apply the theoretical results to the multiple-intermediate piers model developed by Garrione and Gazzola in [78]. We study the distribution of the energy among the longitudinal modes of a suspension bridge with two intermediate piers as the position of the piers vary and we show that asymmetric suspension bridges appear to be more stable than suspension bridges where the piers are symmetric with respect to the center of the deck.

In the third chapter we focus on the analysis of the torsional instability of suspension bridges. In particular, we investigate how the internal resonances, which depend on the bridge structure only, are the source of torsional oscillations. Therefore, since as it was observed during the TNB collapse [6, p. 20] the longitudinal dynamics appeared to concentrate on a single mode of oscillation, we consider a fish-bone model for suspension bridges linearised in a neighbourhood of a purely longitudinal and unimodal dynamics. Since the instability of such structures is mainly originated by the sustaining cables [78, Sec 2] [124], we set ourselves in the hypothesis of rigid hangers. The stability analysis, which turns out to be strongly dependent on the boundary conditions, is carried out by means of standard Floquet theory in the case of Dirichlet boundary conditions and via KAM techniques in the case of Neumann boundary conditions. In both cases, the system is proven to be torsionally stable for a large set of longitudinal initial data.

CHAPTER 2

Asymptotic finite-dimensional approximations for a class of extensible elastic systems

2.1 Introduction

The main purpose of this chapter is to provide a rigorous finite-dimensional approximation for some infinite-dimensional models for suspension bridges. To this end, we set ourselves in a rather general framework suitable to describe a large variety of elastic systems involved in the description of the long-term dynamics of the deck of a suspension bridged undergoing an external force.

Let A^2 be a diagonal, self-adjoint, strictly positive operator, densely defined on a real Hilbert space $(\mathcal{H}, (\cdot,\cdot), \|\cdot\|)$ and we consider the following nonlinear nonlocal evolution equation

$$u_{tt} + \delta u_t + A^2 u + \|A^{\theta/2}u\|^2 A^\theta u = g \quad \text{in } \mathcal{H} \times \mathbb{R}_+ \tag{2.1}$$

where $\theta \in [0,1]$, $\delta > 0$ and $g \in C^0(\mathbb{R}_+, \mathcal{H})$ is a given forcing term.

The purpose of the present chapter is to give a rigorous finite-dimensional approximation of (2.1). To be more precise, we introduce the projection P_N onto the space generated by the first N modes, that is, by the first N eigenvectors of the operator A^2 and we consider the approximated problem

$$u_{tt} + \delta u_t + A^2 u + \|A^{\theta/2}u\|^2 A^\theta u = P_N g \quad \text{in } \mathcal{H} \times \mathbb{R}_+ \tag{2.2}$$

We remark that, by taking $u(0)$ and $u_t(0)$ in $P_N\mathcal{H}$, equation (2.2) can be interpreted as a system of N ODEs. Therefore, equation (2.2) actually provides a finite-dimensional approximation of equation (2.1). We aim to prove that any solution of (2.2) is asymptotically finite-dimensional and to estimate, for any $\varepsilon > 0$, the smallest $N = N(\varepsilon)$ such that the asymptotic distance in the phase space between the solution of (2.1) and the corresponding solution of (2.2) is less than ε. An improvement of the result will be studied for a particular class of forcing terms.

The reduction of infinite-dimensional dynamical systems to finite-dimensional systems of ODEs is a technique which has been widely used in the theoretical and numerical study of PDEs. The idea was first stated by Galerkin [77] and it has been used in many different applied frameworks as well as in the theory of finite-dimensional inertial manifolds (see [51, 55, 58, 64, 121, 151] and the references therein). In particular, it is a fairly common procedure, which we aim to make rigorous, in the study of suspension bridges [30] to approximate the physical system with the dynamics finite number of modes in order to reduce the computational complexity of the model. This approach can be physically justified by observing that *"the higher modes with their shorter waves involve sharper curvature in the truss and, therefore, grater bending moment at a given amplitude and accordingly reflect the influence of the truss stiffness to a greater degree than do the lower modes"* [148, p.11], which means that the dynamics of the higher modes corresponds to a physically irrelevant phenomenon. We remark that our goal would not be achieved just by estimating the dimension of the inertial manifold of our system, since we are interested in providing a finite-dimensional approximation of its asymptotic behavior.

The problem of finding a finite number of natural parameters of a system that uniquely determine its asymptotic behavior was first discussed for the 2D Navier-Stokes equation [75, 117] and to tackle it the concepts of finite-dimensional inertial manifold, determining modes and, later, determining nodes and determining local volume averages were introduced (see [52, Ch. 5], [50] and the references therein). Regarding our problem, Chueshov in [52, Ch. 5, Thm. 7.2] proved that the dynamics of the first N modes of (2.1) completely determines the evolution of the system and Eden and Milani in [65] proved that if the forcing term is $N-$dimensional, then any solution is attracted to an $M-$dimensional manifold with $M \geq N$.

Some particular cases of the damped equation (2.1) have been widely studied in mathematical literature. An ODE version of the problem was investigated by Loud in [122, 123]. Fitouri and Haraux in [73] improved some of the previous results on the ODE case and in [72] they provided a close-to-optimal ultimate bound in the PDE version of the problem. More recently, some sharp stability criteria for the unimodal version of (2.1) and for a related evolution equation were obtained by Haraux in [98] in the case $g = 0$. The case when $\theta = 1$ was studied in a slightly different framework by Holmes and others in [102, 134] as an example of chaotic dynamics (see also [95]) and some undamped versions of (2.1) were studied in the case $\theta = 0$ by Cazenave, Weissler and Haraux in [43–46] in order to obtain a description of the qualitative behavior of more complicated nonlinearities and by Gazzola and Garrione in [78] to study the dynamics of suspension bridges with multiple intermediate piers.

The considered abstract equation was analyzed by many other authors in an even more general framework. Biler [27] and de Brito [61] investigated the decay properties of the unforced problem with weak damping and a more general nonlinear nonlocal term. Da Silvia and Narciso [109, 110] studied an extensible beam model subject to a nonlocal nonlinear parameter-dependent damping and a forcing term. A lot of different variations of (2.1) with a large variety of damping and nonlinear terms has been studied in mathematical literature (see [52–54] and the references therein).

In addition to its mathematical relevance, our study also presents a certain physical and engineering interest. In fact, the considered model is suitable to describe both mono-dimensional and multi-dimensional physical systems. More precisely, some particular cases of (2.1) concerning the dynamics of beams and plates was considered by Holmes and Marsden [100, 101] in order to study the problem of flow-induced oscillations (see also [103, 104]) and in order to provide some more information about the nonlinear structural behavior of suspension bridges. In particular, we expect our results to allow some progress in the study of the structural and torsional instability of plates, to which a vast literature is devoted [16, 17, 22, 23, 80, 82].

If we set $A^2 = \Delta^2$, $\theta = 1$ and $\mathcal{H} = L^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with the smooth boundary $\partial\Omega$, we obtain the equation

$$u_{tt} + \delta u_t + \Delta^2 u - \left(\int_\Omega |\nabla u|^2 \right) \Delta u = g, \qquad \text{in } \Omega \times (0, T).$$

This problem is a special case of the more general model

$$u_{tt} + \Delta^2 u - \phi(\|\nabla u\|^2_{L^2(\Omega)})\Delta u = \mathcal{F}(x, t, u, u_t)$$

that was introduced in 1955 by Berger [25] as a simplification of the von Karman plate equation which describes large deflection of plate. Some related models were later applied to the study of the torsional instability of suspension bridges. In particular, our results apply also to the partially-hinged plate problem discussed in [35, 70]

$$\begin{cases} u_{tt} + \delta u_t + \Delta^2 u + \left(P - S\int_\Omega u_r^2(r,s,t)drds\right) u_{xx} = g & \text{in } \Omega \times (0,T) \\ u = u_{xx} = 0 & \text{on } \{0,\pi\} \times [-l,l] \\ u_{yy} + \sigma u_{xx} = u_{yyy} + (2-\sigma)u_{xxy} = 0 & \text{on } [0,\pi] \times \{-l,l\} \end{cases}$$

where $S > 0$ depends on the elasticity of the material of the deck of the bridge, $l > 0$ represents the width of the bridge and $\sigma > 0$ is the Poisson's ratio of the structure, which is assumed to be, in the case of suspension bridges, between 0 and 0.5. The term P is called "prestressing constant" and it expresses the buckling loads on the plate. In the case of suspension bridges, the compressive forces along the edges are introduced in order to increase the stability of the structure. The abstract prestressed model reads

$$u_{tt} + \delta u_t + A^2 u - PAu + ||A^{\theta/2}u||^2 A^\theta u = g \quad \text{in } \mathcal{H} \times \mathbb{R}_+. \tag{2.3}$$

The behaviour of the prestressed model is strongly dependent by the relationship between P and α_1, that is the first eigenvalue of the operator A^2. The study of this equation will not be discussed in detail since, under the hypothesis $P < \alpha_1^{1/2}$ (weak prestressing), the prestressing term does not modify the qualitative behavior of the system and in the case when $P \geq \alpha_1^{1/2}$ (strong prestressing) our results do not hold. In fact, in a strongly prestressed suspension bridge the linear part of (2.3), which is given by $A^2 - PA$, is not a strictly positive operator anymore and the proofs of the main theorems of this chapter do not work. Nonetheless, further investigations could be devoted to generalize some of our results to a strongly prestressed framework, in the spirit of [95].

Concerning the case where the models describes the dynamics of a mono-dimensional structure, if we take $\mathcal{H} = L^2(I)$ (with $I = [-\pi, \pi]$) and $A = -\partial_{xx}$, we can distinguish three different physically significant cases: $\theta = 0$, $\theta = 1$ and $\theta = 2$.

In the first case, the considered model has been introduced by Garrione and Gazzola [78] in order to describe the behavior of the deck of suspension bridges with two intermediate piers. In the work of Garrione and Gazzola, the deck of the bridge is modeled by a degenerate plate consisting of a beam with a continuum of cross sections free to rotate around the beam. Therefore, the longitudinal dynamics of the bridge is modeled by a beam equation, whose nonlinear term can be interpreted as a representation of *"a stiffened beam where the displacement behaves superquadratically and nonlocally: if the beam is displaced from its equilibrium position in some point, then this increases the resistance to further displacements in all the other points"* [78]. The nonlocal nature of such term is due to the elastic behavior of the components of the bridge, the sustaining cables in particular. This choice of the nonlinear term follows from a comparison between the qualitative behavior of some possible models and the actual behavior of suspension bridges. If we consider $\mathcal{D}(A) = \{v \in H^2(I) \cap H_0^1(I) : v(-\pi) = v(\pi) = v(-a\pi) = v(b\pi) = 0\}$ for $a, b \in (0,1)$, where a and b model the position of the piers along the deck of the bridge, the system reads

$$\begin{cases} u_{tt} + \delta u_t + u_{xxxx} + \|u\|^2_{L^2(I)} u = g(x,t) & \forall t \geq 0, \forall x \in I \\ u(0) = u_0 \in H^2(I) \cap H_0^1(I), u_t(0) = u_1 \in L^2(I) & \\ u(-\pi, t) = u(-\pi b, t) = u(\pi a, t) = u(\pi, t) = 0, & \forall t \geq 0. \end{cases}$$

An analogous equation, in a different functional framework, is involved in the study of the interaction between the cables and the deck of a suspension bridge in the case when the hangers are considered inextensible (see [78, 124]).

The second case ($\theta = 1$) was obtained by Woinowsky-Krieger [155] in 1950 and, independently, by Burgreen [39] in 1951. It models the physical phenomenon that *"if the beam is stretched somewhere, then this increases the resistance to further stretching in all the other points"* [78]. The system has been widely studied in both mathematical and engineering literature (see [65, 94] and the references therein). If we choose $\mathcal{D}(A) = \{v \in H^2(I) \cap H^1_0(I) : v(-\pi) = v(\pi) = v_{xx}(-\pi) = v_{xx}(\pi) = 0\}$, the model becomes

$$\begin{cases} u_{tt} + \delta u_t + u_{xxxx} - \|u_x\|^2_{L^2(I)} u_{xx} = g(x,t) & \forall t \geq 0, \forall x \in I \\ u(0) = u_0 \in H^2(I) \cap H^1_0(I), u_t(0) = u_1 \in L^2(I) \\ u(-\pi,t) = u_{xx}(-\pi,t) = u_{xx}(\pi,t) = u(\pi,t) = 0, & \forall t \geq 0. \end{cases}$$

The case $\theta = 2$ was first introduced in its non-dissipative version by G. Kirchoff in [112] as a simplified model for transversal vibrations of elastic strings and it was later studied in many different frameworks [78, 89, 92, 93]. If we consider $\mathcal{H} = L^2(I)$ and $A = -\partial_{xx}$ as we did before, the nonlinear term $\|u\|^2_\theta A^{\theta/2} u$ reads $\|u_{xx}\|^2_{L^2(I)} u_{xxxx}$ and the corresponding nonlinear equation can be interpreted as a model for *"a stiffened beam with bending energy behaving superquadratically and nonlocally: this means that if the beam is bent somewhere, then this increases the resistance to further bending in all the other points"* [78]. Despite the physical interest of the case $\theta = 2$, due to its technical difficulty, in this work we decided to restrict ourselves to the cases where $\theta \in [0, 1]$.

The results of the chapter are given in three main theorems. First, in Theorem 2.2.1, we prove that if the forcing term is finite-dimensional, i.e. if g is a combination of a finite number N of modes, then any solution is asymptotically finite-dimensional too in a sense that we specify in Definition 2.2. In the case of small oscillations or large damping, our result improves the one of Eden and Milani [65]. The proof is based on an application of a recent work of Haraux [98]. Next, in Theorem 2.2.2 we prove that, under suitable smallness conditions on the nonlinearity and on the forcing term, we are able to give an $M-$dimensional approximation of (2.1). More precisely, we prove that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that the asymptotic distance between a solution of (2.1) and a solution of (2.2) is controlled by ε in the phase space norm. The proof relies on a continuous dependence result and on Theorem 2.2.1. To conclude, in Theorem 2.2.3, fixed $\theta = 0$, we focus on a particular class of forcing terms and we refine the result of Theorem 2.2.2. In particular, under suitable smallness conditions on the solution, we improve the ultimate bounds previously given for general forcing terms in [35, 72] and we estimate how much the dynamics changes as we eliminate a single mode from the dynamics. This latter result represents one of the main novelties of this work since, to the author's knowledge, this is the first statement of this type present in literature.

The chapter is organized as follows. In Section 2.2 we give some definitions and we state the main results of the paper. In Section 2.3, some technical results are given. The proofs of the main results are contained in Section 2.4, Section 2.5 and Section 2.6, which are devoted to the proof of Theorem 2.2.1, Theorem 2.2.2 and Theorem 2.2.3 respectively. In Section 2.7, we present some physical conclusions concerning the application of our results to suspension bridges with multiple intermediate piers.

2.2 Statement of the main results

Let $(\mathcal{H}, (\cdot,\cdot), \|\cdot\|)$ be a Hilbert space and consider a diagonal, self-adjoint and positive-definite operator $A^2 : \mathcal{D}(A^2) \subset \mathcal{H} \to \mathcal{H}$, with eigenvalues $0 < \alpha_1 < \cdots < \alpha_j \nearrow \infty$ and eigenfunctions

e_n, solutions of the problem

$$(Ae_n, Av) = \alpha_n(e_n, v) \qquad \forall v \in \mathcal{D}(A).$$

The sequence $(e_n)_{n\geq 1}$ is a complete orthonormal system of $\mathcal{H}$. For our convenience, we preferred to use A^2 instead of A to build the functional framework of the problem. The operator A^2 defines a family of Hilbert spaces $\mathcal{H}^\sigma = \mathcal{D}(A^{\sigma/2})$ with $\sigma \geq 0$, endowed with the norms $\|\cdot\|_\sigma$ induced by the scalar products

$$\begin{gathered} u, v \in \mathcal{H}^\sigma \implies (u,v)_\sigma := (A^{\frac{\sigma}{2}}u, A^{\frac{\sigma}{2}}v) = \sum_{n=1}^{\infty} \alpha_n^{\sigma/2} u_n v_n, \\ \|u\|_\sigma := \sqrt{(u,u)_\sigma} \end{gathered} \tag{2.4}$$

where $u_n = (u, e_n)$ and $v_n = (v, e_n)$. In particular, $\|\cdot\|_0 = \|\cdot\|$. In the context of this work, we consider the cases when $\sigma \in [-2, 2]$, where for negative s the space $\mathcal{H}^s$ is defined as the dual of $\mathcal{H}^{-s}$. Throughout this chapter, we denote by $\langle\cdot,\cdot\rangle$ the duality product of $\mathcal{H}^2$. It possible to verify that $\mathcal{H}^\rho \hookrightarrow \mathcal{H}^\sigma$ densely whenever $0 \leq \sigma \leq \rho$ and that

$$u \in \mathcal{H}^\rho, \quad 0 \leq \sigma < \rho \implies \|u\|_\rho \geq \alpha_1^{\frac{\rho-\sigma}{4}} \|u\|_\sigma. \tag{2.5}$$

In this framework, for any family of indices $J = \{j_1, \dots, j_n\}$, we define the projection

$$\begin{gathered} P_J : \mathcal{H} \to \langle e_{j_1}, \dots, e_{j_n}\rangle \\ u = \sum_{h=1}^{\infty} u_h e_h \mapsto \sum_{r=1}^{n} u_{j_r} e_{j_r}. \end{gathered}$$

In particular, we denote by P_N and $Q_N := I - P_N$ the orthogonal projections onto $\langle e_1, \dots e_N\rangle$ and onto $\langle e_{N+1}, \dots\rangle$ respectively. In addition, for any $k \in \mathbb{N}$ we introduce the projection $\sqcap_k$ onto the orthogonal complement of e_k given by

$$\sqcap_k := I - P_k Q_{k-1} : \mathcal{H} \to \langle e_k\rangle^\perp.$$

Since A is a diagonal operator, we remark that

$$\forall s \in [0,2], \forall M = \{m_1, \dots, m_n\}, \quad A^s P_M = P_M A^s \text{ and } A^s Q_M = Q_M A^s. \tag{2.6}$$

Moreover, if $u = Q_N u$ for some $N \in \mathbb{N}$, then the estimate (2.5) can be improved by

$$u \in \mathcal{H}^\rho, \quad 0 \leq \sigma < \rho \implies \|u\|_\rho \geq \alpha_{N+1}^{\frac{\rho-\sigma}{4}} \|u\|_\sigma. \tag{2.7}$$

By using the notation in (2.4), problem (2.1) may be rewritten as

$$u_{tt} + \delta u_t + A^2 u + \|u\|_\theta^2 A^\theta u = g \quad \text{ in } \mathcal{H} \times \mathbb{R}_+. \tag{2.8}$$

Let us make clear what is meant by weak solution of (2.8):

Definition 2.1. *Assume that*

$$g \in C_b^0(\mathbb{R}_+, \mathcal{H}) := C^0(\mathbb{R}_+, \mathcal{H}) \cap L^\infty(\mathbb{R}_+, \mathcal{H}). \tag{2.9}$$

A weak solution of (2.8) *is a function*

$$u \in C^0(\mathbb{R}_+, \mathcal{H}^2) \cap C^1(\mathbb{R}_+, \mathcal{H}) \cap C^2(\mathbb{R}_+, \mathcal{H}^{-2})$$

such that

$$\langle u_{tt}, \varphi\rangle + \delta(u_t, \varphi) + (u, \varphi)_2 + \|u\|_\theta^2 (u, \varphi)_\theta = (g, \varphi) \qquad \forall \varphi \in \mathcal{H}^2.$$

We remark that by this definition it follows that $u(0) = u_0 \in \mathcal{H}^2$ and $u_t(0) = u_1 \in \mathcal{H}$. In the case where $\theta \in [0, 2)$, existence and uniqueness of weak solutions follows from an immediate adaptation of the result in [94, Theorem 2.1] (see Theorem 2.3.1). The case where $\theta = 2$ needs stronger assumptions on the initial data because the Galerkin procedure does not allow to control the nonlocal term. In the following, due to technical reasons, we have to further restrict ourselves to the case where $\theta \in [0, 1]$.

First, we prove that if the forcing term if finite-dimensional, i.e. if $g = P_N g$ for some $N \in \mathbb{N}$, then any weak solution of (2.8) is asymptotically finite-dimensional. Actually, we guarantee the validity of the result for a more general family of forcing terms. We introduce the notion of exponentially $N-$dimensional forcing term.

Definition 2.2. *We say that $g \in C_b^0(\mathbb{R}_+, \mathcal{H})$ is exponentially $N-$dimensional if there exists $\eta > 0$ such that*

$$\lim_{t\to\infty} \|Q_N g(t)\| e^{\eta t} = 0.$$

In Section 2.4, we prove the following statement which describes the asymptotic behavior of the solution in the case when the forcing term is exponentially $N-$dimensional.

Theorem 2.2.1. *Assume* (2.9)*, $\delta > 0$ and let $\theta \in [0, 1]$. If g is exponentially $N-$dimensional, there exists $M \geq N$ and $\tilde{\eta} > 0$, both depending on δ, $\limsup_{t\to\infty} \|g(t)\|$, θ, N, η and α_1, i.e. the first eigenvalue of A^2, such that*

$$\lim_{t\to\infty} (\|Q_M u(t)\|_2^2 + \|Q_M u_t(t)\|^2) e^{\tilde{\eta} t} = 0,$$

where u is a weak solution of (2.8).

Motivated by physical arguments (see Section 2.7), we now consider a "separated variables" forcing term such as $g(t) = \mathfrak{g} f(t)$, where $\mathfrak{g} \in \mathcal{H}$ and $f \in C_b^0(\mathbb{R}_+, \mathbb{R})$.

Let us consider a weak solution u of (2.8). Numerical simulations show that for some j we have $\limsup_{t\to\infty} |(u(t), e_j)| \ll \limsup_{t\to\infty} \|u(t)\|$, that is, we have that the asymptotic amplitude of some modes of u seems to be negligible with respect to the overall dynamics (see Figure 2.3). Hence, we expect to be able to neglect such modes both from the forcing term g and the solution u, thus reducing the numerical complexity of the model. Therefore, for any finite family of indices $J = \{j_1, \dots, j_m\}$, we consider the finite-dimensional approximation of (2.8) given by

$$v_{tt} + \delta v_t + A^2 v + \|v\|_\theta^2 A^\theta v = P_J g. \tag{2.10}$$

We remark that in virtue of Theorem 2.2.1, any solution of (2.10) is exponentially finite-dimensional. We prove that under suitable smallness conditions on the forcing term, for an appropriate choice of J, (2.10) is a good approximation of (2.8), i.e for any weak solution u of (2.8), the weak solution v of (2.10) provides a good exponentially finite-dimensional approximation of u. More precisely, in Section 2.5 we prove the following theorem:

Theorem 2.2.2. *Assume $\delta > 0$, $\theta \in [0, 1]$ and $g(t) = \mathfrak{g} f(t)$ with $\mathfrak{g} \in \mathcal{H}$ and $f \in C_b^0(\mathbb{R}_+, \mathbb{R})$. There exists $\bar{g}_\infty = \bar{g}_\infty(\alpha_1, \delta, \theta) > 0$ such that, if*

$$g_\infty := \limsup_{t\to\infty} \|g(t)\| < \bar{g}_\infty,$$

then for every $\varepsilon > 0$ there exists a finite family of indices $J = \{j_1, \dots j_m\}$ depending on α_1, δ, g_∞ and ε such that

$$\limsup_{t\to\infty} (\|u(t) - v(t)\|_2^2 + \|u_t(t) - v_t(t)\|^2) \leq \varepsilon$$

where u is a weak solution of (2.8) *and v is a weak solution of* (2.10).

Moreover, if g is exponentially $N-$dimensional, then there exist $M \geq N$ and $\tilde{\eta} > 0$, both depending on α_1, δ, $\limsup_{t\to\infty} \|g(t)\|$, θ, N and η, such that, if $J = \{1, \dots, M\}$, then

$$\lim_{t\to\infty} (\|P_M u(t) - v(t)\|_2^2 + \|P_M u_t(t) - v_t(t)\|^2) e^{\tilde{\eta} t} = 0.$$

In Section 2.6 we further restrict ourselves to the case when the forcing term is sinusoidal in time and, for the sake of simplicity, we focus on the case when $\theta = 0$, i.e. we study the problem

$$u_{tt} + \delta u_t + A^2 u + \|u\|^2 u = \mathfrak{g} \sin(\omega t). \tag{2.11}$$

For $\|\mathfrak{g}\|$ small enough, Theorem 2.2.2 states that if we replace $\mathfrak{g}$ with $P_M \mathfrak{g}$, we commit an error arbitrarily small as M grows. This suggests to consider the case when $\mathfrak{g} = P_M \mathfrak{g}$ for some $M \in \mathbb{N}$. Let v be a solution of

$$v_{tt} + \delta v_t + A^2 v + \|v\|^2 v = \sqcap_k \mathfrak{g} \sin(\omega t). \tag{2.12}$$

Let us now estimate the distance between u and v. The following theorem holds:

Theorem 2.2.3. *Assume $\delta > 0$, $\theta \in [0, 1]$ and let $g(t) = \mathfrak{g} \sin(\omega t)$ with $\mathfrak{g} = P_M \mathfrak{g}$ for some $M \in \mathbb{N}$. There exists $\bar{\mathfrak{g}} > 0$ depending on δ, ω and α_j with $j = 1, \dots M$, such that, if $\|\mathfrak{g}\| < \bar{\mathfrak{g}}$, then, for any $k \in \{1, \dots, M\}$ and for any u and v weak solutions of* (2.8) *and* (2.12),

$$\limsup_{t\to\infty} (\|\sqcap_k u(t) - v(t)\|_2^2 + \|\sqcap_k u_t(t) - v_t(t)\|^2) \leq \frac{C(\mathfrak{g}, e_k)^4}{((\alpha_k - \omega^2)^2 + \delta^2 \omega^2)^2},$$

where $C = C(\alpha_1, \dots, \alpha_M, \mathfrak{g}, \delta, \omega) > 0$.

The results involved in the proof of Theorem 2.2.3 are the most physically significant in the applications considered (see Section 2.7). In fact, Theorem 2.2.3 relies upon an estimate on the asymptotic amplitude of each mode, that allows us to study the distribution of the energy among the modes (see Figures 2.3 and 2.5) and to obtain a new bound on the asymptotic $\mathcal{H}^2-$norm of u that improves the estimate given in [35, Lemma 22] (see Figure 2.2).

Theorems 2.2.2 and 2.2.3 are not perturbation statements. Indeed, for any fixed $\delta > 0$, an explicit expression of the smallness conditions on g_∞ and $\|\mathfrak{g}\|$ required by the statements of Theorems 2.2.2 and 2.2.3 is obtained in Sections 2.5 and 2.6 respectively. Since the term g models the action of the wind along the deck of the bridge, we physically interpret such smallness conditions on g_∞ as requirements on the aerodynamic load on the structure. In particular, the conditions of Theorems 2.2.2 and 2.2.3 are equivalent to require that the speed of the wind v is below a certain threshold $\bar{v}$. Moreover, we remark that such conditions can not be avoided since even in the ODE case large forcing terms lead to a chaotic dynamics [122, 123] and the behavior of the solutions can be quite complicated, even where the forcing term is periodic in time [79, 136].

Our results are adaptable to more general frameworks. In particular, exploiting the abstract results of Haraux [98] and Chueshov [52], the cases with strong damping terms and with more general nonlinearities such as $A^\theta u_t$ and $M(\|u\|_\theta^2) A^{\theta/2} u$ with $0 \leq \theta \leq 1$ appear to be treatable. On the other hand, our results can not be immediately generalized to evolution equations with nonlinear nonlocal damping terms such as $N(\|u\|_1^2) g(u_t)$, since the linear analysis on which the proof of Theorem 2.2.3 is based seems not to be easily extendable to such case.

We notice that the results of Theorem 2.2.2 and 2.2.3 are not dependent by the initial conditions of (2.8) and (2.10). This is due to the presence of the damping term, as we can observe from Proposition 2.3.3. Nonetheless, if the initial states of (2.8) and (2.10) were close to each other, a uniform estimate on the distance in the phase space between the solutions of the approximated and the exact problem would be expected to hold for any $t \geq 0$. Unfortunately, we were not able to obtain such estimate and the techniques exploited in the proofs of Theorems 2.2.2 and 2.2.3 do not seem suitable to get this result.

2.3 Preliminary results

We start by recalling some basic properties concerning well-posedness and regularity of the solutions.

Theorem 2.3.1. *Let* (2.9) *hold. Then*

1. *(Weak solutions) If* $u(0) = u_0 \in \mathcal{H}^2$ *and* $u_t(0) = u_1 \in \mathcal{H}$*, problem* (2.8) *admits a unique global weak solution such that*
$$u \in C(\mathbb{R}_+, \mathcal{H}^2) \cap C^1(\mathbb{R}_+, \mathcal{H}) \cap C^2(\mathbb{R}_+, \mathcal{H}^{-2});$$
2. *(Regular solutions) If* $u(0) = u_0 \in \mathcal{H}^4$ *and* $u_t(0) = u_1 \in \mathcal{H}^2$*, problem* (2.8) *admits a unique regular solution, that is, a unique global weak solution such that*
$$u \in C(\mathbb{R}_+, \mathcal{H}^4) \cap C^1(\mathbb{R}_+, \mathcal{H}^2) \cap C^2(\mathbb{R}_+, \mathcal{H});$$
3. *(Continuous dependence on initial data) Let* (u_{0n}, u_{1n}) *be any sequence with*
$$(u_{0n}, u_{1n}) \to (u_0, u_1) \quad \textit{in } \mathcal{H}^2 \times \mathcal{H},$$
and let $u_n(t)$ *denote the weak solution of* (2.8) *with initial data* $u_n(0) = u_n$ *and* $u_t(0) = u_{1n}$*. Then for every* $T > 0$ *we have that*
$$(u_n(t), u_{n,t}(t)) \to (u(t), u_t(t)) \textit{ uniformly in } C^0([0, T], \mathcal{H}^2 \times \mathcal{H}).$$

The proof follows from a standard applications of monotone operator theory with locally Lipschitz perturbations. We refer to [54, Theorem 1.5 and Proposition 1.15] and the references therein for a detailed discussion, that we decided to omit. For an alternative approach, see [94, Theorem 2.1] for the global existence and uniqueness of weak solutions and continuous dependence on initial data and [35, Theorem 5] for the global existence and uniqueness of regular solutions.

We remark that in Theorem 2.3.1 we did not introduce the concept of strong or classical solution. This choice is motivated by the fact that in some applications such formulations are not possible, as in the case of the multiple intermediate piers model discussed in the introduction (see [78, Section 4] for a more detailed discussion).

The following proposition gives some ultimate bounds on the Sobolev norms of u. Since the result comes from a straightforward generalization of the estimates proved in Section 7 of [35], we omit the proof.

Proposition 2.3.2. *Assume* (2.9) *and let* u *be a weak solution of* (2.8)*. We introduce the quantities* $g_\infty := \limsup_{t\to\infty} \|g(t)\|$ *and*
$$E_\infty := g_\infty^2 \max\left(\frac{2}{\delta^2}, \frac{1}{2\alpha_1}\right), \qquad \alpha := \begin{cases} \delta/2 & \textit{if} \quad \delta^2 < 4\alpha_1, \\ \delta/2 - \sqrt{\delta^2/4 - \alpha_1} & \textit{if} \quad \delta^2 \geq 4\alpha_1. \end{cases}$$
Then, the following estimates on u *hold:*
$$\limsup_{t\to\infty} \|u(t)\|^2 \leq \frac{4E_\infty}{\sqrt{\alpha_1^2 + 4\alpha_1^\theta E_\infty} + \alpha_1} =: \Phi_0;$$
$$\limsup_{t\to\infty} \|u(t)\|_\theta^2 \leq \frac{4E_\infty + 2\alpha^2\Phi_0}{\sqrt{\alpha_1^{2-\theta} + 2(2E_\infty + \alpha^2\Phi_0)} + \alpha_1^{1-\theta/2}} =: \Phi_\theta;$$
$$\limsup_{t\to\infty} \|u(t)\|_2^2 \leq 2E_\infty + \alpha^2\Phi_0 =: \Phi_2;$$
$$\limsup_{t\to\infty} \|u_t(t)\|^2 \leq \min_{\lambda>0} \frac{1+\lambda}{\lambda}\left(2E_\infty + \max_{s\in[0,\Phi_0]}\left((\lambda+1)\alpha^2 - \alpha_1 s - \frac{1}{2}s^2\right)\right) =: \Phi_v.$$

2.3.1 Continuous dependence on the forcing term

We now prove the continuous dependence of the solutions on the forcing term under suitable smallness conditions on the parameters of the problem.

Proposition 2.3.3. *Let u and v be weak solutions respectively of the problems*

$$u_{tt} + \delta u_t + A^2 u + \|u\|_\theta^2 A^\theta u = g_1, \quad v_{tt} + \delta v_t + A^2 v + \|v\|_\theta^2 A^\theta v = g_2 \tag{2.13}$$

where $g_1, g_2 \in C_b^0(\mathbb{R}_+, \mathcal{H})$. Let $\Upsilon_\mu := \limsup_{t\to\infty} \|(u(t)+v(t))/2\|_\mu^2$ with μ in $[0,2]$. There exists $\mathcal{F}_\theta(\alpha_1, \delta, \Upsilon_\theta, \Upsilon_{2\theta})$ such that, if $\mathcal{F}_\theta < 1$ holds, then there exists $C > 0$ depending on δ and g_∞ such that

$$\limsup_{t\to\infty}(\|u(t)-v(t)\|_2^2 + \|u_t(t)-v_t(t)\|^2) \leq C \limsup_{t\to\infty} \|g_1(t)-g_2(t)\|. \tag{2.14}$$

Moreover, if there exists $\eta > 0$ such that $\limsup_{t\to\infty} \|g_1(t)-g_2(t)\|e^{\eta t} = 0$, then there exists $\eta_1 > 0$ such that

$$\lim_{t\to\infty}(\|u(t)-v(t)\|_2^2 + \|u_t(t)-v_t(t)\|^2)e^{\eta_1 t} = 0. \tag{2.15}$$

In particular, we can take

$$\mathcal{F}_\theta := \frac{2\sqrt{\Upsilon_\theta \Upsilon_{2\theta}}\alpha_1^{-\theta/4} + \Upsilon_\theta}{\alpha_1^{(1-\theta)/2}} \max\left(\frac{1}{\delta}, \frac{1}{2\sqrt{\alpha_1}}\right). \tag{2.16}$$

Proof. The idea of the proof is standard but, for our purposes, it is mandatory to fully report it since we are interested in making the smallness conditions required from our results explicit.

Let $\alpha > 0$. We define

$$\Lambda_\alpha := \frac{1}{2}\|w_t\|^2 + \frac{1}{2}\|w\|_2^2 + \frac{\alpha\delta}{2}\|w\|^2 + \frac{1}{16}\|w\|_\theta^4 + \alpha(w_t, w)$$

and let E be the quantity

$$E := \frac{1}{2}\|w_t\|^2 + \frac{1}{2}\|w\|_2^2 + \frac{1}{4}\|w\|_\theta^4$$

with w a generic function regular enough for Λ_α and E_α to be well-defined. Remark that, by using the Cauchy-Schwarz inequality, the Young inequality and (2.5), we get

$$\Lambda_\alpha \leq \frac{1+\alpha\varepsilon_1^2}{2}\|w_t\|^2 + \frac{\alpha\delta}{2}\|w\|^2 + \frac{\alpha_1 + \alpha/\varepsilon_1^2}{2\alpha_1}\|w\|_2^2 + \frac{1}{16}\|w\|_\theta^4 \leq C_1 E,$$

$$\Lambda_\alpha \geq \frac{1-\alpha\varepsilon_2^2}{2}\|w_t\|^2 + \frac{\alpha\delta}{2}\|w\|^2 + \frac{\alpha_1 - \alpha/\varepsilon_2^2}{2\alpha_1}\|w\|_2^2 + \frac{1}{16}\|w\|_\theta^4 \geq C_2 E, \tag{2.17}$$

where C_1 and C_2 are positive numbers, obtainable for suitable choices of the values of α, ε_1 and ε_2. In particular, to get C_2 we have to require

$$1 - \alpha\varepsilon_2^2 > 0, \qquad \alpha_1 - \frac{\alpha}{\varepsilon_2^2} > 0.$$

Hence, for every α such that $\alpha < \sqrt{\alpha_1}$ we can find ε_2 such that (2.17) holds.

We first consider u and v as regular solutions of the problems in (2.13). We define $w := v - u$ and $r := g_2 - g_1$. The function w is the regular solution of the problem

$$w_{tt} + \delta w_t + A^2 w + \|v\|_\theta^2 A^\theta v - \|u\|_\theta^2 A^\theta u = r. \tag{2.18}$$

We remark that, if $\xi := (u+v)/2$, we have

$$\|v(t)\|_\theta^2 A^\theta v(t) - \|u(t)\|_\theta^2 A^\theta u(t) = 2(\xi(t), w)_\theta A^\theta \xi(t) + \|\xi(t)\|_\theta^2 A^\theta w + \frac{1}{4}\|w\|_\theta^2 A^\theta w. \tag{2.19}$$

From the definition of Λ_α, by using (2.18) and (2.19), since u and v are regular solutions we get

$$\begin{aligned} &\dot{\Lambda}_\alpha + (\delta-\alpha)\|w_t\|^2 + \alpha\|w\|_2^2 + 2(\xi, w)_\theta(A^\theta\xi, w_t) + \|\xi\|_\theta^2(A^\theta w, w_t) + \\ &+ 2\alpha|(\xi,w)_\theta|^2 + \alpha\|\xi\|_\theta^2\|w\|_\theta^2 + \frac{\alpha}{4}\|w\|_\theta^4 = (r, w_t + \alpha w), \end{aligned} \tag{2.20}$$

Let $C_\mu = \sup_{t\geq 0}\|\xi(t)\|_\mu^2$ for any $\mu \in [0,2]$. For a suitable choice of α, by using Cauchy-Schwarz and Young inequality we have that for some positive constants $\bar{\alpha}$ and $\widetilde{\alpha}$

$$\begin{aligned} &(\delta-\alpha)\|w_t\|^2 + \alpha\|w\|_2^2 + 2(\xi,w)_\theta(A^\theta\xi, w_t) + \|\xi\|_\theta^2(A^\theta w, w_t) + 2\alpha|(\xi,w)_\theta|^2 + \\ &+ \alpha\|\xi\|_\theta^2\|w\|_\theta^2 + \frac{\alpha}{4}\|w\|_\theta^4 \geq (\delta-\alpha)\|w_t\|^2 + \\ &+ \alpha\|w\|_2^2 - 2\|\xi\|_\theta\|w\|_\theta\|\xi\|_{2\theta}\|w_t\| - \|\xi\|_\theta^2\|w\|_{2\theta}\|w_t\| + \frac{\alpha}{4}\|w\|_\theta^4 \geq \\ &\geq \left(\delta - \alpha - \frac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}}\right)\|w_t\|^2 + \left(\alpha - \frac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}}\right)\|w\|_2^2 + \\ &+ \frac{\alpha}{4}\|w\|_\theta^4 \geq \bar{\alpha}E \geq \widetilde{\alpha}\Lambda_\alpha. \end{aligned} \tag{2.21}$$

In particular, we choose the parameter α so that

$$\begin{cases} \delta - \alpha - \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}} > 0 \\ \alpha - \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}} > 0, \end{cases} \iff \begin{cases} \delta > \alpha + \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}} \\ \alpha > \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}}. \end{cases}$$

Hence, since $\alpha < \sqrt{\alpha_1}$, if

$$\begin{cases} \delta > \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{\alpha_1^{(1-\theta)/2}}, \\ \sqrt{\alpha_1} > \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}} \end{cases}$$

we can find values of α such that (2.21) holds. Therefore we can find α such that (2.21) is satisfied if

$$\frac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{\alpha_1^{(1-\theta)/2}} \max\left(\frac{1}{\delta}, \frac{1}{2\sqrt{\alpha_1}}\right) < 1. \tag{2.22}$$

Now, for some positive $\widetilde{\alpha}$ and $\widetilde{C}$ we get, from (2.20) and (2.21),

$$\dot{\Lambda}_\alpha + \widetilde{\alpha}\Lambda_\alpha \leq (r, w_t + \alpha w) \leq \widetilde{C}\|r\| =: \widetilde{f}(t). \tag{2.23}$$

By defining

$$M_\alpha(t) = \Lambda_\alpha(t) - \int_{t_0}^t \widetilde{f}(s)e^{\widetilde{\alpha}(s-t)}ds,$$

from (2.23) we obtain

$$\dot{M}_\alpha(t) + \widetilde{\alpha}M_\alpha(t) \leq 0.$$

Hence, from the Gronwall inequality and from the fact that for any $\varepsilon > 0$ there exists $t_0 > 0$ such that $|\widetilde{f}(s)| \le \widetilde{C}(\varepsilon + \limsup_{t\to\infty} \|r(t)\|)$ for any $s \ge t_0$, we get

$$\begin{aligned}\Lambda_\alpha(t) &\le \Lambda_\alpha(t_0)e^{-\widetilde{\alpha}(t-t_0)} + \int_{t_0}^t \widetilde{f}(s)e^{\widetilde{\alpha}(s-t)}ds \le \\ &\le \Lambda_\alpha(t_0)e^{-\widetilde{\alpha}(t-t_0)} + \widetilde{C}(\varepsilon + \limsup_{t\to\infty}\|r(t)\|)e^{-\widetilde{\alpha}t}\frac{e^{\widetilde{\alpha}t} - e^{\widetilde{\alpha}t_0}}{\widetilde{\alpha}}, \quad \forall t \ge t_0.\end{aligned} \tag{2.24}$$

Since we can take ε arbitrarily small as t_0 goes to infinity, from (2.24) we infer that there exists $C > 0$ such that

$$\limsup_{t\to\infty} \Lambda_\alpha(t) \le C \limsup_{t\to\infty} \|r(t)\|. \tag{2.25}$$

Moreover, if there exists $\eta > 0$ such that $\limsup_{t\to\infty} \|r(t)\|e^{\eta t} = 0$, then (2.24) yields that there exists $\eta_1 > 0$ such that

$$\lim_{t\to\infty} \Lambda_\alpha(t)e^{\eta_1 t} = 0. \tag{2.26}$$

From (2.17), there exists a positive constant C_2 such that $\Lambda_\alpha(t) \ge C_2 E(t)$. Therefore, (2.25) and (2.26) imply (2.14) and (2.15) respectively.

We remark that

$$\limsup_{t\to\infty} \|\xi(t)\|_\mu^2 = \Upsilon_\mu.$$

Hence, we can take $C_\mu = \Upsilon_\mu$. Therefore, from (2.22), we get that if

$$\frac{2\sqrt{\Upsilon_\theta \Upsilon_{2\theta}}\alpha_1^{-\theta/4} + \Upsilon_\theta}{\alpha_1^{(1-\theta)/2}} \max\left(\frac{1}{\delta}, \frac{1}{2\sqrt{\alpha_1}}\right) < 1,$$

then the thesis holds for regular solution u and v.

The same conclusions hold for u and v weak solutions of the problems in (2.13) by using a standard density argument. Indeed, since $\mathcal{H}^4$ is dense in $\mathcal{H}^2$ and $\mathcal{H}^2$ is dense in $\mathcal{H}$, setting $(u(0) = u^0, u_t(0) = u^1)$ and $(v(0) = v^0, v_t(0) = v^1)$, there exists two sequences (u_n^0, u_n^1) and (v_n^0, v_n^1) in $\mathcal{H}^4 \times \mathcal{H}^2$ such that

$$(u_n^0, u_n^1) \to (u^0, u^1) \quad \text{and} \quad (v_n^0, v_n^1) \to (v^0, v^1) \qquad \text{in } \mathcal{H}^2 \times \mathcal{H}.$$

Hence, from Theorem 2.3.1 we have the two sequences of regular solutions u_n and v_n with $(u_n(0) = u_n^0, u_{n,t}(0) = u_n^1)$ and $(v_n(0) = v_n^0, v_{n,t}(0) = v_n^1)$ such that, for any $T > 0$,

$$(u_n, u_{n,t}) \to (u, u_t), \quad (v_n, v_{n,t}) \to (v, v_t) \quad \text{uniformly in } C([0,T], \mathcal{H}^2 \times \mathcal{H}).$$

Therefore, since all the calculations hold for u_n and v_n (and the difference $w_n := u_n - v_n$), we get the thesis for the weak solutions u and v passing to the limit when $n \to \infty$. □

2.3.2 Some general stability results

In order to prove Theorem 2.2.1, we give a reformulation of Theorem 4.1 of [98] adapted to our framework.

Proposition 2.3.4. *Let $(H, (\cdot,\cdot), |\cdot|)$ be a Hilbert space and let A^2 be a self-adjoint and strictly positive linear operator on H with dense domain $\mathcal{D}(A)$. We introduce the Hilbert space $V := \mathcal{D}(A)$ endowed with the norm $\|\cdot\|^2 := (A\cdot, A\cdot)$ and we identify the unbounded operator A^2 with its extension in $\mathcal{L}(V, V')$. The duality pairing in $V' \times V$ will be denoted in the same way as the inner product in H.*

We consider $B(t) \in C^1(\mathbb{R}_+, \mathcal{L}(V, H))$ such that for any $v \in V$

$$0 \leq \limsup_{t\to\infty}(B(t)v, v) \leq \lambda\|v\|^2, \quad \limsup_{t\to\infty}(B'(t)v, v) \leq \lambda'\|v\|^2$$

for some positive numbers λ and λ'.

Let u be a bounded solution of

$$u_{tt} + \delta u_t + (A^2 + B(t))u = g$$

where $\delta > 0$, $g \in C(\mathbb{R}_+, H)$ and $\lim_{t\to\infty} |g(t)|e^{c_0 t} = 0$ for some positive constant c_0.

If

$$\frac{\lambda'}{\delta} < 1$$

then there exists $c > 0$ such that

$$\lim_{t\to\infty}(\|u(t)\|^2 + |u_t(t)|^2)e^{ct} = 0.$$

Proof. We proceed as in the proof of Theorem 4.1 of [98] and we define the quadratic form on $V \times H$ given by

$$\Phi(t) = \frac{1}{2}(|u_t|^2 + \|u\|^2) + \frac{\delta}{2}(u, u_t) + \frac{\delta^2}{4}|u|^2 + \frac{1}{2}(B(t)u, u).$$

For any fixed $t_0 > 0$ we have, if $t \geq t_0$,

$$\begin{aligned}\Phi_t &= \frac{1}{2}(B'(t)u, u) - \frac{\delta}{2}|u_t|^2 - \frac{\delta}{2}(B(t)u + A^2u, u) + (g, u_t + \frac{\delta}{2}u) \leq \\ &\leq \frac{1}{2}\sup_{t\geq t_0}(B'(t)u, u) - \frac{\delta}{2}|u_t|^2 - \frac{\delta}{2}\|u\|^2 + Ke^{-c_0 t}.\end{aligned}$$

for some positive constant K. Hence, for t_0 large enough

$$\Phi_t(t) \leq -\frac{\delta}{2}|u_t(t)|^2 - \frac{\delta - \lambda'}{2}\|u(t)\|^2 + Ke^{-c_0 t}$$

Therefore, if $\lambda' < \delta$ we get, for some positive α,

$$\Phi_t(t) + \alpha\Phi(t) \leq Ke^{-c_0 t}$$

for any $t \geq t_0$ and from Gronwall lemma we get the thesis. □

We recall a further stability result due to Haraux for an ODE related to our problem.

Proposition 2.3.5. *[Theorem 2.1 of [98]] Let $\lambda, \delta > 0$, $a \in L^\infty(\mathbb{R}_+)$ with $a(t) \geq 0$ for any $t \geq 0$. Let $x \in C^2(\mathbb{R}_+)$ be a solution of*

$$\ddot{x} + \delta\dot{x} + (\lambda + a(t))x = 0. \tag{2.27}$$

Assume

$$\limsup_{t\to\infty} a(t) < \delta \max(\delta, 2\sqrt{\lambda}).$$

There there are $\eta_1 > 0$ and $M > 0$ such that any bounded solution x of (2.27) satisfies

$$x^2(t) + \dot{x}^2(t) \leq M[x^2(s) + \dot{x}^2(s)]e^{-\eta_1(t-s)}$$

for any $s \leq t$.

With minimal effort, the same statement can be proven for x solving

$$\ddot{x} + \delta\dot{x} + (\lambda + a(t))x = \widetilde{g}.$$

where $\widetilde{g} \in C(\mathbb{R}_+)$ satisfies $\lim_{t\to\infty} \widetilde{g}(t)e^{\eta t} = 0$ for some $\eta > 0$.

2.3.3 Linear analysis

Some preliminary results on the behavior of a damped and forced harmonic oscillator are useful in order to simplify the following study. In particular, we study the equation

$$\ddot{y} + \delta\dot{y} + \lambda y = \Psi, \tag{2.28}$$

where we require Ψ to be antiperiodic. We recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be antiperiodic of antiperiod τ (i.e. $\tau-$antiperiodic) if

$$f(t+\tau) = -f(t), \quad \forall t \in \mathbb{R}.$$

Proposition 2.3.6. *Let us consider $\Psi \in L^2_{loc}(\mathbb{R}_+)$ antiperiodic of anti-period π/ω. We suppose that $\lambda > 0$ and $\delta > 0$. Then there exists an antiperiodic solution z of anti-period π/ω of* (2.28) *and we have that for some $\eta > 0$, for any $y(t)$ solution of* (2.28)*,*

$$\lim_{t\to\infty} (|y(t) - z(t)| + |\dot{y}(t) - \dot{z}(t)|)e^{\eta t} = 0.$$

Proof. Let us consider $\mathcal{A}_\omega \subset L^2([0,\pi/\omega])$ the space of the locally square-integrable antiperiodic functions with anti-period π/ω, endowed with the standard L^2 norm on the interval $[0,\pi/\omega]$. The family $\{e_n = \sqrt{\omega/\pi}e^{(2n+1)i\omega t}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of this space. Hence, we write

$$\Psi(t) = \sqrt{\frac{\omega}{\pi}} \sum_{n\in\mathbb{Z}} \psi_j e^{(2n+1)i\omega t}.$$

Setting

$$z(t) := \sqrt{\frac{\omega}{\pi}} \sum_{n\in\mathbb{Z}} \frac{\psi_n}{-\omega^2(2n+1)^2 + \lambda + i\delta\omega(2n+1)} e^{(2n+1)i\omega t},$$

it is immediate to verify that $z(t)$ is an antiperiodic solution of (2.28). The thesis now follows from the standard theory of ODEs. Indeed, any solution of (2.28) is given by the sum of $z(t)$ with a general solution y_g of the associated homogeneous equation

$$\ddot{y}_g + \delta\dot{y}_g + \lambda y_g = 0,$$

which is given by

$$y_g(t) = e^{-\delta t/2} f(t),$$

with

$$f(t) := \begin{cases} S\sin\left(\frac{t}{2}\sqrt{4\lambda - \delta^2} + \varphi\right), & \text{if} \quad 4\lambda > \delta^2, \\ S\cos(\varphi)t + S\sin(\varphi), & \text{if} \quad 4\lambda = \delta^2, \\ S\sinh\left(\frac{t}{2}\sqrt{\delta^2 - 4\lambda} + \varphi\right), & \text{if} \quad 4\lambda < \delta^2, \end{cases}$$

where the arbitrary constants S and φ are dependent from the initial conditions. We notice that

$$\max(|f(t)|, |f'(t)|) \leq Ce^{\mu t},$$

for some constants $C > 0$ and $0 \leq \mu < \delta/2$. Therefore, since $y(t) = z(t) + y_g(t)$, we get that for a suitable choice of $\eta > 0$

$$\begin{aligned} \lim_{t\to\infty} (|y(t) - z(t)| + |\dot{y}(t) - \dot{z}(t)|)e^{\eta t} &= \lim_{t\to\infty} \left(|f(t)| + \left|f'(t) - \frac{\delta}{2}f(t)\right|\right) e^{(\eta-\delta/2)t} \leq \\ &\leq \frac{\delta+4}{2} C \lim_{t\to\infty} e^{(\eta+\mu-\delta/2)t} = 0, \end{aligned}$$

which is the thesis. □

Proposition 2.3.7. *Let us consider $\Psi \in L^2_{loc}(\mathbb{R}_+)$ antiperiodic of anti-period π/ω and let $y(t)$ satisfy* (2.28). *We suppose $\lambda, \delta > 0$ and $2\sqrt{\lambda} \neq \delta$. We introduce the quantities*

$$w_\lambda^\pm := \frac{\pi^2}{\omega^2}\left(\lambda - \frac{\delta^2}{2} \pm \delta\sqrt{\frac{\delta^2}{4} - \lambda}\right),$$

$$\Omega_\lambda^2 := \frac{\pi^4}{2\omega^4(w_\lambda^+ - w_\lambda^-)}\left(\frac{\tan\left(\frac{\sqrt{w_\lambda^+}}{2}\right)}{\sqrt{w_\lambda^+}} - \frac{\tan\left(\frac{\sqrt{w_\lambda^-}}{2}\right)}{\sqrt{w_\lambda^-}}\right)$$

where, for any $w \in \mathbb{C}$, $\sqrt{w}$ is the complex number z such that

$$z^2 = w \text{ and } z \in \{\zeta : \Re(\zeta) > 0\} \cup \{\zeta : \Re(\zeta) = 0 \text{ and } \Im(\zeta) \geq 0\}.$$

Then the following estimate holds

$$\limsup_{t\to\infty} y(t) \leq \Omega_\lambda \|\Psi\|_{L^\infty([0,\pi/\omega])}. \tag{2.29}$$

Moreover, if $\Psi \in C^2(\mathbb{R}_+)$, then

$$\limsup_{t\to\infty} \dot{y}(t) \leq \Omega_\lambda \|\dot{\Psi}\|_{L^\infty([0,\pi/\omega])}.$$

Proof. From Proposition 2.3.6, equation (2.28) admits an antiperiodic solution $z(t)$ and any solution of $y(t)$ of (2.28) converges exponentially to $z(t)$, which yields that $\limsup_{t\to\infty} y(t) = \limsup_{t\to\infty} z(t)$. Hence, since from the antiperiodicity of $z(t)$ we have that $\limsup_{t\to\infty} z(t) = \|z\|_\infty$, in order to get the result it suffices to estimate the $L^\infty-$norm of $z(t)$. In the notation of Proposition 2.3.6, we have that

$$z(t) := \sqrt{\frac{\omega}{\pi}}\sum_{n\in\mathbb{Z}} \frac{\psi_n}{-\omega^2(2n+1)^2 + \lambda + i\delta\omega(2n+1)} e^{(2n+1)i\omega t},$$

Then, if $c_n = \sqrt{(-\omega^2(2n+1)^2 + \lambda)^2 + \delta^2\omega^2(2n+1)^2}$, from Cauchy-Schwarz inequality we obtain

$$|z(t)| \leq \sqrt{\frac{\omega}{\pi}}\sum_{n\in\mathbb{Z}} \frac{|\psi_n|}{c_n} \leq \sqrt{\frac{\omega}{\pi}}\sqrt{\sum_{n\in\mathbb{Z}} |\psi_n|^2}\sqrt{2\sum_{n\geq 0}\frac{1}{c_n^2}}. \tag{2.30}$$

Moreover, if $\Psi \in C^2(\mathbb{R}_+)$, we have

$$|\dot{z}(t)| \leq \sqrt{\frac{\omega}{\pi}}\sum_{n\in\mathbb{Z}} \frac{|(2n+1)\omega\psi_n|}{c_n} \leq \sqrt{\frac{\omega}{\pi}}\sqrt{\sum_{n\in\mathbb{Z}} |(2n+1)\omega\psi_n|^2}\sqrt{2\sum_{n\geq 0}\frac{1}{c_n^2}}. \tag{2.31}$$

First, we remark that from Parseval's theorem

$$\begin{aligned}\sqrt{\sum_{n\in\mathbb{Z}} |\psi_n|^2} &= \|\Psi\|_{L^2([0,\pi/\omega])} \leq \sqrt{\frac{\pi}{\omega}}\|\Psi\|_{L^\infty([0,\pi/\omega])},\\ \sqrt{\sum_{n\in\mathbb{Z}} |(2n+1)\omega\psi_n|^2} &= \|\dot{\Psi}\|_{L^2([0,\pi/\omega])} \leq \sqrt{\frac{\pi}{\omega}}\|\dot{\Psi}\|_{L^\infty([0,\pi/\omega])}.\end{aligned} \tag{2.32}$$

Then, to conclude the proof, we compute a closed form for the serie

$$\sum_{n\geq 0}\frac{1}{c_n^2}=\sum_{n\geq 0}\frac{1}{\omega^4(2n+1)^4-(2\lambda-\delta^2)(2n+1)^2\omega^2+\lambda^2}. \tag{2.33}$$

We observe that (2.33) becomes

$$\sum_{n\geq 0}\frac{1}{c_n^2}=\sum_{n\geq 0}\frac{\pi^4}{(w_\lambda^+-w_\lambda^-)\omega^4}\left[\frac{1}{(2n+1)^2\pi^2-w_\lambda^+}-\frac{1}{(2n+1)^2\pi^2-w_\lambda^-}\right]. \tag{2.34}$$

We now recall that the Mittag-Leffler expansion for the cotangent function gives

$$\cot(w)=\frac{1}{w}+\sum_{n=1}^{\infty}\frac{2w}{w^2-\pi^2n^2}.$$

Some straightforward computations give

$$\frac{1}{2}\tan\left(\frac{w}{2}\right)=\frac{1}{2}\cot\left(\frac{w}{2}\right)-\cot(w)=\sum_{n=0}^{\infty}\frac{2w}{(2n+1)^2\pi^2-w^2}.$$

Thus, we can infer that

$$\sum_{n\geq 0}\frac{1}{(2n+1)^2\pi^2-w_\lambda}=\frac{\tan\left(\frac{\sqrt{w_\lambda}}{2}\right)}{4\sqrt{w_\lambda}}.$$

Hence, from (2.34) we can conclude that

$$\sum_{n\geq 0}\frac{1}{c_n^2}=\frac{\pi^4}{4\omega^4(w_\lambda^+-w_\lambda^-)}\left(\frac{\tan\left(\frac{\sqrt{w_\lambda^+}}{2}\right)}{\sqrt{w_\lambda^+}}-\frac{\tan\left(\frac{\sqrt{w_\lambda^-}}{2}\right)}{\sqrt{w_\lambda^-}}\right). \tag{2.35}$$

By using (2.32) and (2.35) in (2.30) and (2.31), we obtain the thesis. □

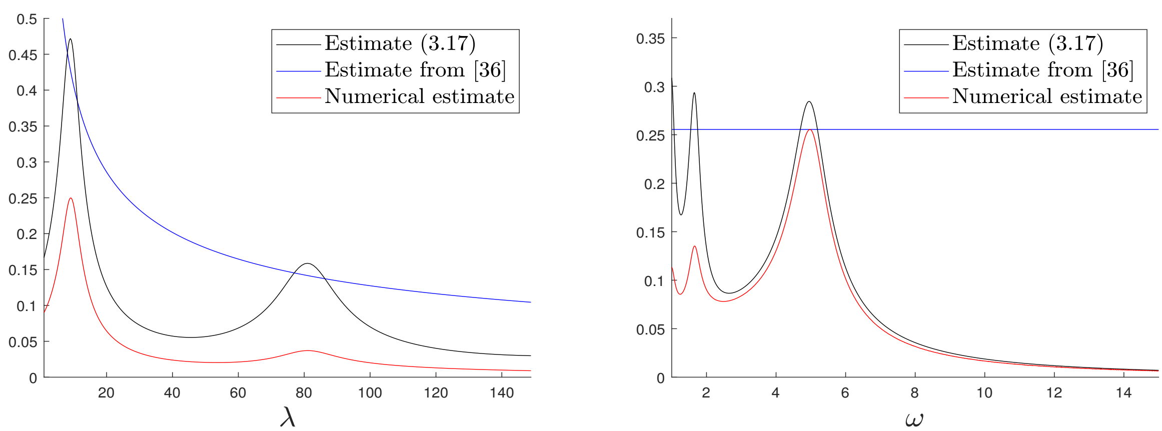

Figure 2.1: *Comparison between the estimates on the* $\|\cdot\|_\infty$*−norm of* y *solution of* (2.28) *given by [97] (blue) and by* (2.29) *(black) with* $\delta=1$ *and* $\omega=3$ *as* λ *vary from* 1 *to* 150 *(left) and with* $\delta=1$ *and* $\lambda=5$ *as* ω *vary from* 1 *to* 15 *(right). In red, we represented the* $\|\cdot\|_\infty$*−norm of the antiperiodic solution of* (2.28) *with* $\Psi(t)=signum(\sin(\omega t))$.

In [97, Theorem 2.1], a result similar to Proposition 2.3.7 is proven. In particular, the maximum value of $\limsup_{t\to\infty} y(t)$ as the forcing term Ψ varies in the unitary ball of $L^\infty(\mathbb{R})$ is determined. On the other hand, for any fixed antiperiodic forcing term Ψ in $C^2(\mathbb{R})$, in Proposition 2.3.7 we estimated $\limsup_{t\to\infty} y(t)$ and $\limsup_{t\to\infty} \dot{y}(t)$. As Figure 2.1 shows, Proposition 2.3.7 almost always gives a better estimate on $\limsup_{t\to\infty} y(t)$.

2.3.4 Structure of the chapter

The remainder of the chapter is organized as follows. First, in Section 2.4 we apply the results of Subsection 2.3.2 in order to prove Theorem 2.2.1. In particular, we apply Proposition 2.3.4 to prove that for N large enough, if g is exponentially $N-$dimensional, then there exists $\bar{N} \geq N$ such that any solution u of (2.8) is exponentially $\bar{N}-$dimensional (see Lemma 2.4.1). After that, fixed $n > N$, we study the asymptotic amplitude of $u_n(t) = (u(t), e_n)$ for any u solution of (2.8) and in Lemma 2.4.2 we determine whether $u_n(t)$ decays exponentially as t goes to infinity. In subsection 2.4.2 we exploit Lemma 2.4.1 and Lemma 2.4.2 in order to get Theorem 2.2.1. We remark that, even though the thesis of Theorem 2.2.1 follows from Lemma 2.4.1, Lemma 2.4.2 is necessary in order to improve the result of Lemma 2.4.1. More precisely, Lemma 2.4.2 provides an improvement of the smallest number $M \geq N$ obtained in Lemma 2.4.1 such that if g is exponentially $N-$dimensional then any solution u is exponentially $M-$dimensional.

Next, by exploiting the continuous dependence of the solution from the forcing term, that is, Proposition 2.3.3, and Theorem 2.2.1, in Section 2.5 we give the proof of Theorem 2.2.2.

In Section 2.5, by proceeding as in a result of Bonheure, Gazzola and Moreira dos Santos [35, Theorem 6], we show that (2.11) admits an antiperiodic solution p. In Lemma 2.6.2 we use Proposition 2.3.7 to estimate, for any $n \in \mathbb{N}$, the asymptotic amplitude of $p_n(t) := (p(t), e_n)$. Such result yields an estimate on the H^s-norms of p (see Lemma 2.6.3) which we numerically verified to be better than the a-priori estimates obtained in [35] (see Figure 2.2). From Proposition 2.3.3, we have that under suitable smallness conditions on $\limsup_{t\to\infty} \|g(t)\|$, any solution u of (2.11) converges to p in the phase space norm. Hence, from Lemma 2.6.2 and Lemma 2.6.3, in Lemma 2.6.4 we get an estimate on the asymptotic amplitude of $u_n(t) = (u(t), e_n)$ and on the H^s-norms of u for any u solution of (2.11). Finally, in Lemma 2.6.5, we exploit the previous results of Section 2.6 in order to get a results for finite-dimensional systems of ODEs and in Subsection 2.2.3 we apply Lemma 2.6.5 and Lemma 2.6.4 to get Theorem 2.2.3.

2.4 Proof of Theorem 2.2.1

2.4.1 Stability of the higher modes

We now apply the results of the previous section to our framework in order to prepare the proof of Theorem 2.2.1.

Lemma 2.4.1. *Let u be a weak solution of* (2.8). *Let g be exponentially $N-$dimensional. If there exists $\bar{N} \geq N$ such that*

$$\limsup_{t\to\infty}\left(\frac{1}{\alpha_1^{1-\theta}}\|u(t)\|_2^2 + \|u_t(t)\|^2\right) < 2\delta\alpha_{\bar{N}+1}^{(2-\theta)/2}$$

then there exists $\widetilde{\eta} > 0$ such that

$$\limsup_{t\to\infty}(\|Q_{\bar{N}}u(t)\|_2^2 + \|Q_{\bar{N}}u_t(t)\|^2)e^{\widetilde{\eta}t} = 0.$$

Proof. Fix $\bar{N} \geq N$ and, for any $s \in [0,2]$, let $\Upsilon_s := \limsup_{t\to\infty} \|u(t)\|_s^2$. We introduce the operator-valued function $B(t) := \|u(t)\|_\theta^2 A^\theta$. By using (2.6), we get that $w = Q_{\bar{N}}u$ solves

$$w_{tt} + \delta w_t + (A^2 + B(t))w = Q_{\bar{N}}g. \tag{2.36}$$

By using (2.7) we remark that for any $v \in \mathcal{H}^2$ such that $Q_{\bar{N}}v = v$

$$\begin{aligned} 0 \leq \limsup_{t\to\infty}(B(t)v,v) &= \limsup_{t\to\infty} \|u(t)\|_\theta^2 \|v\|_\theta^2 \leq \frac{\Upsilon_\theta}{\alpha_{\bar{N}+1}^{(2-\theta)/2}}\|v\|_2^2, \\ \limsup_{t\to\infty}(B'(t)v,v) &= \limsup_{t\to\infty}(u_t(t), A^\theta u(t))\|v\|_\theta^2 \leq \\ &\leq \frac{1}{2\alpha_{\bar{N}+1}^{(2-\theta)/2}} \limsup_{t\to\infty}\left(\frac{1}{\alpha_1^{1-\theta}}\|u(t)\|_2^2 + \|u_t(t)\|^2\right)\|v\|_2^2. \end{aligned} \tag{2.37}$$

We introduce

$$\varphi(t) = \frac{1}{2}(\|u_t(t)\|^2 + \|Au(t)\|^2) + \frac{\delta}{2}(u(t), u_t(t)) + \frac{\delta^2}{4}\|u(t)\|^2.$$

By applying Proposition 2.3.4 to (2.36), from (2.37) we get that if

$$\limsup_{t\to\infty}\left(\frac{1}{\alpha_1^{1-\theta}}\|u(t)\|_2^2 + \|u_t(t)\|^2\right) < 2\delta\alpha_{\bar{N}+1}^{(2-\theta)/2},$$

then $\varphi(t) \to 0$ exponentially as t goes to infinity. This yields that there exists $\tilde{\eta} > 0$ such that

$$\lim_{t\to\infty}(\|Aw(t)\|^2 + \|w_t(t)\|^2)e^{\tilde{\eta}t} = 0.$$

Therefore, since $\|Aw\|^2 = \|w\|_2^2$, we get the thesis. □

We now apply Proposition 2.3.5 to the projection of (2.8) on the $n-$th mode. The following lemma holds.

Lemma 2.4.2. *Let g be exponentially $N-$dimensional. For any weak solution u of* (2.8)*, if*

$$\exists n \geq N+1 \quad \textit{such that} \quad \limsup_{t\to\infty}\|u(t)\|_\theta^2 < \delta\max(2^\theta\delta^{1-\theta}, 2\alpha_n^{(1-\theta)/2}), \tag{2.38}$$

then for any $M \geq n$ there exists $\tilde{\eta} > 0$ such that for any $n \leq \bar{N} \leq M$

$$\lim_{t\to\infty}(|(u(t),e_{\bar{N}})|^2 + |(u_t(t),e_{\bar{N}})|^2)e^{\tilde{\eta}t} = 0.$$

Proof. Fixed $n \geq N+1$, we consider the projection of u on the $n-$th mode, i.e. $u_n := (u, e_n)$. The function u_n satisfies

$$\ddot{u}_n + \delta\dot{u}_n + (\alpha_n + \|u(t)\|_\theta^2\alpha_n^{\theta/2})u_n = (g, e_n).$$

Since $n \geq N+1$, for some $\eta > 0$, $\lim_{t\to\infty}(g(t), e_n)e^{\eta t} = 0$. Let us suppose that $\limsup_{t\to\infty}\|u(t)\|_\theta^2 < \delta\max(2^\theta\delta^{1-\theta}, 2\alpha_n^{(1-\theta)/2})$. Since

$$\max(2^\theta\delta^{1-\theta}, 2\alpha_n^{(1-\theta)/2}) \leq \max\left(\frac{\delta}{\alpha_n^{\theta/2}}, 2\alpha_n^{(1-\theta)/2}\right),$$

we have that

$$\alpha_n^{\theta/2}\limsup_{t\to\infty}\|u(t)\|_\theta^2 < \delta\max(\delta, 2\sqrt{\alpha_n}),$$

which yields that, from Proposition 2.3.5,

$$\lim_{t\to\infty}(|u_n(t)|^2 + |\dot{u}_n(t)|^2)e^{\eta_1 t} = 0.$$

Since $(\alpha_j)_j$ is strictly increasing, $\max(2^\theta\delta^{1-\theta}, 2\alpha_n^{(1-\theta)/2})$ is an increasing sequence. Hence, if (2.38) holds, then for any $\bar{N} \geq n$

$$\limsup_{t\to\infty} \|u(t)\|_\theta^2 \leq \delta \max(2^\theta\delta^{1-\theta}, 2\alpha_{\bar{N}}^{(1-\theta)/2}),$$

that implies that for any $M \geq n$ there exists $\tilde{\eta} > 0$ such that for any $n \leq \bar{N} \leq M$

$$\lim_{t\to\infty}(|u_{\bar{N}}(t)|^2 + |\dot{u}_{\bar{N}}(t)|^2)e^{\tilde{\eta}t} = 0,$$

that is the thesis. □

2.4.2 Completion of the proof of Theorem 2.2.1

Let g be exponentially $N-$dimensional and let u be a weak solution of (2.8). We recall that, from Proposition 2.3.2, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \|u(t)\|_\theta^2 \leq \frac{4E_\infty + 2\alpha^2\Phi_0}{\sqrt{\alpha_1^{2-\theta} + 2(2E_\infty + \alpha^2\Phi_0)} + \alpha_1^{1-\theta/2}} =: \Phi_\theta;\\
&\limsup_{t\to\infty} \|u(t)\|_2^2 \leq 2E_\infty + \alpha^2\Phi_0 =: \Phi_2;\\
&\limsup_{t\to\infty} \|u_t(t)\|^2 \leq \min_{\lambda>0} \frac{1+\lambda}{\lambda}\left(2E_\infty + \max_{s\in[0,\Phi_0]}\left((\lambda+1)\alpha^2 - \alpha_1 s - \frac{1}{2}s^2\right)\right) =: \Phi_v.
\end{aligned} \tag{2.39}$$

We introduce the quantity $\bar{N}$ defined as the smallest integer number greater than N such that

$$\frac{1}{\alpha_1^{1-\theta}}\Phi_2 + \Phi_v < 2\delta\alpha_{\bar{N}+1}^{(2-\theta)/2}. \tag{2.40}$$

From (2.39), (2.40) implies

$$\limsup_{t\to\infty}\left(\frac{1}{\alpha_1^{1-\theta}}\|u(t)\|_2^2 + \|u_t(t)\|^2\right) < 2\delta\alpha_{\bar{N}+1}^{(2-\theta)/2}.$$

Hence, from Lemma 2.4.1, if (2.40) holds then there exists $\eta_1 > 0$ such that

$$\lim_{t\to\infty}(\|Q_{\bar{N}}u(t)\|_2^2 + \|Q_{\bar{N}}u_t(t)\|^2)e^{\eta_1 t} = 0.$$

We introduce the set

$$B := \{n \in \mathbb{N} : n \in [N, \bar{N}] \text{ and } \Phi_\theta < \delta\max(2^\theta\delta^{1-\theta}, 2\alpha_{n+1}^{(1-\theta)/2})\}$$

and we define

$$\underline{N} := \begin{cases} \min B & \text{if } B \neq \varnothing \\ +\infty & \text{if } B = \varnothing. \end{cases}$$

From Proposition 2.3.2 we have that $\limsup_{t\to\infty} \|u(t)\|_\theta^2 \leq \Phi_\theta$. Hence, from Lemma 2.4.2, if $\underline{N} \neq +\infty$, there exists $\eta_2 > 0$ such that

$$\lim_{t\to\infty}(|(u(t), e_{n+1})|^2 + |(\dot{u}(t), e_{n+1})|^2)e^{\eta_2 t} = 0$$

for any $n \in [\underline{N}, \bar{N}] \cap \mathbb{N}$, which yields

$$\lim_{t\to\infty} (\|Q_{\underline{N}} P_{\bar{N}} u(t)\|_2^2 + \|Q_{\underline{N}} P_{\bar{N}} u_t(t)\|^2) e^{\eta_2 t}.$$

Hence, if we set $P_\infty := I$, $Q_\infty := 0$ and $M := \min\{\underline{N}, \bar{N}\}$, for some $\tilde{\eta} > 0$

$$\lim_{t\to\infty} (\|Q_M u(t)\|_2^2 + \|Q_M u_t(t)\|^2) e^{\tilde{\eta} t} = \lim_{t\to\infty} (\|Q_{\underline{N}} P_{\bar{N}} u(t)\|_2^2 + \|Q_{\underline{N}} P_{\bar{N}} u_t(t)\|^2) e^{\tilde{\eta} t} + \\ + \lim_{t\to\infty} (\|Q_{\bar{N}} u(t)\|_2^2 + \|Q_{\bar{N}} u_t(t)\|^2) e^{\tilde{\eta} t} = 0.$$

This concludes the proof of Theorem 2.2.1.

2.5 Proof of Theorem 2.2.2

Let us suppose that

$$\frac{2\sqrt{\Phi_\theta \Phi_2}\alpha_1^{(\theta-2)/4} + \Phi_\theta}{2\alpha_1^{(1-\theta)/2}} \max\left(\frac{1}{\delta}, \frac{1}{\sqrt{\alpha_1}}\right) < 1, \tag{2.41}$$

where Φ_θ and Φ_2 are defined in Proposition 2.3.2. Since Φ_θ and Φ_2 depend on g_∞ and δ, we get that, for any fixed δ, (2.41) translates into $F_\theta(\alpha_1, \delta, g_\infty) < 1$ for some F_θ. Therefore, for any fixed $\delta > 0$, there exists $\bar{g}_\infty > 0$ such that if $g_\infty < \bar{g}_\infty$, then (2.41) holds. We remark that, since the term g models the action of the wind along the deck of the bridge, we physically interpret (2.41) as a requirement on the load exerted on the structure by the wind. In particular, since $\bar{g}_\infty$ in engineering applications (see [68]) is proportional to the speed of the wind v, the relation (2.41) is equivalent to require that $v < \bar{v}$ for some $\bar{v} > 0$.

Let u be a weak solution of (2.8) and for any $J = \{j_1, \ldots, j_m\}$ let v^J be a weak solution of the problem

$$v_{tt}^J + \delta v_t^J + A^2 v^J + \|v^J\|_\theta^2 A^\theta v^J = P_J g.$$

We introduce the quantities $\Upsilon_\mu = \limsup_{t\to\infty} \|(u(t) + v^J(t))/2\|_\mu^2$, where $\mu \in [0, 2]$. From Proposition 2.3.3 with $g_1 = P_J g$ and $g_2 = g = P_J g + Q_J g$, there exists a function $\mathcal{F}_\theta = \mathcal{F}_\theta(\alpha_1, \delta, \Upsilon_\theta, \Upsilon_{2\theta})$, given by (2.16), such that if $\mathcal{F}_\theta < 1$ then there exists a constant $C > 0$ such that

$$\limsup_{t\to\infty} (\|u(t) - v^J(t)\|_2^2 + \|u_t(t) - v_t^J(t)\|^2) \le C \limsup_{t\to\infty} \|Q_J g(t)\|. \tag{2.42}$$

Since $g = \mathfrak{g} f(t)$, for a suitable choice of J, we have that $C \limsup_{t\to\infty} \|Q_J g(t)\| < \varepsilon$. Hence we can conclude that, for a suitable choice of the family J, (2.42) gives

$$\limsup_{t\to\infty} (\|u(t) - v^J(t)\|_2^2 + \|u_t(t) - v_t^J(t)\|^2) \le \varepsilon. \tag{2.43}$$

From Proposition 2.3.2 and (2.5), we have that $\Upsilon_\theta \le \Phi_\theta$ and $\Upsilon_{2\theta} < \alpha_1^{\theta-1}\Phi_2$. Hence, $\mathcal{F}_\theta < 1$ is implied by (2.41). Therefore, fixed δ, if $g_\infty < \bar{g}_\infty$ for some positive constant $\bar{g}_\infty$, where $\bar{g}_\infty$ does not depend by J, then (2.43) holds. This proves the first part of Theorem 2.2.2.

Let now g be exponentially N-dimensional and let $M \ge N$ be obtained from Theorem 2.2.1, i.e. let $M \ge N$ be such that for some $\eta > 0$

$$\lim_{t\to\infty} (\|Q_M u(t)\|_2^2 + \|Q_M u_t(t)\|^2) e^{\eta t} = 0. \tag{2.44}$$

Let u and v be, respectively, weak solutions of (2.8) and

$$v_{tt} + \delta v_t + A^2 v + \|v\|_\theta^2 A^\theta v = P_M g.$$

We remark that u is solution of the following problem

$$u_{tt} + \delta u_t + A^2 u + \|u\|_\theta^2 A^\theta u = g = P_M g + Q_M g.$$

Since we supposed g to be exponentially N-dimensional and $M \geq N$, there exists $\eta > 0$ such that

$$\lim_{t\to\infty} \|P_M g(t) + Q_M g(t) - P_M g(t)\| e^{\eta t} = \lim_{t\to\infty} \|Q_M g(t)\| e^{\eta t} = 0.$$

Therefore, from Proposition 2.3.3 with $g_1 = P_M g$ and $g_2 = g = P_M g + Q_M g$ we have that, fixed δ, if g_∞ is sufficiently small, then there exists $\eta_1 > 0$ such that

$$\lim_{t\to\infty} (\|u(t) - v(t)\|_2^2 + \|u_t(t) - v_t(t)\|^2) e^{\eta_1 t} = 0.$$

Since $v = P_M v$, from (2.44) we get that for some $\tilde{\eta} > 0$

$$\lim_{t\to\infty} (\|P_M u(t) - v(t)\|_2^2 + \|P_M u_t(t) - v_t(t)\|^2) e^{\tilde{\eta} t} = 0.$$

This concludes the proof of Theorem 2.2.2.

2.6 Proof of Theorem 2.2.3

2.6.1 Some preliminary results

In Theorem 2.2.3, we restrict ourselves to the case when the forcing term is antiperiodic in time due to the engineering interest of this case (see Section 2.7). Moreover, for the sake of simplicity, we consider the case $\theta = 0$. The antiperiodicity of the forcing term allows us to provide some more information about the solution of (2.11). In particular, proceeding as in Theorem 6 of [35], where the result was proven in the periodic framework, by using Proposition 2.3.6, we obtain the following statement:

Proposition 2.6.1. *If $g(t)$ is a continuous antiperiodic function of anti-period τ, then there exists a solution of* (2.8) *antiperiodic of anti-period τ.*

Proof. The proof proceeds as in [35, Theorem 6]. First, we fix $n \geq 1$ and we prove the existence of a $\tau-$antiperiodic solution for the problem

$$u_{tt} + \delta u_t + A^2 u + \|u\|^2 u = P_n g. \tag{2.45}$$

Hence, we seek a $\tau-$antiperiodic solution u^n in the form

$$u^n(x,t) := \sum_{k=1}^{n} h_k^n(t) e_k(x).$$

We consider the spaces $C_\tau^2(\mathbb{R})$ and $C_\tau^0(\mathbb{R})$ of C^2 and C^2 $\tau-$antiperiodic functions and in the same notations of [35, Theorem 6] we have that (2.45) is equivalent to

$$L_n(\boldsymbol{h}(t)) + \nabla G_n(\boldsymbol{h}(t)) = \boldsymbol{g}(t),$$

where $\boldsymbol{h} := (h_1^n, \dots, h_n^n)$, $\boldsymbol{g} := (g_1, \dots, g_n)$, L_n is a diagonal operator such that

$$L_n^k(\boldsymbol{h}) := \ddot{h}_k + \delta \dot{h}_k + \alpha_k h_k$$

and

$$G_n(\boldsymbol{h}) := \frac{1}{4} \sum_{j,k=1}^{n} h_j^2 h_k^2.$$

We observe that for any $\boldsymbol{q} \in (C^0_\tau(\mathbb{R}))^n$ from Proposition 2.3.6 there exists a unique $\boldsymbol{h} \in (C^2_\tau(\mathbb{R}))^n$ such that $L_n(\boldsymbol{h}) = \boldsymbol{q}$. Thanks to the compact embedding $(C^2_\tau(\mathbb{R}))^n \subset (C^0_\tau(\mathbb{R}))^n$, we have that the nonlinear map $\Gamma_n : (C^0_\tau(\mathbb{R}))^n \times [0,1] \to (C^0_\tau(\mathbb{R}))^n$ defined by

$$\Gamma_n(\boldsymbol{h}, \nu) = L_n^{-1}(\boldsymbol{g} - \nu \nabla G_n(\boldsymbol{h})), \quad \forall (\boldsymbol{h}, \nu) \in (C^0_\tau(\mathbb{R}))^n \times [0,1]$$

is compact. Moreover, from Proposition 2.3.2 we have that there exists $H_n > 0$ (independent of ν) such that if $\boldsymbol{h} \in (C^0_\tau(\mathbb{R}))^n$ solves $\boldsymbol{h} = \Gamma_n(\boldsymbol{h}, \nu)$, then

$$\|\boldsymbol{h}\|_{(C^0_\tau(\mathbb{R}))^n} \leq H_n.$$

Hence, since the equation $\boldsymbol{h} = \Gamma_n(\boldsymbol{h}, 0)$ from Proposition 2.3.6 admits a unique $\tau-$antiperiodic solution, the Leray-Schauder principle ensures the existence of a solution $\boldsymbol{h} \in (C^0_\tau(\mathbb{R}))^n$ of $\boldsymbol{h} = \Gamma_n(\boldsymbol{h}, 1)$. This proves the existence of a $\tau-$antiperiodic solution of (2.45). The proof the result follows from the existence of a $\tau-$antiperiodic solution of (2.45) exactly as in [35, Theorem 6] by showing that the sequence (u^n) converges to a $\tau-$antiperiodic solution u of (2.11). □

In this section we use the quantities

$$\begin{aligned} w_j^\pm &:= \frac{\pi^2}{\omega^2}\left(\alpha_j - \frac{\delta^2}{2} \pm \delta\sqrt{\frac{\delta^2}{4} - \alpha_j}\right), \\ \Omega_j^2 &:= \frac{\pi^4}{2\omega^4(w_j^+ - w_j^-)}\left(\frac{\tan\left(\frac{\sqrt{w_j^+}}{2}\right)}{\sqrt{w_j^+}} - \frac{\tan\left(\frac{\sqrt{w_j^-}}{2}\right)}{\sqrt{w_j^-}}\right) \end{aligned} \tag{2.46}$$

obtained by replacing λ by α_j in Proposition 2.3.7.

We now apply Proposition 2.3.7 in order to get an estimate on the $j-$th mode of the antiperiodic solution p of (2.11), which we proved to exist in Proposition 2.6.1. In the following, whenever a real-valued function $f(t)$ will be antiperiodic, we will write interchangeably $\limsup_{t\to\infty} f(t)$ and $\|f\|_\infty$.

Lemma 2.6.2. *Let p be an antiperiodic solution of* (2.11)*. If*

$$\max_j \Omega_j \limsup_{t\to\infty} \|p(t)\|^2 < 1 \tag{2.47}$$

where Ω_j is defined in (2.46)*, then, if $\Upsilon_0 := \limsup_{t\to\infty} \|p(t)\|^2$ and $\Upsilon_v := \limsup_{t\to\infty} \|p_t(t)\|^2$,*

$$\frac{g_j}{(1+\Upsilon_0\Omega_j)\sqrt{(\alpha_j - \omega^2)^2 + \delta^2\omega^2}} \leq \limsup_{t\to\infty} |p_j(t)| \leq \frac{g_j}{(1-\Upsilon_0\Omega_j)\sqrt{(\alpha_j - \omega^2)^2 + \delta^2\omega^2}},$$
$$\frac{(\omega(1-\Upsilon_0\Omega_j) - 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j)g_j}{(1-(\Upsilon_0\Omega_j)^2)\sqrt{(\alpha_j - \omega^2)^2 + \delta^2\omega^2}} \leq \limsup_{t\to\infty} |\dot{p}_j(t)| \leq \frac{(\omega(1-\Upsilon_0\Omega_j) + 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j)g_j}{(1-\Upsilon_0\Omega_j)^2\sqrt{(\alpha_j - \omega^2)^2 + \delta^2\omega^2}},$$

where $p_j := (p, e_j)$ and $g_j := \limsup_{t\to\infty}(g(t), e_j) = (\mathfrak{g}, e_j)$.

Proof. We study the $j-$th component of the problem (2.11), namely

$$\ddot{p_j} + \delta\dot{p_j} + \alpha_j p_j + \|p\|^2 p_j = g_j \sin(\omega t). \tag{2.48}$$

We consider the antiperiodic solution v of the problem

$$\ddot{v} + \delta\dot{v} + \alpha_j v = g_j \sin(\omega t). \tag{2.49}$$

It is possible to verify that the general solution of (2.49) is given by

$$v(t) = \frac{g_j}{\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \sin\left(\omega t + \arctan\frac{\delta\omega}{\omega^2-\alpha_j}\right) + Se^{-\delta t/2}\sin\left(\frac{t}{2}\sqrt{4\alpha_j-\delta^2}+\varphi\right),$$

where the constants S and φ are determined by the initial data of (2.49). Hence, it follows that, for any choice of the initial data of (2.49),

$$\limsup_{t\to\infty} v(t) = \frac{g_j}{\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}}, \qquad \limsup_{t\to\infty} \dot v(t) = \frac{\omega g_j}{\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}}. \tag{2.50}$$

If we subtract (2.49) from (2.48), if $w := p_j - v$ we get

$$\ddot w + \delta\dot w + \alpha_j w = -\|p\|^2 p_j.$$

Hence, from Proposition 2.3.7 we get, if $\mathfrak{p}_j^{(0)} := \limsup_{t\to\infty} p_j(t)$, $\mathfrak{p}_j^{(1)} := \limsup_{t\to\infty} \dot p_j(t)$, $\Upsilon_0 := \limsup_{t\to\infty} \|p(t)\|^2$ and $\Upsilon_v := \limsup_{t\to\infty} \|p_t(t)\|^2$,

$$\begin{aligned}
&\limsup_{t\to\infty} |w(t)| \le \Upsilon_0\Omega_j\mathfrak{p}_j^{(0)},\\
&\limsup_{t\to\infty} |\dot w(t)| \le \Omega_j\|2(p(t),p_t(t))p_j(t) + \|p(t)\|^2\dot p_j(t)\|_{L^\infty(0,\pi/\omega)} \le\\
&\qquad\qquad \le 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j\mathfrak{p}_j^{(0)} + \Upsilon_0\Omega_j\mathfrak{p}_j^{(1)}.
\end{aligned} \tag{2.51}$$

Since p and v are both antiperiodic, w is antiperiodic and (2.51) gives

$$\begin{aligned}
&\big|\|v\|_\infty - \|p_j\|_\infty\big| \le \|w\|_\infty \le \Upsilon_0\Omega_j\mathfrak{p}_j^{(0)},\\
&\big|\|\dot v\|_\infty - \|\dot p_j\|_\infty\big| \le \|\dot w\|_\infty \le 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j\mathfrak{p}_j^{(0)} + \Upsilon_0\Omega_j\mathfrak{p}_j^{(1)}.
\end{aligned}$$

We get then

$$\begin{aligned}
\limsup_{t\to\infty} v(t) - \Upsilon_0\Omega_j\mathfrak{p}_j^{(0)} \le \mathfrak{p}_j^{(0)} \le \limsup_{t\to\infty} v(t) + \Upsilon_0\Omega_j\mathfrak{p}_j^{(0)},\\
\limsup_{t\to\infty} \dot v(t) - 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j\mathfrak{p}_j^{(0)} - \Upsilon_0\Omega_j\mathfrak{p}_j^{(1)} \le \mathfrak{p}_j^{(1)} \le \limsup_{t\to\infty} \dot v(t) + \Upsilon_0\Omega_j\mathfrak{p}_j^{(1)} + 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j\mathfrak{p}_j^{(0)}.
\end{aligned}$$

Hence, from (2.50) we get, since hypothesis (2.47) holds,

$$\frac{g_j}{(1+\Upsilon_0\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \le \mathfrak{p}_j^{(0)} \le \frac{g_j}{(1-\Upsilon_0\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}},$$

which yields

$$\frac{(\omega(1-\Upsilon_0\Omega_j) - 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j)g_j}{(1-(\Upsilon_0\Omega_j)^2)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \le \mathfrak{p}_j^{(1)} \le \frac{(\omega(1-\Upsilon_0\Omega_j) + 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j)g_j}{(1-\Upsilon_0\Omega_j)^2\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}}$$

that is the thesis. □

We now apply the results of Lemma 2.6.2 in order to get an estimate on the $\mathcal{H}-$norm and $\mathcal{H}^2-$norm of an antiperiodic solution p of (2.11).

Lemma 2.6.3. *Let p be an antiperiodic solution of* (2.11). *Let us suppose that*

$$\max_j \Omega_j\Phi_0 < 1,$$

where Φ_0 is defined in Proposition 2.3.2. Then the following estimates hold:

$$\limsup_{t\to\infty} \|p(t)\|^2 \le \sum_{j=1}^{\infty} \frac{g_j^2}{(1-\Phi_0\Omega_j)^2((\alpha_j-\omega^2)^2+\delta^2\omega^2)} =: \varphi < \infty, \tag{2.52}$$

$$\limsup_{t\to\infty} \|p_t(t)\|^2 \le \sum_{j=1}^{\infty} \frac{(\omega(1-\Phi_0\Omega_j)+2\sqrt{\Phi_0\Phi_v}\Omega_j)^2 g_j^2}{(1-\Phi_0\Omega_j)^4((\alpha_j-\omega^2)^2+\delta^2\omega^2)} =: \varphi_v < \infty, \tag{2.53}$$

$$\limsup_{t\to\infty} \|p(t)\|_2^2 \le \sum_{j=1}^{\infty} \frac{\alpha_j g_j^2}{(1-\Phi_0\Omega_j)^2((\alpha_j-\omega^2)^2+\delta^2\omega^2)} =: \varphi_2 < \infty. \tag{2.54}$$

Proof. We prove (2.54) only, since the proofs of (2.52) and (2.53) are completely analogous. From Lemma 2.6.2, by using that from Proposition 2.3.2 $\Upsilon_0 := \limsup_{t\to\infty} \|p(t)\|^2 \le \Phi_0$,

$$\limsup_{t\to\infty} \|p(t)\|_2^2 \le \sum_{j=1}^{\infty} \alpha_j \|p_j\|_\infty^2 \le \sum_{j=1}^{\infty} \frac{\alpha_j g_j^2}{(1-\Phi_0\Omega_j)^2((\alpha_j-\omega^2)^2+\delta^2\omega^2)}.$$

We recall that the sequence $(\alpha_j)_j$ is divergent. Therefore, for j large enough, $w_j^- = \overline{w_j^+}$ and $|w_j^+ - w_j^-| = 2\pi^2\delta\sqrt{\alpha_j - \delta^2/4}/\omega^2 \ge \pi^2\delta\sqrt{\alpha_j}/\omega^2$. Hence

$$\left|\Omega_j^2\right| \le \frac{\pi^2}{\delta\omega^2\sqrt{\alpha_j}} \left| \Im\left(\frac{\tan\left(\frac{\sqrt{w_j^+}}{2}\right)}{\sqrt{w_j^+}} \right) \right| \le \frac{\pi^2}{\delta\omega^2\sqrt{\alpha_j}} \frac{\left|\tan\left(\frac{\sqrt{w_j^+}}{2}\right)\right|}{\sqrt{|w_j^+|}}.$$

We remark that

$$|\tan(a+ib)| \le \sqrt{\frac{\sin^2(2a)+\sinh^2(2b)}{(\cos(2a)+\cosh(2b))^2}}.$$

Moreover, from the definition of w_j^+ (see (2.46)), we have that $\Im(w_j^+) \to +\infty$. Hence, we conclude that $\lim_{j\to\infty} |\tan(\sqrt{w_j^+}/2)| = 1$ and consequently

$$\lim_{t\to\infty} \Omega_j = 0.$$

Then, since $\lim_{j\to\infty} \alpha_j = +\infty$ and $\max_j \Omega_j\Phi_0 < 1$, we have that, for some positive constant C, for any $j \in \mathbb{N}$

$$\frac{\alpha_j}{(1-\Phi_0\Omega_j)^2((\alpha_j-\omega^2)^2+\delta^2\omega^2)} < C.$$

Therefore, by using that

$$\sum_{j=1}^{\infty} g_j^2 = \|\mathfrak{g}\|^2 < \infty,$$

we get that

$$\sum_{j=1}^{\infty} \frac{\alpha_j g_j^2}{(1-\Phi_0\Omega_j)^2((\alpha_j-\omega^2)^2+\delta^2\omega^2)} \le \sum_{j=1}^{\infty} C g_j^2 = C\|\mathfrak{g}\|^2 < \infty,$$

that is the thesis. □

We observe that, from Proposition 2.3.3, any solution u of (2.11) exponentially converges to p under suitable smallness conditions on $\|\mathfrak{g}\|$. Hence, Lemma 2.6.2 and Lemma 2.6.3 hold for any weak solution u of (2.11). More precisely, the following lemma holds.

Lemma 2.6.4. *Let u be a weak solution of* (2.11)*. If*

$$\max_j \Omega_j \Phi_0 < 1, \qquad F(\xi_\infty) < 1,$$

where $F(\xi) = 3\xi \max(1/\delta, 1/(2\sqrt{\alpha_1}))/\sqrt{\alpha_1}$ and $\xi_\infty := \big((\sqrt{\Phi_0} + \sqrt{\varphi})/2\big)^2$, then

$$\limsup_{t\to\infty} \|u(t)\|^2 \le \varphi, \quad \limsup_{t\to\infty} \|u(t)\|_2^2 \le \varphi_2, \quad \limsup_{t\to\infty} \|u_t(t)\|^2 \le \varphi_v,$$

and

$$\frac{g_j}{(1+\varphi\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \le \limsup_{t\to\infty} |(u(t),e_j)| \le \frac{g_j}{(1-\varphi\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}},$$
$$\frac{(\omega(1-\varphi\Omega_j)-2\sqrt{\varphi\varphi_v}\Omega_j)g_j}{(1-(\varphi\Omega_j)^2)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \le \limsup_{t\to\infty} |(u_t(t),e_j)| \le \frac{(\omega(1-\varphi\Omega_j)+2\sqrt{\varphi\varphi_v}\Omega_j)g_j}{(1-\varphi\Omega_j)^2\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}},$$

where φ, φ_v and φ_2 are defined in (2.52)*,* (2.53) *and* (2.54) *respectively.*

Proof. Let p be an antiperiodic solution of (2.11). We define $w = p - u$. The function w solves

$$w_{tt} + \delta w_t + A^2 w + \|p\|^2 p - \|u\|^2 u = 0.$$

We proceed as in Proposition 2.3.3 and we get that if

$$F(\limsup_{t\to\infty} \|\xi(t)\|^2) < 1$$

where $\xi = (u+p)/2$, then

$$\lim_{t\to\infty} (\|u(t)-p(t)\|_2^2 + \|u_t(t)-p_t(t)\|^2) = 0. \tag{2.55}$$

Since

$$\limsup_{t\to\infty} \|\xi(t)\| \le \frac{\limsup_{t\to\infty}\|u(t)\| + \limsup_{t\to\infty}\|p(t)\|}{2} \le \frac{\sqrt{\Phi_0}+\sqrt{\varphi}}{2},$$

from the monotonicity of F we get that $F(\xi_\infty) < 1$ implies (2.55). Hence, the thesis follows from Lemma 2.6.2 and Lemma 2.6.3. □

2.6.2 The role of a single mode in the dynamics

Let us consider the finite-dimensional problem

$$\ddot{\underline{x}} + \delta\dot{\underline{x}} + \Lambda\underline{x} + \|\underline{x}\|^2\underline{x} = \underline{g}(t) \tag{2.56}$$

where $\underline{x}(t) = (x_1(t),\dots,x_n(t)) \in \mathbb{R}^n$, $\underline{g}(t) = (g_1(t),\dots,g_n(t))$, $\Lambda = \text{diag}(\alpha_j)_{j=1}^n$ and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. This problem is a finite-dimensional approximation of (2.11).

Here, we estimate how much the evolution of the system changes as we eliminate a single mode from the dynamics. For the sake of simplicity, in the following we consider the case when the higher mode is the one we choose to neglect. We observe that

$$P_{n-1}\ddot{\underline{x}} + \delta\dot{\underline{x}} + \Lambda_{n-1}P_{n-1}\underline{x} + \|P_{n-1}\underline{x}\|^2 P_{n-1}\underline{x} + x_n^2 P_{n-1}\underline{x} = P_{n-1}\underline{g}(t) \tag{2.57}$$

where $P_{n-1}(a_1,\dots,a_n) = (a_1,\dots,a_{n-1})$, $\Lambda_{n-1} = \text{diag}(\alpha_j)_{j=1}^{n-1}$. We consider now the function $\underline{y}(t)$, solution of

$$\ddot{\underline{y}} + \delta\dot{\underline{y}} + \Lambda_{n-1}\underline{y} + \|(\underline{y},0)\|^2\underline{y} = P_{n-1}\underline{g}(t) \tag{2.58}$$

At this point, the question is reduced to estimate the (asymptotic) distance between the solution $\underline{x}$ of (2.56) and the solution $\underline{y}$ of (2.58). To this end, with a slight abuse of notations, we introduce the $\mathbb{R}^n-$norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined by $\|\underline{x}\|_1 = |x_1|+\cdots+|x_n|$ and $\|\underline{x}\|_2 = \sqrt{\alpha_1|x_1|^2+\cdots+\alpha_n|x_n|^2}$. We remark that the result is completely independent of the choice of the mode neglected. The following lemma holds.

Lemma 2.6.5. *Let $\underline{x}$ and $\underline{y}$ be solutions of equations* (2.56) *and* (2.58) *respectively. Let $g = \underline{g}\sin(\omega t)$ with $\underline{g} \in \mathbb{R}^n$ and we suppose that $F(\xi_\infty) < 1$, where ξ_∞ is defined in Lemma 2.6.4 and $F(\xi) = 3\xi\max(1/\delta, 1/(2\sqrt{\alpha_1}))/\sqrt{\alpha_1}$. Moreover, we suppose that*

$$\max_j \Omega_j\Phi_0 < 1, \qquad \max_j \Omega_j\varphi < 1.$$

Then there exists a function S of the parameters of the problem such that if $S < 1$ then we have that

$$\limsup_{t\to\infty} \|P_{n-1}\underline{x}(t) - \underline{y}(t)\|_2 \le C(\underline{\chi})\chi_n^2,$$
$$\limsup_{t\to\infty} \|P_{n-1}\dot{\underline{x}}(t) - \dot{\underline{y}}(t)\| \le C_1(\underline{\chi},\underline{\chi}_v)\chi_n^2 + C_2(\underline{\chi},\underline{\chi}_v)\chi_{n,v}\chi_n$$

where $\underline{\chi} = (\chi_1,\dots\chi_n)$, $\chi_j := \limsup_{t\to\infty}\max(|x_j(t)|,|y_j(t)|)$, $\underline{\chi}_v = (\chi_{1,v},\dots\chi_{n,v})$ and $\chi_{j,v} := \limsup_{t\to\infty}\max(|\dot{x}_j(t)|,|\dot{y}_j(t)|)$.

Proof. First, we remark that as in Lemma 2.6.4, since $F(\xi_\infty) < 1$, we have that there exist two antiperiodic functions $\underline{p_1} \in C^2(\mathbb{R}_+,\mathbb{R}^n)$ and $\underline{p_2} \in C^2(\mathbb{R}_+,\mathbb{R}^{n-1})$ such that

$$\lim_{t\to\infty} \|\underline{x}(t) - \underline{p_1}(t)\|_2^2 + \|\dot{\underline{x}}(t) - \dot{\underline{p_1}}(t)\|^2 = 0,$$
$$\lim_{t\to\infty} \|\underline{y}(t) - \underline{p_2}(t)\|_2^2 + \|\dot{\underline{y}}(t) - \dot{\underline{p_2}}(t)\|^2 = 0.$$

Therefore, since we are interested in the asymptotic behavior of our system, we can restrict ourselves to the case when $\underline{x}$ and $\underline{y}$ are both antiperiodic without loss of generality.

Let us consider the difference between equation (2.57) and (2.58). If we set $\underline{w} := P_{n-1}\underline{x}$ and $\underline{z} := \underline{w} - \underline{y}$, we get

$$\ddot{\underline{z}} + \delta\dot{\underline{z}} + \Lambda_{n-1}\underline{z} = \underline{\Psi}$$

where $\underline{\Psi} = -x_n^2\underline{w} - (\|\underline{w}\|^2 - \|\underline{y}\|^2)\underline{y} - \|\underline{w}\|^2\underline{z}$ and for the sake of simplicity, abusing the notations, we wrote $\|\underline{w}\|$ and $\|\underline{y}\|$ instead of $\|(\underline{w},0)\|$ and $\|(\underline{y},0)\|$ respectively.

We focus on one component, say j, in order to treat only scalar quantities. Hence, we consider the equation

$$\ddot{z}_j + \delta\dot{z}_j + \alpha_j z_j = \Psi_j \tag{2.59}$$

where $\Psi_j = -x_n^2 x_j - (\|\underline{w}\|^2 - \|\underline{y}\|^2)y_j - \|\underline{w}\|^2 z_j = -x_n^2 x_j - (\underline{w}-\underline{y},\underline{w}+\underline{y})y_j - \|\underline{w}\|^2 z_j$. The fact that $\underline{x}$ and $\underline{y}$ are antiperiodic implies that $\underline{\Psi}$ is antiperiodic too. Hence, we can apply Proposition 2.3.7 to (2.59) and, if we introduce the quantities

$$\varphi := \max_{t\ge0}\max(\|\underline{x}(t)\|^2,\|\underline{y}(t)\|^2), \qquad \varphi_v := \max_{t\ge0}\max(\|\dot{\underline{x}}(t)\|^2,\|\dot{\underline{y}}(t)\|^2),$$
$$\chi_j := \max(\|x_j\|_\infty,\|y_j\|_\infty), \qquad \chi_{j,v} := \max(\|\dot{x}_j\|_\infty,\|\dot{y}_j\|_\infty) \quad \text{for } j=1,\dots,n,$$

then, set $\mathcal{Z} := \max_{t\ge0}\|\underline{z}(t)\|$, we have

$$\|z_j\|_\infty \le \Omega_j\|\Psi_j\|_\infty \le \Omega_j(\chi_n^2\chi_j + 2\sqrt{\varphi}\chi_j\mathcal{Z} + \varphi\|z_j\|_\infty).$$

Therefore, set $Z_j := \|z_j\|_\infty$ and $C_j := \Omega_j\varphi$, by requiring that $C_j < 1$ for any $j = 1, \dots n$ we get

$$Z_j \leq \frac{C_j\chi_j}{1-C_j}\left(\frac{\chi_n^2 + 2\sqrt{\varphi}\mathcal{Z}}{\varphi}\right). \tag{2.60}$$

We define the quantity

$$S := \sum_{j=1}^{n-1} \frac{2C_j\chi_j}{(1-C_j)\sqrt{\varphi}}$$

and we suppose $S < 1$.

We remark that for any $\underline{x} \in \mathbb{R}^n$, $\|\underline{x}\| \leq \|\underline{x}\|_1 := |x_1| + \dots |x_n|$ and, for any bounded function $\underline{f} : \mathbb{R} \to \mathbb{R}^n$, $\sup_t \|\underline{f}(t)\|_1 \leq \|f_1\|_\infty + \dots + \|f_n\|_\infty$. Hence we have that $\mathcal{Z} \leq \sum_{j=1}^{n-1} Z_j$. Therefore, by summing (2.60) over j and solving in $\mathcal{Z}$ we get

$$\mathcal{Z} \leq \frac{S}{1-S}\frac{\chi_n^2}{2\sqrt{\varphi}}. \tag{2.61}$$

Next, we remark that for any bounded function $\underline{f} : \mathbb{R} \to \mathbb{R}^n$ we have that $\sup_t \|\underline{f}(t)\|_2 \leq \sqrt{\alpha_1}\|f_1\|_\infty + \dots + \sqrt{\alpha_n}\|f_n\|_\infty$. Hence $\mathcal{Z}_2 := \max_{t\geq 0} \|\underline{z}(t)\|_2 \leq \sum_{j=1}^{n-1} \sqrt{\alpha_j} Z_j$ and from (2.60) and (2.61) it follows that

$$\mathcal{Z}_2 \leq \sum_{j=1}^{n-1} \sqrt{\alpha_j} Z_j \leq \sum_{j=1}^{n-1} \frac{C_j\chi_j\sqrt{\alpha_j}}{1-C_j}\left(\frac{\chi_n^2 + 2\sqrt{\varphi}\mathcal{Z}}{\varphi}\right) \leq \frac{1}{\varphi(1-S)}\sum_{j=1}^{n-1} \frac{C_j\chi_j\sqrt{\alpha_j}}{1-C_j}\chi_n^2. \tag{2.62}$$

In particular, from (2.61) and (2.62) we conclude that there exist two positive constants b and c such that

$$\mathcal{Z} \leq b\chi_n^2, \qquad \mathcal{Z}_2 \leq c\chi_n^2. \tag{2.63}$$

Moreover, from (2.63) and (2.60), there exist constants a_j such that

$$Z_j \leq a_j\chi_n^2 \quad \text{for any } j = 1, \dots, n-1. \tag{2.64}$$

We now define $Z_j^{(1)} := \|\dot{z}_j\|_\infty$ and $\mathcal{Z}^{(1)} := \max_{t\geq 0} \|\dot{\underline{z}}(t)\|$. By applying Proposition 2.3.7 to (2.59) we get

$$Z_j^{(1)} = \limsup_{t\to\infty} |\dot{z}_j(t)| \leq \Omega_j \limsup_{t\to\infty} |\dot{\Psi}_j(t)|. \tag{2.65}$$

Since $\|\underline{w}\|^2 - \|\underline{y}\|^2 = (\underline{w}+\underline{y}, \underline{w}-\underline{y}) = (\underline{w}+\underline{y}, \underline{z})$, we have

$$\begin{aligned} \dot{\Psi}_j = &- 2x_n\dot{x}_n x_j - x_n^2\dot{x}_j - (\dot{\underline{w}} + \dot{\underline{y}}, \underline{z})y_j + \\ &- (\underline{w}+\underline{y}, \dot{\underline{z}})y_j - (\underline{w}+\underline{y}, \underline{z})\dot{y}_j - 2(\underline{w}, \dot{\underline{w}})z_j - \|\underline{w}\|^2\dot{z}_j. \end{aligned} \tag{2.66}$$

Therefore from (2.66) and (2.65) we get

$$Z_j^{(1)} \leq \Omega_j(2\chi_n\chi_{n,v}\chi_j + \chi_n^2\chi_{j,v} + 2\sqrt{\varphi_v}\chi_j\mathcal{Z} + 2\sqrt{\varphi}\chi_j\mathcal{Z}^{(1)} + 2\sqrt{\varphi}\chi_{j,v}\mathcal{Z} + 2\sqrt{\varphi_v\varphi}Z_j + \varphi Z_j^{(1)}).$$

Hence, by using (2.63) and (2.64), if $L_j := \chi_{j,v} + 2\sqrt{\varphi_v\varphi}a_j + 2(\sqrt{\varphi}\chi_{j,v} + \sqrt{\varphi_v}\chi_j)b$ and C_j is defined as before, then

$$Z_j^{(1)} \leq \frac{C_j}{1-C_j}\frac{2\chi_n\chi_{n,v}\chi_j + L_j\chi_n^2 + 2\sqrt{\varphi}\chi_j\mathcal{Z}^{(1)}}{\varphi}.$$

By reasoning as before we conclude that, if $S < 1$, then

$$\mathcal{Z}^{(1)} \leq \frac{1}{1-S}\left(\frac{S}{\sqrt{\varphi}}\chi_n\chi_{n,v} + L\chi_n^2\right)$$

where L is a suitable constant.

We are now able to estimate the asymptotic distance between $\underline{x}$ and $\underline{y}$, since

$$\begin{aligned} &\limsup_{t\to\infty} \|\sqcap_n \underline{x}(t) - \underline{y}(t)\|_2 \leq c\chi_n^2, \\ &\limsup_{t\to\infty} \|\sqcap_n \underline{\dot{x}}(t) - \underline{\dot{y}}(t)\| \leq \frac{S}{(1-S)\sqrt{\varphi}}\chi_n\chi_{n,v} + \frac{L}{1-S}\chi_n^2. \end{aligned} \tag{2.67}$$

We remark that, since we can estimate φ and φ_v in function of $\underline{\chi}$ and $\underline{\chi}_v$, S and L are dependent by $\chi_1, \ldots \chi_n$ and $\chi_{v,1}, \ldots \chi_{v,n}$ only. Therefore, from (2.67) we get the thesis. □

2.6.3 Completion of the proof of Theorem 2.2.3

Since $\mathfrak{g} = P_M \mathfrak{g}$, from Lemma 2.6.4 we get that, if $F(\xi_\infty) < 1$,

$$\lim_{t\to\infty} |(u(t), e_j)| = 0, \quad \lim_{t\to\infty} |(u_t(t), e_j)| = 0 \quad \text{for } j > M.$$

Therefore, we can rewrite (2.11) and (2.12) as finite-dimensional dynamical systems of the form (2.56) and (2.58) respectively.

We introduce the quantities

$$\chi_j := \limsup_{t\to\infty} |(u(t), e_j)|, \quad \chi_{j,v} := \limsup_{t\to\infty} |(u_t(t), e_j)| \quad \text{for } j \leq M.$$

From Lemma 2.6.5, we have that if $\Omega_j \Phi_0 < 1$, $C_j = \Omega_j \varphi < 1$ for any $j \leq M$ and

$$S = \sum_{j=1}^{M} \frac{2C_j\chi_j}{(1-C_j)\sqrt{\varphi}} < 1$$

where Φ_0 and φ are defined in Proposition 2.3.2 and in Lemma 2.6.3, then

$$\begin{aligned} &\limsup_{t\to\infty} \|\sqcap_k u(t) - v(t)\|_2 \leq \frac{1}{\varphi(1-S)} \sum_{j=1}^{M} \frac{C_j\chi_j\sqrt{\alpha_j}}{1-C_j}\chi_k^2, \\ &\limsup_{t\to\infty} \|\sqcap_k u_t(t) - v_t(t)\| \leq \frac{S}{(1-S)\sqrt{\varphi}}\chi_k\chi_{k,v} + \frac{L}{1-S}\chi_k^2, \end{aligned} \tag{2.68}$$

where L is obtained in the proof of Lemma 2.6.5. Fixed δ, we recall that S and L are constants depending on $\chi_1, \ldots \chi_n$ and $\chi_{v,1}, \ldots \chi_{v,n}$. Hence, since from Lemma 2.6.4 we have that

$$\chi_j \leq \frac{g_j}{(1-\varphi\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}}, \qquad \chi_{v,j} \leq \frac{(\omega(1-\varphi\Omega_j) + 2\sqrt{\varphi\varphi_v}\Omega_j)g_j}{(1-\varphi\Omega_j)^2\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}},$$

from (2.68) we obtain that

$$\limsup_{t\to\infty}(\|\sqcap_k u(t) - v(t)\|_2^2 + \|\sqcap_k u(t) - v(t)\|^2) \leq \frac{Cg_k^4}{((\alpha_k-\omega^2)^2+\delta^2\omega^2)^2},$$

where C is a constant depending on A^2, $\mathfrak{g}$ and ω, that is the thesis.

2.7 The intermediate piers model

In this section we show how the analysis performed in this chapter can be useful in order to get some more information about the stability of real world structures such as suspension bridges.

While in the first part of the chapter (Theorem 2.2.1 and Theorem 2.2.2) we study the general case given by (2.8), in the second part (Theorem 2.2.3) we focus in particular on the case when $\theta = 0$ and

$$g = \mathfrak{g}\sin(\omega t).$$

In particular, taking $\mathcal{H} = L^2(I)$ with $I = [-\pi, \pi]$, $A = -\partial_{xx}$ and $\mathcal{D}(A) = \{v \in H^2(I) \cap H_0^1(I) : v(-\pi) = v(\pi) = v(-a\pi) = v(b\pi) = 0\}$ for $a, b \in (0, 1)$, the results of Section 2.6 apply to the system

$$\begin{cases} u_{tt} + \delta u_t + u_{xxxx} + \|u\|^2_{L^2(I)} u = g(x)\sin(\omega t) & \forall t \geq 0, \forall x \in I \\ u(0) = u_0 \in H^2(I) \cap H_0^1(I), u_t(0) = u_1 \in L^2(I) & \\ u(-\pi, t) = u(-\pi b, t) = u(\pi a, t) = u(\pi, t) = 0, & \forall t \geq 0. \end{cases} \tag{2.69}$$

This choice of the forcing term comes from the fact that, in engineering literature (see [96]), the load due to the vortex shedding of the wind along the structure of the bridge is usually modeled in this way with $g(x) \equiv g_\infty \in \mathbb{R}$. The coefficient g_∞ depends on the wind speed and on the geometry of the structure and ω is the frequency at which vortex shedding occurs. More precisely, we have that in engineering applications $g(x, t) = W^2 \sin(\omega t)$, where W is the scalar velocity of the wind blowing on the deck of the bridge and ω can be expressed in terms of the structural constants of the bridge and the aerodynamic parameters of the air. We refer to the European Eurocode [68] (see also [35]) for a more detailed discussion.

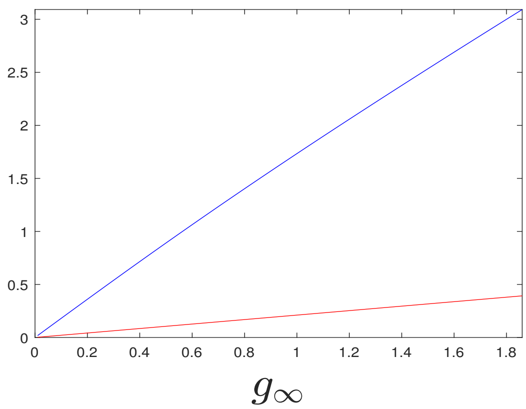

Figure 2.2: *Comparison between the general estimate on* $\limsup_{t\to\infty} \|u(t)\|_2$ *(blue) and the one obtained by using the antiperiodicity of the forcing term (red).*

The peculiar expression of the forcing term allows us to improve the estimate on the asymptotic $\mathcal{H}^2-$norm of the solution of (2.69) that one is able to obtain with no other information on g than the value of $\limsup_{t\to\infty} \|g(t)\|$. A comparison between the general estimate on $\limsup_{t\to\infty} \|u\|_2$ (see Proposition 2.3.2) obtained by using the methods of [35, Lemma 22] and the one obtained by using the antiperiodicity of the forcing term (see Lemma 2.6.4) is given in Figure 2.2. The data considered are $a = b = 14/25$, $\delta = 1.5$, and $\omega = 20$. The maximum value of g_∞ considered represents the largest value of g_∞ such that Lemma 2.6.4 can be applied.

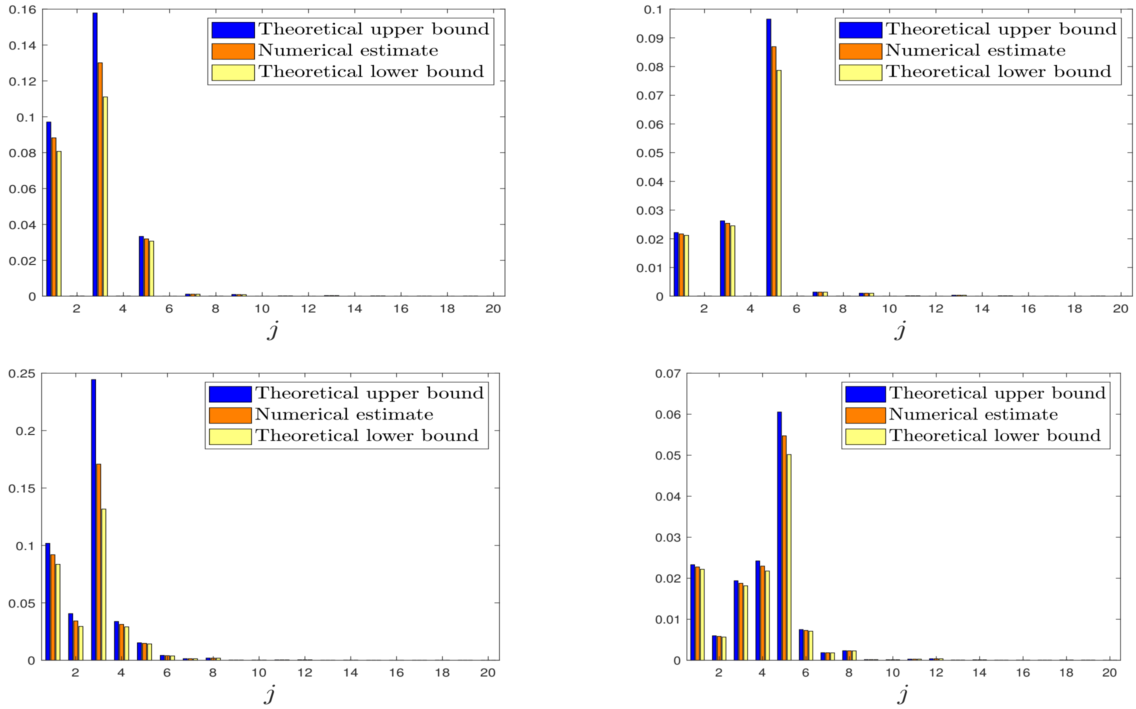

Figure 2.3: *Comparison between the asymptotic estimate on the amplitude of the first* 20 *modes for different values of* ω *and for different configurations of the piers.*

The improvement in the estimates on the asymptotic $\mathcal{H}^2-$norm is obtained by using also ultimate bounds of the asymptotic amplitude of each mode. We represent in Figure 2.3 a comparison between these estimates, obtained in Lemma 2.6.4, and a numerical estimate on the asymptotic amplitude of each of the first 20 modes. Fixed $\delta = 1.5$ and $g_\infty = 1.5$, we considered the cases when $\omega = 5$ (left) and $\omega = 10$ (right). We considered different positions of the piers, namely we chose $a = b = 14/25$ (up) and $(a, b) = (0.51, 0.67)$ (down). Each of these choices respect the hypothesis of Lemma 2.6.4. We remark that the mode with largest amplitude is such that $\sqrt{\alpha_j}/\omega \approx 1$.

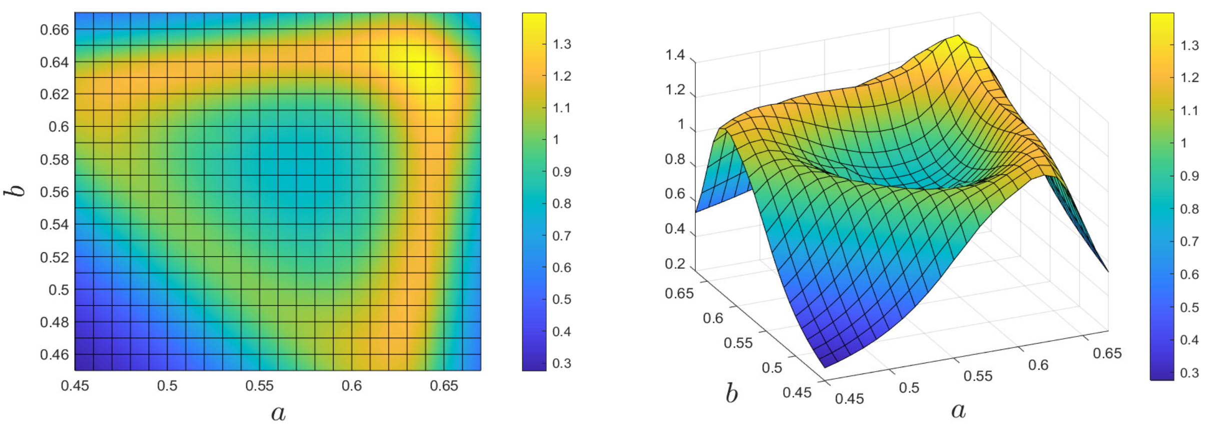

Figure 2.4: *Plot of a theoretical estimate of the asymptotic* $\mathcal{H}^2-$*norm in function of* a *and* b.

The estimates on each single mode of u allow us to study more precisely how the asymptotic $\mathcal{H}^2-$norm of u varies as the position of the piers vary, i.e. as a and b varies (see Lemma 2.6.4). Since most suspension bridges have symmetrical piers with $a = b \in [1/2, 2/3]$, we restrict ourselves to the case where $(a, b) \in [1/2, 2/3] \times [1/2, 2/3]$. We represent in Figure 2.4 the estimate on the asymptotic $\mathcal{H}^2-$norm given by Lemma 2.6.4 in function of a and b, with $\delta = 1.5$, $g_\infty = 1.5$ and $\omega = 10$ fixed. We remark that this figure does not give any information about the stability of the bridge as a and b vary. In fact, the stability of a bridge is more endangered by the concentration of the energy on a single mode than by the generalized oscillation of the structure.

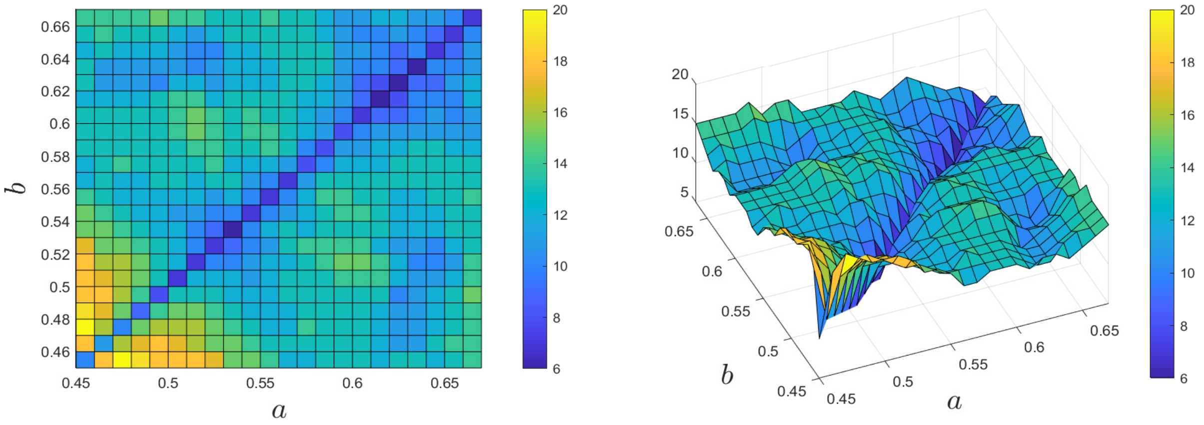

Figure 2.5: *Number of* $0.1-$*prevailing modes in function of* a *and* b.

In order to study the distribution of the $\mathcal{H}^2-$norm among the modes, we introduce the concept of *family of asymptotic* $\eta-$*prevailing modes*.

Definition 2.3. *Let* $0 < \eta < 1$. *We say that a weak solution of* (2.8) *has a family* $S = \{j_1, \dots j_n\}$ *of asymptotic* $\eta-$*prevailing modes if*

$$\limsup_{t\to\infty} \|Q_S u\|_2^2 < \eta^4 \limsup_{t\to\infty} \|P_S u\|_2^2.$$

In Figure 2.5 we plot the number of $\eta-$prevailing modes for $\eta = 0.1$. The value of the parameters is the same as in Figure 2.4, namely $\delta = 1.5$, $g_\infty = 1.5$ and $\omega = 10$. We can observe that the asymptotic $\mathcal{H}^2-$norm concentrates on few modes as $a = b$. Moreover, we notice how the energy turns out to be more dispersed among the modes when $a \neq b$.

In conclusion, we are able to assert that under suitable smallness conditions on the asymptotic amplitude of the forcing term and on the nonlinearity, we are able to perform a rather accurate modal analysis for the nonlinear nonlocal beam equations considered. In particular, Figure 2.5, allows us to conclude that the more stable configurations are achieved when $a \neq b$. This suggests that, according to the model considered, **asymmetric suspension bridges are more stable than suspension bridges where the piers are symmetric with respect to the center of the deck.**

CHAPTER 3

The role of boundary conditions in the torsional stability of suspension bridges

3.1 Introduction

In the previous chapter we focused on studying the distribution of the energy among the fundamental modes of an abstract equation suitable to describe a large variety of extensible elastic models. The results were applied to analyse the longitudinal dynamics of a multiple-intermediate piers model and its stability. Instead, this chapter is devoted to the study of the sudden appearance of torsional movement starting from a purely longitudinal dynamics for the deck of a suspension bridge. This analysis is carried out by mean of a linear model describing the interaction between the longitudinal and the torsional dynamics of the structure. More precisely, we consider the following linear partial differential evolution equation

$$\begin{cases} \theta_{tt} - \partial_x^2\theta + \gamma\|u_\delta\|^2_{L^2(I)}\theta + 2\gamma(u_\delta(t),\theta)_{L^2(I)}u_\delta(t) = 0 \\ \theta(-\pi,t) = \theta(\pi,t) = 0 & \forall t \geq 0 \\ \theta(0,x) = \theta_0(x) \in H^2(I), \quad \theta_t(0,x) = \theta_1(x) \in L^2(I) \end{cases} \tag{3.1}$$

where $\gamma \ll 1$, $I \in [-\pi,\pi]$ and u_δ solves the problem

$$\begin{cases} u_{\delta,tt} + \partial_x^4 u_\delta + \nu\|u_\delta\|^2_{L^2(I)}u_\delta = 0 \\ u_\delta(0) = \delta\mathfrak{e}_n, \quad u_{\delta,t}(0) = 0, \end{cases} \tag{3.2}$$

to be complemented with some boundary conditions; here, $\mathfrak{e}_n$ is the $n-$th eigenfunction of ∂_x^4 in $\mathcal{D}(\partial_x^4) \subset L^2(I)$ and $0 < \nu \ll 1$. Our aim is to study the Lyapunov stability of the trivial equilibrium $\theta \equiv 0$ of (3.1) for different boundary conditions on (3.2) as $\delta > 0$ varies.

This problem comes from a degenerate plate model, the "fish-bone model", composed by a beam representing the midline of the plate, whose displacement from the rest position at a certain time t is given by the function $u(x,t)$, and by cross sections that are free to rotate around the beam, whose angle with respect to the horizontal position is described by the function $\theta(x,t)$, see Figure 1.2. The main contribution to the instability of such structures comes from the sustaining cables [78, Sec 2] [124] and, therefore, we only consider the restoring forces due to their action on the plate. As explained in [78, Sec. 5.2], [23], the interaction between longitudinal and torsional dynamics due to the presence of the cables in the hypotesis of rigid hangers is described by the following nonlinear nonlocal beam-wave system of evolution equations:

$$\begin{cases} u_{tt} + \partial_x^4 u + \gamma \left(\int_I (u^2 + \theta^2) \right) u + 2\gamma \left(\int_I u\theta \right) \theta = 0 \\ \theta_{tt} - \partial_x^2 \theta + \gamma \left(\int_I (u^2 + \theta^2) \right) \theta + 2\gamma \left(\int_I u\theta \right) u = 0 \end{cases} \tag{3.3}$$

where we impose Dirichlet boundary condition on θ. The situation where $u \equiv 0$ in I represents the rest position of the midline of the bridge and $\gamma \ll 1$ is a parameter proportional to the tension at rest of the cables of the structure. Since we are willing to describe how small torsional oscillations may suddenly become large ones because of the nonlinear interaction with the longitudinal dynamics, we consider the linearization of (3.3) in a neighborhood of $\theta \equiv 0$, that is, we consider the problem

$$\begin{cases} u_{tt} + \partial_x^4 u + \gamma \left(\int_{-\pi}^{\pi} u^2 \right) u = 0, \\ \theta_{tt} - \partial_x^2 \theta + \gamma \left(\int_{-\pi}^{\pi} u^2 \right) \theta + 2\gamma \left(\int_{-\pi}^{\pi} u\theta \right) u = 0. \end{cases} \tag{3.4}$$

Motivated by the Federal Report on the Tacoma Narrow Bridge collapse [6, p. 20] where we learn that, in the months prior to the collapse "one principal mode of oscillation prevailed", we restrict our analysis to the class of unimodal longitudinal solutions, that is, we impose the boundary conditions in (3.2). Under such physical assumptions, equation (3.4) may be seen as an evolution problem for θ of the form (3.1)-(3.2).

The engineering interest with respect to the torsional motion of suspension bridges and their instability is more than a hundred years old. From the beginning of the XIX century until today, many suspension bridges have been built, even though their stability with respect to the action of the wind had been questioned quite early [137, p. 161]. Indeed, many bridges suffered unexpected oscillations, sometimes leading to collapses, see e.g. [4, 105]. The failure of the Tacoma Narrow Bridge (TNB), occurred in 1940, raised a particular attention on the topic and the sudden change from a vertical to a torsional mode of oscillation was considered crucial by the board of engineers appointed by the Federal Works Agency to investigate about the accident [6]. Torsional oscillations were also consdered the main culprit for the collapse of other suspension bridges, such as the Brighton Chain Pier (1836) [142], the Wheeling Suspension Bridge (1854) [106] and the Matukiki Suspension Footbridge (1977) [107, Ex. 4.6, p. 180] (see [82] for more details). The sudden appearance of the torsional motion was first attributed by Von Kármán to the vortex shedding [6, p. 31], but this explanation was proven by Scanlan [144] to be incompatible with the phenomenon observed at the TNB by Farquharson [28, p. 120] and the many attempts to provide a purely aeroelastic explanation of the failure of the TNB gave unsatisfactory results. We refer to [9] and the references therein for a detailed discussion of the related controversy. More attempts [7, 8, 22–24, 78, 116] provided a qualitative explanation of the torsional motion in terms of internal resonances and structural instability. The attention has turned to the nonlinear interaction between the different components of the bridge, which is here considered as an isolated systems, that is, neglecting both the aerodynamic forces and dissipation as suggested by Irvine [107]. Instead, the aerodynamic contribution to the dynamics is now introduced through the initial conditions on the longitudinal dynamics. In particular, the model (3.3), which takes into account the nonlinear interaction between the deck of the bridge and its sustaining cables, has been studied by many authors in a variety of different frameworks [3, 23, 78, 82, 86].

From a mathematical point of view, our work fits in the framework of the studies of the stability of linear differential equations with periodic coefficients. The first fundamental results concerning this topic were given in the finite-dimensional case by Hill [99] in order to study the motion of the lunar perigee and, later, by Floquet [74] and Lyapunov [125]. We refer to [38, 47, 48, 126, 149, 150] and the references therein. The infinite-dimensional case appears in general to be much more complex than the finite-dimensional case and various technical difficulties prevented to simply extend the Floquet theory developed in the finite-dimensional case to such framework, as explained in [113]. Nonetheless, the stability of different plate models has been largely studied with a variety of techniques by many authors [10, 11, 35, 43–45, 90, 91, 95]. In this article we apply the KAM machinery which was first developed by Pöschel and Kuksin [114, 115, 138–140] by improving and developing the results of Eliasson [66] and Wayne [154]. More precisely, we follow the KAM reducibility procedure developed in [13, 14], which is actually a small modification of [139], and we adapt it to our framework.

The main novelties presented in this chapter are represented by the the fact that, to the author's knowledge, for the first time the torsional instability of the system considered is not studied in the simplified and approximated case of the interaction between one longitudinal mode and one torsional mode. Indeed, we study the stability of (3.5) with no particular hypotheses on θ. On the other hand, technical difficulties forced us to consider $\gamma < \nu$, at least in Theorem 3.2.1. Moreover, our results do not apply to the multiple intermediate piers model developed in [78], due to the fact that the torsional eigenvalues of such model do not respect the second order Melnikov's condition requested in the KAM iterative scheme we employed. Future works may be devoted to improve our estimates in order to cover the case $\gamma = \nu$ and to apply the recent works of Baldi, Bambusi, Montalto, Langella and others [12, 15, 133] concerning KAM techniques for systems with very weak Melnikov non-resonance conditions to our framework, in order to obtain analogous results also for suspension bridges with multiple intermediate piers.

The results of the chapter are given in two main theorems. First, in Theorem 3.2.1, we prove a stability result in the case when (3.2) is endowed with Dirichlet boundary conditions. The proof is based on the application of a stability result for the Hill's equation due to Burdina [38, 149]. Next, in Theorem 3.2.2 we prove another stability result for "clamped" boundary conditions by adapting, applying and improving the KAM reducibility scheme employed in [13, 139]. Both Theorems 3.2.1 and 3.2.2 provide a threshold γ^*, depending on δ such that for any $\gamma < \gamma^*$ there exists a large measure set of perturbations such that (3.1) is stable, in a sense that we specify in Section 3.2.

The chapter is organized as follows. In Section 3.2 we give some definitions and we state the main results of this part of the thesis. Next, in Section 3.3 we give some preliminary definitions and results necessary for the proof of both Theorem 3.2.1 and Theorem 3.2.2. In Section 3.4 we collect some stability results concerning the Hill's equation, which we employ in Section 3.5 in order to prove Theorem 3.2.1. Then, in Section 3.6 we provide the Hamiltonian machinery required in order to prove Theorem 3.2.2, whose proof is given in Section 3.7. Finally, the appendices contain some technical lemmas we exploited to prove Theorem 3.2.1 and 3.2.2 and to explicitly compute the constants involved in the KAM estimates.

3.2 Statement of the main results

In this section we present our results concerning the problem

$$\begin{cases} \theta_{tt} - \partial_x^2\theta + \gamma\|u_\delta\|^2_{L^2(I)}\theta + 2\gamma(u_\delta(t),\theta)_{L^2(I)}u_\delta(t) = 0 \\ \theta(-\pi,t) = \theta(\pi,t) = 0 & \forall t \geq 0 \\ \theta(0,x) = \theta_0(x) \in H^2(I), \quad \theta_t(0,x) = \theta_1(x) \in L^2(I) \end{cases} \tag{3.5}$$

with u_δ solution of the evolution equation

$$\begin{cases} u_{\delta,tt} + \partial_x^4 u_\delta + \nu\|u_\delta\|^2_{L^2(I)}u_\delta = 0 \\ u_\delta(0) = \delta\mathfrak{e}_n, \quad u_{\delta,t}(0) = 0. \end{cases} \tag{3.6}$$

where $\nu \ll 1$ and $\mathfrak{e}_n$ is the $n-$th eigenfunction of ∂_x^4 in $\mathcal{D}(\partial_x^4) \subset L^2(I)$ corresponding to the eigenvalue λ_n.

We first complement equation (3.6) with the following boundary conditions

$$u(-\pi,t) = u(\pi,t) = u_{xx}(-\pi,t) = u_{xx}(\pi,t) = 0, \quad \forall t \geq 0, \tag{3.7}$$

physically corresponding to “hinged ends”. The following result holds:

Theorem 3.2.1. *Let u and θ solve* (3.6) *and* (3.5) *with boundary conditions* (3.7) *and let us fix $\delta^* > 0$. There exists $\gamma^* = \gamma^*(n,\delta^*) > 0$ such that for any $\delta \in [0,\delta^*]$ and for any $\gamma < \gamma^*$, there exists a positive measure set $\Delta_{n,\gamma,\delta} \subseteq [0,\delta]$ such that for any $\bar{\delta} \in \Delta_{n,\gamma,\delta}$ we have that*

$$\|\theta(t)\|_{H^1(I)} + \|\theta_t(t)\|_{L^2(I)} \leq C(\|\theta(0)\|_{H^1(I)} + \|\theta_t(0)\|_{L^2(I)}), \quad \forall t \geq 0$$

and, moreover,

$$|[0,\delta] \setminus \Delta_{n,\gamma,\delta}| < c\sqrt{\gamma}\delta$$

for some positive constants $c = c(n,\delta^*)$ and $C = C(n,\delta^*)$

The proof of this result is obtained by using a classical stability criterion for the Hill equation due to Burdina [38] (see Section 3.4 and 3.5).

Next, we complement (3.6) with boundary conditions given by

$$u(-\pi,t) = u(\pi,t) = u_x(-\pi,t) = u_x(\pi,t) = 0, \tag{3.8}$$

physically corresponding to “clamped ends”. We obtain the following theorem:

Theorem 3.2.2. *Let u and θ solve* (3.6) *and* (3.5) *with boundary conditions* (3.8) *and let $\delta^* > 0$ be such that $\nu(\delta^*)^2 < 2\lambda_n$. There exists $\gamma^* = \gamma^*(n,\delta^*) > 0$ such that for any $\delta \in [0,\delta^*]$ and for any $\gamma < \gamma^*$, there exists a positive measure set $\Delta_{n,\gamma,\delta} \subseteq [0,\delta]$ such that for any $\bar{\delta} \in \Delta_{n,\gamma,\delta}$ there exists an almost periodic function $\Theta \in C^\infty(\mathbb{R}_+, H^1(I))$ such that*

$$\sup_{t\geq 0}(\|\Theta(t)\|_{H^1(I)} + \|\Theta_t(t)\|_{L^2(I)}) \leq C(\|\theta_0\|_{H^1(I)} + \|\theta_1\|_{L^2(I)})$$

and

$$\sup_{t\geq 0}(\|\theta(t) - \Theta(t)\|_{H^1(I)} + \|\theta_t(t) - \Theta_t(t)\|_{L^2(I)}) \leq c\gamma\delta^2$$

for some positive constants $C := C(n,\delta^*)$ and $c := c(n,\delta^*)$. Moreover, we have that for some $c_\Delta = c_\Delta(n,\delta^*)$

$$|[0,\delta] \setminus \Delta_{n,\gamma,\delta}| \leq c_\Delta\gamma^{1/6}\delta^{1/8}.$$

Due of the techniques employed in their proofs, the stability results of Theorem 3.2.1 and Theorem 3.2.2 are quite different. Indeed, Theorem 3.2.1 provides us information about the distance between θ and the equilibrium $\theta \equiv 0$ in function of the initial state of θ only while Theorem 3.2.2, by using the estimate on Θ, gives a weaker estimate on θ, depending on the initial state of θ and on δ as well. Therefore, in the case of partially clamped boundary conditions, the obtained results are weaker than in the case of partially hinged boundary conditions.

The proof of Theorem 3.2.2 is obtained via KAM techinques. The procedure is essentially an adaptation to our framework of the methods developed by Pöschel, Bambusi and Graffi in [13, 14, 139]. Nevertheless, here we explicitly computed all the constants involved in the KAM estimates and we improved some of them. This allows us to study the applicability of the results to the study of the stability of real world structures such as suspension bridges.

We remark that both Theorem 3.2.1 and Theorem 3.2.2 are susceptible of physical interpretations. Indeed the parameter δ describes the initial amplitude of the oscillations of the structure and γ expresses the tension at rest of the cables of the suspension bridge. In particular, the case $\gamma = 0$ models a suspension bridge with rigid cables. Let us denote by $\Delta_{n,\gamma,\delta}$ the set of initial data $\bar{\delta} \in [0, \delta]$ such that the trivial torsional equilibrium of (3.4) is stable. We observe that for any fixed δ, as γ goes to 0, $\Delta_{n,\gamma,\delta}$ tends to coincide with $[0, \delta]$. This is equivalent to say that the more rigid the cable is taken, less likely are torsional oscillations to appear. Moreover, it is possible to observe from the proofs that the constants $c = c(n, \delta^*)$ in Theorems 3.2.1 and 3.2.2 increases as n grows. Hence, as the mode of longitudinal oscillations grows, the bridge is more likely to present torsional oscillations, which is compatible with the physical observations collected during the accident of the Tacoma Narrow Bridge [6]. To conclude, for any fixed γ, the measure of the set $\Delta_{n,\gamma,\delta}$ shrinks faster as δ grows in the case of clamped boundary conditions than in the case of hinged boundary conditions. Hence, clamped suspension bridges appears to be more likely to develop torsional movements. Nonetheless, we remark that this observation could follow from the technical difficulties of the case with clamped boundary conditions, and further investigations are needed.

3.3 Preliminary definitions and results

3.3.1 The longitudinal dynamics

For any fixed $n \in \mathbb{N}$ and $\nu > 0$, let us consider u_δ the solution of the problem

$$\begin{cases} u_{tt} + \partial_x^4 u + \nu \|u\|^2_{L^2(I)} u = 0 \\ u(0, x) = \delta \mathfrak{e}_n(x), \quad u_t(0, x) \equiv 0. \end{cases}$$

where $\mathfrak{e}_n$ is the $n-$th eigenfunction of ∂_x^4 in $\mathcal{D}(\partial_x^4) \subset L^2(I)$ corresponding to the eigenvalue λ_n. It is possible to verify that $u_\delta(x, t) = U_\delta(t)\mathfrak{e}_n(x)$, where $U_\delta(t)$ is the unique solution of the following ODE

$$\begin{cases} \ddot{U}_\delta + \lambda_n U_\delta + \nu U_\delta^3 = 0 \\ U_\delta(0) = \delta, \quad \dot{U}_\delta = 0. \end{cases}$$

Setting $U_\delta(t) = \sqrt{\lambda_n/\nu}\, Z(\sqrt{\lambda_n} t)$, we get that

$$\begin{cases} \ddot{Z} + Z + Z^3 = 0 \\ Z(0) = \sqrt{\dfrac{\nu}{\lambda_n}}\delta, \quad \dot{Z} = 0, \end{cases}$$

which from [80] is solved by

$$Z(t) = \sqrt{\frac{\nu}{\lambda_n}}\delta \mathrm{cn}\left[t\sqrt{1+\frac{\nu}{\lambda_n}\delta^2}, \frac{\sqrt{\nu}\delta}{\sqrt{2(\lambda_n+\nu\delta^2)}}\right].$$

where $\mathrm{cn}[t, m]$ is the Jacobi elliptic cosine function, which is defined as follows:

$$\mathrm{cn}(t, m) = \cos(\phi), \text{ where } t = \int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1-m^2\sin^2\theta}}. \tag{3.9}$$

This yields that

$$U_\delta(t) = \delta \mathrm{cn}\left[t\sqrt{\lambda_n+\nu\delta^2}, \frac{\sqrt{\nu}\delta}{\sqrt{2(\lambda_n+\nu\delta^2)}}\right].$$

Moreover, the period of the solution U_δ is given by

$$T(\delta) = 4\sqrt{2}\int_0^1 \frac{\mathrm{d}\theta}{\sqrt{(2\lambda_n+\nu\delta^2+\nu\delta^2\theta^2)(1-\theta^2)}}.$$

3.3.2 Frequency-dependent formulation

We introduce the following change of variables

$$\omega(\delta) := \frac{2\pi}{T(\delta)}.$$

Since $T(\delta)$ is a continuous and strictly decreasing function and $T(\delta) \to (2\pi/\sqrt{\lambda_n})^-$ as $\delta \to 0^+$, we have that if $\delta \in [0, \delta^*]$, then $\omega \in [\sqrt{\lambda_n}, \omega^*]$ where $\omega^* := \omega(\delta^*)$ and we can express δ as a function of ω, i.e. we have that

$$\delta(\omega) = T^{-1}\left(\frac{2\pi}{\omega}\right).$$

Let us consider the operator $Q(\delta, t)\theta := \|u_\delta(t)\|^2_{L^2(I)}\theta + 2(u_\delta, \theta)u_\delta$. It is immediate to verify that $Q(\delta, t)\theta = U^2_\delta(t)(1 + 2(e_n, \theta)_{L^2(I)})\theta$. We consider the function

$$V(\omega, \phi) := U^2_{\delta(\omega)}\left(\frac{T(\delta(\omega))}{2\pi}\phi\right) = \delta^2(\omega)\mathrm{cn}^2\left[\frac{T(\delta(\omega))}{2\pi}\phi\sqrt{\lambda_n+\nu\delta^2(\omega)}, \frac{\sqrt{\nu}\delta(\omega)}{\sqrt{2(\lambda_n+\nu\delta^2(\omega))}}\right]$$

and the operator

$$P(\omega, \omega t) := V(\omega, \omega t)(1 + 2(e_n, \theta)_{L^2(I)})\theta.$$

By observing that $P(\omega(\delta), \omega(\delta)t) = Q(\delta, t)$, we rewrite

$$\theta_{tt} - \partial^2_x\theta + \gamma\|u_\delta(t)\|^2_{L^2(I)}\theta + 2\gamma(u_\delta, \theta)u_\delta = 0$$

as the following abstract evolution equation

$$\theta_{tt} + B^2\theta + \gamma P(\omega, \omega t)\theta = 0,$$

where we set $B^2 = -\partial^2_x$. If we introduce the space

$$\mathcal{H} := \{\theta \in L^2(I) : \theta(-\pi) = \theta(\pi)\}, \tag{3.10}$$

the following lemma holds:

Lemma 3.3.1. $P \in \mathit{Lip}([\sqrt{\lambda_n}, \omega^*], C^\infty_{2\pi}(\mathbb{R}_+, \mathcal{B}(\mathcal{H}, \mathcal{H})))$.

Proof. Since P is a "separate variable" time-dependent operator, that is $P(\omega, \phi) = V(\omega, \phi)R$, where V is a real valued function and $R \in \mathcal{B}(\mathcal{H}, \mathcal{H})$, the thesis is equivalent to prove that $V \in \mathrm{Lip}([\sqrt{\lambda_n}, \omega^*], C^\infty_{2\pi}(\mathbb{R}, \mathbb{R}))$.

We observe that $\delta^2(\cdot) \in \mathrm{Lip}([\sqrt{\lambda_n}, \omega^*])$. Indeed, since $T'(\delta) < 0$ for any $\delta > 0$, we have that for any $\varepsilon \in (0, \omega^* - \sqrt{\lambda_n})$, $\delta^2(\cdot) \in \mathrm{Lip}([\sqrt{\lambda_n} + \varepsilon, \omega^*])$. Moreover, if we set $m = \nu\delta^2/(2\lambda_n)$

$$\begin{aligned} T(\delta) &= 4\sqrt{2}\int_0^1 \frac{\mathrm{d}\theta}{\sqrt{(2\lambda_n + \nu\delta^2 + \nu\delta^2\theta^2)(1-\theta^2)}} \\ &= \frac{4}{\sqrt{\lambda_n}}\int_0^1 \frac{\mathrm{d}\theta}{\sqrt{1 + m(1+\theta^2))(1-\theta^2)}} \overset{\theta=\cos(\alpha)}{=} \frac{4}{\sqrt{\lambda_n}}\int_0^{\pi/2} \frac{\mathrm{d}\alpha}{\sqrt{1 + 2m - m\sin^2\alpha}} \\ &= \frac{4}{\sqrt{\lambda_n}\sqrt{1+2m}} K\left(\frac{m}{1+2m}\right), \end{aligned} \tag{3.11}$$

where K is the complete elliptic integral of the first kind, which is defined as

$$K(x) := \int_0^{\pi/2} \frac{\mathrm{d}t}{1 - x\sin^2 t}. \tag{3.12}$$

Hence, since from [2, 17.3.11, p.591] we have that

$$K(x) = \frac{\pi}{2} + \frac{\pi}{8}x + o(x), \qquad \text{as } x \to 0,$$

we get

$$\begin{aligned} \frac{4}{\sqrt{\lambda_n}\sqrt{1+2m}} K\left(\frac{m}{1+2m}\right) &= \frac{4}{\sqrt{\lambda_n}}(1 - m + o(m))\left(\frac{\pi}{2} + \frac{\pi}{8}m + o(m)\right) \\ &= \frac{2\pi}{\sqrt{\lambda_n}} - \frac{3\pi}{2\sqrt{\lambda_n}}m + o(m) \qquad \text{as } m \to 0, \end{aligned}$$

that is, as $\delta \to 0$,

$$T(\delta) = \frac{2\pi}{\sqrt{\lambda_n}} - \frac{3\pi}{4}\frac{\nu\delta^2}{\lambda_n^{3/2}} + o(\delta^2).$$

Therefore, since $\omega = 2\pi/T$, by using the Lagrange's reversion theorem [2, 3.6.25, p.16],

$$\frac{\nu\delta^2(\omega)}{\lambda_n^{3/2}} = \frac{8}{3}(\omega - \sqrt{\lambda_n}) + o(\omega - \sqrt{\lambda_n}), \qquad \text{as } \omega \to \sqrt{\lambda_n}^+.$$

Hence,

$$\frac{\mathrm{d}^+}{\mathrm{d}\omega}\delta^2(\sqrt{\lambda_n}) = \frac{8\lambda_n^{3/2}}{3\nu},$$

which implies that $\delta^2(\cdot) \in \mathrm{Lip}([\sqrt{\lambda_n}, \omega^*])$.

The thesis follows from the regularity properties of δ^2 and of the Jacobi elliptic cosine function. Indeed, from [2, 16.13.2, p. 573] we have that as $m \to 0^+$

$$\mathrm{cn}^2(t, m) = \cos^2(t) + \frac{m}{16}\sin(2t)\left(t - \frac{\sin(2t)}{2}\right)^2 + o(m),$$

which yields that as $\omega \to \sqrt{\lambda_n}^+$, since $\delta(\omega) \to 0^+$, we have for any $x \in \mathbb{R}$

$$\delta^2(\omega)\mathrm{cn}^2\left[x, \frac{\sqrt{\nu}\delta(\omega)}{\sqrt{2(\lambda_n+\nu\delta^2(\omega))}}\right] = \delta^2(\omega)\cos^2(x) + \frac{\sqrt{\nu}\delta^3(\omega)}{16\sqrt{2\lambda_n}}\sin(2x)\left(x - \frac{\sin(2x)}{2}\right)^2 + o(\delta^3(\omega)).$$

Therefore, since $\delta^2(\cdot) \in \mathrm{Lip}([\sqrt{\lambda_n}, \omega^*])$, we get that for any $x \in \mathbb{R}$

$$\delta^2(\cdot)\mathrm{cn}\left[x, \frac{\sqrt{\nu}\delta(\cdot)}{\sqrt{2(\lambda_n+\nu\delta^2(\cdot))}}\right] \in \mathrm{Lip}([\sqrt{\lambda_n}, \omega^*]). \tag{3.13}$$

Moreover, since $T(\delta(\cdot)) \in C^1([\sqrt{\lambda_n}, \omega^*], \mathbb{R})$, we have that $T(\delta(\cdot)) \in \mathrm{Lip}([\sqrt{\lambda_n}, \omega^*])$, which yields that for any $m \in (0,1)$,

$$\delta^2(\cdot)\mathrm{cn}^2\left[\frac{T(\delta(\cdot))}{2\pi}\phi\sqrt{\lambda_n+\nu\delta^2(\cdot)}, m\right] \in \mathrm{Lip}([\sqrt{\lambda_n}, \omega^*]). \tag{3.14}$$

The thesis follows from (3.13) and (3.14). □

3.4 Preliminaries for the proof of Theorem 3.2.1

Let us consider the Hill's equation

$$\ddot{y} + p(t)y = 0, \tag{3.15}$$

where p is a real $T-periodic$ function, Lebesgue-integrable in $[0,T]$. We recall the following stability criterion for (3.15), first proved by Burdina in [38] (see also [149]):

Proposition 3.4.1 (Burdina). *Let $i_1, \dots i_s$ and $j_1, \dots j_s$ be respectively the maximum points and the minimum points of $p(t)$ on the half-open interval $0 \le t < T$. Let*

$$q := \int_0^T \sqrt{p(t)}dt - \frac{1}{2}\ln\left(\frac{p(i_1)\dots p(i_s)}{p(j_1)\dots p(j_s)}\right), \quad Q := \int_0^T \sqrt{p(t)}dt + \frac{1}{2}\ln\left(\frac{p(i_1)\dots p(i_s)}{p(j_1)\dots p(j_s)}\right)$$

If

$$n\pi < q \le Q < (n+1)\pi$$

for some integer $n = 0, 1, 2, \dots$, then the trivial solution of (3.15) is stable.

We conclude this section with the following result concerning an estimate on the bounded solution of a particular family of Hill's equations.

Lemma 3.4.2. *Let $a \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be a $T-$periodic function. Let $y \in C^1(\mathbb{R}_+, \mathbb{R})$ be a bounded solution of*

$$\ddot{y} + (\alpha + a(t))y = 0. \tag{3.16}$$

Then, there exists $\bar{\alpha} = \bar{\alpha}(T, \|a\|_{L^\infty([0,T])}, ||\dot{a}||_{L^\infty([0,T])}) > 0$ such that, if $\alpha > \bar{\alpha}$, then

$$\max_{t\ge 0}(\dot{y}^2(t) + \alpha y^2(t)) \le 8(\dot{y}^2(0) + \alpha y^2(0)).$$

Proof. We introduce the functions

$$E(t) = \frac{1}{2}\dot{y}^2(t) + \frac{\alpha + a(t)}{2}y^2(t), \quad \Lambda(t) = \frac{1}{2}\dot{y}^2(t) + \frac{\alpha}{2}y^2(t).$$

We remark that since for any $t \in \mathbb{R}_+$ we have $a(t) \geq 0$, then $\Lambda(t) \leq E(t)$ for any $t \in [0, T]$. Moreover, if $\alpha \geq 2\|a\|_{L^\infty[0,T]}$, we have that $E(t) \leq 2\Lambda(t)$ for any $t \in [0, T]$. Since y solves (3.16), we have that $\dot{E}(t) = \dot{a}(t)y^2(t)/2$, which yields, for any $t \in [0, T]$,

$$\begin{aligned} \Lambda(t) \leq E(t) &\leq E(0) + \frac{T}{2}||\dot{a}||_{L^\infty([0,T])} \max_{t\in[0,T]} y^2(t) \\ &\leq 2\Lambda(0) + \frac{T}{2}\|\dot{a}\|_{L^\infty([0,T])} \max_{t\in[0,T]} y^2(t). \end{aligned} \tag{3.17}$$

Hence, since $\Lambda(t) \geq \alpha y^2(t)/2$, if $\alpha \geq 2T\|\dot{a}\|_{L^\infty([0,T])}$ we get that

$$\alpha \max_{t\in[0,T]} y^2(t) \leq 8\Lambda(0),$$

which yields, from (3.17),

$$\max_{t\in[0,T]} \dot{y}^2(t) \leq 4\Lambda(0) + \frac{8T\|\dot{a}\|_{L^\infty([0,T])}}{\alpha}\Lambda(0) \leq 8\Lambda(0).$$

Therefore, if $\alpha \geq \bar{\alpha} = 2\max(\|a\|_{L^\infty([0,T])}, T\|\dot{a}\|_{L^\infty([0,T])})$, then

$$\max_{t\in[0,T]} (\dot{y}^2(t) + \alpha y^2(t)) \leq 8(\dot{y}^2(0) + \alpha y^2(0)). \tag{3.18}$$

Let $X(t)$ be the transition matrix of $\dot{\underline{z}}(t) = A(t)\underline{z}(t)$, where

$$\underline{z}(t) = \begin{pmatrix} \dot{y}(t) \\ y(t) \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 0 & 1 \\ -\alpha - a(t) & 0 \end{pmatrix}.$$

It is well-known that $X(T)$ satisfies, for any $t \geq 0$, $\underline{z}(t + T) = X(T)\underline{z}(t)$ and that y is a bounded function if and only if $\rho(X(T)) \leq 1$, where $\rho(X(T))$ denotes the spectral radius of $X(T)$ (see [149, vol.1, p.97]). Therefore, we conclude that for any $t \geq 0$

$$\dot{y}^2(t + T) + \alpha y^2(t + T) = \left| X(T) \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \dot{y}(t) \\ y(t) \end{pmatrix} \right| \leq \dot{y}^2(t) + \alpha y^2(t), \tag{3.19}$$

where we denoted by $|\cdot|$ the Euclidean norm in $\mathbb{R}^2$. From (3.18) and (3.19), we get the thesis. □

3.5 Proof of Theorem 3.2.1

In this section, we consider the problem

$$\begin{cases} \theta_{tt} - \partial_x^2\theta + \gamma\|u_\delta\|^2_{L^2(I)}\theta + 2\gamma(u_\delta, \theta)u_\delta = 0 \\ \theta(-\pi, t) = \theta(\pi, t) = 0, \end{cases} \tag{3.20}$$

where u_δ solves

$$\begin{cases} u_{\delta,tt} + \partial_x^4 u_\delta + \nu\|u_\delta\|^2_{L^2(I)}u_\delta = 0, \\ u_\delta(\pi, t) = u_\delta(-\pi, t) = u_{\delta,xx}(\pi, t) = u_{\delta,xx}(-\pi, t) = 0, \\ u_\delta(x, 0) = \dfrac{\delta}{\sqrt{\pi}} \sin\left(\sqrt[4]{\lambda_n}\,(x + \pi)\right), \quad u_{\delta,t}(x, 0) = 0. \end{cases}$$

with $\delta \in [0, \tilde{\delta}]$ for some $\tilde{\delta} > 0$ and where we now have that

$$\lambda_n := \left(\frac{n}{2}\right)^4.$$

As we showed in subsection 3.3.2, equation (3.20) may be reformulated as an abstract problem

$$\theta_{tt} + B^2\theta + \gamma P(\omega, \omega t)\theta = 0 \tag{3.21}$$

with $\theta \in C^1(\mathbb{R}_+, \mathcal{H})$, where

$$\mathcal{H} := \{\theta \in L^2(I) : \theta(-\pi) = \theta(\pi)\}$$

and

$$P(\omega, \omega t)\theta := V(\omega, \omega t)(\theta + 2(e_n, \theta)_{L^2(I)} e_n),$$

where

$$V(\omega, \phi) = \delta^2(\omega)\mathrm{cn}^2\left[\frac{T(\delta(\omega))}{2\pi}\phi\sqrt{\lambda_n + \nu\delta^2(\omega)}, \frac{\sqrt{\nu}\delta(\omega)}{\sqrt{2(\lambda_n + \nu\delta^2(\omega))}}\right].$$

If we set $\tilde{\omega} = 2\pi/T(\tilde{\delta})$, then $\omega \in [\sqrt{\lambda_n}, \tilde{\omega}]$ and from Lemma 3.3.1 we have that

$$P \in \mathrm{Lip}([\sqrt{\lambda_n}, \tilde{\omega}], C^\infty_{2\pi}(\mathbb{R}_+, \mathcal{B}(\mathcal{H}, \mathcal{H}))).$$

Moreover, since $e_j(x) = \mathfrak{e}_j(x) = \sin(\sqrt[4]{\lambda_j}(x + \pi/2))/\sqrt{\pi}$, where we denoted by $e_n(x)$ the $n-$th normalized eigenfunction for the operator $-\partial_x^2$ in $\mathcal{D}(-\partial_x^2) \subseteq \mathcal{H}$, it follows that

$$\theta(t, x) = \frac{1}{\sqrt{\pi}}\sum_{j=1}^{\infty}\theta_j(t)\sin\left(\sqrt[4]{\lambda_j}\,(x + \pi)\right), \quad u_\delta(t, x) = \frac{1}{\sqrt{\pi}}V(\omega, \omega t)\sin\left(\sqrt[4]{\lambda_n}\,(x + \pi)\right),$$

and it is immediate to verify that

$$P(\omega, \omega t)e_j(x) = \begin{cases} V(\omega, \omega t)e_j(x) \\ 3V(\omega, \omega t)e_j(x). \end{cases}$$

that is, $P(\omega, \omega t)$ is a diagonal and strictly positive operator for any choice of $\omega \in [\sqrt{\lambda_n}, \tilde{\omega}]$ and $t \in \mathbb{R}_+$.

Let us write (3.21) as a pair of first order equations: $\theta_t = \vartheta$, $\vartheta_t = -B^2\theta - \gamma P(\omega, \omega t)\theta$. The solution of this system depends linearly on the initial condition at time zero, thus defining a time-dependent map $\Phi(t)$ via the equation

$$\Theta(t) = \Phi(t)\Theta(0), \quad \Theta(t) = \begin{pmatrix}\theta(t) \\ \vartheta(t)\end{pmatrix}, \qquad t \in \mathbb{R}_+.$$

We say that the equation (3.21) is spectrally stable if the corresponding evolution operator $\Phi(2\pi/\omega)$ has no spectrum outside the unit circle. The following lemma holds:

Lemma 3.5.1. *Let u and θ solve* (3.6) *and* (3.5) *with boundary conditions* (3.7) *and let us fix $\omega^* > 0$. There exists $\gamma^* = \gamma^*(n, \omega^*) > 0$ such that for any $\omega \in [\sqrt{\lambda_n}, \omega^*]$ and for any $\gamma < \gamma^*$, there exists a positive measure set $\Omega_{n,\gamma,\omega} \subseteq [\sqrt{\lambda_n}, \omega]$ such that for any $\bar{\omega} \in \Omega_{n,\gamma,\omega}$ we have that* (3.21) *is spectrally stable.*

Proof. Let θ be a solution of (3.21). We set $p_h(\omega,\omega t) := \gamma(P(\omega,\omega t)e_h, e_h)$ and we observe that, since P is diagonal, (3.21) may be interpreted as an infinite-dimensional dynamical system given by

$$\theta_{h,tt} + \beta_h^2\theta_h + p_h(\omega,\omega t)\theta_h = 0, \qquad h \geq 1. \tag{3.22}$$

Let us define $a_h(\omega,\omega t) = \beta_h^2 + p_h(\omega,\omega t)$ and introduce the quantity

$$\Lambda_h(\omega) = \int_0^{2\pi/\omega} \sqrt{a_h(\omega,\omega t)}dt.$$

From Proposition 3.4.1, since $p_h(\omega,\omega t)$ is $2\pi/\omega-$periodic, the trivial solution of (3.22) is stable if there exists $n \in \mathbb{N}$ such that

$$n\pi < q_h \leq Q_h < (n+1)\pi, \tag{3.23}$$

where, according to the notations of Proposition 3.4.1,

$$q_h := \Lambda_h(\omega) - \frac{1}{2}\ln\left(\frac{a_h(\omega,i_1)\dots a_h(\omega,i_s)}{a_h(\omega,j_1)\dots a_h(\omega,j_s)}\right), \quad Q_h := \Lambda_h(\omega) + \frac{1}{2}\ln\left(\frac{a_h(\omega,i_1)\dots a_h(\omega,i_s)}{a_h(\omega,j_1)\dots a_h(\omega,j_s)}\right).$$

Moreover, we notice that, fixed $\tilde{\omega} > 0$, there exists a positive quantity α such that

$$\forall h \geq 1, \forall \omega \in [\sqrt{\lambda_n}, \tilde{\omega}], \qquad \beta_h^2 \leq a_h(\omega,\omega t) \leq \beta_h^2 + \gamma\alpha,$$

and the Lipschitz dependence of P by ω yields that there exists a positive constant $\alpha^{\mathcal{L}}$ such that

$$\forall h \geq 1, \forall \phi \in \mathbb{R}_+, \forall \omega_1, \omega_2 \in \Omega, \quad |p_h(\omega_1,\phi) - p_h(\omega_2,\phi)| \leq \gamma\alpha^{\mathcal{L}}|\omega_1 - \omega_2|.$$

Since $\omega \mapsto \delta(\omega)$ is a strictly increasing function and for any $m \in (0,1)$ from (3.9) $\mathrm{cn}^2(x,m) \leq 1$, we have that, setting $\tilde{\delta} = \delta(\tilde{\omega})$,

$$\alpha = 3\tilde{\delta}^2, \qquad \alpha^{\mathcal{L}} = 3 \sup_{\substack{\omega_1,\omega_2 \in [\sqrt{\lambda_n},\tilde{\omega}] \\ \phi \in [0,2\pi]}} \left|\frac{V(\omega_1,\phi) - V(\omega_2,\phi)}{\omega_1 - \omega_2}\right|.$$

Therefore, since $a_h(\omega,\phi)$ has 2 maximum points i_1, i_2 and 2 minimum points j_1, j_2 in $[0,2\pi]$, we have that

$$\frac{1}{2}\ln\left(\frac{a_h(\omega,i_1)a_h(\omega,i_2)}{a_h(\omega,j_1)a_h(\omega,j_2)}\right) \leq \frac{3\gamma\tilde{\delta}^2}{\beta_h^2}.$$

Hence, (3.23) is implied by

$$\Lambda_h(\omega) \in \left(n\pi + \frac{3\gamma\tilde{\delta}^2}{\beta_h^2}, (n+1)\pi - \frac{3\gamma\tilde{\delta}^2}{\beta_h^2}\right),$$

which is equivalent to require that

$$\Lambda_h \notin \left[0, \frac{3\gamma\tilde{\delta}^2}{\beta_h^2}\right] \cup \bigcup_{n=1}^{\infty}\left[n\pi - \frac{3\gamma\tilde{\delta}^2}{\beta_h^2}, n\pi + \frac{3\gamma\tilde{\delta}^2}{\beta_h^2}\right],$$

that is, we have that equation (3.22) is stable for any $h \in \mathbb{N}$ if and only if

$$\forall h \geq 1, n \geq 0, \quad |\Lambda_h(\omega) - n\pi| > \frac{3\gamma\tilde{\delta}^2}{\beta_h^2}.$$

We observe that

$$\begin{aligned}\Lambda_h(\omega_1)-\Lambda_h(\omega_2) &= \int_0^{2\pi/\omega_1}\sqrt{a_h(\omega_1,\omega_1 t)}\mathrm{d}t - \int_0^{2\pi/\omega_2}\sqrt{a_h(\omega_2,\omega_2 t)}\mathrm{d}t \\ &= \frac{1}{\omega_1}\int_0^{2\pi}\sqrt{a_h(\omega_1,\phi)}\mathrm{d}\phi - \frac{1}{\omega_2}\int_0^{2\pi}\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi \\ &= \frac{1}{\omega_1}\int_0^{2\pi}\sqrt{a_h(\omega_1,\phi)}-\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi - \frac{\omega_1-\omega_2}{\omega_1\omega_2}\int_0^{2\pi}\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi\end{aligned}$$

which yields

$$|\Lambda_h(\omega_1)-\Lambda_h(\omega_2)| \le \left|\frac{1}{\omega_1}\int_0^{2\pi}\sqrt{a_h(\omega_1,\phi)}-\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi\right| + \left|\frac{\omega_1-\omega_2}{\omega_1\omega_2}\int_0^{2\pi}\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi\right|$$

$$|\Lambda_h(\omega_1)-\Lambda_h(\omega_2)| \ge \left|\left|\frac{1}{\omega_1}\int_0^{2\pi}\sqrt{a_h(\omega_1,\phi)}-\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi\right| - \left|\frac{\omega_1-\omega_2}{\omega_1\omega_2}\int_0^{2\pi}\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi\right|\right|.$$

Since $\beta_h^2 \le a_h(\omega_2,\omega_2 t) \le \beta_h^2 + 3\gamma\tilde{\delta}^2$, we have that

$$\frac{|\omega_1-\omega_2|}{\omega_1\omega_2}\int_0^{2\pi}\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi \le \frac{2\pi\sqrt{\beta_h^2+3\gamma\tilde{\delta}^2}}{\lambda_n}|\omega_1-\omega_2| \tag{3.24}$$

and

$$\frac{|\omega_1-\omega_2|}{\omega_1\omega_2}\int_0^{2\pi}\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi \ge \frac{2\pi\beta_h}{\tilde{\omega}^2}|\omega_1-\omega_2|. \tag{3.25}$$

Moreover, since

$$\begin{aligned}\left|\sqrt{a_h(\omega_1,\phi)}-\sqrt{a_h(\omega_2,\phi)}\right| &= \sqrt{\beta_h^2+p_h(\omega_2,\phi)}\left|\sqrt{1+\frac{p_h(\omega_1,\phi)-p_h(\omega_2,\phi)}{\beta_h^2+p_h(\omega_2,\phi)}}-1\right| \\ &\le \frac{|p_h(\omega_1,\phi)-p_h(\omega_2,\phi)|}{2\beta_h} \le \frac{\gamma\alpha^{\mathcal{L}}|\omega_1-\omega_2|}{2\beta_h}\end{aligned}$$

then

$$\frac{1}{\omega_1}\int_0^{2\pi}(\sqrt{a_h(\omega_1,\phi)}-\sqrt{a_h(\omega_2,\phi)})dt \le \frac{\pi\gamma\alpha^{\mathcal{L}}|\omega_1-\omega_2|}{\beta_h\sqrt{\lambda_n}}. \tag{3.26}$$

Therefore, by combining (3.24) and (3.26),

$$\left|\frac{\Lambda_h(\omega_1)-\Lambda_h(\omega_2)}{\omega_1-\omega_2}\right| \le \frac{2\pi\sqrt{\beta_h^2+\gamma\alpha}}{\lambda_n} + \frac{\pi\gamma\alpha^{\mathcal{L}}}{\beta_h\sqrt{\lambda_n}}.$$

Moreover, if

$$\left|\frac{1}{\omega_1}\int_0^{2\pi}\sqrt{a_h(\omega_1,\phi)}-\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi\right| \le \left|\frac{\omega_1-\omega_2}{\omega_1\omega_2}\int_0^{2\pi}\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi\right|,$$

which is implied, from (3.25) and (3.26), by

$$\frac{\pi\gamma\alpha^{\mathcal{L}}|\omega_1-\omega_2|}{\beta_h\sqrt{\lambda_n}} \le \frac{2\pi\beta_h}{\tilde{\omega}^2}|\omega_1-\omega_2|,$$

that is

$$\gamma < \frac{2\sqrt{\lambda_n}\beta_1^2}{\alpha^{\mathcal{L}}\tilde{\omega}^2} =: \gamma^*(n,\tilde{\omega}),$$

we get that

$$\begin{aligned}\left|\frac{\Lambda_h(\omega_1)-\Lambda_h(\omega_2)}{\omega_1-\omega_2}\right| &\geq \left|\frac{1}{\omega_1\omega_2}\int_0^{2\pi}\sqrt{a_h(\omega_2,\phi)}\mathrm{d}\phi\right| - \left|\frac{1}{\omega_1}\int_0^{2\pi}\frac{\sqrt{a_h(\omega_1,\phi)}-\sqrt{a_h(\omega_2,\phi)}}{\omega_1-\omega_2}\mathrm{d}\phi\right| \\ &\geq \frac{2\pi\beta_h}{\tilde{\omega}^2} - \frac{\pi\gamma\alpha^{\mathcal{L}}}{\beta_h\sqrt{\lambda_n}}.\end{aligned}$$

Hence, since $\beta_j = j/2$, from Corollary A.2 we conclude that if $\tilde{\delta}$ satisfies

$$\gamma < \frac{\sqrt{\lambda_n}}{2\alpha^{\mathcal{L}}\tilde{\omega}^2}, \quad \left(\sum_{j=1}^{\infty}\frac{8(\sqrt{j^2+4\gamma\alpha}/\lambda_n+4\gamma\alpha^{\mathcal{L}}/(j\sqrt{\lambda_n})+2/(\tilde{\omega}-\sqrt{\lambda_n}))}{j^3/\tilde{\omega}^2-4\gamma\alpha^{\mathcal{L}}j/\sqrt{\lambda_n}}\right)3\gamma\tilde{\delta}^2 < 1 \tag{3.27}$$

then there exists a positive measure set $\Omega_{\gamma,n,\tilde{\omega}} \subset [\sqrt{\lambda_n},\tilde{\omega}]$, such that

$$\begin{aligned}|[\sqrt{\lambda_n},\tilde{\omega}]\setminus\Omega_{\gamma,n,\tilde{\omega}}| &\leq \left(\sum_{j=1}^{\infty}\frac{8((\tilde{\omega}-\sqrt{\lambda_n})(\sqrt{j^2+4\gamma\alpha}/\lambda_n+4\gamma\alpha^{\mathcal{L}}/(j\sqrt{\lambda_n}))+2)}{j^3/\tilde{\omega}^2-4\gamma\alpha^{\mathcal{L}}j/\sqrt{\lambda_n}}\right)3\gamma\tilde{\delta}^2 \\ &=: F(\tilde{\delta},n,\gamma)\gamma\tilde{\delta}^2.\end{aligned} \tag{3.28}$$

and

$$\forall\bar{\omega}\in\Omega_{\gamma,n,\tilde{\omega}}, \forall n\in\mathbb{N}, \forall h\geq 1, \quad |\Lambda_h(\bar{\omega})-n\pi| > \frac{3\gamma\tilde{\delta}^2}{\beta_h^2}.$$

Therefore, for any $\bar{\omega} \in \Omega_{\gamma,n,\tilde{\omega}}$, we conclude that for any $h \geq 1$, (3.22) is stable, that is, for any $h \geq 1$, the linear map $\Phi_h(2\pi/\bar{\omega})$ associated with (3.22) has no spectrum outside the unit circle. Hence, since $\sigma(\Phi) = \bigcup_{h\geq 1}\sigma(\Phi_h)$, (3.21) is spectrally stable.

Since from Lemma 3.1 we have that

$$\lim_{\omega\to\sqrt{\lambda_n}^+}\frac{\delta^2(\omega)}{\omega-\sqrt{\lambda_n}} = \frac{8\lambda_n^{3/2}}{3\nu} < +\infty,$$

and $\omega \mapsto \alpha^{\mathcal{L}}(\omega)$ and $\omega \mapsto \delta(\omega)$ are strictly increasing functions, we have that for any fixed $\omega^* > 0$, there exists $\gamma^*(n,\omega^*) > 0$ such that the conditions in (3.27) hold. Hence, we have that for any $\omega < \omega^*$ and for any $\gamma < \gamma^*$, there exists $\Omega_{\gamma,n,\omega} \subseteq [\sqrt{\lambda_n},\omega]$ such that if $\bar{\omega} \in \Omega_{\gamma,n,\omega}$, then (3.21) is spectrally stable, that is the thesis. □

Lemma 3.5.2. *Let u and θ solve* (3.6) *and* (3.5) *with boundary conditions* (3.7) *and let us fix $\delta^* > 0$. There exists $\gamma^* = \gamma^*(n,\delta^*) > 0$ such that for any $\delta \in [0,\delta^*]$ and for any $\gamma < \gamma_0^*$, there exists a positive measure set $\Delta_{n,\gamma,\delta} \subseteq [0,\delta]$ such that for any $\bar{\delta} \in \Delta_{n,\gamma,\omega}$ we have that* (3.21) *is spectrally stable.*

Proof. The proof follows from Lemma 3.5.1. Indeed, we observe that for any fixed $\delta^* > 0$, the applications

$$[0,\delta^*] \ni \delta \mapsto \alpha^{\mathcal{L}}(\omega(\delta)), \quad [0,\delta^*] \ni \delta \mapsto \omega(\delta)$$

are strictly increasing functions, which yields that if $\delta < \delta^*$, $\gamma^*(n,\omega(\delta)) < \gamma^*(n,\omega(\delta^*))$ and, if δ^* satisfies the conditions in (3.27), then, in the notations of Lemma 3.5.1, $F(\delta,n,\gamma) \leq F(\delta^*,n,\gamma)$. Therefore, by reasoning as in Lemma 3.5.1, we have that for any fixed $\delta^* > 0$ there exists $\gamma_0^* =$

$\gamma_0^*(n,\delta^*) > 0$ such that for any $\gamma < \gamma^*$ and for any $\delta \in [0,\delta^*]$, the conditions in (3.27) hold, which yields that there exists $\Omega_{n,\gamma,\delta} \subseteq [\sqrt{\lambda_n}, \omega(\delta)]$ such that for any $\bar{\omega} \in \Omega_{n,\gamma,\delta}$, (3.21) is spectrally stable. Moreover, from (3.28) we have that

$$|[\sqrt{\lambda_n}, \omega] \setminus \Omega_{n,\gamma,\omega}| \leq C_1 \gamma \delta^2$$

for some positive constant $C_1 = C_1(n,\delta^*) > 0$. Since, as we showed in Lemma 3.3.1, $\omega \mapsto \delta(\omega)^2$ is a strictly increasing and Lipschitz function of ω in $[\sqrt{\lambda_n}, \omega^*]$ with Lipschitz constant $L = L(n,\delta^*)$ and $[0,\delta^2] = \delta^2([\sqrt{\lambda_n}, \omega])$, we have that $\delta^2(\Omega_{n,\gamma,\omega}) \subset [0,\delta^2]$ and

$$|[0,\delta^2] \setminus \delta^2(\Omega_{n,\gamma,\delta^2})| < L(n,\delta^*) C_1(n,\delta^*) \gamma \delta^2. \tag{3.29}$$

By using Lemma A.4, if we set $c(n,\delta^*) := \sqrt{L(n,\delta^*) C_2(n,\delta^*)}$ and $\Delta_{\gamma,n,\omega} := \delta(\Omega'_{\gamma,n,\delta})$, from (3.29) we get that

$$|[0,\delta] \setminus \Delta_{n,\gamma,\omega}| \leq c(n,\delta^*) \sqrt{\gamma} \delta. \tag{3.30}$$

We remark that from (3.30) we get that for γ small enough, $\Delta_{\gamma,n,\omega}$ is a set of positive measure. Summarizing, we have that for any $\delta^* > 0$ there exists $\gamma^* = \gamma^*(n,\delta^*) > 0$ such that for any $\gamma < \gamma^*$ and for any $\delta \in [0,\delta^*]$ there exists a positive measure set $\Delta_{n,\gamma,\delta} \subseteq [0,\delta]$ such that for any $\bar{\delta} \in \Omega_{n,\gamma,\delta}$ equation (3.21) is spectrally stable, that is the thesis. □

3.5.1 Completion of the proof

Let θ be a solution of (3.21) and let θ_h be the projection of θ on e_h. From Lemma 3.5.2, there exists $\gamma^* = \gamma^*(n,\delta^*) > 0$ such that for any $\gamma < \gamma^*$ there is a nonempty set $\Delta_{n,\gamma,\delta} \subset [0,\delta]$ such that for any $\bar{\delta} \in \Delta_{n,\gamma,\delta}$, (3.21) is spectrally stable. This is equivalent to say that

$$\forall \bar{\delta} \in \Delta_{n,\gamma,\delta}, \forall h \geq 1, \quad \dot{\theta}_h^2(t) + \beta_h^2 \theta_h^2(t) \leq C_h(\dot{\theta}_h^2(0) + \beta_h^2 \theta_h^2(0))$$

for some positive constant C_h. From Lemma 3.4.2, we have that for any $\bar{\delta} \in \Delta_{n,\gamma,\delta}$ there exists $\tilde{h}(\bar{\delta}) = \tilde{h}(\bar{\delta}, n, \nu)$ such that for any $h \geq \tilde{h}$ we can take $C_h \equiv 8$. Hence, if $\bar{h} := \max_{\delta \in [0,\delta^*]} \tilde{h}(\delta)$, we have that

$$\forall \omega \in \Omega_\gamma, \forall h \geq \bar{h}, \quad \dot{\theta}_h^2(t) + \beta_h^2 \theta_h^2(t) \leq 8(\dot{\theta}_h^2(0) + \beta_h^2 \theta_h^2(0)).$$

Next, let us consider $h \leq \bar{h}$. If we set $\bar{C} := \max_{h \leq \bar{h}} C_h$, we have that

$$\forall \omega \in \Omega_\gamma, \forall 1 \leq h \leq \bar{h}, \quad \dot{\theta}_h^2(t) + \beta_h^2 \theta_h^2(t) \leq \bar{C}(\dot{\theta}_h^2(0) + \beta_h^2 \theta_h^2(0)).$$

Hence, if we set $C = \max(\bar{C}, 8)$, we conclude that for any $\delta^* > 0$ there exists $\gamma^* = \gamma^*(n,\delta^*)$ such that if $\gamma < \gamma^*$, for any $\delta < \delta^*$ there exists $\Delta_{n,\gamma,\delta}$ such that for any $\bar{\delta} \in \Delta_{n,\gamma,\delta}$, for any $h \geq 1$,

$$\dot{\theta}_h^2(t) + \beta_h^2 \theta_h^2(t) \leq C(\dot{\theta}_h^2(0) + \beta_h^2 \theta_h^2(0)) \tag{3.31}$$

and by summing (3.31) over h we get the thesis.

3.6 Preliminaries for the proof of Theorem 3.2.2

3.6.1 Hamiltonian machinery

Let us consider the formal infinite polynomials

$$\sum_{i,j=1}^{\infty} F_{ij} \eta_i \xi_j, \quad \sum_{i,j=1}^{\infty} F_{ij} \eta_i \eta_j \quad \sum_{i,j=1}^{\infty} F_{ij} \xi_i \xi_j. \tag{3.32}$$

We introduce the spaces

$$Y^s_C := \{(\xi,\eta) \in \ell^2(\mathbb{N},\mathbb{C}) \times \ell^2(\mathbb{N},\mathbb{C}) : \|(\xi,\eta)\|^2_{Y^s} := \frac{1}{2}\sum_{j\in\mathbb{N}} j^{2s}(|\xi_j|^2 + |\eta_j|^2) < \infty\}.$$

and the spaces $Y^s \subseteq Y^s_C$, where

$$Y^s := \{(\xi,\eta) \in Y^s_C : \eta_j = \bar{\xi}_j \quad \forall j \in \mathbb{N}\}. \tag{3.33}$$

In the following, we denote $Y := Y^{1/2}$. Moreover, for any $s \geq 0$, we introduce the spaces

$$(\mathcal{H}^s := H^s(I) \cap \mathcal{H}, \|\cdot\|_s)$$

where $H^s(I)$ is the standard Sobolev space and $\|\theta\|_s := 2\|(-\partial_x^2)^{s/2}\theta\|_{L^2(I)}$. We define for any $s > 0$, $\mathcal{H}^{-s}$ as the dual of $\mathcal{H}^s$ and we remark that $\mathcal{H}^0 = \mathcal{H}$. The following lemma holds.

Lemma 3.6.1. *Let $F \in \mathcal{B}(\mathcal{H}^{1/2}, \mathcal{H}^{1/2+k})$ with $k \geq 0$ and let $F_{ij} := (Fe_i, e_j)$. Then*

$$\sum_{i,j=1}^{\infty} F_{ij}\eta_i\xi_j : Y^{1/2} \mapsto \mathbb{C}.$$

Proof. We observe that for any $k \geq 0$ we have $\mathcal{B}(\mathcal{H}^{1/2}, \mathcal{H}^{1/2+k}) \subset \mathcal{B}(\mathcal{H}^{1/2}, \mathcal{H}^{1/2})$. Moreover, $F \in \mathcal{B}(\mathcal{H}^{1/2}, \mathcal{H}^{1/2})$ if and only if $B^{1/2}FB^{-1/2} \in \mathcal{B}(\mathcal{H},\mathcal{H})$, that is, if and only if the operator M with matrix $M_{ij} = i^{1/2}F_{ij}j^{-1/2}$ belongs to $\mathcal{B}(\ell^2,\ell^2)$. Therefore, by Cauchy-Schwarz inequality we have that, if we take $x \in \ell^2$ with $x_i = 1/i$,

$$\begin{aligned}
\left|\sum_{i,j=1}^{\infty} F_{ij}\eta_i\xi_j\right| &= \sum_{i,j} i^{1/2}F_{ij}j^{-1/2}i^{-1}i^{1/2}\eta_i j^{1/2}\xi_j \\
&\leq \left(\sum_{j=1}^{\infty} j\xi_j^2\right)^{1/2} \left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} i^{1/2}F_{ij}j^{-1/2}i^{-1}i^{1/2}\eta_i\right)^2\right)^{1/2} \\
&\leq \left(\sum_{j=1}^{\infty} j\xi_j^2\right)^{1/2} \left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} i^{1/2}F_{ij}j^{-1/2}i^{-1}\right)^2 \left(\sum_{i=1}^{\infty} i^{1/2}\eta_i\right)^2\right)^{1/2} \\
&\leq \left(\sum_{j=1}^{\infty} j\xi_j^2\right)^{1/2} \left(\sum_{i=1}^{\infty} i\eta_i^2\right)^{1/2} \left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} i^{1/2}F_{ij}j^{-1/2}i^{-1}\right)^2\right)^{1/2} \\
&\leq \|(\xi,\eta)\|^2_{Y^{1/2}}\|Mx\|_{\ell^2} < +\infty.
\end{aligned}$$

□

By proceeding as in Lemma 3.6.1, it is possible to prove that if $F \in \mathcal{B}(\mathcal{H}^{1/2}, \mathcal{H}^{1/2+k})$ with $k \geq 0$, then all the expressions in (3.32) are well-defined $\mathbb{C}-$valued functions from $Y^{1/2}$. We define the spaces

$$H^{k_-,k_+} := \left\{ F : Y^{1/2} \xrightarrow{C^1} \mathbb{C} \;\middle|\; \begin{array}{l} F(\xi,\eta) := \sum_{i,j\in\mathbb{N}_0} F^-_{ij}\eta_i\xi_j + \frac{1}{2}\sum_{i,j\in\mathbb{N}_0} \overline{F^+_{ij}}\xi_i\xi_j + \frac{1}{2}\sum_{i,j\in\mathbb{N}_0} F^+_{ij}\eta_i\eta_j, \\ \text{where } F^\pm \in \mathcal{B}(\mathcal{H}^{1/2}, \mathcal{H}^{1/2+k_\pm}) \text{ and } F^\pm_{ij} := (F^\pm e_i, e_j) \\ \text{with } F^- = (F^-)^\dagger \text{ and } F^+ = (\overline{F^+})^\dagger \end{array} \right\}$$

endowed with the norm

$$\|F\|_{H^{k_-,k_+}} := \|F^-\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+k_-\wedge k_+})} + \|F^+\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+k_-\wedge k_+})}, \text{ where } k_-\wedge k_+ := \min(k_-,k_+). \tag{3.34}$$

For the sake of brevity, in the following we set $H^s := H^{s,s}$ and we call Hamiltonian function any function in H^{k_-,k_+}. From Lemma 3.6.1, we conclude that the elements of H^{k_-,k_+} are well-defined functions from Y to $\mathbb{C}$ for any $k_-, k_+ \geq 0$.

We consider $\zeta = (\zeta_s)_{s\in\mathbb{Z}}$ with $\zeta_s = (\xi_s, \eta_s)^t$ and for any $F \in H^{h,k}$ we denote by X_F the linear operator defined by

$$X_F\zeta := \begin{pmatrix} -i\nabla_\eta F(\zeta) \\ i\nabla_\xi F(\zeta) \end{pmatrix}. \tag{3.35}$$

For any $F, G \in H^0$ we define the Hamiltonian $\{F, G\}$ as the unique Hamiltonian such that

$$X_{\{F,G\}} = X_G X_F - X_F X_G =: -[X_F, X_G].$$

By direct computation, it is possible to verify that $\{\cdot,\cdot\}$ is the Poisson bracket associated to the symplectic form $i\sum_s \mathrm{d}\xi_s \wedge \mathrm{d}\eta_s$, that is

$$\{F, G\} = \sum_{s\in\mathbb{N}_0} i\frac{\partial F}{\partial \xi_s}\frac{\partial G}{\partial \eta_s} - i\frac{\partial F}{\partial \eta_s}\frac{\partial G}{\partial \xi_s},$$

which yields that

$$\begin{aligned} \{F,G\}^- &= i(F^-G^- - G^-F^- + G^+\overline{F^+} - F^+\overline{G^+}), \\ \{F,G\}^+ &= i(F^-G^+ + G^+\overline{F^-} - F^+\overline{G^-} - G^-F^+). \end{aligned} \tag{3.36}$$

Lemma 3.6.2. *Let $F \in H^s$. Then $X_F : Y^s \to Y^s$ and $\|X_F\|_{\mathcal{B}(Y^s,Y^s)} \leq \|F\|_{H^s}$.*

Proof. We remark that, since

$$\begin{matrix} \psi_1 = \sum_{j\in\mathbb{N}} \xi_j e_j \in \mathcal{H}^s \\ \psi_2 = \sum_{j\in\mathbb{N}} \eta_j e_j \in \mathcal{H}^s \end{matrix} \iff \zeta = (\xi,\eta) \in Y^s_C$$

and we have that

$$\frac{1}{2}(\|\psi_1\|_s^2 + \|\psi_2\|_s^2) = \|\zeta\|^2_{Y^s},$$

it follows that

$$\begin{aligned} \sup_{\|\zeta\|^2_{Y^s}=1} \|X_F\zeta\|_{Y^s} &= \frac{1}{2}\sup_{(\|\psi\|_s^2+\|\bar\psi\|_s^2)/2=1} \left(\|F^-\psi + F^+\bar\psi\|_s + \|\overline{F^-\psi} + \overline{F^+}\psi\|_s\right) \\ &\leq \sup_{\|\psi\|_s=1} (\|F^-\psi\|_s + \|F^+\bar\psi\|_s) \leq \sup_{\|\psi\|_s=1}\|F^-\psi\|_s + \sup_{\|\bar\psi\|_s=1}\|F^+\bar\psi\|_s = \|F\|_{H^s}, \end{aligned}$$

that is the thesis. □

3.6.2 Lie transform

For any $F \in H^{k_-,k_+}$, let us denote by ϕ^t_F the Hamiltonian flow of F with respect to the symplectic structure $i\sum_s \mathrm{d}\xi_s \wedge \mathrm{d}\eta_s$, that is $\phi^t_F(x) = (y^1(t), y^2(t))$ where $y(t)$ solves

$$\dot y = X_F y \quad \text{with } y(0) = x.$$

Lemma 3.6.3. *$F \in H^0$ if and only if $\phi_F^1 : Y \to Y$. Moreover, if $\|F\|_{H^0} \le 5/4$, we have that*

$$\|id - \phi_F^1\|_{\mathcal{B}(Y,Y)} \le 2\|F\|_{H^0}.$$

Proof. We have that $\phi_F^1 : Y \to Y$ if and only if for any $\zeta \in Y$ we have that $X_F\zeta \in Y$. Since, if $\zeta \in Y$,

$$(X_F\zeta)_s = \begin{pmatrix} -i\sum_j F^-_{sj}\xi_j - i\sum_j F^+_{sj}\eta_j \\ i\sum_j F^-_{js}\eta_j + i\sum_j \overline{F^+}_{sj}\xi_j \end{pmatrix} \in Y \iff F^- = (F^-)^\dagger, F^+ = (\overline{F^+})^\dagger$$

we conclude that $F \in H^0$ if and only if $X_F\zeta \in Y$ for any $\zeta \in Y$, that is, $F \in H^0$ if and only if $\phi_F^1 : Y \to Y$. Next, we observe that by definition

$$\frac{d}{dt}\phi_F^t = X_F\phi_F^t,$$

which yields that

$$\phi_F^1 = \sum_{n=0}^{\infty} \frac{X_F^n}{n!}.$$

Therefore, from Lemma 3.6.2, if $\|F\|_{H^0} \le 5/4$ we have that

$$\|\mathrm{id} - \phi_F^1\|_{\mathcal{B}(Y,Y)} \le \sum_{n=1}^{\infty} \frac{\|X_F\|^n_{\mathcal{B}(Y,Y)}}{n!} \le \sum_{n=1}^{\infty} \frac{\|F\|^n_{H^0}}{n!} \le \exp(\|F\|_{H^0}) - 1 \le 2\|F\|_{H^0}.$$

□

Definition 3.1. *Let $F \in H^{k_1,h_1}$ and $G \in H^{k_2,h_2}$ with $k_2, h_2 \ge 0$. Then the Lie transform of F generated by G is the unique Hamiltonian function $Lie_G F$ such that*

$$X_{Lie_G F} := \phi_G^1 X_F \phi_G^{-1}.$$

By proceeding exactly as in [13, Lemma 3.2], we obtain the following lemma.

Proposition 3.6.4. *Let $F \in H^{h_1,k_1}$ and $Z \in C^1(\mathbb{R}_+, H^{h_2,k_2})$. Assume that $\zeta(t) = (\xi(t), \eta(t))^T$ fulfills the equation*

$$\dot{\zeta} = X_F\zeta$$

Then Ξ defined by

$$\Xi = \phi_Z^1\zeta$$

satisfies

$$\dot{\Xi} = X_{T_Z F}\Xi$$

with

$$T_Z F := Lie_Z F - Y_Z, \qquad Y_Z := \int_0^1 (Lie_{(1-\varepsilon)Z}\dot{Z})d\varepsilon.$$

In particular, we remark that if $F = G + H$, then $T_Z F = Lie_Z G + Lie_Z H - Y_Z$.

In the following, we shall consider Hamiltonian functions $F(\omega, \phi, \zeta) \in H^{k_-,k_+}$ depending analitically and periodically on ϕ. We recall that any analytic and periodic function F can be identified, for some positive r, with an holomorphic and bounded function $\tilde{F}$ defined in $S_r^1 := \{\Phi = \alpha + i\xi :$

$\alpha \in S^1, \xi \in \mathbb{R}$ with $|\xi| < r\}$ such that $\tilde{F}|_{\mathbb{R}} = F$. Slightly abusing the notation, we will denote the holomorphic extension of F to S^1_r by the same letter. In addition, we require F to be Lipschitz-dependent by ω, that is

$$\sup_{\omega_1 \neq \omega_2 \in \Omega} \| \frac{F(\omega_1, \phi) - F(\omega_2, \phi)}{\omega_1 - \omega_2} \|_{h,k} < \infty, \quad \forall \phi \in S^1_r.$$

For our convenience, we introduce the space

$$\mathcal{B}^{h,k}_r(\Omega) := \mathrm{Lip}(\Omega, C^\infty(S^1_r, H^{h,k}))$$

endowed with the norm $|\cdot|_{h\wedge k,r}$ and the seminorm $|\cdot|^{\mathcal{L}}_{h\wedge k,r}$ defined by

$$|F|_{h\wedge k,r} := \sup_{\omega\in\Omega} \sup_{\phi \in S^1_r} \|F(\omega,\phi)\|_{H^{h\wedge k}} \qquad |F|^{\mathcal{L}}_{h\wedge k,r} := \sup_{\omega_1\neq\omega_2\in\Omega} \left| \frac{F(\omega_1) - F(\omega_2)}{\omega_1 - \omega_2} \right|_{h,k}.$$

For the sake of simplicity, in the following we write $\mathcal{B}^h_r(\Omega) := \mathcal{B}^{h,h}_r(\Omega)$ and we set $|\cdot|_{2,r} = |\cdot|_r$ and $|\cdot|^{\mathcal{L}}_{2,r} = |\cdot|^{\mathcal{L}}_r$. Moreover, abusing the notations, for any real periodic analytic function $a(\omega, \phi)$ we define $|a|_r := |\tilde{a} \sum_i \beta_i^2 \xi_i \eta_i|_r$ and $|a|^{\mathcal{L}}_r := |\tilde{a} \sum_i \beta_i^2 \xi_i \eta_i|_r$, where $\tilde{a}$ is the holomorphic extension of a to S^1_r. By further abusing the notations for any $F \in H^0$ we define

$$|X_F|_r := \sup_{\omega\in\Omega} \sup_{\phi\in S^1_r} \|X_F(\omega,\phi)\|_{\mathcal{B}(Y,Y)}.$$

In this notation, from Lemma 3.6.3 we obtain the following corollary:

Corollary 3.6.5. *Let $F \in \mathcal{B}^0_r(\Omega)$. Then, if $|F|_r \leq 5/4$,*

$$\left| id - \phi^1_F \right|_r \leq 2\,|F|_r\,.$$

3.7 Proof of Theorem 3.2.2

We now consider the problem

$$\begin{cases} \theta_{tt} - \partial_x^2 \theta + \gamma \|u_\delta(t)\|^2_{L^2(I)} \theta + 2\gamma (u_\delta(t), \theta) u_\delta(t) = 0 \\ \theta(-\pi, t) = \theta(\pi, t) = 0, \end{cases} \tag{3.37}$$

where u_δ solves

$$\begin{cases} u_{\delta,tt} + \partial_x^4 u_\delta + \nu \|u_\delta\|^2_{L^2(I)} u_\delta = 0, \\ u_\delta(\pi,t) = u_\delta(-\pi,t) = u_{\delta,x}(\pi,t) = u_{\delta,x}(-\pi,t) = 0, \\ u_\delta(x,0) = \delta \mathfrak{e}_n(x), \quad u_{\delta,t}(x,0) = 0. \end{cases}$$

for some fixed $\nu > 0$, with $\delta \in [0, \tilde{\delta}]$ for some $\tilde{\delta} > 0$ and where we now have

$$\mathfrak{e}_n(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \left(\dfrac{\cos(\lambda^+_{(n+1)/2} x)}{\cos(\lambda^+_{(n+1)/2} \pi)} - \dfrac{\cosh(\lambda^+_{(n+1)/2} x)}{\cosh(\lambda^+_{(n+1)/2} \pi)} \right) & \text{if } n \text{ is odd,} \\ \dfrac{1}{\sqrt{2\pi}} \left(\dfrac{\sin(\lambda^-_{n/2} x)}{\sin(\lambda^-_{n/2} \pi)} - \dfrac{\sinh(\lambda^-_{n/2} x)}{\sinh(\lambda^-_{n/2} \pi)} \right) & \text{if } n \text{ is even,} \end{cases}$$

with $\lambda_n^{\pm}$ defined as the $n-th$ positive solution of the equation

$$\tan(\lambda^4\pi) \pm \tanh(\lambda^4\pi) = 0.$$

In the following, we refer to the eigenvalue corresponding to $\mathfrak{e}_n$ as λ_n. By reasoning as in subsection 3.3.2, we have that equation (3.37) may be reformulated as follows:

$$\theta_{tt} + B^2\theta + \gamma P(\omega, \omega t)\theta = 0$$

where $\theta \in C^1(\mathbb{R}_+, \mathcal{H})$, $\mathcal{H}$ is defined as in (3.10), $\omega \in \Omega := [\sqrt{\lambda_n}, \tilde{\omega}]$, $\tilde{\omega} := \omega(\tilde{\delta})$, $B^2 := -\partial_x^2$ and

$$P(\omega, \omega t) := V(\omega, \omega t)(\theta + 2(\mathfrak{e}_n, \theta)_{L^2(I)}\mathfrak{e}_n,$$

where

$$V(\omega, \phi) = \delta^2(\omega)\mathrm{cn}^2\left[\frac{T(\delta(\omega))}{2\pi}\phi\sqrt{\lambda_n + \nu\delta^2(\omega)}, \frac{\sqrt{\nu}\delta(\omega)}{\sqrt{2(\lambda_n + \nu\delta^2(\omega))}}\right].$$

We remark that if we set $a(\omega, \phi) := V(\omega, \phi)/2$ and $R(\omega, \phi)\theta := 2V(\omega, \phi)(\mathfrak{e}_n, \theta)_{L^2(I)}\mathfrak{e}_n$, we have that

$$P(\omega, \phi) := 2a(\omega, \phi)\mathrm{id} + R(\omega, \phi).$$

where id is the identity operator. Since from Lemma 3.3.1, $\delta^2 \in \mathrm{Lip}(\Omega)$, we have that $a \in \mathrm{Lip}(\Omega, C^\infty_{2\pi}(\mathbb{R}_+, \mathbb{R}_+))$ and, since for any $n \in \mathbb{N}$ we have that $\mathfrak{e}_n \in \mathcal{H}^2$, $R \in \mathrm{Lip}(\Omega, C^\infty_{2\pi}(\mathbb{R}_+, \mathcal{B}(\mathcal{H}, \mathcal{H}^2)))$. We observe that, since the function $\mathrm{cn}(z, \mathfrak{m})$ has a simple pole in $z = K(1-\mathfrak{m})$, where K is defined in (3.12), from (3.11) we have that $V(\omega, \phi)$ has a pole in

$$\phi_p(m) = \frac{\pi}{2}\frac{K\left(1 - \frac{\sqrt{m}}{\sqrt{1+m}}\right)}{K\left(\frac{m}{1+2m}\right)}, \quad \text{where } m := \frac{\nu\delta^2}{2\lambda_n}. \tag{3.38}$$

Since if $\nu\delta^2 < 2\lambda_n$, then $m \in [0, 1]$ and $\min_{m\in[0,1]} \phi_p(m) \approx 1.54$, we get that there exists $r > 1$ such that $\tilde{P}(\omega, z) = \mathrm{cn}(z, m)$ is holomorphic on S^1_r and the restriction of $P(\omega, z)$ to the real line coincides with $\tilde{P}(\omega, z)$ for any choice of ω.

3.7.1 Complex formulation

Following [26, 67], we reformulate

$$\theta_{tt} + B^2\theta + 2\gamma a(\omega, \omega t)\theta + \gamma R(\omega, \omega t) = 0, \tag{3.39}$$

as a complex Hamiltonian system. Setting $p = \theta_t$ and

$$\psi := \frac{\sqrt{2}}{2}(B^{1/2}\theta + iB^{-1/2}p),$$

from (3.39) we get for the complex function $\psi(t, x)$ the equation

$$\dot{\psi} = -iB\psi - i\gamma(\mathcal{D}(\omega, \omega t) + \mathcal{R}(\omega, \omega t))(\psi + \bar{\psi}), \tag{3.40}$$

where

$$\mathcal{D}(\omega, \omega t) := a(\omega, \omega t)B^{-1}, \quad \mathcal{R}(\omega, \omega t) := \frac{1}{2}B^{-1/2}R(\omega, \omega t)B^{-1/2}.$$

We remark that, since $\theta \in \mathcal{H}^1$ and $p \in \mathcal{H}$, we have that $\psi \in \mathcal{H}^{1/2}_{\mathbb{C}}$, where for any s we denote by $\mathcal{H}^s_{\mathbb{C}}$ the complexification of $(\mathcal{H}^s, (\cdot, \cdot)_s)$. In the following, for the sake of simplicity the complexified space will again be denoted by $(\mathcal{H}^s, (\cdot,\cdot)_s)$. The convention adopted here is that the scalar product $(\cdot,\cdot)_s$ is linear in the first argument and antilinear in the second. We set

$$\psi = \sum_{s\in\mathbb{N}} \xi_s e_s, \quad \bar{\psi} = \sum_{s\in\mathbb{N}} \eta_s e_s.$$

We have that (3.40) is equivalent to the Hamiltonian system

$$\begin{cases} \dot{\xi}_s = -i\partial H/\partial\eta_s \\ \dot{\eta}_s = i\partial H/\partial\xi_s \end{cases} \qquad \text{where } s \in \mathbb{N}, \tag{3.41}$$

restricted to the space $Y := Y^{1/2} \subseteq Y_C^{1/2}$ endowed with the complex symplectic structure $i\sum_s \mathrm{d}\xi_s \wedge \mathrm{d}\eta_s$, where the Hamiltonian function H is defined by $H(\xi,\eta) = \Lambda(\xi,\eta) + \gamma D(\xi,\eta) + \gamma S(\xi,\eta)$ with

$$\Lambda(\xi,\eta) := \sum_{j\in\mathbb{N}} \beta_j \xi_j \eta_j, \qquad D(\xi,\eta) := \sum_{i,j\in\mathbb{N}} \mathcal{D}_{ij}\xi_i\eta_j + \frac{1}{2}\sum_{i,j\in\mathbb{N}} \mathcal{D}_{ij}(\eta_i\eta_j + \xi_i\xi_j),$$
$$S(\xi,\eta) := \sum_{i,j\in\mathbb{N}} \mathcal{R}_{ij}\eta_i\xi_j + \frac{1}{2}\sum_{i,j\in\mathbb{N}} \mathcal{R}_{ij}(\eta_i\eta_j + \xi_i\xi_j)$$

where $\mathcal{D}_{ij} := (\mathcal{D}e_i, e_j)$ and $\mathcal{R}_{ij} := (\mathcal{R}e_i, e_j)$. In the notation of subsection 3.6.1, we can write (3.41) in the form

$$\dot{\zeta} = X_H \zeta. \tag{3.42}$$

We proceed in two steps. First, we regularize the perturbation $D + S$ into a smoother perturbation R_0. Then, we proceed by KAM techniques in order to conclude the proof.

3.7.2 Smoothing the perturbation

The following lemma is completely standard. Nonetheless, we decided to insert its proof for the sake of completeness and in order to explicitly compute all the constants involved in the estimates needed.

Lemma 3.7.1. *Let*

$$a(\omega,\phi) = \sum_{k\in\mathbb{Z}} a_k(\omega) e^{ik\phi} \in \textbf{\textit{Lip}}(\Omega, C^\infty(S^1_r, \mathbb{C})).$$

We consider the functions

$$\hat{a}(\omega,\phi) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{a_k(\omega)}{ik\omega} e^{ik\phi}, \qquad \bar{a}(\omega) := \frac{1}{2\pi}\int_{S^1} a(\omega,\phi)d\phi.$$

Then $\hat{a}$ and $\bar{a}$ are Lipschitz functions of the frequencies and $\hat{a}$ is an analytic function on $S^1_{r-\sigma}$ for any positive $\sigma < r$.

Proof. We observe that

$$\begin{aligned} |a_k(\omega_1) - a_k(\omega_2)| &= \left| \frac{1}{2\pi}\int_{S^1} (a(\omega_1,\phi) - a(\omega_2,\phi)) e^{-ik\phi} d\phi \right| \\ &\overset{\phi'=\phi+ir\,\mathrm{sign}(k)}{=} \left| \frac{1}{2\pi}\int_{S^1} (a(\omega_1,\phi'-ir) - a(\omega_2,\phi'-ir)) e^{-ik\phi'} e^{-|k|r} d\phi' \right| \\ &\le |a(\omega_1) - a(\omega_2)|_r\, e^{-|k|r} \le |a|_r^{\mathcal{L}}\, e^{-|k|r} |\omega_1 - \omega_2|, \end{aligned}$$

that implies $|a_k|^{\mathcal{L}} \leq |a|_r^{\mathcal{L}} e^{-|k|r}$. In particular, since $\bar{a} = a_0$, we get that $|\bar{a}|^{\mathcal{L}} \leq |a|_r^{\mathcal{L}}$. By proceeding as above, we obtain that $\max_{\omega\in\Omega} |a_k(\omega)| \leq |a|_r e^{-|k|r}$. Moreover,

$$\hat{a}(\omega_1, \phi) - \hat{a}(\omega_2, \phi) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{a_k(\omega_1) - a_k(\omega_2)}{ik\omega_1} e^{ik\phi} + \sum_{k\in\mathbb{Z}\setminus\{0\}} a_k(\omega_2) \frac{\omega_2 - \omega_1}{ik\omega_1\omega_2} e^{ik\phi}.$$

Fixed $0 < \sigma < r$, if $\phi \in S^1_{r-\sigma}$, then $|e^{ik\phi}| \leq e^{|k|(r-\sigma)}$. Therefore, we have that

$$\begin{aligned} |\hat{a}(\omega_1, \phi) - \hat{a}(\omega_2, \phi)| &\leq \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} |a|_r^{\mathcal{L}} \frac{e^{-|k|\sigma}}{|k|\omega_1} + \sum_{k\in\mathbb{Z}\setminus\{0\}} |a|_r \frac{e^{-|k|\sigma}}{|k|\omega_1\omega_2} \right) |\omega_1 - \omega_2| \\ &\leq C_0(\sigma) \left(\frac{|a|_r^{\mathcal{L}}}{\omega_m} + \frac{|a|_r}{\omega_m^2} \right) |\omega_1 - \omega_2| \end{aligned}$$

where

$$C_0(\sigma) := 2 \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n}.$$

Therefore we have that

$$|\hat{a}|_{r-\sigma}^{\mathcal{L}} \leq \frac{C_0(\sigma)}{\omega_m} |a|_r^{\mathcal{L}} + \frac{C_0(\sigma)}{\omega_m^2} |a|_r$$

and, in the same way, we obtain that

$$|\hat{a}|_{r-\sigma} \leq \frac{C_0(\sigma)}{\omega_m} |a|_r .$$

□

We are now ready to prove the following lemma:

Lemma 3.7.2. *Let $\omega \in \Omega$ and suppose $\max(|S|_{1+\sigma_0}, |a|_{1+\sigma_0}) = \varepsilon_0$, $\max(|S|^{\mathcal{L}}_{1+\sigma_0}, |a|^{\mathcal{L}}_{1+\sigma_0}, \varepsilon_0) = \varepsilon_1$ for some positive constants σ_0, ε_0, ε_1. There exists a $\gamma_* = \gamma_*(\varepsilon_0, \sigma_0) > 0$ such that if $\gamma < \gamma_*$, then there exists an Hamiltonian $Z \in \mathcal{B}^1_{1+\sigma_0}(\Omega)$ such that if ζ is a solution of (3.42), then $\varphi := \phi^1_{\gamma Z}\zeta$ solves*

$$\varphi_t = X_{H_0}\varphi, \qquad \textit{where } H_0 = \Lambda_0(\omega) + \gamma R_0(\omega, \omega t) \tag{3.43}$$

with $\Lambda_0 := \sum_{j\in\mathbb{N}} \lambda_{0,j}\eta_j\xi_j$, $\lambda_{0,j} := \beta_j + \gamma\bar{a}(\omega)\beta_j^{-1}$ and $R_0 \in \mathcal{B}^{3,2}_1(\Omega)$. Moreover

$$|R_0|_1 \le |S|_1 + |a|_{1+\sigma_0} + K_0\gamma\varepsilon_0^2, \qquad |R_0|^{\mathcal{L}}_1 \le |S|^{\mathcal{L}}_1 + |a|^{\mathcal{L}}_{1+\sigma_0} + K_0^{\mathcal{L}}\gamma\varepsilon_1^2 \tag{3.44}$$

for some positive constants $K_0 = K_0(\omega_m, \omega_M, \sigma_0)$ and $K_0^{\mathcal{L}}(\omega_m, \omega_M, \sigma_0)$ and

$$\left|\phi^1_{\Xi_0} - id\right|_1 \le K_1\gamma\varepsilon_0 \tag{3.45}$$

with $K_1 = K_1(\omega_m, \omega_M, \sigma_0)$.

Proof. We consider the Hamiltonian $Z(\omega, \omega t)$ defined by

$$Z^-(\omega, \omega t) := \hat{a}(\omega, \omega t)B^{-1}, \qquad Z^+(\omega, \omega t) := \frac{i}{2}a(\omega, \omega t)B^{-2}.$$

We observe that from Lemma 3.7.1 we have that

$$\begin{aligned} |Z|_{0,1} &= \sup_{\substack{\omega\in\Omega\\ \phi\in S^1_r}} \|Z(\omega,\phi)\|_{H^0} = \sup_{\substack{\omega\in\Omega\\ \phi\in S^1_1}} (\|Z^-(\omega,\phi)\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2})} + \|Z^+(\omega,\phi)\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2})}) \\ &\le |\hat{a}|_1 \beta_1^{-1} + \frac{\beta_1^{-2}}{2}|a|_1 \le \frac{2C_0(\sigma_0)\beta_1 + \omega_m}{2\omega_m}\beta_1^{-2}|a|_{1+\sigma_0} \end{aligned} \tag{3.46}$$

where β_1 is the first eigenvalue of B, and, analogously,

$$|Z|^{\mathcal{L}}_{0,1} = |\hat{a}|^{\mathcal{L}}_1 \beta_1^{-1} + \frac{\beta_1^{-2}}{2}|a|^{\mathcal{L}}_1 \le \frac{C_0(\sigma_0)}{\omega_m^2}\beta_1^{-1}|a|_{1+\sigma_0} + \frac{2C_0(\sigma_0)\beta_1 + \omega_m}{2\omega_m}\beta_1^{-2}|a|^{\mathcal{L}}_{1+\sigma_0}$$

where $C_0(\sigma_0)$ is defined in Lemma 3.7.1. From Proposition 3.6.4, $\varphi := \phi^1_{\gamma Z}\zeta$ solves $\varphi_t = X_{H_0}\varphi$ with

$$H_0 = \Lambda + \gamma(D + \{\Lambda, Z\} - \dot{Z}) + \gamma\tilde{R}_0 \tag{3.47}$$

where $\gamma\tilde{R}_0 = \mathrm{Lie}_{\gamma Z}\Lambda - \Lambda - \gamma\{\Lambda, Z\} + Y_{\gamma Z} + \gamma\dot{Z} + \mathrm{Lie}_{\gamma Z}D - D + \mathrm{Lie}_{\gamma Z}S$ and $\{\Lambda, Z\}$ is given by, according to (3.36),

$$\begin{aligned} \{\Lambda, Z\}^- &= i(\Lambda^-Z^- - Z^-\Lambda^- + \Lambda^+\overline{Z^+} - Z^+\overline{\Lambda^+}) = 0 \\ \{\Lambda, Z\}^+ &= i(\Lambda^-Z^+ + Z^+\overline{\Lambda^-} - \Lambda^+\overline{Z^-} - Z^-\Lambda^+) = -a(\omega,\omega t)B^{-1}. \end{aligned}$$

Therefore, we have that

$$D + \{\Lambda, Z\} - \dot{Z} = \left(a - \frac{d}{dt}\hat{a}\right)\sum_{j\in\mathbb{N}_0}\beta_j^{-1}\xi_j\eta_j + \frac{1}{2}\sum_{j\in\mathbb{N}_0}\frac{i}{2}\frac{d}{dt}a\beta_j^{-2}(\xi_j^2 - \eta_j^2).$$

Let us observe that

$$a(\omega,\omega t) - \frac{d\hat{a}(\omega,\omega t)}{dt} = a(\omega,\omega t) - \sum_{k\in\mathbb{Z}\setminus\{0\}} a_k(\omega)e^{ik\omega t} = a_0(\omega) = \bar{a}(\omega).$$

It follows that (3.47) reads

$$H_0 = \sum_{j\in\mathbb{N}} (\beta_j + \gamma\bar{a}(\omega)\beta_j^{-1})\xi_j\eta_j + \gamma R_0$$

where

$$R_0 := \tilde{R}_0 + \frac{i}{4}\sum_{j\in\mathbb{N}_0} \frac{d}{dt} a\beta_j^{-2}(\xi_j^2 - \eta_j^2).$$

We notice that $Z \in \mathcal{B}_1^{1,2}(\Omega)$ and the transformation $\varphi = \phi^1_{\Xi_0}\zeta$ is canonical. Let us suppose that $\gamma\left|Z\right|_{0,1} < 5/8$, which from (3.46) is implied by

$$\frac{2C_0(\sigma_0)\beta_1 + \omega_m}{2\omega_m}\beta_1^{-2}\gamma\left|a\right|_{1+\sigma_0} < \frac{5}{8}, \tag{3.48}$$

which is equivalent to require that $\gamma < \gamma_*(\varepsilon_0, \sigma_0)$.
Since

$$\{\{\Lambda, Z\}, Z\} = -\sum_{i\in\mathbb{N}_0} a^2\beta_i^{-3}\eta_i\xi_i + \frac{i}{2}\sum_{i\in\mathbb{N}_0} 2\hat{a}a\beta_i^{-2}(\eta_i^2 - \xi_i^2) \in \mathcal{B}^{3,2}_{1+\sigma_0}(\Omega),$$

which yields that

$$\left|\{\{\Lambda, Z\}, Z\}\right|_1 \le \left|a\right|_1^2\beta_1^{-1} + 2\left|\hat{a}\right|_1\left|a\right|_1 \le \left(\beta_1^{-1} + \frac{2C_0(\sigma_0)}{\omega_m}\right)\left|a\right|^2_{1+\sigma_0}.$$

Therefore, from Lemma B.5, if (3.48) holds, then

$$\left|\mathrm{Lie}_{\gamma Z}\Lambda - \Lambda - \gamma\{\Lambda, Z\}\right|_1 \le \gamma^2\left|\{\{\Lambda, Z\}, Z\}\right|_1 \le \left(\beta_1^{-1} + \frac{2C_0(\sigma_0)}{\omega_m}\right)\gamma^2\left|a\right|^2_{1+\sigma_0},$$

Analogously, by proceeding as in Lemma B.5, if (3.48) holds, we get that

$$\left|\mathrm{Lie}_{\gamma Z}D - D\right|_1 \le 2\gamma\left|\{Z, D\}\right|_1 \le 4\left(\beta_1^{-1} + \frac{C_0(\sigma_0)}{\omega_m}\right)\gamma\left|a\right|^2_{1+\sigma_0},$$

$$\left|\mathrm{Lie}_{\gamma Z}S - S\right|_1 \le 2\gamma\left|\{Z, S\}\right|_1 \le 4\gamma\left|Z\right|_{0,1}\left|S\right|_1 \le 2\beta_1^{-2}\left(1 + \frac{2C_0(\sigma_0)}{\omega_m}\beta_1\right)\gamma\varepsilon_0^2.$$

In the exact same way, we obtain that

$$\left|\mathrm{Lie}_{\gamma Z}\Lambda - \Lambda - \gamma\{\Lambda, Z\}\right|_1^{\mathcal{L}} \le \frac{2C_0(\sigma_0)}{\omega_m^2}\gamma^2\left|a\right|^2_{1+\sigma_0} + 2\beta_1^{-2}\left(1 + \frac{2C_0(\sigma_0)}{\omega_m}\beta_1\right)\left|a\right|_{1+\sigma_0}\gamma^2\left|a\right|^{\mathcal{L}}_{1+\sigma_0},$$

$$\left|\mathrm{Lie}_{\gamma Z}D - D\right|_1^{\mathcal{L}} \le \frac{4C_0(\sigma_0)}{\omega_m^2}\gamma\left|a\right|^2_{1+\sigma_0} + 8\beta_1^{-2}\left(1 + \frac{2C_0(\sigma_0)}{\omega_m}\beta_1\right)\gamma\left|a\right|_{1+\sigma_0}\left|a\right|^{\mathcal{L}}_{1+\sigma_0},$$

$$\left|\mathrm{Lie}_Z S - S\right|_1^{\mathcal{L}} \le 2\gamma(\left|Z\right|^{\mathcal{L}}_{0,1}\left|S\right|_1 + \left|Z\right|_{0,1}\left|S\right|^{\mathcal{L}}_1).$$

We remark that

$$\{Z, \dot{Z}\} = \left(a\frac{d\hat{a}}{dt} - \hat{a}\frac{da}{dt}\right)\sum_{i\in\mathbb{N}_0}\beta_i^{-3}(\xi_i^2 + \eta_i^2).$$

Since

$$\left|\frac{d\hat{a}}{dt}\right|_1 = \left|a - \bar{a}\right|_1 \le 2\left|a\right|_{1+\sigma_0}, \qquad \left|\frac{d\hat{a}}{dt}\right|_1^{\mathcal{L}} = \left|a - \bar{a}\right|_1^{\mathcal{L}} \le 2\left|a\right|^{\mathcal{L}}_{1+\sigma_0}$$

and that by Cauchy's integral formula

$$\frac{da}{dt}(\omega,\omega t) = \frac{\omega}{2\pi i}\oint_{\gamma_{\sigma_0}(\omega t)} \frac{a(\omega,\xi)}{(\xi-\omega t)^2}\mathrm{d}\xi,$$

where $\gamma_{\sigma_0}(\phi)$ is the circle of center ϕ and radius σ_0 in the complex plane, which yields that

$$\left|\frac{da}{dt}\right|_1 \le \frac{\omega_M}{\sigma_0}|a|_{1+\sigma_0}, \qquad \left|\frac{da}{dt}\right|_1^{\mathcal{L}} \le \frac{1}{\sigma_0}|a|_{1+\sigma_0} + \frac{\omega_M}{\sigma_0}|a|_{1+\sigma_0}^{\mathcal{L}},$$

we have that

$$|\{Z,\dot Z\}|_1 \le \left(|a|_1\left|\frac{d\hat a}{dt}\right|_1 + \left|\frac{da}{dt}\right|_1|\hat a|_1\right)\beta_1^{-1} \le 2\left(1+\frac{\omega_M}{\omega_m\sigma_0}C_0(\sigma_0)\right)\beta_1^{-1}|a|_{1+\sigma_0}^2$$

$$\begin{aligned}|\{Z,\dot Z\}|_1^{\mathcal{L}} &\le \left(|a|_1^{\mathcal{L}}\left|\frac{d\hat a}{dt}\right|_1 + \left|\frac{da}{dt}\right|_1^{\mathcal{L}}|\hat a|_1 + |a|_1\left|\frac{d\hat a}{dt}\right|_1^{\mathcal{L}} + \left|\frac{da}{dt}\right|_1|\hat a|_1^{\mathcal{L}}\right)\beta_1^{-1}\\ &\le \left(2|a|_1^{\mathcal{L}}|a|_1 + \frac{1}{\sigma_0}(|a|_{1+\sigma_0} + \omega_M|a|_{1+\sigma_0}^{\mathcal{L}})|\hat a|_1 + 2|a|_1|a|_1^{\mathcal{L}} + \frac{\omega_M}{\sigma_0}|a|_{1+\sigma_0}|\hat a|_1^{\mathcal{L}}\right)\beta_1^{-1}\\ &\le \left(\frac{C(\sigma_0)}{\sigma_0}\left(\frac{1}{\omega_m}+\frac{\omega_M}{\omega_m^2}\right)|a|_{1+\sigma_0}^2 + 2\left(2+\frac{\omega_M}{\sigma_0\omega_m}C(\sigma_0)\right)|a|_{1+\sigma_0}^{\mathcal{L}}|a|_{1+\sigma_0}\right)\beta_1^{-1}.\end{aligned}$$

Therefore, from Lemma B.5

$$\begin{aligned}&|Y_{\gamma Z}+\gamma\dot Z|_1 \le \gamma^2|\{Z,\dot Z\}|_1 \le C_1\gamma^2\varepsilon_0^2,\\ &|Y_{\gamma Z}+\gamma\dot Z|_1^{\mathcal{L}} \le \gamma^2|\{Z,\dot Z\}|_1^{\mathcal{L}} + \gamma^2|Z|_{0,1}^{\mathcal{L}}|\{Z,\dot Z\}|_1 \le C_2\gamma^2\varepsilon_1^2,\end{aligned}$$

for some positive constants $C_1 = C_1(\omega_m,\omega_M,\sigma_0)$ and $C_2 = C_2(\omega_m,\omega_M,\sigma_0)$. Hence,

$$\begin{aligned}\gamma|R_0|_1 \le\ & |\mathrm{Lie}_{\gamma Z}\Lambda - \Lambda - \gamma\{\Lambda,Z\}|_1 + \gamma|S|_1 + \gamma|\mathrm{Lie}_{\gamma Z}S - S|_1\\ &+|Y_{\gamma Z}+\gamma\dot Z|_1 + \gamma|\mathrm{Lie}_{\gamma Z}D - D|_1 \le \gamma|S|_1 + K_0\gamma^2\varepsilon_0^2,\\ \gamma|R_0|_1^{\mathcal{L}} \le\ & |\mathrm{Lie}_{\gamma Z}\Lambda - \Lambda - \gamma\{\Lambda,Z\}|_1^{\mathcal{L}} + \gamma|S|_1^{\mathcal{L}} + \gamma|\mathrm{Lie}_{\gamma Z}S - S|_1^{\mathcal{L}} +\\ &+|Y_{\gamma Z}+\gamma\dot Z|_1^{\mathcal{L}} + \gamma|\mathrm{Lie}_{\gamma Z}D - D|_1^{\mathcal{L}} \le \gamma|S|_1^{\mathcal{L}} + K_0^{\mathcal{L}}\gamma^2\varepsilon_1^2,\end{aligned}$$

where K and $K^{\mathcal{L}}$ can be explicitly computed by exploiting the previous results. Moreover, since $\{\{\Lambda,Z\},Z\}$, $\{Z,D\}$, S, and $\{Z,\dot Z\}$ belong to $\mathcal{B}_1^{3,2}(\Omega)$, from Remark B.4 we have that $R_0 \in \mathcal{B}_1^{3,2}(\Omega)$.

To conclude, we observe that if $\gamma|Z|_{0,1} \le 5/4$, which is implied by (3.48), then, from Corollary 3.6.5

$$\left|\mathrm{id} - \phi_{\gamma Z}^1\right|_1 \le 2\gamma|Z|_{0,1} < \frac{2C_0(\sigma_0)\beta_1 + \omega_m}{\omega_m}\beta_1^{-2}\gamma|a|_{1+\sigma_0}.$$

□

Corollary 3.7.3. *Let $\delta^* > 0$. There exists $\gamma_0^* = \gamma_0^*(n,\delta^*) > 0$ such that for any $\delta \in [0,\delta^*]$ and for any $\gamma < \gamma_0^*$, there exists a canonical transformation Ξ such that if $\varphi = \Xi\zeta$, then φ solves*

$$\varphi_t = X_{H_0}\varphi, \quad \text{where } H_0 = \Lambda_0(\omega) + \gamma R_0(\omega,\omega t), \tag{3.49}$$

with $\Lambda_0 := \sum_{j\in\mathbb{N}}\lambda_{0,j}\eta_j\xi_j$, $\lambda_{0,j} = \beta_j + \gamma\bar a(\omega)\beta_j^{-1}$ and $R_0 \in \mathcal{B}_1^{3,2}(\Omega)$. Moreover we have that

$$|R_0|_1 \le C(n,\delta^*)\delta^2, \qquad |R_0|_1^{\mathcal{L}} \le C^{\mathcal{L}}(n,\delta^*)|V|_{4/3}^{\mathcal{L}}. \tag{3.50}$$

and that

$$|\Xi - id|_1 \le \tilde K_1(n,\delta^*)\gamma\delta^2 \tag{3.51}$$

Proof. The thesis follows immediately from Lemma 3.7.2 by observing that from (3.38), $R(\omega,\phi)$ and $S(\omega,\phi)$ are holomorphic in ϕ in the strip $\{\phi \in \mathbb{C} : \Im(\phi) < 3/2\}$ and Lipschitz dependent by $\omega \in \Omega := \omega([0,\bar\delta]) = [\sqrt{\lambda_n},\omega^*]$, where $\omega^* := \omega(\delta^*)$. Indeed, in the notation of Lemma 3.7.2, setting $\sigma_0 = 1/3$, since $\varepsilon_0 = \varepsilon_0(n,\delta)$ and $\varepsilon_1 = \varepsilon_1(n,\delta)$ and

$$\delta \mapsto \varepsilon_0(n,\delta), \qquad \delta \mapsto \varepsilon_1(n,\delta)$$

are increasing functions. Therefore, we get that for any $\delta^* > 0$ there exists $\gamma^*(n,\delta^*) > 0$ such that for any $\gamma < \gamma^*$, $\delta < \delta^*$, if we define $\Xi := \phi^1_{\gamma Z}$ and $\varphi := \Xi\zeta$ as in Lemma 3.7.2, φ satisfies (3.49). Moreover, from (3.44) and (3.45), since $\varepsilon_0 \le C_0(n,\delta^*)\delta^2$, which yields that for any $r \in [0,4/3]$ and for any $\delta < \delta^*$,

$$|S|_r \le 2|V|_r \le C_1(r,\delta^*)\delta^2, \quad |a|_r \le |V|_r \le C_2(r,\delta^*)\delta^2,$$

we get (3.50) and (3.51). □

3.7.3 The KAM step

The procedure comes from a quite straightforward adaptation of the KAM procedure shown in [13] to our framework. For any Hamiltonian function $A(\omega,\omega t)$, we define the Hamiltonian

$$[A](\omega) := \sum_{i\in\mathbb{N}_0} \tilde{A}^-_{ii}(\omega)\eta_i\xi_i \quad \text{where } \tilde{A}^-_{ii}(\omega) := \frac{1}{2\pi}\int_{-\pi}^{\pi} A^-_{ii}(\omega,\phi)\mathrm{d}\phi.$$

Let $\lambda = (\lambda_j)_j \subset \mathrm{Lip}(\Omega,\mathbb{R})$. For any fixed $\mu > 0$, we introduce the set of frequencies $\Omega_\mu(\lambda) \subseteq \Omega$ given by

$$\Omega_\mu(\lambda) := \left\{\omega \in \Omega : \forall h,k \in \mathbb{N}, \forall l \in \mathbb{Z}, |\lambda_h(\omega) \pm \lambda_k(\omega) + l\omega| > \mu\frac{\langle h \pm k\rangle}{1+l^4}\right\}.$$

From Lemma 3.7.2, there exists a canonical transformation such that (3.42) is transformed into

$$\phi_t = X_{H_0}\phi \tag{3.52}$$

where $H_0 = \Lambda_0(\omega)\phi + R_0(\omega,\omega t)$ and $\Lambda_0 := \sum_{j\in\mathbb{N}_0}\lambda_{0,j}$, $\lambda_{0,j} := \beta_j + \gamma\bar{a}(\omega)\beta_j^{-1}$ and $R_0 \in \mathcal{B}^{3,2}_1(\Omega)$.

For any $a : \Omega \to \mathbb{R}$, we introduce the quantities

$$|a|_\infty = \sup_{\omega\in\Omega}|a(\omega)|, \quad |a|^{\mathcal{L}} = \sup_{\omega 1,\omega_2\in\Omega}\left|\frac{a(\omega_1)-a(\omega_2)}{\omega_1-\omega_2}\right|.$$

The following lemma holds:

Lemma 3.7.4. *Fix $\mu > 0$ and $r > 0$ and let $\sigma = 1/2^m < r$ for some $m \in \mathbb{N}$ such that $m \ge N(\omega_m)$, where $N(\omega_m)$ is defined in Lemma C.3. Let us suppose that*

$$\begin{aligned} &\gamma(|a|^{\mathcal{L}}/2 + 4|a|_\infty) < 1, \qquad \gamma|R_0|_1 \le \frac{5\sigma^4}{8\mathfrak{C}},\\ &\mu < \min\left(\left(\frac{\omega_M-\omega_m}{C_\Omega}\right)^{3/2}, \frac{1}{4} - \gamma|a|_\infty\right), \end{aligned} \tag{3.53}$$

where $\mathfrak{C} := C(4)$ and C_Ω are defined in Lemma C.1 and in Lemma A.3 respectively. Then there exists a Lipschitz-analytic canonical transformation $\Xi_1 \in \mathcal{B}^0_1(\Omega_\mu(\lambda_0))$ such that if ϕ_0 is a solution of (3.43)*, then $\phi_1 := \Xi_1(\omega,\omega t)\phi_0$ solves*

$$\phi_{1,t} = (\Lambda_1(\omega) + \gamma^2 R_1(\omega,\omega t))\phi_1 \tag{3.54}$$

where $\Lambda_1 := \Lambda_0 + \gamma[R_0]$ *and* $R_1 \in \mathcal{B}^{3,2}_{1-\sigma}(\Omega_\mu)$. *Moreover,*

$$|R_1|_{1-\sigma} \leq \frac{\mathfrak{K}}{\sigma^9} |R_0|_1^2, \qquad |R_1|^{\mathcal{L}}_{r-\sigma} \leq \frac{\mathfrak{K}^{\mathcal{L}}}{\sigma^9}\left(|R_0|^{\mathcal{L}}_1 + \frac{|R_0|_1}{\sigma^4}\right)|R_0|_1,$$

$$|\Xi_1 - id|_{1-\sigma} \leq \frac{2\mathfrak{C}}{\sigma^4}|R_0|_1, \quad |\Xi_1 - id|^{\mathcal{L}}_{1-\sigma} \leq \frac{2\mathfrak{C}}{\sigma^4}|R_0|^{\mathcal{L}}_1 + \frac{2\mathfrak{C}^{\mathcal{L}}}{\sigma^8}|R_0|_1.$$

where $\mathfrak{C}^{\mathcal{L}} := C^{\mathcal{L}}(4)$ *and* $\mathfrak{K} := \mathfrak{K}(\mathfrak{C}, \mathfrak{C}^{\mathcal{L}})$ *and* $\mathfrak{K}^{\mathcal{L}} := \mathfrak{K}^{\mathcal{L}}(\mathfrak{C}, \mathfrak{C}^{\mathcal{L}})$.

Proof. We observe that, since $\beta_j = j/2$ and $\bar{a} \in \mathrm{Lip}(\Omega, \mathbb{R})$, we have that there exist $\mathcal{K}_{00} > 0$, $\mathcal{G}_{00}$ and $\mathcal{K}_{10} > 0$ such that

$$\mathcal{K}_{00}|h \pm k| \leq |\lambda_{0,h}(\omega) \pm \lambda_{0,k}(\omega)| \leq \mathcal{G}_0|h \pm k|, \qquad \left|\frac{\Delta(\lambda_{0,h}(\omega) \pm \lambda_{0,k}(\omega))}{\Delta\omega}\right| \leq \mathcal{K}_{10}. \tag{3.55}$$

In particular, since we can take

$$\mathcal{G}_0 = \frac{1}{2} + 2\gamma|a|_\infty, \qquad \mathcal{K}_{00} = \frac{1}{2} - 2\gamma|a|_\infty, \qquad \mathcal{K}_{10} = 2\gamma|a|^{\mathcal{L}},$$

where

$$|a|_\infty = \sup_{\substack{\omega\in\Omega\\ \phi\in S^1}} |a(\omega,\phi)|, \qquad |a|^{\mathcal{L}} \sup_{\substack{\omega_1,\omega_2\in\Omega\\ \phi\in S^1}} \left|\frac{a(\omega_1,\phi) - a(\omega_1,\phi)}{\omega_1 - \omega_2}\right|,$$

from (3.53), we have that $\mathcal{K}_{00} \geq \mathcal{K}_{10}/8$, $\mu < \mathcal{K}_{00}/2$ and $\mathcal{G}_0 \leq 5/8 < 2$. Therefore, since from Lemma A.3, $|\Omega \setminus \Omega_\mu(\lambda_0)| \leq C_\Omega \mu^{2/3}$ for some positive constant $C_\Omega > 0$, if (3.53) hold, then $\Omega_\mu \neq \varnothing$.

We introduce the Hamiltonian X defined by

$$X^{\pm}_{hk}(\omega,\phi) = -i\sum_{l\in\mathbb{Z}} \frac{R^{\pm}_{0,hkl}(\omega)}{\lambda_{0,h}(\omega) \pm \lambda_{0,k}(\omega) + l\omega} e^{-il\phi}.$$

We remark that since $\omega \in \Omega_\mu(\lambda_0)$, then X is well defined. Moreover, if we define $\phi_1 := \phi^1_{\gamma X}\phi_0$, from Proposition 3.6.4 we have that ϕ_1 solves

$$\phi_{1,t} = X_{H_1}\phi_1$$

with $H_1 := \Lambda_0 + \gamma R_0 + \gamma\{\Lambda_0, X\} - \gamma\dot{X} + \gamma R_1$ where $R_1 := \mathrm{Lie}_{\gamma X}\Lambda_0 - \Lambda_0 - \gamma\{\Lambda_0, X\} + Y_{\gamma X} + \gamma\dot{X} + \mathrm{Lie}_{\gamma X} R_0 - R_0$. Since

$$\{\Lambda_0, X\}^-_{hk} = -i(\lambda_{0,h} - \lambda_{0,k})X^-_{hk}, \quad \{\Lambda_0, X\}^+_{hk} = -i(\lambda_{0,h} + \lambda_{0,k})X^+_{hk}$$

it follows that ϕ_1 solves (3.54) with $R_0 + \{\Lambda_0, X\} - \dot{X} = [R_0]$. Moreover, we observe that $\{\{\Lambda_0, X\}, X\}$, $\{X, \dot{X}\}$ and $\{R_0, X\}$ belong to $\mathcal{B}^{3,2}(\Omega_\mu(\lambda_0))$. Therefore, from Remark B.4, $R_1 \in \mathcal{B}^{3,2}(\Omega_\mu(\lambda_0))$.

From Lemma C.3 we have that

$$|X|_{1-\sigma} \leq \frac{\mathfrak{C}}{\sigma^4}|R_0|_1, \qquad |X|^{\mathcal{L}}_{1-\sigma} \leq \frac{\mathfrak{C}}{\sigma^4}|R_0|^{\mathcal{L}}_1 + \frac{\mathfrak{C}^{\mathcal{L}}}{\sigma^8}|R_0|_r. \tag{3.56}$$

Hence, from Proposition B.2, if $\gamma|X|_{1-\sigma} < 5/8$, which is implied by

$$\gamma|R_0|_r \leq \frac{5\sigma^4}{8\mathfrak{C}}, \tag{3.57}$$

we have that, from (B.4), (B.6) and (B.9), since $\beta_1 = 1/2$,

$$
\begin{aligned}
|R_1|_{1-\sigma} &\le |\mathrm{Lie}_{\gamma X}\Lambda_0 - \Lambda_0 - \gamma X|_{1-\sigma} + |Y_{\gamma X} + \gamma \dot{X}|_{1-\sigma} + |\mathrm{Lie}_{\gamma X} R_0 - R_0|_{1-\sigma} \\
&\le 16\gamma |X|_{1-\sigma}\left(\frac{2\omega_M}{\sigma}\gamma |X|_{1-\sigma/2} + |R_0|_{1-\sigma}\right) + \frac{32\omega_M}{\sigma}\gamma^2 |X|_{1-\sigma/2}|X|_{1-\sigma} + 16\gamma |X|_{1-\sigma}|R_0|_1 \\
&\le 32\gamma |X|_{1-\sigma}|R_0|_1 + \frac{64\omega_M}{\sigma}\gamma^2 |X|_{1-\sigma}|X|_{1-\sigma/2},
\end{aligned}
$$

and, analogously, from (B.5), (B.16) and (B.10)

$$
\begin{aligned}
|R_1|^{\mathcal{L}}_{1-\sigma} &\le |\mathrm{Lie}_{\gamma X}\Lambda_0 - \Lambda_0 - \gamma X|^{\mathcal{L}}_{1-\sigma} + |Y_{\gamma X} + \gamma \dot{X}|^{\mathcal{L}}_{1-\sigma} + |\mathrm{Lie}_{\gamma X} R_0 - R_0|^{\mathcal{L}}_{1-\sigma} \\
&\le \frac{32}{\sigma}\gamma^2 |X|_{1-\sigma/2}|X|_{1-\sigma} + \frac{32\omega_M}{\sigma}\gamma^2(|X|_{1-\sigma/2}|X|^{\mathcal{L}}_{1-\sigma} + |X|_{1-\sigma}|X|^{\mathcal{L}}_{1-\sigma/2}) + 16\gamma |X|_{1-\sigma}|R_0|^{\mathcal{L}}_1 \\
&\quad + 16\gamma |X|^{\mathcal{L}}_{1-\sigma}|R_0|_1 + \frac{32}{\sigma}\gamma^2 |X|_{1-\sigma/2}|X|_1 + \frac{32\omega_M}{\sigma}\gamma^2 |X|_{1-\sigma}|X|^{\mathcal{L}}_{1-\sigma/2} \\
&\quad + \frac{32\omega_M}{\sigma}\gamma^2 |X|_{1-\sigma/2}|X|^{\mathcal{L}}_1 + 16\gamma |X|_{1-\sigma}|R_0|^{\mathcal{L}}_1 + 28\gamma |X|^{\mathcal{L}}_{1-\sigma}|R_0|_1 \\
&\le 44\gamma |X|^{\mathcal{L}}_1 |R_0|_r + 32\gamma |X|_{1-\sigma}|R_0|^{\mathcal{L}}_1 + \frac{64}{\sigma}\gamma^2 |X|_{1-\sigma}|X|_{1-\sigma/2} \\
&\quad + \frac{64\omega_M}{\sigma}\gamma^2(|X|_{1-\sigma}|X|^{\mathcal{L}}_{1-\sigma/2} + |X|_{1-\sigma/2}|X|^{\mathcal{L}}_{1-\sigma})
\end{aligned}
$$

which yields, from (3.56) and from $\sigma \le 1/4$,

$$
\begin{aligned}
|R_1|_{1-\sigma} &\le 4\gamma \frac{8\sigma^5\mathfrak{C} + 256\omega_M\gamma\mathfrak{C}^2}{\sigma^9}|R_0|_1^2 \le \frac{\mathfrak{K}}{\sigma^9}\gamma |R_0|_1^2, \\
|R_1|^{\mathcal{L}}_{1-\sigma} &\le \frac{4\gamma}{\sigma^9}|R_0|_1\left((512\mathfrak{C}^2\gamma\omega_M + 19\mathfrak{C}\sigma^5)|R_0|^{\mathcal{L}}_1 + \frac{11\mathfrak{C}^{\mathcal{L}}\sigma^5 + 256\mathfrak{C}^2\gamma\sigma^4 + 4352\mathfrak{C}\mathfrak{C}^{\mathcal{L}}\gamma\omega_M}{\sigma^4}|R_0|_1\right) \\
&\le \frac{\mathfrak{K}^{\mathcal{L}}}{\sigma^9}\gamma |R_0|_1\left(|R_0|^{\mathcal{L}}_1 + \frac{1}{\sigma^4}|R_0|_1\right),
\end{aligned}
$$

where we set

$$
\begin{aligned}
&\mathfrak{K} = 256\omega_M\gamma\mathfrak{C}^2 + 8/4^5\mathfrak{C}, \\
&\mathfrak{K}^{\mathcal{L}} := \max(512\gamma\mathfrak{C}^2\omega_M + 19\mathfrak{C}4^{-5}, 11\mathfrak{C}^{\mathcal{L}}4^{-5} + 256\gamma\mathfrak{C}^2 4^{-4} + 4352\gamma\mathfrak{C}\mathfrak{C}^{\mathcal{L}}\omega_M).
\end{aligned}
\tag{3.58}
$$

To conclude, from Corollary 3.6.5 we obtain that, from Lemma C.1 and Lemma C.2,

$$
\left|\mathrm{id} - \phi^1_{\gamma X}\right|_{1-\sigma} \le 2\gamma |X|_{1-\sigma} \le \frac{2\mathfrak{C}}{\sigma^4}\gamma |R_0|_1, \qquad \left|\mathrm{id} - \phi^1_{\gamma X}\right|^{\mathcal{L}}_{1-\sigma} \le \frac{2}{\sigma^4}\gamma\left(\mathfrak{C}|R_0|^{\mathcal{L}}_1 + \frac{\mathfrak{C}^{\mathcal{L}}}{\sigma^4}|R_0|_1\right).
$$

□

Lemma 3.7.5. *Under the assumption of Lemma 3.7.4, the eigenvalues of $\Lambda_1(\omega)$ fulfill* (3.55) *with new constants given by*

$$
\mathcal{K}_{01} := \mathcal{K}_{00} - 2\gamma |R_0|_1, \quad \mathcal{K}_{11} := \mathcal{K}_{10} + 2\gamma |R_0|^{\mathcal{L}}_1, \quad \mathcal{G}_1 := \mathcal{G}_0 + 2\gamma |R_0|_1
$$

Let us introduce two positive constants $\mathcal{K}_0$ and $\mathcal{K}_1$ such that

$$
\mathcal{K}_1 \le \frac{\mathcal{K}_0}{8}, \quad \mathcal{K}_{01} \ge \mathcal{K}_0, \quad \mathcal{K}_1 \le \mathcal{K}_{11}. \tag{3.59}
$$

If we consider $K > 0$ *such that* $2\gamma(1+K^4) < \mu/|R_0|_1$, *then we have that*

$$|\Omega_\mu(\lambda_0) \setminus \Omega_{\mu_1}(\lambda_1)| \le C'_\Omega K^{-1}\mu_1,$$

where $C'_\Omega = C'_\Omega(\mathcal{K}_0, K, \omega_M)$ *and*

$$\mu_1 = \mu - 2\gamma(1+K^4)\,|R_0|_1\,.$$

Proof. The proof proceeds exactly as the proof of Lemma 6.4 of [13], with $\tau = 4$ and $\kappa = 3$. In particular, the equation (6.40) of [13] gives

$$\left|\bigcup_{i,j}\mathcal{R}_{ijk}\left(\lambda_1, \frac{\mu_1}{1+k^4}\right)\right| \le \sum_{i,j}\frac{4}{\mathcal{K}_0}\frac{\mu_1}{1+k^4} \le \frac{4}{\mathcal{K}_0}\frac{2\omega_M}{\mathcal{K}_0-\mu}|k|\left(\frac{\mathcal{K}_0}{4}\right)^{1/3}|k|\frac{\mu_1}{1+k^4} \le \frac{2\cdot 4^{1/3}\omega_M}{\mathcal{K}_0^{5/3}}\mu_1\frac{1}{k^2}$$

which yields that

$$|\Omega_\mu(\lambda_0)\setminus\Omega_{\mu_1}(\lambda_1)| \le \frac{2\cdot 4^{1/3}\omega_M}{\mathcal{K}_0^{5/3}}\mu_1\sum_{|k|>K}\frac{1}{k^2} \le \frac{2\cdot 4^{5/3}\omega_M}{\mathcal{K}_0^{5/3}}K^{-1}\mu_1.$$

□

Lemma 3.7.6. *Let* $\mathcal{K}_{00}$, $\mathcal{K}_{10}$ *and* $\mathcal{G}_0$ *be defined as in Lemma 3.7.4 and let* $N = N(\omega_m)$ *be defined in Lemma C.3. Let us introduce the following sequences:*

$$\varepsilon_1^{(l+1)} := \frac{\mathfrak{K}}{\sigma_{l+1}^9}(\varepsilon_1^{(l)})^2, \quad \varepsilon_2^{(l+1)} := \frac{\mathfrak{K}^{\mathcal{L}}}{\sigma_{l+1}^9}\varepsilon_1^{(l)}\left(\varepsilon_2^{(l)} + \frac{\varepsilon_1^{(l)}}{\sigma_{l+1}^4}\right), \quad \textit{where } \sigma_l := \frac{1}{2^{l+N}},$$
$$K^{(l)} = (\varepsilon_1^{(l)})^{-1/8}, \quad \mu^{(l+1)} = \mu^{(l)} - (\varepsilon_1^{(l)})^{1/2},$$
$$\mathcal{K}_{0(l+1)} = \mathcal{K}_{0(l)} - 2\varepsilon_1^{(l)}, \quad \mathcal{K}_{1(l+1)} = \mathcal{K}_{1(l)} + 2\varepsilon_2^{(l)}, \quad \mathcal{G}_{(l+1)} = \mathcal{G}_{(l)} + 2\varepsilon_1^{(l)}$$

where $\mathfrak{K}$ *and* $\mathfrak{K}^{\mathcal{L}}$ *are defined in* (3.58) *and we set* $\mathcal{K}_{0(0)} := \mathcal{K}_{00}$, $\mathcal{K}_{1(0)} := \mathcal{K}_{10}$ *and* $\mathcal{G}_{(0)} := \mathcal{G}_0$. *Let* $\varepsilon_1^{(0)}$ *and* $\varepsilon_2^{(0)}$ *be such that* $\varepsilon_1^{(l)}$ *and* $\varepsilon_2^{(l)}$ *are decreasing sequences. Moreover, let us suppose that*

$$\varepsilon_1^{(0)} < \frac{5\sigma_1^9}{8\mathfrak{C}} = \frac{5}{2^{9N+11}\mathfrak{C}} \tag{3.60}$$

and that for any l

$$\mu^{(l)} \ge \Gamma, \quad \mathcal{K}_{0(l)} \ge \mathcal{K}_0, \quad \mathcal{K}_{1(l)} \le \mathcal{K}_1, \quad \mathcal{G}_{(l)} \le 2$$

where $\mathcal{K}_0$ *and* $\mathcal{K}_1$ *satisfy* (3.59), *then for any* l *there exists a positive sequence* $\lambda_l = (\lambda_{l,j}(\omega))_{j=1}^\infty$ *and a Lipschitz analitic map* $\Xi_l(\omega,\omega t)$ *defined on* $\Omega_{\mu^{(l)}}(\lambda_l)$ *such that the function* ϕ_l *defined by the sequence*

$$\phi_{j+1} = \Xi_j\phi_j, \quad \textit{with } \phi_0 \textit{ solution of } (3.43)$$

solves

$$\phi_{l,t} = X_{H_l(\omega,\omega t)}\phi_l, \qquad \textit{with} \quad H_l(\omega,\omega t) = \Lambda_l(\omega) + \gamma_l R_l(\omega,\omega t)$$

where $\Lambda_j(\omega) = \sum_{j=1}^\infty \lambda_j(\omega)\xi_j\eta_j$, $R_l \in \mathcal{B}^{3,2}_{r_l}(\Omega_{\mu^{(l)}}(\lambda_l))$, $\gamma_l = \gamma^{2^l}$ *and* $r_l := 1 - \sum_{j=1}^l \sigma_j$. *Moreover* *we have that*

$$
\begin{aligned}
&|\Omega_{\mu^{(l-1)}}(\lambda_{l-1}) \setminus \Omega_{\mu^{(l)}}(\lambda_l)| \le C'_\Omega \mu^{(l)} (\varepsilon_1^{(l)})^{1/8},\\
&\mathcal{K}_{0l}|i \pm j| \le |\lambda_{l,i} \pm \lambda_{l,j}| \le \mathcal{G}_l |i \pm j|, \qquad \left|\frac{\Delta(\lambda_{l,i} \pm \lambda_{l,j})}{\Delta\omega}\right| \le K_{1l}|i \pm j|,\\
&|\lambda_{l,i} \pm \lambda_{l,j} + \omega k| \ge \frac{\mu^{(l)}}{1+k^4}\langle i-j\rangle, \quad |i-j|+|k| \ne 0,\\
&\gamma_l |R_l|_{r_l} \le \varepsilon_1^{(l)}, \quad \gamma_l |R_l|_{r_l}^{\mathcal{L}} \le \varepsilon_2^{(l)},\\
&|id - \Xi_l|_{r_l} \le \frac{2\mathfrak{C}}{\sigma_l^4}\gamma_l |R_l|_{r_l}, \quad |id - \Xi_l|_{r_l}^{\mathcal{L}} \le \frac{2\mathfrak{C}}{\sigma_l^4}\gamma_l |R_l|_{r_l}^{\mathcal{L}} + \frac{2\mathfrak{C}^{\mathcal{L}}}{\sigma_l^8}\gamma_l |R_l|_{r_l}.
\end{aligned}
$$

Proof. The proof immediately follows from an iterate application of Lemma 3.7.4 and Lemma 3.7.5. For the sake of clarity, we show how the condition in (3.60) is sufficient to the validity of the result. For Lemma 3.7.4 to hold for any $l \ge 1$, we have to require that (3.57) holds for any $l \ge 1$, that is

$$
\gamma_l |R_l|_{r_l} \le \frac{5\sigma_l^9}{8\mathfrak{C}}. \tag{3.61}
$$

Since

$$
\gamma_l \frac{|R_l|_{r_l}}{\sigma_l^9} \le \frac{\varepsilon_1^{(l)}}{\sigma_l^9} = \sqrt{\mathfrak{K}\varepsilon_1^{(l+1)}}\sigma_{l+1} \searrow 0 \quad \text{as } l \to +\infty,
$$

we have that (3.60) implies (3.61), that is, (3.60) is sufficient to guarantee the validity of the thesis for any $l \ge 0$. □

3.7.4 The KAM result

From Lemma 3.7.4 and Lemma 3.7.5, the following proposition holds

Proposition 3.7.7. *Let $\mathcal{K}_0$ and $\mathcal{K}_1$ be defined as in Lemma 3.7.4. Assume that for some positive ζ, we have that*

$$
\max(|R_0|_1, |R_0|_1^{\mathcal{L}}) \le \zeta
$$

Let us suppose that

$$
\gamma < \frac{1}{2^{8(N+2)}\max(\mathfrak{K}, 2\mathfrak{K}^{\mathcal{L}})\zeta} \tag{3.62}
$$

where $N = N(\omega_m)$ is defined in Lemma C.3 and the constants $\mathfrak{K}$ and $\bar{\mathfrak{K}}$ are defined in Lemma 3.7.4. Moreover, for some positive constants C_0, C_Γ and Γ,

$$
\mathcal{K}_{00} - C_0\gamma\zeta > \mathcal{K}_0, \quad \mathcal{K}_{10} + C_0\gamma\zeta < \mathcal{K}_1, \quad \mathcal{G}_0 + C_0\gamma\zeta \le 2 \quad \mu_0 - C_\Gamma \gamma^{1/2}|R_0|_1^{1/2} \ge \Gamma, \tag{3.63}
$$

where $\mathcal{K}_{00}$ and $\mathcal{K}_{10}$ are defined in Lemma 3.7.4 and $\mathcal{K}_0$ and $\mathcal{K}_1$ satisfy (3.59). *Then there exists a positive sequence $\lambda_\infty = (\lambda_{\infty,j}(\omega))_{j=1}^\infty$ and a Lipschitz analitic map $U(\omega, \omega t)$ defined on $\Omega_\Gamma(\lambda_\infty)$ such that the transformation $U(\omega,\omega t)\phi' = \phi$ transform* (3.52) *into*

$$
\phi'_t = X_{\Lambda_\infty}\phi', \quad \text{with } \Lambda_\infty = \sum_{i,j\in\mathbb{N}_0} \lambda_{\infty,j}\xi_j\eta_j.
$$

Futhermore, the following estimates hold:

$$
|\lambda_{\infty,i} \pm \lambda_{\infty,j}| \ge (\mathcal{K}_0 - C_0\zeta)|i \pm j|, \qquad \left|\frac{\Delta(\lambda_{\infty,i} \pm \lambda_{\infty,j})}{\Delta\omega}\right| \le (\mathcal{K}_1 + C_0\zeta)|i \pm j|, \tag{3.64}
$$

$$
|\Omega \setminus \Omega_{\mu_\infty}^\infty| \le C_\Omega^\infty \gamma^{1/3}, \tag{3.65}
$$

$$
|id - U|_r \le C_U\gamma\zeta. \tag{3.66}
$$

for some $r = r(\omega_m)$ and for some positive constants C_0 and C_Ω^∞ dependent by γ, Ω, a and R_0.

Proof. The proof immediately follows from Lemma C.1 and Lemma C.2 by proceeding as in [13, Theorem 6.6]. In particular, let us introduce the sequences

$$\varepsilon_1^{(l+1)} := \frac{\mathfrak{K}}{\sigma_{l+1}^9}(\varepsilon_1^{(l)})^2, \quad \varepsilon_2^{(l+1)} := \frac{\mathfrak{K}^{\mathcal{L}}}{\sigma_{l+1}^9}\varepsilon_1^{(l)}\left(\varepsilon_2^{(l)} + \frac{\varepsilon_1^{(l)}}{\sigma_{l+1}^4}\right), \quad \text{where } \sigma_l := \frac{1}{2^{l+N}}.$$

with $N = N(\omega_m)$ defined in Lemma C.3. By setting $\varepsilon_1^{(l)} = \varepsilon_2^{(l)} := \zeta^{(l)}$ and $\zeta^{(0)} := \gamma\zeta$, we obtain that, setting $\bar{\mathfrak{K}} = \max(\mathfrak{K}, 2^4\mathfrak{K}^{\mathcal{L}})$,

$$\zeta^{(l+1)} \leq 2^{13(N+1)}\gamma\bar{\mathfrak{K}}2^{8l}(\zeta^{(l)})^2.$$

Hence, from Lemma D.5 we have that if (3.62) holds, then $\zeta^{(l)}$ goes to zero and

$$\sum_{l=0}^{\infty}\zeta^{(l)} \leq 2\zeta^{(0)} = 2\gamma\zeta \tag{3.67}$$

From Lemma 3.7.5, if we set $\mu^{(l+1)} := \mu^{(l)} - 2(\varepsilon_1^{(l)})^{1/2}$ and $\mu_0 = \mu^{(0)} := c_\mu\gamma^{1/2}$ for a suitable constant c_μ such that (3.63) holds, we have that

$$|\Omega_{\mu^{(l+1)}}(\lambda_{l+1}) \setminus \Omega_{\mu^{(l)}}(\lambda_l)| \leq C'_\Omega\mu^{(l+1)}(\varepsilon_1^{(l)})^{1/8} \leq C'_\Omega\mu^{(0)}(\varepsilon_1^{(l)})^{1/8} \leq C'_\Omega c_\mu\gamma^{1/2}(\zeta^{(l)})^{1/8}$$

which yields that, by using Lemma A.3,

$$\begin{aligned}|\Omega \setminus \Omega_{\mu_\infty}| = |\Omega \setminus \Omega_\gamma| + \sum_{l=0}^{\infty}|\Omega_{\mu^{(l)}} \setminus \Omega_{\mu^{(l+1)}}| &\leq (C_\Omega c_\mu^{2/3} + C'_\Omega\gamma^{1/6}\sum_{l=0}^{\infty}(\varepsilon_1^{(l)})^{1/8})\gamma^{1/3} \\ &\leq (C_\Omega c_\mu^{2/3} + 2^{1/8}\gamma^{1/6}C'_\Omega(\varepsilon_1^{(0)})^{1/8})\gamma^{1/3}.\end{aligned}$$

Moreover, from Lemma 3.7.5, we have that

$$|\lambda_{\infty,i} \pm \lambda_{\infty,j}| \geq (\mathcal{K}_0 - 2\sum_{l=0}^{\infty}\varepsilon_1^{(l)})|i \pm j| \geq (\mathcal{K}_0 - 2\sum_{l=0}^{\infty}\zeta^{(l)})|i \pm j| \geq (\mathcal{K}_0 - 2^2\gamma\zeta)|i \pm j|$$

and, analogously,

$$\left|\frac{\Delta(\lambda_{\infty,i} \pm \lambda_{\infty,j})}{\Delta\omega}\right| \leq (\mathcal{K}_1 + 2\sum_{l=0}^{\infty}\varepsilon_2^{(l)})|i \pm j| \leq (\mathcal{K}_1 + 2^2\gamma\zeta)|i \pm j|.$$

Hence from (3.67) we obtain that (3.64) and (3.65) hold with

$$C_0 := 2^2, \quad C_\Omega^\infty := C_\Omega c_\mu^{2/3} + 2^{1/8}\gamma^{1/6}C'_\Omega(\varepsilon_1^{(0)})^{1/8}.$$

To conclude, we observe that, if $\Xi_n := \phi^1_{\gamma_n X_n}$ where

$$X^{\pm}_{n,hk}(\omega,\phi) = -i\sum_{l\in\mathbb{Z}}\frac{R^{\pm}_{n,hkl}(\omega)}{\lambda_{n,h}(\omega) \pm \lambda_{n,k}(\omega) + l\omega}e^{-il\phi},$$

setting $U_n = \Xi_n\Xi_{n-1}\dots\Xi_0$ and $r_{n+1} = r_n - \sigma_{n+1}$ with $r_0 = 1$, we have that

$$\begin{aligned}|\mathrm{id} - U_n|_{r_n} &\leq |\mathrm{id} - \Xi_n|_{r_n} + |\Xi_n(\mathrm{id} - U_{n-1})|_{r_n} \\ &\leq 2^5\mathfrak{C}2^{4n}\zeta^{(n)} + \left(1 + 2^5\mathfrak{C}2^{4n}\zeta^{(n)}\right)|\mathrm{id} - U_{n-1}|_{r_{n-1}},\end{aligned} \tag{3.68}$$

where we used that from Lemma 3.7.4

$$|\mathrm{id} - \Xi_n|_{r_n} \le \frac{2\mathfrak{C}}{\sigma_n^9}\varepsilon_1^{(n)} = 2^{9N+1}\mathfrak{C}2^{9n}\zeta^{(n)}.$$

We introduce the following sequences:

$$u_n := |\mathrm{id} - U_n|_{r_n}, \quad x_n := 2^{9N+5}\mathfrak{C}2^{9n}\zeta^{(n)}.$$

Moreover, for any $n \in \mathbb{N}$, we define the sequence

$$p_{n,j+1} = (1 + x_{n-j})p_{n,j}, \quad \text{with } p_{n,0} = 1$$

and we remark that, from Lemma D.5,

$$p_{n,j} \le \prod_{j=0}^{\infty}(1 + x_j) \le \exp\left(\sum_{j=0}^{\infty} x_j\right) \le \exp(2^{9N+6}\mathfrak{C}\zeta^{(0)}) =: p.$$

From (3.68) we get that

$$\begin{aligned} u_n \le x_n + (1 + x_n)u_{n-1} &\le \sum_{j=0}^{n} p_{n,j}x_{n-j} + p_n a_0 \le p\left(u_0 + \sum_{j=0}^{\infty} x_j\right) \\ &\le \exp(2^{9N+6}\mathfrak{C}\zeta^{(0)})(2^{9N+5}\mathfrak{C}\zeta^{(0)} + 2^{9N+6}\mathfrak{C}\zeta^{(0)}) \\ &\le 2^{9N+5}(1+2)\mathfrak{C}\exp(2^{9N+6}\mathfrak{C}\gamma\zeta)\gamma\zeta =: C_U\gamma\zeta. \end{aligned}$$

Therefore, passing to the limit $n \to \infty$, we get (3.66). □

3.7.5 Completion of the proof

Let us fix $\delta^* > 0$. We denote by $e_n(x)$ the $n-$th normalized eigenfunction in $\mathcal{H}$ for the operator $-\partial_x^2$ and we write

$$\theta(x,t) = \sum_{j\in\mathbb{N}} \theta_j(t)e_j(x).$$

We introduce the variables

$$\xi_j = \frac{\beta_j^{1/2}\theta_j + i\beta_j^{-1/2}\theta_{j,t}}{\sqrt{2}}, \quad \eta_j = \frac{\beta_j^{1/2}\theta_j - i\beta_j^{-1/2}\theta_{j,t}}{\sqrt{2}}, \quad \zeta := \begin{pmatrix}\xi \\ \eta\end{pmatrix}$$

where $\beta_j^2 = (j/2)^2$ is the $n-$th eigenvalue of $B^2 = -\partial_{xx}$ and we reformulate (3.37) by using the notation in (3.35) as an abstract problem

$$\dot{\zeta} = X_H\zeta.$$

The Hamiltonian function H is defined as $H(\zeta) = \Lambda(\zeta) + D(\zeta) + S(\zeta)$ with

$$\begin{aligned} \Lambda(\zeta) &:= \sum_{j\in\mathbb{N}} \beta_j\xi_j\eta_j, \quad D(\zeta) := \sum_{j\in\mathbb{N}} a\beta_j^{-1}\xi_j\eta_j + \frac{1}{2}\sum_{j\in\mathbb{N}} a\beta_j^{-1}(\eta_j^2 + \xi_j^2), \\ S(\zeta) &:= \sum_{i,j\in\mathbb{N}} \mathcal{R}_{ij}\eta_i\xi_j + \frac{1}{2}\sum_{i,j\in\mathbb{N}} \mathcal{R}_{i,j}(\eta_i\eta_j + \xi_i\xi_j), \end{aligned}$$

where we set $\mathcal{R}_{i,j} := (B^{-1/2}RB^{-1/2}e_i, e_j)_{L^2(I)}/2$. From Corollary 3.7.3, we have that there exists $\gamma_{1,*}(\delta^*, n) > 0$ such that, if $\gamma < \gamma_{1,*}$ and $\delta < \delta^*$, then there exists a canonical transformation Ξ such that, if $\varphi = \Xi\zeta$, then φ solves

$$\varphi_t = X_{H_0}\varphi \quad \text{where } H_0 = \Lambda_0(\omega) + \gamma R_0(\omega, \omega t) \tag{3.69}$$

with $\Lambda_0 := \sum_{j\in\mathbb{N}} \lambda_{0,j}\eta_j\xi_j$, $\lambda_{0,j} = \beta_j + \gamma\bar{a}(\omega)\beta_j^{-1}$ and $R_0 \in \mathcal{B}_1^{3,2}(\Omega)$. Moreover, we have that

$$|R_0|_1 \le C(n,\delta)\delta^2, \qquad |R_0|_1^{\mathcal{L}} \le C^{\mathcal{L}}(n,\delta)|V|_{4/3}^{\mathcal{L}}.$$

From (3.51), since Ξ is invertible and bounded we get that for some $K_1(n,\delta) > 0$,

$$\begin{aligned}\sup_{t\ge0}\|\varphi(\omega,\omega t) - \zeta(\omega,\omega t)\|_Y &= \sup_{t\ge0}\|(\Xi(\omega,\omega t) - \mathrm{id})\Xi^{-1}(\omega,\omega t)\varphi(\omega,\omega t)\|_Y \\ &\le \left|(\Xi(\omega,\omega t) - \mathrm{id})\Xi^{-1}(\omega,\omega t)\right|_1 \sup_{t\ge0}\|\varphi(\omega,\omega t)\|_Y \\ &\le K_1(n,\delta)\sup_{t\ge0}\|\varphi(\omega,\omega t)\|_Y.\end{aligned}$$

We now apply Theorem 3.7.7 to equation (3.69). We observe that, setting

$$\tilde{\zeta} := \max(|R_0|_1, |R_0|_1^{\mathcal{L}})$$

we have that $\tilde{\zeta} = \tilde{\zeta}(n,\nu,\delta,\delta^*)$ and that, since $\omega_m = \sqrt{\lambda_n}$, in the notation of Proposition 3.7.7 we have that $N = N(\lambda_n)$. Therefore, by reasoning as in Corollary 3.7.3, from (3.62) and from the conditions in (3.63) we get that there exists $\gamma_{2,*}(n,\nu,\delta^*) > 0$ such that, if $\gamma < \gamma_{2,*}$, then for any $\omega < \omega^* := \omega(\delta^*)$ there exists a Lipschitz analitic map $U(\omega,\phi)$ defined on $\Omega_{\gamma,n,\omega} \subseteq \Omega = [\sqrt{\lambda_n}, \omega]$ such that for any $\bar{\omega} \in \Omega_{\gamma,n,\omega}$ if $U(\bar{\omega}, \bar{\omega}t)\phi = \varphi$, then ϕ satisfies

$$\phi_t = X_{\Lambda_\infty}\phi, \quad \text{with } \Lambda_\infty = \sum_{i,j\in\mathbb{N}} \lambda_{\infty,j}\xi_j\eta_j.$$

We remark that, since from Proposition 3.7.7 we have that $\lambda_{\infty,j} = \beta_j + \gamma g_j(\bar{\omega})\beta_j^{-1}$ for some $g_j(\bar{\omega})$ uniformly bounded in ω with respect to j, it follows that for some positive constants c_β and C_β we have that, if $\gamma < \gamma_{3,*}$ small enough,

$$c_\beta\beta_j \le \lambda_{\infty,j} \le C_\beta\beta_j. \tag{3.70}$$

Moreover, from (3.65), in the notations of Proposition 3.7.7

$$|\Omega \setminus \Omega_{\gamma,n,\omega}| \le (C_\Omega c_\mu^{2/3} + 2^{1/8}\gamma^{1/6}C'_\Omega(\varepsilon_1^{(0)})^{1/8})\gamma^{1/3} \tag{3.71}$$

and from (3.66) we have that

$$\begin{aligned}\sup_{t\ge0}\|\varphi(\omega,\omega t) - \phi(\omega,\omega t)\|_Y &= \sup_{t\ge0}\|(U(\omega,\omega t) - \mathrm{id})\phi\|_Y \\ &\le |U(\omega,\omega t) - \mathrm{id}|_1 \sup_{t\ge0}\|\phi(\omega,\omega t)\|_Y \\ &\le K_2(n,\tilde{\delta})\sup_{t\ge0}\|\phi(\omega,\omega t)\|_Y.\end{aligned}$$

Since from Lemma A.3 we have that

$$C_\Omega \le C_\omega(n,\delta^*)\delta^{1/3}$$

and we can estimate $\varepsilon_1^{(0)} \le C_\varepsilon \delta^2$ for some positive constant C_ε, from (3.71) we get for some positive constant $K_\omega = K_\omega(n, \delta^*)$

$$|\Omega \setminus \Omega_{n,\gamma,\omega}| \le K_\omega \gamma^{1/3} \delta^{1/4}.$$

Hence, by considering $\gamma < \gamma^* = \min(\gamma_{1,*}, \gamma_{2,*}, \gamma_{3,*})$ and by reasoning as in Lemma 3.5.2, we conclude that for any $\delta^* > 0$ and for any $\gamma < \gamma^*$ there exists $\Delta_{n,\gamma,\delta} \subseteq [0, \delta^*]$ such that

$$|[0,\delta] \setminus \Delta_{n,\gamma,\delta}| \le K_\Delta \gamma^{1/6} \delta^{1/8} \tag{3.72}$$

for some positive constant $K_\Delta = K_\Delta(n, \delta^*)$

From Lemma 3.6.3 and from (3.70), since the transformations Ξ and U are obtained by combining transformations in the form ϕ_Z^1 for some Hamiltonian $Z \in H^0$, we get that $\phi = (\phi_1, \phi_2) \in Y^{1/2}$, where $Y^{1/2} = Y$ is defined in (3.33). The real function

$$\Theta(x,t) = \sum_{j\in\mathbb{N}} \Theta_j(t) e_j(x), \qquad \text{where } \Theta_j(t) = \frac{\phi_{1,j}(t) + \bar{\phi}_{1,j}(t)}{\sqrt{2\beta_j}}$$

solves the equation

$$\Theta_{tt} + \Lambda_\infty^2 \Theta = 0, \qquad \text{where } \Lambda_\infty^2 = \text{diag}(\beta_{\infty,j}^2)$$

and we have that

$$\underline{c}_\beta \beta_j^2 \le \beta_{\infty,j}^2 \le \bar{c}_\beta \beta_j^2 \tag{3.73}$$

for some positive constant $\bar{c}_\beta$ and $\underline{c}_\beta$. We remark that

$$\|\Theta_t(t)\|^2_{L^2(I)} + \|\Lambda_\infty \Theta(t)\|^2_{L^2(I)} = \|\Theta_t(0)\|^2_{L^2(I)} + \|\Lambda_\infty \Theta(0)\|^2_{L^2(I)} \tag{3.74}$$

and we observe that

$$\begin{aligned}\|\Theta_t(t)\|^2_{L^2(I)} + \|\Lambda_\infty \Theta(t)\|^2_{L^2(I)} &= \sum_{j\in\mathbb{N}} \left| \frac{\beta_{j,\infty}\phi_{1,j} - \beta_{j,\infty}\overline{\phi}_{1,j}}{\sqrt{2\beta_j}} \right|^2 + \beta_{\infty,j}^2 \left| \frac{\phi_{1,j} + \overline{\phi}_{1,j}}{\sqrt{2\beta_j}} \right|^2 \\ &= \sum_{j\in\mathbb{N}} \frac{\beta_{j,\infty}^2}{\beta_j} |\phi_{1,j}|^2\end{aligned}$$

which yields, from (3.73), since $\beta_j = j/2$,

$$\underline{c}_\beta^2 \|\phi(t)\|_Y^2 \le \|\Theta_t(t)\|^2_{L^2(I)} + \|\Lambda_\infty \Theta(t)\|^2_{L^2(I)} \le \bar{c}_\beta^2 \|\phi(t)\|_Y^2. \tag{3.75}$$

By observing that, since for any $f \in H^1(I)$ by using (3.73)

$$\underline{c}_\beta \|f\|_{H^1(I)} \le \|\Lambda_\infty f\|_{L^2(I)} \le \bar{c}_\beta \|f\|_{H^1(I)},$$

equation (3.74) together with (3.75) implies that

$$\|\Theta_t(t)\|^2_{L^2(I)} + \|\Theta(t)\|^2_{H^1(I)} \le C_\beta \|\phi(0)\|_Y$$

with $C_\beta > 0$ a positive constant dependent on $\underline{c}_\beta$ and $\bar{c}_\beta$. Since $\phi(0) = U^{-1}(\omega(\delta), 0)\Xi(\omega(\delta), 0)\zeta(0)$, it follows that

$$\|\phi(0)\|_Y \le \|U^{-1}(\omega(\delta), 0)\Xi(\omega(\delta), 0)\|_{\mathcal{B}(Y,Y)} \|\zeta(0)\|_Y \le K(\|\theta_t(0)\|^2_{L^2(I)} + \|\theta(0)\|^2_{H^1(I)})$$

with $K = K(n, \delta^*) > 0$. Therefore, we conclude that for some constant $C = C(n, \delta^*)$, we have that

$$\|\Theta_t(t)\|^2_{L^2(I)} + \|\Theta(t)\|^2_{H^1(I)} \leq C(\|\theta_t(0)\|^2_{L^2(I)} + \|\theta(0)\|^2_{H^1(I)}). \tag{3.76}$$

Moreover, since both Ξ and U are bounded in $\mathcal{B}(Y, Y)$ and near to the identity operators, that is from (3.45) and from (3.66)

$$\|\mathrm{id} - \Xi\|_{\mathcal{B}(Y,Y)} \leq C_\Xi \gamma \delta^2, \qquad \|\mathrm{id} - U\|_{\mathcal{B}(Y,Y)} \leq C_U \gamma \delta^2,$$

for some $C_\Xi = C_\Xi(n, \delta^*)$ and $C_U = C_U(n, \delta^*)$, it follows that

$$\|\theta_t(t) - \Theta_t(t)\|_{L^2(I)} + \|\theta(t) - \Theta(t)\|_{H^1(I)} \leq c\gamma\delta^2 \tag{3.77}$$

for some positive constant $c = c(n, \delta^*)$.

Summarizing, we have that for any $\delta^* > 0$ there exists $\gamma^* = \gamma^*(n, \delta^*) > 0$ such that for any $\gamma < \gamma^*$ and for any $\delta \in [0, \delta^*]$ there exists a positive measure set $\Delta_{n,\gamma,\delta} \subseteq [0, \delta]$ such that (3.72) holds and for any $\bar{\delta} \in \Omega_{n,\gamma,\delta}$, (3.76) and (3.77) hold, that is the thesis.

Appendices

A Some measure estimates

Lemma A.1. *Let $\omega_M > \omega_m > 0$. Let us introduce the following sequences:*

- *$\lambda = (\lambda_j)_{j=1}^\infty$ such that $\lambda_j > 0$, $\lambda_j \leq \lambda_{j+1}$ for any $j \geq 1$ and $\lambda_j \sim cj^d$ for some $d > 1$;*
- *$\alpha^- = (\alpha_j^-)_{j=1}^\infty$ such that $\alpha_j^- > 0$ and $\alpha_j^- \leq \alpha_{j+1}^-$ for any $j \geq 1$;*
- *$\alpha^+ = (\alpha_j^+)_{j=1}^\infty$ such that $\alpha_j^- < \alpha_j^+ < c_1\alpha_j^-$ for some $c_1 > 0$, for any $j \geq 1$;*
- *$\Lambda = (\Lambda_j)_{j=1}^\infty \subset Lip([\omega_m, \omega_M], \mathbb{R}_+)$ such that*

$$\forall \omega_1 \neq \omega_2 \in [\omega_m, \omega_M], \forall j \geq 1, \qquad \alpha_j^- < \left|\frac{\Lambda_j(\omega_1) - \Lambda_j(\omega_2)}{\omega_1 - \omega_2}\right| < \alpha_j^+. \tag{A.1}$$

Let us introduce the set

$$\Omega_\mu := \left\{\omega \in [\omega_m, \omega_M] : \forall n \geq 0, \forall j \geq 1, \quad |\Lambda_j(\omega) - n\pi| > \frac{\mu}{\lambda_j}\right\}$$

Then, if $\mu < \lambda_1\pi$, we have that $|[\omega_m, \omega_M] \setminus \Omega_\mu| < c\mu$ where $c = c(\alpha^+, \alpha^-, \omega_M, \omega_m, \lambda_j) > 0$.

Proof. Let us consider the sets

$$\Omega_{\mu,j} = \{\omega \in [\omega_m, \omega_M] : \exists n \in \mathbb{N} \text{ such that } |\Lambda_j(\omega) - n\pi| \leq \mu/\lambda_j\}.$$

We observe that $[\omega_m, \omega_M] \setminus \Omega_\mu = \bigcup_{j\geq 1} \Omega_{\mu,j}$ and

$$\Omega_{\mu,j} = [\omega_m, \omega_M] \cap \bigcup_{n=0}^\infty \Lambda_j^{-1}\left(\left[n\pi - \frac{\mu}{\lambda_j}, n\pi + \frac{\mu}{\lambda_j}\right]\right).$$

Let $n_j = \#\{n \in \mathbb{N} : [\omega_m, \omega_M] \cap \Lambda_j^{-1}([n\pi - \mu/\lambda_j, n\pi + \mu/\lambda_j]) \neq \varnothing\}$. From (A.1) we have

$$\left|\Lambda_j^{-1}\left(\left[n\pi - \frac{\mu}{\lambda_j}, n\pi + \frac{\mu}{\lambda_j}\right]\right)\right| \leq \frac{|(n\pi - \mu/\lambda_j, n\pi + \mu/\lambda_j)|}{\alpha_j^-} = \frac{2\mu}{\alpha_j^- \lambda_j} \tag{A.2}$$

and, if $\mu < \lambda_1 \pi$,

$$n_j \leq \frac{|\Lambda_j(\omega_M) - \Lambda_j(\omega_m)|}{\pi} + 2 = \frac{|\Lambda_j(\omega_M) - \Lambda_j(\omega_m)| + 2\pi}{\pi}. \tag{A.3}$$

Therefore, by observing that $|\Omega_{\mu,j}| \leq n_j \max_{n\geq 0} |\Lambda_j^{-1}([n\pi - \mu/\lambda_j, n\pi + \mu/\lambda_j])|$, we get, by using (A.1), (A.2) and (A.3)

$$|\Omega_{\mu,j}| \leq \frac{2\mu(|\Lambda_j(\omega_M) - \Lambda_j(\omega_m)| + 2\pi)}{\alpha_j^- \lambda_j \pi} \leq \frac{2\mu(\alpha_j^+(\omega_M - \omega_m) + 2\pi)}{\alpha_j^- \lambda_j \pi}$$

Therefore, since $\lambda_j \sim c_\lambda j^d$ for some $d > 1$ and $\alpha_j^+ < c_1 \alpha_j^-$ for some $c_1 > 0$, we get that

$$|[\omega_m, \omega_M] \setminus \Omega_\mu| = \left| \bigcup_{j\geq 1} \Omega_{\mu,j} \right| \leq \sum_{j=1}^{\infty} \frac{2\mu(\alpha_j^+(\omega_M - \omega_m) + 2\pi)}{\alpha_j^- \lambda_j \pi} = c\mu \tag{A.4}$$

for some positive constant $c = c(\alpha^+, \alpha^-, \omega_m, \omega_M, \lambda_j)$. □

From Lemma A.1, we immediately obtain the following corollary:

Corollary A.2. *Let λ, α^-, α^+, Λ and Ω_μ be defined as in Lemma A.1. If $\mu < \lambda_1 \pi$ and*

$$\sum_{j=1}^{\infty} \frac{2(\alpha_j^+(\omega_M - \omega_m) + 2\pi)}{\alpha_j^- \lambda_j \pi} < \frac{\omega_M - \omega_m}{\mu}, \tag{A.5}$$

then $|\Omega_\mu| > 0$.

Proof. From equation (A.4), we have that

$$|[\omega_m, \omega_M] \setminus \Omega_\mu| \leq \sum_{j=1}^{\infty} \frac{2\mu(\alpha_j^+(\omega_M - \omega_m) + 2\pi)}{\alpha_j^- \lambda_j \pi}.$$

Therefore, since $|\Omega_\mu| > 0$ is equivalent to require that $|[\omega_m, \omega_M] \setminus \Omega_\mu| < \omega_M - \omega_m$, we get that if (A.5) holds, then $|\Omega_\mu| > 0$.

□

Lemma A.3. *Let us consider a sequence $\lambda(\omega)$ given by*

$$\lambda_i(\omega) := \frac{i}{2} + \gamma \frac{2\bar{a}(\omega)}{i} + \gamma \frac{\nu_i(\omega)}{i^3},$$

where $\nu_i \in Lip(\Omega, \mathbb{R})$ and $\bar{a}(\omega)$ is defined in Lemma 3.7.1. We require that there exists $\varepsilon > 0$ such that

$$|\nu_i|_\infty := \sup_{\omega \in \Omega} |\nu_i| \leq \varepsilon, \quad \forall i \in \mathbb{N}.$$

Moreover, we suppose that there exists $\mathcal{K}_0, \mathcal{K}_1 > 0$ such that

$$|\lambda_i(\omega) \pm \lambda_j(\omega)| \geq \mathcal{K}_0 |i \pm j|, \qquad \left| \frac{\Delta(\lambda_i(\omega) \pm \lambda_j(\omega))}{\Delta\omega} \right| \leq \mathcal{K}_1, ,$$

and we introduce the set

$$\Omega_\mu(\lambda) := \left\{ \omega \in \Omega : \forall i, j \in \mathbb{N}, \forall k \in \mathbb{Z}, |\lambda_i(\omega) \pm \lambda_j(\omega) + k\omega| > \mu \frac{\langle i \pm j \rangle}{1 + k^4} \right\},$$

where $\langle m\rangle := \max(1,|m|)$. *Then, if*

$$\mathcal{K}_1 < \frac{\mathcal{K}_0}{8}, \qquad \mu \leq \frac{\mathcal{K}_0}{2} \tag{A.6}$$

we have that

$$|\Omega\setminus\Omega_\mu| \leq C_\Omega \mu^{2/3}$$

for some positive constant $C_\Omega = C(\mathcal{K}_0, \omega_M, \mu, \gamma, \lambda)$.

Proof. We have that $\Omega_\mu = \Omega_\mu^+ \cap \Omega_\mu^-$, where

$$\Omega_\mu^\pm = \left\{\omega\in\Omega : \forall i,j\in\mathbb{N}, \forall k\in\mathbb{Z}, |\lambda_i(\omega)\pm\lambda_j(\omega)+k\omega| > \mu\frac{\langle i\pm j\rangle}{1+k^4}\right\}.$$

Let us suppose that $i>j$. Hence, $i=j+m$ with $m>0$. Then, since $|\bar a(\omega)|\leq |a|_\infty$,

$$|\lambda_i-\lambda_j| \leq m/2+\delta_j m, \text{ where } \delta_j := \gamma\frac{2|a|_\infty}{j^2}+\gamma\frac{2\varepsilon}{j^3}.$$

By proceeding as in [13, Lemma 5.2] (see also [139, Lemma 8]), introducing

$$\mathcal{R}^\pm_{ijk}(\lambda,\alpha) := \left\{\omega\in\Omega : |\lambda_i(\omega)\pm\lambda_j(\omega)+k\omega| < \alpha\langle i\pm j\rangle\right\},$$
$$\mathcal{Q}_{mjk} := \left\{\omega\in\Omega : |m+\omega k| < \frac{\mu m}{1+k^4}+m\delta_j\right\}.$$

and observing that $\mathcal{R}_{ijk}\subset\mathcal{Q}_{mjk}$ and $\mathcal{Q}_{mjk}\subset\mathcal{Q}_{mj'k}$ if $j>j'$, we get that

$$\bigcup_{i>j}\mathcal{R}^-_{ijk}\left(\lambda,\frac{\mu}{1+k^4}\right) \subset \left(\bigcup_{i-j=m, j<j_*}\mathcal{R}^-_{ijk}\left(\lambda,\frac{\mu}{1+k^4}\right)\right)\cup\left(\bigcup_{m\in\mathbb{N}}\mathcal{Q}_{mj_*k}\right).$$

Therefore, by taking some $j_*>0$ and fixing k, since from [13, Lemma 5.2, Lemma A.2] we have that

$$\left|\bigcup_{i-j=m, j<j_*}\mathcal{R}^-_{ijk}\left(\lambda,\frac{\mu}{1+k^4}\right)\right| \leq \frac{4}{\mathcal{K}_0}j_* m\frac{\mu}{1+k^4} \leq \frac{4\omega_M}{\mathcal{K}_0(\mathcal{K}_0-\mu)}\frac{\mu j_*|k|}{1+k^4}$$
$$\left|\bigcup_{m\in\mathbb{N}}\mathcal{Q}_{mjk}\right| \leq \frac{4\omega_M}{\mathcal{K}_0(\mathcal{K}_0-\mu)}\left(\frac{\mu}{1+k^4}+\delta_{j_*}\right)|k|$$

which yields that

$$\left|\bigcup_{i,j}\mathcal{R}^-_{ijk}\left(\lambda,\frac{\mu}{1+k^4}\right)\right| \leq \frac{8\omega_M}{\mathcal{K}_0(\mathcal{K}_0-\mu)}\left(\frac{\mu(j_*+1)}{1+k^4}+\frac{2\gamma(|a|_\infty+\varepsilon)}{j_*^2}\right)|k|.$$

We take

$$j_* := \left(\frac{\gamma(|a|_\infty+\varepsilon)}{\mu}(1+k^4)\right)^{1/3},$$

and, observing that $\mu<1$ we obtain that

$$\begin{aligned}\left|\bigcup_{i,j}\mathcal{R}^-_{ijk}\left(\lambda,\frac{\mu}{1+k^4}\right)\right| &\leq \frac{8\omega_M}{\mathcal{K}_0(\mathcal{K}_0-\mu)}\left(\frac{\mu(j_*+1)}{1+k^4}+\frac{2\gamma(|a|_\infty+\varepsilon)}{j_*^2}\right)|k| \\ &= \frac{8\omega_M}{\mathcal{K}_0(\mathcal{K}_0-\mu)}\left(\frac{\mu(j_*+1)}{1+k^4}+\frac{2(\gamma(|a|_\infty+\varepsilon))^{1/3}\mu^{2/3}}{(1+k^4)^{2/3}}\right)|k|\end{aligned}$$

By proceeding as in [13, Lemma 5.2], we get that, since for k large enough we have that $j_* > 1$, which yields that $j_* + 1 \le 2j_*$, for a certain constant $C = C(\gamma, |a|_\infty, \varepsilon)$

$$|\Omega \setminus \Omega_\mu^-| \le \frac{16C\omega_M}{\mathcal{K}_0(\mathcal{K}_0-\mu)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|k|}{(1+k^4)^{2/3}} (\gamma(|a|_\infty + \varepsilon))^{1/3}\mu^{2/3} \le \frac{56C\omega_M}{\mathcal{K}_0(\mathcal{K}_0-\mu)}(\gamma(|a|_\infty+\varepsilon))^{1/3}\mu^{2/3}.$$

Since if $\mathcal{R}_{ij}^+ \neq \varnothing$, then

$$\omega_M|k| \ge |\omega k| \ge |\lambda_i + \lambda_j| - \frac{\mu}{1+k^4}|i+j| \ge (\mathcal{K}_0 - \mu)|i+j|,$$

which yields that

$$|i+j| \le \frac{\omega_M}{\mathcal{K}_0 - \mu}|k|,$$

by reasoning as before, from [13, Lemma A.2, Lemma A.4] we obtain that

$$\left|\bigcup_{i,j,k} \mathcal{R}_{ijk}^+\left(\lambda, \frac{\mu}{1+k^4}\right)\right| \le \sum_{k\in\mathbb{Z}} \sum_{i,j:\mathcal{R}_{ijk}^+\neq\varnothing} \frac{4}{\mathcal{K}_0}\frac{\mu}{1+k^4} \le \frac{2\omega_M^2\mu}{\mathcal{K}_0(\mathcal{K}_0-\mu)^2} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{k^2}{1+k^4} = \frac{5\omega_M^2}{\mathcal{K}_0(\mathcal{K}_0-\mu)^2}\mu.$$

Therefore, we obtain that

$$|\Omega \setminus \Omega_\mu| \le |\Omega \setminus \Omega_\mu^-| + |\Omega \setminus \Omega_\mu^+| \le C_\Omega \mu^{2/3},$$

where, from (A.6),

$$C_\Omega := \frac{56C\omega_M}{\mathcal{K}_0(\mathcal{K}_0-\mu)}(\gamma(|a|_\infty + \varepsilon))^{1/3} + \frac{5\omega_M^2\mu^{1/3}}{\mathcal{K}_0(\mathcal{K}_0-\mu)^2}$$

□

Lemma A.4. *Let $A \subset \mathbb{R}_+$ be a measurable set and let $B := \{x \in \mathbb{R}_+ : x^2 \in A\}$. Then we have that*

$$|B| \le \sqrt{|A|},$$

where $|X|$ denotes the Lebesgue measure of $X \subseteq \mathbb{R}$.

Proof. The proof proceeds by direct computation.

$$\begin{aligned}|B| &= \int_{\{x\in\mathbb{R}_+:x^2\in A\}} \mathrm{d}x = \int_{\{t\in\mathbb{R}_+:t\in A\}} \frac{1}{2\sqrt{t}}\mathrm{d}t = \frac{1}{2}\int_A \frac{1}{\sqrt{t}}\mathrm{d}t = \frac{1}{2}\int_0^\infty \left|\left\{x \in A : \frac{1}{\sqrt{x}} \ge \lambda\right\}\right| \mathrm{d}\lambda \\ &= \frac{1}{2}\int_0^{1/\sqrt{|A|}} \left|\left\{x \in A : \frac{1}{\sqrt{x}} \ge \lambda\right\}\right| \mathrm{d}\lambda + \frac{1}{2}\int_{1/\sqrt{|A|}}^\infty \left|\left\{x \in A : \frac{1}{\sqrt{x}} \ge \lambda\right\}\right| \mathrm{d}\lambda \\ &\le \frac{1}{2}\int_0^{1/\sqrt{|A|}} |A|\mathrm{d}\lambda + \frac{1}{2}\int_{1/\sqrt{|A|}}^\infty \lambda^{-2}\mathrm{d}\lambda = \sqrt{|A|}.\end{aligned}$$

□

B Some norm estimates

Lemma B.1. *Let $Z, F \in \mathcal{B}_r^2(\Omega)$. Then*

$$|\{Z,F\}|_r \le 2\beta_1^{-2}\,|Z|_r\,|F|_r \tag{B.1}$$

$$|\{Z,F\}|_r^{\mathcal{L}} \le 2\beta_1^{-2}(|Z|_r^{\mathcal{L}}\,|F|_r + |Z|_r\,|F|_r^{\mathcal{L}}), \tag{B.2}$$

Proof. A direct computation shows that

$$\{Z,F\} = \sum_{l=1}^{\infty} i\frac{\partial F}{\partial \xi_l}\frac{\partial Z}{\partial \eta_l} - i\frac{\partial F}{\partial \xi_l}\frac{\partial Z}{\partial \eta_l} =$$
$$= \sum_{i,j\in\mathbb{N}_0} (\{Z,F\}^-)_{ij}\eta_i\xi_j + \frac{1}{2}\sum_{i,j\in\mathbb{N}_0} (\overline{\{Z,F\}^+})_{ij}\xi_i\xi_j + \frac{1}{2}\sum_{i,j\in\mathbb{N}_0} (\{Z,F\}^+)_{ij}\eta_i\eta_j$$

where

$$\{Z,F\}^- := i(Z^-F^- - F^-Z^- + F^+\overline{Z^+} - Z^+\overline{F^+})$$
$$\{Z,F\}^+ := i(Z^-F^+ - F^+\overline{Z^-} + F^-Z^+ - Z^+\overline{F^-}).$$

Since $\mathcal{H}^{1/2+2} \subseteq \mathcal{H}^{1/2}$ and we have that $\|u\|_{\mathcal{H}^{1/2}} \le \beta_1^{-2}\|u\|_{\mathcal{H}^{1/2+2}}$, then

$$\|Z\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2})} = \sup_{u\in\mathcal{H}^{1/2}} \frac{\|Zu\|_{\mathcal{H}^{1/2}}}{\|u\|_{\mathcal{H}^{1/2}}} \le \beta_1^{-2} \sup_{u\in\mathcal{H}^{1/2}} \frac{\|Zu\|_{\mathcal{H}^{1/2+2}}}{\|u\|_{\mathcal{H}^{1/2}}} \le \beta_1^{-2}\|Z\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+2})},$$
$$\|Z\|_{\mathcal{B}(\mathcal{H}^{1/2+2},\mathcal{H}^{1/2+2})} = \sup_{u\in\mathcal{H}^{1/2+2}} \frac{\|Zu\|_{\mathcal{H}^{1/2+2}}}{\|u\|_{\mathcal{H}^{1/2+2}}} \le \beta_1^{-2} \sup_{u\in\mathcal{H}^{1/2}} \frac{\|Zu\|_{\mathcal{H}^{1/2+2}}}{\|u\|_{\mathcal{H}^{1/2}}} \le \beta_1^{-2}\|Z\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+2})}.$$

Therefore, for any $A, B \in \mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+2})$ we have that

$$\|AB\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+2})} \le \|A\|_{\mathcal{B}(\mathcal{H}^{1/2+2},\mathcal{H}^{1/2+2})}\|B\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+2})}$$
$$\le \beta_1^{-2}\|A\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+2})}\|B\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+2})},$$

which yields that

$$\|AB\| \le \beta_1^{-2}\|A\|\|B\|,$$

where we set $\|\cdot\| := \|\cdot\|_{\mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+2})}$. Hence, we get that

$$||Z^-F^- - F^-Z^- + F^+\overline{Z^+} - Z^+\overline{F^+}|| \le 2\beta_1^{-2}(||Z^-||\cdot||F^-|| + ||Z^+||\cdot||F^+||),$$
$$||Z^-F^+ - F^+\overline{Z^-} + F^-Z^+ - Z^+\overline{F^-}|| \le 2\beta_1^{-2}(||Z^-||\cdot||F^+|| + ||Z^+||\cdot||F^-||).$$

Therefore, by using (3.36) and the notation in (3.34),

$$\|\{Z,F\}\|_{H^2} \le 2\beta_1^{-2}(\|Z^-\| + \|Z^+\|)(\|F^-\| + \|F^+\|) = 2\beta_1^{-2}\|Z\|_{H^2}\|F\|_{H^2},$$

which yields (B.1). The estimate (B.2) follows as above by observing that for any $A, B \in \text{Lip}(\Omega, \mathcal{B}(\mathcal{H}^{1/2},\mathcal{H}^{1/2+2}))$

$$\|AB\|^{\mathcal{L}} \le \beta_1^{-2}(\|A\|^{\mathcal{L}}\|B\| + \|A\|\|B\|^{\mathcal{L}}).$$

□

Proposition B.2. *Let $Z, F \in \mathcal{B}_r^2(\Omega)$. If*

$$|Z|_{r-\sigma} < \frac{5}{8}\beta_1^2, \tag{B.3}$$

then

$$|Lie_Z F - F|_{r-\sigma} \le 4\beta_1^{-2}\,|Z|_{r-\sigma}\,|F|_r\,, \tag{B.4}$$
$$|Lie_Z F - F|_{r-\sigma}^{\mathcal{L}} \le 4\beta_1^{-2}\,|Z|_{r-\sigma}\,|F|_r^{\mathcal{L}} + 7\beta_1^{-2}\,|Z|_{r-\sigma}^{\mathcal{L}}\,|F|_r \tag{B.5}$$

and

$$|Y_Z + \dot{Z}|_{r-\sigma} \leq 2\beta_1^{-2} \, |Z|_{r-\sigma} \, |\dot{Z}|_{r-\sigma}, \tag{B.6}$$

$$|Y_Z + \dot{Z}|^{\mathcal{L}}_{r-\sigma} \leq 2\beta_1^{-2} \, |Z|_{r-\sigma} \, |\dot{Z}|^{\mathcal{L}}_{r-\sigma} + 2\beta_1^{-2} \, |Z|^{\mathcal{L}}_{r-\sigma} \, |\dot{Z}|_{r-\sigma}. \tag{B.7}$$

Moreover, if $\Lambda \in \mathcal{B}^1_r(\Omega)$ *and*

$$R + \{\Lambda, Z\} - \dot{Z} = [R], \tag{B.8}$$

where

$$[R] := \sum_{j\in\mathbb{N}} R^-_{jj}\xi_j\eta_j$$

then

$$|Lie_Z\Lambda - \Lambda - \{\Lambda, Z\}|_{r-\sigma} \leq 4\beta_1^{-2} \, |Z|_{r-\sigma} \left(\frac{2\omega_M}{\sigma} \, |Z|_{r-\sigma/2} + |R|_{r-\sigma} \right), \tag{B.9}$$

$$\begin{aligned} |Lie_Z\Lambda - \Lambda - \{\Lambda, Z\}|^{\mathcal{L}}_{r-\sigma} &\leq \frac{8}{\sigma}\beta_1^{-2} \, |Z|_{r-\sigma/2} \, |Z|_{r-\sigma} + 4\beta_1^{-2} \, |Z|_{r-\sigma} \, |R|^{\mathcal{L}}_r \\ &+ 4\beta_1^{-2} \, |Z|^{\mathcal{L}}_{r-\sigma} \, |R|_r + \frac{8\omega_M}{\sigma}\beta_1^{-2}(|Z|_{r-\sigma/2} \, |Z|^{\mathcal{L}}_{r-\sigma} + |Z|_{r-\sigma} \, |Z|^{\mathcal{L}}_{r-\sigma/2}). \end{aligned} \tag{B.10}$$

Proof. Since

$$X_{\mathrm{Lie}_Z F} = \phi^1_Z X_F \phi^{-1}_Z = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{d\varepsilon^k}\bigg|_{\varepsilon=1} \phi^{\varepsilon}_Z X_F \phi^{-\varepsilon}_Z = \sum_{k=0}^{\infty} \frac{1}{k!} X_{F_k},$$

where

$$F_0 := F, \quad F_{k+1} = \{Z, F_k\},$$

at least formally we have that

$$\mathrm{Lie}_Z F = \sum_{k=0}^{\infty} \frac{F_k}{k!}.$$

Therefore observing as in [13] that from (B.1) we have

$$|F_k|_{r-\sigma} \leq (2\beta_1^{-2} \, |Z|_{r-\sigma})^k \, |F|_r \, , \tag{B.11}$$

we obtain that if (B.3) holds, then we have that

$$|\mathrm{Lie}_Z F - F|_{r-\sigma} \leq (e^{2\beta_1^{-2}|Z|_{r-\sigma}} - 1) \, |F|_r \, . \tag{B.12}$$

Hence, since $e^x - 1 \leq 2x$ if $x \leq -(2W_{-1}(-1/(2\sqrt{e})) + 1)/2$ where $W_{-1}(x)$ is the Lambert W function (see [56]), by observing that $-(2W_{-1}(-1/(2\sqrt{e})) + 1)/2 \geq 5/4$, we get that, if (B.3) holds, from (B.12) we get

$$|\mathrm{Lie}_Z F - F|_{r-\sigma} \leq 4\beta_1^{-2} \, |Z|_{r-\sigma} \, |F|_r \, .$$

By proceeding as in Lemma B.2 of [13], by using (B.2), we obtain that

$$|F_k|^{\mathcal{L}}_{r-\sigma} \leq (2\beta_1^{-2} \, |Z|_{r-\sigma})^k \, |F|^{\mathcal{L}}_{r-\sigma} + 2\beta_1^{-2} \, |Z|^{\mathcal{L}}_{r-\sigma} \, k(2\beta_1^{-2} \, |Z|_{r-\sigma})^{k-1} \, |F|_{r-\sigma} \, , \tag{B.13}$$

which gives, by using (B.3),

$$\begin{aligned} |\mathrm{Lie}_Z F - F|^{\mathcal{L}}_{r-\sigma} &\leq (e^{2\beta_1^{-2}|Z|_{r-\sigma}} - 1) \, |F|^{\mathcal{L}}_r + 2\beta_1^{-2}e^{2\beta_1^{-2}|Z|_{r-\sigma}} \, |Z|^{\mathcal{L}}_{r-\sigma} \, |F|_r \\ &\leq 4\beta_1^{-2} \, |Z|_{r-\sigma} \, |F|^{\mathcal{L}}_r + 2\beta_1^{-2}(1 + 4\beta_1^{-2} \, |Z|_{r-\sigma}) \, |Z|^{\mathcal{L}}_{r-\sigma} \, |F|_r \\ &\leq 4\beta_1^{-2} \, |Z|_{r-\sigma} \, |F|^{\mathcal{L}}_r + 7\beta_1^{-2} \, |Z|^{\mathcal{L}}_{r-\sigma} \, |F|_r \, . \end{aligned}$$

By using Cauchy's integral formula we have that

$$\dot{Z}(\omega t, \omega) = \omega \partial_\phi Z|_{\phi=\omega t} = \frac{\omega}{2\pi i} \oint_{\gamma_\sigma(\omega t)} \frac{Z(\xi, \omega)}{(\xi - \omega t)^2} d\xi,$$

where $\gamma_\sigma(\omega t)$ is the circle in the complex plane $|z - \omega t| = \sigma/2$ in the clockwise direction. This yields that

$$|\dot{Z}|_{r-\sigma} \leq \frac{4\omega_M}{\sigma} |Z|_{r-\sigma/2}, \quad |\dot{Z}|^{\mathcal{L}}_{r-\sigma} \leq \frac{4}{\sigma} |Z|_{r-\sigma/2} + \frac{4\omega_M}{\sigma} |Z|^{\mathcal{L}}_{r-\sigma/2}. \tag{B.14}$$

Hence, we observe that if Z satisfies (B.8) then

$$|\{\Lambda, Z\}|_{r-\sigma} = |\dot{Z} + [R] - R|_{r-\sigma} \leq \frac{4\omega_M}{\sigma} |Z|_{r-\sigma/2} + 2 |R|_r =: b,$$
$$|\{\Lambda, Z\}|^{\mathcal{L}}_{r-\sigma} = |\dot{Z} + [R] - R|^{\mathcal{L}}_{r-\sigma} \leq \frac{4}{\sigma} |Z|_{r-\sigma/2} + \frac{4\omega_M}{\sigma} |Z|^{\mathcal{L}}_{r-\sigma/2} + 2 |R|^{\mathcal{L}}_r =: b^{\mathcal{L}}.$$

Therefore, following the procedure of Lemma B.3 of [13], we obtain that, if we set $\mu := 2\beta_1^{-2} |Z|_{r-\sigma}$, by using (B.11) and (B.13),

$$|\mathrm{Lie}_Z\Lambda - \Lambda - \{\Lambda, Z\}|_{r-\sigma} \leq \sum_{k=2}^{\infty} \frac{1}{k!} |\Lambda_k|_{r-\sigma} \leq \sum_{k=2}^{\infty} \frac{\mu^{k-1}}{k!} |\Lambda_1|_{r-\sigma} \leq \frac{e^\mu - \mu - 1}{\mu} b,$$

$$|\mathrm{Lie}_Z\Lambda - \Lambda - \{\Lambda, Z\}|^{\mathcal{L}}_{r-\sigma} \leq \sum_{k=2}^{\infty} \frac{1}{k!} |\Lambda_k|^{\mathcal{L}}_{r-\sigma} \leq \sum_{k=2}^{\infty} \frac{\mu^{k-1}}{k!} |\Lambda_1|^{\mathcal{L}}_{r-\sigma} + 2 |Z|^{\mathcal{L}}_{r-\sigma} \sum_{k=2}^{\infty} \frac{(k-1)\mu^{k-2}}{k!} |\Lambda_1|_{r-\sigma}$$

$$\leq \frac{e^\mu - \mu - 1}{\mu} b^{\mathcal{L}} + 2 |Z|^{\mathcal{L}}_{r-\sigma} \frac{e^\mu(\mu - 1) + 1}{\mu^2} b.$$

By observing that if (B.3) holds then

$$e^\mu - 1 - \mu \leq \mu^2, \qquad e^\mu(\mu - 1) + 1 \leq \mu^2, \tag{B.15}$$

we get (B.9) and (B.10). To conclude, (B.6) and (B.7) follow exactly as in [13, Lemma B.4] by proceeding as above, observing that

$$Y_Z = -\sum_{k \geq 0} \frac{1}{(k+1)!} Y_k, \qquad \text{where} \quad Y_0 = \dot{Z}, \quad Y_{k+1} = \{Z, Y_k\}$$

and by using the inequalities in (B.11), (B.13) and (B.15).

□

Corollary B.3. *Let $Z, F \in \mathcal{B}^2_r(\Omega)$. If* (B.3) *holds, then*

$$|Y_Z + \dot{Z}|_{r-\sigma} \leq \frac{8\beta_1^{-2}\omega_M}{\sigma} |Z|_{r-\sigma} |Z|_{r-\sigma/2},$$
$$|Y_Z + \dot{Z}|^{\mathcal{L}}_{r-\sigma} \leq \beta_1^{-2} \left(\frac{8}{\sigma} |Z|_{r-\sigma/2} |Z|_{r-\sigma} + \frac{8\omega_M}{\sigma} |Z|_{r-\sigma} |Z|^{\mathcal{L}}_{r-\sigma/2} + \frac{8\omega_M}{\sigma} |Z|_{r-\sigma/2} |Z|^{\mathcal{L}}_{r-\sigma} \right). \tag{B.16}$$

Proof. The inequalities in (B.16) immediately follow from (B.6) and (B.7) by using (B.14). □

Remark B.4. *By reasoning as in Proposition B.2 and Corollary B.3, we have that if $\{Z, F\}$, $\{Z, \dot{Z}\}$, $\{Z, \{Z, \Lambda\}\} \in \mathcal{B}^{h,k}_r(\Omega)$ for some $h, k \geq 0$, then the Hamiltonians*

$$\mathit{Lie}_Z F - F, \quad \mathit{Lie}_Z\Lambda - \Lambda - \{\Lambda, Z\}, \quad Y_Z + \dot{Z}$$

belong to $\mathcal{B}^{h,k}_{r-\sigma}(\Omega)$ for any $\sigma \in (0, r)$.

Lemma B.5. *Let $Z \in \mathcal{B}^0_r(\Omega)$ be a bounded and diagonal operator and let $A, B \in \mathcal{B}^2(\Omega)$ be such that $\{Z, A\}, \{Z, \{Z, B\}\} \in \mathcal{B}^2_r(\Omega)$. Then if $|Z|_{0,r-\sigma} < 5/8$,*

$$
\begin{aligned}
&|\mathit{Lie}_Z A - A|_{r-\sigma} \leq 2\,|\{Z, A\}|_{r-\sigma} \\
&|\mathit{Lie}_Z B - B - \{Z, B\}|_{r-\sigma} \leq |\{Z, \{Z, B\}\}|_{r-\sigma} \\
&|\mathit{Lie}_Z A - A|^{\mathcal{L}}_{r-\sigma} \leq 2\,|\{Z, A\}|^{\mathcal{L}}_{r-\sigma} + 2\,|Z|^{\mathcal{L}}_{0,r-\sigma}\,|\{Z, A\}|_{r-\sigma}\,, \\
&|\mathit{Lie}_Z B - B - \{Z, B\}|^{\mathcal{L}}_{r-\sigma} \leq |\{Z, \{Z, B\}\}|^{\mathcal{L}}_{r-\sigma} + |Z|^{\mathcal{L}}_{0,r-\sigma}\,|\{Z, \{Z, B\}\}|_{r-\sigma}
\end{aligned}
$$

Moreover, if $\{Z, \dot{Z}\} \in \mathcal{B}^2_r(\Omega)$, then

$$
|Y_Z + \dot{Z}|_{r-\sigma} \leq |\{Z, \dot{Z}\}|_{r-\sigma}, \qquad |Y_Z + \dot{Z}|^{\mathcal{L}}_{r-\sigma} \leq |\{Z, \dot{Z}\}|^{\mathcal{L}}_{r-\sigma} + |Z|^{\mathcal{L}}_{0,r-\sigma}\,|\{Z, \dot{Z}\}|_{r-\sigma}
$$

Proof. Since for any diagonal operator $A : H \to H$ where H is an Hilbert space with basis given by $\{e_1, \ldots, e_n, \ldots\}$, if for any $n \in \mathbb{N}$, $Ae_n = \alpha_n$, we have that

$$
\|A\|_{\mathcal{B}(H,H)} = \sup_n |\alpha_n|,
$$

it follows that for any $s \in \mathbb{R}$ we have that

$$
|Z|_{s,r} = |Z|_{0,r}\,.
$$

Therefore, by reasoning as in Lemma B.1, we get that

$$
\begin{aligned}
&|\{Z, F\}|_{0,r} \leq |Z|_{0,r}\,|F|_r \\
&|\{Z, F\}|^{\mathcal{L}}_{0,r} \leq (|Z|^{\mathcal{L}}_{0,r}\,|F|_r + |Z|_{0,r}\,|F|^{\mathcal{L}}_r)
\end{aligned}
$$

and by proceeding as in Proposition B.2 we get the thesis. □

C Estimates of an operator

Let us introduce the sequence $(\lambda_j)_{j\in\mathbb{N}} \subset \mathrm{Lip}(\Omega, \mathbb{R}_+)$. For any choice of $\omega, \omega' \in \Omega$ and for any real valued function $f = f(\omega)$, we introduce the notation

$$
\Delta f = f(\omega) - f(\omega').
$$

We suppose that there exists three positive constants $\mathcal{G}_0, \mathcal{K}_0$ and $\mathcal{K}_1$ such that

$$
\mathcal{K}_0|i - j| \leq |\lambda_i(\omega) - \lambda_j(\omega)| \leq \mathcal{G}_0|i - j|, \qquad \left|\frac{\Delta(\lambda_i - \lambda_j)}{\Delta\omega}\right| \leq \mathcal{K}_1.
$$

Moreover, we assume that $\mathcal{K}_0 \leq 1, \mathcal{K}_1 \leq 1$ and $\mathcal{G}_0 \leq 2$. For any operator $F \in \mathrm{Lip}(\Omega, C^\infty(S^1_r, \mathcal{B}(\ell^2, \ell^2)))$, by abusing the notation we define

$$
|F|_r := \sup_{\omega\in\Omega}\sup_{\phi\in S^1_r} \|F(\omega, \phi)\|_{\mathcal{B}(\ell^2,\ell^2)} \qquad |F|^{\mathcal{L}}_r := \sup_{\omega_1\neq\omega_2\in\Omega}\left|\frac{F(\omega_1) - F(\omega_2)}{\omega_1 - \omega_2}\right|_r.
$$

The following lemma holds

Lemma C.1. *Let $0 < r \leq 1$ and let $H \in C^\infty(S^1_r, \mathcal{B}(\ell^2, \ell^2))$. For any fixed $\mu > 0$, let $\omega \in \Omega_\mu$, where*

$$
\Omega_\mu = \left\{\omega \in \Omega : \forall k \in \mathbb{Z}, \forall i, j \in \mathbb{N},\ \text{s.t.}\ |k| + |i - j| \neq 0, \quad |k\omega + \lambda_i - \lambda_j| > \frac{\mu\langle i - j\rangle}{1 + |k|^\tau}\right\}
$$

for some $\tau \geq 1$ *and* $\mu < 1$, *where* $\langle x \rangle := \max(|x|, 1)$. *We introduce the operator* $W \in C^\infty(S^1_{r-\sigma}, \mathcal{B}(\ell^2, \ell^2))$ *defined by*

$$W_{ij}(z) := \sum_{\substack{k \in \mathbb{Z} \\ |k|+|i-j| \neq 0}} \frac{H_{ijk}}{k\omega + \lambda_i - \lambda_j} e^{ikz}.$$

Then, if $0 < \sigma \leq \sigma_*$ *for a certain constant* $\sigma_* = \sigma_*(\omega_m, \mu, \mathcal{K}_0, \tau) < r$, *there exists a positive constant* $C(\tau)$ *such that*

$$|W|_{r-\sigma} \leq \frac{C(\tau)}{2\sigma^\tau} |H|_r .$$

Proof. Let us consider $\omega \in \Omega_\mu$. For any $i, j \in \mathbb{N}$, fixed $c > 0$, let us define $\mathbb{Z}^{i,j}_\infty, \mathbb{Z}^{i,j}_\# \subseteq \mathbb{Z}$ by

$$\mathbb{Z}^{i,j}_\infty = \{k \in \mathbb{Z} : |k\omega + \lambda_i - \lambda_j| > c\sigma^{\tau-1}\},$$
$$\mathbb{Z}^{i,j}_\# = \left\{k \in \mathbb{Z} : |k\omega + \lambda_i - \lambda_j| \leq c\sigma^{\tau-1}, \quad |k| + |i-j| \neq 0\right\}.$$

It is immediate to observe that $\mathbb{Z}^{i,j}_\infty \cap \mathbb{Z}^{i,j}_\# = \varnothing$ and that $\mathbb{Z}^{i,j}_\infty \cup \mathbb{Z}^{i,j}_\# = \{k \in \mathbb{Z} : |k| + |i-j| \neq 0\}$. Therefore, we can write $W = W^\infty + W^\#$, where

$$W^\infty_{ij} = \sum_{k \in \mathbb{Z}} W^\infty_{ijk} e^{ikz}, \quad W^\#_{ij} = \sum_{k \in \mathbb{Z}} W^\#_{ijk} e^{ikz},$$

with

$$W^\infty_{ijk} = \begin{cases} \dfrac{H_{ijk}}{k\omega + \lambda_i - \lambda_j} & \text{if } k \in \mathbb{Z}^{i,j}_\infty \\ 0 & \text{if } k \notin \mathbb{Z}^{i,j}_\infty \end{cases}, \qquad W^\#_{ijk} = \begin{cases} \dfrac{H_{ijk}}{k\omega + \lambda_i - \lambda_j} & \text{if } k \in \mathbb{Z}^{i,j}_\# \\ 0 & \text{if } k \notin \mathbb{Z}^{i,j}_\# \end{cases}.$$

By denoting with $|\cdot|$ the $\mathcal{B}(\ell^2, \ell^2)$ norm, since for any $H \in \mathcal{B}(\ell^2, \ell^2)$ we have that

$$\sum_{i=1}^\infty |H_{ij}|^2 \leq |H| .$$

Hence, by using Cauchy-Schwarz inequality, if $c\sigma^{\tau-1} < 1$, then

$$\begin{aligned}\sum_{i=1}^\infty |W^\infty_{ijk}| &= \sum_{i : k \in \mathbb{Z}^{i,j}_\infty} \frac{|H_{ijk}|}{|k\omega + \lambda_i - \lambda_j|} \leq |H_k| \sqrt{\sum_{i : k \in \mathbb{Z}^{i,j}_\infty} \frac{1}{|k\omega + \lambda_i - \lambda_j|^2}} \\ &\leq |H_k| \sqrt{2 \sum_{i=0}^\infty \frac{1}{(c\sigma^{\tau-1} + \mathcal{K}_0 i)^2}} \leq \sqrt{2}\, |H_k| \sqrt{\frac{\zeta(2)}{\mathcal{K}_0{}^2} + \frac{1}{c^2\sigma^{2\tau-2}}} \leq \frac{\sqrt{2}\sqrt{\zeta(2)+1}}{\mathcal{K}_0 c\sigma^{\tau-1}} |H_k|\end{aligned}$$

where we used that, since $|k\omega + \lambda_i - \lambda_j| > c\sigma^{\tau-1}$, for any fixed $k \in \mathbb{Z}^{i,j}_\infty$ there exists $h \in \mathbb{Z}$ such that for any $i \in \mathbb{N}$, we have that $|k\omega + \lambda_i - \lambda_j| > c\sigma^{\tau-1} + \mathcal{K}_0(h+i)$, which yields

$$\begin{aligned}\sum_{i : k \in \mathbb{Z}^{i,j}_\infty} \frac{1}{|k\omega + \lambda_i - \lambda_j|^2} &\leq \sum_{i : k \in \mathbb{Z}^{i,j}_\infty} \frac{1}{(c\sigma^{\tau-1} + \mathcal{K}_0(h+i))^2} \leq \sum_{i \in \mathbb{Z}} \frac{1}{(c\sigma^{\tau-1} + \mathcal{K}_0(h+i))^2} \\ &\leq \frac{2}{c^2\sigma^{2\tau-2}} + \frac{2}{\mathcal{K}_0^2} \sum_{i=1}^\infty \frac{1}{i^2}.\end{aligned}$$

Therefore, by proceeding as in [139, Lemma A.1], we conclude that

$$|W_k^\infty| \leq \frac{\sqrt{2}\sqrt{\zeta(2)+1}}{c\mathcal{K}_0\sigma^{\tau-1}}\,|H_k|\,,$$

which implies that, if $z \in S^1_{r-\sigma}$,

$$\begin{aligned}|W^\infty(z)| &\leq \sum_{k\in\mathbb{Z}} |W_k^\infty|\,|e^{ikz}| \leq \frac{\sqrt{2}\sqrt{\zeta(2)+1}}{c\mathcal{K}_0\sigma^{\tau-1}}\sum_{k\in\mathbb{Z}}|H_k|\,|e^{ikz}|\\ &\leq \frac{\sqrt{2}\sqrt{\zeta(2)+1}}{c\mathcal{K}_0\sigma^{\tau-1}}\sum_{k\in\mathbb{Z}} e^{-|k|\sigma}\,|H|_r \leq \frac{5\sqrt{2}\sqrt{\zeta(2)+1}}{2c\mathcal{K}_0\sigma^{\tau}}\,|H|_r\,,\end{aligned}$$

where we used that from Cauchy's integral formula $|H_k| \leq |H|_r\, e^{-|k|r}$ and that

$$\sum_{k\in\mathbb{Z}_\infty} e^{-|k|\sigma} < \frac{5}{2\sigma}$$

for any $\sigma < 1$.

Next, we remark that, if we consider $z \in S^1_{r-\sigma}$,

$$\begin{aligned}|W_{ij}^\#(z)| &\leq \sum_{k\in\mathbb{Z}_\#^{i,j}} \frac{|H_{ijk}|}{|k\omega+\lambda_i-\lambda_j|}|e^{ikz}| \leq \frac{1}{\mu}\sum_{k\in\mathbb{Z}_\#^{i,j}}\frac{|H_{ijk}|}{\langle i-j\rangle}(1+|k|^\tau)|e^{ikz}|\\ &\leq \frac{1}{\mu}\sum_{k\in\mathbb{Z}_\#^{i,j}}\frac{|H_k|}{\langle i-j\rangle}(1+|k|^\tau)|e^{ikz}| \leq \frac{1}{\mu\langle i-j\rangle}\,|H|_r\sum_{k\in\mathbb{Z}_\#^{i,j}}(1+|k|^\tau)e^{-\sigma|k|},\end{aligned}$$

where we used that

$$|H_{ijk}| \leq |H_k| \leq |H|_r\, e^{-|k|r}.$$

Hence,

$$\begin{aligned}\sum_{j=1}^{\infty}|W_{ij}^\#| &\leq \frac{1}{\mu}\,|H|_r\sum_{j=1}^{\infty}\frac{1}{\langle i-j\rangle}\sum_{k\in\mathbb{Z}_\#^{i,j}}(1+|k|)^\tau e^{-\sigma|k|} \overset{\sigma<1}{\leq}\\ &\leq \frac{1}{\mu\sigma^{\tau+1}}\,|H|_r\sum_{j=1}^{\infty}\frac{1}{\langle i-j\rangle}\sum_{k\in\mathbb{Z}_\#^{i,j}}(1+|\sigma k|^\tau)e^{-\sigma|k|}\sigma.\end{aligned} \tag{C.1}$$

We remark that, if $k \in \mathbb{Z}_\#^{i,j}$ we have that

$$\frac{\mathcal{K}_0|i-j|-c\sigma^{\tau-1}}{\omega} \leq \frac{|\lambda_i-\lambda_j|-c\sigma^{\tau-1}}{\omega} \leq |k| \leq \frac{|\lambda_i-\lambda_j|+c\sigma^{\tau-1}}{\omega} \leq \frac{\mathcal{G}_0|i-j|+c\sigma^{\tau-1}}{\omega} \tag{C.2}$$

Therefore, if $c\sigma^{\tau-1}/\omega < 1/2$, that is

$$\sigma < \left(\frac{\omega}{2c}\right)^{\frac{1}{\tau-1}}, \tag{C.3}$$

then for any $i \in \mathbb{N}$ we have $\mathbb{Z}_\#^{i,i} = \varnothing$, which yields that we can write

$$\sum_{j=1}^{\infty}\frac{1}{\langle i-j\rangle}\sum_{k\in\mathbb{Z}_\#^{i,j}}(1+|\sigma k|^\tau)e^{-\sigma|k|}\sigma = \sum_{j=1}^{\infty}\frac{1}{|i-j|}\sum_{k\in\mathbb{Z}_\#^{i,j}}(1+|\sigma k|^\tau)e^{-\sigma|k|}\sigma. \tag{C.4}$$

Moreover, since for any $k \in \mathbb{Z}_{\#}^{i,j}$,

$$\frac{\mu|i-j|}{1+|k|^{\tau}} \leq c\sigma^{\tau-1},$$

if we require $c\sigma^{\tau-1} < \mu/2$, that is

$$\sigma < \left(\frac{\mu}{2c}\right)^{\frac{1}{\tau-1}},$$

we get

$$|k|^{\tau} \geq \frac{\mu|i-j| - c\sigma^{\tau-1}}{c\sigma^{\tau-1}} \geq \frac{\mu}{2c}\frac{|i-j|}{\sigma^{\tau-1}}.$$

Hence, if $\mathbb{Z}_{\#}^{i,j} \neq \varnothing$, from (C.2) since $\mathcal{G}_0 \leq 2$, if $c\sigma^{\tau-1} < 1$,

$$\frac{\mu}{2c}\frac{|i-j|}{\sigma^{\tau-1}} \leq |k|^{\tau} \leq \left(\frac{\mathcal{G}_0|i-j| + c\sigma^{\tau-1}}{\omega}\right)^{\tau} \leq \frac{3^{\tau}}{\omega}|i-j|^{\tau},$$

that gives

$$\mathbb{Z}_{\#}^{i,j} \neq \varnothing \implies |i-j| \geq \left(\frac{\mu\omega}{3^{\tau}2c}\right)^{1/(\tau-1)}\frac{1}{\sigma} =: \frac{K}{\sigma}.$$

Therefore, from (C.4) we get

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{1}{\langle i-j\rangle}\sum_{k\in\mathbb{Z}_{\#}^{i,j}}(1+|\sigma k|^{\tau})e^{-\sigma|k|}\sigma &\leq \sum_{j:|i-j|>K/\sigma}\frac{1}{|i-j|}\sum_{k\in\mathbb{Z}_{\#}^{i,j}}(1+|\sigma k|^{\tau})e^{-\sigma|k|}\sigma \\ &\leq \left(\frac{3^{\tau}2c}{\mu\omega}\right)^{1/(\tau-1)}\sigma\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}_{\#}^{i,j}}(1+|\sigma k|^{\tau})e^{-\sigma|k|}\sigma\end{aligned} \tag{C.5}$$

We remark that from (C.3), (C.2) gives that $\#\mathbb{Z}_{\#}^{i,j} \leq 1$. Let $\kappa_{i,j}$ be the unique integer such that $\kappa_{i,j} \in [(|\lambda_i - \lambda_j| - c\sigma^{\tau-1})/\omega, (|\lambda_i - \lambda_j| + c\sigma^{\tau-1})/\omega]$. Hence, since for any fixed i there exist at most two integers j_1 and j_2 such that $\kappa_{i,j_1} = \kappa_{i,j_2}$ we can estimate

$$\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}_{\#}^{i,j}}(1+|\sigma k|^{\tau})e^{-\sigma|k|}\sigma \leq \sum_{j=1}^{\infty}f(\sigma\kappa_{i,j})\sigma \leq 2\sum_{k=0}^{\infty}f_{\tau}(\sigma k)\,\sigma, \tag{C.6}$$

where $f_{\tau}(x) = (1+|x|^{\tau})e^{-|x|}$. Moreover, from Proposition D.3 if $\sigma < \sigma_f$ for a certain $\sigma_f > 0$, we have that

$$2\sum_{k=0}^{\infty}f_{\tau}(\sigma k)\,\sigma \leq 3\int_0^{\infty}f_{\tau}(x)dx =: \alpha_{\tau}. \tag{C.7}$$

Thus, from (C.1) and (C.5), by using (C.6) and (C.7) we obtain that

$$\begin{aligned}\sum_{j=1}^{\infty}|W_{ij}^{\#}| &\leq \frac{1}{\mu\sigma^{\tau+1}}\left|H\right|_r\sum_{j=1}^{\infty}\frac{1}{\langle i-j\rangle}\sum_{k\in\mathbb{Z}_{\#}^{i,j}}(1+|\sigma k|^{\tau})e^{-\sigma|k|}\sigma \\ &\leq \frac{1}{\sigma^{\tau}}\left|H\right|_r\left(\frac{3^{\tau}2c}{\mu^{\tau}\omega}\right)^{1/(\tau-1)}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}_{\#}^{i,j}}(1+|\sigma k|^{\tau})e^{-\sigma|k|}\sigma \leq \left(\frac{3^{\tau}2c}{\mu^{\tau}\omega}\right)^{1/(\tau-1)}\alpha_{\tau}\frac{1}{\sigma^{\tau}}\left|H\right|_r,\end{aligned}$$

which yields that, by proceeding as in [139, Lemma A.1],

$$\left|W^{\#}\right|_{r-\sigma} \leq \left(\frac{3^\tau 2c}{\mu^\tau \omega}\right)^{1/(\tau-1)} \alpha_\tau \frac{1}{\sigma^\tau} |H|_r .$$

Hence, if we set

$$c = \frac{\mu\omega_m^{1/\tau}}{2^{1/\tau}3}\left(\frac{5\sqrt{2}\sqrt{\zeta(2)+1}(\tau-1)}{\mathcal{K}_0\alpha_\tau}\right)^{\frac{\tau-1}{\tau}}$$

we obtain that

$$|W|_{r-\sigma} \leq \frac{2^{1/\tau}3\alpha_\tau^{(\tau-1)/\tau}}{\mu\omega_m^{1/\tau}}\left(\frac{1}{(\tau-1)^{(\tau-1)/\tau}} + (\tau-1)^{1/\tau}\right)\left(\frac{5\sqrt{2}\sqrt{\zeta(2)+1}}{\mathcal{K}_0}\right)^{1/\tau}\frac{1}{\sigma^\tau}|H|_r .$$

□

Lemma C.2. *Let* $0 < \sigma_{\mathcal{L}} < r \leq 1$ *and let* $H \in Lip(\Omega_\mu, C^\infty(S^1_r, \mathcal{B}(\ell^2, \ell^2)))$. *There exists a positive* $\sigma_{\mathcal{L}} = \sigma_*(\omega_m, \mu, \mathcal{K}_0, \tau) < r$ *such if* $\sigma < \sigma_{\mathcal{L}}$, *there exists a positive constant* $C^{\mathcal{L}}(\tau)$ *such that*

$$|W|^{\mathcal{L}}_{r-\sigma} \leq \frac{C(\tau)}{2\sigma^\tau}|H|^{\mathcal{L}}_r + \frac{C^{\mathcal{L}}(\tau)}{2\sigma^{2\tau}}|H|_r .$$

Proof. By reasoning as in [13, Lemma 6.2] we have that

$$\begin{aligned}
|\Delta W_{ijk}| &= \left|\frac{\Delta H_{ijk}}{\omega k + \lambda_i - \lambda_j}\right| + \left|\frac{k\Delta\omega + \Delta(\lambda_i - \lambda_j)}{(\omega k + \lambda_i(\omega) - \lambda_j(\omega))(\omega' k + \lambda_i(\omega') - \lambda_j(\omega'))}H_{ijk}\right| \\
&\leq \left|\frac{\Delta H_{ijk}}{\omega k + \lambda_i - \lambda_j}\right| + \frac{(k+\mathcal{K}_1)|\Delta\omega|}{|\omega k + \lambda_i(\omega) - \lambda_j(\omega)||\omega' k + \lambda_i(\omega') - \lambda_j(\omega')|}|H_{ijk}| \\
&= |W_{1,ijk}| + |W_{2,ijk}||\Delta\omega|
\end{aligned}$$

where we set

$$W_{1,ijk} := \frac{\Delta H_{ijk}}{\omega k + \lambda_i(\omega) - \lambda_j(\omega)}, \quad W_{2,ijk} = \frac{k+\mathcal{K}_1}{(\omega k + \lambda_i(\omega) - \lambda_j(\omega))(\omega' k + \lambda_i(\omega') - \lambda_j(\omega'))}H_{ijk}.$$

From Lemma C.1 we get that

$$|W_1|_{r-\sigma} \leq \frac{C(\tau)}{2\sigma^\tau}|\Delta H|_r .$$

In order to estimate $|W_2|_{r-\sigma}$, we introduce the sets

$$\begin{aligned}
\mathbb{Z}^{i,j}_\infty &= \{k \in \mathbb{Z} : \min(|k\omega + \lambda_i - \lambda_j|, |k\omega' + \lambda_i - \lambda_j|) > c\sigma^{\tau-1}\}, \\
\mathbb{Z}^{i,j}_\# &= \left\{k \in \mathbb{Z} : \max(|k\omega + \lambda_i - \lambda_j|, |k\omega' + \lambda_i - \lambda_j|) \leq c\sigma^{\tau-1}, \quad |k| + |i-j| \neq 0\right\}, \\
\mathbb{Z}^{i,j}_\chi &= \mathbb{Z} \setminus (\mathbb{Z}^{i,j}_\# \cup \mathbb{Z}^{i,j}_\infty).
\end{aligned}$$

By proceeding with the same notations of Lemma C.1, we write $W_2 = W_2^\infty + W_2^\# + W_2^\chi$. The estimates of $|W_2^\infty|_{r-\sigma}$ and of $\left|W_2^\#\right|_{r-\sigma}$ are obtained by proceeding as in Lemma C.1. In particular, we have that

$$\sum_{i=1}^\infty |W^\infty_{2,ijk}| \leq |H_k|\sqrt{\frac{2}{c^4\sigma^{4\tau-4}} + 2\sum_{i=1}^\infty \frac{1}{(c\sigma^{\tau-1} + \mathcal{K}_0 i)^4}} \leq \frac{\sqrt{2}\sqrt{\zeta(4)+1}}{c\mathcal{K}_0^2\sigma^{2\tau-2}}|H_k| .$$

Hence, if $z \in S^1_{r-\sigma}$,

$$|W_2^{\infty}(z)| \leq \frac{\sqrt{2}\sqrt{\zeta(4)+1}}{c\mathcal{K}_0^2\sigma^{2\tau-2}} \sum_{k\in\mathbb{Z}} e^{-|k|\sigma} |H|_r \leq \frac{5\sqrt{2}\sqrt{\zeta(4)+1}}{2c\mathcal{K}_0^2\sigma^{2\tau-1}} |H|_r \,.$$

Next we have that, by requiring $\sigma < \sigma_*$ for a certain $\sigma_* > 0$,

$$\begin{aligned}\sum_{j=1}^{\infty} |W^{\#}_{2,ij}| &\leq \frac{1}{\mu\sigma^{2\tau+2}} |H|_r \sum_{j:|i-j|>K/\sigma} \frac{1}{|i-j|^2} g_\tau(\sigma\xi_{i,j})\sigma \\ &\leq \left(\frac{3^\tau 2c}{\mu\omega}\right)^{2/(\tau-1)} \frac{1}{\mu^2\sigma^{2\tau}} |H|_r \sum_{j=1}^{\infty} g(\sigma\xi_{i,j})\sigma \leq \left(\frac{3^\tau 2c}{\mu^\tau\omega}\right)^{2/(\tau-1)} \lambda_\tau \frac{1}{\sigma^{2\tau}} |H|_r \,,\end{aligned}$$

where we now set

$$g_\tau(x) := (1+|x|)(1+|x|^\tau)^2 e^{-|x|}, \quad \lambda_\tau := 3\int_0^\infty g_\tau(x)dx.$$

Therefore, we obtain that

$$|W_2^{\#}|_{r-\sigma} \leq \left(\frac{3^\tau 2c}{\mu^\tau\omega}\right)^{2/(\tau-1)} \lambda_\tau \frac{1}{\sigma^{2\tau}} |H|_r \,.$$

Lastly, we observe that $\mathbb{Z}^{i,j}_{\chi} = \mathbb{Z}^{i,j}_{\chi,1} \cup \mathbb{Z}^{i,j}_{\chi,2}$, where

$$\begin{aligned}\mathbb{Z}^{i,j}_{\chi,1} &:= \{k \in \mathbb{Z} : |k\omega' + \lambda_i - \lambda_j| > c\sigma^{\tau-1}, \quad |k\omega + \lambda_i - \lambda_j| \leq c\sigma^{\tau-1}\}, \\ \mathbb{Z}^{i,j}_{\chi,2} &:= \{k \in \mathbb{Z} : |k\omega' + \lambda_i - \lambda_j| \leq c\sigma^{\tau-1}, \quad |k\omega + \lambda_i - \lambda_j| > c\sigma^{\tau-1}\}.\end{aligned}$$

Then, we have that $W_2^{\chi} = W_2^{\chi,1} + W_2^{\chi,2}$, where

$$W^{\chi,h}_{2,ij} = \sum_{k\in\mathbb{Z}^{i,j}_{\chi,h}} \frac{k+\mathcal{K}_1}{(\omega k + \lambda_i(\omega) - \lambda_j(\omega))(\omega' k + \lambda_i(\omega') - \lambda_j(\omega'))} H_{ijk}.$$

We have that,

$$|W^{\chi,1}_{2,ij}(z)| \leq \frac{1}{c\sigma^{\tau-1}} \sum_{k\in\mathbb{Z}^{i,j}_{\chi,1}}^{\infty} \frac{1+|k|}{|\omega k + \lambda_i - \lambda_j|} |H_{ijk}| \leq \frac{1}{\mu c\sigma^{\tau-1}} \frac{1}{\langle i-j\rangle} |H|_r \sum_{k\in\mathbb{Z}^{i,j}_{1,\chi}}^{\infty} (1+|k|)(1+|k|^\tau)e^{-\sigma|k|},$$

which yields, by proceeding as in Lemma C.1 and requiring $\sigma < \sigma_{**}$ for a certain $\sigma_{**} > 0$

$$\sum_{j=1}^{\infty} |W^{\chi}_{2,ij}| \leq \frac{2}{\mu c\sigma^{2\tau+1}} \sum_{j:|i-j|>K/\sigma}^{\infty} \frac{1}{|i-j|} h_\tau(\sigma\kappa_{i,j})\sigma \leq \left(\frac{3^\tau 2c}{\mu^\tau\omega}\right)^{1/(\tau-1)} 2\psi_\tau \frac{1}{\sigma^{2\tau}} |H|_r \,, \quad \text{(C.8)}$$

where we set

$$h_\tau(x) = (1+|x|)(1+|x|^\tau)e^{-|x|}, \qquad \psi_\tau := 3\int_0^\infty h_\tau(x)dx.$$

Inequality (C.8) gives

$$|W_2^{\chi}|_{r-\sigma} \leq \left(\frac{3^\tau 2c}{\mu^\tau\omega}\right)^{1/(\tau-1)} 2\psi_\tau \frac{1}{\sigma^{2\tau}} |H|_r \,.$$

Hence, since for any $\omega \neq \omega'$, setting $\rho_\tau := \lambda_\tau + \psi_\tau$,

$$|\Delta W|_{r-\sigma} \leq |W_1|_{r-\sigma} + (|W_2^\infty|_{r-\sigma} + |W_2^\#|_{r-\sigma} + |W_2^\chi|_{r-\sigma})|\Delta\omega|$$
$$\leq \frac{C(\tau)}{2\sigma^\tau}|\Delta H|_r + \left(\frac{5\sqrt{2}\sqrt{\zeta(4)+1}}{2\mathcal{K}_0^2 c} + \left(\frac{3^\tau 2c}{\mu^\tau\omega}\right)^{1/(\tau-1)}\rho_\tau\right)\frac{1}{\sigma^{2\tau}}|\Delta\omega|\,|H|_r$$

if we set

$$c = \frac{\mu\omega_m^{1/\tau}}{2^{1/\tau}3}\left(\frac{5\sqrt{2}\sqrt{\zeta(4)+1}(\tau-1)}{2\mathcal{K}_0^2\rho_\tau}\right)^{\frac{\tau-1}{\tau}},$$

for any $\sigma < \sigma_{\mathcal{L}} := \min(\sigma_*, \sigma_{**})$, we obtain that

$$|W|^{\mathcal{L}}_{r-\sigma} \leq \frac{C(\tau)}{2\sigma^\tau}|H|^{\mathcal{L}}_r + \frac{C^{\mathcal{L}}(\tau)}{2\sigma^{2\tau}}|H|_r\,,$$

where

$$C^{\mathcal{L}}(\sigma) := \frac{3\rho_\tau^{\frac{\tau-1}{\tau}}}{\mu\omega_m^{1/\tau}}\left(\frac{1}{(\tau-1)^{(\tau-1)/\tau}} + (\tau-1)^{1/\tau}\right)\left(\frac{5\sqrt{2}\sqrt{\zeta(4)+1}}{\mathcal{K}_0^2}\right)^{1/\tau}.$$

□

Lemma C.3. *Let $\sigma = 1/2^n$ for some $n \geq N(\omega_m)$ and let $\mu < \omega_m$, $\sigma < r \leq 1$ and $\mathcal{K}_0 \geq 1/4$. Let $H \in \mathcal{B}_r^{h,k}(\Omega_\mu)$ with $h, k > 0$, where*

$$\Omega_\mu = \left\{\omega \in \Omega : \forall k \in \mathbb{Z}, \forall i,j \in \mathbb{N},\ \textit{s.t.}\ |k| + |i-j| \neq 0, \quad |k\omega + \lambda_i \pm \lambda_j| > \frac{\mu\langle i \pm j\rangle}{1+|k|^4}\right\}$$

for some $\mu < 1$, where $\langle x\rangle := \max(|x|, 1)$. We introduce the Hamiltonian function $W \in \mathcal{B}^0(\Omega_\mu)$ defined by

$$W_{ij}^{\pm}(z) := \sum_{\substack{k\in\mathbb{Z}\\ |k|+|i-j|\neq 0}} \frac{H_{ijk}^{\pm}}{k\omega + \lambda_i - \lambda_j}e^{ikz}.$$

Then we have that

$$|W|_{r-\sigma} \leq \frac{\mathfrak{C}}{\sigma^4}|H|_r\,, \quad |W|_{r-\sigma} \leq \frac{\mathfrak{C}}{\sigma^4}|H|^{\mathcal{L}}_r + \frac{\mathfrak{C}^{\mathcal{L}}}{\sigma^8}|H|_r\,.$$

where $\mathfrak{C} := C(4)$ and $\mathfrak{C}^{\mathcal{L}} := C^{\mathcal{L}}(4)$ are defined in Lemma C.1 and C.2.

Proof. We remark that $F \in \mathcal{B}(\mathcal{H}^{1/2}, \mathcal{H}^{1/2+h})$ if and only if the operator $\mathcal{F}$ with matrix $F_{ij} = i^{1/2+h}F_{ij}j^{-1/2}$ belongs to $\mathcal{B}(\ell^2, \ell^2)$. Therefore, from Lemma C.1 we have that if $\sigma < \sigma_*$, then

$$\left|W^-\right|_{r-\sigma} \leq \frac{\mathfrak{C}}{2\sigma^4}\left|H^-\right|_r \leq \frac{\mathfrak{C}}{2\sigma^4}|H|_r\,. \tag{C.9}$$

More precisely, by recollecting the conditions on σ that we required in the proof of Lemma C.1, we have that if

$$c_1\sigma^3 < 1, \quad \sigma < 1, \quad c_1\sigma^3 < \frac{\omega}{2}, \quad c_1\sigma^3 < \frac{\mu}{2},$$

where

$$c_1 := \frac{\mu\omega_m^{1/4}}{2^{1/4}3}\left(\frac{15\sqrt{2}\sqrt{\zeta(2)+1}}{\mathcal{K}_0\alpha_4}\right)^{\frac{3}{4}},$$

which are equivalent to, since $\mu < \omega_m$ and $\mu < 1$,

$$c_1\sigma^3 < \frac{\mu}{2},$$

which is implied by, since $\mathcal{K}_0 \geq 1/4$,

$$\sigma^3 \leq \frac{1.13}{\omega_m^{1/4}} \tag{C.10}$$

then (C.9) holds. We remark that (C.10) and $\sigma \leq 1/4$ hold with $\sigma = 1/2^n$ if and only if

$$n \geq \max(\log_2(\omega_m^{1/4}/1.13), 2) =: N(\omega_m).$$

By proceeding as in the proof of Lemma C.1, it is possible to show that if $n \geq N(\omega_m)$ then

$$\left|W^+\right|_{r-\sigma} \leq \frac{\mathfrak{C}}{2\sigma^4}\left|H^+\right|_r \leq \frac{\mathfrak{C}}{2\sigma^4}\left|H\right|_r,$$

which yields that, together with (C.9),

$$|W|_{r-\sigma} \leq \left|W^-\right|_{r-\sigma} + \left|W^+\right|_{r-\sigma} \leq \frac{\mathfrak{C}}{\sigma^4}|H|_r.$$

To conclude, by reasoning as above, from Lemma C.2 we obtain that if (C.10) holds, then

$$|W|_{r-\sigma} \leq \frac{\mathfrak{C}}{\sigma^4}|H|_r^{\mathcal{L}} + \frac{\mathfrak{C}^{\mathcal{L}}}{\sigma^8}|H|_r.$$

□

D Some technical lemmas

Lemma D.1. *Let $f(x) \in L^1(\mathbb{R}_+,\mathbb{R}_+) \cap C^1(\mathbb{R}_+,\mathbb{R}_+)$ and let $\sigma \in (0,1)$ be such that*

$$\max_{x\in\sigma[k,k+1]}|f'(x)| \leq \frac{2}{3\sigma}\min_{x\in\sigma[k,k+1]} f(x), \quad \forall k \geq 0. \tag{D.1}$$

Then

$$\sum_{k=0}^{\infty} f(\sigma k)\sigma \leq \frac{3}{2}\int_0^{\infty} f(x)dx.$$

Proof. By using Lagrange's theorem, if we set $I_k = [\sigma k, \sigma(k+1)]$, we get

$$\begin{aligned}\left|\sum_{k=0}^{\infty}\left(f(\sigma k)\sigma - \int_{I_k} f(x)dx\right)\right| &\leq \sum_{k=0}^{\infty}\int_{I_k}|f(\sigma k) - f(x)|dx \leq \sum_{k=0}^{\infty}\max_{y\in I_k}|f'(y)|\int_{I_k}|x-\sigma k|\mathrm{d}x\\ &\leq \sum_{k=0}^{\infty}\frac{2}{3\sigma}\min_{y\in I_k} f(y)\frac{\sigma^2}{2} \leq \frac{1}{3}\sum_{k=0}^{\infty} f(\sigma k)\sigma\end{aligned}$$

which yields

$$\sum_{k=0}^{\infty} f(\sigma k)\sigma \leq \frac{3}{2}\sum_{k=0}^{\infty}\int_{I_k} f(x)dx = \frac{3}{2}\int_0^{\infty} f(x)dx.$$

□

Lemma D.2. *Let $p(x)$ be a strictly positive polynomial function such that* $\lim_{x\to+\infty} p(x) = +\infty$ *and let $f(x) := p(x)e^{-x}$. Then there exists $\sigma_f > 0$ such that for any $\sigma < \sigma_f$,* (D.1) *holds.*

Proof. Let $M > 0$ be such that f and $|f'|$ are decreasing, $p(x)$ is increasing and $p(x) > p'(x)$ on $[M, +\infty)$. For M large enough, we have that

$$\frac{p(y) - p'(y)}{p(y+\sigma)} \leq \frac{p(y) - p'(y)}{p(y)} \leq 1,$$

which yields that, for σ such that for any $\sigma < 2e^{-\sigma}/3$, i.e. for any $\sigma < W(2/3) \approx 0.43$ where W is the Lambert function,

$$p(y) - p'(y) \leq \frac{2}{3}\frac{e^{-\sigma}}{\sigma} p(y+\sigma). \tag{D.2}$$

Since for $x \in (M, +\infty)$, $|f'(x)| = (p(x) - p'(x))e^{-x}$, it is immediate to verify that for any $I = [y, y+\sigma] \subset [M, +\infty)$, if $\sigma < 3e^{-\sigma}/2$ from (D.2) we have that

$$\max_{x\in I} |f'(x)| = (p(y) - p'(y))e^{-y} < \frac{2}{3}\frac{1}{\sigma} p(y+\sigma)e^{-y-\sigma} = \frac{2}{3\sigma} \min_{x\in I} f(x).$$

By considering

$$\sigma \leq \frac{2}{3}\text{argmin}_{\sigma\leq 0.43} \min_{\substack{I\subseteq[0,M]\\ |I|=\sigma}} \frac{\min_{x\in I} f(x)}{\max_{x\in I} |f'(x)|} =: \sigma_f, \tag{D.3}$$

we get that for any $I \subset [0, M]$ such that $|I| < \sigma_f$

$$\max_{x\in I} |f'(x)| \leq \frac{2}{3\sigma} \min_{x\in I} |f(x)|.$$

Therefore, if $\sigma < \sigma_f$, then (D.1) holds for any interval $I \subset \mathbb{R}$ such that $|I| = \sigma$, that is the thesis □

By combining Lemma D.1 and Lemma D.2, we get the following proposition.

Proposition D.3. *Let $p(x)$ be a strictly positive polynomial function such that $\lim_{x\to+\infty} p(x) = +\infty$ and let $f(x) := p(x)e^{-x}$. Then there exists $\sigma_f > 0$ depending on f such that for any $\sigma < \sigma_f$ we have that*

$$\sum_{k=0}^{\infty} f(\sigma k)\sigma \leq \frac{3}{2}\int_0^{+\infty} f(x)dx.$$

Corollary D.4. *Let us consider the functions*

$$f_1(x) = (1+x^4)e^{-x}, \quad f_2(x) = (1+x)(1+x^4)e^{-x}, \quad f_3(x) = (1+x)(1+x^4)^2e^{-x}.$$

Then, for any $n \geq 2$ and for any $j = 1, 2, 3$, we have that

$$\sum_{k=0}^{\infty} f_j\left(\frac{k}{2^n}\right)\frac{k}{2^n} \leq \frac{3}{2}\int_0^{+\infty} f_j(x)dx.$$

Proof. The thesis immediately follows from Proposition D.3 by computing σ_{f_j} from (D.3). Indeed, in the notation used in the proof of Lemma D.2, if we take $M_j = \max(S_j)$ where S is the set of the positive zeros of f_j, it is possible to verify by using a software that $\sigma_{f_1} \geq 1/4$, $\sigma_{f_2} \geq 1/4$ and $\sigma_{f_3} \in [1/16, 1/8)$. Nevertheless, since

$$\sum_{k=0}^{\infty} f_3\left(\frac{k}{4}\right)\frac{k}{4} \leq \frac{3}{2}\int_0^{+\infty} f_3(x)dx, \quad \sum_{k=0}^{\infty} f_3\left(\frac{k}{8}\right)\frac{k}{8} \leq \frac{3}{2}\int_0^{+\infty} f_3(x)dx,$$

we get the thesis for any $j = 1, 2, 3$ and for any $n \geq 2$.

□

Lemma D.5. *For $\nu \geq 0$, define, for $a > 1$,*

$$\zeta_{\nu+1} = c_1 2^{a\nu} \zeta_\nu^2.$$

Then one has

$$\zeta_\nu = \frac{1}{c_1 2^{a(\nu+1)}} (2^a c_1 \zeta_0)^{2^\nu}.$$

Moreover, if $2^a c_1 \zeta_0 < 1$ and $2^{1-a} b < 1$, for any $b > 0$ then

$$\sum_{\nu \geq k} b^\nu \zeta_\nu \leq \frac{b^k}{2^{a(k+1)-1} c_1} (2^a c_1 \zeta_0)^{2^k}.$$

Proof. The proof proceeds exactly as in [13, Lemma C.3]. □

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