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Executive Summary of the Thesis

Characterization of insurance premium principles: the extension to dynamic risk models

Laurea Magistrale in Mathematical Engineering - Ingegneria Matematica

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1. Introduction

One of the most relevant topic in actuarial literature is the definition of the premium principles, namely rules that assign a price to an insurance risk. The thesis aims to extend the treatment already existent for the static context to the dynamic one, considering thus the loss of an insurance portfolio as a stochastic process. Therefore, in the first part of the work, risks are modeled as single random variables, and some premium principles are shown with their properties (developing the notions contained in [5]). Then, there is a theoretical part in which many mathematical tools are reviewed, with a particular care with respect to point processes and marked point processes (see [1] for details), in order to have the instruments for approaching the dynamic risk models. Eventually, the Cramér-Lundberg model and a risk model with the loss distributed as a Compound Hawkes are discussed, and the premium principles presented in the first section are discussed again with a study of their properties in this context. This last part represents the most original contribute of the thesis to the literature, since it provides a new treatment and a theoretical basis to the concept of premium rate.

2. Premiums in static context

Let $(\Omega, \mathcal{F}, P)$ be a probability space in which Ω is the set of states of the world or possible outcomes, $\mathcal{F} \subseteq 2^{\Omega}$ is a σ-algebra that is the collection of events in Ω and P is a probability measure. Let χ be the set of non-negative random variables measurable in the probability space defined before, a premium principle is defined as a functional from the set of insurances to the set of real non-negative numbers, therefore:

$$H : \chi \to [0, +\infty)$$

2.1. Catalog and properties of the premium principles

In this framework, the desired properties for a "good" premium principle are:

1. Conditional state dependence: For a given market condition, the premium for a risk X depends only on its ddf.
2. Monotonicity: Let X and Y be in χ, if $X(\omega) \leq Y(\omega)$ a.s., then $H(x) \leq H(y)$
3. Comonotonic additivity: If X and Y are in χ and comonotonic,then: $H(X+Y) = H(X) + H(Y)$
4. Continuity: If $X \in \chi$ and $d \geq 0$, then: $\lim_{d\to 0+} H((X-d)_+) = H(X)$ and $\lim_{d\to\infty} H(\min(X,d)) = H(X)$

5. Risk loading: $H(X) \geq E[X]$ for all $X \in \chi$.
6. No unjustified risk loading: if $X = c$, $X \in \chi$ and c constant, then $H(X) = c$.
7. Maximal loss: $H(X) \leq esssup(X)$ for all $X \in \chi$
8. Translation equivariance or invariance: $H(X + a) = H(X) + a$ for all $X \in \chi$ and $a \geq 0$.
9. Scale equivariance or invariance: $H(bX) = bH(X)$ for all $X \in \chi$ and $b \geq 0$.
10. Additivity: $H(X+Y) = H(X)+H(Y)$ for all $X, Y \in \chi$.
11. Subadditivity: $H(X+Y) \leq H(X)+H(Y)$ for all $X, Y \in \chi$.
12. Superadditivity: $H(X + Y) \geq H(X) + H(Y)$ for all $X, Y \in \chi$.
13. Additivity for independent risks:$H(X + Y) = H(X) + H(Y)$ for all $X, Y \in \chi$ independents.
14. Preserves FSD ordering: If $S_X(t) \leq S_Y(t)$ for all $t \geq 0$, then $H(X) \leq H(Y)$.
15. Preserves stop-loss ordering:$E[X - d]_+ \leq E[Y - d]_+$ for all $d \geq 0$, then $H(X) \leq H(Y)$.

The following, instead, are the premium principles discussed:

- Net premium principle: $H(X) = E[X]$;
- Expected value premium principle: $H(X) = (1 + \theta)E[X]$, $\theta > 0$;
- Variance premium principle: $H(X) = E[X] + \alpha Var(X)$, $\alpha > 0$;
- Standard deviation premium principle: $H(X) = E[X] + \alpha\sqrt{Var(X)}$, $\alpha > 0$;
- Wang's premium principle: $H(X) = \int_0^\infty g(S_x(t))dt$;
  In particular, $S_x(t)$ represents the survival function of the risk X, while $g : [0, 1] \to [0, 1]$ is non-decreasing with $g(0) = 0$ and $g(1) = 1$.

For each premium, the properties are proved or denied with a counterexample.

2.2. Decomposition of the premium functional into risk and deviation measures

Another central result for the analysis of the premium principles in the static context, due to Nendel, Riedel and Schmeck (see [4] for further details), is the following:

Theorem 2.1. *H is a premium principle normed and translation invariant $\iff H(X) = R(X) + D(X)$ such that $R(X)$ is a well-defined monetary risk measure and $D(X)$ is a well-defined deviation measure.*

The theorem 2.1 offers a new point of view for the comprehension of the premiums, indeed the variance premium principle and the standard deviation one have the first term in the sum which is a risk measure, and the second, conversely, which is a deviation measure. The net value premium, the expected value one and the Wang's one are instead only monetary risk measures. This is easily proved observing that they enjoy the property of monotonicity, and thus they have deviation part null.

3. Premiums in dynamic context

The risk models provided allow to treat the dynamics of the portfolio surplus of an insurance company, which is subjucted to a loss process, and collects premiums with a rate c over time. The aim of the analysis is to define the quantity c according to the principles of the previous section and show its properties.

3.1. Cramér-Lundberg model

Let consider the Cramér-Lundberg model:

$$R_t = R_0 + ct - \sum_{n=1}^{N_t} Z_n \tag{1}$$

in which:

- $R_0 \geq 0$ represents the starting value of the portfolio.
- $c > 0$ is the premium rate, namely the model assumes that the premium income is continuous over time, and, therefore, proportional in any time interval to the interval length.
- $\sum_{n=1}^{N_t} Z_n$ is a compound Poisson with intensity λ. Since the sequence of $\{Z_n\}$ represent the claims, they must have positive support.

The model assumes as hypothesis that N_t and Z_n are independent $\forall t \geq 0\ \forall n \geq 1$. It is possible to compute the mean and the variance of the loss in (1) with the purpose of extending the premium principles listed before to this model. Thus, observing that defining $L_t = \sum_{n=1}^{N_t} Z_n$, one can have:

$$E[L_t] = \lambda t E[Z]$$
$$Var(L_t) = \lambda t E[Z^2];$$

Knowing these quantities, the estimation of the premium rate c exploiting the premium principles listed follows straightforwardly, indeed, considering for each functional H described above the equation $ct = H(L_t)\ \forall t$, it is possible to have:

- Net premium principle: $c = \lambda E[Z]$;
- Expected value premium principle: $c = (1+\theta)\lambda E[Z]$, $\theta > 0$;
- Variance premium principle: $c = \lambda E[Z] + \alpha\lambda E[Z^2]$, $\alpha > 0$;
- Standard deviation premium principle: $c = \lambda E[Z] + \alpha\sqrt{\frac{\lambda}{t}E[Z^2]}$, $\alpha > 0$;

The study of the properties of the premiums in this context allows to achieve surprising results. Indeed, all the properties valid in the static case keep holding true, and other ones hold in the dynamic case. In particular, it is important the property of monotonicity for the variance and standard deviation premium rates, because, by the result of decomposition of the premium previously explained, they can be now interpreted as monetary risk measures.

3.2. Compound Hawkes risk model

Many of the theoretical concepts presented on the Hawkes processes are developed also in ([3]). An Hawkes process is a point process $\mathcal{F}_t$-adapted such that, defined its conditional intensity function as:

$$\lambda_t^* = \lim_{h\to 0} \frac{E[N_{t+h} - N_t|\mathcal{F}_t]}{h}$$

then, it holds:

$$P(N_{t+h} - N_t = m|\mathcal{F}_t) = \begin{cases} \lambda_t^* h + o(h), m = 1 \\ o(h), m > 1 \\ 1 - \lambda_t^* h + o(h), m = 0 \end{cases}$$

with the conditional intensity function of the form:

$$\lambda_t^* = \lambda + \int_{-\infty}^{t} \mu(t-s)dN_s$$

The treatment is done in particular for the exponentially decaying Hawkes process, namely:

$$\mu(t) = \alpha e^{-\beta t}$$

with $\alpha, \beta > 0$ and $\alpha < \beta$ for guaranteeing non-explosivity.
Eventually, following the procedure contained in ([2]), after some computations, it is known the mean of the process, which is:

$$E[N_t] = \frac{\lambda\beta(-1 + e^{(\alpha-\beta)t} - (\alpha-\beta)t)}{(\alpha-\beta)^2} + \\ + \frac{\lambda(-1 + e^{(\alpha-\beta)t})}{\alpha-\beta} \tag{2}$$

In this framework, it is possible to define the following risk model:

$$R_t = R_0 + ct - \sum_{n=1}^{N_t} Z_n \tag{3}$$

in which:

- R_0 and $c > 0$ have the same meaning in (1)
- $\sum_{n=1}^{N_t} Z_n = L_t$ represents as usual the loss of the portfolio. $\{Z_n\}$ is a sequence of random variable i.i.d with positive support, while N_t is an exponentially decaying Hawkes process.

As in (1) the counting process and the claims are independent. The main difference with the Cramér-Lundberg model is that the Compound Hawkes one has not constant intensity in the arrival of the claims, which represents one of the principal drawbacks of (1), modeling thus possible clustering of events.
The theory on marked point processes states that, under the hypothesis of independence between the counting process and the sequence of random variables:

$$E\left[\sum_{n=1}^{N_t} Z_n\right] = E[N_t]E[Z]$$

Exploiting (2), it is possible to extend the net value and expected value principle to the model

(3), therefore:

$$\begin{aligned} c =&(1+\theta)\bigg(\frac{\lambda\beta(-1+e^{(\alpha-\beta)t}-(\alpha-\beta)t)}{(\alpha-\beta)^2 t}+\\ &+\frac{\lambda(-1+e^{(\alpha-\beta)t})}{(\alpha-\beta)t}\bigg)E[Z]; \end{aligned}$$

The net premium rate is obtained imposing $\theta = 0$. There are two possible choices: the first consists in fixing a time horizon t for covering the risks in $[0, t]$, the second one instead is recomputing the premium rates continuously over time, so updating in t_n $\forall n = 0, ..., N-1$, following the same principle with $0 = t_0 < t_1 < ... < t_n < ... < t_N = t$. Even though (3) represents a much more complicated choice with respect to (1), the study of c in this framework shows that the premium rate has the same properties which enjoys in the Cramér-Lundberg model.

Then, the work deals with the possible error committed in a context which presents clustering, therefore the real dynamics of the portfolio surplus follows (3), but the premium rate is computed following the expected value principle of the model (1). Considered R_t as in (3), and the quantity:

$$\widetilde{R}_t = R_0 + \widetilde{c}t - \sum_{n=1}^{N_t} Z_n$$

where $\widetilde{c} = (1+\theta)\lambda E[Z]$ is as in the compound Poisson case, but the loss is a Compound Hawkes with N_t exponentially decaying Hawkes process with parameters α, β, λ, and defined the error as:

$$\epsilon_t = R_t - \widetilde{R}_t = (c - \widetilde{c})t$$

it is easy to show that the quantity ϵ_t grows up linearly with respect to t in both the case with fixed time horizon and with discrete monitoring. This means that an insurance company which collects premium using as reference the Cramér-Lundberg model, but in a situation in which the loss presents the clustering property as in the Compound Hawkes one, suffers on average an unexpected loss that explodes with $t \longrightarrow +\infty$

Lastly, it is considered the case when the premium rate is defined in order to be proportional to the intensity of the counting process as in the case of the Compound Poisson, even though this does not respect the definition of the net premium principle, thus:

$$c = \lambda_t E[Z]$$

The goal of the treatment is to show that this is however a reasonable choice for defining the premium rate. Let consider the condition for which the insurance company does not lose money on average over time with the rate presented above:

$$E[ct - L_t] \geq 0 \tag{4}$$

With some computations, one can notice that:

$$\begin{aligned} &\lim_{t\longrightarrow 0^+} E[ct - L_t] = 0 \\ &\lim_{t\to+\infty} E[ct - L_t] = -\frac{\lambda}{\alpha-\beta}E[Z] > 0 \end{aligned}$$

Moreover, the study of the derivative of the quantity $E[ct - L_t]$ proves that it is increasingly monotone. This makes (4) respected $\forall t$, thus an insurance company which uses the premium rate based on the intensity λ_t gains money on average.

4. Tables

In the sequel are reported the tables which resume the properties of the principles in the different cases. As one can observe, in the dynamic case the expected value premium principle gains the maximal loss property. The variance and standard deviation one, instead, obtain the monotonicity, which implies also the preserving of the first stochastic dominance and stop-loss orderings, and the maximal loss property, with respect to the static context.

5. Conclusions

The present work achieves two different goals: the first is providing a complete and rigorous treatment of the modeling of premiums in the static case, gathering the existing literature in a single book. The second is extending the concepts of the first part to dynamic risk models, defining with a solid mathematical background the premium rates and showing their properties.

Properties	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Net pp	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Expected value pp	Y	Y	Y	Y	Y	N	N	N	Y	Y	Y	Y	Y	Y	Y
Variance pp	Y	N	N	Y	Y	Y	N	Y	N	N	N	N	Y	N	N
Std pp	Y	N	N	Y	Y	Y	N	Y	Y	N	N	N	N	N	N
Wang pp	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	N	N	Y	Y

Table 1: Properties of the premium principles treated in the static context.

Properties	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Net pp	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Expected value pp	Y	Y	Y	Y	Y	N	Y	N	Y	Y	Y	Y	Y	Y	Y
Variance pp	Y	Y	N	Y	Y	N	Y	Y	N	N	N	N	Y	Y	Y
Std pp	Y	Y	N	Y	Y	N	Y	Y	Y	N	N	N	N	Y	Y

Table 2: Properties of the premium rates in the Cramér-Lundberg model. The net premium and the expected value premium present the same properties also in the Compound Hawkes risk model.

The notions developed can be useful for further works in which classical problems of the actuarial sector are solved considering the risk models with the premium rates studied here.

References

[1] Bremaud. *Point Processes and Queues: Martingale Dynamics*. Springer, 1981.

[2] da Fonseca and Zaatour. Hawkes process: fast calibration, application to trade clustering and diffusive limit. *The Journal of Futures Markets, vol. 34*, (6):548–579, 2013.

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[4] Nendel, Riedel, and Schmeck. A decomposition of general premium principles into risk and deviation. *Insurance: Mathematics and Economics*, (100):193–209, 2021.

[5] Young. Premium principles. *Encyclopedia of Actuarial Science*, 2004.