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Funding Value Adjustment and Wrong-Way Risk: the Interest Rate Swap case

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Abstract

In response to the tenor basis emergence and credit and funding spread rise during the 2007/2008 crisis, financial institutions undertook drastic actions to limit the credit and liquidity risks. In particular, it has become standard practice the use of price adjustments for OTC derivatives. One of them is the Funding Value Adjustment, namely a price adjustment related to the funding risk. The Funding Value Adjustment intends to cover the risk caused by the fluctuations of the funding rate. Many institutions assume this rate deterministic or independent of the market risk factors. This assumption significantly simplifies simulations and implementations in internal banking systems. Nevertheless, this hypothesis neglects the Wrong-Way Risk. In this context, the Wrong-Way Risk corresponds to the effects on the Funding Value Adjustment due to the dependence between the credit/funding risk and the market risks.

The proposed analysis will offer a deep study on the relationship between Funding Value Adjustment and Wrong-Way Risk. In particular, to clarify the Funding Value Adjustment meaning, the historical context of its introduction will be first examined. After that, a review of the literature will be proposed to identify some of the most modern techniques regarding the modeling and management of the Funding Value Adjustment. Furthermore, one of the aims of this document is to find a pragmatic method to quantify the Wrong-Way Risk impact. Thus, some of the existing models on the matter will be analyzed and then the author's proposal will be presented. The hypothesis of uncorrelation between the funding rate and market risks will be removed to quantify its impact on the Funding Value Adjustment. Finally, the importance of a correct inclusion and management of the Wrong-Way Risk will be proved through a numerical example.

Sommario

Dopo la comparsa delle basi su tenor e degli spread di credito/finanziamento, a fronte della crisi 2007/2008, sono state adottate misure molto più rigide per contenere i rischi di credito e di liquidità. In particolare, è diventata pratica comune l'utilizzo di aggiustamenti di prezzo per i derivati OTC. Tra di essi si vede la nascita del Funding Value Adjustment, ovvero un adeguamento di prezzo relativo al rischio di finanziamento.
Ogni banca finanzia i suoi flussi di cassa facendosi prestare denaro ad un determinato tasso di interesse, specifico per ogni istituzione. Il Funding Value Adjustment ha l'obbiettivo di coprire il rischio dovuto alle fluttuazioni di tale tasso. Molte istituzioni assumono il tasso di finanziamento deterministico o indipendente dai fattori di rischio di mercato. Ciò semplifica molto simulazioni e implementazioni nel sistema interno della banca. Tuttavia, questa ipotesi trascura ciò che in letteratura viene chiamato Wrong-Way Risk. In questo contesto, si identifica con Wrong-Way Risk la variazione nel valore del Funding Value Adjustment dovuta alla dipendenza che sussiste tra i rischi di credito/finanziamento e i rischi di mercato.

Nell'analisi proposta verranno investigati i rapporti esistenti tra il Funding Value Adjustment e il Wrong-Way Risk. In particolare, verrà analizzato dapprima il Funding Value Adjustment, prestando particolare attenzione al contesto storico in cui venne introdotto per chiarificarne il significato. Secondariamente, sarà proposta una revisione della letteratura atta ad identificare alcune delle tecniche più moderne riguardo la modellizzazione e gestione del Funding Value Adjustment. Inoltre, sarà ricercato un metodo pragmatico per quantificare l'impatto del Wrong-Way Risk. Verranno quindi analizzati alcuni modelli presenti in letteratura, per poi presentare la personale proposta dell'autore. Verrà rimossa l'ipotesi di incorrelazione tra tasso di finanziamento e rischi di mercato per valutare l'impatto sul Funding Value Adjustment. Infine, sarà dimostrata l'importanza di una corretta inclusione e gestione del Wrong-Way Risk tramite un esempio numerico.

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Milano, 28 Aprile 2021 Nicola

1. Synopsis

The purposes of this document are to present the Funding Value Adjustment and study its Wrong Way Risk component.
Especially, we will remove the independence assumption between the funding component and the market risk factors to study the Funding Value Adjustment behavior.
For this reason, we will first present (chapter 2) a historical introduction to explain the context in which the price adjustments were introduced, and the reason why banks have to adopt them during an OTC pricing process. In particular, we will report the main remarks about the 2007/2008 credit crisis by summarizing some publications (*e.g.* [Gre15a; Gre15b; Sia16; Sav17]).
We will propose a general overview of the Funding Value Adjustment (FVA), which includes its mathematical definition. Especially, we will describe it and we will analyse its components and its nature; thereafter, we will mention the debate about the use of this adjustment during a pricing process, which to some extent is still open. We will also mention the double counting problem highlighted by Hull and White, Morini and Prampolini in [HW12b] and [MP11] respectively. Then, we will exploit the argumentation proposed by Ruiz in [Rui13] to clarify the debate and to point out our position. Then, we will introduce the Wrong-Way Risk (WWR) by reporting the banks' common practice which assumes constant funding spread.

After that, we will perform a review of the literature to identify some of the most modern techniques regarding the modeling and management of the Funding Value Adjustment. Especially, in chapter 3, we will derive some models and methods to compute the Funding Value Adjustment during a pricing process.
The first model, proposed by Garcıa Muñoz ([Gar13]), highlights the implicit FVA component during a hedging process by including as many realistic assumptions as possible. There also exists a possible generalization, which considers the stochastic funding spread. Furthermore, it extends the results of Garcıa Muñoz to the other Values Adjustment. Nevertheless, we prefer to avoid its presentation because the extension is straightforward, and Antonov and McClelland report it in [AM14].
We will propose the second model described in [Fri11] for its interesting parallelism between the funding world and the multi-currency one, which leads to a possible inclusion of the Wrong-Way Risk. Actually, in the following chapters, we will take inspiration from it to propose our model of the Wrong-Way Risk estimation. Moreover, thanks to the similarities that Fries shows, we will use an

already existing model for FVA purpose.
Instead, the last method relies on a different paradigm. According to [Sia16], we will start from the NSFR regulatory constraint, and then we will derive the FVA value in order to satisfy it. In this way, we will highlight a possible different use of the FVA, which permits to quantify the cost of regulatory satisfaction.

In chapter 4, we will deeply explore the Wrong-Way Risk through a review of the literature. We will start from its definition, and then we will report the common methods used to quantify it within the Credit Value Adjustment, by following the argumentation of Ruiz, Pachón, and Del Boca in [RPD13]. In this way, we will highlight the problems of a possible FVA extension.
After that, we will propose the method of Moni, which estimates the implicit WWR approximation into the internal system of a bank by exploiting the methodology of [Mon14].

In chapter 2-4, we will review the existing models, and our contributions will be simple observations about the utility and the possible extensions for WWR purposes. Instead, in chapter 5, we will describe our proposal for Wrong-Way Risk computation, and we will explain the rationale behind the choices made.
In particular, after a brief recap of the previous results, we will propose our approach to quantify the Wrong-Way Risk impact by paying attention to the references from which we took a cue. We will introduce our specific model to quantify the WWR within the Funding Value Adjustment of an interest rate derivative. We point out the considered plain vanilla interest rate derivative has no credit support annex (CSA) to easily understand the results, while a possible CSA management is described by Piterbarg in [Pit10]. We will start from the FVA definition, we will clarify the necessary assumptions, and then we will identify the market risk factors. Since we will consider an IR derivative, these market risk factors will be the discount rate and the Forward rate, built as discounting plus spread (basis tenor).
Therefore, we will also introduce the rate models adopted. Especially, we will divide the funding spread into two components to isolate the idiosyncratic funding part which drives the FVA. Finally, we will derive the (possibly extendable) correlation structure, and we will identify the WWR component.

In chapter 6, we will show the numerical results of the WWR study for the Funding Value Adjustment of an interest rate derivatives, namely for an Interest Rate Swap (IRS). We will test the model proposed in chapter 5 by calibrating it through real market data, and by computing the WWR impact. In this chapter, we will also pay particular attention to the calibration process and the market curves selected for it.
We will estimate a Wrong-Way Risk contribute. Therefore, we will compute also the results for different IRS moneyness to validate our thesis. Moreover, we will report the corresponding FVA value, which assumes a constant funding spread,

to operate a better comparison with the common banks' practice. We will conclude the chapter by highlighting the proof of our hypothesis: the Funding Value Adjustment is a fundamental price component, which must be included during the pricing process together with the Wrong-Way Risk.

The last chapter reports the conclusions that we achieved during this study. We will wrap up the results and discuss the outcome.
We will outline the main points covered by this document, and we will stress again about the FVA importance and the need for the WWR inclusion. We will conclude the study with our personal opinion about these topics.
Finally, in the appendix, we will present some insights about the topics covered during this study, and some proofs which are not central for the main discussion.

2. Introduction

The 2007/2008 crisis highlighted the presence of the liquidity and credit risk. Liquidity and funding issues has to be carefully considered. Liquidity shortage may increase the risk of default even for large banks.
Since the probability of default for all financial players was perceived as non-zero, the deposit market has been affected. In particular, the loan maturity has started to influence its rate and banks have observed the onset of basis among the different tenors.
For example, lending money for three months and rolling the deposit over the following three months has became different from lending money for six months, despite the benchmark rate was the same for the two strategies (see fig. 2.1). This fact was even more evident for overnight loans and eventually led to the multi-curve framework [BT20].

Figure 2.1 shows this new situation. Tenors have been divided from each others, causing the need for a different curve for each of them.
Forwarding money started to be a multi-curve problem, hence the basis became crucial. They permit to distinguish among tenors through a specific add-on.
In particular, after choosing the reference curve, the basis is added; in this way the representation of a specific tenor is performed.

Roughly speaking, there exist two types of rates: LIBOR[1] and OIS (secured). The LIBOR rate refers to short unsecured loans, while the OIS rate refers to secured overnight loans [Sav17]. Before the crisis, the spread between LIBOR rate and OIS rate was negligible, but it has exploded during the crisis. Indeed, it reached a value greater than 200 basis point in the last quarter of 2008 (see fig. 2.2).
This should not be surprising, it was a direct consequence of the meaning of the two rates[Sav17]. Therefore, the different rates reaction was a natural effect of the general awareness that no one is risk-free, especially over medium-long horizons. For many financial market analysts, this was the principal cause of the 2007/2008 global crisis [Sia16].

A consequence of the crisis was the employment of collateral transactions [BT20]. The main difference between the collateralized and uncollateralized transaction is the presence of the collateral, *i.e.* a guarantee that the lender keeps if there are issues with the debtor. Since collateralized transactions are very close to risk-

[1]We refer to LIBOR rate as the Interbank Offered Rate types. Historically, the first Interbank Offered Rate was the London one, so the literature refers to them as LIBOR rates.

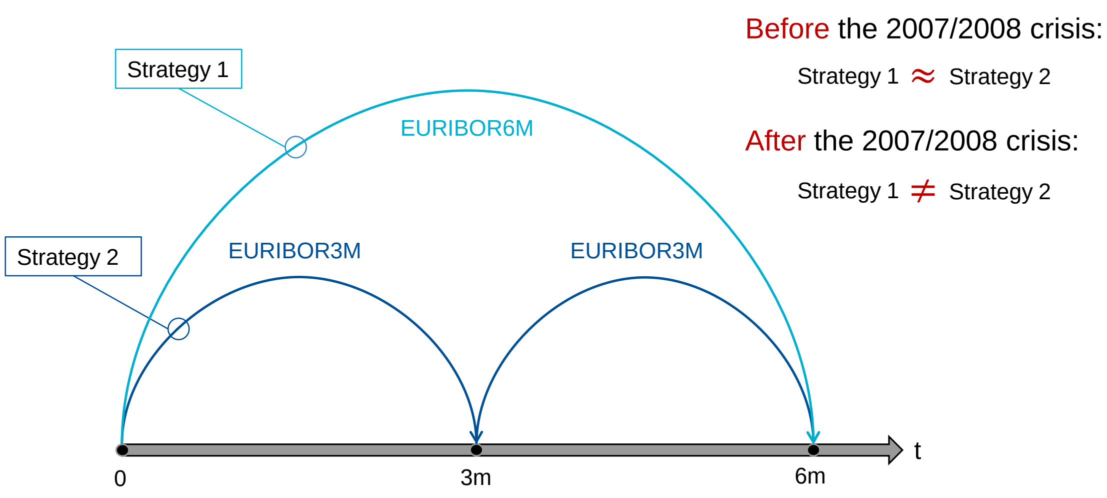


Figure 2.1.: Representation of two different strategies with the same time horizon and the same reference rate (EURIBOR), but different value due to the tenor basis.

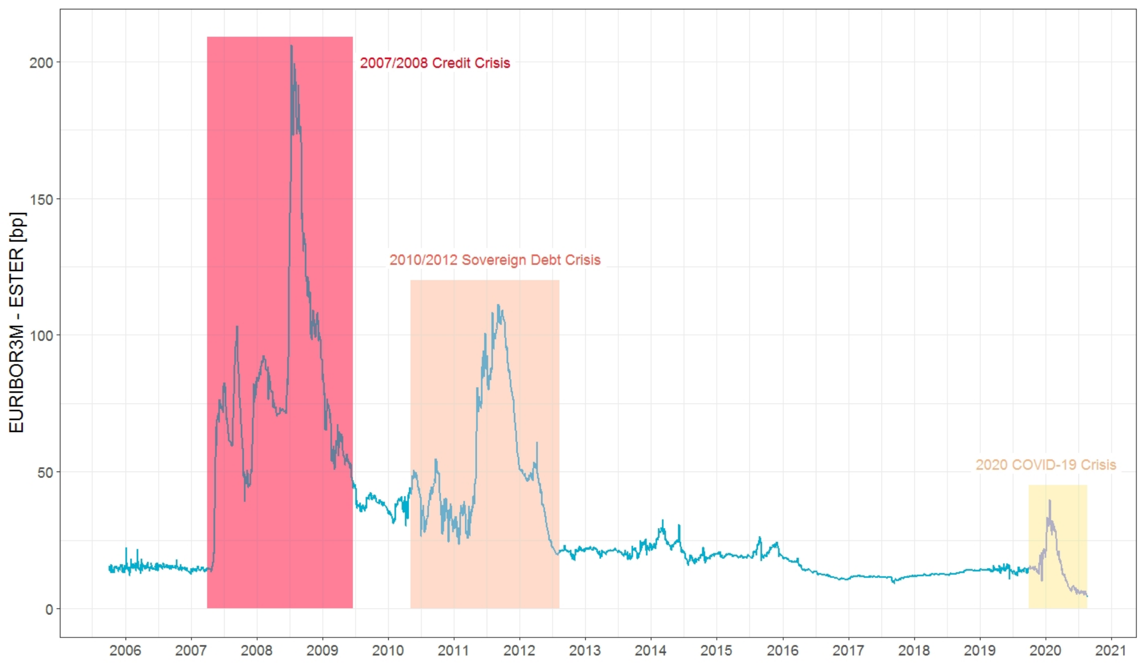


Figure 2.2.: Historical spread between the 3-month EURIBOR and the 3-month ESTER.

free operations, and since the OIS rate is a good estimator of the risk-free rate, banks started using OIS as the discount rate for collateral transactions [Sav17]. Instead, most of the players kept using the LIBOR rate (as a proxy of the funding rate [AP10]) for the uncollateralized transactions.

Before the crisis, banks used LIBOR as a benchmark for discount purposes: it was nearly risk-free, since there were no spread, and it was also a great approximation of banks unsecured funding costs [Gre15b]. Afterward, they realized that they could not borrow at LIBOR rate anymore and therefore it did no longer represent the funding rate.[Sia16]
So, this implied the aforementioned transition to the OIS-based lending; nowadays, there is only a negligible part of the market which still uses LIBOR for short term lending purpose. Moreover, it is common practice to use LIBOR for OTC derivatives discounting rate, besides being used as a curve to estimate forwards (*e.g.* for the floating rate in an interest rate swap or caps). While the OIS rate is chosen as discount for hedged positions [Sav17].

Notice that, in the 2020 COVID-19 crisis, in particular in march 2020, it was also observed a high LIBOR-OIS spread (see fig. 2.2). The reason why this divergence was less evident than the 2007/2008 crisis was probably due to two facts: the more strict regulation which was imposed to the banks (see section 2.1), and the decrease of uncollateralized transactions [BSV20].
Nevertheless, in the 2010/2012 there was another high spread due to the Sovereign crisis which affected the Europe. Since in fig. 2.2 we are considering the ESTER as OIS rate, and since the new legislations were not in force in these years, the spreads exploded again.
By comparing the three historic crisis, we could suppose that the new adopted practices were actually effective.

Finally, the birth of tenor basis had a natural impact also on mathematical finance. Indeed, different techniques to manage them were conceived: from the simplest ones, that considers constant basis, to the more complex ones, that uses stochastic processes[2].
In the following chapters, we will apply stochastic processes to deal with this issue.

2.1. The birth of the Adjustments

During the 2007/2008 crisis, financial institutions observed a dramatic factor: how the financial economy is strictly correlated to the real economy. The whole world saw the tragic collapse of some financial institutions, *e.g.* Lehman Brothers, September 15, 2008.

[2] The literature talks about Multi-Curve World (see [BM07; Gre15a]).

This highlighted the need of a more strict and accurate regulation about risk management and capital requirement.

Through the Third Basel Accord (Basel III), regulators imposed the new standards and strengthen the already existing ones. Roughly speaking, they asked financial institutions to guarantee a minimum capital charge in order to hedge the credit and liquidity risks for every positions they have.
In this context, financial institutions have started to apply price adjustments. The first one was the Credit Value Adjustment (CVA), immediately followed by the Debt Value Adjustment (DVA).

The CVA is usually defined as the market price of the credit risk of a contract [Gre15a]. Therefore, it is the price adjustment made on an OTC contract due to the possibility for the counterparty to became insolvent, *i.e.* the credit risk. Instead, the DVA is the corresponding CVA in the counterparty view, *i.e.* it is the adjustment due to the own credit risk. Observe that, the reason why regulators and market practices introduced the DVA was to reduce the valuation difference between the two parts involved in a trade. Hence, the price of an OTC contract is

$$\mathsf{Price} = \mathsf{Price}_{\mathsf{riskfree}} - \mathsf{CVA} - \mathsf{DVA}. \tag{2.1}$$

To give a mathematical description of the CVA and the DVA, the following quantities are defined.

- $D(0, t)$ is the discount factor from t to the present time;
- $\mathsf{EPE}(t)$ $\left(\mathsf{ENE}(t)\right)$ is the Expected Positive (Negative) Exposure of the trade;
- τ_c $\left(\tau_i\right)$ is the default time of the counterparty (institution);
- LGD_c $\left(\mathsf{LGD}_i\right)$ is the Loss Given Default of the counterparty (institution).

Since the CVA role is to cover the credit risk, it is defined as the average lost in response to the counterparty default. On the other hand, the DVA is the possible counterparty lost in response to the institution default. Mathematically:

$$\begin{aligned} \mathsf{CVA} &= \mathbb{E}\left[\mathsf{LGD}_c D(0,\tau_c)\mathsf{EPE}(\tau_c)\mathbb{1}_{\tau_i>\tau_c}\right] \\ \mathsf{DVA} &= \mathbb{E}\left[\mathsf{LGD}_i D(0,\tau_i)\mathsf{ENE}(\tau_i)\mathbb{1}_{\tau_c>\tau_i}\right], \end{aligned} \tag{2.2}$$

where $\mathbb{1}$ is the indicator function.
We assume, without loss of generality, that the Loss Given Default are constant quantities, while the default times are stochastic variables described by two density functions p^c and p^i. Therefor, eq. (2.2) becomes

$$\begin{aligned} \mathsf{CVA} &= \mathsf{LGD}_c \int_0^{+\infty} D(0,t)\mathsf{EPE}(t)\mathbb{P}\left(\tau_i > \tau_c \middle| \tau_c = t\right) p^c(dt) \\ \mathsf{DVA} &= \mathsf{LGD}_i \int_0^{+\infty} D(0,t)\mathsf{ENE}(t)\mathbb{P}\left(\tau_c > \tau_i \middle| \tau_i = t\right) p^i(dt). \end{aligned} \tag{2.3}$$

Other adjustments were introduced shortly afterwards. Among them, it was the Funding Value Adjustment (FVA).
The aim of the Funding Value Adjustment is to cover the funding risk, which identifies the possible costs the institution has to face during the lifetime of a trade.
Therefore, eq. (2.1) becomes

$$\mathrm{Price} = \mathrm{Price}_{riskfree} - CVA - DVA - FVA. \tag{2.4}$$

A contract usually generates some cashflows which have to be funded:

- if they are positive, they will generate a benefit at rate FS^B;
- if they are negative, they will generate a cost at rate FS^C.

Clearly, the Expected Positive (Negative) Exposure of the contract is the sum of its expected positive (negative) cashflows. Mathematically, we define

$$FVA = \int_0^{+\infty} D(0,t)\left[EPE(t)FS^B(t) + ENE(t)FS^C(t)\right]\mathbb{P}(\tau_i > t)\,\mathbb{P}(\tau_c > t)\,dt. \tag{2.5}$$

Moreover, since financial derivatives are OTC contracts, financial institutions hedge them through market trades. Unfortunately, contrary to the market operations, OTC contract are usually uncollateralized.
This implies a misalignment between the derivative and the market position which in turn causes more funding cost. Figure 2.3 shows a typical situation for an investment bank which sells a derivative to a corporate counterpart, while he/she hedges it on the market, causing the arising of the funding costs.
In summary, there are two sources of funding costs: collateral asymmetry and derivative cashflows. [3] [Rui13].

Some authors use sophisticated models to include funding costs. For example, Pallavicini, Perini, and Brigo gives a deep description of the funding process in [PPB11] to derive a recursive form for the FVA. They prove that the funding cost could not be a price add-on only. Nevertheless, we prefer to keep the modelling framework as simple as possible. Our study focuses on the FVA importance and the Wrong-Way Risk impact; therefore, we do not need unnecessary complexity, either in the adjustments definition neither in the modelling sections.

Notice that, OTC contracts are discounted to the LIBOR rate because they are unsecured. Instead, hedging contracts are linked to the OIS rate because they are collateralized. During the 2007/2008 crisis, it took place the aforementioned divergence of the two rates, so financial institutions have realized the importance

[3] Actually, there are many other secondary sources of funding costs, such as profit margins, intermediation costs, and changings in CSA agreements [Gre15b]. However, they are usually neglected in the literature.

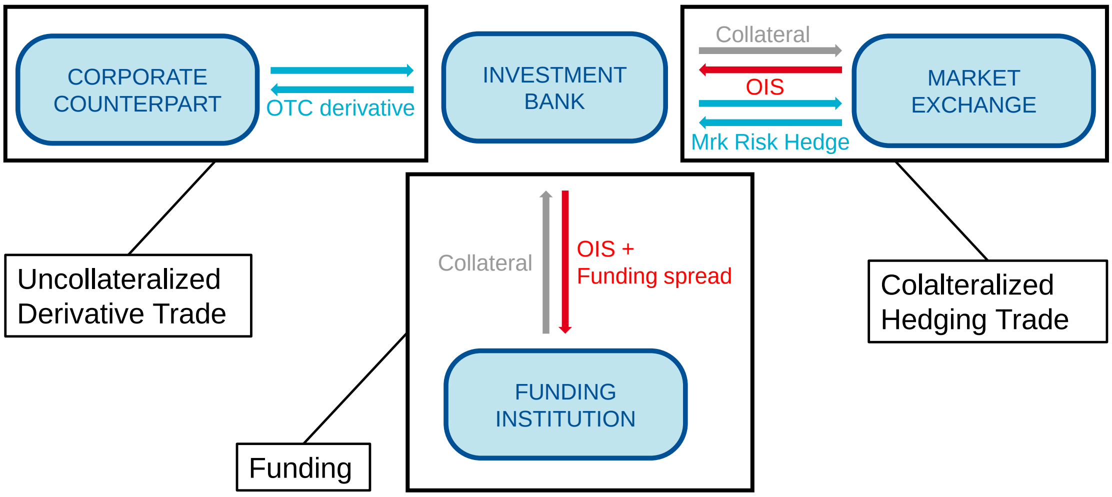


Figure 2.3.: Typical situation for an investment bank which sells a derivative to a corporate counterpart, while he/she hedges it on the market, arising funding costs.

of the funding cost.
However, the funding risk did not arise from the crisis. What the crisis has caused was the spread of the two rates, previously negligible. Thus, the funding cost has always been present, but it was hidden since this spread was too small [Sia16].

2.2. The discussion around FVA

The Funding Value Adjustment is not mandatory as balanche sheet item, so there is a debate about its inclusion in the pricing process.
There are two main stances on this topic:

- someone thinks that funding adjustment is necessary - since the trading desk pays the funding costs during the lifetime of the contract, it must be considered in its price;
- others argue that considering the funding adjustment is wrong - the funding cost causes a misalignment between the two parties involved because they have different funding curves.

The debate is still open nowadays, but in general there is a growing consensus in including the FVA in the balance sheets.

In order to clarify the argument, we analyze hereafter the distinction between the price and the value of the deal [Rui13]. The price takes into account the hedging cost in the risk-free world. Therefore, it can be expressed by eq. (2.1), *i.e.*

$$\mathsf{Price} = \mathsf{Price}_{\mathsf{riskfree}} - \mathsf{CVA} - \mathsf{DVA}. \tag{2.6}$$

While the value is the profit that a transaction generates for the company. So it considers all the costs which the contract produces during its life.

$$\mathsf{Value} = \mathsf{Price} - \mathsf{FVA}. \tag{2.7}$$

If the Value is less then zero, the company has a loss, and it is not convenient to enter in this contract. Therefore, the trading desk has to use the FVA to value the transaction from a profitable point of view. Moreover, financial institutions could employ it to determine some incentive for traders [Rui13].
The debate revolves around the wrong question. Banks should not ask themselves if they should include FVA in the price, but they should ask themselves, what is the best price for the deal to make it profitable and acceptable for the client [Rui13].

Another issue regarding the Funding Value Adjustment is the double-counting problem. Since the DVA and the FVA are both related to the risks of the institution, it is argued that using them together would hedge these risks twice [HW12b; MP11].
However, this is false because they cover different events:

- the DVA represents the institution default, and it is related to his/her credit risk;
- the FVA represents the possible fluctuations of the institution borrow rate, hence it is related to his/her funding risk.

Moreover, their components are also different [Sia16]:

- Loss Given Default and Expected Negative Exposure drive the DVA;
- Funding costs/benefits spread and Expected Exposure ($\mathsf{ENE} + \mathsf{EPE}$) drive the FVA.

Last but not least, the standardised approach for counterparty credit risk (SA-CCR) defines the CVA/DVA on netting sets to take into account the hedging strategies [Bas14b]. Instead, banks compute the FVA at portfolio level since the benefits arising from a transaction could be used to cover costs arising from another[Rui13; Rui14].
Therefore, the DVA and the FVA are clearly two different things. The DVA refers to the default, while the FVA to the funding; so there is no double-counting.

Finally, we conclude the argument with an example. Let's consider a portfolio composed by some uncollateralized trades. If the financial institution decides to delta-hedge[4] them through the corresponding market plain vanilla options, the CVA/DVA will remain the same. This is natural since the CVA/DVA concerns only the default events, which are independent from this hedging strategy. However, the FVA becomes maximal due to the misalignment between portfolio and hedging collaterals [Rui13]. Table 2.1 summarizes all possible cases.

Collateralized Portfolio	**Market Hedging**	**CVA/DVA**	**FVA**
✓	✓	min	min
✓	x	min	MAX
x	✓	MAX	MAX
x	x	MAX	min

Table 2.1.: The impact on the CVA/DVA and FVA of different collateralized and hedged situation.

It is clear that there is no double-counting problem. The real issue seems to be how to define the DVA and the FVA, and how they are used [Sia16].
It seems that the root of the discussion may not be a fundamental quantitative finance problem, but rather a semantic one: each side of the discussion is using the word 'price' for a different thing; theorists for 'fair value', while practitioners for 'value to me' [Rui13].

2.3. The Wrong-Way Risk

The complexity of real banks portfolios led to even more sophisticated models because simple ones do not catch all their features. Unfortunately, these models are difficult to calibrate and to interpret, and they usually have too many parameters. Actually, large portfolio usually have many risk factors and netting sets, which depend from each other. Therefore, the common practice is to add some simplification hypothesis when it is possible.
One of them is the independence assumption between the market risks and the credit/funding risks, which leads to the following CVA, DVA and FVA simplified formulas.

$$\begin{aligned}
\mathrm{CVA} &= \mathrm{LGD}_c \int_0^{+\infty} D(0,t)\mathrm{EPE}(t)dt \int_0^{+\infty} \mathbb{P}\left(\tau_i > \tau_c \middle| \tau_c = t\right) p^c(dt) \\
\mathrm{DVA} &= \mathrm{LGD}_i \int_0^{+\infty} D(0,t)\mathrm{ENE}(t)dt \int_0^{+\infty} \mathbb{P}\left(\tau_c > \tau_i \middle| \tau_i = t\right) p^i(dt)
\end{aligned} \tag{2.8}$$

[4] Delta-hedge refers to the practice of immunization of a position with respect to the market risks associated to it.

$$
\begin{aligned}
\mathrm{FVA} = \Bigg(& \int_0^{+\infty} D(0,t)\mathrm{EPE}(t)dt \int_0^{+\infty} \mathrm{FS}^{B}(t)dt + \\
& + \int_0^{+\infty} D(0,t)\mathrm{ENE}(t)dt \int_0^{+\infty} \mathrm{FS}^{C}(t)dt \Bigg) \int_0^{+\infty} \mathbb{P}(\tau_i > t)\,\mathbb{P}(\tau_c > t)\, dt.
\end{aligned}
\tag{2.9}
$$

Unfortunately, this approach neglects the Wrong-Way Risk (WWR) and the Right-Way Risk (RWR).

The WWR/RWR materializes when the exposures are not independent from credit and funding.
In the CVA framework, the Wrong-Way Risks refers to the negative correlation between exposure and counterparty credit quality. When this correlation is positive, we deal with Right-Way Risk. Actually, it is better to speak about unfavourable and favourable dependency because WWR(RWR) identifies a loss(gain) [Gre15b]. Indeed, CVA is a charge for a bank, so WWR increases its value, while RWR decreases it [Gre15a].
From now on, we use WWR to both identify Wrong-Way Risk and Right Way Risk to keep the nomenclature easier.

Financial institutions usually manage the WWR during the CVA computation because of its impact [RPD13; Mon14; HW12a]. Instead, it is usually neglected in the FVA context, where its contribute is uncertain.
In practice, many banks assume the independence between funding and market risk factors. This assumption is usually made to simplify the implementation into the banking system. Actually, it is common to already have the expected exposure for the CVA and the DVA. Therefore, if financial institutions assume the independence, they do not need a revaluation of the whole portfolio, which implies a non-negligible computational effort.
Practically, they usually assume constant and equal funding spread for benefits and costs, which leads eq. (2.9) to

$$
\mathrm{FVA} = \mathrm{FS} \int_0^{+\infty} D(0,t)\mathrm{EE}(t)dt \int_0^{+\infty} \mathbb{P}(\tau_i > t)\,\mathbb{P}(\tau_c > t)\, dt, \tag{2.10}
$$

where EE is the Expected Exposure ($\mathrm{EE} = \mathrm{ENE} + \mathrm{EPE}$) and $\mathrm{FS} = \mathrm{FS}^{B} = \mathrm{FS}^{C}$.

In the following chapters, besides some examples of the numerical computation of FVA in a mock portfolio, we will investigate about the WWR impact on the Funding Value Adjustment. Moreover, we will conclude this study with a numerical example which will prove the need to include the FVA in the pricing process, and the importance of a proper WWR management.

3. The Funding Value Adjustment

In this chapter, we present some methodologies to include the Funding Value Adjustment in the pricing process. Actually, we have performed a review of the literature and we have selected some possible different approaches. Naturally, there are many possible choices, and we report only some of them. Ideally, we have selected the ones with particular features or interesting applications. Furthermore, we also investigate about the importance of the Funding Value Adjustment, which we will furthermore investigate, and hopefully confirm, in the next chapters.

In section 3.1, we try to include as many realistic assumptions as possible to highlight the implicit FVA component during a hedging process of an equity derivative.
We report this approach for two main reasons: it proposes a possible FVA model of an equity derivative, and it shows the implicit existence of the Funding Value Adjustment. Actually, we show the importance of the FVA through a mathematical discussion, which leads to the following conclusion: if we did not include the FVA in the pricing process, we would not obtain the fair price in generating a loss for the bank.

Instead, the second method employs a parallelism between the funding world and the multi-currency one (section 3.2). We have exploited these similarities to adapt an already existing model to FVA purpose.
Unlike the previous approach, this method is independent of the financial instrument considered. Actually, it permits to compute the FVA for all possible cashflows.
It is not the simplest approach, but it is very useful to include the WWR. Indeed, in chapter 5 we take a cue from it to derive our own model through some suitable simplifications to highlight WWR aspects.

Last but not least, in section 3.3, we propose a method which derives the FVA value to satisfy a regulatory constraint. In particular, regulators require the Net Stable Funding Ratio (NSFR) greater or equal then one. Therefore, we use this constrain to compute the minimum FVA value which guarantees it.
Thanks to this approach, we show another proof about the importance of the FVA: it permits to evaluate the costs of regulatory constraint satisfaction. Moreover, it shows another use of the Funding Value Adjustment, different from the conventional ones.

This notwithstanding, it would be more difficult (and not fully adherent to market practice) to base computation and hedging on such grounds, since it is derived outside the usual no-arbitrage framework under which a trading desk is normally managed.

3.1. A real world model approach

In this section we present an elementary way to model the real pricing process of an equity derivative, with the purpose to highlight the FVA component. To do this, we follow the methodology proposed in [Gar13].
Actually, [Gar13] explains how to incorporate the Funding Value Adjustment by considering the hedger perspective. They employ some reasonable assumptions to keep the results closer to reality, as a stochastic credit curve for both the hedger and the investor. Finally, they compute the derivative price which guarantees the risks hedging. Their argument ends by highlighting the presence of CVA and FVA in the obtained price.
Naturally, there exist some generalizations, closer to reality which includes other adjustments. For example, the approach of Antonov and McClelland also considers a stochastic funding curve [AM14]. Nevertheless, we prefer to provide this real but simpler version in order to focus only on the FVA by avoiding misleading add-on. Clearly, the generalization of Antonov and McClelland could be optimal to include the Wrong-Way Risk. However, we decide to avoid it because we will study the WWR in the next chapters, so we keep the topics separate for sake of clarity.

Since our interest is in the Funding Value Adjustment only, we report the García Muñoz's approach by avoiding the counterparty risk to show the implicit FVA presence in the hedging procedure. In this way, we point out the importance of the FVA, which is implicitly present if a derivative would be hedged. Moreover, it is a natural approach to include it during a pricing procedure. We end up this discussion by observing how the FVA denial can lead the bank to a loss.

The assumptions to construct a model close to reality are the following:

- The derivative prices must include all costs given by the hedging strategy;
- The no-hedgeable price components, *i.e.* the ones which generate profit, are not modeled;
- The only hedging costs included in the model are the costs arising from the hedger part; indeed it is common to have a part which takes the risk and another one that sells it during an uncollateralized transaction;
- The hedging process only considers fluctuations of the prices; so default is not hedged. Indeed, the part who takes the risks does not usually want to get also the default risk;

- There is no external FVA and CVA adjustment for a fully collateralize transaction.

There is also the need to include some market assumptions:

- The bonds issued by the hedger have a liquid curve;
- The underlying of the derivatives is usable for a REPO agreement at overnight rate;
- There are no trading costs and no bid-offer spread; there is unlimited liquidity and continuous hedging is possible;

Last but not least, we list hereafter the model assumptions:

- The hedger could default, but the investor is risk free because the aim is to highlight only the FVA component;
- A diffusion process describes the underlying asset dynamics (S_t);
- the hedger default does not affect the underlying asset;
- A diffusion process models the hedger credit spreads (h_t);
- A deterministic interest rate curve is adopted, although the results may be extended.

The approach is naturally applied to equity payoffs, and in principle an extension to FX derivatives can be done easily, while substantial work should be done to extend to Interest Rates derivatives.
The only unreal assumption is the non-defautable investor. Nevertheless, since the derivative is bought by the investor, the price is not too affected by its default. Furthermore, we want to highlight only the FVA component, so it is useless to add unnecessary complexity.
In order to clarify the deal, the hedger sells a derivative to the investor and includes in the price all the hedging costs he/she carries out. According to fig. 2.3, the hedger represents the investment bank, while the investor is the corporate counterpart. Instead, the hedging process takes place in the market exchange, which produces the funding costs/benefits.

According to the model assumptions, we present the dynamics of the underlying asset (S_t) and the short term CDS spread of the derivative's hedger (h_t) under the real world measure $\mathbb{P}$.

$$dS_t = \mu_t^S S_t dt + \sigma_t^S S_t dW_t^{S,\mathbb{P}}, \tag{3.1a}$$

$$dh_t = \mu_t^H dt + \sigma_t^H dW_t^{H,\mathbb{P}}, \tag{3.1b}$$

where μ_t^S, μ_t^H are the real world drifts, σ_t^S, σ_t^H are the real world volatilities and $W_t^{S,\mathbb{P}}$, $W_t^{H,\mathbb{P}}$ are correlated Brownian motion under the real world measure.
A time dependent function identifies this correlation:

$$\rho_t^{S,H} dt = dW_t^{S,\mathbb{P}} dW_t^{H,\mathbb{P}}. \tag{3.2}$$

Furthermore, there is the need of an indicator processes ($N_t^{\mathbb{P}}$) to model the default event of the hedger. Thus, $N_t^{\mathbb{P}} = \mathbb{1}_{\tau \leq t}$, where τ is the hedger default time. $N_t^{\mathbb{P}}$ has also an associate real world intensity λ_t.

We consider a derivative from the hedger perspective, which is a function of all the risks described above: $V_t = V_t(S_t, h_t, N_t^{\mathbb{P}})$.
Itô's Lemma for jump diffusion processes provides us the dynamics of V_t between t and t + dt. Naturally, there is the implicit assumption about the hedger survival till time t.

$$dV_t = \frac{\partial V_t}{\partial S_t} dS_t + \frac{\partial V_t}{\partial h_t} dh_t + \Delta V_t dN_t^{\mathbb{P}} + O(dt), \tag{3.3}$$

where ΔV_t identifies the loss in the derivative value due to the hedger default at time t.

The next issue is to hedge all risks of the aforementioned derivative V_t.
First of all, the market risk associated to the underlying fluctuations $\frac{\partial V_t}{\partial S_t} dS_t$ is considered. To hedge this risk, the hedger has to trade an α_t number of the underlying stock. Moreover, he/she has to use a REPO agreement to avoid extra management of cashflows. The REPO agreement is feasible because we have assumed the REPO market existence.

The second component we have to manage is the hedger credit risk, *i.e.* $\frac{\partial V_t}{\partial h_t} dh_t$. The usual practice to hedge credit risk is to buy a credit default swap. Unfortunately, no one could buy a CSD written on himself/herself.
Therefore, another way to proceed could be to modify the hedger debt structure. In particular he/she has to issue some short debt ($\bar{D}(t, t + dt)$) and buy back some long one($\bar{D}(t, T)$), or vice-versa. Especially, $\bar{D}(t, t + dt)$ is a bond issued by the hedger with maturity $t + dt$ which pays an interest of $r(t) + FS^C(t)$. Where $r(t)$ represents the approximation of the risk-free rate (*e.g.* OIS rate), while $FS^C(t)$ is the funding spread of the hedger. Naturally, $FS^C(t)$ is different from h_t which represents the CDS spread.
Notice that $\bar{D}(t, T)$ is the risky-hedger bond, so it is function of h_t.

Moreover, we made implicitly the self financing portfolio assumption: every replication of the derivative cannot have different cashflow from the one generating from the derivative itself. Therefore, the number of short debt bought must be equal to $\frac{\bar{D}(t,T)}{\bar{D}(t,t+dt)}$ for any long debt sold during the replication process.
By identifying with γ_t the number of $\bar{D}(t, T)$ bought, we need $\gamma_t \frac{\bar{D}(t,T)}{\bar{D}(t,t+dt)}$ of

$\bar{D}(t, t + dt)$ sold.

Usually the derivative has a cost at time zero ($V_0 > 0$), which is payed by the investor.
While, at every future time t, if V_t is less then zero, the hedger pays some borrowing interest $r(t) + FS^C(t)$, *i.e.* the risk-free rate plus the hedger funding spread to finance his/her cash-outflows.
Instead, whenever V_t is greater then zero, the hedger will have a gain. He/she has to reinvest this gain in a no-risky position, otherwise it generates another unhedged risk. For example, he/she could lend it in a collateralized REPO agreement, or he/she could leave it as a collateral in a fully collateralized derivative transaction, or he/she could stop paying the borrowing interest rate [Gar13].
In any situation described above, the gain generated by $V_t > 0$ produces an interest rate received by the hedger $\Omega^B(t)$, and the loss given by $V_t < 0$ produces an interest rate payed by the hedger $\Omega^C(t) = r(t) + FS^C(t)$.

Notice that, in all scenarios where the hedger is defaulted, the value of the derivative V_t is not appreciated by the hedger. Therefore, the hedger default is not covered. Actually, we are considering this situation, but without posting a protection on it.
This approach is very intuitive because it is useless to cover a scenario in which the hedger is not alive. There are no gains for the hedger, with or without an hedging strategy.
For this reason, $\Delta V_t^H dN_t^{H,\mathbb{P}}$ does not affect the replication process, and the methodology used remains close to reality.

Following the hedging procedure described above, we can replicate the derivative value at time t as:

$$V_t = \alpha_t S_t + \beta_t + \gamma_t \left(\bar{D}(t, T) - \frac{\bar{D}(t, T)}{\bar{D}(t, t + dt)} \bar{D}(t, t + dt) \right), \tag{3.4}$$

where β_t identifies the hedger credits/debts arising from the fluctuations of V_t. Hence, its dynamics is $\Omega^C(t)V_t$ if $V_t < 0$, or $\Omega^B(t)V_t$ if $V_t > 0$, minus the interest ν_t generated by the component $\alpha_t S_t$. Therefore we have[1]:

$$d\beta_t = \Omega^B(t)V_t^+ dt + \Omega^C(t)V_t^- dt - \nu_t \alpha_t S_t dt. \tag{3.5}$$

Moreover, we can apply Itô's Lemma on the other components of V_t, and following the dependencies described above we obtain:

$$d\bar{D}(t, t + dt) = (r(t) + FS^C(t))\bar{D}(t, t + dt)dt, \tag{3.6a}$$

$$d\bar{D}(t, T) = \frac{\partial \bar{D}(t, T)}{\partial t} dt + \frac{\partial \bar{D}(t, T)}{\partial h_t} dh_t + \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 \bar{D}(t, T)}{\partial h_t^2} dt. \tag{3.6b}$$

[1]We use $^+$ and $^-$ to identify positive and negative part respectively.

We refine eq. (3.3) by exploiting the $O(dt)$ term:

$$\begin{aligned} dV_t =& \frac{\partial V_t}{\partial t} dt + \frac{\partial V_t}{\partial h_t} dh_t + \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 V_t}{\partial h_t^2} dt + \\ &+ \frac{\partial V_t}{\partial S_t} dS_t + \frac{1}{2}(\sigma_t^S S_t)^2 \frac{\partial^2 V_t}{\partial S_t^2} dt + \sigma_t^H \sigma_t^S \rho_t^{H,s} S_t \frac{\partial^2 V_t}{\partial S_t \partial h_t} dt. \end{aligned} \tag{3.7}$$

We have omitted the dN_t component because the hedger does not cover the scenarios in which he/she is defaulted, as we have stressed before.

We could also derive the dynamics of V_t by differentiating the equation (3.4):

$$\begin{aligned} dV_t &= d\left(\alpha_t S_t + \beta_t + \gamma_t \left(\bar{D}(t,T) - \frac{\bar{D}(t,T)}{\bar{D}(t,t+dt)} \bar{D}(t,t+dt)\right)\right) = \\ &= \alpha_t dS_t + \alpha_t q_t S_t dt + \Omega^B(t) V_t^+ dt + \Omega^C(t) V_t^- dt - \nu_t \alpha_t S_t dt + \\ &+ \gamma_t \left(\frac{\partial \bar{D}(t,T)}{\partial t} dt + \frac{\partial \bar{D}(t,T)}{\partial h_t} dh_t + \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 \bar{D}(t,T)}{\partial h_t^2} dt\right) + \\ &- \gamma_t \frac{\bar{D}(t,T)}{\bar{D}(t,t+dt)} (r(t) + FS^C(t)) \bar{D}(t,t+dt) dt, \end{aligned} \tag{3.8}$$

where q_t comes from the possible dividends paid by the asset S_t.

The next step is to compare the equation (3.7) with the equation (3.8).

$$\begin{aligned} &\left(\alpha_t - \frac{\partial V_t}{\partial S_t}\right) dS_t + \left(\gamma_t \frac{\partial \bar{D}(t,T)}{\partial h_t} - \frac{\partial V_t}{\partial h_t}\right) dh_t + \\ &+ \left[\gamma_t \left(\frac{\partial \bar{D}(t,T)}{\partial t} + \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 \bar{D}(t,T)}{\partial h_t^2} - \frac{\bar{D}(t,T)}{\bar{D}(t,t+dt)} (r(t) + FS^C(t)) \bar{D}(t,t+dt)\right) + \right. \\ &\left. + \left(\alpha_t (q_t - \nu_t) S_t + \Omega^B(t) V_t^+ + \Omega^C(t) V_t^-\right) \right] dt = \\ &= \left(\frac{\partial V_t}{\partial t} + \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 V_t}{\partial h_t^2} + \frac{1}{2}(\sigma_t^S S_t)^2 \frac{\partial^2 V_t}{\partial S_t^2} + \sigma_t^H \sigma_t^S \rho_t^{H,s} S_t \frac{\partial^2 V_t}{\partial S_t \partial h_t}\right) dt. \end{aligned} \tag{3.9}$$

Now we impose the stochastic dynamic equal to zero, *i.e.* the dS_t and the dh_t terms.
Therefore, the derivative is hedged if

$$\alpha_t = \frac{\partial V_t}{\partial S_t} \quad \text{and} \quad \gamma_t = \frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial \bar{D}(t,T)}{\partial h_t}}. \tag{3.10}$$

Then, the partial differential equation that $\bar{D}(t,T)$ has to satisfy is reported in eq. (3.11) and derived in appendix C.

$$\begin{aligned}
&\frac{\partial \bar{D}(t,T)}{\partial t} + \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 \bar{D}(t,T)}{\partial h_t^2} = \\
&= -(\mu_t^H - \sigma_t^H \eta(t)) \frac{\partial \bar{D}(t,T)}{\partial h_t} - \frac{h_t}{1-R} \Delta \bar{D}(t,T) + r^{B,T}(t) \bar{D}(t,T),
\end{aligned} \tag{3.11}$$

where R is the recovery rate of the hedger, $r^{B,T}(t)$ is the REPO rate with maturity T and $\bar{D}(t,T)$ as underlying. $\Delta\bar{D}(t,T)$ identifies the variation of $\bar{D}(t,T)$ caused by the hedger default, and $\eta(t)$ is the Market Price of Credit Risk of the hedger, which we derive in appendix C too.

Similarly to $r^{B,T}(t)$, it exists a short term REPO rate $r^{B,t+dt}(t)$ whose underlying is $\bar{D}(t, t+dt)$. This rate has a no-arbitrage relation with the hedger credit spread h_t and the funding cost rate $\Omega^C(t)$.
We report this relation in eq. (3.12), while we prove it in appendix B.

$$\Omega^C(t) = r(t) + FS^C(t) = h_t + r^{B,t+dt}(t). \tag{3.12}$$

Therefore, if we impose the hedged condition (3.10) and we substitute eq. (3.11), (3.12) into eq. (3.9), we obtain:

$$\begin{aligned}
&\left(\frac{\partial V_t}{\partial S_t}(q_t - \nu_t) S_t + \Omega^B(t) V_t^+ + \Omega^C(t) V_t^- \right) dt + \\
&+ \Bigg[\left(-\frac{h_t}{1-R} \Delta \bar{D}(t,T) + r^{B,T}(t) \bar{D}(t,T) - (h_t + r^{B,t+dt}(t)) \bar{D}(t,T) \right) + \\
&- (\mu_t^H - \sigma_t^H \eta(t)) \frac{\partial \bar{D}(t,T)}{\partial h_t} \Bigg] \frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial \bar{D}(t,T)}{\partial h_t}} dt = \\
&= \left(\frac{\partial V_t}{\partial t} + \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 V_t}{\partial h_t^2} + \frac{1}{2}(\sigma_t^S S_t)^2 \frac{\partial^2 V_t}{\partial S_t^2} + \sigma_t^H \sigma_t^S \rho_t^{H,s} S_t \frac{\partial^2 V_t}{\partial S_t \partial h_t} \right) dt.
\end{aligned} \tag{3.13}$$

Next step is to impose the equivalence between the deterministic parts, *i.e.* the dt terms.
Thus, if we assume:

- the independence between short term REPO rate and its underlying ($r^{B,T}(t) = r^{B,t+dt}(t) = r^B(t)$);
- that the hedger default implies a jump of $\bar{D}(t,T)$ to $R\bar{D}(t,T)$ (so $\Delta\bar{D}(t,T) = -(1-R)\bar{D}(t,T)$).

Equation (3.13) becomes

$$\begin{aligned}
&\frac{\partial V_t}{\partial S_t}(q_t - \nu_t) S_t + \Omega^B(t) V_t^+ + \Omega^C(t) V_t^- - \frac{\partial V_t}{\partial h_t}(\mu_t^H - \sigma_t^H \eta(t)) = \\
&= \frac{\partial V_t}{\partial t} + \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 V_t}{\partial h_t^2} + \frac{1}{2}(\sigma_t^S S_t)^2 \frac{\partial^2 V_t}{\partial S_t^2} + \sigma_t^H \sigma_t^S \rho_t^{H,s} S_t \frac{\partial^2 V_t}{\partial S_t \partial h_t}.
\end{aligned} \tag{3.14}$$

After a rearrangement, the final partial differential equation for V_t is

$$\begin{aligned}\frac{\partial V_t}{\partial t} &+ (\mu_t^H - \sigma_t^H \eta(t))\frac{\partial V_t}{\partial h_t} + \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 V_t}{\partial h_t^2} + (\nu_t - q_t)S_t \frac{\partial V_t}{\partial S_t} + \\ &+ \frac{1}{2}(\sigma_t^S S_t)^2 \frac{\partial^2 V_t}{\partial S_t^2} + \sigma_t^H \sigma_t^S \rho_t^{H,s} S_t \frac{\partial^2 V_t}{\partial S_t \partial h_t} = \Omega^B(t)V_t^+ + \Omega^C(t)V_t^-.\end{aligned} \tag{3.15}$$

By identifying with g the payoff of the option, we impose the boundary condition at maturity, *i.e.* $V_T = g(S_T)$. So the solution of eq. (3.15) is

$$\begin{aligned}V_t = &\underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} V_T \middle| \mathcal{F}_t\right]}_{\text{Fully collateralized price}} + \\ &\underbrace{- \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(h)dh}(FS^B(s)V_s^+ + FS^C(s)V_s^-)ds \middle| \mathcal{F}_t\right]}_{\text{Funding value adjustment}},\end{aligned} \tag{3.16}$$

where $FS^B(t)$ represents the funding benefit spread[2] at time t which is equal to $\Omega^B(t) - r(t)$.

We have identified with $\mathbb{Q}$ the measure such that

$$dS_t = (\nu_t - q_t)S_t dt + \sigma_t^S S_t dW_t^{S,\mathbb{Q}}, \tag{3.17a}$$

$$dh_t = (\mu_t^H - \eta(t)\sigma_t^H)dt + \sigma_t^H dW_t^{H,\mathbb{Q}}. \tag{3.17b}$$

The approach used to solve equation (3.15) is to set $X_t = V_t e^{-\int_0^t r(s)ds}$, to apply Itô Lemma to compute dXt, to integrate between t and T, and finally take the conditional expectation under the filtration $\mathcal{F}_t$. We report a more detailed proof in the appendix D.

Notice that, the second term of eq. (3.16) is similar to eq. (2.5). $e^{-\int_t^s r(h)dh}$ is the discount factor, V_s^+ (V_s^-) the Expected Positive (Negative) Exposure and there is no default probability because we assume no credit risk.
Therefore, it shows how a derivatives price implicitly includes the FVA component, even if there are no a priori assumptions about it. This means that, if financial institutions follow a close-market approach to model the pricing process, the results incorporate the funding cost too.
Therefore, the FVA could be interpreted as the price to edge the own spread risk.

Furthermore, eq. (3.16) divides the price of V_t in two components: the risk free price and its FVA. Therefore, by following this method, banks could estimate the

[2]Costs and benefits have different rates, otherwise we would have $FS^B = FS^C = FS$ and $\Omega^B = \Omega^C = \Omega$.

FVA of an equity derivative. In this sense, we propose a possible approach to quantify it.
Moreover, this implies that: if we neglect the Funding Value Adjustment, we would not obtain the fair price of the option, therefore we would lead to a loss for the bank.

Observe that, we have modeled only the hedger's risk. We have assumed the investor non-defaultable because the aim of this study is to highlight the implicit FVA component during an hedging process, which originates from the hedger's risk.
The model extension, which also includes the investor credit risk, produces, in addition, the CVA component. This extension is presented in [Gar13]. While, the generalization which concerns the stochastic funding curve is described in [AM14].

In summary, we have kept the model close to the market through the aforementioned realistic assumptions. Then, we have derived the hedger derivative dynamic according to the real hedging process. So, we have obtained the derivative price which includes the hedging costs by following the Garcıa Muñoz's approach.
We have highlighted the implicit FVA component in the price to prove its importance, and to stress its meaning. Indeed, we have shown in details the composition of the Funding Value Adjustment, which is naturally included if a hedging procedure is adopted.

3.2. The Cross-Currency parallelism

In this section, we present the approach proposed by Fries for modelling the funding spread as a stochastic variable. Fries reports a parallelism between the cross-Currency world and the funding one. He describes their similarity in order to use some already existing results for funding needs. Actually, his aim is to study the implication of a stochastic funding curve, by paying particular attention to the valuation and hedging procedure. He also reports how to compute the correspondent sensitivities.

We are interested in the FVA modelling, therefore we present only the parallelism and a possible application with a LIBOR market model. We found in the approach proposed by Fries an interesting and realistic application, according to which the presence of the funding cost requires the adoption of different curves for discounting, forwarding and funding purposes [Pit10]. Moreover, this approach could be a starting point to assess the effects of Wrong-Way Risk. We show how to define a stochastic model for rates and spreads, and we highlight where the WWR could be managed. Unfortunately, the model is not very tractable, so we take a cue from it to define our own model approach in the next chapters.

Notice that, since the Cross-Currency instruments are very common in the everyday bank activities, there exist a lot of literature about them. Therefore, thanks to this parallelism we could use any existing models to funding purposes, which is a very useful tool. Furthermore, we could take into account the Wrong-Way Risk through the correlation which subsists between rates (currencies). So, this is the reason why this approach inspires the simplified ones we use to obtain the numerical results.

As we have explained in the previous chapters, FVA is the price adjustment to compensate the funding cost. Therefore, to discount a cashflow there are two different Bonds: a risk-free one which ignores the funding cost, and a risky one which includes it. The risk-free interest rate drives the first Bond, while the risky interest rate drives the second one. The aim of this section is to propose a methodology to model the spread between these two rates.

The idea proposed by Fries in [Fri11] is to consider the funding rate as a rate in an other currency. Hence, we can model the two rates, the market riskless rate and the funding rate, with a cross-currency approach by exploiting its models and its results. In particular, we present a cross-currency LIBOR market model.

First of all, we need to introduce some preliminary results about the cross-currency framework.
We denote with N^{dom} and N^{for} two numeraires in the domestic and the foreign currency respectively. Therefore, the following equation relates them

$$N^{for}(t) = FX(t)N^{dom}(t), \tag{3.18}$$

where FX identifies the foreign exchange rate ($[FX] = 1\frac{for}{dom}$).
Let's call $V^{dom}(T)$ a future cashflow at time T in the domestic currency. From the Asset Pricing Theory, the value of this cashflow at time t is equal to

$$V^{dom}(t) = N^{dom}(t)\mathbb{E}^{\mathbb{Q}^{N^{dom}}}\left[\frac{V^{dom}(T)}{N^{dom}(T)}\middle|\mathcal{F}_t\right], \tag{3.19}$$

where $\mathbb{Q}^{N^{dom}}$ expresses the measure under which $\frac{V^{dom}(t)}{N^{dom}(t)}$ is a martingale.
The same cashflow is valued by a foreign investor in the foreign currency, by using the foreign numeraire:

$$V^{for}(t) = V^{dom}(t)FX(t) = N^{for}(t)\mathbb{E}^{\mathbb{Q}^{N^{for}}}\left[\frac{V^{dom}(T)FX(T)}{N^{for}(T)}\middle|\mathcal{F}_t\right], \tag{3.20}$$

where $\mathbb{Q}^{N^{for}}$ is the measure under which $\frac{V^{dom}(t)FX(t)}{N^{for}(t)}$ is a martingale.

Observe that the two martingale measures are equivalent because

$$\begin{gathered}\frac{V^{dom}(t)}{N^{dom}(t)} \text{ is a } \mathbb{Q}^{N^{dom}}\text{-martingale } \Longleftrightarrow \\ \frac{V^{dom}(t)FX(t)}{N^{for}(t)} = \frac{V^{dom}(t)FX(t)}{N^{dom}(t)FX(t)} = \frac{V^{dom}(t)}{N^{dom}(t)} \text{ is a } \mathbb{Q}^{N^{for}}\text{-martingale.}\end{gathered} \tag{3.21}$$

Therefore, if we know the dynamic of a process under the $\mathbb{Q}^{N^{dom}}$ measure, we also know its dynamics under the $\mathbb{Q}^{N^{for}}$ measure.

In a foreign investor perspective, we price a domestic cashflow $V^{dom}(T)$ in the foreign currency as

$$V^{q}(t) = N^{for}(t)\mathbb{E}^{\mathbb{Q}^{N^{for}}}\left[\frac{V^{dom}(T)1\frac{for}{dom}}{N^{for}(T)}\middle|\mathcal{F}_t\right]. \tag{3.22}$$

After that, we use the equivalence between the two martingale measures (3.21) to obtain

$$V^{q}(t) = N^{for}(t)\mathbb{E}^{\mathbb{Q}^{N^{dom}}}\left[\frac{V^{dom}(T)1\frac{for}{dom}}{N^{for}(T)}\middle|\mathcal{F}_t\right]. \tag{3.23}$$

The literature refers to $V^{q}(t)$ as a Quantos, which represents a payment denominated in a currency but payed in an other.
Notice that what we prove is not a change of measure, but it is a change of numeraire during a valuation of a Quantos.

All instruments to construct the parallelism between cross-currency and funding world were defined. Hence, in the funding perspective, there are:

- a virtual market in which funding is at the free-risk rate, with its associated numeraire N (N^{dom});
- the real market in which we operate, with our funding rate and its relative numeraire N^{fd} (N^{for}).

In the real world, we cannot adjust cashflows to consider our funding costs when we trade with the market. Therefore, a Quantos represents our replicated value $V^{fd}(t)$ of some market value $V(T)$.

$$V^{fd}(t) = N^{fd}(t)\mathbb{E}^{\mathbb{Q}^{N}}\left[\frac{V(T)}{N^{fd}(T)}\middle|\mathcal{F}_t\right]. \tag{3.24}$$

Notice that, we are operating under the same currency, in the funding view. Therefore the unit conversion $1\frac{for}{dom}$ is simply one. Table 3.1 summarises the presented framework.

Cross-currency		Funding
Domestic rate	$\longleftrightarrow$	Risk-free rate
Foreign rate	$\longleftrightarrow$	Funding rate
Domestic cashflow	$\longleftrightarrow$	Market cashflow
Quantoed Domestic Cash Flow in Foreign Currency	$\longleftrightarrow$	Funded Market Cash Flow

Table 3.1.: Comparison between cross-currency and funding frameworks. Source: [Fri11]

From the parallelism explained above, we can choose any already existing model for cross-currency and adapt it to funding purposes. In particular, we report some results using the cross-currency LIBOR market model.
This model is an extension of the well-known LIBOR market model, which describes the evolution of a Bond through its LIBOR rate dynamics.
In its extension, there are two different LIBOR rates - one for the domestic currency and another for the foreign one - and a foreign exchange rate which connects them.

Notice that domestic quantities represent market objects, while foreign ones are the funding components. This means that, we could manage the Wrong-Way Risk through the relationship between domestic and foreign quantities. Since there exist very accurate Cross-Currency models, we could guarantee a precise description of the WWR.

In the following part, we present the main relation of the LIBOR market model and its extension to the cross-currency one. However, we do not derive them because it is out of our purposes. There is a lot of literature about LIBOR market model, *e.g.* [BM07; BT20], while for the extended version the reader could find a deeper presentation, and its derivation, in [Fri07].

The LIBOR market model (LMM) was born to price exotic interest rate derivatives, and it has evolved in different formulations (*e.g.* Shifted LMM, SABR LMM, Stochastic volatility LMM and Multi-curve LMM)[BT20; BM07; AP10].
The fundamental variables, of the classical formulation, are a discrete number $\bar{N}$ of forward rate agreements (FRA), *i.e.* the LIBOR rate from which the model takes its name. We assume FRA rates evolving as Geometric Brownian motion[3] and to be martingale under the T_i-forward measure, *i.e.* the measure whose numeraire is the bond maturing in T_i.
It is common practice to operate a change of measure during the definition of these processes because the rates are usually used to price products depending on multiple forward rates [BT20]. However, they must evolve under the same

[3] Actually, in a negative interest rate world, the lognormal component is the shifted rate (Shifted LMM).

measure.
Let's call $\mathbb{Q}^{T_k}$ the selected T_k-forward measure, and L_i the LIBOR rate valued in t and maturing between T_i and T_{i+1}. So, there are the following results:

$$\begin{aligned} dL_i(t) = \mu_i^k(t)L_i(t)dt + \sigma_i(t)L_i(t)dW_i^k(t), \qquad \forall i \in \{1, \bar{N}\} \\ dW_i^k(t)dW_j^k(t) = \rho_{ij}(t)dt, \qquad \forall i \neq j \in \{1, \bar{N}\} \end{aligned} \tag{3.25}$$

where $\{W_i^k(t)\}_{i=1}^{\bar{N}}$ is a vector of Brownian motion under the T_k-forward measure with correlations $\{\rho_{ij}(t)\}_{i,j=1}^{\bar{N}}$, $\{\sigma_i\}_{i=1}^{\bar{N}}$ are the time-dependent instantaneous volatilities, and $\mu_i^k(t)$ is the drift which depends on the T_k chosen. In particular, Brigo and Mercurio (in [BM07]) prove that

$$\mu_i^k(t) = \begin{cases} \sigma_i(t) \sum_{j=k+1}^{i} \dfrac{\rho_{ij}(t)\sigma_j(t)L_j(t)\delta_j}{1 + L_j(t)\delta_j}, \text{ if } k < i, t < T_k, \\ 0, \text{ if } k = i, t < T_{i-1}, \\ -\sigma_i(t) \sum_{j=i+1}^{k} \dfrac{\rho_{ij}(t)\sigma_j(t)L_j(t)\delta_j}{1 + L_j(t)\delta_j}, \text{ if } k > i, t < T_{i-1}. \end{cases} \tag{3.26}$$

Observe that, LIBOR rate is a compounded interest rate, therefore the relation which has with the discount factor D is:

$$L_i(t) = L_i(t; T_{i-1}, T_i) = \frac{1}{\delta_i}\left[\frac{D(t, T_{i-1})}{D(t, T_i)} - 1\right] \tag{3.27}$$

where δ_i identifies the year fraction between T_{i-1} and T_i.

We can now extend the LMM to our funding needs by following the Fries' proposal. The first step is to choose a market numeraire N related to the market forward rates L_i. The second step is to define the dynamics of the funding forward rates L_i^{fd} under the numeraire $N^{fd} := FXN$, and its associated measure $\mathbb{Q}^{N^{fd}}$. Then, the third step is to use the equivalence between the two measures proved before ($\mathbb{Q}^{N^{fd}} = \mathbb{Q}^N$) to move the dynamics of L_i^{fd} under the measure $\mathbb{Q}^N$. In the end, we performe the pricing process with the FVA as a Quantos.

The market numeraire proposed in [Fri11] is the spot measure numeraire

$$N(t) = D(T_{m(t)}; t) \prod_{j=0}^{m(t)} (1 + L_j(T_j)\delta_j) \tag{3.28}$$

where $m(t) := \max(i : T_i < t)$.
As we have mentioned,there are two LIBOR rates ($L_i(t)$, $L_i^{fd}(t)$) under the same martingale measure $\mathbb{Q}^N$, and a foreign exchange rate FX(t) which connects them.

We assume that the FX rate evolves as a lognormal process; therefore, mathematically:

$$\begin{aligned} dL_i(t) &= \mu_i(t)L_i(t)dt + \sigma_i(t)L_i(t)dW_i^{\mathbb{Q}^N}(t) \\ dFX(t) &= \mu_{FX}(t)FX(t)dt + \sigma_{FX}(t)FX(t)dW_{FX}^{\mathbb{Q}^N}(t) \\ dL_i^{fd}(t) &= \tilde{\mu}_i(t)L_i^{fd}(t)dt + \tilde{\sigma}_i(t)L_i^{fd}(t)d\tilde{W}_i^{\mathbb{Q}^N}(t) \end{aligned} \tag{3.29}$$

where $W_i^{\mathbb{Q}^N}(t)$, $\tilde{W}_i^{\mathbb{Q}^N}(t)$ and $W_{FX}^{\mathbb{Q}^N}(t)$ are Brownian motion under the martingale measure $\mathbb{Q}^N$, which are correlated by

$$\begin{aligned} dW_i^{\mathbb{Q}^N}(t)dW_j^{\mathbb{Q}^N}(t) &= \rho_{ij}(t)dt \\ d\tilde{W}_i^{\mathbb{Q}^N}(t)d\tilde{W}_i^{\mathbb{Q}^N}(t) &= \tilde{\rho}_{ij}(t)dt \\ dW_i^{\mathbb{Q}^N}(t)dW_{FX}^{\mathbb{Q}^N}(t) &= \rho_{i,FX}(t)dt \\ d\tilde{W}_i^{\mathbb{Q}^N}(t)dW_{FX}^{\mathbb{Q}^N}(t) &= \tilde{\rho}_{i,FX}(t)dt. \end{aligned} \tag{3.30}$$

Notice that, ρ-parameters are the correlations between the different Brownian motions which describe this processes. If we considered the WWR, we would set it through this correlation and the FX rate.
We have taken a cue from this possible WWR inclusion to define the one we propose in chapter 5.

Observe that $\mu_i(t)$ is the same of classical LMM (eq. (3.26)), while $\tilde{\mu}_i(t)$ and $\mu_{FX}(t)$ derive from the cross-currency extensions. They map the relation between the two rates as

$$\begin{aligned} \tilde{\mu}_i(t) &= \tilde{\sigma}_i(t) \sum_{j=m(t)+1}^{i} \frac{\tilde{\rho}_{ij}(t)\tilde{\sigma}_j(t)L_j^{fd}(t)\delta_j}{1 + L_j^{fd}(t)\delta_j} - \tilde{\sigma}_i(t)\sigma_{FX}(t)\tilde{\rho}_{i,FX}(t) \\ \int_{T_i}^{T_{i+1}} \mu_{FX}(t)dt &= \log\left(\frac{1 + L_i(T_{i-1})}{1 + L_i(T_{i-1})}\right). \end{aligned} \tag{3.31}$$

We recall that, in [Fri07] there is the derivation of these quantities .

We finally pass to the funding spread rate by observing that

$$FS_i(t) := L_i^{fd}(t) - L_i(t). \tag{3.32}$$

Therefore we could model FS_i instead of L_i^{fd}. Notice that, we assume equal funding rate for benefits and costs because if we divided it, we would need a second FX rate. So this extension is not so trivial.
Instead, to avoid negative spread, we can choose a log normal process for it, consistently with the extended LMM:

$$dFS_i(t) = \mu_i^S(t)FS_i(t)dt + \sigma_i^S(T)FS_i(t)dW_{S,i}^{\mathbb{Q}^N}(t). \tag{3.33}$$

which is the dynamics we will select in chapter 5.
Moreover, thanks to eq. (3.32), we derive the relations which the model parameters have to satisfy.

$$\tilde{\mu}_i(t) = \frac{\mu_i^S(t)FS_i(t) + \mu_i(t)L_i(t)}{FS_i(t) + L_i(t)} \tag{3.34}$$

$$\Big(FS_i(t) + L_i(t)\Big)\tilde{\sigma}_i(t)d\tilde{W}_i^{\mathbb{Q}^N}(t) = \sigma_i(t)L_i(t)dW_i^{\mathbb{Q}^N}(t) + \sigma_i^S(T)S_i(t)dW_{S,i}^{\mathbb{Q}^N}(t). \tag{3.35}$$

Finally, the FVA is simply the difference between the market price and the funding price given by the Quantos parallelism (eq. (3.24) and table 3.1).

Summing up, we have explored the parallelism reported by Fries in order to consider a stochastic funding curve. Moreover, we have highlighted a possible solution to consider the WWR into the model.
Unfortunately, this approach is very intensive from a computational point of view, since it admits too many parameters. So, it is difficult to calibrate and to interpret. We have reported the method to focus on the main idea it proposes which is very useful if the WWR is considered. Actually, in the numerical example, we will adopt a simplify version of it because our aim is to investigate the WWR impact. In particular, we will decompose the Funding Spread into two components: the systemic funding spread (EURIBOR3M-ESTER), and the funding spread over the LIBOR (see chapter 5). Therefore, thanks to this study we show the natural extension of the model that we will use to obtain the numerical results.
Finally, after the WWR importance is proven, banks could use this approach to include it in internal systems.

3.3. An alternative computation for the FVA

In the previous section we proposed some models to compute FVA and to highlight its meaning. Instead in this section we show how to derive the FVA component to satisfy the BCBS (Basel Committee of Banking Supervision) regulatory. In particular, we derive the FVA in order to guarantee the Net Stable Funding Ratio greater then one. To do that, we follow the approach proposed by Siadat in [Sia16].
Siadat presents this new method to explore a possible different definition of the Funding Value Adjustment. Indeed, we use her results to show a different application of it. The purpose is to highlight the importance of the FVA, which could have many applications. Conventionally, it is a price adjustment, which covers the funding risk. Moreover, banks could use it to quantify incentives for the trading desks [Rui13]. Finally, we see how to use it for valuing the regulatory constraint. The proposed method is parallel to the others presented, and intends to give an additional proof about the importance of the FVA.
However, we point out that, this is an alternative approach based on liquidity management conditions from a regulatory point of view. Therefore, it is very

different from the usual method which exploits the no-arbitrage relations.

First of all, we present the Net Stable Funding Ratio (NSFR), which is an insolvency indicator introduced by the Basel Committee of Banking Supervision (see [Bas14a]) in 2014[4]. NSFR measures the capability of a bank to face a stressed scenario over one year time horizon caused by a lack of liquidity. In particular, it is computed as the ratio

$$\mathrm{NSFR} = \frac{\mathrm{ASF}}{\mathrm{RSF}} \tag{3.36}$$

where ASF refers to the available stable funding, and RSF is the required stable funding. ASF is the value of equity and liability that should provide a stable source of funding, while RSF is the funding needs arising from assets, both in one year time horizon. The BCBS regulatory imposes to banks to maintain NSFR greater or equal than one, so that banks might be able to face unexpected costs.
Observe that, NSFR is a bank personal value because it is strictly correlated to its portfolio and its activities.

The idea of Siadat, proposed in [Sia16], is to model FVA in such a way that it considers some information which guides NSFR. In this way, we could impose $\mathrm{NSFR} = 1$ to satisfy regulators and derive the corresponding FVA.
In a simplified view, since ASF is the funding raising and RSF the funding needed from an activity, if we impose their equivalence we can derive the missing part. This missing quantity represents the funding costs/benefits we need/gain, *i.e.* the FVA related to that activity.

Following the regulatory, every cashflows refer to ASF or RSF with a specific weight (ω) according to its origin, its rating, its maturity and its type (liability or assets). Therefore, if we focus on a collateralized derivative transaction - for which we want to compute its FVA component - we could call $\{V_i\}_i$ its cashflows. Then, we define $L = \{i : V_i \text{ is a liability}\}$ and $A\{i : V_i \text{ is an asset}\}$. So, by calling $\bar{D}$ the market debt used to finance this collateralized derivative transaction and α its associated NFSR weight, we obtain:

$$\mathrm{NFSR} = \frac{\sum_{i \in L} \omega_i V_i + \alpha \bar{D}}{\sum_{i \in A} \omega_i V_i}. \tag{3.37}$$

Notice that, the derivative contract defines all the derivative cashflows $\{V_i\}_i$. Therefore, the only term we could manage is the market debt and its maturity.
In particular, we impose $\mathrm{NFSR} = 1$, to satisfy the regulators, and we assume to operate on a single maturity, to obtain a fixed value for α. So, in this settings, we have

$$\bar{D}(0,t) = \frac{\sum_{i \in A} \omega_i V_i(t) - \sum_{i \in L} \omega_i V_i(t)}{\alpha}, \tag{3.38}$$

[4]Actually, it is introduced in 2010, published in 2014 and a became standard in 2018.

where we explicit the the time dependence of V_i. We point out again that, $\bar{D}(0,t)$ is the debt bought from the market which guarantees $NSFR = 1$, *i.e.* it is the regulatory satisfaction for the derivative transaction considered.

Let's call $C(t)$ the collateral value at time t. So, we might have two different situations:

- $\bar{D}(0,t) \geq C(t)$, *i.e.* the debt is enough to cover the posted collateral;
- $\bar{D}(0,t) < C(t)$, *i.e.* it is needed some other debt to cover the posted collateral.

Therefore, we have to compute the funding value adjustment by taking the second possibility into account. To do so, we split FVA in two parts: FVA_1 and FVA_2. The first part considers the fixed bought debt $\bar{D}(0,t)$. While the second part raises from the need of more money to cover the collateral.

$$FVA = FVA_1 + FVA_2 \tag{3.39}$$

Finally, banks could choose the way they prefer to discount the debts, according to the aforementioned distinction.

We identify with τ the default time of the bank, which could be impose equal to infinity in case of no-default assumption, or modeled, for example, as a Poisson process. We call $FS(t)$ the funding spread between the short funding rate of the bank $r^B(t)$ and the short risk-free rate $r(t)$.
Observe that financial institution could model $FS(t)$ as they prefer. For example, they could model it through the Cross-Currency parallelism described in section 3.2, or through the approach presented in section 3.1 (if they have to manage an equity derivative).
We impose the same rate for costs and benefits ($FS^B = FS^C = FS$) to keep the model as simple as possible, but the extension is straightforward. From what we said, we have

$$\begin{aligned} FVA_1 &= \mathbb{E}\left[\int_0^\tau e^{-\int_0^t r^B(s)ds} FS(t)\bar{D}(0,t)dt\right] \\ FVA_2 &= \mathbb{E}\left[\int_0^\tau e^{-\int_0^t r^B(s)ds} FS(t)\left(C(t) - \bar{D}(0,t)\right)^+ dt\right]. \end{aligned} \tag{3.40}$$

Clearly the obtained value will be greater then the correspondence one obtained from traditional approaches because it considers also the possible additional regulatory charge (FVA_2) [Sia16].
We use this difference to estimate the impact of the regulatory constraint. Actually, the financial institution needs an additional debt of $(C(t) - \bar{D}(0,t))^+$ to satisfy the regulators. Therefore, he/she has an additional funding adjustment of FVA_2. Since FVA_2 is the cost to funding the additional debt, we can use it as a metrics of

the regulatory constraint.

In conclusion, we reported a simple and flexible alternative approach to compute the FVA component, which takes into account the NSFR constraint that a bank has to satisfy. Moreover, we used this new method to quantify the cost of regulatory satisfaction. So, we obtained an additional proof about the importance of the FVA, which we will finally prove through the numerical application presented in chapter 6.
Nevertheless, this approach has weaknesses concerning the time horizon, the portfolio application and the liquidity management. Indeed, NSFR greather then one is a constrain with one year time horizon, but the considered derivative could have different maturity. This difference may imply a wrong liquidity management, by adding an unnecessary stress on the short period.
Furthermore, it is a theoretical approach which would be difficult to apply to real trading activity, since it would be more correct to apply it to the entire portfolio by considering all the assets and liabilities.
Finally, each bank has internal liquidity management methodologies that differ from NSFR. Therefore, the approach is difficult to implement, and it has only a theoretical application.

4. The Wrong-Way Risk

In the previous chapter we have shown how to give a mathematical description of the Funding Value Adjustment and how to include the Wrong-Way Risk. However, we do not already know if it is negligible. The possibility that the WWR inclusion will cause a significant change in the FVA is not certain.
Actually, the Wrong-Way Risk has an evident impact on the Credit Value Adjustments, as we have already noticed in section 2.3 (see also [Rui14; RPD13; Mon14]). Therefore, we will analyze if the same holds for the Funding Value Adjustment.
Some authors have already faced this topic (see for example [Mon14; RPD13; Tur13]), so we will exploit their results to deeply investigate it. We will study the correlation between market risk factors and funding cost which are the driving factors of the WWR.

A common practice is to use a constant funding spread (see section 2.3) in the internal models of a bank. Therefore, the zero-correlation assumption is implicitly made.
In the following sections, we will remove this assumption by considering a stochastic funding curve, and we will study its impact.
To reach our purpose, we first present the main existing methodologies to estimate WWR in the CVA framework, and then we derive the ones that could be used with FVA. Moreover, in the next chapter, we show a specific approach for an interest rate derivative that we have used in the numerical example.

4.1. From CVA to FVA Wrong-Way Risk

Many models to estimate WWR in the CVA context are present in the literature because of its impact [Rui14; RPD13; HW12a]. For example, Ruiz, Pachón, and Del Boca explain in [RPD13] the most popular methodologies that we list hereafter.

1. To satisfy regulators, banks have to compute the Exposure at Default (EAD), which is the sum of the Replacement cost[1] and the potential future exposure[2], multiplied by a constant equal to 1.4 [Bas14b]. Thanks to this constant, we have the simplest way to take into account WWR. Actually, this is not a

[1]Replacement cost identifies the loss produced by the close out strategy induced by the counterparty default.
[2]Financial institutions compute the Replacement cost at the present time, but there is a time lag between the computation and the effective close out action; the Potential future exposure captures this potential change in value.

price adjustment but it is a capital charge justified by the correlation between exposure and counterparty credit quality.

2. Another way to consider the WWR could be by operating a change of measure on the process which describes the exposure. Practically, through the Girsanov Theorem, we change the drift of the risk factors evolution to consider the correlation with credit quality.

3. This method suppose that the institution already has a Monte Carlo engine to simulate the default of its counterparty and the market risk factors. Therefore, the idea is to simulate the defaulted scenario and consider only the ones in which the counterparty default. Then, the next step is to run a second Montecarlo on these scenarios to obtain the risk factors simulations. From these simulations the expected exposure is computed. So, this method derives the expected exposure value by conditioning the risk factors evolution to the default of the counterparty.

4. Instead of simulating default probability as in the previous method, financial institutions could use some analytic formulas driven from market factors. In this way they will improve the computational speed. Moreover, since the market factor are usually quoted instruments (*e.g.* stock prices, or FX rate), the obtained model correctly represents the market. For example, Hull and White, Rosen and Saunders show a possible application to this approach ([HW12a; RS12]).

5. Last but not least, we propose to compute the expected exposure through a particular stressed market scenario, *e.g.* with some sovereign defaults. Then, the computation are repeated under the normal market conditions. If the WWR is present, the stressed Expected Exposure will have the greatest value.

In [RPD13], the authors expresses their preference in favor of method 4 because it is the closer to the market data, it does not have an over complicated mathematical description and it does not use unobservable variables, which need a justification. According to these observations, in the following parts, we propose two different approaches to highlight the WWR component in the FVA computation.

Approach 1 is the easiest because it uses a simple multiplier to include the WWR. Moreover, financial institutions also can decrease this multiplier until 1.2 according to the riskiness and their internal model. For example, Cespedes et al. in [Ces+10] propose a practical and robust methodology to compute the capital requirement and the alpha multiplier. Nevertheless, as we have stressed before, this is a capital charge, so we could not use it as a price adjustment. Therefore, this method is not suitable for our FVA purposes.

The change of measure technique is a very common and useful practice. Actually, it is the same method that we proposed in section 3.1 to derive the FVA component. Unfortunately, for the WWR case there are not enough real justified assumptions, so the drift change would be completely arbitrary. Therefore, approach 2 is not the best way to estimate and quantify WWR [RPD13], and we do not use it for our FVA adaptations.

In [Tur13] there is an application of the last proposal (approach 5). Turlakov includes the Wrong-Way Risk in its model through tail events given by sovereign defaults, so it is also a suitable possibility for the FVA needs. Nevertheless, it has a very marginal application, and it does not identify all WWR causes and implications. Moreover, since the impact of a sovereign default is not easy to simulate, the calibration becomes very difficult[RPD13]. Hence, we prefer to avoid this method and to examine a more understandable one.

The other methods (3,4) are centered on the default event of the counterparty, therefore we could not adapt them to the FVA computation. The reason is very intuitive: as we explain in section 2.1, FVA is conceived to contemplate the risk of the bank. However, they could not consider scenarios in which they have defaulted, because they could not gain/loss anything after their default. In some sense, the FVA includes default probability of the financial institution, but avoid its default.

From what we said, we need ad-hoc models to estimate the WWR impact on FVA. Especially, we will present in the next section an approach which highlights the implicit Wrong-Way Risk component in an already existing model. Actually, we will take a cue from approach 4 by exploiting the internal system of the bank. Finally, we will show a specific method which quantifies the WWR impact in an Interest Rate derivative in the next chapter. In particular, we will use a model which guarantees an easy interpretation, closeness to the market data, and a reasonable extension of the most popular FVA practices. Then, we will conclude our argument through a numerical example in chapter 6.

4.2. The implicit WWR within a model

In this section we present the method, proposed in [Mon14], to identify the WWR component into the FVA. Actually, Moni quantifies the contribution of the main characters into the FVA. It assumes an already existing engine for the FVA computation, and splits its result highlighting the correlation impact. He employs a separate model to include the dependency assumption. This permits a general application since the knowledge about the bank specific FVA model is unnecessary. Moreover, the simulation procedure does not need an additional simulation of risk factors.

We report this study because it is simple and easy to implement. It guarantees an accurate estimation of the WWR component and it is defined through the FVA definition (see section 2.1). Finally, we focus on the WWR component, which is the subject of study of this document.
Unfortunately, we will not use this method in the proposed numerical example because we can not report specific FVAs of any institutions. And, if we had assumed an arbitrary value, we would have lost the reality closeness, which is one of the main properties we want to keep. Therefore, in the next chapter, we propose our model to quantify the WWR impact, which does not need any sensitive information.

The main idea, presented by Moni, is to use a simple model to estimate the impact of WWR within the current system of the bank for a general portfolio. In particular Moni employs two stochastic processes for funding spread and the portfolio value. Then, he assumes to have an internal model, which generates the funding spread as a deterministic function of time. Finally he uses the Delta-Gamma method to extrapolate the information searched. Actually, the Delta-Gamma method is applied on the FVA generated by the stochastic model which he imposes equal to the internal one.
Notice that, this approach is suitable for any model which assume deterministic funding curve. Moreover, it does not require the exact composition of the portfolio, but only its risk factors identification.

We consider a general portfolio which value $V(t)$ is stochastic at each time t. Moreover, we assume equal funding spread for costs and benefits ($FS^B = FS^C = FS$).
Since we want to estimate the Wrong-Way Risk we also assume, without loss of generality, that the survival probability of the institution and the counterparty is one, *i.e.* $\mathbb{P}(\tau_i > t) = \mathbb{P}(\tau_c > t) = 1\ \forall t$.
Therefore, eq. (2.5) becomes

$$FVA = \int_0^{+\infty} D(0,t)\mathbb{E}\left[FS(t)V(t)\right] dt. \tag{4.1}$$

Observe that the Expected Exposure is the portfolio value V, and the expectation $\mathbb{E}$ is necessary since $FS(t)$ and $V(t)$ are stochastic variables.

For each time t, the stochastic and the internal models produce the instantaneous Funding Value Adjustment which we impose to be equal, *i.e.*

$$\mathbb{E}\left[FS(t)V(t)\right] = FVA(t) = \tilde{FVA}(t) = \tilde{FS}(t)\mathbb{E}\left[V(t)\right] \tag{4.2}$$

where we identify with $\sim$ the quantities deriving from the deterministic funding spread used into the internal system.
Naturally, the (final) Funding Value Adjustment will be the integral of the dis-

counted instantaneous FVA, *i.e.* $FVA = \int_0^{+\infty} D(0,t)FVA(t)dt = \int_0^{+\infty} D(0,t)\tilde{FVA}(t)dt$.

Observe that, $V(t)$ is stochastic, but we do not know its law. However, we could compute its expectation thanks to eq. (4.2):

$$\mathbb{E}\left[V(t)\right] = \frac{\tilde{FVA}(t)}{\tilde{FS}(t)}. \tag{4.3}$$

Then, we apply the Delta-Gamma method (see appendix E) by using a suitable selection of market risk factors $\{x_i(t)\}_i$ (*e.g.* the stock price of portfolio components, or the underlying price of the portfolio derivatives). Equations 4.4 and 4.5 report the results.

$$V(t) \approx a(t) + \sum_{i=0}^{N-1} b_i(t)y_i(t) + \frac{1}{2}\sum_{i,j=0}^{N-1} D_{ij}\left(y_i(t)y_j(t) - \mathbb{E}[y_i(t)y_j(t)]\right) \tag{4.4}$$

$$\begin{aligned}
\bar{x}_i &:= \mathbb{E}[x_i(t)] \\
y_i(t) &:= x_i(t) - \bar{x}_i \\
a(t) &:= \mathbb{E}[V(t)] \\
b_i(t) &:= \mathbb{E}\left[\frac{\partial V(t)}{\partial \bar{x}_i}\right] \\
D_{ij} &:= \mathbb{E}\left[\frac{\partial^2 V(t)}{\partial \bar{x}_i \partial \bar{x}_j}\right]
\end{aligned} \tag{4.5}$$

Therefore, eq. (4.6) approximates the instantaneous FVA value.

$$\begin{aligned}
&FVA(t) = \mathbb{E}\left[FS(t)V(t)\right] \approx \\
&\approx \mathbb{E}\left[FS(t)\left(a(t) + \sum_{i=0}^{N-1} b_i(t)y_i(t) + \frac{1}{2}\sum_{i,j=0}^{N-1} D_{ij}\left(y_i(t)y_j(t) - \mathbb{E}[y_i(t)y_j(t)]\right)\right)\right]
\end{aligned} \tag{4.6}$$

We split the terms of eq. (4.6) to obtain the following relations:

$$\begin{aligned}
\mathbb{E}\left[FS(t)a(t)\right] &= \mathbb{E}\big[FS(t)\mathbb{E}[V(t)]\big] = \frac{\mathbb{E}\big[FS(t)\mathbb{E}[V(t)\tilde{FS}(t)]\big]}{\tilde{FS}(t)} = \frac{\mathbb{E}\left[FS(t)\tilde{FVA}(t)\right]}{\tilde{FS}(t)} = \\
&= \tilde{FVA}(t)\left(\underbrace{\frac{\mathbb{E}\big[FS(t)\big]}{\tilde{FS}(t)} - 1}_{:=\alpha(t)} + 1\right) = \tilde{FVA}(t)\big(\alpha(t)+1\big),
\end{aligned} \tag{4.7}$$

$$\begin{aligned}\mathbb{E}\left[\mathrm{FS}(t)y_i(t)b_i(t)\right] &= \mathbb{E}\left[\mathrm{FS}(t)y_i(t)\mathbb{E}\left[\frac{\partial V(t)}{\partial \bar{x}_i}\right]\right] = \frac{\mathbb{E}\left[\mathrm{FS}(t)y_i(t)\mathbb{E}[\frac{\partial V(t)}{\partial \bar{x}_i}\tilde{\mathrm{FS}}(t)]\right]}{\tilde{\mathrm{FS}}(t)} = \\ &= \frac{\mathbb{E}\left[\mathrm{FS}(t)y_i(t)\frac{\partial \mathrm{F\tilde{V}A}(t)}{\partial \bar{x}_i}\right]}{\tilde{\mathrm{FS}}(t)} = \frac{\partial \mathrm{F\tilde{V}A}(t)}{\partial \bar{x}_i}\underbrace{\frac{\mathbb{E}\left[\mathrm{FS}(t)y_i(t)\right]}{\tilde{\mathrm{FS}}(t)}}_{:=\beta_i(t)} = \frac{\partial \mathrm{F\tilde{V}A}(t)}{\partial \bar{x}_i}\beta_i(t),\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\mathbb{E}\left[\mathrm{FS}(t)D_{ij}\left(y_i(t)y_j(t) - \mathbb{E}[y_i(t)y_j(t)]\right)\right] = \\ &= \mathbb{E}\left[\mathrm{FS}(t)\mathbb{E}\left[\frac{\partial^2 V(t)}{\partial \bar{x}_i \partial \bar{x}_j}\right]\left(y_i(t)y_j(t) - \mathbb{E}[y_i(t)y_j(t)]\right)\right] = \\ &= \frac{\partial^2 \mathrm{F\tilde{V}A}(t)}{\partial \bar{x}_i \partial \bar{x}_j}\underbrace{\frac{\mathbb{E}\left[\mathrm{FS}(t)\left(y_i(t)y_j(t) - \mathbb{E}[y_i(t)y_j(t)]\right)\right]}{\tilde{\mathrm{FS}}(t)}}_{:=\gamma_{ij}(t)} = \frac{\partial^2 \mathrm{F\tilde{V}A}(t)}{\partial \bar{x}_i \partial \bar{x}_j}\gamma_{ij}(t).\end{aligned} \tag{4.9}$$

Then, by replacing equations (4.7), (4.8) and (4.9) into eq. (4.6), we derive:

$$\mathrm{FVA}(t) \approx \mathrm{F\tilde{V}A}(t)\big(\alpha(t) + 1\big) + \sum_{i=0}^{N-1}\frac{\partial \mathrm{F\tilde{V}A}(t)}{\partial \bar{x}_i}\beta_i(t) + \sum_{i,j=0}^{N-1}\frac{\partial^2 \mathrm{F\tilde{V}A}(t)}{\partial \bar{x}_i \partial \bar{x}_j}\gamma_{ij}(t). \tag{4.10}$$

In this way, we have approximated the instantaneous FVA (and therefore the FVA) of the new model trough the already implemented one.
More interesting is the decomposition we have obtained. Every coefficients $\alpha(t)$, $\beta_i(t)$ and $\gamma_{ij}(t)$ have a specific meaning. In particular, $\alpha(t)$ is a first order Funding Spread term. While the correlation between the Funding Spread $\mathrm{FS}(t)$ and the market risk factor $x_i(t)$ drives $\beta_i(t)$. Therefore, $\{\beta_i(t)\}_{i=1}^N$ identifies the correlation subsisting between $\mathrm{FS}(t)$ and $V(t)$; while $\gamma_{ij}(t)$ is a higher order term, similar to $\beta_i(t)$.

What we achieve is an estimation of the correlation which subsists between market factors and funding component.
According to the new model and the internal one, we could compute $\beta_i(t) = \frac{\mathbb{E}\left[\mathrm{FS}(t)y_i(t)\right]}{\tilde{\mathrm{FS}}(t)}$, to obtain the desired implicit Wrong-Way Risk quantification.
Finally, observe that, we did not make any assumption regarding the correlation structure of the model. Therefore, the WWR we found was the implicit one present in the internal FVA system of the bank.
Since the FVA is generally a portfolio computation, the Wrong-Way Risk effect could be small [Rui14]. Actually, thanks to this approach, a financial institution could quantify it, specifically for his/her model. Then, he/she could decide to manage it or not.
Nevertheless, since β_i refers to the single x_i, this approach estimates the WWR on single market factors and not on its whole. Moreover it requires sensitive information about the institution specific FVA.

Hence, in the following chapters, we propose our method to give our personal answer about the WWR consideration question not leveraging on this general framework but on specific example where we can see "hands on" the role of the correlations.

5. Modelling the WWR component of the FVA in an interest rate derivative

In the previous chapters, we have performed a review of the literature to emphasize two points:

- how to compute the Funding Value Adjustment with different approaches;
- how to consider the Wrong-Way Risk.

In this section we propose a method to compute the FVA for a plain vanilla interest rate derivative by also considering the WWR. Actually, we want to verify the presence of the Wrong-Way Risk.

As we have mentioned in section 2.3, to simplify the implementation into banking systems, financial institutions usually assume the independence between funding and market risk factors. Therefore, the independence assumption allows a leaner computation, leveraging on exposures that might be available out of CVA and CCR computation.

Although the complexity of real portfolios justifies the use of sophisticated models, the impact of WWR could be better understood by using few risk factors and simpler models. Hence, contrary to the CVA simulations, in which all risks factors were considered, we prefer to use few of them.
Indeed, the CVA simulations use all the risk factors which influence the portfolio. However, in this way, it is difficult to appreciate real effects and dependencies.
The aim of this thesis is to investigate the WWR effect, hence

- we reduce the number of the risk factors - we will use three risk factors (discount factor, forward rate and funding spread);
- we simplify the model by limiting the number of stochastic drivers;
- we simplify the payoff by considering plain vanilla interest rate derivatives (in particular, in chapter 6 we will study the Interest Rate Swap case);
- we simplify the correlation structure by identifying three stochastic drivers - it is difficult to fully analyze and understand the main features of a large matrix with respect to a 3x3 one.

Therefore, we take a cue from the literature to construct a simplified model with three stochastic factors of which we compute the correlation.
In particular, we start from the FVA formulation, and then we propose a model for market and funding factors. We explain how to capture the correlation between them, and finally, we identify the WWR contribute.

For sake of simplicity, we select a plain vanilla interest rate derivative without the credit support annex. In order to identify the Wrong-Way Risk impact, we prefer to avoid equity derivative and collateralized transaction which need more sophisticated treatments. For example, Piterbarg describes a possible model approach in [Pit10], where the CSA presence are managed. Nevertheless, in this study, we prefer to avoid unnecessary complexity, like the other choices we made. The understandability remains one of the primary purposes of this document.

Taking a cue from section 4.2, we assume the same rate for costs and benefits ($\mathsf{FS} = \mathsf{FS}^C = \mathsf{FS}^B$) because we prefer to keep the model simple without adding unnecessary complications. We want to investigate about the WWR impact altogether, without the distinction between Right-Way and Wrong-Way Risk, which could be a possible future work. Moreover, we use a stochastic process to model the funding curve.
Finally, we identify as market risks the discount factor and the forward rate, since we are considering an interest rate derivative.
Therefore, the FVA is

$$\mathsf{FVA} = \int_0^{+\infty} D(0,t)\mathbb{E}\left[\mathsf{FS}(t)V(t)\right] dt, \tag{5.1}$$

where the derivative value V is function of the forward rate and the future discount.
Thus, we have reached the simplification purpose by including only three risk factors:

- two for the market risks - the discount and the forward rate;
- one for the funding risk - the funding rate.

The idea is to model the WWR as driven by a correlation between this three risk factors. Naturally, the chosen rate models influences the correlation structure. Therefore, we fix these models to present some numeric results.
Notice that this is a reduced version of the Cross-Currency model (see section 3.2). Indeed, we use the correlation between processes to induce a relation between market and funding components instead of the correspondent foreign exchange rate. Moreover, we reduce the stochastic drivers to three, rather then one for each rates maturity. Actually, this is the most simplified version of the Cross-Currency approach.

In the next sections, we will show the rates models that we have chosen, and how to set their correlation to quantify the Wrong-Way Risk impact.

5.1. Rates modelling

In the following part, we show how to model the chosen risk factors: funding spread, forward rate and discount factor.
In particular, we decide to model the short rate $r(t)$ of the discount factor $D(t,T)$ according to the 1-factor Hull & White model (HW). Beyond historic reason, this choice is due to different aspects. The HW model is easy to understand and implement, and it permits negative rates. Moreover, it has a close formula for the plain vanilla interest rates derivatives, which is an useful tool for calibration purposes. Finally, it has only one stochastic driver.

The model choice adopted for the forward and funding rates is not so intuitive. As we have mentioned before, we search a model as less stochastic drivers as possible. Moreover, we want a model which guarantees a simple interpretation, otherwise we may fail to investigate about the WWR impact. For example, the Cross-Currency approach, proposed in section 3.2, would be too difficult to interpret. Therefore, as we said before, we prefer to avoid it, although we have taken a cue from it.

We choose to model the spread of the forward rate through a Geometric Brownian motion (GBM). The spreads modelling is natural, since they were born during the 2007/2008 credit crisis (see chapter 2). Actually, it is common practice to use spreads over a rate in order to capture different time risks due to different maturities[1] (tenor basis decomposition) [BT20; Gre15a; Gre15b].
Therefore, we set the forward rate defined in t maturing between s and T as

$$F(t,s,T) = zR(t,s,T) + \Phi(t,s,T), \tag{5.2}$$

where $zR(t,s,T)$ is the zero rate corresponding to the discount factor

$$D(t,s,T) = \frac{D(t,T)}{D(t,s)} = e^{-zR(t,s,T)(T-s)}. \tag{5.3}$$

The reason behind the GBM choice is due to the need of positive spreads. Indeed, the forward spread over an OIS rate was almost always positive during the past twelve-thirteen years[2], therefore we decide to use a positive defined stochastic process to model it.

[1] For example EURIBOR3M is different from EURIBOR6M, despite of they have the same meaning.

[2] The forward spread over the ESTER was always positive, while the forward spread over the EONIA was almost always positive. Since in chapter 6 we use the ESTER curve (see section 6.1), we model it as a positive quantity without loss of generality.

Moreover, by considering the LIBOR decommission[3], which will involve deterministic spreads, we prefer to use a constant process to which we add a volatility, *i.e.* the Geometric Brownian motion centered on current data. If this possibility becomes real, it will not be necessary to change the model, but it will be enough to impose the volatility equal to zero.

Instead, we decide to divide the funding spread FS in two components: Φ and χ.

$$FS(t, s, T) = \Phi(t, s, T) + \chi(t, s, T) \tag{5.4}$$

where t represents the time in which the rate is computed, while $[s, T]$ is the time interval in which it matures ($t < s < T$).
Notice that, Φ identifies the forward spread, which is a market factor, common for all institutions; while, we use χ as a specific institution funding component. In this way, we identify two funding elements:

- the systemic component Φ - which refers to the market, it is common for all financial institution, and it has an associated market risk;
- the idiosyncratic component χ - which is different for all financial institution, and it drives its specific FVA.

Therefore, different institutions have different idiosyncratic factor, thus, χ identifies the funding risk.
Clearly, the idiosyncratic/systemic division is a working assumption made for practical purposes, to simplify the picture and understand the interplay of the different components. If the two components were correlated, we would incur in the Wrong-Way Risk.
For this reason, we do not assume a priori the uncorrelation, and we will verify if this hypothesis is actually correct.

Notice that, this is an unusual decomposition, but is useful to derive a real funding spread replication. Moreover, we have taken inspiration from Rosen and Saunders in [RS12]. They use an idiosyncratic/systemic decomposition to model the credit event. Unfortunately, we do not need a credit description, since we are interested in the Funding Value Adjustment, so we use their idea for our funding spread model.

Since the idiosyncratic component χ is an add-on due to the funding risk, we need a positive definite stochastic process. Hence, we chose a second Geometric Brownian motion.
This seems the most natural choice because the funding spread is almost always

[3]The LIBOR decommissioning (fallback) refers to the transition to a new reference rate. All operation linked to LIBOR will probably move to a constant spread. So, over time, there will be no more operations connected to LIBOR. For more information about the LIBOR fallback, refer to [Hen19; Hen18; The19a].

assumed constant. So, as for the forward spread, we use a constant process on which we add a volatility.
Finally, we need a model which permits to identify the correlation with as less parameters as possible to deeply understand its effects. Therefore, this is the most suitable choice.

If we adopted a two-factor Gaussian model (G2++, [BM07]), we would not always get the characteristics that we need, *i.e.* positive spreads between forwarding and discounting. Instead, the trifactorial model with the two GBM components guarantees these properties.
A reasonable alternative is a LIBOR Market Model (LMM, [BM07]), but we would lose tractability and simplicity.
Although it is not a standard approach, we decide to use this trifactorial model (with two GBM components) because it has the best trade-off about simplicity, tractability, positivity, adaptability and number of parameters.

Let's introduce the 1-factor Hull & White model. We call a the mean reversion parameter, and σ_r the volatility. From theory ([BM07]), the short rate is equal in law to

$$r(t) = \mathbb{E}^T\left[r(t)\middle|\mathcal{F}_s\right] + \sqrt{Var^T\left(r(t)\middle|\mathcal{F}_s\right)}\mathcal{Z}^r, \tag{5.5}$$

where $\mathcal{Z}^r$ is a Standard Normal random variable, and T identifies the T-forward measure. While, the others terms are

$$\begin{aligned} \mathbb{E}^T\left[r(t)\middle|\mathcal{F}_s\right] &= \left(r(s) - \alpha(s)\right)e^{-a(t-s)} - M^T(s,t) + \alpha(t), \\ Var^T\left(r(t)\middle|\mathcal{F}_s\right) &= \frac{\sigma_r^2}{2a}\left[1 - e^{-2a(t-s)}\right], \end{aligned} \tag{5.6}$$

$$\begin{aligned} \alpha(t) &= f^M(0,t) + \frac{\sigma_r^2}{2a^2}(1 - e^{-at})^2, \\ M^T(s,t) &= \frac{\sigma_r^2}{a^2}\left[1 - e^{-a(t-s)}\right] - \frac{\sigma_r^2}{2a^2}\left[e^{-a(T-t)} - e^{-a(T+t-2s)}\right]. \end{aligned} \tag{5.7}$$

$f^M(0,T) = -\frac{\partial \ln(D^M(0,T))}{\partial T}$ is the market instantaneous forward rate at time 0 for the maturity T, and $D^M(0,T)$ is the market discount factor for the maturity T.

We could obtain the short rate $r(t)$ by simulating $\mathcal{Z}^r$. Therefore, we use it to get the future discount factor from T to t ($T > t > 0$)

$$D(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{5.8}$$

where

$$\begin{aligned} A(t,T) &= \frac{D^M(0,T)}{D^M(0,t)} exp\left\{B(t,T)f^M(0,t) - \frac{\sigma_r^2}{4a}(1 - e^{-2at})B(t,T)^2\right\}, \\ B(t,T) &= \frac{1}{a}\left[1 - e^{-a(T-t)}\right]. \end{aligned} \tag{5.9}$$

Finally, we present the GBM evolution. Let's call μ_Φ and μ_χ the constant drift parameters, σ_Φ and σ_χ the positive volatilities, while Φ_0 and χ_0 the initial conditions. Therefore, the dynamics for Φ and χ are

$$\begin{cases} d\Phi(t,s,T) = \mu_\Phi \Phi(t,s,T)dt + \sigma_\Phi \Phi(t,s,T)dW_t^\Phi \\ \hfill \Phi(0,s,T) = \Phi_0, \end{cases} \tag{5.10}$$

$$\begin{cases} d\chi(t,s,T) = \mu_\chi \chi(t,s,T)dt + \sigma_\chi \chi(t,s,T)dW_t^\chi \\ \hfill \chi(0,s,T) = \chi_0. \end{cases} \tag{5.11}$$

Thanks to Itô Lemma ([Hul09; Bjö09]) we derive the solution of equations 5.10 and 5.11, which are equal in law to

$$\begin{aligned} \Phi(t,t,T) &= \Phi_0 e^{(\mu_\Phi - \frac{\sigma_\Phi^2}{2})t + \sigma_\Phi \mathcal{Z}^\Phi \sqrt{t}}, \\ \chi(t,t,T) &= \chi_0 e^{(\mu_\chi - \frac{\sigma_\chi^2}{2})t + \sigma_\chi \mathcal{Z}^\chi \sqrt{t}}, \end{aligned} \tag{5.12}$$

where $\mathcal{Z}^\Phi$ and $\mathcal{Z}^\chi$ are Standard Normal random variable.

Notice that, the drifts permit to center the processes on the market observable quantities (see section 6.2.2). Therefore, we could use these two processes to simulate a fluctuation on the observable rates, which is exactly our aim.

5.2. Correlation structure

In section 5.1, we have presented the models for the discount, funding and forward factors. Thanks to eq. (5.2), (5.4), (5.5) and eq. (5.12), we have simple formulas to simulate them. Moreover, we have identified only three stochastic drivers ($\mathcal{Z}^r$, $\mathcal{Z}^\Phi$ and $\mathcal{Z}^\chi$). Therefore, we could study and understand the correlation impact more precisely.

The simulation is achieved by extracting a sample from a standard normal distribution for each rate.
We did not assume anything about the correlation of these three processes ($D(t,T)$, $\Phi(t,t,T)$ and $\chi(t,t,T)$) because we will use it to model the Wrong-Way Risk component in the Funding Value Adjustment.
In particular, we impose the WWR through the correlation of their stochastic part, which are three Standard Normal variables.

If we were in a no-Wrong-Way Risk world, these processes would be uncorrelated, but, since we want to estimate the WWR impact, they are not.
Instead of simulate three separate random variables $\mathcal{Z}^r$, $\mathcal{Z}^\Phi$ and $\mathcal{Z}^\chi$, we simulate

a single multivariate Standard Normal variable $\mathcal{Z}$ with three dimensions.
After that, each component of $\mathcal{Z}$ will be associated to one process: $\mathcal{Z} = \begin{bmatrix} \mathcal{Z}^r \\ \mathcal{Z}^\Phi \\ \mathcal{Z}^\chi \end{bmatrix}$.
To obtain the desired effect, the correlation through the components of $\mathcal{Z}$ is imposed.

$$\mathcal{Z} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \Sigma\right)$$
$$\Sigma = \begin{bmatrix} 1 & c_{r,\Phi} & c_{r,\chi} \\ c_{r,\Phi} & 1 & c_{\Phi,\chi} \\ c_{r,\chi} & c_{\Phi,\chi} & 1 \end{bmatrix} \tag{5.13}$$

Thanks to matrix Σ, we can variate the WWR weight in the model. We could impose some reasonable values to fit an economic features, we could calibrate it from market or historical data, or we could set it equal to the identity matrix if we want to assume no-Wrong-Way Risk in the model.
Notice the parallelism with the Cross-Currency approach (see section 3.2). Instead of the Foreign Exchange rate and the correlation among Brownian motions, we model the WWR through the covariance matrix of the random components of the rates.

The aim of this study is to quantify the WWR impact. Therefore we will calibrate Σ from historical series, and then, we will compare the resulted FVA with the ones obtained by varying the two components ($c_{r,\chi}$ and $c_{\Phi,\chi}$) in [-1,1].
A more detailed explanation about the calibration procedure is given in section 6.2.3.

We could also estimate the WWR impact by setting Σ as the identity matrix ($\Sigma = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$). So, the Wrong-Way Risk component will be the difference in the FVA value of the two results obtained

$$WWR = FVA_\Sigma - FVA_I. \tag{5.14}$$

Moreover, if we wanted to assume deterministic spreads, we would impose the respective volatilities to zero. Hence, after computing the correspondence FVA value (FVA_{det}), the Wrong-Way Risk would be the difference

$$WWR = FVA_\Sigma - FVA_{det}. \tag{5.15}$$

In chapter 6, we will compare the results with the constant idiosyncratic funding spread case. In particular, we will reproduce the usual banks' practice by setting χ as a constant and equal to its market spot value; then we will test this approximation to understand if it is reliable.

The strength of these approaches is their simplicity, since we select only three stochastic drivers. Actually, they are easy to understand, to calibrate, and to implement.

Finally, the core of these methods is in the correlation structure of the standard normal random component of these three processes. Therefore, one could change the model of some rates and use the same methodology. The only constraint is that the new processes must be equal in law to a standard normal transformation.

In the next chapter, we will compute two FVA values: through the historical calibrated Σ, and through the constant funding spread. Then, we will compare the results by varying the correlation structure. In this way, we will achieve a complete analysis of the WWR.

6. Numerical Results

In the previous chapters, we have presented some Funding Value Adjustment models and two methodologies to estimate the WWR component in the FVA. In the following part, instead, we show some numeric results.
We adopt the approach explained in chapter 5. In particular, we compute the Funding Value Adjustment by considering the Wrong-Way risk for a receiver Interest Rate Swap (IRS) through a Monte Carlo method [Gla13].

We assume the 19th of November 2020 as the market date. So we consider a receiver IRS with the following settings:

- settlement date: 20th of November 2020
- maturity: T=10y
- fixed leg:
  - fixed rate = At-The-Money (ATM)
  - day count convention = Act/360
  - payment frequency = annually
- floating leg:
  - floating rate = EURIBOR3M
  - day count convention = 30/365
  - payment frequency = quarterly

We have decided to use a plain vanilla IRS as case study because it is the simplest interest rate derivatives. Therefore, we could better understand the WWR impact. Indeed, exotic payoffs may distort the results, and they might need more risk factors. In addition, we avoid optionality to keep the symmetry between positive and negative exposures. Thus, we have adopted this naive choice by considering an Interest Rate Swap.

We recall that we use the 1-factor Hull & White (HW) for the discount short rate and a Geometric Brownian Motion (GBM) for the zero rates of the forward spread and the funding idiosyncratic component. Moreover, we use the following market curves to calibrate them:

- discount curve: ESTER;

- forward spread: EURIBOR3M-ESTER;
- funding idiosyncratic component: BND_BBB[1].

In section 6.1 we will give a deeply explanation about the choice of the market curves.

Although we model the three curves as continuous objects, they are a discrete set of values. In practice, we have to select a pillar structure of Zero Coupon Rates. Then, we use linear interpolation to extract the rates that we need. So, we choose ten pillars yearly spaced.

Since our aim is to compute the FVA value by using a Monte Carlo approach, we need a time grid in which the future rates are simulated. We decide to use an eight time steps grid. It is useless to simulate rates after the nine-th year because the Swap maturity is ten year, therefore the FVA value will be zero after the ten-th year.
So, the chosen time grid is $t_1 = 1y < t_2 = 2y < ... < t_9 = 9y$. For every simulation time t_i, for $i \in \{1, ..., 9\}$, we have set the number of simulation N_{sim} equals to 10^4, in order to cover the majority of possible scenarios. 10^4 simulation is a fair compromise between accuracy and feasibility: already with 10^4 we get a fair representation of the tails, while increasing the simulation number would imply too-large computational times.

In the following parts, we will present the motivation behind the market curves chosen, the calibration procedure, the simulation algorithm, and the obtained results.

6.1. Curve selection

In this section, we analyze the market curve alternatives for our specific purpose. In particular, we try to understand the reason behind our choice.

We recall that we decide to use three different market curves for the calibration: the ESTER curve for the discount, the EURIBOR3M-ESTER curve for the forward spread and the BND_BBB curve for the funding spread.

The choice that we made for the forward spread market curve is the most natural one because we are considering an Interest Rate Swap indexed to the EURIBOR3M. Since we are modelling the forward spread, and since we have chosen the ESTER curve for the discount, we have simply obtained the forward spread curve as the difference between the EURIBOR3M and the ESTER curves

[1]With BND_BBB we refer to the Markit curve obtained from the average spread of the European financial institutions rated BBB (see section 6.1).

(EURIBOR3M-ESTER).
It is clear that if the floating leg had been indexed to another market rate, we would have used that market rate instead of the EURIBOR3M.

For the discount market curve we have to use the collateral rate. In this exercise we consider an unmargined contract. We choose to discount with risk free rates anyway, following the standard market practice. Since we are interested in the Euro-zone panel, possible choices are EONIA or ESTER rates. They refer to the weighted average rate of overnight unsecured transactions, but the ESTER rate is conceived to replace the EONIA one (see [The19b; The19a]), therefore, according to the current practical methodologies, we prefer the ESTER choice.

Last but not least, there is the idiosyncratic component of the funding spread curve, which we calibrate through the BOND_BBB market curve. In particular, we call BOND_BBB the Markit curve given by the average bond spread of the European financial institutions rated BBB. Actually, it is a proxy for the funding spread of a typical corporate/financial entity.
We decide to adopt this curve because it represents a generic financial institution, so the results would be not specific. Our aim is to investigate the presence of the Wrong-Way Risk component, and it is not to estimate the FVA of a particular institution. Naturally, if we were interested in a specific bank, we would have used its spread curve.
Moreover, this curve guarantees a not too low starting level of funding spread, which could invalidate our analysis. Therefore, we think that our choice is fully consistent.

What we did is one of the most common practice, and permits us to investigate about the rates correlations. Actually, we are identifying into the funding spread two distinct components, one driven by the market (EURIBOR3M-ESTER) and the other driven by the financial entity (BOND_BBB). Thanks to this choice, we can study the market influence on the funding factor, *i.e.* the correlation of the funding idiosyncratic component with the discount factor, and with the forward spread.

6.2. Calibration

We adopt a calibration which is a mixture between risk neutral and historical. In particular, we use:

- a risk neutral calibration for the discount short rate (HW) parameters and for the two GBM drifts (for the forward spread and the funding idiosyncratic component);

- an historical calibration for the correlation structure and the GBM volatilities (for the forward spread and the funding idiosyncratic component).

The reasons why we chose this mixture approach is due to the lack of liquid instruments in the market. Clearly, using liquid instruments to calibrate the model is always the best choice because it produces a more reliable replication of the market. Unfortunately, there are not enough of them to calibrate all the parameters. We will present a detailed explanation in the following parts.

Furthermore, we decide to cap the historical calibration to three years because we do not want to bias the results with too old data. Moreover, in a three year sample, there is enough information to obtain significant results and to avoid Credit and Sovereign Debt crises, which could mislead the results. Actually, if we used more data, we would not obtain a reliable representation of the present market scenario. On the other hand, if we used less data, some parameterization like correlation could be less stable.
A possible choice is the weighted historical calibration, but it would introduce an unnecessary complication to the study. Therefore, three years is a trade-off between reliable representation, simplicity and volatility containment.
Finally, notice that the sampling of the historical time series is daily, but there are other possible alternatives (*e.g.* weekly or monthly). Since this choice is completely arbitrary, we used the daily one, which guarantees the widest calibration set.

We will explain in details the calibration procedure in the following parts.
The output of the calibration is:

$$\begin{aligned} a &= 0.25808169 \\ \sigma_r &= 0.01065507 \end{aligned} \tag{6.1}$$

$$\begin{aligned} \{\sigma_\Phi(p)\}_{p=1y}^{10y} &= \{\, 0.242\ 0.194\ 0.169\ 0.151\ 0.139\ 0.132\ 0.126\ 0.122\ 0.119\ 0.117 \,\} \\ \{\sigma_\chi(p)\}_{p=1y}^{10y} &= \{\, 0.235\ 0.232\ 0.231\ 0.228\ 0.217\ 0.199\ 0.179\ 0.162\ 0.148\ 0.137 \,\} \end{aligned} \tag{6.2}$$

$$\Sigma = \begin{bmatrix} 1.00 & 0.0836 & 0.0127 \\ 0.0836 & 1.00 & 0.5157 \\ 0.0127 & 0.5157 & 1.00 \end{bmatrix} \tag{6.3}$$

6.2.1. Short rate calibration

As we presented before, we decide to model the discount short rate as a 1-factor Hull & White process. In the following part, we present the reasons behind the calibration choice, we show how to calibrate the model and the results that we have obtained.

To calibrate a 1-factor Hull & White model through a risk neutral approach, it means to find the value of the two parameters a and σ_r which best fits the market data (notation of section 5.1). In particular, we use ATM Swaptions to reproduce

the market behaviour because they are some of the most liquid instruments traded in the market.
Moreover, we avoid Cap, Floor and others interest rate derivatives because our focus is on an Interest Rate Swap, and the best practice is to select some derivatives defined on it. Since the only liquid and quoted derivatives with this property are the Swaptions, we select them.

Notice that the calibration is complete by setting the Swap rate parameters to reproduce the prices of the derivatives defined on it. Theoretically, a Swaption depends on the discount and the forward curves. Unfortunately, we have only one type of quoted Swaption, therefore, we decide to use it for the discount factor. Ideally, this is equivalent to assume a constant spread for the forward rate. Hence, for FVA purposes, we add to it a noise driven by the historical-calibrated volatility (see section 6.2.2). Clearly, this is an approximation, but it catches the implicit dynamic of the forward rate.
The adopted model choice is made necessary also because there are no instrument (*e.g.* basis swaptions) that could provide implied volatilities for the remaining parameters.

Jamshidian proves that exists a close formula for the Swaption price ([BM07; Jam89]), which is the derivative type used to calibrate the short rate model. So our aim will be to find the best value for the parameters a and σ_r, which gives the close formula price more close to the market price.
In the market ATM Swaptions are quoted for a set of maturities and tenors. It is indifferent to consider the payer or the receiver option since they have the same price[2]. Nevertheless, we have to choose a type for calibration purpose, in order to fix the formulas. So, we consider the payer one.
Therefore, eq. (6.4) report the price of an ATM payer Swaption at time t with maturity T, with notional equal to 1 and defined on an ATM Swap ([BM07]). The underlying IRS has the fixed payments performed in $\{t_i\}_{i=1}^N$ ($t_i > t_{i-1} > T$ $\forall i \in \{2, ..., N\}$).

$$PS(t, T, K, \{t_i\}_{i=1}^N) = \sum_{i=1}^{N} c_i PB_{eu}(t, T, t_i, K_i), \tag{6.4}$$

where K is the ATM strike, $c_i = \begin{cases} K\tau_i & i<N \\ 1+K\tau_i & i=N \end{cases}$, and τ_i is the year fraction between t_{i-1} and t_i (imposing $t_0 = T$). While $K_i = A(T, t_i)e^{-B(T,t_i)r^*}$, and r^* satisfies the relation[3]

$$\sum_{i=1}^{N} c_i A(T, t_i)e^{-B(T,t_i)r^*} = 1. \tag{6.5}$$

[2]Payer and receiver Swaptions ATM have the same price (considering equal maturity, tenor and underlying).
[3]r^* is computed using Brent's method through the `uniroot` command in R. We set a tolerance of 10^{-7}, while the lower and the upper bounds equal to -1 and 1 respectively.

Finally, through the Jamshidian formula ([Jam89]), we obtain $PB_{eu}(t, T, s, X)$, *i.e.* the price of an European Put option at time t with strike X, maturity T and defined on a bond with maturity s.

$$PB_{eu}(t, T, s, X) = XD(t, T)\mathcal{N}(-h + \sigma_p) - D(t, s)\mathcal{N}(-h), \tag{6.6}$$

where $\mathcal{N}$ is the cumulative density function of the standard Gaussian distribution, and

$$\begin{aligned} \sigma_p &= \sigma_r \sqrt{\frac{1 - e^{-2a(T-t)}}{2a}} B(T, s), \\ h &= \frac{1}{\sigma_p} \ln \frac{D(t, s)}{D(t, T)X} + \frac{\sigma_p}{2}, \end{aligned} \tag{6.7}$$

while $D(t, T)$, $A(t, T)$ and $B(t, T)$ derive from eq. (5.8) and eq. (5.9).

To simplify the notation, we call PS_j^{mrk} the j-th quoted Payer Swaption price and $PS_j(a, \sigma_r)$ the corresponding price given by eq. (6.4).
Therefore, to find the parameters a and σ_r, we use a Leas Square approach, which consists in solving the following minimization problem[4]:

$$\arg \min_{a>0, \sigma_r>0} \sum_{j=1}^{J} \left(PS_j^{mrk} - PS_j(a, \sigma_r) \right)^2, \tag{6.8}$$

where J is the number of quoted Swaption used in the calibration procedure.

Observe that, according to section 5.1 notation, the market discount factor $D^M(0, T)$ is the ESTER market curve, which we have selected to represent the discount curve.
Moreover, we report the chosen Swaptions in table 6.1.
We point out that, these Swaptions have fixed and floating underlying IRS payments every six months. And, we decide to use them because they have a maturity plus tenor equal to ten, which is our interested time window (the IRS maturity).

Maturity	2y	3y	4y	5y
Tenor	8y	7y	6y	5y
Price (€)	0.01611	0.01773	0.01754	0.01647

Table 6.1.: Swaption market data used for calibrating the 1-factor Hull & White model.

Notice that, it would be useless to include other Swaptions in the calibration procedure since they would generate only a slight adjustment. On the other

[4] We solve problem 6.8 using the Nelder–Mead method through the `optim` function on R. We set max number of iterations to 10^4, the tolerance to 10^{-6}, while the starting parameters values to (0.5,0.05) for a and σ_r respectively.

hand, widening the calibration set produces a sensible slowdown of the program. Therefore, these four Swaptions have the best trade-off between efficiency and accuracy. Moreover, they full represent our interested time window.

Finally, according to eq. (6.8), we obtain the following calibrated parameter:

$$\begin{aligned} a &= 0.25808169 \\ \sigma_r &= 0.01065507 \end{aligned} \tag{6.9}$$

6.2.2. Forward Rate and Funding Spread calibration

As we explained in section 5.1, we model the forward rate through its spread. While we divide the funding spread in the sum of a systemic and an idiosyncratic component. Equations (5.2) and (5.4) report these division.
We model the forward spread and funding idiosyncratic component through two different Geometric Brownian Motions, which we want to centre on their observable market spot value ([Hul09; Bjö09]).
In particular, we use the Girsanov theorem ([Bal17]) to change the measure of the stochastic differential equations 5.10 and 5.11. In this way, we move to the measure under which the drifts are zeros. In other words, we are assuming that the two processes are martingale under another different measure:

$$\begin{cases} d\Phi(t,s,T) = \sigma_\Phi \Phi(t,s,T) d\bar{W}_t^\Phi \\ \Phi(0,s,T) = \dfrac{zR_1(T)T - zR_1(s)s}{T-s} \end{cases} \tag{6.10}$$

$$\begin{cases} d\chi(t,s,T) = \sigma_\chi \chi(t,s,T) d\bar{W}_t^\chi \\ \chi(0,s,T) = \dfrac{zR_2(T)T - zR_2(s)s}{T-s} \end{cases} \tag{6.11}$$

where $\bar{W}_t^\Phi$ and $\bar{W}_t^\chi$ are two Brownian Motions (possibly correlated), while $zR_1(T)$ and $zR_2(T)$ represent the EURIBOR3M-ESTER and the BOND_BBB market curves respectively.
Notice that, by considering the forward spread, the relation between eq. (5.10) and eq. (6.10) is

$$dW_t^\Phi = -\frac{\mu_\Phi}{\sigma_\Phi} dt + d\bar{W}_t^\Phi \tag{6.12}$$

which is exactly the Girsanov theorem application.
Roughly speaking, we have calibrated the drift of Φ through its initial condition and the market spot values[5]:

$$\mu_\Phi = \frac{1}{t} \log\left(\frac{\Phi(0,t,T)}{\Phi_0}\right). \tag{6.13}$$

[5]Mathematically, this is not rigorously correct, since we are using as solutions of eq. (5.10) and eq. (6.10) their equivalences in law. Therefore, the μ_Φ relation is an abuse of writing. Nevertheless, we report it because it clearly represents the parallelism with the risk neutral calibration.

Naturally, the same holds for the funding idiosyncratic component χ.
Then, we are actually calibrating the real drift through the market values, obtaining as solution of eq. (6.10) and eq. (6.11):

$$\begin{aligned} \Phi(t,t,T) &= \Phi(0,t,T)e^{-\frac{\sigma_\Phi^2}{2}t+\sigma_\Phi \mathcal{Z}^\Phi \sqrt{t}}, \\ \chi(t,t,T) &= \chi(0,t,T)e^{-\frac{\sigma_\chi^2}{2}t+\sigma_\chi \mathcal{Z}^\chi \sqrt{t}}. \end{aligned} \tag{6.14}$$

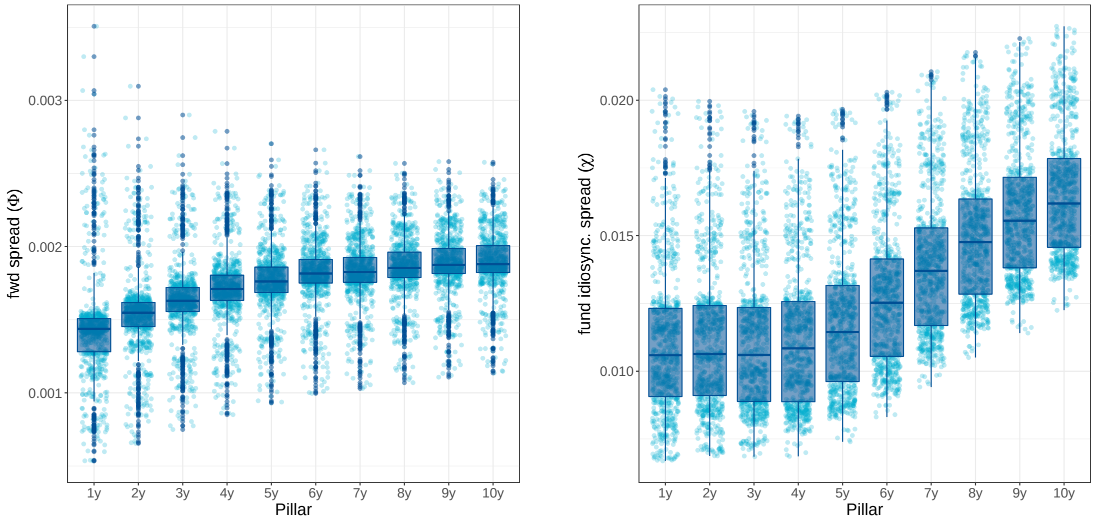
In this way we center the processes on the real observable quantities. This is actually a risk neutral calibration.

The other parameters, that we have to calibrate, are the volatilities of these two processes: σ_Φ and σ_χ.
As we said, we decided to compute them through the historical approach.
Since we have decided to discretize the market curves into ten pillars, we prefer to calibrate ten volatilities: one for each pillar. In this way, every simulated rates are closer to the real observable ones. We are imposing a very intuitive constraint: rates with different maturity have different volatilities.

Therefore, we select for each pillar $p \in \{1y, ..., 10y\}$ the correspondent three years historical market series from which we extract its volatility as a standard deviation.
Actually, since we are assuming a Geometric Brownian Motion evolution, rates are distributed as a Log-Normal; so the volatilities, that we have to calibrate, must be computed on the logarithmic transformations of the market series (log($\mathrm{EURIBOR3M} - \mathrm{ESTER}$) and log($\mathrm{BOND\_BBB}$)).
For example, let us consider the five year pillar (p=5y) for the forward spread. We first select the historical market curve EURIBOR3M-ESTER, then we filter its five-year rates over a three-year time span, and then we take the standard deviation of its logarithmic transformation to obtain $\sigma_\Phi(5y)$.
In the same way we calibrate the other volatilities, obtaining $\sigma_\Phi(p)$ and $\sigma_\chi(p)$ $\forall p \in \{1y, ..., 10y\}$.

The reason why we choose this mixture approach for the two processes is due to the lack of liquid instruments in the market. In particular, we use the risk neutral approach to fit the present term structure, and to obtain a perfect replication of the present rates, which is the best property that we could achieve. While, unfortunately, we have to adopt the historical approach for the volatilities because there are not enough liquid quoted instrument for our purpose.
One could use some Basis Swaption for the forward spread volatilities, but the results could be unreliable for the aforementioned liquidity problem.

We report in fig. 6.1 the box-plot related to the historical data used to calibrate the volatilities. We observe higher values in the long term, but more dispersion in the short one, for both Φ and χ. The short terms seem to have a higher uncertainty compared to the long one, causing higher volatilities. Moreover, the forward spread has a semi-constant



(a) Forward spread. (b) Funding idiosyncratic component.

Figure 6.1.: Box-plot of the historical observations related to the simulated pillar.

mean over the different time horizons, which could identify more stability of the market than financial crisis periods. Instead, the funding systemic component increases its mean over time. And this is natural since χ is institution-specific, it represents the institution risk, which increases during the time.
Finally, according to the explained procedure, we report the resulting calibrated volatilities in table 6.2.

Pillar	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
σ_Φ	0.242	0.194	0.169	0.151	0.139	0.132	0.126	0.122	0.119	0.117
σ_χ	0.235	0.232	0.231	0.228	0.217	0.199	0.179	0.162	0.148	0.137

Table 6.2.: Volatilities computed according to the historical calibration approach.

6.2.3. Correlation calibration

As we have discussed in the section 5.2, we use the correlation between the simulated processes to set the WWR component of the model. For this reason, the calibration approach used for the correlation matrix will be crucial. Even more if we consider the impact of the Wrong-Way Risk.

We recall that, we are considering the covariance matrix Σ which guides the dynamics of the three random components ($\mathcal{Z}^r$, $\mathcal{Z}^\Phi$ and $\mathcal{Z}^\chi$) of the three processes $D(t,T)$, $\Phi(t,t,T)$ and $\chi(t,t,T)$.
In particular Σ is a three by three matrix with the form $\Sigma = \begin{bmatrix} 1 & c_{r,\Phi} & c_{r,\chi} \\ c_{r,\Phi} & 1 & c_{\Phi,\chi} \\ c_{r,\chi} & c_{\Phi,\chi} & 1 \end{bmatrix}$. Where, for example, $c_{r,\Phi}$ identifies the correlation with the short rate process and the forward spread.

As we mentioned before, we decide to calibrate the correlation through the historical approach. As we notice for the GBM volatilities, there are not enough liquid instruments in the market to adopt a risk neutral approach.
Actually, there are Spread Options and Constant Maturity Swap (CMS), but, due to their nature, it is very difficult to use them for calibration purposes.

We use the three years historical market data for the three market curves that we have selected (ESTER, EURIBOR3M-ESTER and BOND_BBB).
Since we want a three by three correlation matrix, we need to select a reasonable pillar $\bar{p} \in \{1y, ..., 10y\}$ for the calibration. Then, we filter the three historical market curves at $\bar{p}$, and we use the resulting rates in the correlation computation.
Actually, as we have done for the volatility, we have to apply the logarithmic transformations to the market series ($\log(EURIBOR3M - ESTER)$ and $\log(BOND\_BBB)$) to extract their Gaussian component.

The selected $\bar{p}$ is the five year one because it is usually the most liquid pillar. In addition, five year can be considered a reasonable average duration of a portfolio of swaps with a typical corporate counterparty.

Let us consider for example the component $c_{r,\Phi}$. We first select the relative market curve ESTER and EURIBOR3M-ESTER, then we apply the logarithmic transformation to obtain $\log(EURIBOR3M - ESTER)$. We filter its five-year rates till three years have passed, and finally we compute its correlation to obtain $c_{r,\Phi}$.
In the same way, we calibrate the other correlations $c_{r,\chi}$ and $c_{\Phi,\chi}$.

Notice that, in the proposed model, the short rates are Gaussian. In particular, thanks to eq. (5.5), we have that $r(t) \sim \mathcal{N}\left(\mathbb{E}^T\left[r(t) \middle| \mathcal{F}_s\right], \sqrt{Var^T\left(r(t) \middle| \mathcal{F}_s\right)}\right)$.
Since short rates are Gaussian, also zero rates will be. In fact, zero rates are a linear transformation of the short rate (in the 1-factor Hull & White model framework): $zR(t,T) = r(t)\frac{B(t,T)}{T-t} - \frac{\log(A(t,T))}{T-t}$.
Therefore, this is the reason why we do not apply the logarithmic transformations to the market curve ESTER, which we use as zero rates of the discount factor.

According to the calibration procedure that we have presented, the resulting calibrated correlation matrix is

$$\Sigma = \begin{bmatrix} 1.00 & 0.0836 & 0.0127 \\ 0.0836 & 1.00 & 0.5157 \\ 0.0127 & 0.5157 & 1.00 \end{bmatrix}. \tag{6.15}$$

Finally, we report in figure fig. 6.2 the historical series that we have considered to calibrate the correlation matrix. The ESTER curve seems to be less correlated to the others, according to the resulting calibrated correlation. While the EURIBOR3M-ESTER and the BOND_BBB curves appear to have the same pattern. Moreover, to highlight the obtained results, we propose in fig. 6.3 a comparison between the Gaussian component of these three processes. Especially, we apply the log transformation to the EURIBOR3M-ESTER and the BOND_BBB curve to extract the Gaussian component. And then we normalize all of them, to obtain comparable graphical results.

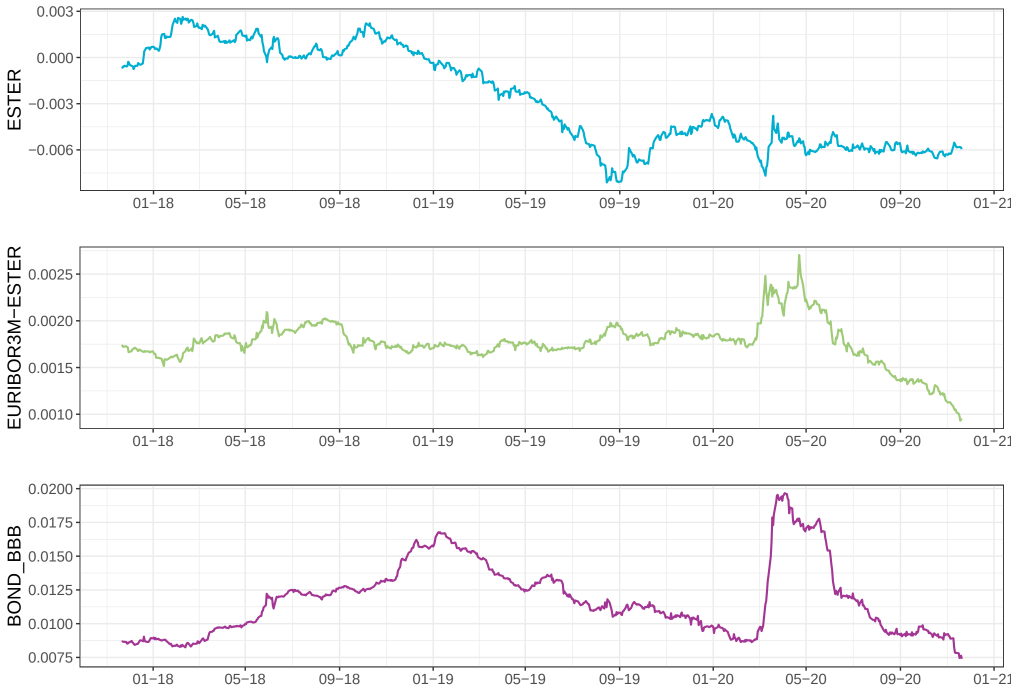


Figure 6.2.: Time series of market curves used to calibrate the correlation matrix.

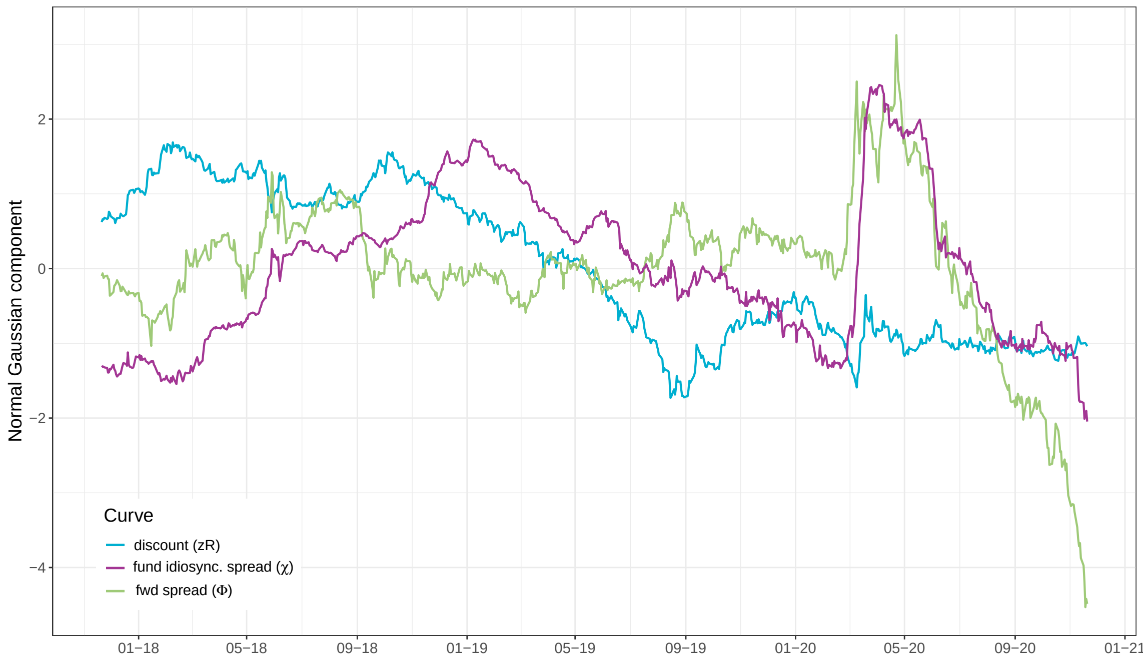


Figure 6.3.: Comparison between the Gaussian component of the historical series used to calibrate the correlation matrix.

Figure 6.3 underlines the Gaussian behaviour of the curves.
This results is actually opposed to our hypothesis. Since the EURIBOR3M-ESTER represents the funding systemic component, while the BOND_BBB the idiosyncratic funding part, we will expect no correlation. Nevertheless, we will search a quantitative confirmation in the next section (see section 6.4) since the historical data could be correlated, but their correlation may not impact severely the FVA.

6.3. Simulation

In the following section, we briefly recall the formulas used for the simulation and the adopted procedure.

We recall that the Hull & White model requires, at each simulation time, to choose a starting time s and a T-forward measure. For sake of simplicity, we decide to set s equal to the settlement date (*i.e.* 20th of November 2020) and T equal to the current simulated time t_i. In this way, $r(s)$ will be zero, and the numeraire[6] will be 1 ($D(t_i, T) = D(t_i, t_i) = 1$).

As we mentioned before, we use a Monte Carlo method with an eight time step grid: $t_1 = 1y < ... < t_9 = 9y$. Then, we set $N = 10$ and $t_N = t_{10} = 10y$, which is the maturity of the IRS.
At every simulation time t_i for $i = 1, 2, ..., 9$, we simulate $N_{sim} = 10^4$ market scenarios, and then we compute the correspondent FVA value according to the the formulation presented in chapter 5.
In particular, let's assume that we have already calibrated the model parameters (a, σ_r, $\sigma_\Phi(p)$, $\sigma_X(p)$ $\forall p \in \{1y, ..., 10y\}$, Σ), and let us consider a generic simulation time t_i ($i \in \{1, ..., 9\}$):

- we first simulate the random component $Z \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \Sigma\right)$ by extracting N_{sim} samples from a multivariate standard normal with covariance matrix equal to Σ;
- for every scenario $j \in \{1, ..., N_{sim}\}$, we compute the discount short rate as

$$r_j(t_i) = \big(r(s) - \alpha(s)\big)e^{-a(t_i - s)} - M^T(s, t_i) + \alpha(t_i) + \frac{\sigma_r^2}{2a}\left[1 - e^{-2a(t_i - s)}\right] z_j^r$$
$$\alpha(t) = f^M(0, t) + \frac{\sigma_r^2}{2a^2}(1 - e^{-at})^2 \quad (6.16)$$
$$M^T(s, t) = \frac{\sigma_r^2}{a^2}\left[1 - e^{-a(t-s)}\right] - \frac{\sigma_r^2}{2a^2}\left[e^{-a(T-t)} - e^{-a(T+t-2s)}\right],$$

where z_j^r is the first component of the j-th simulated normal Z, while $f^M(0, t)$ is the market instantaneous forward rate at time 0 for the maturity t. Then, for every pillar $p \in \{1y, ..., 10y\}$ we derive the discount factor ($D_j(t_i, t_{i+p})$) according to eq. (5.8) and eq. (5.9), and the correspondent zero rate

$$zR_j(t_i, t_{i+p}) = -\frac{\log\big(D_j(t_i, t_{i+p})\big)}{t_i + p - t_i}; \quad (6.17)$$

[6] The T-forward measure identifies the measure under which the T maturity discount factor $D(t, T)$ is the numeraire [BM07; Bjö09]. Therefore, by assuming $T = t$, we obtain a no-numeraire adjustment in the expectation value because it will be simply one.

- for every scenario $j \in \{1, ..., N_{sim}\}$, for every pillar $p \in \{1y, ..., 10y\}$, we compute the forward spread and funding idiosyncratic component as

$$\begin{aligned}\Phi_j(t_i, t_i, t_i + p) &= \Phi(0, t_i, t_i + p) e^{-\frac{\sigma_\Phi(p)^2}{2} t_i + \sigma_\Phi(p) z_j^\Phi \sqrt{t_i}} \\ \chi_j(t_i, t_i, t_i + p) &= \chi(0, t_i, t_i + p) e^{-\frac{\sigma_F S(p)^2}{2} t_i + \sigma_F S(p) z_j^\chi \sqrt{t_i}},\end{aligned} \tag{6.18}$$

where z_j^Φ and z_j^χ are the second and third components of the j-th simulated normal Z respectively, and

$$\begin{aligned}\Phi(0, t_i, t_i + p) &= \frac{zR_1(t_i + p)(t_i + p) - zR_1(t_i)t_i}{p} \\ \chi(0, t_i, t_i + p) &= \frac{zR_2(t_i + p)(t_i + p) - zR_2(t_i)t_i}{p}.\end{aligned} \tag{6.19}$$

While zR_1 and zR_2 represent the EURIBOR3M-ESTER and the BOND_BBB market curves respectively;

- for every scenario $j \in \{1, ..., N_{sim}\}$, we compute the forward and funding curves as

$$\begin{aligned}F_j(t_i) &:= \{F_j(t_i, t_i, t_i + p)\}_{p=1y}^{10y} \\ F_j(t_i, t_i, t_i + p) &= zR_j(t_i, t_i + p) + \Phi_j(t_i, t_i, t_i + p) \\ FS_j(t_i, t_i, t_i + p) &= \Phi_j(t_i, t_i, t_i + p) + \chi_j(t_i, t_i, t_i + p)\end{aligned} \tag{6.20}$$

for every $p \in \{1y, ..., 10y\}$;

- we compute the FVA value relative to the i-th simulated grid point as

$$FVA_i = \frac{\sum_{j=1}^{N_{sim}} \left[FS_j(t_i, t_i, t_{i+1}) V\left(t_i, r_j(t_i), F_j(t_i)\right)\right]}{N_{sim}}, \tag{6.21}$$

where $V\left(t_i, r_j(t_i), F_j(t_i)\right)$ is the interest rate derivatives value at time t_i in the j-th scenario, which is in our case an Interest Rate Swap (see appendix F).

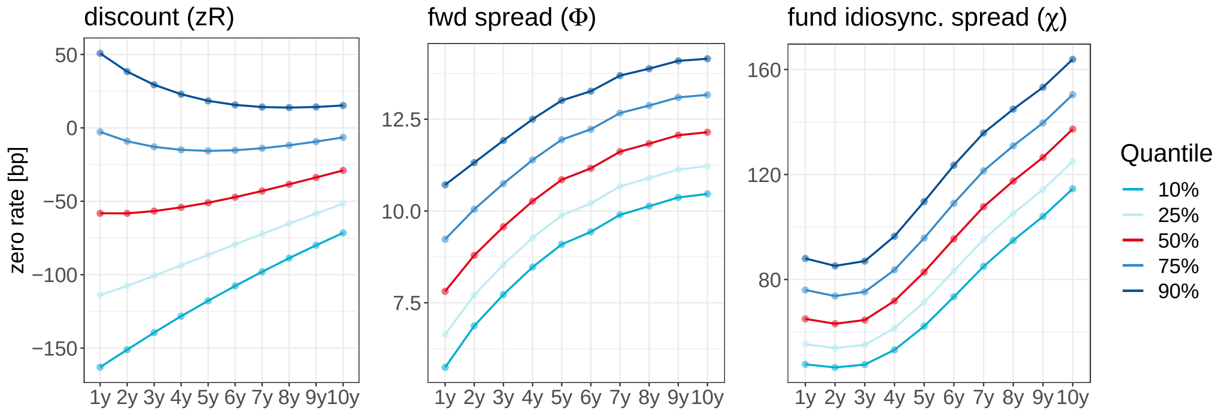
Finally, after we calculate FVA_i for every simulation time $\{t_i\}_{i=1}^{9}$, we compute the Funding value Adjustment as the discretization of the integral 5.1:

$$FVA = \sum_{i=1}^{N-1} D(0, t_i) FVA_i \delta_{t_i, t_{i+1}} \tag{6.22}$$

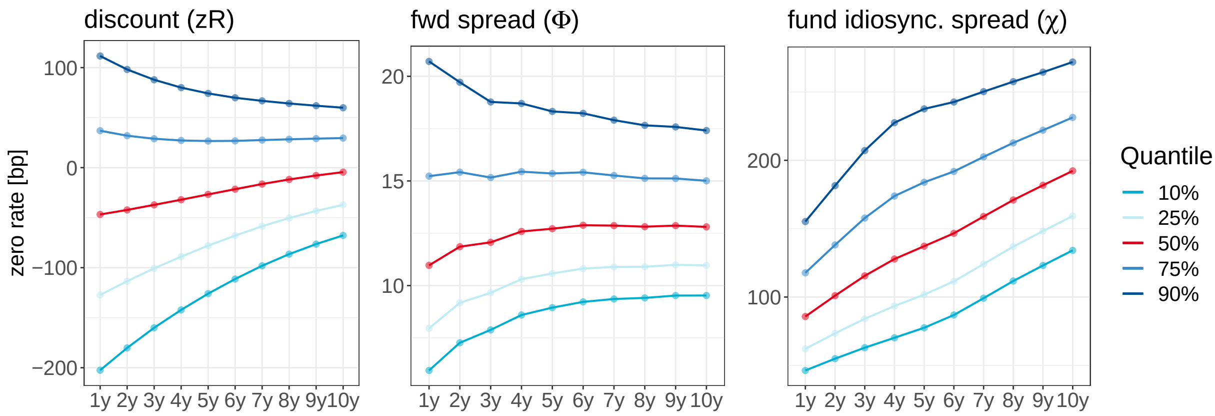
where $\delta_{t_i, t_{i+1}}$ is the year fraction between t_i and t_{i+1}.

Figure 6.4 shows the quantiles for the first, fourth and ninth simulated grid points, to highlight the behaviours of the simulated curves in the short, medium and long period respectively. In particular, we compute the distribution quantile at 10%, 25%, 50%, 75% and 90% for any simulated pillar $p \in \{1y, ..., 10y\}$.

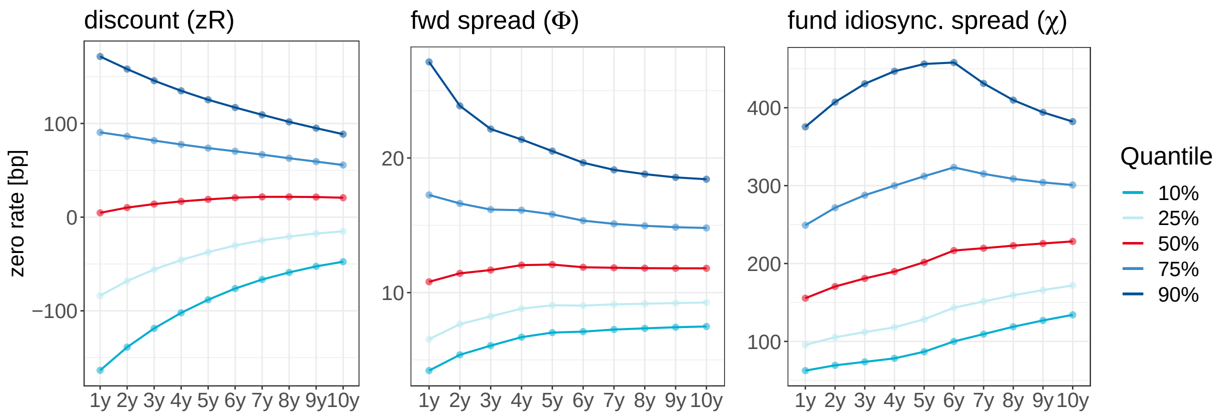
We point out, as a final remark, that the used Standard Normal simulations (z_j^Φ and z_j^χ $\forall j \in \{1, ..., N_{sim}\}$) are the same for every pillar $p \in \{1y, ..., 10y\}$. We re-scale the Standard Normal samples according to the pillar volatility ($\sigma_\Phi(p)$ and $\sigma_\chi(p)$). In this way, the stochastic drivers remain three, although maturity differentiation is performed, achieving the desired simplification.



(a) 1-st year simulated.



(b) 4-th year simulated.



(c) 9-th year simulated.

Figure 6.4.: Quantiles for discount, forward spread and funding spread.

6.4. Results

This section presents the numerical results that we have obtained by following the approach, and using the market data, described in this chapter.
R is the software employed to obtain the numerical results and the charts.

We have adopted the calibration procedure and the simulation algorithm presented in this chapter to compute the FVA value of the examined IRS.
The obtained result is:

$$\mathrm{FVA} = -0.00127€ \tag{6.23}$$

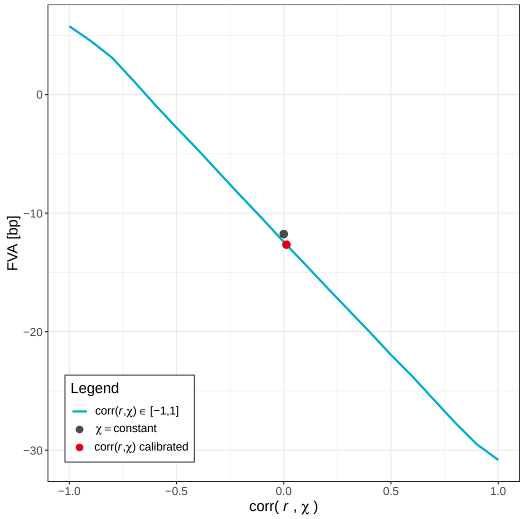
As we were expecting, the FVA value is not negligible, especially if we consider that the price of the aforementioned ATM IRS is zero.
Moreover, we observe that, entering in this IRS is a cost for the bank, because the present market condition will generate an adverse future scenario (in the average cases), and therefore the FVA of the IRS will be negative.

Figure 6.5 also reports the FVA response to a variation in the correlation structure. In particular, we have explored all possible scenarios by imposing some correlation value in the $[-1, 1]$ interval. We have defined an equispaced correlation grid $\{\mathrm{corr}_i\}_{i=1}^{11} = \{-1, -0.9, ..., 0, ..., 0.9, 1\}$. Thus, we have used corr_i, instead of $c_{r,\chi}$ and $c_{\Phi,\chi}$, for every $i \in \{1, ..., 21\}$, to obtain fig. 6.5a and fig. 6.5b respectively.
Since varying the correlation between discount factor and forward spread ($c_{r,\Phi}$) does not affect the WWR, we have not considered it. It could be important and relevant, but since it is a market-market correlation we prefer to keep it fixed and avoid an unnecessary and misleading study.

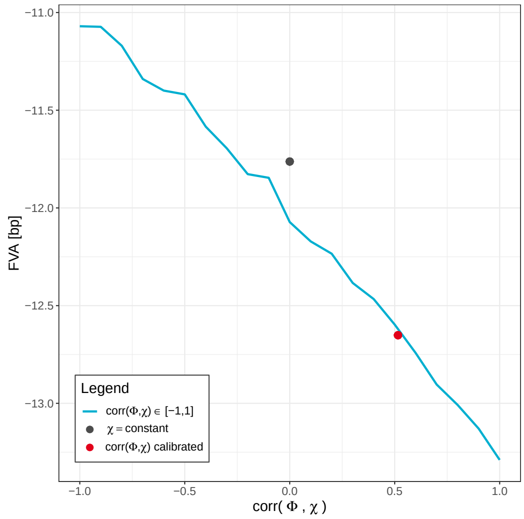
As we were expecting, the correlation between the market risk and the funding component (*i.e.* the Wrong-Way Risk) is not negligible. In fact, we have observed an impact of 289% and 18% in the first and the second cases respectively.
It seems that the correlation with the discount factor is dominant, and it guides the FVA value. Instead, the correlation with the forward spread gives a slight adjustment, which is more sensible to the random effect of the simulation procedure.
It seem that, the result confirms our hypothesis of uncorrelation between the idiosyncratic and systemic components.

We have also computed the FVA obtained by the current methodologies, *i.e.* by imposing a constant funding idiosyncratic component. In particular, we have set its volatility (σ_χ) to zero. This leads to an FVA value (grey bullet) equal to $-11.7\mathrm{bp}$, which compared to the calibrated value (red bullet), introduces an absolute error of 8%.
This seems a reasonable approximation, since the correlation with the discount factor is very small. Clearly, it would not represent the real FVA anymore if this correlation will change.

We also report another case to compare the results. In particular, we have followed the same procedure, but we have used EURIBOR3M as the discount curve. This is actually a simplified case because in this model framework the forward spread is zero, and thus also the funding systemic component.

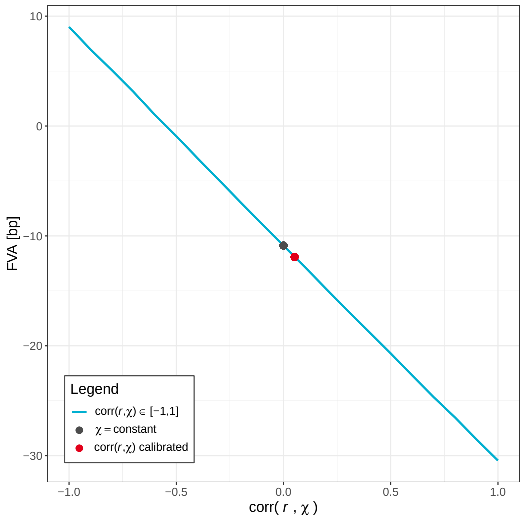


(a) Correlation between discount and funding idiosyncratic component.

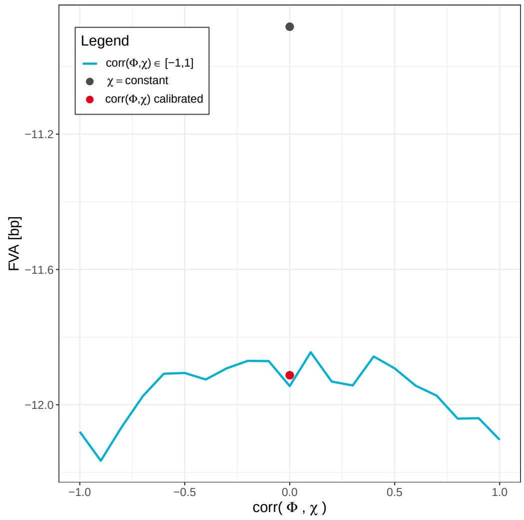


(b) Correlation between forward spread and funding idiosyncratic component.

Figure 6.5.: Variation of the FVA value caused by a variation in the correlation structure.



(a) Correlation between discount and funding idiosyncratic component.



(b) Correlation between forward spread and funding idiosyncratic component.

Figure 6.6.: Variation of the FVA value caused by a variation in the correlation structure (EURIBOR3M as discount curve).

Figure 6.6 shows the results, and it is observable the same behaviour of the previous case. As we have mentioned, the forward spread is zero; in fact, we have not obtained any effect in fig. 6.6b. The reason why the FVA value is not constant is due to the random component of the simulation, which introduces a fluctuation of 1.7%. So, thanks to this simplified case, an estimation of the simulation error is about 1-2%.
Thanks to this comparison, we get a second confirmation of our hypothesis, and we notice that the correlation between the discount factor and the funding idiosyncratic component drives the Wrong-Way Risk. In fact, it generates the greatest impact.

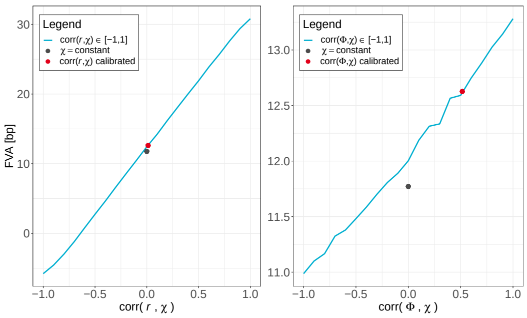
To conclude the analysis, we also report the obtained results for the In-The-Money (ITM), Out-of-The-Money (OTM), payer and receiver cases. In this way, we have explored all possibilities, and we could better understand the FVA and the WWR importance. In particular, we have added a perturbation of ±1% at the ATM fixed rate. Then, we have considered all possible combinations: ATM payer, ATM receiver, ITM payer, ITM receiver, OTM payer and OTM receiver.
By following the same procedure described above, we have obtained the results reported in table 6.3 and the graphical comparison of fig. 6.7. Clearly, we have used the general curve framework: ESTER, EURIBOR3M-ESTER and BOND_BBB.

	Fixed rate	**Price (€)**	**FVA [min, MAX]**	**Variation**	**Impact [min, MAX]**	**Figure**
ATM payer	-0.0027	0.00	[-6, 31] bp	289%	-	6.7a
ATM receiver	-0.0027	0.00	[-31, 6] bp	289%	-	6.7b
ITM payer	-0.0127	0.1043	[48, 83] bp	53%	[5%, 8%]	6.7c
ITM receiver	0.0073	0.1043	[21, 60] bp	97%	[2%, 6%]	6.7d
OTM payer	0.0073	-0.1043	[-60, -22] bp	95%	[2%, 6%]	6.7e
OTM receiver	-0.0127	-0.1043	[-84, -48] bp	55%	[5%, 8%]	6.7f

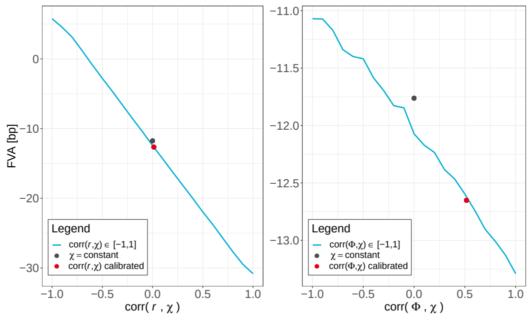
$$Variation = |(MAX(FVA) - min(FVA))/FVA_{cal}|$$
$$Impact[min, MAX] = [min(FVA)/Price, MAX(FVA)/Price]$$

Table 6.3.: FVA comparison between different IRS. Where min and MAX are computed by varying the correlation structure, and FVA_{cal} refers to the FVA with the calibrated correlation.

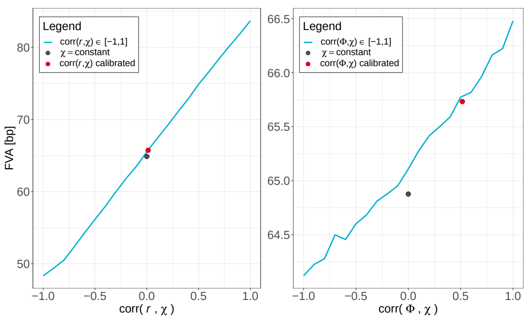
Payer and receiver types generate different reactions. It seems that, the FVA of the payer IRS has a positive linear relation with the correlation. While the receiver one shows a possible negative linear relation.
Actually, against the same positive variation in the correlation structure, the payer IRS increases its FVA, whereas the receiver one decreases it. We talk about correlation structure because the reaction behaviour is the same for both $c_{r,\chi}$ and $c_{\Phi,\chi}$. The different reaction of payer and receiver option is intuitive, since they have opposite sign. Instead, the linear relation is a consequence of the FVA formula used (eq. (5.1)) and the IRS computation method (see appendix F). In particular, the IRS value has a linear relation with the discount factor and the forward spread. And, since it is multiplied to the Funding Spread, this linear relation is inherited.



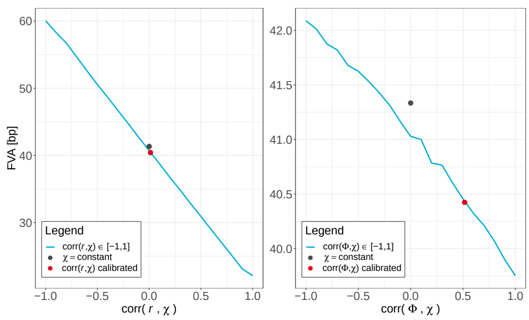
(a) IRS ATM payer.



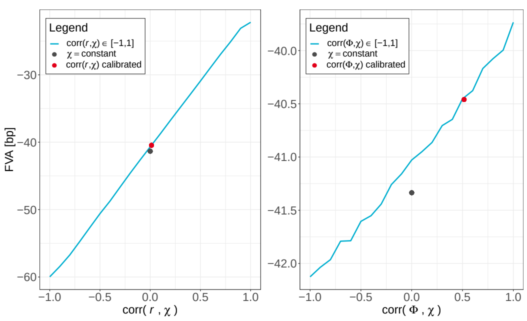
(b) IRS ATM receiver.



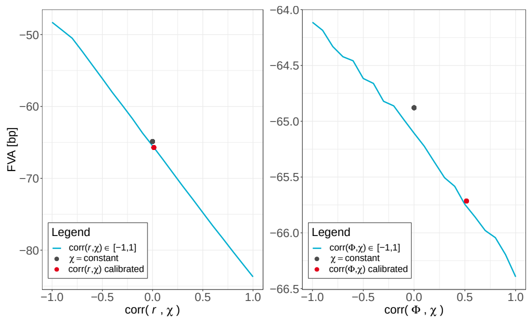
(c) IRS ITM payer.



(d) IRS ITM receiver.



(e) IRS OTM payer.



(f) IRS OTM receiver.

Figure 6.7.: FVA comparison among different IRS.

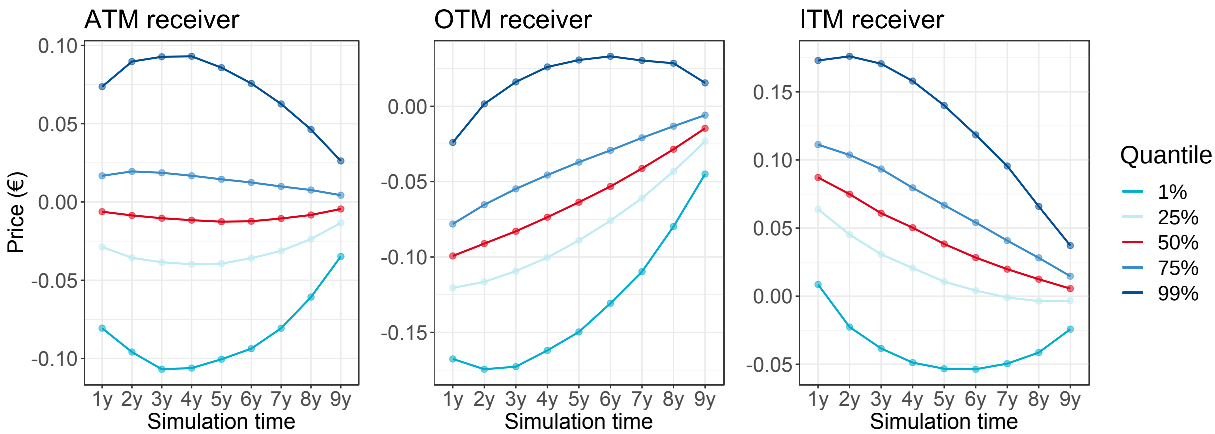


Figure 6.8.: Simulated quantiles for ATM, INT and OTM Interest Rate Swap.

The second remark is that, the Funding Value Adjustment is coherent with the Interest Rate Swap price. In particular, when the IRS is In-The-Money, the FVA is positive and vice-versa. This means that, the ITM IRS price has more probability to remain positive in the future scenarios. Therefore, the FVA catches this feature and generates a funding benefit. This is an intuitive consequence: since we have a positive valued object, the market expectation is that it will generate a gain, therefore it will also generate a funding benefit.
We show a second proof of that in fig. 6.8, where we report the simulated quantile for the ATM, ITM and OTM IR Swap receiver. As we said, in the majority of the simulated scenarios, the IRS preserves its moneyness. Furthermore, the tail events concerning the moneyness changing are also included.
Mathematically, eq. (5.1) explains the property of the FVA sign. In particular, since the IRS expectation value is integrated without a sign change, and since the integral is a linear operator, the FVA sign must remain the same.

As we expect, the more the fixed rate is far from the ATM value, the higher is the FVA. Indeed, ATM trades tend to compensate in and out cashflows.
Moreover, the FVA variations are about 35bp in absolute value, and this is due to the correlation effect. The sign depends on the moneyness of the trade, but the variation is nearly the same. Therefore, we have proven that, the Wrong-Way Risk has an impact of 35bp in the considered IRS.

In the whole examined cases, we observe a small contribution of the forward spread. This is a natural consequence of its meaning since EURIBOR3M represents the average unsecured funds rate offered by the bank, which is common for all financial institutions. So, it could be associated to a systemic component. Therefore, this result is expected and confirms our hypothesis: the funding spread model adopted is actually an idiosyncratic/systemic division.
The WWR improvement, due to forward spread, is about of 18% compared to the FVA; so it has the order of 2-3bp. For this reason, we conclude that its contribution

(in the proposed Wrong-Way Risk analysis) is negligible.
Mathematically, the explanation comes from the model adopted. The spot forward spread is too low, and the calibrated volatility is too small to generate rates comparable to those of the discount. In fact, they have one order of magnitude of difference. Therefore, although the correlation is changed, and it has generated different simulated rates, the impact on the FVA remains too small compared to that obtained by varying the discount.

We have again compared the results with the approximation currently adopted, *i.e.* constant funding spread (grey bullet), and we have reported the results in table 6.4.
Even in the perturbed cases, the approximation is still reliable, since the calibrated correlation has not changed. It would seem that imposing the idiosyncratic component constant does not cause an appreciable error. Nevertheless, we believe that it is wrong to adopt such a strong assumption which would produce a dramatic consequences in response to a shift in the rates correlation.
Moreover, this result is drastically sensible to the calibrated correlation and to the approach used to obtain it. Thus, the sampling method of the historical curves could influence the result too. And since there exists different reasonable procedures, we believe that the comparison with the approximated value is secondary for this analysis. The impact of the correlation on the FVA is clearly more relevant.

	FVA Σ **calibrated**	**FVA** χ **constant**	**Error**	**Figure**
ATM payer	12.7 bp	11.6 bp	9%	6.7a
ATM receiver	-12.7 bp	-11.7 bp	8%	6.7b
ITM payer	65.8 bp	64.7 bp	2%	6.7c
ITM receiver	40.3 bp	41.4 bp	3%	6.7d
OTM payer	-40.4 bp	-41.4 bp	3%	6.7e
OTM receiver	-65.7 bp	-64.7 bp	2%	6.7f

Table 6.4.: FVA model comparison between different IRS.

Thanks to this study, we also observe the impact of the FVA value. From the results that we obtained, we have an average FVA of about 5% in comparison to the IRS price. Therefore, according to the literature [Gre15a; Gre15b; Rui13], we confirm the Funding Value Adjustment importance.
Moreover, we observe an oscillation of approximately ±35bp in the FVA, due to the Wrong-Way Risk effect. Hence, it is a non negligible contribute, since it has the same order of magnitude of the FVA.
Finally, we have proven our thesis: the Funding Value Adjustment is a fundamental price component, which must be included during the pricing process together with the Wrong-Way Risk.

7. Conclusions

In this thesis, we have presented a thorough analysis of the Funding Value Adjustment. In particular, we have described the historical context in which the adjustments were introduced for the first time. We have also mentioned the debate about the use of FVA, which is still open, and we have introduced the main elements to model it.
Then we have performed a review of the literature to highlight the models and the methodologies behind the Funding Value Adjustment in order to clarify its meaning. Particular attention was paid to show the importance of its use. In our opinion, it is a necessary price component which can not be excluded, and the reason why the debate is still open follows from a misleading definition.
We have defined the Wrong-Way Risk, and we have introduced two different approaches to estimate its impact. Especially, the first one quantifies the implicit error that a deterministic model produces; while the second one is our personal proposal which is used in the numerical example.

We have removed the independence assumption between funding rate and market risk factors, and we have proved the importance of the WWR component through a case study.
By considering a stochastic funding rate and two different market risks, we have computed the FVA for an Interest Rate Swap ATM. We have studied all possible scenarios to conclude that $\pm 35bp$ is the Wrong-Way Risk contribute, which compared to the FVA value, gives a possible 290% add-on.
Therefore, we have obtained the expected conclusion: the correlation between the market risk factors and the funding component must be considered because it sensibly influences the result.

Moreover, we have observed the dependency of the results on the rate models and the market curves selected. Nevertheless, if we would have chosen them differently, the conclusions would have been the same. Indeed, we have achieved this significant impact by using simple rate models and real market curves.

Finally, we have observed a low level of historical correlation which leads a slight adjustment with respect to the current FVA methodologies (constant funding spread). Especially, we have obtained $-12.7bp$ against $-11.7bp$ of the current practice. Hence, we accept the independence approximation, but within the considered market condition and the adopted calibration procedure. We prefer a more reliable approach because of the uncertainty of the market; if the rates diverged (*e.g.* during the 2007/2008 crisis, chapter 2), their correlation would change, and then the FVA would modify accordingly up to a possible ±35bp. Therefore, a correct Wrong-Way Risk estimation and management is necessary to incorporate in the final price all the elements. This is becoming standard practice for CVA modelling; so we believe it should be the same for FVA.

A. Glossary

Basel Committee on Banking Supervision (BCBS): it is the world's leading standards-setting body for the prudential regulation of banks.
Basis Swaption: it is a Swaption indexed on a Basis Swap, which is a IRS with two floating legs. The Basis Swap permits to each parts to exchange floating rate for floating rate.
CDS (Credit Default Swap): it is a swap that has the function of transferring credit risk. The insured part has to pay a fixed amount till the maturity of the contract or the default of the company underlying. If the default happens, the counterparty has to pay to the insured the loss rate multiplied by the nominal amount (nominal value*(1-recovery rate)).
Collateralized transaction: it is a transaction exposed to a credit risk, in which the counterparty, or a third party on behalf of the counterparty, poses a collateral to hedge it.
Constant Maturity Swap (CMS): it is a Swap in which the purchaser could fix the duration of received flows.
CSA (Credit Support Annex): it is a legal document that governs the credit support for derivative transactions, in particular, it regulates the management of collateral.
Derivative: it indicates a financial contract which derives its value from another financial asset or from an index, called the underlying.
EONIA (Euro OverNight Index Average): it represents the weighted average of the overnight rates of all unsecured financing transactions in the interbank market by the main European banks.
Expected Exposure: it is the mean of the distribution of exposures. An exposure represents the possible loss in a specific market scenario and a future date.
ESTER (Euro Short Term Rate): it represents the new interest rate on short-term loans, and it was born to substitute the EONIA rate. It is an overnight and unsecured rate which will complement the existing reference rates produced by the private sector. The difference from the oldest benchmark rates is that it is based on actual transactions and not on surveys.
EURIBOR: it is a reference rate, daily computed, which indicates the average interest rate of financial transactions in euro between the main European banks.
It acts as a forum for the banks which participate, and as interlocutor with European regulators for the legislations aspects.
European Central Bank (ECB): it is the central bank responsible for the implementation of the monetary policy for the 19 countries of the European Union that joined the single currency (forming the so-called Euro-zone), as well as for

the supervision of the credit institutions. Its tasks are: to define and implement monetary policy for the euro area, managing foreign exchange operations, maintaining the payment system, issuing banknotes and maintaining financial stability with a prudential policy.
Forward rate agreement (FRA) rate: it is an OTC contract between parties which specifies the rate of interest to be paid on an agreed maturity date in the future. The notional amount is not exchanged, but rather a cash amount given by the rate differentials and the notional.
Hedging: it is a strategy aimed at protecting an investment from possible unexpected events such as currency or price fluctuations.
LIBOR (London Interbank Offered Rate): it is the European reference rate which is used by the banks to lend money to each other. It is less than the discount rate that lenders pay for a loan to the central bank. Libor can be used as an index of the short-term cost of money which is commonly used as a basis for calculating interest rates on many financial transactions.
Mark-to-market (MTM): it is the expression used to qualify the valuation method according to which the value of a financial instrument or contract is systematically adjusted according to the current market prices.
Markit: IHS Markit Ltd is a global information provider based in London.
Martingale: a stochastic process X is said to be martingale under a probability measure $\mathbb{Q}$ if its expected value at time T, given the filtration at time t, is equal to the value of the process at time t for any $0 < t < T$; mathematically: $\mathbb{E}_{\mathbb{Q}}\left[X(T)\middle|\mathcal{F}_t\right] = X(t), \forall 0 < t < T$.
Netting set: it is a collection of transactions, derivatives or other financial instruments that could be grouped to simplify real cash flows; *e.g.* if a bank has two open positions with the same counterparty in which it has to pay something in the first one and receive something else in the second one, the only cash movement is the sum with the sign of the two transactions, in favour to the counterparty who has to receive the greatest amount.
Numeraire: it is the unit of measure to quantify the value and it is used to normalized the price of different objects, given a common standard. It is fundamental in pricing theory to obtain the fair value.
OIS (Overnight Indexed Swap): it is an agreement between two parties who undertake to exchange, for a certain predefined period, a series of daily payments at a variable rate, in return for a fixed rate (OIS).
OTC (Over The Counter) contract: it is a contract between two parties with minimal intermediation or regulation; it does not have standardized terms and it is not quoted on an exchange.
Plain vanilla option: it is a contract between two parties which gives to the holder the right to buy or sell at maturity time the underlying.
Recovery rate: it is the percentage of debt repaid after the default of the company associated to it.
REPO: the repurchase agreement (REPO) is a loan guaranteed by a bond. The counterparty which receives the cash has to give as a guarantee a bond. At matu-

rity, the loan is repaid and the bond returned, if the counterparty defaults before the maturity, the creditor could keep the bond.
Spread Options: it is a derivative where its payoff is based on the price difference of the two underlying assets.
Uncollateralized (Unmargined) transaction: it is a transaction with no-collateral associated to it to cover the credit risk.

B. Relation between REPO rate and CDS spread

In this section, we investigate the relation between the short term REPO rate $r^B(t)$ and the CDS spread h_t by following the approach proposed by Garcıa Muñozl in [Gar13].

First of all, we consider two different strategies:

- to sell a CDS contract at time t with maturity in $t + dt$ and an at-par rate h_t;
- to buy at time t a bond with maturity in $t + dt$, throght a REPO agreement maturing in $t + dt$.

Naturally, the CDS underlying must be the same bond used for the REPO transaction.
The at-par rate is the rate for which a CDS has zero value at time t, therefore the first strategy has zero cost. We highlight that, what we model in section 3.1 is the CDS spread, *i.e.* the at-par rate.
A REPO agreement is a loan guaranteed by a bond. The counterparty which receives the cash has to give as a guarantee a bond. At maturity, the loan is repaid, the bond is returned, but the creditor could keep the bond if the counterparty default before the maturity. In the second strategy we enter in a REPO agreement by posting as collateral the bond bought with the REPO loan. In this way, the cost of the second strategy is zero.
At time $t + dt$ the CDS seller receives the premium $h_t d_t$ and, in case of default of the underlined company, he/she has to pay $(1 - R)$; where R identifies its recovery rate. On the other hand, in the second strategy we have to pay back the REPO loan $1 + r^B(t)dt$ and we receive the interest on the bond $1 + \Omega^C(t)dt$ in case of no-default, or the recovery R if the default happened.
Therefore, following the no-arbitrage principle, the two strategies must have the same value in $t + dt$, therefore

$$\begin{aligned} h_t dt - (1 - R)\mathbb{1}_{\tau \le t+dt} &= (1 + \Omega^C(t)dt)\mathbb{1}_{\tau > t+dt} + R\mathbb{1}_{\tau \le t+dt} - (1 + r^B(t)dt) \\ &= (1 + \Omega^C(t)dt) - (1 + r^B(t)dt) + (1 - R + \Omega^C(t)dt)\mathbb{1}_{\tau \le t+dt} \\ &= (\Omega^C(t) - r^B(t))dt - (1 - R)\mathbb{1}_{\tau \le t+dt}. \end{aligned} \tag{B.1}$$

So we obtain the relation

$$h_t = \Omega^C(t) - r^B(t). \tag{B.2}$$

C. Long Term Debt PDE

In this section, we derive the partial differential equation (PDE) which a bond $\bar{D}(t, T)$ has to satisfy. In doing so, we will follow the approach described by [Gar13].

First of all, we consider a credit derivatives E_t which depends on the short term credit default swap h_t, whose dynamics is

$$dh_t = \mu_t^{\mathbb{P}} dt + \sigma_t dW_t^{\mathbb{P}}, \tag{C.1}$$

where $W_t^{\mathbb{P}}$ is a Brownian process under the real measure $\mathbb{P}$. E_t also depends on the default of the counterparty, *i.e.* it depends on $N_t^{\mathbb{P}} = \mathbb{1}_{\tau \leq t}$, where τ is the default time. Therefore, by applying Itô Lemma for jump diffusion process, we have

$$dE_t = \frac{\partial E_t}{\partial t} dt + \frac{\partial E_t}{\partial h_t} dh_t + \frac{1}{2} \sigma_t^2 \frac{\partial^2 E_t}{\partial h_t^2} dt + \Delta E_t dN_t^{\mathbb{P}}, \tag{C.2}$$

where ΔE_t represents the change in value of E_t due to a default of the counterparty. In order to hedge E_t, we have to trade two others credit derivatives: a short term at-par CDS with dynamics

$$dCDS(t, t + dt) = h_t dt - (1 - R) dN_t^{\mathbb{P}} \tag{C.3}$$

and a second collateralized credit derivatives H_t which follow the same dynamics of E_t given by (C.2). As usual, we use R to identify the recovery rate.
So, the hedging equation is

$$E_t = \alpha_t H_t + \gamma_t CDS(t, t + dt) + \beta_t, \tag{C.4}$$

where α_t and γ_t are the numbers of credit instrument bought, and β_t is the cash held in collateral accounts. By assuming the two derivatives E_t and H_t collateralized in cash, we have that β_t evolves as

$$d\beta_t = r(t) E_t dt - r(t) \alpha_t H_t dt \tag{C.5}$$

with the OIS rate $r(t)$. Therefore, if we differentiate the equation C.4 and impose the equivalence with the E_t dynamics (eq. (C.2)), we obtain

$$\begin{aligned} &\frac{\partial E_t}{\partial t}dt + \frac{\partial E_t}{\partial h_t}dh_t + \frac{1}{2}\sigma_t^2\frac{\partial^2 E_t}{\partial h_t^2}dt + \Delta E_t dN_t^{\mathbb{P}} = \\ &= \alpha_t\left(\frac{\partial H_t}{\partial t}dt + \frac{\partial H_t}{\partial h_t}dh_t + \frac{1}{2}\sigma_t^2\frac{\partial^2 H_t}{\partial h_t^2}dt + \Delta H_t dN_t^{\mathbb{P}}\right) + \\ &+ \gamma_t\left(h_t dt - (1-R)dN_t^{\mathbb{P}}\right) + r(t)E_t dt - r(t)\alpha_t H_t dt. \end{aligned} \tag{C.6}$$

We impose now the hedging condition, *i.e.* the stochastic terms (dh_t and $dN_t^{\mathbb{P}}$) equal to zero, obtaining

$$\alpha_t = \frac{\frac{\partial E_t}{\partial h_t}}{\frac{\partial H_t}{\partial h_t}} \quad \text{and} \quad \gamma_t = \alpha_t\frac{\Delta H_t}{1-R} - \frac{\Delta E_t}{1-R}. \tag{C.7}$$

By substituting (C.7) in (C.6) after a rearrangement, we obtain

$$\frac{\frac{\partial E_t}{\partial t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 E_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta E_t - r(t)E_t}{\frac{\partial E_t}{\partial h_t}} = \frac{\frac{\partial H_t}{\partial t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 H_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta H_t - r(t)H_t}{\frac{\partial H_t}{\partial h_t}}. \tag{C.8}$$

If we add $\mu_t^{\mathbb{P}}$ and we divide by σ_t both sides of the equation (C.8), we could interpret it as the expected excess return of the derivatives over the collateral rate, divided by the volatility of the derivatives. We call this quantity the Market Price of Credit Risk, and we identify it with $\eta(t, h_t)$. Since E_t and H_t are generic credit derivatives, $\eta(t, h_t) = \eta(t)$ does not depend on their payoffs and

$$\eta(t) = \frac{\frac{\partial E_t}{\partial t} + \mu_t^{\mathbb{P}}\frac{\partial E_t}{\partial h_t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 E_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta E_t - r(t)E_t}{\sigma_t\frac{\partial E_t}{\partial h_t}} \tag{C.9}$$

is true for any credit derivatives.
This means that the PDE which describes the dynamics of any credit derivatives is

$$\frac{\partial E_t}{\partial t} + (\mu_t^{\mathbb{P}} - \sigma_t\eta(t))\frac{\partial E_t}{\partial h_t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 E_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta E_t - r(t)E_t = 0. \tag{C.10}$$

In order to use eq. (C.10) for a defaultable bond in a REPO agreement, we observe that the collateral rate used to remunerate collateral accounts in cash is the REPO rate (instead the OIS rate $r(t)$). Therefore we have

$$\begin{aligned} &\frac{\partial\bar{D}(t,T)}{\partial t} + (\mu_t^{\mathbb{P}} - \sigma_t\eta(t))\frac{\partial\bar{D}(t,T)}{\partial h_t} + \frac{1}{2}\sigma_t^2\frac{\partial^2\bar{D}(t,T)}{\partial h_t^2} + \\ &+ \frac{h_t}{1-R}\Delta\bar{D}(t,T) - r^{B,T}(t)\bar{D}(t,T) = 0. \end{aligned} \tag{C.11}$$

Where $\bar{D}(t,T)$ represents the bond value at time t and $r^{B,T}(t)$ is the short term REPO rate for bond $\bar{D}(t,T)$. Notice that $r^{B,T}(t)$ depends on the bonds maturity, then could be different for a bond maturing in a different time.

D. Solution of the hedging PDE

In this section we present the procedure used by Garcıa Muñoz in [Gar13] to solve the hedging PDE (3.15) reported below.

$$\begin{aligned}\frac{\partial V_t}{\partial t} &+ (\mu_t^H - \sigma_t^H \eta(t)^H)\frac{\partial V_t}{\partial h_t} + \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 V_t}{\partial h_t^2} + (\nu_t - q_t)S_t \frac{\partial V_t}{\partial S_t} + \\ &+ \frac{1}{2}(\sigma_t^S S_t)^2 \frac{\partial^2 V_t}{\partial S_t^2} + \sigma_t^H \sigma_t^S \rho_t^{H,s} S_t \frac{\partial^2 V_t}{\partial S_t \partial h_t} = \Omega^B(t) V_t^+ + \Omega^C(t) V_t^-\end{aligned} \tag{D.1}$$

We recall that V_t is the derivative price at time t with payoff at time T given by $V_T = g(S_T)$. $\eta(t)$ is the market price of the credit risk, h_t is the CDS spread of the hedger and S_t is the derivative underlying. Under the real word probability measure $\mathbb{P}$, their dynamics are

$$dS_t = \mu_t^S S_t dt + \sigma_t^S S_t dW_t^{S,\mathbb{P}} \tag{D.2a}$$

$$dh_t = \mu_t^H dt + \sigma_t^H dW_t^{H,\mathbb{P}}, \tag{D.2b}$$

where μ_t^S, μ_t^H are the real world drifts, σ_t^S, σ_t^H are the real world volatilities and $W_t^{S,\mathbb{P}}$, $W_t^{H,\mathbb{P}}$ are correlated Brownian motion under the real world measure.
We identify this correlation with a time dependent function

$$\rho_t^{S,H} dt = dW_t^{S,\mathbb{P}} dW_t^{H,\mathbb{P}}. \tag{D.3}$$

Last terms are: ν_t, *i.e.* the interest generated by S_t; q_t, *i.e.* the dividend paid by S_t; $\Omega^B(t)$ and $\Omega^C(t)$, *i.e.* the interest rate given by V_t in case of positive or negative value respectively.

Let's define the operator $\mathcal{L}$ to simplify the notation:

$$\begin{aligned}\mathcal{L}V_t =& \frac{\partial V_t}{\partial t} + (\mu_t^H - \sigma_t^H \eta(t)^H)\frac{\partial V_t}{\partial h_t} + (\nu_t - q_t)S_t \frac{\partial V_t}{\partial S_t} + \\ &+ \frac{1}{2}(\sigma_t^H)^2 \frac{\partial^2 V_t}{\partial h_t^2} + \frac{1}{2}(\sigma_t^S S_t)^2 \frac{\partial^2 V_t}{\partial S_t^2} + \sigma_t^H \sigma_t^S \rho_t^{H,s} S_t \frac{\partial^2 V_t}{\partial S_t \partial h_t}.\end{aligned} \tag{D.4}$$

So, we use the measure $\mathbb{Q}$ under which S_t and h_t follow the dynamics[1]

$$dS_t = (\nu_t - q_t)S_t dt + \sigma_t^S S_t dW_t^{S,\mathbb{Q}}, \tag{D.5a}$$

$$dh_t = (\mu_t^H - \eta(t)\sigma_t^H)dt + \sigma_t^H dW_t^{H,\mathbb{Q}}. \tag{D.5b}$$

[1] Thanks to the Girsanov theorem [Bal17], we derive the relation between the two Brownian motion.

After that, we can apply Itô Lemma to compute the dynamics of V_t under the real world measure $\mathbb{Q}$:

$$dV_t = \mathcal{L}V_t dt + \frac{\partial V_t}{\partial S_t}\sigma_t^S S_t dW_t^{S,\mathbb{Q}} + \frac{\partial V_t}{\partial h_t}\sigma_t^H dW_t^{H,\mathbb{Q}}. \tag{D.6}$$

The second step is to define the discounted process X_t as

$$X_t = V_t \exp\left\{-\int_0^t r(s)ds\right\}. \tag{D.7}$$

Then, we derive its dynamics from eq. (D.6).

$$\begin{aligned} dX_t = e^{-\int_0^t r(s)ds}\Big(& - r(t)V_t dt + \mathcal{L}V_t dt + \\ & + \frac{\partial V_t}{\partial S_t}\sigma_t^S S_t dW_t^{S,\mathbb{Q}} + \frac{\partial V_t}{\partial h_t}\sigma_t^H dW_t^{H,\mathbb{Q}}\Big). \end{aligned} \tag{D.8}$$

We rearrange the partial differential equation (D.1)

$$\begin{aligned} \mathcal{L}V_t &= \Omega^B(t)V_t^+ + \Omega^C(t)V_t^- = (\Omega^B(t) - r(t))V_t^+ + (\Omega^C(t) - r(t))V_t^- + r(t)V_t = \\ &= FS^B(t)V_t^+ + FS^C(t)V_t^- + r(t)V_t \end{aligned} \tag{D.9}$$

and we define $FS^B(t) = \Omega^B(t) - r(t)$ as the funding benefit spread. Moreover, we identify the funding cost spread with $FS^C(t) = \Omega^C(t) - r(t)$.
In this way we can substitute $\mathcal{L}V_t$ in (D.8) to obtain

$$\begin{aligned} dX_t =& e^{-\int_0^t r(s)ds}\Big(FS^B(t)V_t^+ dt + FS^C(t)V_t^- dt + \\ & + \frac{\partial V_t}{\partial S_t}\sigma_t^S S_t dW_t^{S,\mathbb{Q}} + \frac{\partial V_t}{\partial h_t}\sigma_t^H dW_t^{H,\mathbb{Q}}\Big). \end{aligned} \tag{D.10}$$

Now we integrate dX_t between t and T

$$\begin{aligned} V_T e^{-\int_t^T r(s)ds} - V_t = X_T - X_t = \int_t^T dX_s = \\ = \int_t^T e^{-\int_h^s r(h)dh}\Big((FS^B(s)V_s^+ + FS^C(s)V_s^-)\Big)ds + \\ + \int_t^T e^{-\int_h^s r(h)dh}\left(\frac{\partial V_s}{\partial S_s}\sigma_s^S S_s dW_s^{S,\mathbb{Q}} + \frac{\partial V_s}{\partial h_s}\sigma_s^H dW_s^{H,\mathbb{Q}}\right) \end{aligned} \tag{D.11}$$

and we take the conditional expectation under the filtration $\mathcal{F}_t$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[V_T e^{-\int_t^T r(s)ds}|\mathcal{F}_t - V_t = \mathbb{E}^{\mathbb{Q}}\Bigg[\int_t^T e^{-\int_h^s r(h)dh}\Big((FS^B(s)V_s^+ + \\ + FS^C(s)V_s^-)ds + \frac{\partial V_s}{\partial S_s}\sigma_s^S S_s dW_s^{S,\mathbb{Q}} + \frac{\partial V_s}{\partial h_s}\sigma_s^H dW_s^{H,\mathbb{Q}}\Big)\Big|\mathcal{F}_t\Bigg]. \end{aligned} \tag{D.12}$$

Observe that, the last two left terms are Itô integral. Therefore they have zero mean, so (D.12) becomes

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[V_T e^{-\int_t^T r(s)ds} | \mathcal{F}_t\right] + \\ - \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_h^s r(h)dh}\Big((FS^B(s)V_s^+ + FS^C(s)V_s^-)ds\Big)\Big|\mathcal{F}_t\right], \tag{D.13}$$

that is the value of the derivative we searched.

E. Delta-Gamma approximation procedure

In this chapter we present the Delta-Gamma approach, used in section 4.2, by following the procedure proposed in [Mon14].

The idea of the Delta-Gamma method is to approximate a random variable V with a second order polynomial in N random variable $x = \{x_i\}_{i=1}^{N} - 1$. Assume that V is function of $M > N$ Gaussian vector $\tilde{x}$, *i.e.* $V = f(\tilde{x})$, where $\tilde{x}_i = x_i \; \forall i < N$. We also assume that we could compute $\mathbb{E}[V] = \mathbb{E}[f(\tilde{x})]$ after we change the mean μ of $\tilde{x}$ by an amount $\tilde{\varepsilon} \in \mathbb{R}^M$.
Now we decompose $\tilde{x}$ as:

$$\begin{aligned} \tilde{x}_i &= \mu_i + \eta_i, i < N \\ \tilde{x}_i &= \theta'_{i-N} x + z_{i-N}, i \geq N, \end{aligned} \tag{E.1}$$

where $\eta = \{\eta_i\}_{i=0}^{N-1}$ is a N-dimensional Gaussian vector with zero mean, $\theta_i \in \mathbb{R}^N$ $\forall i = 1, ..., M - N$ and $z = \{z_i\}_{i=0}^{M-N}$ is an $(M - N)$-dimensional Gaussian vector orthogonal to x.
So, we can divide the contribution into V of the two components x and z

$$V = f(\tilde{x}) = g(x, z), \tag{E.2a}$$
$$g(x, z) := f(\tilde{x}(x, z)), \tag{E.2b}$$
$$\tilde{x}_i(x, z) = \tilde{x}_i, i < N, \tag{E.2c}$$
$$\tilde{x}_i(x, z) = \theta'_{i-N} x + z_{i-N}, i \geq N. \tag{E.2d}$$

Then, by using the centered Taylor approximation, we obtain

$$g(x, z) \approx a + b'(x - d) + \frac{1}{2}\left((x - d)'D(x - d) + e\right), \tag{E.3}$$

where $D \in \mathbb{R}^{N \times N}$, $a := \mathbb{E}[V]$, $d := \mathbb{E}[x]$ and $e := \mathbb{E}[(x - d)'D(x - d)] = \sum_{i,j=0}^{N-1} D_{ij}\mathbb{E}[\eta_i \eta_j]$.
We first observe that the approximation of $g(x, z)$ preserves its mean (a), and secondly that if eq. (E.3) was an exact equivalence, b_i and D_{ij} would satisfy

$$\begin{aligned} \frac{\partial \mathbb{E}[g(x, z)]}{\partial \mu_i} = \mathbb{E}\left[\frac{\partial g(x, z)}{\partial x_i}\right] = \mathbb{E}\left[b_i + \sum_{j=0}^{N-1} D_{ij}(x_j - d_j)\right] \\ \frac{\partial^2 \mathbb{E}[g(x, z)]}{\partial \mu_i \partial \mu_j} = \mathbb{E}\left[\frac{\partial^2 g(x, z)}{\partial x_i \partial x_j}\right] = D_{ij}. \end{aligned} \tag{E.4}$$

Observe that, the parameters a, d and e are not stochastic, and they do not change if we have a variation in μ. Therefore, Moni proposes in [Mon14] to set b_i and D_{ij} according to eq. (E.4), to obtain the approximation (E.3).
Since we have assumed that we could compute $\mathbb{E}[f(\tilde{x})]$ after a changing in μ by an arbitrary value $\tilde{\epsilon}$, we could also compute $\mathbb{E}[g(x, z)]$ after a changing in μ_i of ϵ_i (thanks to eq. (E.2b)).
In particular, we change μ_i by ϵ_i $\forall i < N$, and by $\theta_i' \epsilon$ for all other $i \in [N, M]$. Where $\epsilon = \{\epsilon_i\}_{i=0}^{N-1}$. In this way, we satisfy eq. (E.2d).

Finally, we observe that, if we estimate b and D in this way, we do not need to know d and e; so this is a useful method to approximate a random variable with a second order polynomial.

F. Interest Rate Swap pricing

In the following chapter, we present the pricing procedure used to price an Interest Rate Swap (IRS) in a multi curve world.

An IRS is an interest rate derivatives (OTC contract), which consists in the exchange of cashflows between two counterparties. The payer counterpart has to pay a fixed rate R and he/she receives a floating rate $r(t)$ (*e.g.* EURIBOR) each time specified by the contract. Instead, the receiver counterpart has to pay the fixed rate and he/she receives the floating one.

We call $T_{fix} = \{t_i\}_{i=1}^{N}$ and $T_{float} = \{t_j\}_{j=1}^{M}$ the sets of times in which the fixed and the floating rate are respectively payed. Then, we observe that the two time sets could be different, *i.e.* $N \neq M$; *e.g.* in chapter 6 we consider an IRS with fixed payment spaced by one year each and a floating payment spaced by three months each.

Since we assume a multi curve framework, we have two different curves for the discount factor and the forward rate. Therefore, let us define

- $z(t)$ the zero rate corresponding to the discount factor with maturity t, *i.e.* $D(0,t) = e^{-t(t)t}$;
- $F(s,t)$ the forward rate maturing between s and t with the associating zero rate curve $z_F(t)$, *i.e.* $F(s,t) = \frac{z_F(t)t - z_F(s)s}{t-s}$.

In fig. F.1 there is a simple representation of a payer IRS.

Thanks to the no-arbitrage principle, we have that the price of a IRS is the sum of its discounted cashflows, *i.e.* its net present value (NPV).
By considering the payer part, the NPV will be the difference between the fixed leg and the floating one. Actually, this difference is computed on their NPV, *i.e.*

$$NPV_{IRS}^{payer} = NPV_{float} - NPV_{fix}, \tag{F.1}$$

$$\begin{aligned} NPV_{fix} &= \sum_{i=1}^{N} D(0,t_i) R \delta_{i-1,i}^{fix}, \\ NPV_{float} &= \sum_{j=1}^{M} D(0,t_j) F(t_{j-1},t_j) \delta_{j-1,j}^{float}. \end{aligned} \tag{F.2}$$

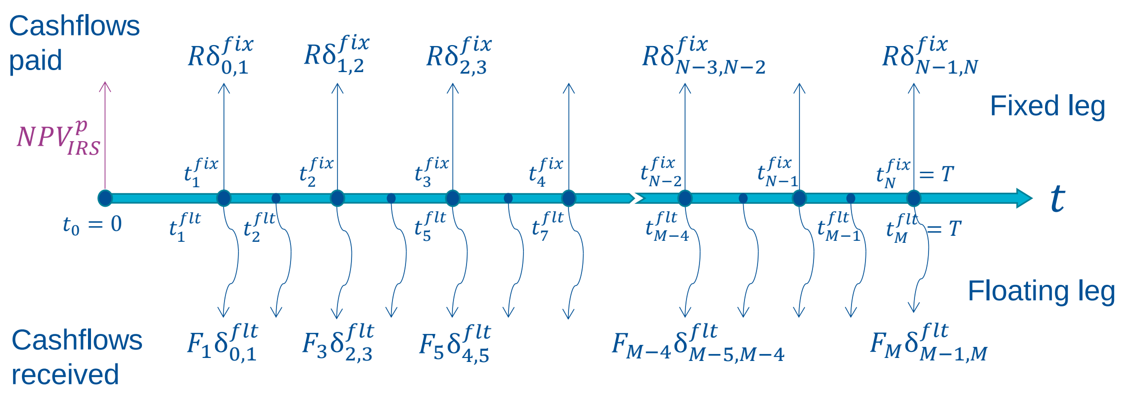


Figure F.1.: Representation of a payer Interest Rate Swap.

Where we assume $t_0 = 0$, and we call $\delta_{i-1,i}^{fix}$ ($\delta_{i-1,i}^{float}$) the year fraction between t_{i-1} and t_i computed using the fixed (floating) leg convention described in the contract.

Naturally, by considering the receiver part, the IRS price is

$$NPV_{IRS}^{receiver} = NPV_{fix} - NPV_{float}. \tag{F.3}$$

We point out that, an IRS is said to be at-the-money (ATM) when its price is zero ($NPV_{fix} = NPV_{float}$). Therefore, we could compute its ATM rate as

$$R_{ATM} = \frac{\sum_{j=1}^{M} D(0,t_j)F(t_{j-1},t_j)\delta_{j-1,j}^{float}}{\sum_{i=1}^{N} D(0,t_i)\delta_{i-1,i}^{fix}}. \tag{F.4}$$

Moreover, we observe that, in a single curve framework ($z(t) = z_F(t)\ \forall t$), we could use the telescopic sum simplification to compute NPV_{float}, *i.e.*

$$NPV_{float} = 1 - D(0,t_M). \tag{F.5}$$

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